Analytic Hierarchy Process

Wm C Bauldry

MAT 4340/5340

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Contents

What is AHP ...

Analytic Hierarchy Process (AHP) ...

is a Multi-Criteria Decision Analysis method developed by Thomas Saaty in 1980. AHP is a method to derive **ratio scales** from paired comparisons. Inputs are obtained from objective measurements and/or from subjective preferences. AHP allows small inconsistencies in subjective assessments. The ratio scales are taken from normalized maximal eigenvectors; the **consistency index** is calculated from the maximal eigenvalue.

[1] ASU Library access to JStor, Science Direct, and SpringerLink.

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Sources[1]

1. "How to make a decision: The Analytic Hierarchy Process," Thomas L. Saaty, *Euro. J. Oper. Res.*, No. 48, (1990), pp. 9-26.
2. "The Analytic Hierarchy Process: A Survey of the Method and Its Applications," F. Zahedi, *Interfaces*, Vol. 16, No. 4 (Jul-Aug, 1986), pp. 96-108.
3. "Aggregating individual judgments and priorities with the Analytic Hierarchy Process," E. Forman and K. Peniwati, *Euro. J. Op. Res.*, No. 108 (1998), pp. 165-169.

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Canonical First Example

Choosing Which Car to Buy

Objective Selecting a car

Criteria Style, Reliability, Fuel Economy

Alternatives Civic Coupe, Saturn Coupe, Ford Escort, Mazda Miata

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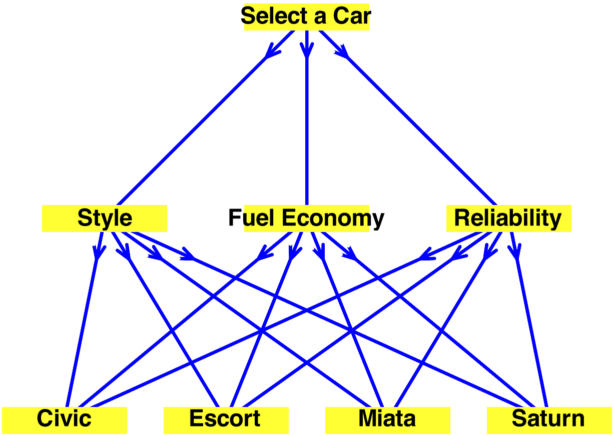
How to compare the cars? How to choose which one to purchase?

Canonical First Example



Decision Hierarchy

Pairwise Criteria Comparisons

Compare...

Using the scale

Criterion$_1$				*versus*				*Criterion*$_2$
9	7	5	3	1	3	5	7	9
extreme importance	very strong	strong	moderate	equal	moderate	strong	very strong	extreme importance

rate the relative importance of the criteria

[2] For Excel, c.f., the *power method* and the *Rayleigh quotient*.

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rate the relative importance of the criteria to create a *positive reciprocal priority matrix*.

over	Style	Reliability	Fuel Economy
Style	1	$\square$	$\square$
Reliability	___	1	$\square$
Fuel Economy	___	___	1

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Now

1. Find[2] the dominant eigenvalue λ_m. Run Maple

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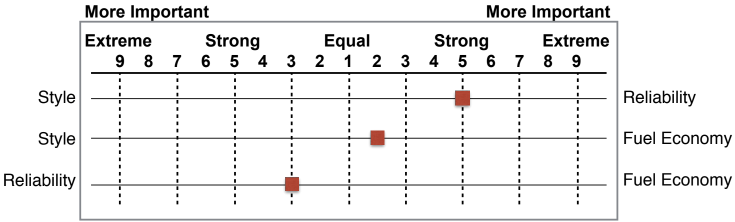
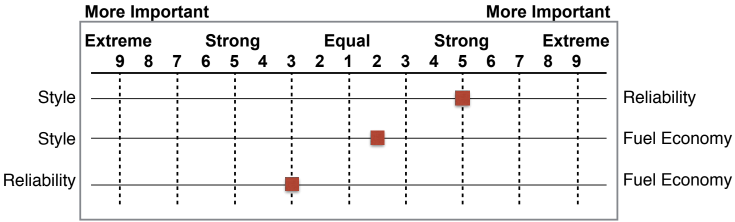
1. Find[2] the dominant eigenvalue λ_m. ▸ *Run Maple*
2. Find and normalize (so $\sum v_i = 1$) λ_m's eigenvector.

[2]For Excel, c.f., the *power method* and the *Rayleigh quotient*.

