

Analytic Hierarchy Process

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What is AHP ...

Analytic Hierarchy Process (AHP) ...

is a Multi-Criteria Decision Analysis method developed by Thomas Saaty in 1980. AHP is a method to derive **ratio scales** from paired comparisons. Inputs are obtained from objective measurements and/or from subjective preferences. AHP allows small inconsistencies in subjective assessments. The ratio scales are taken from normalized maximal eigenvectors; the **consistency index** is calculated from the maximal eigenvalue.

Look at a Google Scholar Search for AHP.

Sources¹

1. "How to make a decision: The Analytic Hierarchy Process," Thomas L. Saaty, *Euro. J. Oper. Res.*, No. 48, (1990), pp. 9-26.
2. "The Analytic Hierarchy Process: A Survey of the Method and Its Applications," F. Zahedi, *Interfaces*, Vol. 16, No. 4 (Jul-Aug, 1986), pp. 96-108.
3. "Aggregating individual judgments and priorities with the Analytic Hierarchy Process," E. Forman and K. Peniwati, *Euro. J. Op. Res.*, No. 108 (1998), pp. 165-169.

¹ASU Library access to JStor, Science Direct, and SpringerLink.

Canonical First Example

Choosing Which Car to Buy

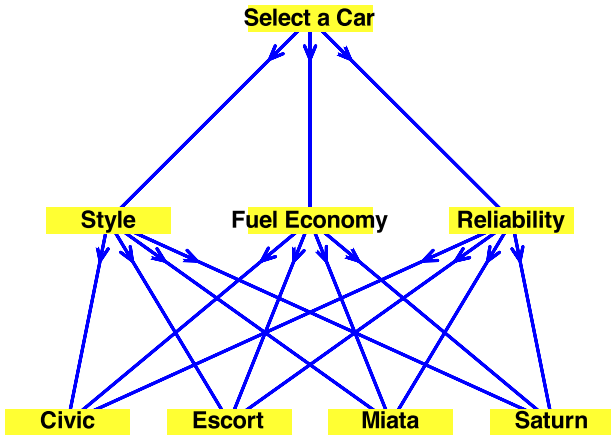
Objective Selecting a car

Criteria Style, Reliability, Fuel Economy

Alternatives Civic Coupe, Saturn Coupe, Ford Escort, Mazda Miata

How to compare the cars? How to choose which one to purchase?

Canonical First Example



Decision Hierarchy

Pairwise Criteria Comparisons

Compare...

Using the scale

Criterion ₁				versus					Criterion ₂
9	7	5	3	1	3	5	7	9	
extreme importance	very strong	strong	moderate	equal	moderate	strong	very strong	extreme importance	

rate the relative importance of the criteria to create a *positive reciprocal priority matrix*.

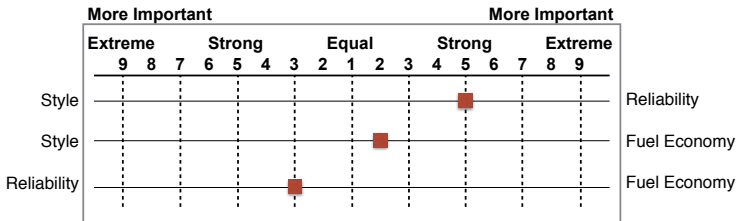
	over	Style	Reliability	Fuel Economy
Style		1	<input type="text"/>	<input type="text"/>
Reliability		—	1	<input type="text"/>
Fuel Economy		—	—	1

Now

1. Find² the dominant eigenvalue λ_m . [▶ Run Maple](#)
2. Find and normalize (so $\sum v_i = 1$) λ_m 's eigenvector.

²For Excel, c.f., the *power method* and the *Rayleigh quotient*.

