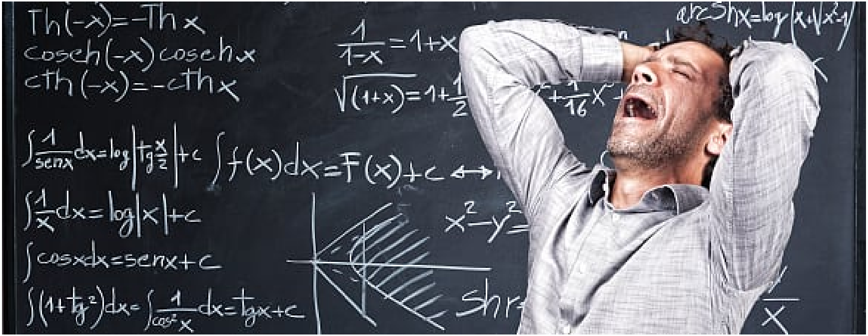
2WBB0 Calculus for BCS

Dr. Björn Baumeier

Technische Universiteit
Eindhoven
University of Technology

Why do we (=you) need Calculus?



Why do we (=you) need Calculus?

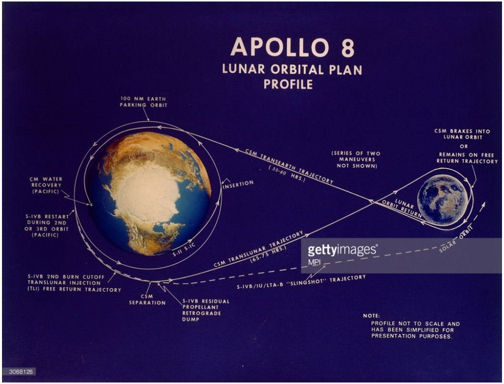
December 21, 1968? Shorter clip

Why do we (=you) need Calculus?

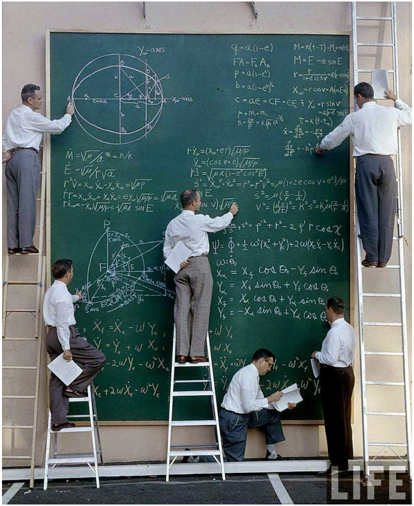
December 21, 1968? Shorter clip

Why do we (=you) need Calculus?

December 21, 1968? Shorter clip



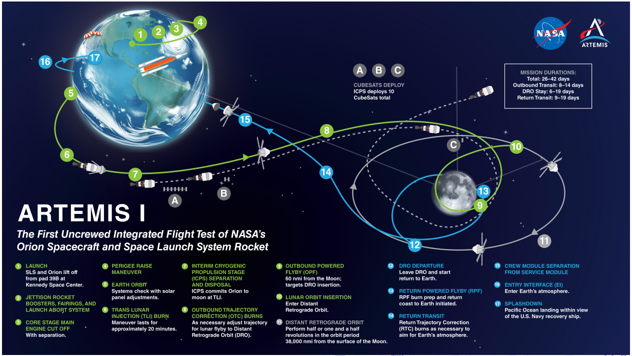
December 21, 1968?



Why do we (=you) need Calculus?

December 21, 1968?

November 16, 2022

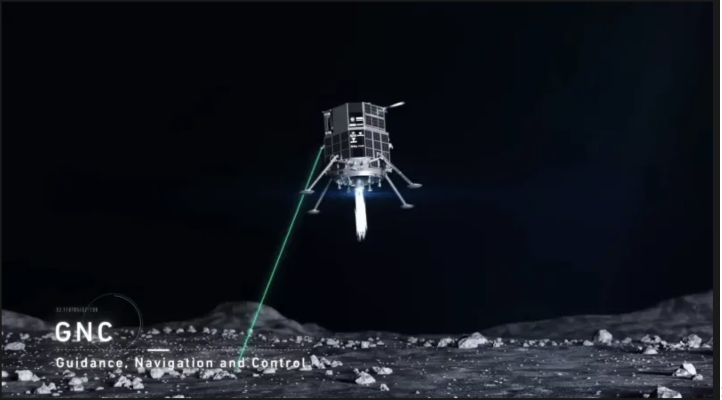


April 26, 2023

THE GLITCH THAT BROUGHT DOWN JAPAN'S LUNAR LANDER

by: Matthew Carlson   37 Comments

June 3, 2023



Why do we (=you) need Calculus?

April 26, 2023

When a computer crashes, it usually doesn't leave debris. But when a computer happens to be descending towards the lunar surface and glitches out, that's a very different story. Turns out that's what happened on April 26th, as the Japanese Hakuto-R Lunar lander made its mark on the Moon…by crashing into it. [Scott Manley] dove in to try and understand the software bug that caused an otherwise flawless mission to go splat.

The lander began the descent sequence as expected at 100 km above the surface. However, as it descended, the altitude sensor reported the altitude as much lower than it was. It thought it was at zero altitude once it reached about 5 km above the surface. Confused by the fact it hadn't yet detected physical contact with the surface, the craft continued to slowly descend until it ran out of fuel and plunged to the surface.

Ultimately it all came down to sensor fusion. The lander merges several noisy sensors, such as accelerometers, gyroscopes, and radar, into one cohesive source of truth. The craft passed over a particularly large cliff that caused the radar altimeter to suddenly spike up 3 km. Like good filtering software, the craft reasons that the sensor must be getting spurious data and filters it out. It was now just estimating its altitude by looking at its acceleration. As anyone who has tried to track an object through space using just gyros and accelerometers alone can attest, errors accumulate, and suddenly you're not where you think you are.

Why do we (=you) need Calculus?

There are 20 people in the room. They all need to say good bye to each other, and shake hands. How many handshakes?

Why do we (=you) need Calculus?

There are 20 people in the room. They all need to say good bye to each other, and shake hands. How many handshakes?

Without Calculus

```
Readln('how many people?', X);
cnt := 0;
for i := 1 to X do
     for j := 1 to X do
           if j>i then inc(cnt);
Writeln(cnt);
```

Why do we (=you) need Calculus?

There are 20 people in the room. They all need to say good bye to each other, and shake hands. How many handshakes?

Without Calculus

```
Readln('how many people?', X);
cnt := 0;
for i := 1 to X do
     for j := 1 to X do
          if j>i then inc(cnt);
Writeln(cnt);
```

With Calculus

```
Readln('how many people?', X);
Writeln(X*(X-1)/2);
```

Why do we (=you) need Calculus?

There are 20 people in the room. They all need to say good bye to each other, and shake hands. How many handshakes?

Without Calculus

```
Readln('how many people?', X);
cnt := 0;
for i := 1 to X do
     for j := 1 to X do
           if j>i then inc(cnt);
Writeln(cnt);
```

With Calculus

```
Readln('how many people?', X);
Writeln(X*(X-1)/2);
```

- For 20 people, nobody can tell the difference.

