2WBB0 Calculus for BCS

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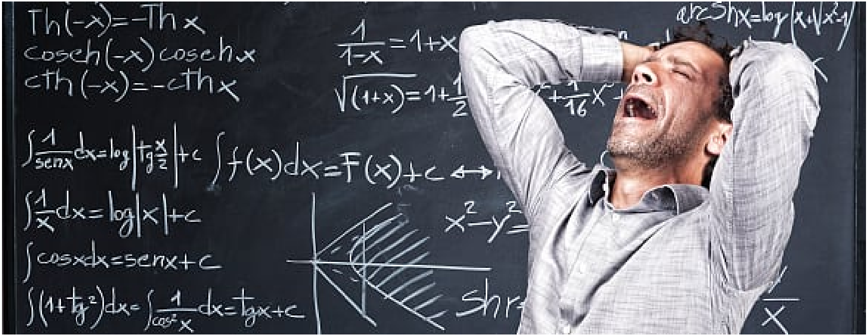
Why do we (=you) need Calculus?



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December 21, 1968? Shorter clip

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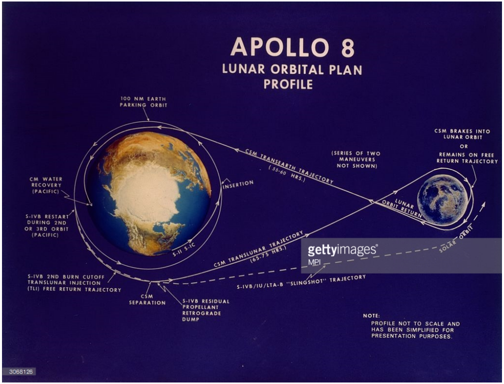

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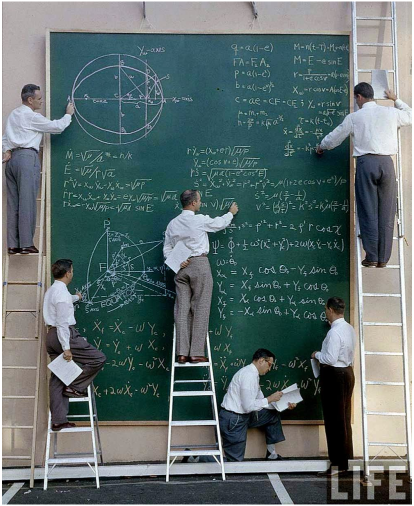

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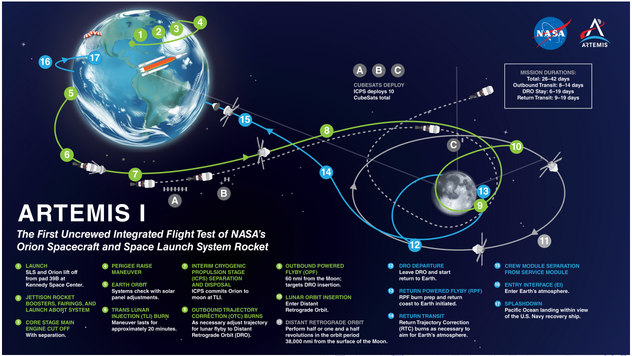
Why do we (=you) need Calculus?

November 16, 2022



Why do we (=you) need Calculus?

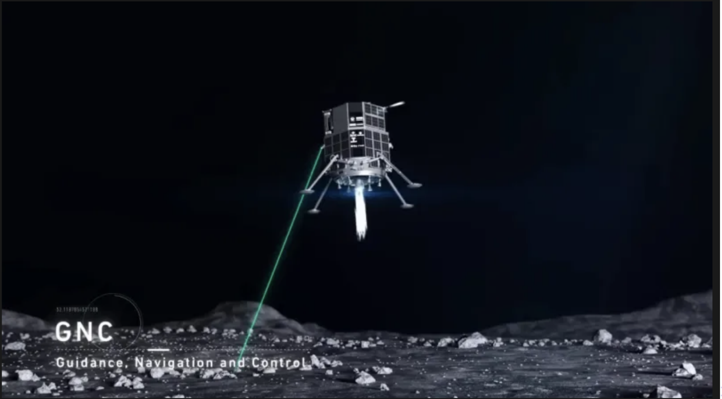
April 26, 2023

THE GLITCH THAT BROUGHT DOWN JAPAN'S LUNAR LANDER

by: Matthew Carlson

37 Comments

June 3, 2023



Why do we (=you) need Calculus?

April 26, 2023

When a computer crashes, it usually doesn't leave debris. But when a computer happens to be descending towards the lunar surface and glitches out, that's a very different story. Turns out that's what happened on April 26th, as the Japanese Hakuto-R Lunar lander made its mark on the Moon...by crashing into it. [Scott Manley] dove in to try and understand the software bug that caused an otherwise flawless mission to go splat.

The lander began the descent sequence as expected at 100 km above the surface. However, as it descended, the altitude sensor reported the altitude as much lower than it was. It thought it was at zero altitude once it reached about 5 km above the surface. Confused by the fact it hadn't yet detected physical contact with the surface, the craft continued to slowly descend until it ran out of fuel and plunged to the surface.

Ultimately it all came down to sensor fusion. The lander merges several noisy sensors, such as accelerometers, gyroscopes, and radar, into one cohesive source of truth. The craft passed over a particularly large cliff that caused the radar altimeter to suddenly spike up 3 km. Like good filtering software, the craft reasons that the sensor must be getting spurious data and filters it out. It was now just estimating its altitude by looking at its acceleration. As anyone who has tried to track an object through space using just gyros and accelerometers alone can attest, errors accumulate, and suddenly you're not where you think you are.

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Without Calculus

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for i := 1 to X do
     for j := 1 to X do
           if j>i then inc(cnt);
Writeln(cnt);
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Readln('how many people?', X);
Writeln(X*(X-1)/2);
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```

- For 20 people, nobody can tell the difference.
- For one million people in the room, you get to do 10^{14} loops. You need a 64 bit machine for storing the counter. At 3 GHz, that takes 166 seconds (way more in real life). With Calculus: instant!

Calculus in Computer Science

For video game programming.
Calculus, when taught alongside Physics, gives you the background necessary to understand how objects can interact and move based on forces. Even if you don't ever write your own game engine, you'll be a better game programmer if you have a mastery of calculus.

For machine learning.
Basically machine learning is mathematics meets computer science meets data. This is one of the hottest topics in CS right now, but you have to know your mathematics to succeed with ML.

For scientific applications.
A lot of computer scientists work alongside scientists like biologists, physicists, and chemists. CS people help them solve their data-analysis problems using large clusters of computers, and anytime you have to understand how equations work, it is hard to escape calculus.

In research.
Researchers such as Professors and PhD students use mathematics when writing papers. If you want to one day pursue an advanced degree, you'll have to decipher these papers and then write some of your own. I would not recommend starting out a career in research without at least a basic command over calculus.

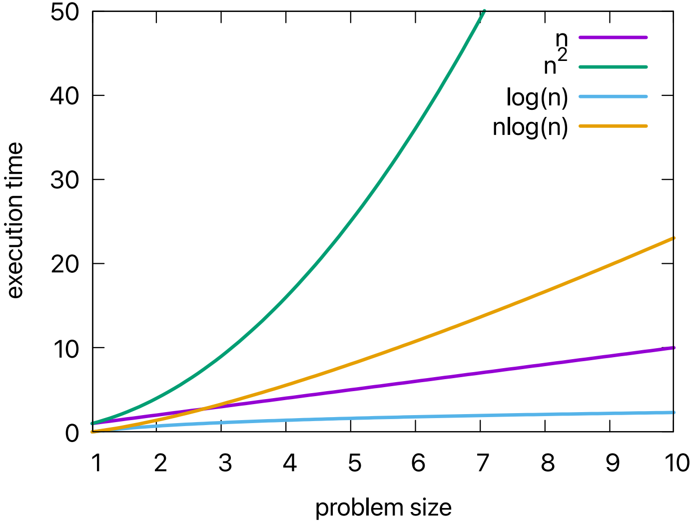
Analyze a Computer Algorithm

Imagine you are a programmer of an automatic, autonomous guidance system (think of cars, rockets). You have an algorithm that takes data collected from a number of input sources (sensors) and processes it to adjust the control of the vehicle. It takes 50ms to obtain the adjustment after the acquisition of data (reaction time).

Now, the engineers plan to double the number of sensors to improve the guidance. The maximal acceptable reaction time for the vehicle is 150ms.

Can your algorithm deal with double the amount of data within the 150ms?

Different scaling of (parts of complex) algorithms



Example: Bubble Sort

Sorting an unordered list of five numbers by swapping adjacent elements that are out of place:

17	**12**	8	3	7
12	**17**	**8**	3	7
12	8	**17**	**3**	7
12	8	3	**17**	**7**
12	8	3	7	**17**

How many comparisons? $n - 1$

Example: Bubble Sort

Now, do it again

12	**8**	3	7	17
8	**12**	**3**	7	17
8	3	**12**	**7**	17
8	3	7	**12**	**17**

How many comparisons? $n - 2$

And again:

8	**3**	7	12	17
3	**8**	**7**	12	17
3	7	**8**	**12**	**17**

How many comparisons? $n - 3$

Example: Bubble Sort

Finally:

3 7 8 12 17

How many comparisons? $n - 4$
So all in all:

$$(n-1) + (n-2) + (n-3) + \ldots + 1 = \frac{n(n-1)}{2}$$

Ranking Different Algorithms

- How to compare different algorithms?
- Two algorithms

$$\text{Algorithm 1} \rightarrow f(n) = n$$
$$\text{Algorithm 2} \rightarrow g(n) = n^2$$

- Which one is "faster"?
- Algorithm 1, because it grows at a slower rate with n

Ranking Different Algorithms

- How to formulate this more generally?
  $\rightarrow$ Limits
- $$\lim_{n\to\infty} \frac{f}{g} = 0 \qquad g \text{ grows faster}$$
  $$\lim_{n\to\infty} \frac{f}{g} = c \qquad g \text{ grows same as } f$$
  $$\lim_{n\to\infty} \frac{f}{g} = \infty \qquad g \text{ grows slower}$$

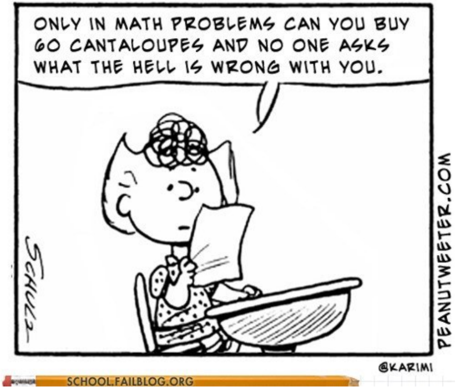
Limits not always straightforward

- $f(n) = n\log(n)$
- $g(n) = n^{3/2}$
- $$\lim_{n\to\infty} \frac{n^{3/2}}{n\log(n)} = ?$$
- To evaluate we will need:
  - derivatives
  - Taylor series/polynomials
  - Exponential functions and Logarithms
  - rational functions
  - ...

Overview

- W1: Numbers, Functions, (In-)equalities, Polynomials, Rational functions
- W2: Trigonometric functions/Vectors in 2 and 3 dimensions
- W3: Limits, continuity, differentiation I
- W4: Differentiation II and Inverse functions
- W5: Exponential function and logarithm/Taylor series/Limits II
- W6: Integration/Integration techniques I
- W7: Integration techniques II/1st order differential equations
- W8: Exam and W1 – W7 summary (no new material)

Shall we begin?



Topics in Week 1

- Real numbers and the real line
- Graphs of quadratic equations
- Polynomials and rational functions
- Powers and roots
- Cartesian coordinates in the plane
- More functions and their graphs
- Combining functions to make new functions

Real numbers

$$\mathbb{N} \underset{\substack{\uparrow \\ \text{is a subset of}}}{\subset} \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$\mathbb{N}$ natural numbers: $0, 1, 2, \ldots$ (sometimes without 0)
$\mathbb{Z}$ integer numbers: $\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$ rational numbers = fractions: $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$
$\mathbb{R}$ real numbers: $\mathbb{Q}$ and π, $\sqrt{2}$, etc
$\mathbb{C}$ complex numbers: $2 + 3\mathbf{i}$, with $\mathbf{i} \cdot \mathbf{i} = \mathbf{i}^2 = -1$ (not in curriculum)

Decimal Expansion

For rational numbers, the “decimal expansion” has a repeating pattern, for irrational numbers, this is not the case:

$$
\begin{aligned}
\tfrac{1}{2} &= 0.50000|0|\cdots \\
\tfrac{1}{3} &= 0.33333|3|\cdots \\
\tfrac{1}{11} &=
\end{aligned}
$$

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\pi &= 3.141592653589793\cdots
\end{aligned}
$$

Intervals

Notation for intervals:

- [,]: end points are part of the interval
- (,): end points are NOT part of the interval

interval	alternative notation	type	type
$(0,1)$	$0 < x < 1$	open	finite
$[0,1)$	$0 \leq x < 1$	half open	finite
$(0,1]$	$0 < x \leq 1$	half open	finite
$[0,1]$	$0 \leq x \leq 1$	closed	finite
$[1,\infty)$	$1 \leq x < \infty$		infinite
$(-\infty,3)$	$-\infty < x < 3$		infinite
$(-\infty,\infty)$	$-\infty < x < \infty$		infinite

Symbols

$\cup$	union
$\cap$	intersection
$\in$	element of
$\vee$	or
$\wedge$	and
(A) $\implies$ (B)	from (A) follows (B)
(A) $\Leftrightarrow$ (B)	from (A) follows (B) and from (B) follows (A)
	in other words: they are equivalent
	(A) “if and only if” (B) (iff)

For two sets A and B:

- $A - B$
- $A \backslash B$

represents alle $x \in A$ for which $x \notin B$.

Polynomials

A polynomial is a "many term" construct

The degree (order) of a polynomial is the highest ocurring power

Ex: $p(x) = x^3 - 2x^2 + 1$ is of degree 3

Ex: $p(x) = 0 \cdot x^3 - 2x^2 + 1$ is of degree 2

Ex: $p(x) = (x+1)^2 - (x-1)^2$ is of degree 1

Ex: $p(x) = 3 = 3 \cdot x^0$ is of degree 0

Computers can only do

- additions/subtractions
- multiplications

All other operations, such as division or calculation of a root have be performed using only these basis operations – and then cost 3 to 5 times more calculational time

Frequently appearing polynomials

Degree	Name	Example
0	constant	$p(x) = 2$
1	linear	$p(x) = 2x + 3$
2	quadratic	$p(x) = 5x^2 + 4x + 5$
3	cubic	$p(x) = 2x^3 + 1$

Equations with polynomials of degree 1 always have a solution:

Ex: $4x - 3 = 0$

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Equations with polynomials of degree 1 always have a solution:

Ex: $4x - 3 = 0 \implies 4x = 3 \implies x = \frac{3}{4}$

Theorem:

> Equations of polynomials of degree n have a maximum of n real solutions

Equations with a polynomial of degree 2

Quadratic functions are more than algebraic curiosities – they are widely used in science, business, and engineering:

- trajectories of water jets in a fountain or of a bouncing ball
- parabolic reflectors that form the base of satellite dishes and car headlights
- forecast business profit and loss
- data interpolation

Equations of quadratic polynomials can always be written into the general form:

$$ax^2 + bx + c = 0$$

Equations with a polynomial of degree 2

Equations of quadratic polynomials can always be written into the general form:

$$ax^2 + bx + c = 0$$

Our task is to find the solutions (or zeros) of this equation.

3 different ways to find the solution:

- completing the square
- abc-formula (quadratic formula)
- "Sum-product"-formula

Use the way that you like best **for this purpose** but learn the concept of **completing the square**! It will be useful later.

Idea behind Completing the Square

Ex: : $x^2 = 3$

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This is easy (and also an equation with a quadratic polynomial)!