Pairwise Comparisons Chart



Pairwise Comparisons Chart



$$
\begin{array}{c c}
 & \begin{array}{ccc} S & R & FE \end{array} \\
\begin{array}{c} S \\ R \\ FE \end{array} & \begin{bmatrix} 1 & 1/5 & 1/2 \\ 5 & 1 & 3 \\ 2 & 1/3 & 1 \end{bmatrix}
\end{array}
$$

Pairwise Comparisons Eigenvector

The Criteria Ranking

The normalized eigenvector gives the relative ranking of the criteria.

$$\begin{array}{r} \text{Style} \\ \text{Reliability} \\ \text{Fuel Economy} \end{array} = \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} = ev_{\mathit{Criteria}}$$

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Consistency Index and Consistency Ratio

Measure the consistency of the comparisons with the formulas

$$CI(\lambda_m) = \frac{\lambda_m - n}{n-1} \qquad \text{and} \qquad CR = CI(\lambda_m)/RI$$

where RI is from the following table listing *random indices*.

n	3	4	5	6	7	8	9	10
Random Matrix Consistency Index	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

A ratio $CR < 0.1$ is good. *(Larger values mean the comparisons need revision.)*

Pairwise Alternatives Comparisons

Alternatives Comparisons

For each criterion, construct the pairwise comparison matrix

Style

over	Civic	Saturn	Escort	Miata
Civic	1	$\square$	$\square$	$\square$
Saturn	___	1	$\square$	$\square$
Escort	___	___	1	$\square$
Miata	___	___	___	1

Reliability

over	Civic	Saturn	Escort	Miata
Civic	1	$\square$	$\square$	$\square$
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$$ev_S = [\square, \square, \square, \square]$$

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$$ev_R = [\boxed{}, \boxed{}, \boxed{}, \boxed{}]$$

Fuel Economy is an objective value; it needs no comparison matrix. Normalize so the sum is 1 and use for its 'eigenvector':

$$[34,\ 27,\ 24,\ 28] \implies ev_{FE} = [0.30,\ 0.24,\ 0.21,\ 0.25]$$

Comparison Matrix

AHP Comparison Matrix

- Form the *comparison matrix* from the eigenvectors

$$CM = [ev_S,\ ev_R,\ ev_{FE}] = \begin{bmatrix} \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} & \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} & \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} \end{bmatrix}$$

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- Multiply the comparison matrix by the eigenvector of priority rankings for the criteria $\vec{R} = CM \cdot ev_{\mathit{Criteria}}$ to produce the *rankings vector* $\vec{R}$

$$R = \left[\begin{array}{cccc} \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{array}\right]$$

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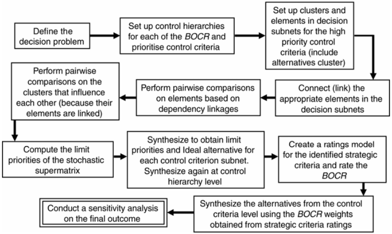
$$R = \begin{bmatrix} \Box & \Box & \Box & \Box \end{bmatrix}$$

- To include *cost*:
  1. Normalize the cost vector so the sum is 1.
  2. Divide respective rankings by cost to generate the *cost-benefit vector*.

AHP Workflow Charts



From: T. L. Saaty, "Decision making with the analytic hierarchy process," Int. J. Services Sciences, Vol. 1, No. 1, 2008. Figure 2, pg. 94.

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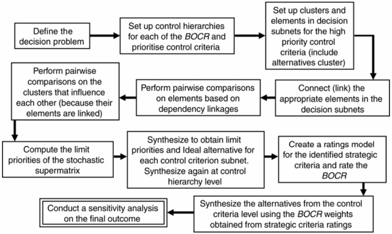

Step 1 Create the decision hierarchy by breaking the decision problem into interrelated decision elements

Step 2 Collect input data with pairwise comparisons of decision elements

Step 3 Use eigenvalues to estimate the relative weights of decision elements

Step 4 Aggregate the relative weights of decision elements to rate the decision alternatives

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From: F. Zahedi, "The Analytic Hierarchy Process: A Survey of the Method and Its Applications," *Interfaces*, Vol. 16, No. 4 (Jul-Aug, 1986), pg. 96.

▸ *Open Excel Example*

Combining Multiple Assessments

Assessments from Multiple Agents: First Method

1. Each individual ranks all pairs of criteria.

[3] See The HM-GM-AM-QM Inequalities by P. Gwanyama

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2. Average the rankings with a *geometric mean*[3].

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5. Use the matrices of geometric means in AHP.

[3] See The HM-GM-AM-QM Inequalities by P. Gwanyama

Combining Multiple Assessments, II

Assessments from Multiple Agents: Second Method

1. Each agent completes an AHP producing their final ranking
2. Use the geometric mean of the individual final rankings to create an overall final ranking.

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Assessments from Multiple Agents: Third Method

1. Each agent completes an AHP producing their final ranking
2. If the individual criteria priority matrices are very different, raise the final ranking to the power of the criteria priorities.
3. Use the geometric mean of the individual, power-adjusted, final rankings to create an overall final ranking.

Hundreds of examples are in *The Hierarchon: A Dictionary of Hierarchies* by Saaty.