Why do we (=you) need Calculus?

There are 20 people in the room. They all need to say good bye to each other, and shake hands. How many handshakes?

Without Calculus

```
Readln('how many people?', X);
cnt := 0;
for i := 1 to X do
     for j := 1 to X do
           if j>i then inc(cnt);
Writeln(cnt);
```

With Calculus

```
Readln('how many people?', X);
Writeln(X*(X-1)/2);
```

- For 20 people, nobody can tell the difference.
- For one million people in the room, you get to do 10^{14} loops. You need a 64 bit machine for storing the counter. At 3 GHz, that takes 166 seconds (way more in real life). With Calculus: instant!

Calculus in Computer Science

For video game programming.
Calculus, when taught alongside Physics, gives you the background necessary to understand how objects can interact and move based on forces. Even if you don't ever write your own game engine, you'll be a better game programmer if you have a mastery of calculus.

For machine learning.
Basically machine learning is mathematics meets computer science meets data. This is one of the hottest topics in CS right now, but you have to know your mathematics to succeed with ML.

For scientific applications.
A lot of computer scientists work alongside scientists like biologists, physicists, and chemists. CS people help them solve their data-analysis problems using large clusters of computers, and anytime you have to understand how equations work, it is hard to escape calculus.

In research.
Researchers such as Professors and PhD students use mathematics when writing papers. If you want to one day pursue an advanced degree, you'll have to decipher these papers and then write some of your own. I would not recommend starting out a career in research without at least a basic command over calculus.

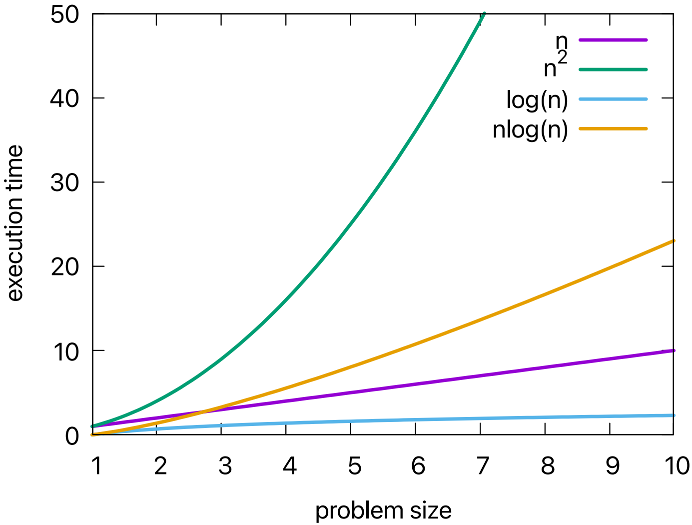
Analyze a Computer Algorithm

Imagine you are a programmer of an automatic, autonomous guidance system (think of cars, rockets). You have an algorithm that takes data collected from a number of input sources (sensors) and processes it to adjust the control of the vehicle. It takes 50ms to obtain the adjustment after the acquisition of data (reaction time).

Now, the engineers plan to double the number of sensors to improve the guidance. The maximal acceptable reaction time for the vehicle is 150ms.

Can your algorithm deal with double the amount of data within the 150ms?

Different scaling of (parts of complex) algorithms



Example: Bubble Sort

Sorting an unordered list of five numbers by swapping adjacent elements that are out of place:

17	**12**	8	3	7
12	**17**	**8**	3	7
12	8	**17**	**3**	7
12	8	3	**17**	**7**
12	8	3	7	**17**

How many comparisons? $n - 1$

Example: Bubble Sort

Now, do it again

12	**8**	3	7	17
8	**12**	**3**	7	17
8	3	**12**	**7**	17
8	3	7	**12**	**17**

How many comparisons? $n - 2$

And again:

8	**3**	7	12	17
3	**8**	**7**	12	17
3	7	**8**	**12**	**17**

How many comparisons? $n - 3$

Example: Bubble Sort

Finally:

3 7 8 12 17

How many comparisons? $n - 4$
So all in all:

$$(n-1) + (n-2) + (n-3) + \ldots + 1 = \frac{n(n-1)}{2}$$

Ranking Different Algorithms

- How to compare different algorithms?
- Two algorithms

$$\text{Algorithm } 1 \rightarrow f(n) = n$$
$$\text{Algorithm } 2 \rightarrow g(n) = n^2$$

- Which one is "faster"?
- Algorithm 1, because it grows at a slower rate with n

Ranking Different Algorithms

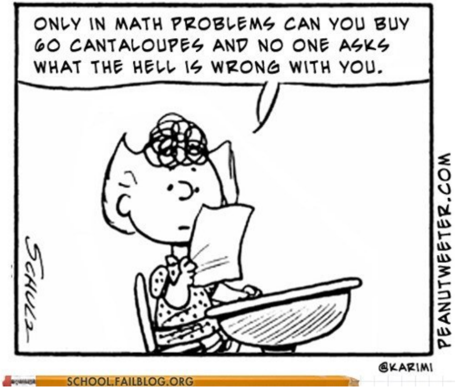
- How to formulate this more generally?
  $\rightarrow$ Limits
- $$\lim_{n\to\infty} \frac{f}{g} = 0 \qquad g \text{ grows faster}$$
  $$\lim_{n\to\infty} \frac{f}{g} = c \qquad g \text{ grows same as } f$$
  $$\lim_{n\to\infty} \frac{f}{g} = \infty \qquad g \text{ grows slower}$$

Limits not always straightforward

- $f(n) = n\log(n)$
- $g(n) = n^{3/2}$
- $$\lim_{n\to\infty} \frac{n^{3/2}}{n\log(n)} = ?$$
- To evaluate we will need:
  - derivatives
  - Taylor series/polynomials
  - Exponential functions and Logarithms
  - rational functions
  - ...

Overview

- W1: Numbers, Functions, (In-)equalities, Polynomials, Rational functions
- W2: Trigonometric functions/Vectors in 2 and 3 dimensions
- W3: Limits, continuity, differentiation I
- W4: Differentiation II and Inverse functions
- W5: Exponential function and logarithm/Taylor series/Limits II
- W6: Integration/Integration techniques I
- W7: Integration techniques II/1st order differential equations
- W8: Exam and W1 – W7 summary (no new material)


ONLY IN MATH PROBLEMS CAN YOU BUY 60 CANTALOUPES AND NO ONE ASKS WHAT THE HELL IS WRONG WITH YOU.
SCHULZ
PEANUTWEETER.COM
@KARIMI
SCHOOL.FAILBLOG.ORG

Topics in Week 1

- Real numbers and the real line
- Graphs of quadratic equations
- Polynomials and rational functions
- Powers and roots
- Cartesian coordinates in the plane
- More functions and their graphs
- Combining functions to make new functions