So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.

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Ex: $(x-1)^2 = 9$

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This is easy (and also an equation with a quadratic polynomial)!

So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.

Ex: $(x-1)^2 = 9 \Leftrightarrow x - 1 = \pm 3 \Leftrightarrow x = 1 \pm 3$

> Rewrite a quadratic equation to a form with a complete square on the left-hand side
>
> $$(x \pm R)^2 = T$$
>
> and take the square root on both sides, if possible.

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{array}{rcl} (n+m)^2 & = & n^2 + 2nm + m^2 \\ (n-m)^2 & = & n^2 - 2nm + m^2 \end{array}}$$

and that one can also use them from right to left!

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Ex: $x^2-4x+4=(x-2)^2$

Ex: $4x^2+12x+9=(\underbrace{2x}_{=n})^2+\underbrace{2}_{\text{binom. form.}}\cdot\underbrace{2x}_{=n}\cdot\underbrace{3}_{=m}+(\underbrace{3}_{=m})^2=(2x+3)^2$

Those examples are complete squares. They can be rewritten directly into one of the two binomial formulas.

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But what about x^2-4x+3? We are missing $+1$ for a complete square!

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$$x^2-4x+3=x^2-4x+3+0=x^2-4x+3+1-1=x^2-4x+4-1=(x-2)^2-1$$

Completing the square to solve quadratic equations

Solve:

$$ax^2 + bx + c = 0$$

1. Identify n and m in the expression.
   - n is the square root of the term with x^2
     $ax^2 \to n = \sqrt{a}x$
   - m is the linear term in x, divided by $2n$
     $m = bx/(2 \cdot \sqrt{a}x) = b/(2\sqrt{a})$
2. Use binomial formula
   $(n+m)^2 = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = ax^2 + bx + \frac{b^2}{4a}$
3. what is missing/too much
   $ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 - \frac{b^2}{4a} + c = 0$
4. isolate the square, take roots, simplify
   $(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = \frac{b^2}{4a} - c \Leftrightarrow \sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$
   $$\Leftrightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = 0$ via abc-formula (I)

2nd degree polynomial $ax^2 + bx + c = 0$ with $a \neq 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How many solutions are possible?

Discriminant: $D = b^2 - 4ac$

$D > 0$

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$D < 0$	

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$D = 0$ 1 double solution (2 solutions $x_1 = x_2$)

$D < 0$ no (real) solution

The zeros determine the factoring of the polynomial:

$$ax^2 + bx + c = (x - x_1)(x - x_2).$$

Pay attention to the minus sign!

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions $(x_1 \neq x_2)$
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0$

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions ($x_1 \neq x_2$)
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0 \implies x^2 = -1 \implies$ no solution
indeed: $D = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0 \implies$ no solution

Ex: $x^2 + 2x + 1 = 0$

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

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$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
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indeed: $D = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0 \implies$ no solution

Ex: $x^2 + 2x + 1 = 0$: $D = 2^2 - 4 \cdot 1 \cdot 1 = 0 \implies$ double solution
indeed: $x^2 + 2x + 1 = 0 \implies (x+1)^2 = 0$
$\implies x + 1 = 0$ or $x + 1 = 0 \implies$ double solution $x = -1$

$ax^2 + bx + c = 0$ via "Sum-product"

Set $a = 1$:

$$x^2 + \underbrace{(r+s)}_{=b(\text{sum})} x + \underbrace{rs}_{=c(\text{product})} = (x+r)(x+s)$$

Ex: $x^2 + 5x + 6 =$

$ax^2 + bx + c = 0$ via "Sum-product"

Set $a = 1$:

$$x^2 + \underbrace{(r+s)}_{=b(\text{sum})} x + \underbrace{rs}_{=c(\text{product})} = (x+r)(x+s)$$

Ex: $x^2 + 5x + 6 = (x+2)(x+3)$

Because $b = 5 = 2 + 3$ and $c = 6 = 2 \cdot 3$

Ex: $x^2 - 9 = (x-3)(x+3)$

Attention! $x^2 = 9 \not\Longrightarrow x = 3$, but :

$$x^2 = 9 \implies x^2 - 9 = 0 \implies (x-3)(x+3) = 0 \implies$$
$$x - 3 = 0 \text{ or } x + 3 = 0 \implies x = 3 \quad \text{or} \quad x = -3$$

Examples: Quadratic Equations

1. Solve: $x^2 - 3x + 2 = 0$
2. Factorize: $x^2 = 2$
3. Solve: $x^2 + 2 = 0$
4. Complete the square: $x^2 + 3x + \frac{1}{4}$
5. Complete the square: $4x^2 + 12x + 1$

Equations with a 3rd degree polynomial

1. Guess first zero. Try out first $a = 0, 1, -1, 2, -2, \ldots$
2. Divide by factor $x - a$
3. The rest must be 0 (if not, you made a mistake)
4. Find solutions of the remaining 2nd degree polynomial equation, if they exist

Ex: $x^3 + 2x^2 - x - 2 = 0$
Try:

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Ex: $x^3 + 2x^2 - x - 2 = 0$
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Then is $x^3 + 2x^2 - x - 2 = (x - 1) \cdot (\text{2nd degree polynomial})$

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And then $x^2 + 3x + 2 = (x+2)(x+1)$ via abc-formula
Conclusion $x^3 + 2x^2 - x - 2 = (x-1)(x+2)(x+1)$
Thus solution of $x^3 + 2x^2 - x - 2 = 0$:

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Conclusion $x^3 + 2x^2 - x - 2 = (x-1)(x+2)(x+1)$
Thus solution of $x^3 + 2x^2 - x - 2 = 0$: $\;x = 1$ or $x = -1$ or $x = -2$

Equations with 3rd, 4th, 5th, ...-degree polynomials

Principle: guess, divide, guess, etc, until 2nd-degree remains

1. First, have a look if there is not a special easier case, as $p(x) = 0$ with $p(x) = x^3 - 8$ or $p(x) = x^4 + 2x^2 + 1$
2. If not, guess a zero, say $x = a$, $a = 0, 1, -1, 2, -2, \ldots$
3. Divide out by Long Division: $p(x)/(x - a)$
   (here **must** have rest 0),
   this yields a new polynomial $q(x)$
4. If the degree of $q(x)$ is still larger than 2,
   repeat the above
5. If the degree of $q(x)$ is equal to 2,
   try with sum-product technique or the abc-formula

Degree 4, 5, . . .: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$

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$$p^2 - 50p + 49 = 0 \implies (p - 49)(p - 1) = 0 \implies p = 49 \text{ of } p = 1$$
$$\underset{p=x^2}{\implies} \implies x^2 = 49 \text{ or } x^2 = 1 \implies x = \pm 7 \text{ or } x = \pm 1$$

Higher degree but only produts of lower degrees

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$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$

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$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$
$$\implies x = 0 \text{ or } x - 2 = 0$$

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Higher degree but only produts of lower degrees

$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$
$$\implies x = 0 \text{ or } x - 2 = 0 \implies x = 0 \text{ or } x = 2$$

(respectively 4-fold and 2-fold)

Factoring and roots (zeros) of polynomials

Writing a polynomial as a product of two or more polynomials is called factoring.

Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2)$
The polynomials $x-3$ and x^2-2 are factors with degree 1 and 2.

Factoring breaks up a complicated polynomial into easier, lower degree pieces.

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Factoring breaks up a complicated polynomial into easier, lower degree pieces.

In the example, we can do more:
Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2) = (x-3)(x-\sqrt{2})(x+\sqrt{2})$

This polynomial is factored into three **linear** polynomials. We can't do ant better, the polynomial is **factored completely**.

> Knowing all roots of a polynomial $\Leftrightarrow$ Knowing the factoring of a polynomial

Polynomials: When to let factors stand as they are?

When a polynomial consists of one term of factors:
$p(x) = (x+1)^3$ or $p(x) = (x-1)(x+2)$

Most of the times, it is useful to let these stand, because:

- it is easy, no extra effort
- you see the zeros at once
- sometimes this is easier to differentiate/integrate (later in Calculus) (eg derivative or primitive of $(x+1)^{40}$)
- useful for limits (eg $\lim_{x \to 1^+} (x-1)^3$ etc)

Polynomials: When to write factors out?

When a polynomial consists of several terms (of factors), eg:
$p(x) = (x+1)^3 - (x-1)^3$

Most of the times, it is useful to write these out, because:

- Terms can cancel each other out, eg:
  $(x+1)^3 - (x-1)^3 = 6x^2 + 2$ shows to be of degree 2 (not 3)

Examples: Higher order polynomial equations

1. Solve $3x^4 = 9x^2$
2. Solve $x^3 + 9x^2 + 26x + 24 = 0$
3. Solve $x^4 - 26x^2 + 25 = 0$
4. What are all solutions to $(x^2 - 1)^4(x + 3)^7 = 0$
5. Which of these polynomials has roots $x = 1 \vee x = 2 \vee x = 3/4$
   a) $p(x) = (x + 1)(x + 2)(x + 3/4)$
   b) $p(x) = (x - 1)(x - 2)(x - 3/4)$

Inequalities with polynomials

Finding solutions to inequalities of the kind

$$p(x) > d \qquad p(x) \geq d \qquad p(x) < d \qquad p(x) \leq d$$

is done in the following steps:

1. Replace $<$, $\leq$, $>$ or $\geq$ by $=$
2. Get a 0 to the right hand side
3. Mark the solutions (zeros) on the x-axis
4. Determine the signs for each interval (e.g., by filling in an arbitrary point
5. Determine the solution of the inequality

Simple example: solve $x^2 > 9$

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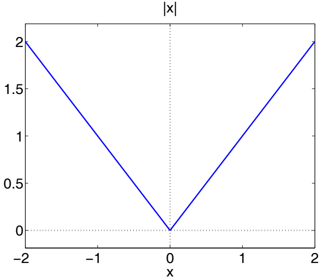
4. Determine the signs for each interval (e.g., by filling in an arbitrary point
5. Determine the solution of the inequality

Simple example: solve $x^2 > 9 \Longleftrightarrow x \in (-\infty, -3) \cup (3, \infty)$

Absolute value

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



"Distance to 0". Properties:

- $|x| = |-x|$
- $|xy| = |x| \cdot |y|$
- $|x + y| \leq |x| + |y|$
- $|x - a|$ "distance to a"

For example:

- $|(-2)(-4)| = |-2| \cdot |-4|$
- $|2 + 4| = |2| + |4|, \qquad |2 - 4| < |2| + |-4|$

Absolute values: Examples

Attention!

$$(\sqrt{a})^2 = a \text{ but } \sqrt{a^2} = |a|$$

Ex: $\sqrt{(-3)^2} = 3$

Only if you know that $a \geq 0$, then $\sqrt{a^2} = |a| = a$, eg

$$\sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2 + 1)^2} = \left|x^2 + 1\right| = x^2 + 1$$

The last step only because $x^2 + 1$ always ≥ 0 (even ≥ 1)

Equations and inequalities with absolute values

1. Equations
   split into the two parts and solve separately
2. Inequalities

- Just as for polynomial inequalities: replace $<$ or $\leq$ by $=$
- Solve and mark solutions on the x-axis with 0s
- Determine sign by filling in arbitrary points within the intervals
- Determine the final solution

Fractions

Addition: make denominators same, then you can add

$$\frac{1}{p}+\frac{1}{q}=\frac{q}{pq}+\frac{p}{pq}=\frac{p+q}{pq}$$

Up to "smallest common multiple":

$$\tfrac{1}{4}+\tfrac{1}{12}=\tfrac{3}{12}+\tfrac{1}{12}=\tfrac{4}{12}=\tfrac{1}{3} \qquad \tfrac{1}{6}+\tfrac{1}{8}=\tfrac{4}{24}+\tfrac{3}{24}=\tfrac{7}{24}$$

Multiplication: Multiply numerators and denominators

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

Simplify: divide numerator and denominator by the same factor

$$\tfrac{144}{30}=\tfrac{72}{15}=\tfrac{24}{5}$$

$$\tfrac{2x+6}{4}=\tfrac{x+3}{2}=\tfrac{1}{2}x+\tfrac{3}{2} \quad \text{but} \quad \tfrac{4}{2x+6}=\tfrac{2}{x+3}\neq\tfrac{2}{x}+\tfrac{2}{3}\ !$$

Division by a fraction is multiplication with the inverse

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\Big/\frac{c}{d}=\tfrac{a}{b}\cdot\tfrac{d}{c}=\tfrac{ad}{bc}$$

Equations with rational functions

Rational functions $= \frac{\text{polynomial}}{\text{polynomial}}$ $\qquad R(x) = \frac{P(x)}{Q(x)}$

Two strategies:

(1) Crosswise multiplication

(2) Bring everything on the same denominator

$$(1): \quad \frac{3}{x+1} = \frac{2}{x+2}$$

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$$(1):\ \frac{3}{x+1} = \frac{2}{x+2} \implies 3\,(x+2) = 2\,(x+1) \text{ and } x \neq -1 \text{ and } x \neq -2$$

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$$(1): \quad \frac{3}{x+1} = \frac{2}{x+2} \implies 3\,(x+2) = 2\,(x+1) \text{ and } x \neq -1 \text{ and } x \neq -2$$

$$\implies x = -4 \quad (\text{ and } x \neq -1 \text{ and } x \neq -2)$$

(however the condition $x \neq -1$ and $x \neq -2$ is thus not relevant)

$$(2): \frac{3\,(x+2)}{(x+1)(x+2)} - \frac{2\,(x+1)}{(x+1)(x+2)} = 0 \implies \frac{x+4}{(x+1)(x+2)} = 0$$

Thus numerator $= 0$: $\quad x = -4$

and denominator $\neq 0$: $\quad x \neq -1$ and $x \neq -2$, but that does not apply here

Division with rest

Rewrite a fraction into an integer + "something with a numerator smaller than the denominator": $\frac{7}{4} = 1 + \frac{3}{4}$:

This also works with rational functions:

$$\boxed{\frac{\text{polynomial}}{\text{polynomial}} = \text{polynomial} + \frac{\text{polynomial P}}{\text{polynomial Q}}}$$

so that the degree of P $<$ degree of Q.

Previous example: $x = 1$ was zero of $x^3 + 2x^2 - x - 2$ and

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

with rest-term $\frac{\text{polynomial P}}{\text{polynomial Q}}$ equal to zero.

Division with rest, different formulation

Target: Write "rational function $=$ polynomial $+$ rational function" so that the degree of numerator $<$ degree of denominator

Ex: $\dfrac{x^3+2}{x^2+x}$

Procedure: The same as for long division for finding zeros.
However, the rest does not have to be 0.

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$x^2 + x \quad / \quad x^3 \qquad\qquad +2 \quad \backslash$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x - 1 + \frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

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Target: rational function = polynomial+other rational function
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$$\begin{array}{lccccc} & x^2+x & / & x^3 & +2 & \backslash \quad x \\ x\cdot & (x^2+x) & \rightarrow & x^3+x^2 & & \end{array}$$

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\begin{array}{rrlll}
 & x^2+x & / \ x^3 \qquad +2 & \backslash & x \\
x\cdot & (x^2+x) \to & x^3+x^2 & & \\
 & - & \overline{\qquad\qquad\qquad} & & \\
 & & \quad -x^2 \quad +2 & &
\end{array}
$$

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\begin{array}{rlll}
 & x^2 + x \;\; / & x^3 \qquad\qquad +2 & \backslash \;\; x - 1 \\
x \cdot & (x^2 + x) \to & x^3 + x^2 & \\
 & - & \overline{} & \\
 & & \quad -x^2 \qquad +2 & \\
-1 \cdot & (x^2 + x) \to & \quad -x^2 - x &
\end{array}
$$

Conclusion

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\begin{array}{rrl}
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 & - & \overline{\qquad\qquad\qquad} \\
 & & \;\; -x^2 \quad +2 \\
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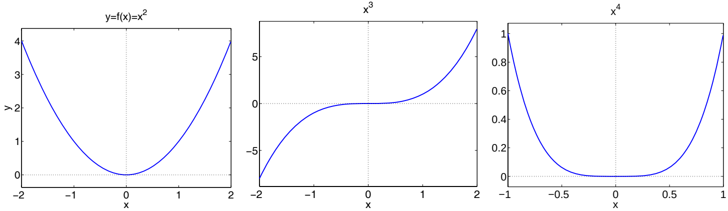
Inequalities, abs. values, rational funct.