Pairwise Comparisons Chart



$$\begin{array}{c}
 S \quad R \quad FE \\
 S \quad \begin{bmatrix} 1 & 1/5 & 1/2 \end{bmatrix} \\
 R \quad \begin{bmatrix} 5 & 1 & 3 \end{bmatrix} \\
 FE \quad \begin{bmatrix} 2 & 1/3 & 1 \end{bmatrix}
 \end{array}$$

Pairwise Comparisons Eigenvector

The Criteria Ranking

The normalized eigenvector gives the relative ranking of the criteria.

$$\begin{array}{l} \text{Style} \\ \text{Reliability} \\ \text{Fuel Economy} \end{array} = \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} = ev_{\text{Criteria}}$$

Consistency Index and Consistency Ratio

Measure the consistency of the comparisons with the formulas

$$CI(\lambda_m) = \frac{\lambda_m - n}{n - 1} \quad \text{and} \quad CR = CI(\lambda_m) / RI$$

where RI is from the following table³ listing *random indices*.

n	3	4	5	6	7	8	9	10
Random Matrix Consistency Index	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

A ratio $CR < 0.1$ is good. (*Larger values mean the comparisons need revision.*)

³From *Decision Making for Leaders*, Saaty, RWS Pub, Pittsburgh, 2001.

Pairwise Alternatives Comparisons

Alternatives Comparisons

For each criterion, construct the pairwise comparison matrix

	<i>Style</i>			
<i>over</i>	Civic	Saturn	Escort	Miata
Civic	1	<input type="text"/>	<input type="text"/>	<input type="text"/>
Saturn	—	1	<input type="text"/>	<input type="text"/>
Escort	—	—	1	<input type="text"/>
Miata	—	—	—	1

$$ev_S = [\text{, , , }]$$

	<i>Reliability</i>			
<i>over</i>	Civic	Saturn	Escort	Miata
Civic	1	<input type="text"/>	<input type="text"/>	<input type="text"/>
Saturn	—	1	<input type="text"/>	<input type="text"/>
Escort	—	—	1	<input type="text"/>
Miata	—	—	—	1

$$ev_R = [\text{, , , }]$$

Fuel Economy is an objective value; it needs no comparison matrix.
 Normalize so the sum is 1 and use for its 'eigenvector':

$$[34, 27, 24, 28] \implies ev_{FE} = [0.30, 0.24, 0.21, 0.25]$$

Comparison Matrix

AHP Comparison Matrix

- Form the *comparison matrix* from the eigenvectors

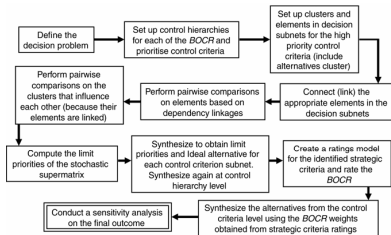
$$CM = [ev_S, ev_R, ev_{FE}] = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

- Multiply the comparison matrix by the eigenvector of priority rankings for the criteria $\vec{R} = CM \cdot ev_{Criteria}$ to produce the *rankings vector* \vec{R}

$$R = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

- To include *cost*:
 - Normalize the cost vector so the sum is 1.
 - Divide respective rankings by cost to generate the *cost-benefit vector*.

AHP Workflow Charts



- Step 1 Create the decision hierarchy by breaking the decision problem into interrelated decision elements
- Step 2 Collect input data with pairwise comparisons of decision elements
- Step 3 Use eigenvalues to estimate the relative weights of decision elements
- Step 4 Aggregate the relative weights of decision elements to rate the decision alternatives

From: T. L. Saaty, "Decision making with the analytic hierarchy process," *Int. J. Services Sciences*, Vol. 1, No. 1, 2008. Figure 2, pg. 94.

From: F. Zahedi, "The Analytic Hierarchy Process: A Survey of the Method and Its Applications," *Interfaces*, Vol. 16, No. 4 (Jul-Aug, 1986), pg. 96.

► [Open Excel Example](#)

Combining Multiple Assessments

Assessments from Multiple Agents: First Method

1. Each individual ranks all pairs of criteria.
2. Average the rankings with a *geometric mean*⁴.
3. Create the positive reciprocal priority matrix from the geometric means of the rankings.
(*Unlike arithmetic means, geometric means preserve reciprocals.*)
4. Follow the same procedure with each individual ranking each of the alternatives.
5. Use the matrices of geometric means in AHP.