Real numbers

$$\mathbb{N} \underset{\substack{\uparrow \\ \text{is a subset of}}}{\subset} \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$\mathbb{N}$	natural numbers: $0, 1, 2, \ldots$ (sometimes without 0)
$\mathbb{Z}$	integer numbers: $\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	rational numbers = fractions: $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$
$\mathbb{R}$	real numbers: $\mathbb{Q}$ and π, $\sqrt{2}$, etc
$\mathbb{C}$	complex numbers: $2 + 3\mathbf{i}$, with $\mathbf{i} \cdot \mathbf{i} = \mathbf{i}^2 = -1$ (not in curriculum)

Decimal Expansion

For rational numbers, the "decimal expansion" has a repeating pattern, for irrational numbers, this is not the case:

$$\begin{aligned}
\tfrac{1}{2} &= 0.50000|0|\cdots \\
\tfrac{1}{3} &= 0.33333|3|\cdots \\
\tfrac{1}{11} &=
\end{aligned}$$

Decimal Expansion

For rational numbers, the “decimal expansion” has a repeating pattern, for irrational numbers, this is not the case:

$$
\begin{aligned}
\tfrac{1}{2} &= 0.50000|0|\cdots \\
\tfrac{1}{3} &= 0.33333|3|\cdots \\
\tfrac{1}{11} &= 0.0909|09|\cdots \\
\tfrac{1}{7} &=
\end{aligned}
$$

Decimal Expansion

For rational numbers, the “decimal expansion” has a repeating pattern, for irrational numbers, this is not the case:

$$
\begin{aligned}
\tfrac{1}{2} &= 0.50000|0|\cdots \\
\tfrac{1}{3} &= 0.33333|3|\cdots \\
\tfrac{1}{11} &= 0.0909|09|\cdots \\
\tfrac{1}{7} &= 0.142857|142857|\cdots \\
\pi &=
\end{aligned}
$$

Decimal Expansion

For rational numbers, the "decimal expansion" has a repeating pattern, for irrational numbers, this is not the case:

$$
\begin{aligned}
\tfrac{1}{2} &= 0.50000|0|\cdots \\
\tfrac{1}{3} &= 0.33333|3|\cdots \\
\tfrac{1}{11} &= 0.0909|09|\cdots \\
\tfrac{1}{7} &= 0.142857|142857|\cdots \\
\pi &= 3.141592653589793\cdots
\end{aligned}
$$

Intervals

Notation for intervals:

- [,]: end points are part of the interval
- (,): end points are NOT part of the interval

interval	alternative notation	type	type
$(0,1)$	$0 < x < 1$	open	finite
$[0,1)$	$0 \leq x < 1$	half open	finite
$(0,1]$	$0 < x \leq 1$	half open	finite
$[0,1]$	$0 \leq x \leq 1$	closed	finite
$[1,\infty)$	$1 \leq x < \infty$		infinite
$(-\infty,3)$	$-\infty < x < 3$		infinite
$(-\infty,\infty)$	$-\infty < x < \infty$		infinite

Symbols

$\cup$	union
$\cap$	intersection
$\in$	element of
$\vee$	or
$\wedge$	and
(A) $\Longrightarrow$ (B)	from (A) follows (B)
(A) $\Leftrightarrow$ (B)	from (A) follows (B) and from (B) follows (A)
	in other words: they are equivalent
	(A) "if and only if" (B) (iff)

For two sets A and B:

- $A - B$
- $A \backslash B$

represents alle $x \in A$ for which $x \notin B$.

Polynomials

A polynomial is a “many term” construct

The degree (order) of a polynomial is the highest ocurring power

Ex: $p(x) = x^3 - 2x^2 + 1$ is of degree 3

Ex: $p(x) = 0 \cdot x^3 - 2x^2 + 1$ is of degree 2

Ex: $p(x) = (x+1)^2 - (x-1)^2$ is of degree 1

Ex: $p(x) = 3 = 3 \cdot x^0$ is of degree 0

Computers can only do

- additions/subtractions
- multiplications

All other operations, such as division or calculation of a root have be performed using only these basis operations – and then cost 3 to 5 times more calculational time

Frequently appearing polynomials

Degree	Name	Example
0	constant	$p(x) = 2$
1	linear	$p(x) = 2x + 3$
2	quadratic	$p(x) = 5x^2 + 4x + 5$
3	cubic	$p(x) = 2x^3 + 1$

Equations with polynomials of degree 1 always have a solution:

Ex: $4x - 3 = 0$

Frequently appearing polynomials

Degree	Name	Example
0	constant	$p(x) = 2$
1	linear	$p(x) = 2x + 3$
2	quadratic	$p(x) = 5x^2 + 4x + 5$
3	cubic	$p(x) = 2x^3 + 1$

Equations with polynomials of degree 1 always have a solution:

Ex: $4x - 3 = 0 \implies 4x = 3 \implies x = \frac{3}{4}$

Theorem:

> Equations of polynomials of degree n have a maximum of n real solutions

Equations with a polynomial of degree 2

Quadratic functions are more than algebraic curiosities – they are widely used in science, business, and engineering:

- trajectories of water jets in a fountain or of a bouncing ball
- parabolic reflectors that form the base of satellite dishes and car headlights
- forecast business profit and loss
- data interpolation

Equations of quadratic polynomials can always be written into the general form:

$$ax^2 + bx + c = 0$$

Equations with a polynomial of degree 2

Equations of quadratic polynomials can always be written into the general form:

$$ax^2 + bx + c = 0$$

Our task is to find the solutions (or zeros) of this equation.

3 different ways to find the solution:

- completing the square
- abc-formula (quadratic formula)
- "Sum-product"-formula

Use the way that you like best **for this purpose** but learn the concept of **completing the square**! It will be useful later.

Idea behind Completing the Square

Ex: : $x^2 = 3$

Idea behind Completing the Square

Ex: : $x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$

Idea behind Completing the Square

Ex: : $x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$

This is easy (and also an equation with a quadratic polynomial)!

So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.

Idea behind Completing the Square

Ex: : $x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$
This is easy (and also an equation with a quadratic polynomial)!
So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.
Ex: $(x-1)^2 = 9$

Idea behind Completing the Square

Ex: : $x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$

This is easy (and also an equation with a quadratic polynomial)!

So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.

Ex: $(x-1)^2 = 9 \Leftrightarrow x - 1 = \pm 3 \Leftrightarrow x = 1 \pm 3$

> Rewrite a quadratic equation to a form with a complete square on the left-hand side
>
> $$(x \pm R)^2 = T$$
>
> and take the square root on both sides, if possible.

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{aligned} (n+m)^2 &= n^2 + 2nm + m^2 \\ (n-m)^2 &= n^2 - 2nm + m^2 \end{aligned}}$$

and that one can also use them from right to left!

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{array}{rcl} (n+m)^2 & = & n^2 + 2nm + m^2 \\ (n-m)^2 & = & n^2 - 2nm + m^2 \end{array}}$$

and that one can also use them from right to left!

Ex: $x^2 - 4x + 4 = (x-2)^2$

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{aligned}(n+m)^2 &= n^2+2nm+m^2\\ (n-m)^2 &= n^2-2nm+m^2\end{aligned}}$$

and that one can also use them from right to left!