1. $x^2 - 3x \geq -2$
2. $\frac{x+1}{x} \geq 2$
3. Simplify $\sqrt{x^4 + 6x^2 + 9}$
4. $|2x - 5| < 6$

Roots $\sqrt[n]{a}$

$$
\begin{array}{llll}
x^2 = a & \Longrightarrow & x = \pm\sqrt{a} & \text{here must be } a \geq 0 \\
x^3 = a & \Longrightarrow & x = \sqrt[3]{a} & \text{here } a \text{ can be everything } (a \in \mathbb{R}) \\
x^4 = a & \Longrightarrow & x = \pm\sqrt[4]{a} & \text{here must be } a \geq 0 \\
x^5 = a & \Longrightarrow & x = \sqrt[5]{a} & \text{here } a \text{ can be everything } (a \in \mathbb{R})
\end{array}
$$

etc; these are standard results that you can use in exams



Ex: $x^3 = 8$ thus $x^3 - 8 = 0$: you could say right away $x = 2$

but you could also solve this using the recipe for higher order polynomials

- guess solution: $x = 2$
- long division: $\frac{x^3-8}{x-2} = x^2 + 2x + 4$
- the discriminant of $x^2 + 2x^2 + 4 = -12 < 0$

  thus no further solutions

$$x^3 = \quad 27$$

n-th power roots: examples

$$x^3 = \quad 27 \qquad\qquad x = \sqrt[3]{27} =$$

n-th power roots: examples

$$\begin{aligned} x^3 &= 27 \qquad\qquad x = \sqrt[3]{27} = 3 \\ x^3 &= -27 \end{aligned}$$

n-th power roots: examples

$$\begin{aligned} x^3 &= 27 & \qquad x &= \sqrt[3]{27} = 3 \\ x^3 &= -27 & \qquad x &= \sqrt[3]{-27} = -3 \\ x^4 &= 4 \end{aligned}$$

n-th power roots: examples

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n-th power roots: examples

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x^4 = & 4 & \qquad x = \pm\sqrt[4]{4} = \pm \sqrt[2]{\sqrt[2]{4}} = \pm\sqrt[2]{2} = \pm\sqrt{2} \\
 & & \qquad \text{different notation: } \pm 4^{\frac{1}{4}} = \pm(4^{\frac{1}{2}})^{\frac{1}{2}} = \pm 2^{\frac{1}{2}} \\
x^4 = & -4 &
\end{array}
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x^3 &= 27 & x &= \sqrt[3]{27} = 3 \\
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& & & \text{different notation: } \pm 4^{\frac{1}{4}} = \pm(4^{\frac{1}{2}})^{\frac{1}{2}} = \pm 2^{\frac{1}{2}} \\
x^4 &= -4 & & \text{no solutions}
\end{aligned}
$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} =$$

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Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} =$$

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$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} =$$

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"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} = \frac{1}{\sqrt{a} - 2} \cdot \frac{\sqrt{a} + 2}{\sqrt{a} + 2} = \frac{\sqrt{a} + 2}{(\sqrt{a})^2 - 2^2} = \frac{\sqrt{a} + 2}{a - 4}$$

$$\frac{1}{\sqrt{a} + \sqrt{b}} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} = 15\sqrt{5}$$

Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{2}}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} = \tfrac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a}-2} = \frac{1}{\sqrt{a}-2} \cdot \frac{\sqrt{a}+2}{\sqrt{a}+2} = \frac{\sqrt{a}+2}{(\sqrt{a})^2 - 2^2} = \frac{\sqrt{a}+2}{a-4}$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{b}} \cdot \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a})^2 - (\sqrt{b})^2} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

$=$ and $<, \geq$ with roots

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x - 1$$

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\begin{aligned}\sqrt{4x+1} = x-1 \implies 4x+1 = (x-1)^2 \implies x^2 - 6x = 0 \\ \implies x\,(x-6) = 0 \implies x = 0 \text{ or } x = 6\end{aligned}$$

Check:

$=$ and $<, \geq$ with roots

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x-1 \implies 4x+1 = (x-1)^2 \implies x^2 - 6x = 0$$
$$\implies x\,(x-6) = 0 \implies x = 0 \text{ or } x = 6$$

Check: $x = 0$ is no solution, $x = 6$ is

- Equations: solve by taking squares (evt several times)
  check answers always at the end
- Inequalities: solve the equality
  also pay attention that for the solutions (intervals) all arguments under the roots have to be ≥ 0

Algebraic skills: frequently made mistakes

FAIL: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

GOOD: $\sqrt{xy} = \sqrt{x}\sqrt{y}$ (if $x \geq 0, \ \ y \geq 0$)

FAIL: $x\,(x-2) = 1 \implies x = 1$ or $x - 2 = 1$

GOOD: $x\,(x-2) = 0 \implies x = 0$ or $x - 2 = 0$

FAIL: $x^2 = 3x \implies x = 3$

GOOD: $x^2 = 3x \implies x = 3$ or $x = 0$ because $x\,(x-3) = 0$

FAIL: $\sqrt{x^2} = x$

GOOD: $\sqrt{x^2} = |x|$

FAIL: $x^2 = a \implies x = \sqrt{a}$

GOOD: $x^2 = a \implies x = \pm\sqrt{a}$

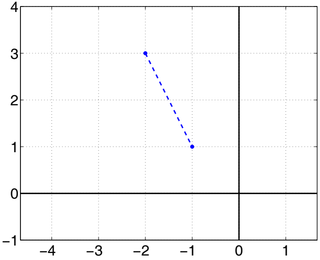
FAIL: $\frac{2}{x+3} = \frac{2}{x} + \frac{2}{3}$

GOOD: $\frac{x+3}{2} = \frac{x}{2} + \frac{3}{2}$

The plane $\mathbb{R}^2$, distance, lines

$\mathbb{R}^2$: notation for a 2-dimensional space

Distance d between 2 points $(x_1, y_1) = (-1, 1)$ adn $(x_2, y_2) = (-2, 3)$



- $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2$
- $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + 4} = \sqrt{5}$ (Pythagoras! [A B12])

Slope m of the line between the points $(-1, 1)$ and $(-2, 3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{2}{-1} = -2$$ [A P.2 E7]

Lines (I)

The line through 2 points (x_1, y_1) and (x_2, y_2)

two-point equation: $(y - y_1)\Delta x = (x - x_1)\Delta y$

(Remember: $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$).

point-slope equation: $y = m(x - x_1) + y_1$

Formula is correct because filling in yields:

- (x_1, y_1): $0 \cdot \Delta x = 0 \cdot \Delta y$
- (x_2, y_2): $(y_2 - y_1)\Delta x = (x_2 - x_1)\Delta y \Longrightarrow \Delta y \Delta x = \Delta x \Delta y$

Lines (II)

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$
\begin{aligned}
(y - y_1)\underbrace{\Delta x}_{=-1} = (x - x_1)\underbrace{\Delta y}_{=2} \quad &\implies \quad (y-1)\cdot -1 = (x+1)\cdot 2 \\
&\implies \quad (y-1) = -2(x+1) \\
&\implies \quad y = -2x - 1
\end{aligned}
$$

because $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2.$
Straight lines thus have the form $ax + by = c$. Special cases:

- $a = 0$: horizontal line
- $b = 0$: vertical line

Ex: Set point $(-2, 3)$
Horizontal line through the point has the eq.:

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Horizontal line through the point has the eq.: $y = 3$ $(b = 1)$
Vertical line through this point has the eq.:

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Horizontal line through the point has the eq.: $y = 3$ $(b = 1)$
Vertical line through this point has the eq.: $x = -2$ $(a = 1)$

Lines (III)

Intersections with the x- and y-axes are called "intercepts"

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$y = -2x - 1$$

Where does this line intersect the x-axis?

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$$y = -2x - 1$$

Where does this line intersect the x-axis? Then $y = 0$ thus

$0 = -2x - 1 \implies -2x = 1 \implies x = -\frac{1}{2}$ (thus "x-intercept" $= -\frac{1}{2}$)

Where does this line intersect the y-axis?

Lines (III)

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Where does this line intersect the y-axis?

Then $x = 0$ thus $y = -2 \cdot 0 - 1 = -1$ (thus "y-intercept" $= -1$)

Do [A P.2 T38]

Vertical and general lines (I)

A line is vertical if $x_1 = x_2$, then it follows

$$(y - y_1)\Delta x = (x - x_1)\Delta y \implies 0 = (x - x_1)\Delta y \implies 0 = x - x_1$$

A vertical line in $\mathbb{R}^2$ is given by $x = x_1$.

Other lines: $x_1 \neq x_2 \implies \Delta x \neq 0$ thus

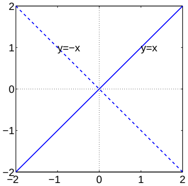
$$(y - y_1)\Delta x = (x - x_1)\Delta y \implies (y - y_1) = \frac{\Delta y}{\Delta x}(x - x_1) \implies$$
$$(y - y_1) = m(x - x_1)$$

A general line in $\mathbb{R}^2$ is described by the equation $y = mx + b$

with $b = m \cdot x_1 + y_1$ the y-intercept and $m = \Delta y / \Delta x$ the slope.

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:

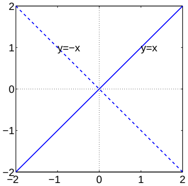


Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope $\frac{1}{2}$
because $-2 \cdot \frac{1}{2} = -1$
Thus $y = \frac{1}{2}x + b$, now fill in $(1, 1)$:
$1 = \frac{1}{2} + b \implies b = \frac{1}{2}$ so the line $y = \frac{1}{2}x + \frac{1}{2}$. [Do A P.2 T49]

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$



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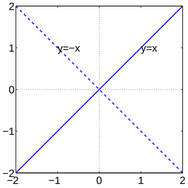

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The given line has the slope

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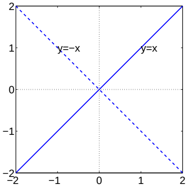

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Thus the sought line has the slope

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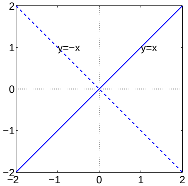

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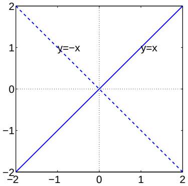

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Perpendicular lines

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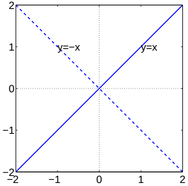

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Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

Circle with center $(0,0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

Distance between (x, y) and $(1, 2)$ is

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

Distance between (x, y) and $(1, 2)$ is $\sqrt{(x-1)^2 + (y-2)^2}$

If the distance is equal to r is then is holds

$\sqrt{(x-1)^2 + (y-2)^2} = r \iff (x-1)^2 + (y-2)^2 = r^2.$

General equation of a circle in $\mathbb{R}^2$ with center (a, b) and radius r:

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle with center (1,2) and radius 3:

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

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General equation of a circle in $\mathbb{R}^2$ with center (a, b) and radius r:

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle with center (1,2) and radius 3:

$(x-1)^2 + (y-2)^2 = 3^2$

Disc, circle and “hole”

The circle with radius r and center (a, b) consists of all points (x, y) for which it holds

$(x-a)^2 + (y-b)^2 = r^2$	circle line

The disk with radius r and center (a, b) consists of all points (x, y) inside the circle for which it holds

$(x-a)^2 + (y-b)^2 < r^2$	circle interior $\rightarrow$ open disk
$(x-a)^2 + (y-b)^2 \leq r^2$	circle interior $+$ line $\rightarrow$ closed disk

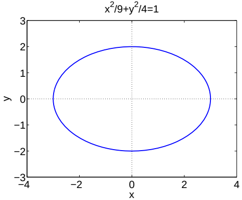
The “hole” with radius r and center (a, b) consists of alle points (x, y) outside the circle for which it holds

$(x-a)^2 + (y-b)^2 > r^2$	circle exterior

Ellipse with center $(0, 0)$ and "radii" r_x and r_y

We saw:
Circle with radius r and center $(0, 0)$:

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$



ellipse $\boxed{\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 = 1}$

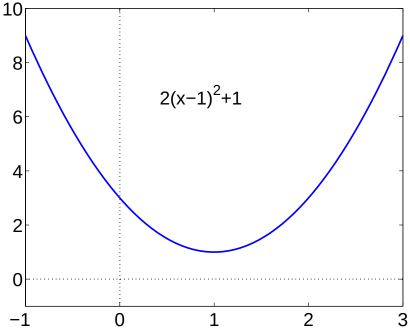
is thus of the form $cx^2 + dy^2 = e$

General ellipse with center (a, b) and semi-axes r_x and r_y

$$\left(\frac{x-a}{r_x}\right)^2 + \left(\frac{y-b}{r_y}\right)^2 = 1$$

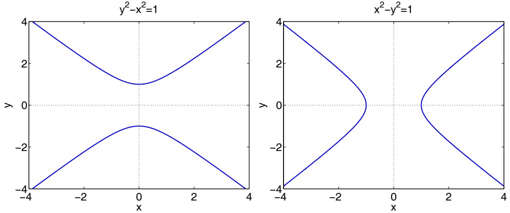
Parabolas

Parabolas with the origin as vertex and the y-axis as axis of symmetry:
$y = ax^2$ with $a \neq 0$.
Parabolas with (x_v, y_v) as vertex and the y-axis as axis of symmetry:

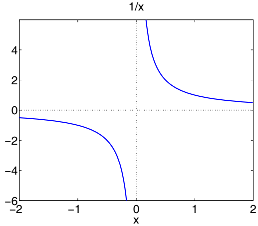


$$y = a\,(x - x_V)^2 + y_V$$

For example: $\left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 = 1$ and $\left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 = -1$



Also: $xy = c$ thus $y = \frac{c}{x}$



Overview of quadratic equations

Standard form has "$= 1$" on the right hand side

Circle ex: $$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1 \quad \text{or} \quad x^2 + y^2 = r^2$$

Ellipse ex: $$\left(\frac{x-a}{r_x}\right)^2 + \left(\frac{y-b}{r_y}\right)^2 = 1 \quad \text{or} \quad cx^2 + dy^2 = e$$

Parabola ex: $$\left(\frac{x-a}{r_x}\right)^2 + \frac{y-b}{r_y} = 1 \quad \text{or} \quad y = cx^2 + d$$

Hyperbola ex: $$\left(\frac{x-a}{r_x}\right)^2 - \left(\frac{y-b}{r_y}\right)^2 = \pm 1 \quad \text{or} \quad xy = c$$

A curve is an equation of the form $f(x, y) = 0$.

Ex:

- $f(x, y) = x^2 - (2x - y^2) = 0$
  $\rightarrow x^2 = 2x - y^2$
- $f(x, y) = 2y - 3x + 2 = 0$
  $\rightarrow 2y = 3x - 2$ (straight line)
- $f(x, y) = y - (3x + 2)$
  $\rightarrow y = 3x + 2$ (straight line)

What kind of curve is $x^2 = 2x - y^2$?
ellipse, parabola, ...?