⁴See The HM-GM-AM-QM Inequalities by P. Gwanyama

Combining Multiple Assessments, II

Assessments from Multiple Agents: Second Method

1. Each agent completes an AHP producing their final ranking
2. Use the geometric mean of the individual final rankings to create an overall final ranking.

Assessments from Multiple Agents: Third Method

1. Each agent completes an AHP producing their final ranking
2. If the individual criteria priority matrices are very different, raise the final ranking to the power of the criteria priorities.
3. Use the geometric mean of the individual, power-adjusted, final rankings to create an overall final ranking.

Hundreds of examples are in *The Hierarchon: A Dictionary of Hierarchies* by Saaty.

Combining Multiple Assessments Example

Simple Example: First Method

Given three assessment matrices

$$PM1 = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1/4 \\ 1/3 & 4 & 1 \end{bmatrix}, \quad PM2 = \begin{bmatrix} 1 & 3 & 1/2 \\ 1/3 & 1 & 2 \\ 2 & 1/2 & 1 \end{bmatrix}, \quad PM3 = \begin{bmatrix} 1 & 1/2 & 5 \\ 2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}$$

form the assessment matrix $PM = [pm_{i,j}]$ with

$$pm_{i,j} = \text{GeometricMean}(PM1_{i,j}, PM2_{i,j}, PM3_{i,j})$$

$$PM = \begin{bmatrix} 1 & \sqrt[3]{3} & \sqrt[3]{\frac{15}{2}} \\ \sqrt[3]{\frac{1}{3}} & 1 & \sqrt[3]{\frac{3}{2}} \\ \sqrt[3]{\frac{2}{15}} & \sqrt[3]{\frac{2}{3}} & 1 \end{bmatrix} \approx \begin{bmatrix} 1.0 & 1.44 & 1.96 \\ 0.693 & 1.0 & 1.14 \\ 0.511 & 0.872 & 1.0 \end{bmatrix}$$

The critical property of geometric means needed is

$$\text{GeometricMean}(1/x_i) = 1 / \text{GeometricMean}(x_i).$$

Aggregation Diagram

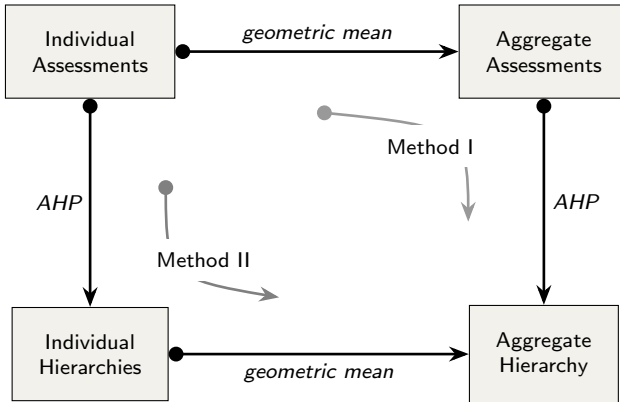


Figure: Aggregating Multiple Agents: Two Methods

Axioms of the Analytic Hierarchy Process

The Basic Principle of AHP⁵

Principle of Hierarchic Composition. Multiply local priorities of elements in a cluster by the “global” priority of the parent element producing global priorities throughout the hierarchy.

The Axioms of AHP

Reciprocal Axiom If $P(A,B)$ is the paired comparison of A to B , then $P(B,A) = 1/P(A,B)$.

Homogeneity Axiom Elements being compared do not differ by *too much*.
(Over an order of magnitude leads to judgment errors.)

Synthesis Axiom Judgments about the priorities of the elements in a hierarchy do not depend on lower level elements.

⁵See “The Analytic Hierarchy Process: An Exposition,” Forman & Gass, *Operations Research*, Vol. 49, No. 4 (Jul. - Aug., 2001), pp. 469-486.