Ex: $x^2-4x+4=(x-2)^2$

Ex: $4x^2+12x+9=(\underbrace{2x}_{=n})^2+\underbrace{2}_{\text{binom. form.}}\cdot\underbrace{2x}_{=n}\cdot\underbrace{3}_{=m}+(\underbrace{3}_{=m})^2=(2x+3)^2$

Those examples are complete squares. They can be rewritten directly into one of the two binomial formulas.

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{aligned}(n+m)^2 &= n^2+2nm+m^2\\(n-m)^2 &= n^2-2nm+m^2\end{aligned}}$$

and that one can also use them from right to left!

Ex: $x^2-4x+4=(x-2)^2$

Ex: $4x^2+12x+9=(\underbrace{2x}_{=n})^2+\underbrace{2}_{\text{binom. form.}}\cdot\underbrace{2x}_{=n}\cdot\underbrace{3}_{=m}+(\underbrace{3}_{=m})^2=(2x+3)^2$

Those examples are complete squares. They can be rewritten directly into one of the two binomial formulas.

But what about x^2-4x+3? We are missing $+1$ for a complete square!

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{array}{rcl}(n+m)^2 & = & n^2+2nm+m^2 \\ (n-m)^2 & = & n^2-2nm+m^2\end{array}}$$

and that one can also use them from right to left!

Ex: $x^2 - 4x + 4 = (x-2)^2$

Ex: $4x^2 + 12x + 9 = (\underbrace{2x}_{=n})^2 + \underbrace{2}_{\text{binom. form.}} \cdot \underbrace{2x}_{=n} \cdot \underbrace{3}_{=m} + (\underbrace{3}_{=m})^2 = (2x+3)^2$

Those examples are complete squares. They can be rewritten directly into one of the two binomial formulas.

But what about $x^2 - 4x + 3$? We are missing $+1$ for a complete square!

$$x^2-4x+3 = x^2-4x+3+0 = x^2-4x+3+1-1 = x^2-4x+4-1 = (x-2)^2-1$$

Completing the square to solve quadratic equations

Solve:

$$ax^2 + bx + c = 0$$

1. Identify n and m in the expression.
   - n is the square root of the term with x^2
     $ax^2 \to n = \sqrt{a}x$
   - m is the linear term in x, divided by $2n$
     $m = bx/(2 \cdot \sqrt{a}x) = b/(2\sqrt{a})$
2. Use binomial formula
   $(n+m)^2 = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = ax^2 + bx + \frac{b^2}{4a}$
3. what is missing/too much
   $ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 - \frac{b^2}{4a} + c = 0$
4. isolate the square, take roots, simplify
   $(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = \frac{b^2}{4a} - c \Leftrightarrow \sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$
   $\Leftrightarrow x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0$ via abc-formula (I)

2nd degree polynomial $ax^2 + bx + c = 0$ with $a \neq 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How many solutions are possible?

Discriminant: $D = b^2 - 4ac$

$D > 0$

$ax^2 + bx + c = 0$ via abc-formula (I)

2nd degree polynomial $ax^2 + bx + c = 0$ with $a \neq 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How many solutions are possible?

Discriminant: $D = b^2 - 4ac$

$D > 0$ 2 different solutions ($x_1 \neq x_2$)

$D = 0$

$ax^2 + bx + c = 0$ via abc-formula (I)

2nd degree polynomial $ax^2 + bx + c = 0$ with $a \neq 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How many solutions are possible?

Discriminant: $D = b^2 - 4ac$

$D > 0$ 2 different solutions ($x_1 \neq x_2$)

$D = 0$ 1 double solution (2 solutions $x_1 = x_2$)

$D < 0$

$ax^2 + bx + c = 0$ via abc-formula (I)

2nd degree polynomial $ax^2 + bx + c = 0$ with $a \neq 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How many solutions are possible?

Discriminant: $D = b^2 - 4ac$

- $D > 0$ 2 different solutions ($x_1 \neq x_2$)
- $D = 0$ 1 double solution (2 solutions $x_1 = x_2$)
- $D < 0$ no (real) solution

The zeros determine the factoring of the polynomial:

$$ax^2 + bx + c = (x - x_1)(x - x_2).$$

Pay attention to the minus sign!

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions ($x_1 \neq x_2$)
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0$

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions $(x_1 \neq x_2)$
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0 \implies x^2 = -1 \implies$ no solution
indeed: $D = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0 \implies$ no solution

Ex: $x^2 + 2x + 1 = 0$

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions ($x_1 \neq x_2$)
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0 \implies x^2 = -1 \implies$ no solution
indeed: $D = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0 \implies$ no solution

Ex: $x^2 + 2x + 1 = 0$: $D = 2^2 - 4 \cdot 1 \cdot 1 = 0 \implies$ double solution
indeed: $x^2 + 2x + 1 = 0 \implies (x+1)^2 = 0$
$\implies x + 1 = 0$ or $x + 1 = 0 \implies$ double solution $x = -1$

$ax^2 + bx + c = 0$ via "Sum-product"

Set $a = 1$:

$$x^2 + \underbrace{(r+s)}_{=b(\text{sum})} x + \underbrace{rs}_{=c(\text{product})} = (x+r)(x+s)$$

Ex: $x^2 + 5x + 6 =$

$ax^2 + bx + c = 0$ via "Sum-product"

Set $a = 1$:

$$x^2 + \underbrace{(r+s)}_{=b(\text{sum})} x + \underbrace{rs}_{=c(\text{product})} = (x+r)(x+s)$$

Ex: $x^2 + 5x + 6 = (x+2)(x+3)$

Because $b = 5 = 2 + 3$ and $c = 6 = 2 \cdot 3$

Ex: $x^2 - 9 = (x-3)(x+3)$

Attention! $x^2 = 9 \not\Longrightarrow x = 3$, but :

$$x^2 = 9 \implies x^2 - 9 = 0 \implies (x-3)(x+3) = 0 \implies$$
$$x - 3 = 0 \text{ or } x + 3 = 0 \implies x = 3 \quad \text{or} \quad x = -3$$

Examples: Quadratic Equations

1. Solve: $x^2 - 3x + 2 = 0$
2. Factorize: $x^2 = 2$
3. Solve: $x^2 + 2 = 0$
4. Complete the square: $x^2 + 3x + \frac{1}{4}$
5. Complete the square: $4x^2 + 12x + 1$

Equations with a 3rd degree polynomial

1. Guess first zero. Try out first $a = 0, 1, -1, 2, -2, \ldots$
2. Divide by factor $x - a$
3. The rest must be 0 (if not, you made a mistake)
4. Find solutions of the remaining 2nd degree polynomial equation, if they exist

Ex: $x^3 + 2x^2 - x - 2 = 0$
Try:

Equations with a 3rd degree polynomial

1. Guess first zero. Try out first $a = 0, 1, -1, 2, -2, \ldots$
2. Divide by factor $x - a$
3. The rest must be 0 (if not, you made a mistake)
4. Find solutions of the remaining 2nd degree polynomial equation, if they exist

Ex: $x^3 + 2x^2 - x - 2 = 0$
Try: $x = 1$ is a solution
Then is $x^3 + 2x^2 - x - 2 = (x - 1) \cdot (\text{2nd degree polynomial})$