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$x^2 - 2x + y^2 = 0$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$x^2 - 2x + y^2 = 0$$
$$\implies x^2 - 2x + 1 - 1 + y^2 = 0$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$
\begin{aligned}
& x^2 - 2x + y^2 = 0 \\
\implies\ & x^2 - 2x + 1 - 1 + y^2 = 0 \\
\implies\ & (x-1)^2 + y^2 = 1
\end{aligned}
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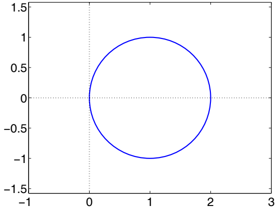
\end{aligned}
$$

Circle with radius 1 and center (1,0)



See [A page 18], do [A, P.2 T9,11] and [A P.3 T7,15]

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

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Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$

$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$
$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$
$$\implies (x - \tfrac{3}{2})^2 + y^2 = (\tfrac{3}{2})^2$$

Determining the type of curve

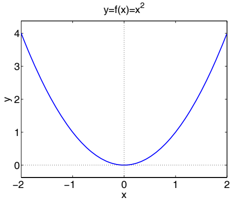
What does $x^2 = 3x - y^2$ represent?

Complete the square

$$\begin{aligned} & x^2 - 3x + y^2 = 0 \\ \implies & x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0 \\ \implies & (x - \tfrac{3}{2})^2 + y^2 = (\tfrac{3}{2})^2 \end{aligned}$$

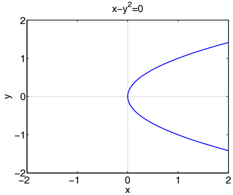
Circle with radius 3/2 and center (3/2,0)

Function f is a rule that:
prescribes for each $x \in D$ (domain) exactly one $f(x)$
Notation: $y = f(x)$
x is the independent, y is the dependent variable
Function or not?

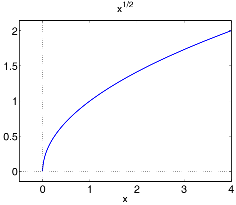


$y = f(x) = x^2$
function

Do [A P.4 T7]



$y = \pm\sqrt{x}$
no function
but curve



$y = \sqrt{x}$
function on $[0, \infty)$
no function on $\mathbb{R}$

Function $y = g(x)$ is described by the curve

$$y - g(x) = 0$$

that means by $f(x, y) = 0$ with $f(x, y) = y - g(x)$. Each function “is” therefore a curve.

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

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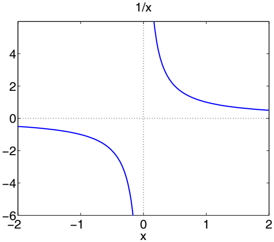

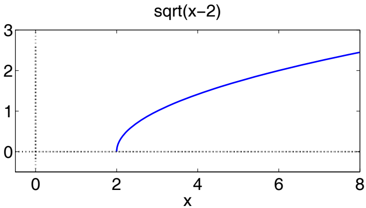

Do [A P.4 T6] and [A P.6 T16]

Even/odd function I

Even function

- $f(x) = f(-x)$ for all x
- means: graph of f is the same as if it is mirrored at the y-axis
- ex: $f(x) = x^2$, $f(x) = x^6 + x^2$, $f(x) = 1$, $f(x) = \cos(x)$
- sums of even powers in x are even

Odd function

- $f(x) = -f(-x)$ for all x
- means: graph of f is the same as first being mirrored at the y-axis, then at x-axis
- ex: $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^7 + x^3$, $f(x) = \sin(x)$
- sums of odd powers in x are odd

Ex: $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 + (-x) = x^2 - x$
Is function even? Is $x^2 - x = x^2 + x$ for all x?

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Ex: $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 + (-x) = x^2 - x$
Is function even? Is $x^2 - x = x^2 + x$ for all x? No
Is function odd? Is $x^2 - x = -(x^2 + x)$ for all x? No
A function does NOT have to be even or odd!

Symmetries

Even and odd are special cases of (anti-)symmetry:
namely around $x = 0$
One can also have symmetries around $x = 1$:

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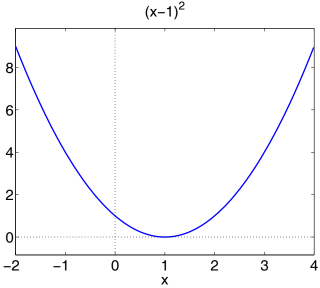

This can easily be seen by subsituting $w = x - 1$: $y = w^2$, even function:
symmetric around $w = 0$,
this coincides with $x = 1$
Ex: function which is anti-symmetric around $x = 2$?

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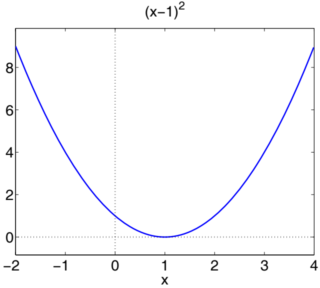

This can easily be seen by subsituting $w = x - 1$: $y = w^2$, even function:
symmetric around $w = 0$,
this coincides with $x = 1$
Ex: function which is anti-symmetric around $x = 2$?
$f(x) = x - 2$ or $f(x) = (x - 2)^3$ or ...

- Give domain and range of $f(x) = -x^2 + 2x - 7$
- Give domain and range of for $f(x) = \sqrt{9 - x^2}$
- Draw the graph of $y = -x^2$. How does the equation change for shifting by 7 left, right, up, down?
- Give function and domain for shifting $f(x) = \sqrt{x}$ 1 to the left
- Even, odd or nothing?
  $\sqrt{1 - x^2}, \quad x + 1, \quad x^3 + x$

Addition and Multiplication

If f and g functions are, then also sums and products of them are functions:

operation			example
$(fg)(x)$	$=$	$f(x) \cdot g(x)$	$x^2 \sin(x)$
$(f+g)(x)$	$=$	$f(x) + g(x)$	$x^2 + \sin(x)$
$(f-g)(x)$	$=$	$f(x) - g(x)$	$x^2 - \sin(x)$
$\left(\frac{f}{g}\right)(x)$	$=$	$\frac{f(x)}{g(x)}$	$x^2/\sin(x)$

Do [A P.5 T2]

Composite functions – one after the other

Composite functions: $(f \circ g)(x) = f(g(x))$ "f after g"

- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$
  $(f \circ g)(x) =$

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  $$(f \circ g)(x) = 4\,(x^2 - 1) - 1$$
  $(g \circ f)(x) =$

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- $\sqrt{\sin(x^2)} = f(g(h(x)))$ with:

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  $(f \circ g)(x) = 4\,(x^2 - 1) - 1$
  $(g \circ f)(x) = (4x - 1)^2 - 1$
- $\sqrt{\sin(x^2)} = f(g(h(x)))$ with:
  $f(x) = \sqrt{x},\ \ g(x) = \sin(x),\ \ h(x) = x^2$

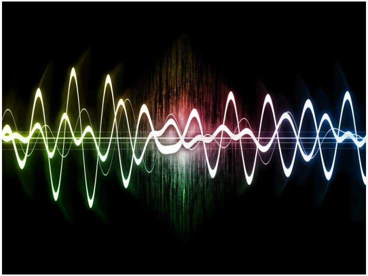
$\mathrm{D}_{f \circ g}$ consists of all $x \in D_g$ for which $g(x) \in D_f$

Ex: $f(x) = 1/x$, $g(x) = 1/x$. Then is $(f \circ g)(x) = x$ and $\mathrm{D}_{f \circ g} = \mathbb{R} - \{0\}$.
Do [A P.5 T6,7,9,13,15]

Week 2: Trigonometrics, powers, and vectors

- Trigonometric functions (sin, ...)
- Powers
- Vectors in 2 and 3 dimensions
- Lines and planes
- Distances, angles and the inner product
- The Cross Product

Sound



Sound, water, electro-magnetic and other waves are a sum of elementary waves: $A\sin(f \cdot 2\pi \cdot t)$

- $\sum_{k=1}^{n} A_k \sin(2\pi \cdot f_k \cdot t)$

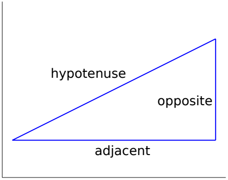
where the frequency f determines the pitch and
the amplitude A the intensity of the sound. sin is the sine function

Trigonometric functions: sin, cos, tan

$\sin = \frac{\text{opp}}{\text{hyp}}$, $\cos = \frac{\text{adj}}{\text{hyp}}$, $\tan = \frac{\text{opp}}{\text{adj}}$

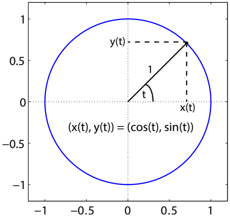
Relation degrees – radians:

$30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, $90^\circ = \frac{\pi}{2}$



Radians = the length of a piece of the circle

Complete circle $= 2\pi = 360^\circ$



Point on a circle with radius 1 has the coordinates $(\cos(t), \sin(t))$

This agrees with $\sin^2(t) + \cos^2(t) = (\sin(t))^2 + (\cos(t))^2 = 1$

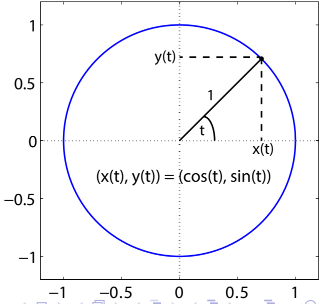
Similar: point on the circle with radius a has the coordinates $(a\cos(t), a\sin(t)) = a(\cos(t), \sin(t))$

The 4 quadrants

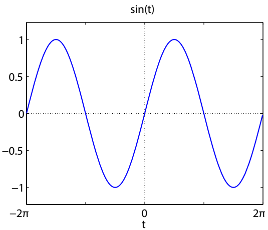
1. $t \in [0, \pi/2] \Longrightarrow (cos(t), sin(t))$ is in the 1st quadrant
2. $t \in [\pi/2, \pi] \Longrightarrow (cos(t), sin(t))$ is in the 2nd quadrant
3. $t \in [\pi, 3/2 \cdot \pi] \Longrightarrow (cos(t), sin(t))$ is in the 3rd quadrant
4. $t \in [3/2 \cdot \pi, 2\pi] \Longrightarrow (cos(t), sin(t))$ is in the 4th quadrant

so that we define

1. the 1st quadrant: $t \in [0, \pi/2]$
2. the 2nd quadrant: $t \in [\pi/2, \pi]$
3. the 3rd quadrant: $t \in [\pi, 3/2 \cdot \pi]$
4. the 4th quadrant: $t \in [3/2 \cdot \pi, 2\pi]$



Sine, Cosine, and Tangens functions



$\sin(t+2\pi)=\sin(t)$

odd function:

$\sin(-t)=-\sin(t)$

$\sin(t)=0$

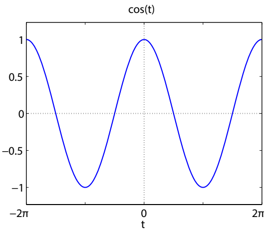
$\implies \quad t=k\pi \qquad (k\in\mathbb{Z})$

$\sin(t)=1$

$\implies \quad t=\frac{\pi}{2}+2k\pi$

$\sin(t)=-1$

$\implies \quad t=\frac{3\pi}{2}+2k\pi$



$\cos(t+2\pi)=\cos(t)$

even function:

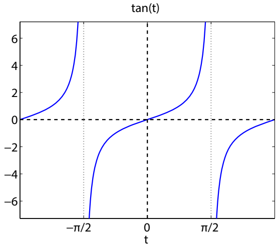
$\cos(-t)=\cos(t)$

$\cos(t)=0$

$\implies \quad t=\frac{\pi}{2}+k\pi$

$\sin(t)=\cos(\frac{\pi}{2}-t)$

$\cos(t)=\sin(\frac{\pi}{2}-t)$



$\tan(t)=\frac{\sin(t)}{\cos t}$

odd function:

$\tan(-t)=-\tan(t)$

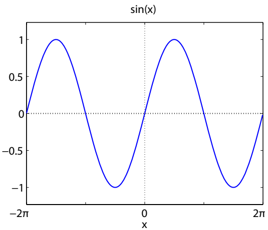
$\tan(t)=0$

$\implies \quad t=k\pi \qquad (k\in\mathbb{Z})$

vertical asymptote in

$\frac{\pi}{2}+k\pi, \quad k\in\mathbb{Z}$

Sine, Cosine, and Tangens functions



$\sin(x+2\pi) = \sin(x)$

odd function:

$\sin(-x) = -\sin(x)$

$\sin(x) = 0$

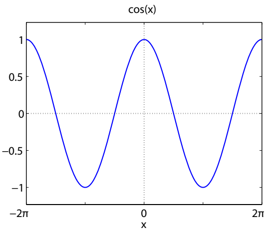
$\implies \quad x = k\pi \qquad (k \in \mathbb{Z})$

$\sin(x) = 1$

$\implies \quad x = \frac{\pi}{2} + 2k\pi$

$\sin(x) = -1$

$\implies \quad x = \frac{3\pi}{2} + 2k\pi$



$\cos(x+2\pi) = \cos(x)$

even function:

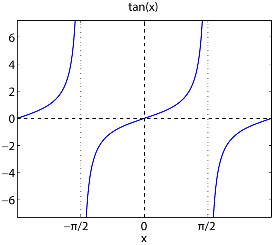
$\cos(-x) = \cos(x)$

$\cos(x) = 0$

$\implies \quad x = \frac{\pi}{2} + k\pi$

$\sin(x) = \cos(\frac{\pi}{2} - x)$

$\cos(x) = \sin(\frac{\pi}{2} - x)$



$\tan(x) = \frac{\sin(x)}{\cos x}$

odd function:

$\tan(-x) = -\tan(x)$

$\tan(x) = 0$

$\implies \quad x = k\pi \qquad (k \in \mathbb{Z})$

vertical asymptote in

$\frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

Trigonometric identities

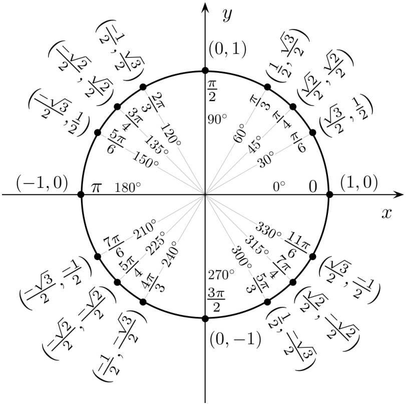
Know at least the red ones by heart!

$$
\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x)
\end{aligned}
$$

Furthermore, we can use the last two lines to rewrite $\sin^2(x)$ and $\cos^2(x)$

$$\sin^2(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \qquad \cos^2(x) = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)$$

Angles on the unit circle



Do we need to remember all of those?!?

No, just those in the 1st quandrant:

x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$\sin(x)$	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
$\cos(x)$	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$
$\tan(x)$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	ng

$$\begin{aligned}
\sin(\pi - x) &= \sin(\pi + (-x)) \\
&\underset{\sin(x+y)=\ldots}{=} \sin(\pi)\cos(-x) + \cos(\pi)\sin(-x) \\
&= 0 \cdot \cos(-x) + -1 \cdot \sin(-x) \\
&= \sin(x)
\end{aligned}$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [\frac{\pi}{2}, 2\pi]$ (I)

Use the formulas (that you know by heart)

- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

one can relate angles in Q2-4 to one in Q1!

Very practical: use π and 2π!

$$\begin{aligned}
\sin(x) &= \sin(\pi - x) & \cos(x) &= -\cos(\pi - x) & \tan(x) &= -\tan(\pi - x)\\
\sin(x) &= -\sin(x - \pi) & \cos(x) &= -\cos(x - \pi) & \tan(x) &= \tan(x - \pi)\\
\sin(x) &= -\sin(-x) & \cos(x) &= \cos(-x) & \tan(x) &= -\tan(-x)
\end{aligned}$$

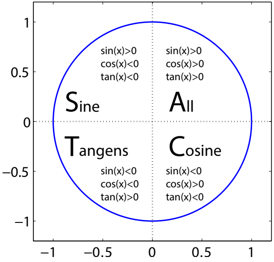
Ex: Calculate $\sin(x)$ for $x = \frac{5}{6}\pi$.
$\frac{5}{6}\pi$ is in the 2nd quadrant and
$\pi - \frac{5}{6}\pi$ in the 1st: $\sin(\frac{5}{6}\pi) = \sin(\pi - \frac{5}{6}\pi) = \sin(\frac{1}{6}\pi) = \frac{1}{2}$.