Combining Multiple Assessments Example

Simple Example

Given three assessment matrices

$$PM1 = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 1/4 \\ 1/3 & 4 & 1 \end{bmatrix}, \quad PM2 = \begin{bmatrix} 1 & 3 & 1/2 \\ 1/3 & 1 & 2 \\ 2 & 1/2 & 1 \end{bmatrix}, \quad PM3 = \begin{bmatrix} 1 & 1/2 & 5 \\ 2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix},$$

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form the assessment matrix $PM = [pm_{i,j}]$ with

$$pm_{i,j} = \text{GeometricMean}(PM1_{i,j}, PM2_{i,j}, PM3_{i,j})$$

$$PM = \begin{bmatrix} 1 & \sqrt[3]{3} & \sqrt[3]{\frac{15}{2}} \\ \sqrt[3]{\frac{1}{3}} & 1 & \sqrt[3]{\frac{3}{2}} \\ \sqrt[3]{\frac{2}{15}} & \sqrt[3]{\frac{2}{3}} & 1 \end{bmatrix} \approx \begin{bmatrix} 1.0 & 1.44 & 1.96 \\ 0.693 & 1.0 & 1.14 \\ 0.511 & 0.872 & 1.0 \end{bmatrix}$$

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The critical property of geometric means needed is

$$\text{GeometricMean}(1/x_i) = 1/\text{GeometricMean}(x_i).$$

Axioms of the Analytic Hierarchy Process

The Basic Principle of AHP[4]

Principle of Hierarchic Composition. Multiply local priorities of elements in a cluster by the “global” priority of the parent element producing global priorities throughout the hierarchy.

[4] See “The Analytic Hierarchy Process: An Exposition,” Forman & Gass, *Operations Research*, Vol. 49, No. 4 (Jul. - Aug., 2001), pp. 469-486.

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(Over an order of magnitude leads to judgment errors.)

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Synthesis Axiom Judgments about the priorities of the elements in a hierarchy do not depend on lower level elements.

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The Power Method for the Maximal Eigenvector

Power Method[5]

Suppose that $\mathbf{M}$ is a square matrix that has a dominant eigenvalue $|\lambda_1| > |\lambda_i|$ for $i = 2\ldots$

1. Choose $\vec{x}_0 \neq \vec{0}$. In AHP, we normally use $\vec{x}_0 = \langle 1, 1, 1 \rangle$
2. Compute the "scaled power sequence"

$$\begin{aligned}
\vec{x}_1 &= \tfrac{1}{\|\mathbf{M}\cdot\vec{x}_0\|_\infty} \cdot \mathbf{M} \cdot \vec{x}_0 \\
\vec{x}_2 &= \tfrac{1}{\|\mathbf{M}\cdot\vec{x}_1\|_\infty} \cdot \mathbf{M} \cdot \vec{x}_1 = \tfrac{1}{\|\mathbf{M}\cdot\vec{x}_1\|_\infty \cdot \|\mathbf{M}\cdot\vec{x}_0\|_\infty} \cdot \mathbf{M}^2 \cdot \vec{x}_0 \\
\vec{x}_3 &= \tfrac{1}{\|\mathbf{M}\cdot\vec{x}_2\|_\infty} \cdot \mathbf{M} \cdot \vec{x}_2 = \tfrac{1}{\|\mathbf{M}\cdot\vec{x}_2\|_\infty \cdot \|\mathbf{M}\cdot\vec{x}_1\|_\infty \cdot \|\mathbf{M}\cdot\vec{x}_0\|_\infty} \cdot \mathbf{M}^3 \cdot \vec{x}_0 \\
&\vdots
\end{aligned}$$

3. Continue the sequence until $\|\vec{x}_{k+1} - \vec{x}_k\| < \varepsilon$.

[5] See also "Power Iteration" on Wikipedia.

The Rayleigh Quotient for the Maximal Eigenvalue

The Rayleigh Quotient[6]

Let $\vec{x}$ be a nonzero vector and $\mathbf{M}$ be a square matrix. Define the *Rayleigh quotient of* $\vec{x}$ to be

$$R(\vec{x}) = \frac{\vec{x} \cdot (\mathbf{M} \cdot \vec{x})}{\vec{x} \cdot \vec{x}}$$

If $\vec{x}$ is an eigenvector, then $\mathbf{M} \cdot \vec{x} = \lambda \vec{x}$, so $R(\vec{x})$ simplifies to λ, the eigenvalue of the eigenvector $\vec{x}$.

Estimating the Maximal Eigenvalue

Let $\{\vec{x}_k\}$ be vectors generated via the power method and set $r_k = R(\vec{x}_k)$.

If $\vec{x}_k \to \vec{x}$, the dominant eigenvector, then $r_k \to \lambda$ the eigenvalue of $\vec{x}$.

(Nota bene: $\mathbf{M} \cdot \vec{x}_k \neq \vec{x}_{k+1}$. The $\vec{x}_k$s are scaled.)

[6] See also "Spectral Radius" on Wikipedia.