$x^3 + 2x^2 - x - 2 = (x-1) \cdot$ (2nd degree polynomial)

To determine the 2nd order polynomial, divide out (see later):

$$\frac{x^3 + 2x^2 - x - 2}{x-1} = x^2 + 3x + 2$$

And then $x^2 + 3x + 2 =$

$x^3 + 2x^2 - x - 2 = (x - 1) \cdot$ (2nd degree polynomial)

To determine the 2nd order polynomial, divide out (see later):

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

And then $x^2 + 3x + 2 = (x + 2)(x + 1)$ via abc-formula
Conclusion $x^3 + 2x^2 - x - 2 = (x - 1)(x + 2)(x + 1)$
Thus solution of $x^3 + 2x^2 - x - 2 = 0$:

$x^3 + 2x^2 - x - 2 = (x-1) \cdot$ (2nd degree polynomial)

To determine the 2nd order polynomial, divide out (see later):

$$\frac{x^3 + 2x^2 - x - 2}{x-1} = x^2 + 3x + 2$$

And then $x^2 + 3x + 2 = (x+2)(x+1)$ via abc-formula
Conclusion $x^3 + 2x^2 - x - 2 = (x-1)(x+2)(x+1)$
Thus solution of $x^3 + 2x^2 - x - 2 = 0$: $x = 1$ or $x = -1$ or $x = -2$

Equations with 3rd, 4th, 5th, ...-degree polynomials

Principle: guess, divide, guess, etc, until 2nd-degree remains

1. First, have a look if there is not a special easier case, as $p(x) = 0$ with $p(x) = x^3 - 8$ or $p(x) = x^4 + 2x^2 + 1$
2. If not, guess a zero, say $x = a$, $a = 0, 1, -1, 2, -2, \ldots$
3. Divide out by Long Division: $p(x)/(x - a)$ (here **must** have rest 0), this yields a new polynomial $q(x)$
4. If the degree of $q(x)$ is still larger than 2, repeat the above
5. If the degree of $q(x)$ is equal to 2, try with sum-product technique or the abc-formula

Degree 4, 5, . . .: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$

Ex: $x^4 - 50x^2 + 49 = 0$
4th degree but 2nd degree in x^2! Rename $p = x^2$
It holds that $(x^2)^2 = x^2 \cdot x^2 = x^{2+2} = x^4$

$$p^2 - 50p + 49 = 0$$

Degree 4, 5, . . .: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$
4th degree but 2nd degree in x^2! Rename $p = x^2$
It holds that $(x^2)^2 = x^2 \cdot x^2 = x^{2+2} = x^4$

$$p^2 - 50p + 49 = 0 \implies (p - 49)(p - 1) = 0 \implies p = 49 \text{ of } p = 1$$
$$\underset{p=x^2}{\implies} \implies x^2 = 49 \text{ or } x^2 = 1 \implies x = \pm 7 \text{ or } x = \pm 1$$

Higher degree but only produts of lower degrees

$$x^4(x - 2)^2 = 0$$

Degree 4, 5, ...: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$
4th degree but 2nd degree in x^2! Rename $p = x^2$
It holds that $(x^2)^2 = x^2 \cdot x^2 = x^{2+2} = x^4$

$$p^2 - 50p + 49 = 0 \implies (p-49)(p-1) = 0 \implies p = 49 \text{ of } p = 1$$
$$\underset{p=x^2}{\implies} \implies x^2 = 49 \text{ or } x^2 = 1 \implies x = \pm 7 \text{ or } x = \pm 1$$

Higher degree but only produts of lower degrees

$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$

Degree 4, 5, . . .: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$
4th degree but 2nd degree in x^2! Rename $p = x^2$
It holds that $(x^2)^2 = x^2 \cdot x^2 = x^{2+2} = x^4$

$$p^2 - 50p + 49 = 0 \implies (p-49)(p-1) = 0 \implies p = 49 \text{ of } p = 1$$
$$\underset{p=x^2}{\implies} \implies x^2 = 49 \text{ or } x^2 = 1 \implies x = \pm 7 \text{ or } x = \pm 1$$

Higher degree but only produts of lower degrees

$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$
$$\implies x = 0 \text{ or } x - 2 = 0$$

Degree 4, 5, ...: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$
4th degree but 2nd degree in x^2! Rename $p = x^2$
It holds that $(x^2)^2 = x^2 \cdot x^2 = x^{2+2} = x^4$

$$p^2 - 50p + 49 = 0 \implies (p-49)(p-1) = 0 \implies p = 49 \text{ of } p = 1$$
$$\underset{p=x^2}{\implies} \implies x^2 = 49 \text{ or } x^2 = 1 \implies x = \pm 7 \text{ or } x = \pm 1$$

Higher degree but only produts of lower degrees

$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$
$$\implies x = 0 \text{ or } x - 2 = 0 \implies x = 0 \text{ or } x = 2$$

(respectively 4-fold and 2-fold)

Factoring and roots (zeros) of polynomials

Writing a polynomial as a product of two or more polynomials is called factoring.

Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2)$
The polynomials $x-3$ and x^2-2 are factors with degree 1 and 2.

Factoring breaks up a complicated polynomial into easier, lower degree pieces.

Factoring and roots (zeros) of polynomials

Writing a polynomial as a product of two or more polynomials is called factoring.

Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2)$
The polynomials $x-3$ and x^2-2 are factors with degree 1 and 2.

Factoring breaks up a complicated polynomial into easier, lower degree pieces.

In the example, we can do more:
Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2) = (x-3)(x-\sqrt{2})(x+\sqrt{2})$

This polynomial is factored into three **linear** polynomials. We can't do ant better, the polynomial is **factored completely**.

> Knowing all roots of a polynomial $\Leftrightarrow$ Knowing the factoring of a polynomial

Polynomials: When to let factors stand as they are?

When a polynomial consists of one term of factors:
$p(x) = (x+1)^3$ or $p(x) = (x-1)(x+2)$

Most of the times, it is useful to let these stand, because:

- it is easy, no extra effort
- you see the zeros at once
- sometimes this is easier to differentiate/integrate (later in Calculus) (eg derivative or primitive of $(x+1)^{40}$)
- useful for limits (eg $\lim_{x\to 1^+} (x-1)^3$ etc)

Polynomials: When to write factors out?