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [\frac{\pi}{2}, 2\pi]$ (II)

Who can (and wants to) remember all 9 relations?
Instead, remember to change the argument in the trigonometric function according to:

- if x is in 2nd quadrant $[\frac{1}{2}\pi, \pi]$: $x \to \pi - x$
- if x is in 3rd quadrant $[\pi, \frac{3}{2}\pi]$: $x \to x - \pi$
- if x is in 4th quadrant $[\frac{3}{2}\pi, 2\pi]$: $x \to 2\pi - x$

and the CAST rule to check for a changing sign.



Ex: Calculate $\sin(x)$ for $x = \frac{7}{6}\pi$.
$\frac{7}{6}\pi$ is in the 3rd quadrant.
$\frac{7}{6}\pi \to \frac{7}{6}\pi - \pi = \frac{1}{6}\pi$
sin(x) changes sign from Q3 to Q1
so that $\sin(\frac{7}{6}\pi) = -\sin(\frac{7}{6}\pi - \pi) = -\sin(\frac{1}{6}\pi) = -\frac{1}{2}$.

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\sin\left(\frac{\pi}{6}\right) \quad =$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$
\begin{aligned}
\sin\left(\tfrac{\pi}{6}\right) &= \tfrac{1}{2} \\
\cos\left(\tfrac{2}{3}\pi\right) &=
\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\sin(\tfrac{\pi}{6}) = \tfrac{1}{2}$$

$$\cos(\tfrac{2}{3}\pi) = -\cos(\tfrac{1}{3}\pi) =$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\begin{aligned} \sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\ \cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) = -\tfrac{1}{2} \qquad \text{qu 2, } \cos(\pi - x) \\ \tan(-\tfrac{\pi}{4}) &= \end{aligned}$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$
\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &&= -\tfrac{1}{2} && \text{qu 2, } \cos(\pi - x) \\
\tan(-\tfrac{\pi}{4}) &= -\tan(\tfrac{\pi}{4}) &&=
\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

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\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &&= -\tfrac{1}{2} && \text{qu 2, } \cos(\pi - x) \\
\tan(-\tfrac{\pi}{4}) &= -\tan(\tfrac{\pi}{4}) &&= -1 && \text{qu 1, tan is odd} \\
\sin(\tfrac{4\pi}{3}) &=
\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

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\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$
\begin{array}{llll}
\sin(\frac{\pi}{6}) & = \frac{1}{2} & & \\
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\cos(\tfrac{7}{6}\pi) &= -\cos(\tfrac{1}{6}\pi) &&= -\tfrac{1}{2}\sqrt{3} && \text{qu 3, } \cos(x - \pi) \\
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\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

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\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} & & \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &= -\tfrac{1}{2} & \quad \text{qu 2, } \cos(\pi - x) \\
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\tan(\tfrac{11}{6}\pi) &= -\tan(\tfrac{1}{6}\pi) &= &
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Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

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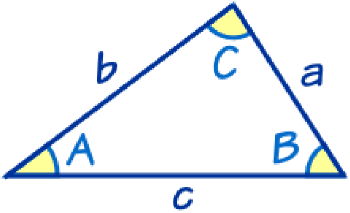
\tan(\frac{11}{6}\pi) & = -\tan(\frac{1}{6}\pi) & = -\frac{1}{3}\sqrt{3} & \text{qu 4, } \tan(x - 2\pi)
\end{array}
$$

Do [A P.7 T5]

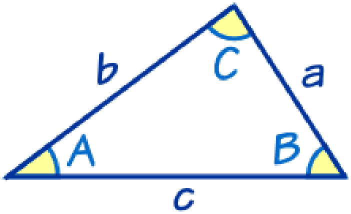
Sine and Cosine Rules



Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Sine and Cosine Rules



Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine rule

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

Do [A P.7 T31,37].

Trigonometric Equations and Inequalities

$\sin(x) = \sin(a)$

Trigonometric Equations and Inequalities

$$\begin{array}{r} \sin(x) = \sin(a) \\ \cos(x) = \cos(a) \end{array} \quad \Longrightarrow \quad x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi$$

Trigonometric Equations and Inequalities

$$
\begin{aligned}
\sin(x) = \sin(a) &\implies x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\
\cos(x) = \cos(a) &\implies x = \pm a + 2k\pi \\
\tan(x) = \tan(a) &
\end{aligned}
$$

Trigonometric Equations and Inequalities

$$
\begin{aligned}
\sin(x) = \sin(a) &\quad\Longrightarrow\quad x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\
\cos(x) = \cos(a) &\quad\Longrightarrow\quad x = \pm a + 2k\pi \\
\tan(x) = \tan(a) &\quad\Longrightarrow\quad x = a + k\pi \\
\sin(x) = \cos(a) &
\end{aligned}
$$

Trigonometric Equations and Inequalities

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\sin(x) = \sin(a) \quad &\Longrightarrow \quad x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\
\cos(x) = \cos(a) \quad &\Longrightarrow \quad x = \pm a + 2k\pi \\
\tan(x) = \tan(a) \quad &\Longrightarrow \quad x = a + k\pi \\
\sin(x) = \cos(a) \quad &\Longrightarrow \quad \text{rewrite cos to sin :} \\
&\qquad\ \cos(a) \underset{\sin(x+y)=\ldots}{=} \sin(\tfrac{\pi}{2} - a)
\end{aligned}$$

Recipe: Trigonometric Equations/Inequalities

1. Use substitution to rewrite to an equation with (powers of) only 1 unknown, eg $\sin(x)$
   and with only 1 argument (so no mix of $\sin(x)$ and $\sin(2x)$)
   or products of different functions (avoid roots as much as possible).
2. Maybe use substitution, eg $y = \sin(x)$, and solve the equation / inequality
3. Rewrite y back to eg $\sin(x)$, and solve, eg
   $\sin(x) = \frac{1}{2} \Longrightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$
4. Pay attention to "$+k\,2\pi$" (sin and cos) or "$+k\,\pi$" for tan

Exercises

1. For $\sin(x) = -\frac{5}{13}$ and $x \in [3/2\pi, 2\pi]$, determine $\cos(x)$ and $\tan(x)$.
2. For $\tan(x) = -\frac{7}{24}$, and x in the 4th quadrant, determine $\sin(x)$ and $\cos(x)$.
3. Calculate $\sin(\frac{5\pi}{12})$. Hint: make use of the $\sin(x + y)$ identity.
4. Solve $3\cos^2(x) - \sin^2(x) = 0$ for $x \in (0, 2\pi)$.
5. Solve $\cos^2(x) + \cos(x) = \sin^2(x)$.

Exercises

1. For $\sin(x) = -\frac{5}{13}$ and $x \in [3/2\pi, 2\pi]$, determine $\cos(x)$ and $\tan(x)$.
   $\cos(x) = \frac{12}{13}$ and $\tan(x) = -\frac{5}{12}$
2. For $\tan(x) = -\frac{7}{24}$, and x in the 4th quadrant, determine $\sin(x)$ and $\cos(x)$.
   $\cos(x) = \frac{24}{25}$ and $\sin(x) = -\frac{7}{25}$
3. Calculate $\sin(\frac{5\pi}{12})$. Hint: make use of the $\sin(x+y)$ identity.
   $\frac{1}{4}(\sqrt{6}+\sqrt{2})$
4. Solve $3\cos^2(x) - \sin^2(x) = 0$ for $x \in (0, 2\pi)$.
   $x = \frac{\pi}{3} \vee x = \frac{4\pi}{3} \vee x = \frac{2\pi}{3} \vee x = \frac{5\pi}{3}$
5. Solve $\cos^2(x) + \cos(x) = \sin^2(x)$.
   $x = \pi + 2n\pi \vee x = \frac{\pi}{3} + 2n\pi \vee x = \frac{5\pi}{3} + 2n\pi$ with $n \in \mathbb{Z}$

Powers and roots

Reminder: Rules for calculating powers and roots

$$a^p a^q = a^{p+q} \qquad (ab)^p = a^p b^p \qquad a^{-p} = \frac{1}{a^p}$$

$$a^{1/2} = \sqrt{a} \qquad a^{1/n} = \sqrt[n]{a} \qquad a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

$$(a^p)^q = a^{pq} \qquad a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1}{\sqrt[q]{a^p}} \qquad \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

Useful to rewrite expressions into more simplified forms:

Ex: $$\frac{(3a^{-\frac{2}{3}}b)^2}{2a^2b^{-\frac{1}{3}}} = \frac{3^2(a^{-\frac{2}{3}})^2b^2}{2a^2b^{-\frac{1}{3}}} = \frac{9a^{-\frac{4}{3}}b^2}{2a^2b^{-\frac{1}{3}}} = \frac{9}{2}a^{-\frac{10}{3}}b^{\frac{7}{3}}$$

Equations with powers

Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

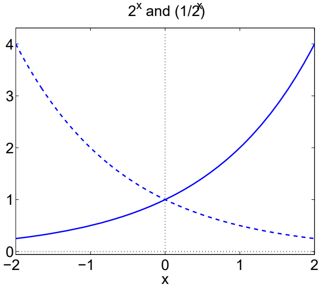
Equations with powers

Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

- $f(x) = a^x$ everywhere increasing function if $a > 1$
- $f(x) = a^x$ everywhere decreasing function if $0 < a < 1$



Ex:

$2^{x+1} + 2^{x+3} = 320$

Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

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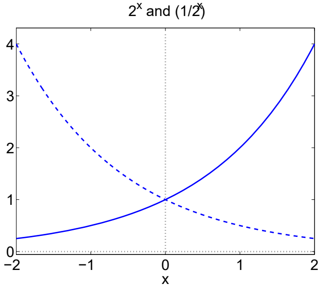

Ex:

$$2^{x+1} + 2^{x+3} = 320 \implies 2^x(2+2^3) = 320 \implies 2^x = 32 = 2^5 \implies x = 5$$

Ex: $16^{3x+3} = 8^{x^2+4}$

Equations with powers

Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

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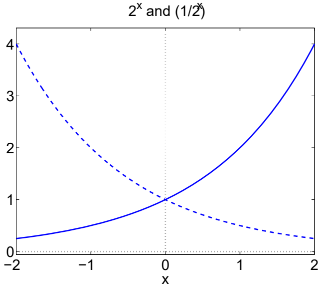

Ex:

$$2^{x+1} + 2^{x+3} = 320 \implies 2^x(2+2^3) = 320 \implies 2^x = 32 = 2^5 \implies x = 5$$

Ex: $16^{3x+3} = 8^{x^2+4} \implies (2^4)^{3x+3} = (2^3)^{x^2+4} \implies 2^{12x+12} = 2^{3x^2+12}$
$\implies 12x + 12 = 3x^2 + 12 \implies 3x(x-4) = 0 \implies x = 0$ or $x = 4$

Inequalities with powers

Again: get the same base number, then compare the powers
Remember:

- $f(x) = a^x$ is increasing everywhere if $a > 1$
- $f(x) = a^x$ is decreasing everywhere if $0 < a < 1$

Ex: the base number: is it < 1 or > 1?

- $f(x) = 2^x$ is increasing everywhere because $2 > 1$
- $f(x) = 2^{-x}$

Inequalities with powers

Again: get the same base number, then compare the powers
Remember:

- $f(x) = a^x$ is increasing everywhere if $a > 1$
- $f(x) = a^x$ is decreasing everywhere if $0 < a < 1$

Ex: the base number: is it < 1 or > 1?

- $f(x) = 2^x$ is increasing everywhere because $2 > 1$
- $f(x) = 2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$ is decreasing everywhere because $0 < 1/2 < 1$

Ex: Inequalities

- $2^x > 2^4 \implies x > 4$ because 2^x is increasing
- $(\frac{1}{2})^x > (\frac{1}{2})^4 \implies x < 4$ because $(\frac{1}{2})^x$ is decreasing

Quiz: Powers

1. Solve $e^{x^2} = 5e^{2x-1}$.

2. Solve $\frac{2^{2x}-8}{2^x-4} \leq 0$

3. $f(x) = 5^x$ is
   (a) increasing everywhere
   (b) decreasing everywhere
   (c) increasing for $x > 0$, decreasing for $x < 0$
   (d) increasing for $x < 0$, decreasing for $x > 0$

4. If $x < 5$, then
   (a) $5^5 > 5^x$ (b) $5^x \leq 5^5$ (c) $5^5 < 5^x$ (d) $5^x \geq 5^5$

Quiz: Powers

1. Solve $e^{x^2} = 5e^{2x-1}$.
   $x = 1 + \pm\sqrt{\ln(5)}$
2. Solve $\frac{2^{2x}-8}{2^x-4} \leq 0$
   $\frac{3}{2} \leq x < 2$
3. $f(x) = 5^x$ is
   (a) increasing everywhere
   (b) decreasing everywhere
   (c) increasing for $x > 0$, decreasing for $x < 0$
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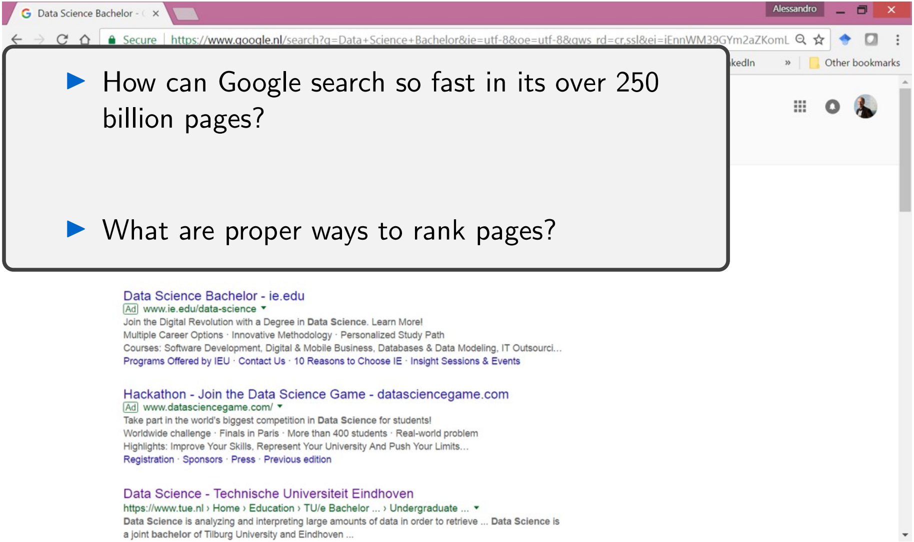
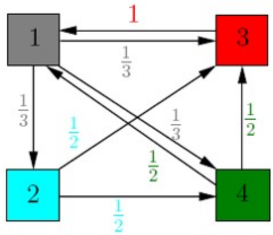
Google Search

- How can Google search so fast in its over 250 billion pages?
- What are proper ways to rank pages?



Mathematical Model for Weblinks



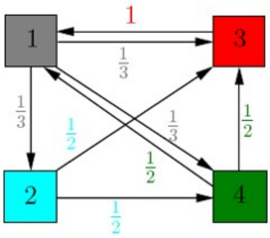
Model the WWW as a **graph**:

- Squares (nodes, vertices) denote web pages
- Arrows (edges) denote links from one page to another

> Relevance is translated into number of incoming links weighted by the importance of referring pages.

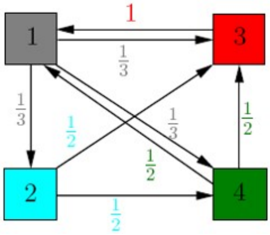
(What comes first?)