The Power Method for the Maximal Eigenvector

Power Method⁶

Suppose that \mathbf{M} is a square matrix that has a dominant eigenvalue $|\lambda_1| > |\lambda_i|$ for $i = 2, \dots$

1. Choose $\vec{x}_0 \neq \vec{0}$. In AHP, we normally use $\vec{x}_0 = \langle 1, 1, 1 \rangle$
2. Compute the “scaled power sequence”

$$\vec{x}_1 = \frac{1}{\|\mathbf{M} \cdot \vec{x}_0\|_\infty} \cdot \mathbf{M} \cdot \vec{x}_0$$

$$\vec{x}_2 = \frac{1}{\|\mathbf{M} \cdot \vec{x}_1\|_\infty} \cdot \mathbf{M} \cdot \vec{x}_1 = \frac{1}{\|\mathbf{M} \cdot \vec{x}_1\|_\infty \cdot \|\mathbf{M} \cdot \vec{x}_0\|_\infty} \cdot \mathbf{M}^2 \cdot \vec{x}_0$$

$$\vec{x}_3 = \frac{1}{\|\mathbf{M} \cdot \vec{x}_2\|_\infty} \cdot \mathbf{M} \cdot \vec{x}_2 = \frac{1}{\|\mathbf{M} \cdot \vec{x}_2\|_\infty \cdot \|\mathbf{M} \cdot \vec{x}_1\|_\infty \cdot \|\mathbf{M} \cdot \vec{x}_0\|_\infty} \cdot \mathbf{M}^3 \cdot \vec{x}_0$$
$$\vdots$$

3. Continue the sequence until $\|\vec{x}_{k+1} - \vec{x}_k\| < \varepsilon$.

⁶See also “Power Iteration” on Wikipedia.

The Rayleigh Quotient for the Maximal Eigenvalue

The Rayleigh Quotient⁷

Let \vec{x} be a nonzero vector and \mathbf{M} be a square matrix. Define the *Rayleigh quotient of \vec{x}* to be

$$R(\vec{x}) = \frac{\vec{x} \cdot (\mathbf{M} \cdot \vec{x})}{\vec{x} \cdot \vec{x}}$$

If \vec{x} is an eigenvector, then $\mathbf{M} \cdot \vec{x} = \lambda \vec{x}$, so $R(\vec{x})$ simplifies to λ , the eigenvalue of the eigenvector \vec{x} .

Estimating the Maximal Eigenvalue

Let $\{\vec{x}_k\}$ be vectors generated via the power method and set $r_k = R(\vec{x}_k)$.

If $\vec{x}_k \rightarrow \vec{x}$, the dominant eigenvector, then $r_k \rightarrow \lambda$ the eigenvalue of \vec{x} .

(Nota bene: $\mathbf{M} \cdot \vec{x}_k \neq \vec{x}_{k+1}$. The \vec{x}_k s are scaled.)

⁷See also “Spectral Radius” on Wikipedia.

Readings

References

1. K Teknomo, *Analytic Hierarchy Process (AHP) Tutorial*, (website) 2007.
2. T Saaty, "How to Make a Decision: The Analytic Hierarchy Process," *Interfaces*, Vol 24, No 6 (Nov-Dec, 1994), pp 19-43.
3. T Saaty, "Decision-making with the AHP: Why is the principal eigenvector necessary," *Euro J Operational Research*, 145 (2003), pp 85-91.
4. T Saaty, "Decision making with the analytic hierarchy process," *Int J Services Sciences*, Vol 1, No 1, 2008, pp 83-98.
5. M Liberatore and R Nydick, "Group Decision Making in Higher Education Using the Analytic Hierarchy Process," *Research in Higher Education*, Vol 38, No 5 (Oct, 1997), pp 593-614.
6. S Shiraishi, T Obata, and M Daigo, "Properties of a Positive Reciprocal Matrix and Their Application to AHP," *J Operations Research Soc of Japan*, Vol 41, No 3, Sep 1998, pp 404-414.