When a polynomial consists of several terms (of factors), eg:
$p(x) = (x+1)^3 - (x-1)^3$

Most of the times, it is useful to write these out, because:

- Terms can cancel each other out, eg:
  $(x+1)^3 - (x-1)^3 = 6x^2 + 2$ shows to be of degree 2 (not 3)

Examples: Higher order polynomial equations

1. Solve $3x^4 = 9x^2$
2. Solve $x^3 + 9x^2 + 26x + 24 = 0$
3. Solve $x^4 - 26x^2 + 25 = 0$
4. What are all solutions to $(x^2 - 1)^4(x + 3)^7 = 0$
5. Which of these polynomials has roots $x = 1 \vee x = 2 \vee x = 3/4$
   a) $p(x) = (x + 1)(x + 2)(x + 3/4)$
   b) $p(x) = (x - 1)(x - 2)(x - 3/4)$

Inequalities with polynomials

Finding solutions to inequalities of the kind

$$p(x) > d \qquad p(x) \geq d \qquad p(x) < d \qquad p(x) \leq d$$

is done in the following steps:

1. Replace $<$, $\leq$, $>$ or $\geq$ by $=$
2. Get a 0 to the right hand side
3. Mark the solutions (zeros) on the x-axis
4. Determine the signs for each interval (e.g., by filling in an arbitrary point
5. Determine the solution of the inequality

Simple example: solve $x^2 > 9$

Inequalities with polynomials

Finding solutions to inequalities of the kind

$$p(x) > d \qquad p(x) \geq d \qquad p(x) < d \qquad p(x) \leq d$$

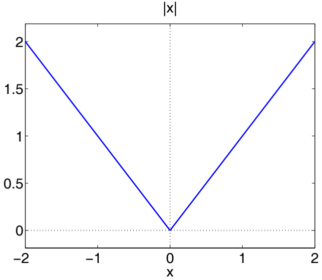
is done in the following steps:

1. Replace $<$, $\leq$, $>$ or $\geq$ by $=$
2. Get a 0 to the right hand side
3. Mark the solutions (zeros) on the x-axis
4. Determine the signs for each interval (e.g., by filling in an arbitrary point
5. Determine the solution of the inequality

Simple example: solve $x^2 > 9 \iff x \in (-\infty, -3) \cup (3, \infty)$

Absolute value

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



"Distance to 0". Properties:

- $|x| = |-x|$
- $|xy| = |x| \cdot |y|$
- $|x + y| \leq |x| + |y|$
- $|x - a|$ "distance to a"

For example:

- $|(-2)(-4)| = |-2| \cdot |-4|$
- $|2 + 4| = |2| + |4|, \qquad |2 - 4| < |2| + |-4|$

Attention!

$$(\sqrt{a})^2 = a \text{ but } \sqrt{a^2} = |a|$$

Ex: $\sqrt{(-3)^2} = 3$

Only if you know that $a \geq 0$, then $\sqrt{a^2} = |a| = a$, eg

$$\sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2+1)^2} = \left|x^2 + 1\right| = x^2 + 1$$

The last step only because $x^2 + 1$ always ≥ 0 (even ≥ 1)

Equations and inequalities with absolute values

1. Equations
   split into the two parts and solve separately
2. Inequalities

- Just as for polynomial inequalities: replace $<$ or $\leq$ by $=$
- Solve and mark solutions on the x-axis with 0s
- Determine sign by filling in arbitrary points within the intervals
- Determine the final solution

Fractions

Addition: make denominators same, then you can add

$$\frac{1}{p}+\frac{1}{q}=\frac{q}{pq}+\frac{p}{pq}=\frac{p+q}{pq}$$

Up to "smallest common multiple":

$$\tfrac{1}{4}+\tfrac{1}{12}=\tfrac{3}{12}+\tfrac{1}{12}=\tfrac{4}{12}=\tfrac{1}{3} \qquad \tfrac{1}{6}+\tfrac{1}{8}=\tfrac{4}{24}+\tfrac{3}{24}=\tfrac{7}{24}$$

Multiplication: Multiply numerators and denominators

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

Simplify: divide numerator and denominator by the same factor

$$\tfrac{144}{30}=\tfrac{72}{15}=\tfrac{24}{5}$$

$$\tfrac{2x+6}{4}=\tfrac{x+3}{2}=\tfrac{1}{2}x+\tfrac{3}{2} \quad \text{but} \quad \tfrac{4}{2x+6}=\tfrac{2}{x+3}\neq\tfrac{2}{x}+\tfrac{2}{3}\ !$$

Division by a fraction is multiplication with the inverse

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\Big/\frac{c}{d}=\tfrac{a}{b}\cdot\tfrac{d}{c}=\tfrac{ad}{bc}$$

Equations with rational functions

Rational functions $= \frac{\text{polynomial}}{\text{polynomial}}$ $\qquad R(x) = \frac{P(x)}{Q(x)}$

Two strategies:

(1) Crosswise multiplication
(2) Bring everything on the same denominator

(1) : $\dfrac{3}{x+1} = \dfrac{2}{x+2}$

Equations with rational functions

Rational functions $= \frac{\text{polynomial}}{\text{polynomial}}$ $\qquad R(x) = \frac{P(x)}{Q(x)}$

Two strategies:

(1) Crosswise multiplication

(2) Bring everything on the same denominator

$$(1): \quad \frac{3}{x+1} = \frac{2}{x+2} \implies 3\,(x+2) = 2\,(x+1) \text{ and } x \neq -1 \text{ and } x \neq -2$$

Equations with rational functions

Rational functions $= \frac{\text{polynomial}}{\text{polynomial}}$ $\qquad R(x) = \frac{P(x)}{Q(x)}$

Two strategies:

(1) Crosswise multiplication
(2) Bring everything on the same denominator

$$(1): \ \frac{3}{x+1} = \frac{2}{x+2} \implies 3\,(x+2) = 2\,(x+1) \text{ and } x \neq -1 \text{ and } x \neq -2$$

$$\implies x = -4 \quad (\text{ and } x \neq -1 \text{ and } x \neq -2)$$

(however the condition $x \neq -1$ and $x \neq -2$ is thus not relevant)

$$(2): \frac{3\,(x+2)}{(x+1)(x+2)} - \frac{2\,(x+1)}{(x+1)(x+2)} = 0 \implies \frac{x+4}{(x+1)(x+2)} = 0$$

Thus numerator $= 0$: $\ x = -4$
and denominator $\neq 0$: $\ x \neq -1$ and $x \neq -2$, but that does not apply here

Division with rest

Rewrite a fraction into an integer + "something with a numerator smaller than the denominator": $\frac{7}{4} = 1 + \frac{3}{4}$:

This also works with rational functions:

$$\boxed{\frac{\text{polynomial}}{\text{polynomial}} = \text{polynomial} + \frac{\text{polynomial P}}{\text{polynomial Q}}}$$

so that the degree of P $<$ degree of Q.

Previous example: $x = 1$ was zero of $x^3 + 2x^2 - x - 2$ and

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

with rest-term $\frac{\text{polynomial P}}{\text{polynomial Q}}$ equal to zero.

Division with rest, different formulation

Target: Write "rational function $=$ polynomial $+$ rational function" so that the degree of numerator $<$ degree of denominator

Ex: $$\frac{x^3+2}{x^2+x}$$

Procedure: The same as for long division for finding zeros. However, the rest does not have to be 0.