System of Linear Equations



$$
\begin{aligned}
x_1 &= 1 \cdot x_3 + \frac{1}{2} \cdot x_4 \\
x_2 &= \frac{1}{3} \cdot x_1 \\
x_3 &= \frac{1}{3} \cdot x_1 + \frac{1}{2} \cdot x_2 + \frac{1}{2} \cdot x_4 \\
x_4 &= \frac{1}{3} \cdot x_1 + \frac{1}{2} \cdot x_2
\end{aligned}
$$

System of Linear Equations - Matrix and Vectors



$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} = A$$

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{31} \cdot \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

Information about page ranks is determined as a **vector** of page relevance (solutions of linear systems).

More Matrix-Vector problems

Alice	4			4
Bob		5	4	
Joe		5	4	
Sam	5			5

- User preferences/histories are encoded as vector data.
- How similar is your vector to those of others?
- completely similar = "parallel"
- completely different = "perpendicular"
- Trigonometric functions (just done) used with vector calculus (coming up)

Introduction: Vectors in Analytic Geometry

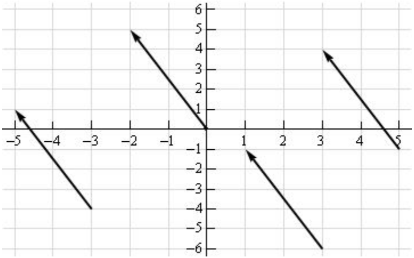
We will introduce vectors and their calculus in a specific application: Analytic Geometry

- describe directions in two or three space dimensions
- addition and scaling of vectors
- linear combinations
- lines and line segments in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Planes in $\mathbb{R}^3$
- Orthogonality, angle between vectors
- Intersections of lines, lines and planes, planes

Vectors in 2 and 3 dimensions

A vector is represented in plane or space by an arrow with direction and length (magnitude).

Vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (2 to the left, 5 up)



These vectors are equivalent, even though they have different starting points.

Vectors in 2 and 3 dimensions: Notation

In a plane (2D): 2 to the left and 5 up is a vector:

- Notation: Write $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ or $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (lectures: latter notation)
- Sometimes "also": $<-2, 5>$: eg in books to save space...
- $\mathbf{v}$, $\underline{v}$, $\vec{v}$, or simply v (on the slides $\mathbf{v}$, on paper, I write v of $\underline{v}$)
- Sketch: Arrow (a combination of a line segment and directions)

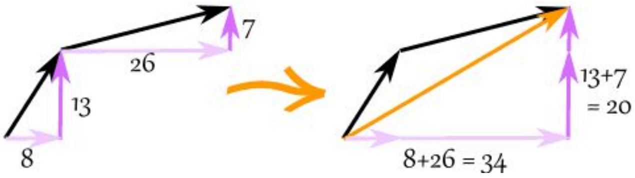
A vector has a direction and length (here $\sqrt{(-2)^2 + 5^2}$).

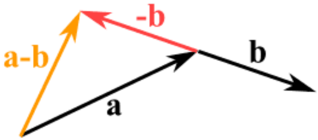
Vectors: Addition and subtraction

- add or subtract component-wise:

$$\begin{bmatrix} 8 \\ 13 \end{bmatrix} + \begin{bmatrix} 26 \\ 7 \end{bmatrix} = \begin{bmatrix} 34 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

- graphically: parallellogram





Vectors from point A to point B

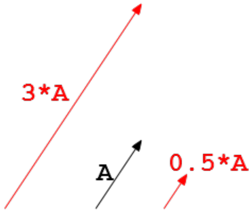
Because $A = (x_1, y_1)$ and $B = (x_2, y_2)$ then the vector from A to B is:

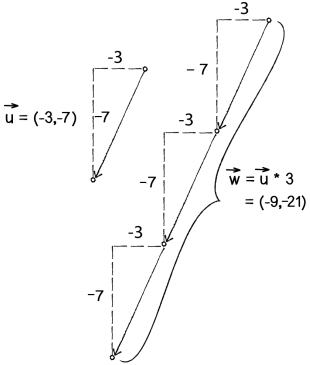
$$B - A = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Notation: AB or $\vec{AB}$.
Do [V 5 T1]

Scaling of vectors

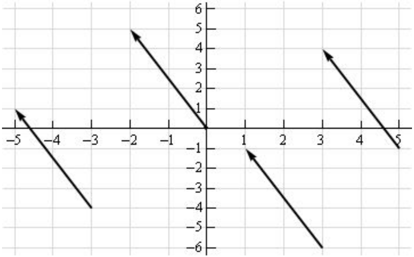
Scaling (scalar multiplication) changes the length and evt the direction:





- scalar means using (real) number
- multiply component-wise: $2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Scaling of vectors



- multiplication by 2:
  vector becomes 2 times as long, keeps the same direction
- multiplication by -2:
  vector becomes 2 times as long, in opposite direction

(Later for more about length)

Subtraction of vectors

Convention: Subtraction is addition of the vector with opposite direction:

$$\mathbf{b} - \mathbf{a} = \mathbf{b} + ((-1) \cdot \mathbf{a})$$

Ex: $\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Length of a vector

Notation:

- $\|\mathbf{x}\|$ (chosen)
- or sometimes also $|\mathbf{x}|$

Calculate: if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then

$$\boxed{\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} \quad \text{thus} \quad \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\boxed{\text{Distance between two points } \mathbf{x} \text{ and } \mathbf{y}: \quad \|\mathbf{x} - \mathbf{y}\|}$$

Ex: Distance between $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \sqrt{5}$

is also the length of the difference vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\left(\text{and } \begin{bmatrix} -1 \\ -2 \end{bmatrix}\right)$

Length of a vector

Ex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length

Length of a vector

Ex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $\sqrt{2}$

Sometimes we seek a vector in the same direction, but with length 1

Ex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sqrt{2} \cdot \underbrace{\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}}_{\text{length } 1}$$

Angle between vectors

Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$

Inner product in 2D: $\boxed{\mathbf{a} \cdot \mathbf{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2}$

Inner product nD (so also for $n = 3$):

$$\boxed{\mathbf{a} \cdot \mathbf{b} = (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \cdots + a_n b_n}$$

Useful for:

- length: $\boxed{\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2}}$
- angle: $\boxed{\cos(\mathbf{a}, \mathbf{b}) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}} \Leftarrow \boxed{\underbrace{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\mathbf{a}, \mathbf{b})}_{\text{inner product rule}}}$

Thus also $\boxed{\mathbf{a} \cdot \mathbf{b} = \cos(\alpha) \, \|\mathbf{a}\| \, \|\mathbf{b}\|, \quad \alpha = \angle(\mathbf{a}, \mathbf{b})}$

Different notation for inner product: $\boxed{\langle \mathbf{a}, \mathbf{b} \rangle}$ and $\boxed{\mathbf{a}^T \mathbf{b}}$

Angle between vectors

Ex: Calculate the angle θ between $\underbrace{\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}}_{\mathbf{v}}$ and $\underbrace{\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}}_{\mathbf{w}}$: Inner product rule:

$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} = \left| \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right| \cdot \left| \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \right| \cdot \cos\theta$$

or (length of both vectors is $\sqrt{(\pm 1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$):

$$\underbrace{1 \cdot -1 + \sqrt{3} \cdot \sqrt{3}}_{\mathbf{v} \cdot \mathbf{w}} = \underbrace{2}_{|\mathbf{v}|} \cdot \underbrace{2}_{|\mathbf{w}|} \cdot \cos\theta$$

and $2 = 4 \cdot \cos\theta \implies \cos\theta = \frac{1}{2} \implies \theta = \pm\pi/3$ (solution with minus sign goes away)

(the angle between two vectors in maximal π!)

Distances, angles and the inner product

- $\mathbf{x} \cdot \mathbf{x} \geq 0$

  logical, is same as $\|\mathbf{x}\|^2$, thus ≥ 0

- $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$

  only the $\mathbf{0}$-vector has length 0

- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

  just as for the multiplication of numbers, the order does not matter

- $(\lambda \mathbf{x}) \cdot \mathbf{y} = \lambda (\mathbf{x} \cdot \mathbf{y})$

  if $\mathbf{x}$ eg 2 times as long, then the inner product is 2 times as big

  logical, since $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\alpha)$

Orthogonal vectors

Vectors $\mathbf{a}$ and $\mathbf{b}$ are called $\boxed{\text{orthogonal (or perpendicular) if } \mathbf{a}\cdot\mathbf{b}=0}$

Notation: $\mathbf{a} \perp \mathbf{b}$

Works with: $\cos(\mathbf{a},\mathbf{b}) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$

because then the numerator is 0, so $\cos(\mathbf{a},\mathbf{b}) = 0$, thus $\angle(\mathbf{a},\mathbf{b}) = \frac{\pi}{2}$

Ex: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are indeed orthogonal

but also $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Orthogonal vectors

In $\mathbb{R}^2$ the vectors that are orthogonal to a given vector can be calculated

Ex: Which vectors are $\perp (1, 2)$?
We need to have: $(a_1, a_2) \cdot (1, 2) = a_1 + 2a_2 = 0$

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switch the components, and give one an extra minus sign, thus:

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- $(2,-1)$, because $(2,-1) \cdot (1,2) = 0$
- $(-2,1)$, because $(-2,1) \cdot (1,2) = 0$
- and also all multiples of $(-2,1)$ and $(2,-1)$:
  $(-2\lambda, \lambda) = \lambda(-2,1)$ for $\lambda \in \mathbb{R}$

(In $\mathbb{R}^2$ a line of vectors is perpendicular on a given vector!)

Quiz: Vector calculus

1. Given vectors $\mathbf{v} =< -2, 3 >$ and $\mathbf{u} =< 4, 6 >$, find
   $\mathbf{v} + 2\mathbf{u}$ and $\mathbf{u} - 4\mathbf{u}$

2. Given vectors $\mathbf{v} =< 1, -2 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $\mathbf{v} + 3\mathbf{u} = 0$

3. Given vectors $\mathbf{v} =< 4, 1 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $2\mathbf{v} - 3\mathbf{u} = 0$

4. Calculate the distance between the points $A(3, 1, 7)$ and $B(5, -1, 3)$

5. Calculate the angle between the vectors $\mathbf{v} =< 1, 1, 1 >$ and $\mathbf{w} =< 4, 0, -3 >$

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   $\mathbf{v} + 2\mathbf{u}$ and $\mathbf{u} - 4\mathbf{u}$
   $\mathbf{v} + 2\mathbf{u} =< 6, 15 >$ and $\mathbf{u} - 4\mathbf{u} =< -12, -18 >$
2. Given vectors $\mathbf{v} =< 1, -2 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $\mathbf{v} + 3\mathbf{u} = 0$
   $\mathbf{u} =< -\frac{1}{3}, \frac{2}{3}$
3. Given vectors $\mathbf{v} =< 4, 1 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $2\mathbf{v} - 3\mathbf{u} = 0$
   $\mathbf{u} =< \frac{8}{3}, \frac{2}{3} >$
4. Calculate the distance between the points $A(3, 1, 7)$ and $B(5, -1, 3)$
   $d = 2\sqrt{5}$
5. Calculate the angle between the vectors $\mathbf{v} =< 1, 1, 1 >$ and $\mathbf{w} =< 4, 0, -3 >$
   $\cos(\phi) = \frac{1}{5\sqrt{2}}$

Linear combination of vectors

In linear algebra, we say for vectors $\mathbf{v}_1, \ldots \mathbf{v}_n$ and real numbers $a_1, \ldots, a_n$ that the sum (-vector) $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$
is a linear combination.

Let's get to work with that.

Linear functions and combinations

A function is called linear if:

- in 1 variable: $f(x) = ax$ (or sometimes also with "$+b$")
- in 2 variables: $f(x, y) = ax + by$
- in n variables: $f(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$
- in n vectors: $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$

Thus linear is always:

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Thus linear is always: sum of constants times variables
What is not-linear?

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- in n vectors: $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$

Thus linear is always: sum of constants times variables
What is not-linear? Everything else, eq:

- everything containing multiplications of variables, eg x^2, xy, etc
- sin, cos, tan, e^x, $\ln(x)$, etc etc

Read examples [V 1.11]

Lines: An alternative to $ax + by = c$ in $\mathbb{R}^2$ and also $\mathbb{R}^3$

We have seen: Every line in $\mathbb{R}^2$ is described by:

All (x, y) for which: $ax + by = c$
Different a, b, c represent possibly different lines.
But it must not: $2x - 3y = 1$ is the same line as $3y - 2x = -1$ (multiply left/right with -1).

In $\mathbb{R}^3$

$$ax + by + cz = d$$

does not represent a line.

Lines: An alternative to $ax + by = c$ in $\mathbb{R}^2$ and also $\mathbb{R}^3$

Ex: What does $ax + by + cz = d$ describe in $\mathbb{R}^3$? Let's take an example:

$$0 \cdot x + 0 \cdot y + 1 \cdot z = 0$$

$(a = b = d = 0$ and $c = 1)$ – or $z = 0$. This is the xy-plane!: Table:

x	y	z	$0 \cdot x + 0 \cdot y + z$
0	0	0	$0 + 0 + 0 = 0$
1	0	0	$0 \cdot 1 + 0 + 0 = 0$
0	1	0	$0 + 0 \cdot 1 + 0 = 0$
1	1	0	$0 \cdot 1 + 0 \cdot 1 + 0 = 0$
$\vdots$	$\vdots$	0	$0 = 0$

The 4 points $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$ are the corners of a square in the xy-plane. It cannot be a straight line!
We need: **An expression for a line that works in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$ ...**

Parameter representation of a Line

Let $\mathbf{a} = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} x_B \\ y_B \end{bmatrix}$ be two coordinate vectors of the points A and B. (the same for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$). It holds that

$$\mathbf{x} = \mathbf{x}(\lambda) = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) \text{ for all } \lambda \in \mathbb{R}$$

describes all points on the line through A and B. Specifically

- $\lambda = 0$: $\mathbf{x} = \mathbf{a} + 0 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{a}$
- $\lambda = 1$: $\mathbf{x} = \mathbf{a} + 1 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{b}$

The representation $\mathbf{x} = \mathbf{x}(\lambda) = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$ with the parameter λ is called parameter representation or vector representation. Definitions:

- $\mathbf{a}$ is the support vector or the start point
- $\mathbf{b} - \mathbf{a}$ is the direction vector

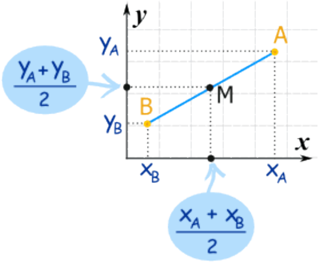
Parameter representation of a Line, now in $\mathbb{R}^3$

Ex: Parameter representation of a line through points $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$:

$$
\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2-1 \\ 3-1 \\ 1-1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\text{direction vector}} \\
&= \begin{bmatrix} 1+\lambda \\ 1+2\lambda \\ 1+0\cdot\lambda \end{bmatrix} = \begin{bmatrix} 1+\lambda \\ 1+2\lambda \\ 1 \end{bmatrix}
\end{aligned}
$$

Lines, line segments, straight lines

For all $\lambda \in \mathbb{R}$ is $\mathbf{x} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$ on the line through A en B:



- $\lambda = 0$: $\mathbf{x} = \mathbf{a} + 0 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{a}$ (point A)
- $\lambda = 1$: $\mathbf{x} = \mathbf{a} + 1 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{b}$ (point B)
- $\lambda = \frac{1}{2}$: $\mathbf{x} = \mathbf{a} + \frac{1}{2} \cdot (\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ middle between $\mathbf{a}$ en $\mathbf{b}$
- $\lambda = \frac{1}{4}$: $\mathbf{x} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$ middle between M and A.

Complete lines, half lines and line segments

Parameter representation $\boxed{\mathbf{x} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})}$
yields for each $\lambda \in \mathbb{R}$
one point $\mathbf{x}$ on the line through points B and A
Depending on what λ can be, this is a line segment, line etc:

- $\lambda \in [0, 1]$: line segment between A and B including these points
- $\lambda \geq 0$: half line from A through B and further
- $\lambda \in \mathbb{R}$: (complete) line through points A and B

Note that $\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$ is a linear combination because
$\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a} = \lambda\mathbf{b} + (1 - \lambda)\mathbf{a}$ is of the form: scalar times vector plus scalar times vector.