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$x^2 + x \;\; / \;\; x^3 \qquad\qquad +2 \;\; \backslash$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x - 1 + \frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$\begin{array}{rlllll} & x^2+x & / & x^3 & +2 & \backslash \quad x \\ x\cdot & (x^2+x) & \to & x^3+x^2 & & \end{array}$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x - 1 + \frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$\begin{array}{rcll} & x^2+x \;/ & x^3 \qquad\quad +2 \;\backslash & x \\ x\cdot & (x^2+x) \to & x^3+x^2 & \\ & - & \overline{\qquad\qquad\qquad} & \\ & & -x^2 \quad +2 & \end{array}$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x - 1 + \frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$
\begin{array}{rlllll}
 & x^2+x & / & x^3 & +2 & \backslash \quad x-1 \\
x\cdot & (x^2+x) & \to & x^3+x^2 & & \\
 & & - & \overline{} & & \\
 & & & -x^2 & +2 & \\
-1\cdot & (x^2+x) & \to & -x^2-x & &
\end{array}
$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x-1+\frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$
\begin{array}{rrlll}
 & x^2+x & / \; x^3 & \quad +2 & \backslash \quad x-1 \\
x\cdot & (x^2+x) \rightarrow & x^3+x^2 & & \\
 & - & \overline{} & & \\
 & & -x^2 & \quad +2 & \\
-1\cdot & (x^2+x) \rightarrow & -x^2-x & & \\
 & - & \overline{} & & \\
 & & & \quad x+2 &
\end{array}
$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x-1+\frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

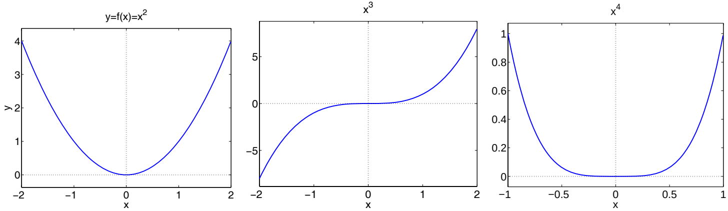
Inequalities, abs. values, rational funct.

1. $x^2 - 3x \geq -2$
2. $\frac{x+1}{x} \geq 2$
3. Simplify $\sqrt{x^4 + 6x^2 + 9}$
4. $|2x - 5| < 6$

Roots $\sqrt[n]{a}$

$$
\begin{array}{llll}
x^2 = a & \Longrightarrow & x = \pm\sqrt{a} & \text{here must be } a \geq 0 \\
x^3 = a & \Longrightarrow & x = \sqrt[3]{a} & \text{here } a \text{ can be everything } (a \in \mathbb{R}) \\
x^4 = a & \Longrightarrow & x = \pm\sqrt[4]{a} & \text{here must be } a \geq 0 \\
x^5 = a & \Longrightarrow & x = \sqrt[5]{a} & \text{here } a \text{ can be everything } (a \in \mathbb{R})
\end{array}
$$

etc; these are standard results that you can use in exams



Ex: $x^3 = 8$ thus $x^3 - 8 = 0$: you could say right away $x = 2$

but you could also solve this using the recipe for higher order polynomials

- guess solution: $x = 2$
- long division: $\frac{x^3-8}{x-2} = x^2 + 2x + 4$
- the discriminant of $x^2 + 2x^2 + 4 = -12 < 0$

  thus no further solutions

n-th power roots: examples

$$x^3 = \quad 27$$

n-th power roots: examples

$$x^3 = \quad 27 \qquad\qquad x = \sqrt[3]{27} =$$

n-th power roots: examples

$$
\begin{aligned}
x^3 &= 27 \qquad && x = \sqrt[3]{27} = 3 \\
x^3 &= -27 &&
\end{aligned}
$$

n-th power roots: examples

$$
\begin{aligned}
x^3 &= 27 & x &= \sqrt[3]{27} = 3 \\
x^3 &= -27 & x &= \sqrt[3]{-27} = -3 \\
x^4 &= 4
\end{aligned}
$$

n-th power roots: examples

$$\begin{aligned} x^3 &= 27 & \qquad x &= \sqrt[3]{27} = 3 \\ x^3 &= -27 & \qquad x &= \sqrt[3]{-27} = -3 \\ x^4 &= 4 & \qquad x &= \pm\sqrt[4]{4} = \end{aligned}$$

$$
\begin{aligned}
x^3 &= 27 & \qquad x &= \sqrt[3]{27} = 3 \\
x^3 &= -27 & \qquad x &= \sqrt[3]{-27} = -3 \\
x^4 &= 4 & \qquad x &= \pm\sqrt[4]{4} = \pm\sqrt[2]{\sqrt[2]{4}} = \pm\sqrt[2]{2} = \pm\sqrt{2} \\
& & & \text{different notation: } \pm 4^{\frac{1}{4}} = \pm(4^{\frac{1}{2}})^{\frac{1}{2}} = \pm 2^{\frac{1}{2}} \\
x^4 &= -4 & &
\end{aligned}
$$

n-th power roots: examples

$$
\begin{array}{rclcl}
x^3 &=& 27 & \qquad & x = \sqrt[3]{27} = 3 \\
x^3 &=& -27 & & x = \sqrt[3]{-27} = -3 \\
x^4 &=& 4 & & x = \pm\sqrt[4]{4} = \pm\sqrt[2]{\sqrt[2]{4}} = \pm\sqrt[2]{2} = \pm\sqrt{2} \\
 & & & & \text{different notation: } \pm 4^{\frac{1}{4}} = \pm(4^{\frac{1}{2}})^{\frac{1}{2}} = \pm 2^{\frac{1}{2}} \\
x^4 &=& -4 & & \text{no solutions}
\end{array}
$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} = 15\sqrt{5}$$

Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} = 15\sqrt{5}$$

Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{2}}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} = \tfrac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} = 15\sqrt{5}$$

Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{2}}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} = \tfrac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} = \frac{1}{\sqrt{a} - 2} \cdot \frac{\sqrt{a} + 2}{\sqrt{a} + 2} = \frac{\sqrt{a} + 2}{(\sqrt{a})^2 - 2^2} = \frac{\sqrt{a} + 2}{a - 4}$$

$$\frac{1}{\sqrt{a} + \sqrt{b}} =$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} = 15\sqrt{5}$$

Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{2}}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} = \tfrac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} = \frac{1}{\sqrt{a} - 2} \cdot \frac{\sqrt{a} + 2}{\sqrt{a} + 2} = \frac{\sqrt{a} + 2}{(\sqrt{a})^2 - 2^2} = \frac{\sqrt{a} + 2}{a - 4}$$

$$\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{1}{\sqrt{a} + \sqrt{b}} \cdot \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{(\sqrt{a})^2 - (\sqrt{b})^2} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$$

$=$ and $<, \geq$ with roots

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x - 1$$

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\Longrightarrow$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x-1 \implies 4x+1 = (x-1)^2 \implies x^2 - 6x = 0$$
$$\implies x\,(x-6) = 0 \implies x = 0 \text{ or } x = 6$$

Check:

$=$ and $<, \geq$ with roots

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x-1 \implies 4x+1 = (x-1)^2 \implies x^2 - 6x = 0$$
$$\implies x\,(x-6) = 0 \implies x = 0 \text{ or } x = 6$$

Check: $x = 0$ is no solution, $x = 6$ is

- Equations: solve by taking squares (evt several times)
  check answers always at the end
- Inequalities: solve the equality
  also pay attention that for the solutions (intervals) all arguments under the roots have to be ≥ 0

Algebraic skills: frequently made mistakes

FAIL: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

GOOD: $\sqrt{xy} = \sqrt{x}\sqrt{y}$ $\quad$ (if $x \geq 0, \;\; y \geq 0$)