Lines, line segments, straight lines

Ex: A line has several parameter representations:

Parameter representations are not unique! $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$ just take $\mu = -\frac{1}{2}\lambda$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$ just take $\mu = -\frac{1}{2}\lambda - \frac{1}{2}$

If your neighbor gets a different answer, it does not mean anything! ...
But: All direction vectors are "the same" except for a minus sign.

Change from function to parameter representation:

Ex: $y = 2x + 3$ goes through $(0, 3)$ and $(1, 5)$, thus

parameter representation $\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \lambda \\ 3 + 2\lambda \end{bmatrix}$

Ex: Let a parameter representation be $\begin{bmatrix} 1 + 2\lambda \\ 2 + 3\lambda \end{bmatrix}$

then the line goes through $(1, 2)$ (for $\lambda = 0$) and $(3, 5)$ (for $\lambda = 1$)

thus it's the equation $y = \frac{3}{2}x + \frac{1}{2}$

Lines in $\mathbb{R}^3$

Can be given as in $\mathbb{R}^2$ by parametrisation:
starting point and direction vector

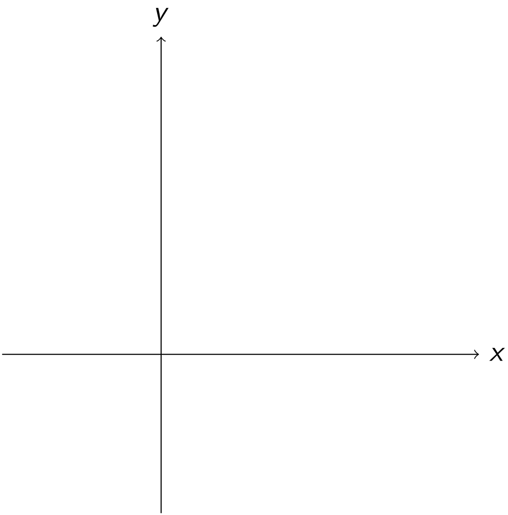
$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

A line can also be given by the intersection of two planes

Lines

- Determine a parametric representation of the lines through:
  - $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$
  - $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- Give an equation for the line:
  $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
- Find a parametric representation for the following lines in $\mathbb{R}^2$
  $3x - 4y + 7 = 0$

Vectors in $\mathbb{R}^2$



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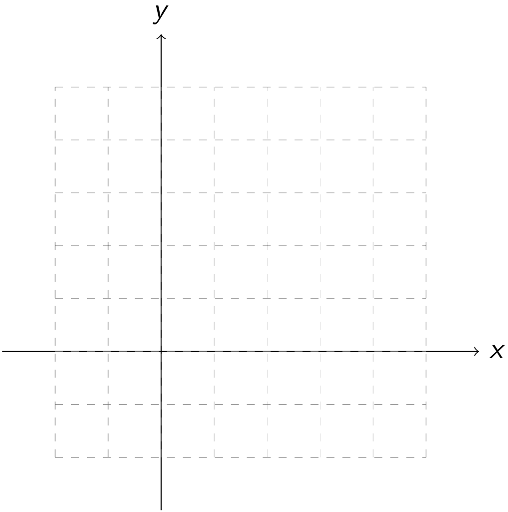

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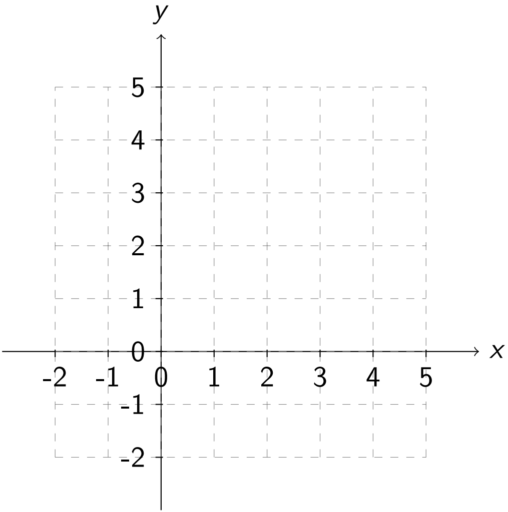

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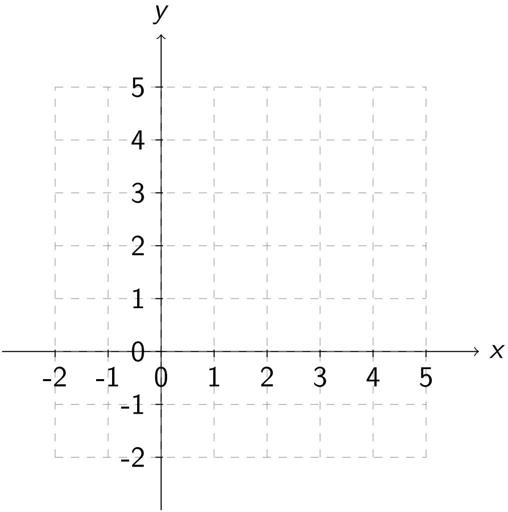

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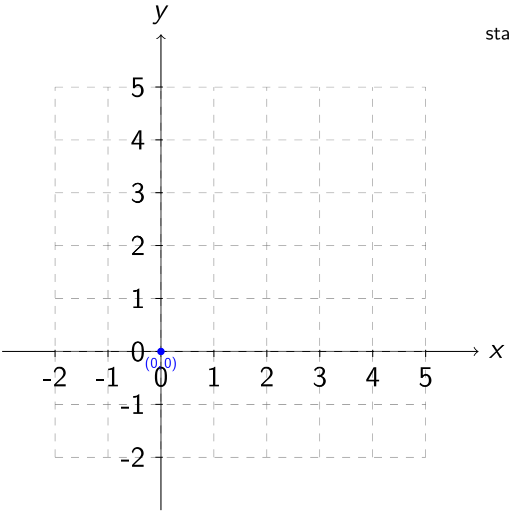
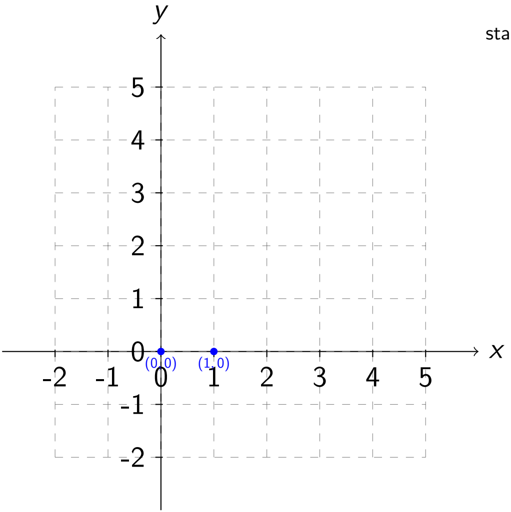


start point $\mathbf{p} = (0, 0)$

Vectors in $\mathbb{R}^2$



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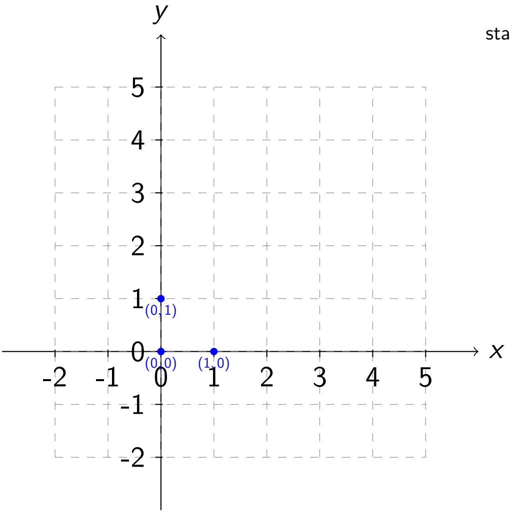

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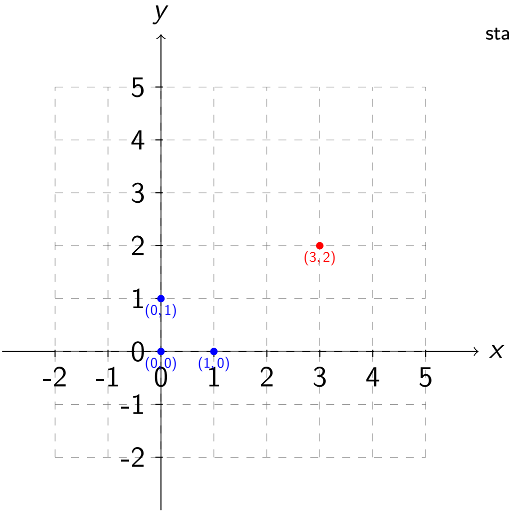
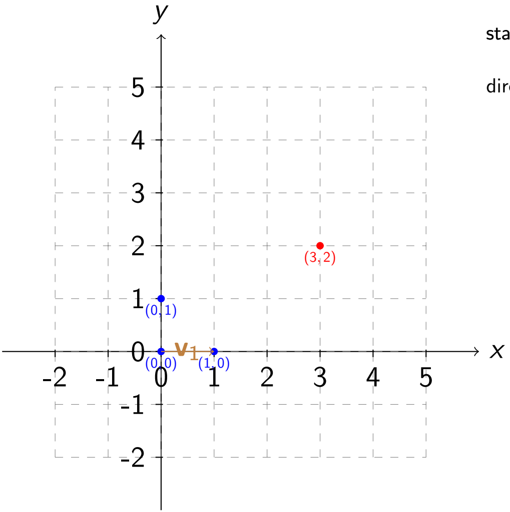


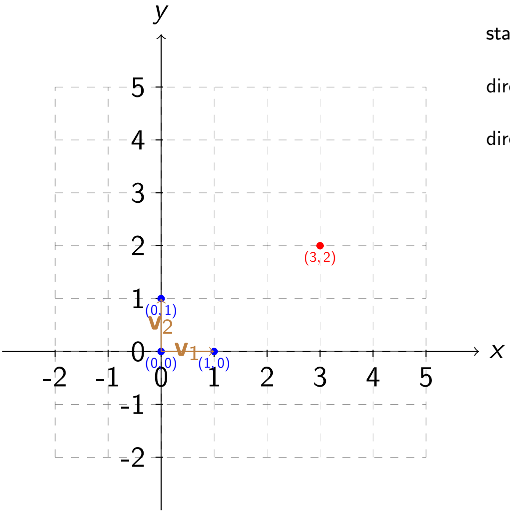
start point $\mathbf{p} = (0, 0)$, to reach $(3, 2)$



start point $\mathbf{p} = (0, 0)$, to reach $(3, 2)$

direction $\mathbf{v}_1 = (1, 0) - (0, 0) = (1, 0)$

Vectors in $\mathbb{R}^2$

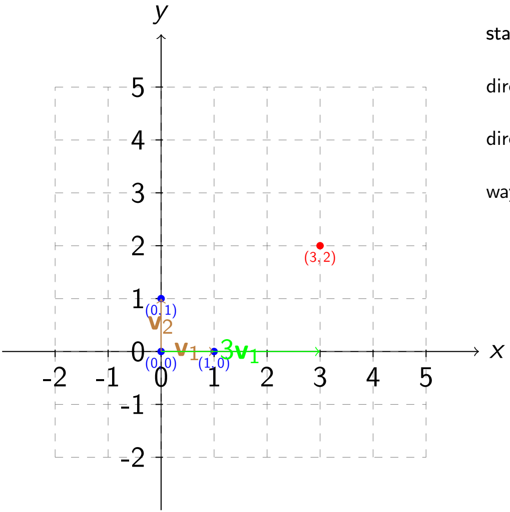


start point $\mathbf{p} = (0, 0)$, to reach $(3, 2)$

direction $\mathbf{v}_1 = (1, 0) - (0, 0) = (1, 0)$

direction $\mathbf{v}_2 = (0, 1) - (0, 0) = (0, 1)$

Vectors in $\mathbb{R}^2$



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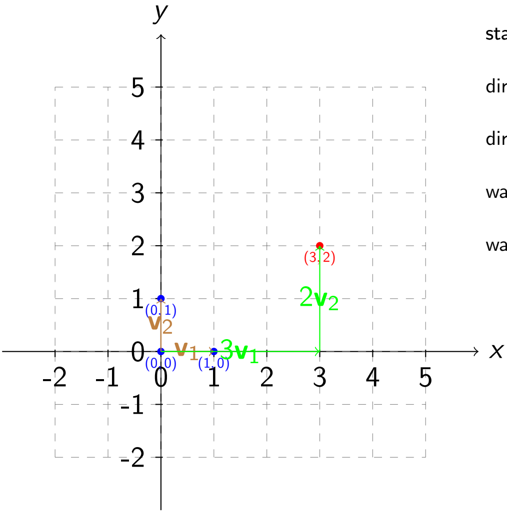
way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

Vectors in $\mathbb{R}^2$



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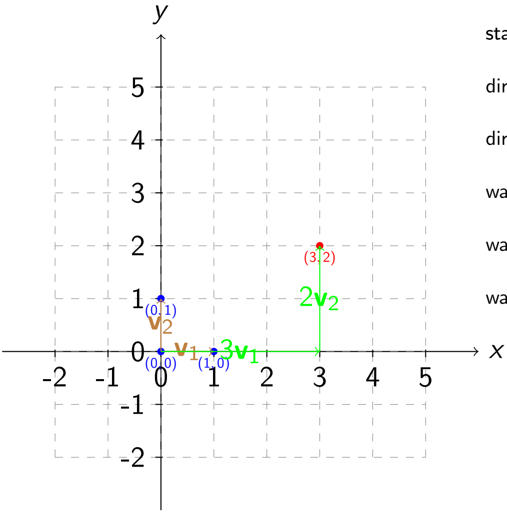
way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Vectors in $\mathbb{R}^2$



start point $\mathbf{p} = (0, 0)$, to reach $(3, 2)$

direction $\mathbf{v}_1 = (1, 0) - (0, 0) = (1, 0)$

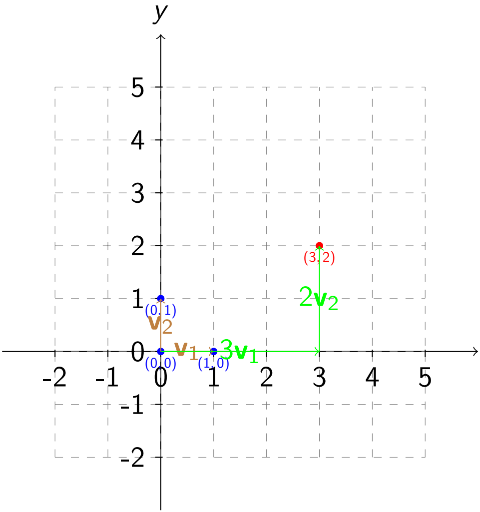
direction $\mathbf{v}_2 = (0, 1) - (0, 0) = (0, 1)$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{p} + 3\mathbf{v}_1 + 2\mathbf{v}_2$

Vectors in $\mathbb{R}^2$



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direction $\mathbf{v}_1 = (1,0) - (0,0) = (1,0)$

direction $\mathbf{v}_2 = (0,1) - (0,0) = (0,1)$

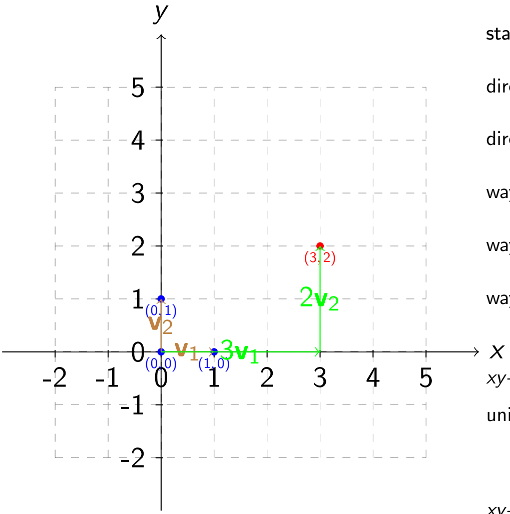
way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{p} + 3\mathbf{v}_1 + 2\mathbf{v}_2$

xy-plane: Each point (x, y) determined by unique λ, μ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Vectors in $\mathbb{R}^2$



start point $\mathbf{p} = (0,0)$, to reach $(3,2)$

direction $\mathbf{v}_1 = (1,0) - (0,0) = (1,0)$

direction $\mathbf{v}_2 = (0,1) - (0,0) = (0,1)$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

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xy-plane: Each point (x, y) determined by unique λ, μ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

xy-plane: $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{p} + \lambda\mathbf{v}_1 + \mu\mathbf{v}_2$

Why the parameter representation?

The 2D notation $y = ax + b$ does not work for equations in the 3D space!

Ex: What object is $x - y + z = 1$ in $\mathbb{R}^3$?
Answer: That depends on the set of points (x, y, z) that satisfies the equation.
Let's try to find a description for that:

- 1 equation with 3 unknowns. Chose 2 arbitrarily:
- $z = \lambda$ and $y = \mu$ and calculate x:
- $x = 1 - \lambda + \mu$

So for all $\lambda, \mu \in \mathbb{R}$ the resulting $\mathbf{x}$ satisfies $x - y + z = 1$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - \lambda + \mu \\ \mu \\ \lambda \end{bmatrix}$$

Why the parameter representation?

Different notation:

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1-\lambda+\mu \\ \mu \\ \lambda \end{bmatrix} \\
&= \begin{bmatrix} 1+(-1)\cdot\lambda+1\cdot\mu \\ 0+0\cdot\lambda+1\cdot\mu \\ 0+1\cdot\lambda+0\cdot\mu \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

represents all linear combinations of 2 directions from 1 support point.
Such an object is always a plane!
Summarized:

- in 2D: $ax + by = c$ is the equation of a line
- in 3D: $ax + by = c \iff ax + by + 0 \cdot z = c$ is the equation of a plane!

Planes in $\mathbb{R}^3$

Planes can be given by a parametrisation:
start point and 2 direction vectors:

$$\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) + \mu\,(\mathbf{c} - \mathbf{a}) \text{ for all } \lambda, \mu \in \mathbb{R}$$

λ and μ are the parameters

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} + \mu \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

Ex: the xy-plane: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The parameter representation of the *xy*-plane

Addition yields $\begin{bmatrix}0\\0\\0\end{bmatrix} + \lambda \begin{bmatrix}1\\0\\0\end{bmatrix} + \mu \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\lambda\\\mu\\0\end{bmatrix}$ that the points $(\lambda, \mu, 0)$ are described for $\lambda, \mu \in \mathbb{R}$.

Ex: The equation $z = 0$ describes the same *xy*-plane:

- 1 equation with 3 unknowns: $0 \cdot x + 0 \cdot y + z = 0$. Choose 2 arbitrarily:
- $x = \lambda$ and $y = \mu$ and calculate z:
- $z = 0$.

So that for all $\lambda, \mu \in \mathbb{R}$ the resulting $\mathbf{x}$ satisfy $z = 0$:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\lambda\\\mu\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} + \lambda \begin{bmatrix}1\\0\\0\end{bmatrix} + \mu \begin{bmatrix}0\\1\\0\end{bmatrix}$$

The parameter representation is not unique

Take V the xy-plane:

$$V: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}} + \mu \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{w}}$$

All points $P = (a, b, 0)$ are in V:

$$V: \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We obtain the point P by choosing $\lambda = a$ and $\mu = b$

The parameter representation is not unique

Take V the xy-plane:

$$V: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{r}} + \mu \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{s}}$$

All points $P = (a, b, 0)$ are in V:

$$V: \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}a \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + (b + \frac{1}{2}a) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We obtain the point P by choosing $\lambda = \frac{1}{2}a$ and $\mu = b + \frac{1}{2}a$
A plane can be described by many different directions
Here $\mathbf{r} = 2\mathbf{v} - \mathbf{w}$ and $\mathbf{s} = \mathbf{w}$

Orthogonal vectors

In $\mathbb{R}^3$

Orthogonal vectors

In $\mathbb{R}^3$ a plane of vectors is orthogonal to a given vector.

Ex: Which vectors $\mathbf{v} \in \mathbb{R}^3$ are perpendicular on $\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$? For such a vector

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \text{ it must hold: } \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} = 0 \iff -3v_x + 4v_y + v_z = 0$$

- 1 equation with 3 unknown. Choose 2 arbitrarily:
- $v_x = \lambda$ and $v_y = \mu$ and calculate v_z:
- $v_z = 3\lambda - 4\mu$

which is a plane through the origin:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 3\lambda - 4\mu \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$$

Orthogonal vectors

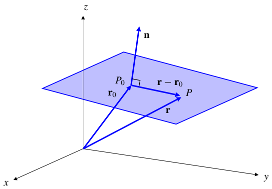
Ex: For which numbers $a \in \mathbb{R}$ is $\begin{bmatrix} a \\ 4 \\ 1 \end{bmatrix}$ orthogonal to $\begin{bmatrix} a \\ a \\ 3 \end{bmatrix}$?

We must have:

$$
\begin{aligned}
(a, 4, 1) \cdot (a, a, 3) &= 0 \\
\iff a^2 + 4a + 3 &= 0 \\
\iff (a+1)(a+3) &= 0 \\
\iff a = -1 \vee a = -3 &
\end{aligned}
$$

Planes and normals (see also [A 10.4])

Let $P_0 = (x_0, y_0, z_0)$ be an arbitrary point in the plane
and $\mathbf{n} = (n_1, n_2, n_3)$ a normal(vector) on this plane (length does not matter):



normal vector = vector that is orthogonal to the plane

The plane consists of all points $P = (x, y, z)$ for which it holds:
$(x - x_0,\ y - y_0,\ z - z_0)$ is orthogonal to $\mathbf{n}$
Thus for (x, y, z) in the plane:
$0 = \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0)$
So $n_1 x + n_2 y + n_3 z = \underbrace{n_1 x_0 + n_2 y_0 + n_3 z_0}_{d}$

Normal vector form of a plane: $\mathbf{n} \cdot \mathbf{x} = d$

Planes and normals

Earlier, we saw that the plane V with equation $x - y + z = 1$ has the vector representation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and normal } \mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The normal $\mathbf{n}$ must be orthogonal to both directions V!

Check:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot -1 + -1 \cdot 0 + 1 \cdot 1 = 0 \text{ works}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot 1 + -1 \cdot 1 + 1 \cdot 0 = 0 \text{ works too}$$

Planes and normals

The plane V through point $P = (1, 2, 3)$ with normal $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$?

has the equation $n_1x + n_2y + n_3z = d$

Thus $-x + 0 \cdot y + z = d \Longrightarrow -x + z = d$

Point P lies in V: Filling in yields $-1 + 3 = 2 = d$.

The plane V through point $P = (1,2,3)$ with normal $\mathbf{n} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$?

has the equation $-x + z = 2$ (previous slide)

But also the equation $-2x + 2z = 4$ with normal $\mathbf{n} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

and equation $x - z = -2$ with normal $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

The normals: All parallel!

Equations for a plane and normal on plane

$$n_1 x + n_2 y + n_3 z = \underbrace{n_1 x_0 + n_2 y_0 + n_3 z_0}_{d}$$

Formula for d unimportant, but:

- From the equation of the plane $n_1 x + n_2 y + n_3 z = d$ you can see:
  normal $= (n_1, n_2, n_3)$
- And vice versa: from the normal $= (n_1, n_2, n_3)$ you know the equation of the plane: $n_1 x + n_2 y + n_3 z = d$ where you still need to determine d via

Equations for a plane and normal on plane

$$n_1 x + n_2 y + n_3 z = \underbrace{n_1 x_0 + n_2 y_0 + n_3 z_0}_{d}$$

Formula for d unimportant, but:

- From the equation of the plane $n_1 x + n_2 y + n_3 z = d$ you can see: normal $= (n_1, n_2, n_3)$
- And vice versa: from the normal $= (n_1, n_2, n_3)$ you know the equation of the plane: $n_1 x + n_2 y + n_3 z = d$ where you still need to determine d via filling in 1 point P in the plane

Lines in $\mathbb{R}^3$

Reminder: 2 options: given by

- parameter representation
- or 2 equations: intersection of 2 planes

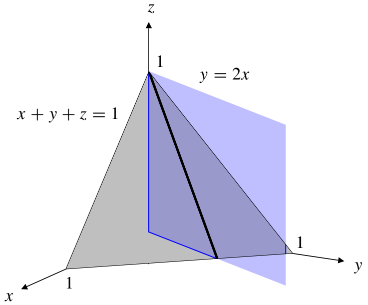
Parameter representation: $(x, y, z) = (x_1, y_1, z_1) + t\,(a, b, c), \quad t \in \mathbb{R}$
support point + $t \cdot$ direction vector

Ex: $\begin{aligned} x &= 2+t \\ y &= 3 \\ z &= -4t \end{aligned}$ is rewritten as $(x, y, z) = (2, 3, 0) + t\,(1, 0, -4)$

Ex: Line through $(1, -2, 3)$ orthogonal to plane $x - 2y + 4z = 5$ has parameter representation
$(x, y, z) = (1, -2, 3) + t\,(1, -2, 4) = (1 + t,\ -2 - 2t,\ 3 + 4t)$

Lines in $\mathbb{R}^3$

Also possible by 2 equations: intersection of 2 planes



One can make a parameter representation out of this by finding 2 points, eg:

choose $x = 0$ then $y = 0$ and $z = 1$

choose $x = 1$ then $y = 2$ and $z = -2$

A parameter representation is then

$(0, 0, 1) + t\,((1, 2, -2) - (0, 0, 1)) = (0, 0, 1) + t\,(1, 2, -3)$

Planes

- Give a parameter representation of the line through $(5, -2, -5)$ parallel to a vector $(7, 7, 10)$

- The same for the line through $(-8, -7, 2)$ orthogonal to the plane $3x + 3y + 3z = 8$

- Give the equation for the plane through $(3, 5, 5)$ and normal to the vector $(8, 6, 9)$

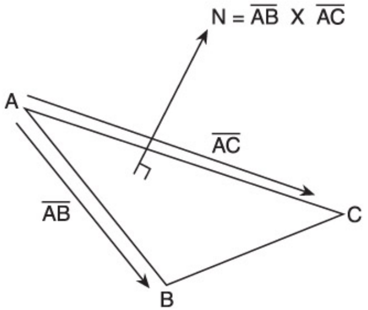
- Where does the line $x = -8 + 3t, \;\; y = 4 + 7t, \;\; z = -9 - 8t$ intersect the plane $9x - 6y + 3z = -11$

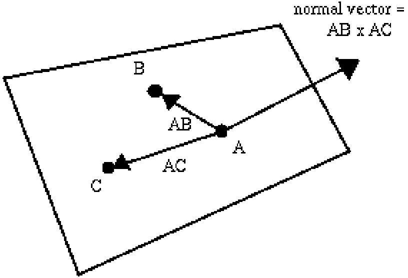
Line in $\mathbb{R}^2$ and plane in $\mathbb{R}^3$

Note the similarities and differences:
Line in $\mathbb{R}^2$: $ax + by = c$
(a, b, c fixed, x, y variable)
Plane in $\mathbb{R}^3$: $ax + by + cz = d$
(a, b, c, d fixed, x, y, z variable)

The cross product vector

Cross product vector: A vector orthogonal to two different vectors. Three points A, B and C define a start point (eg A) and two directions from A ($\vec{AB} = \mathbf{b} - \mathbf{a}$ and $\vec{AC} = \mathbf{c} - \mathbf{a}$) and with that a plane with vector representation: $\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$.
The normal N (vector $\mathbf{n}$) is orthogonal to both vectors.





and can be calculated with the so called cross product.

Definition of the cross product

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

Ex: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Ex: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Ex: $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ Ex: $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

What is noteworthy about these examples for $\mathbf{v} \times \mathbf{w}$?

Definition of the cross product

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

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What is noteworthy about these examples for $\mathbf{v} \times \mathbf{w}$?

- Cross product is orthogonal to the 2 vectors $\mathbf{v}$ and $\mathbf{w}$
- Length is the same as the surface of a parallellogram formed by $\mathbf{v}$ and $\mathbf{w}$

This always turns out to be true.

Cross product and normal vector

The cross product is useful for calculating the normal vector on the plane

Ex: Plane with parametrisation $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

this is the

Cross product and normal vector

The cross product is useful for calculating the normal vector on the plane

Ex: Plane with parametrisation $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

this is the (x, y)-plane

Normal is

Cross product and normal vector

The cross product is useful for calculating the normal vector on the plane

Ex: Plane with parametrisation $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

this is the (x, y)-plane

Normal is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Cross product: properties

- $\mathbf{x} \times \mathbf{x} = \mathbf{0}$
  cross product of a vector with itself is the **0**-vector
  logical, because then the parallellogram has the surface 0
- $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{x} = 0$
  cross product of one vector with another is perpendicular to the other
  this is 1 of the 2 basic properties of the cross product
- $\mathbf{y} \times \mathbf{x} = -(\mathbf{x} \times \mathbf{y})$
  when you swap the order, you get the opposite direction
  "corkscrew-rule"
- $(\lambda\mathbf{x}) \times \mathbf{y} = \lambda\left(\mathbf{x} \times \mathbf{y}\right)$
  if **x** is eg 2 times longer, then the cross product is 2 times as long
  logical, because the length of the cross product is surface of a parallellogram

Distance of a point to a plane

Distance of point (p, q, r) to a plane $ax + by + cz = d$

1. Set up the eq. for the line through point orthogonal to plane:
   $(x, y, z) = (p, q, r) + \lambda\,(a, b, c)$
2. Calculate the intersection point of the line with the plane: fill in $x = p + \lambda a, \quad y = q + \lambda b, \quad z = r + \lambda c$ into the plane equation
3. Calculate distance between (p, q, r) and intersection point

Distance of a point to a line

Distance of the point to a line $\mathbb{R}^3$

1. Set up the eq. of a plane through point orthogonal to the line;
   $ax + by + cz = d$, with (a, b, c) the direction vector of the line
2. Calculate d by filling in the point
3. calculate intersection of line with plane
4. calculate the distance between point and intersection

Surface of a triangle in plane and space

Surface triangle (a, b, c), (d, e, f), (g, h, i) in $\mathbb{R}^3$

1. calculate cross product of $(d, e, f) - (a, b, c)$ and $(g, h, i) - (a, b, c)$
2. surface $= \frac{1}{2} \cdot$ length of this vector

Surface triangle (a, b), (c, d), (e, f) in $\mathbb{R}^2$

1. Calculate cross product of $(c, d, 0) - (a, b, 0)$ and $(e, f, 0) - (a, b, 0)$
2. surface $= \frac{1}{2} \cdot$ length of this vector

Note: there are also other methods you can use.

When inner product/when cross product

Let $\mathbf{u}, \mathbf{v}$ be two vectors.

Inner product:

- To see if $\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0$
- Calculation of the angle θ between $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$

Cross product:

- Calculation of a vector $\mathbf{n}$ orthogonal to $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{n} = \mathbf{u} \times \mathbf{v}$
- To see if $\mathbf{u} \parallel \mathbf{v} \iff \mathbf{u} \times \mathbf{v} = 0$
- Calculation of the surface O of the parallellogram spanned by $\mathbf{u}$ and $\mathbf{v}$: $O = |\mathbf{u} \times \mathbf{v}|$

Week 2: We have seen

- Trigonometric functions sin, . . .
- Powers e^x 2^x, . . .
- Vectors in 2 and 3 dimensions
- Lines and planes
- Distance, angle, inner product
- Cross product