FAIL: $x\,(x-2) = 1 \implies x = 1$ or $x - 2 = 1$

GOOD: $x\,(x-2) = 0 \implies x = 0$ or $x - 2 = 0$

FAIL: $x^2 = 3x \implies x = 3$

GOOD: $x^2 = 3x \implies x = 3$ or $x = 0$ $\quad$ because $x\,(x-3) = 0$

FAIL: $\sqrt{x^2} = x$

GOOD: $\sqrt{x^2} = |x|$

FAIL: $x^2 = a \implies x = \sqrt{a}$

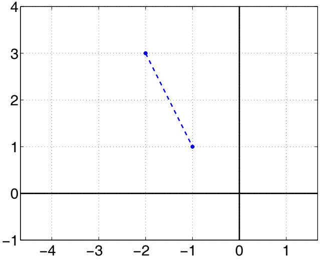
GOOD: $x^2 = a \implies x = \pm\sqrt{a}$

FAIL: $\frac{2}{x+3} = \frac{2}{x} + \frac{2}{3}$

GOOD: $\frac{x+3}{2} = \frac{x}{2} + \frac{3}{2}$

$\mathbb{R}^2$: notation for a 2-dimensional space

Distance d between 2 points $(x_1, y_1) = (-1, 1)$ adn $(x_2, y_2) = (-2, 3)$



- $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2$
- $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1+4} = \sqrt{5}$ (Pythagoras! [A B12])

Slope m of the line between the points $(-1, 1)$ and $(-2, 3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{2}{-1} = -2 \qquad \text{[A P.2 E7]}$$

Lines (I)

The line through 2 points (x_1, y_1) and (x_2, y_2)

two-point equation: $(y - y_1)\Delta x = (x - x_1)\Delta y$

(Remember: $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$).

point-slope equation: $y = m(x - x_1) + y_1$

Formula is correct because filling in yields:

- (x_1, y_1): $0 \cdot \Delta x = 0 \cdot \Delta y$
- (x_2, y_2): $(y_2 - y_1)\Delta x = (x_2 - x_1)\Delta y \Longrightarrow \Delta y \Delta x = \Delta x \Delta y$

Lines (II)

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$(y - y_1)\underbrace{\Delta x}_{=-1} = (x - x_1)\underbrace{\Delta y}_{=2} \implies (y-1)\cdot -1 = (x+1)\cdot 2$$

$$\implies (y-1) = -2(x+1)$$

$$\implies y = -2x - 1$$

because $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2.$
Straight lines thus have the form $ax + by = c$. Special cases:

- $a = 0$: horizontal line
- $b = 0$: vertical line

Ex: Set point $(-2, 3)$
Horizontal line through the point has the eq.:

Lines (II)

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$
\begin{aligned}
(y - y_1)\underbrace{\Delta x}_{=-1} = (x - x_1)\underbrace{\Delta y}_{=2} \quad &\implies \quad (y-1)\cdot -1 = (x+1)\cdot 2 \\
&\implies \quad (y-1) = -2(x+1) \\
&\implies \quad y = -2x - 1
\end{aligned}
$$

because $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2.$
Straight lines thus have the form $ax + by = c$. Special cases:

- $a = 0$: horizontal line
- $b = 0$: vertical line

Ex: Set point $(-2, 3)$
Horizontal line through the point has the eq.: $y = 3$ $(b = 1)$
Vertical line through this point has the eq.:

Lines (II)

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$\begin{aligned}(y - y_1)\underbrace{\Delta x}_{=-1} = (x - x_1)\underbrace{\Delta y}_{=2} \quad &\Longrightarrow \quad (y-1)\cdot -1 = (x+1)\cdot 2 \\ &\Longrightarrow \quad (y-1) = -2(x+1) \\ &\Longrightarrow \quad y = -2x - 1\end{aligned}$$

because $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2.$
Straight lines thus have the form $ax + by = c$. Special cases:

- $a = 0$: horizontal line
- $b = 0$: vertical line

Ex: Set point $(-2, 3)$
Horizontal line through the point has the eq.: $y = 3$ $(b = 1)$
Vertical line through this point has the eq.: $x = -2$ $(a = 1)$

Lines (III)

> Intersections with the x- and y-axes are called "intercepts"

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$y = -2x - 1$$

Where does this line intersect the x-axis?

Lines (III)

> Intersections with the x- and y-axes are called "intercepts"

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$y = -2x - 1$$

Where does this line intersect the x-axis? Then $y = 0$ thus

$0 = -2x - 1 \implies -2x = 1 \implies x = -\frac{1}{2}$ (thus "x-intercept" $= -\frac{1}{2}$)

Where does this line intersect the y-axis?

Lines (III)

Intersections with the x- and y-axes are called "intercepts"

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$y = -2x - 1$$

Where does this line intersect the x-axis? Then $y = 0$ thus

$0 = -2x - 1 \implies -2x = 1 \implies x = -\frac{1}{2}$ (thus "x-intercept" $= -\frac{1}{2}$)

Where does this line intersect the y-axis?

Then $x = 0$ thus $y = -2 \cdot 0 - 1 = -1$ (thus "y-intercept" $= -1$)

Do [A P.2 T38]

Vertical and general lines (I)

A line is vertical if $x_1 = x_2$, then it follows

$$(y - y_1)\Delta x = (x - x_1)\Delta y \Longrightarrow 0 = (x - x_1)\Delta y \Longrightarrow 0 = x - x_1$$

A vertical line in $\mathbb{R}^2$ is given by $x = x_1$.

Other lines: $x_1 \neq x_2 \Longrightarrow \Delta x \neq 0$ thus

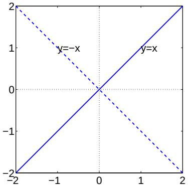
$$(y - y_1)\Delta x = (x - x_1)\Delta y \Longrightarrow (y - y_1) = \frac{\Delta y}{\Delta x}(x - x_1) \Longrightarrow$$
$$(y - y_1) = m(x - x_1)$$

A general line in $\mathbb{R}^2$ is described by the equation $y = mx + b$

with $b = m \cdot x_1 + y_1$ the y-intercept and $m = \Delta y / \Delta x$ the slope.

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:

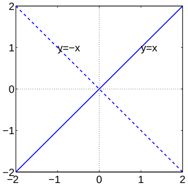


Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope $\frac{1}{2}$
because $-2 \cdot \frac{1}{2} = -1$
Thus $y = \frac{1}{2}x + b$, now fill in $(1, 1)$:
$1 = \frac{1}{2} + b \implies b = \frac{1}{2}$ so the line $y = \frac{1}{2}x + \frac{1}{2}$. [Do A P.2 T49]

Perpendicular lines

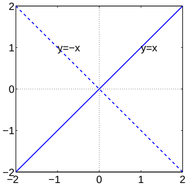
For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$



Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$

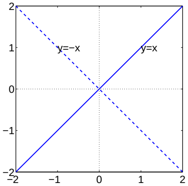


Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$

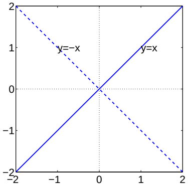


Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$

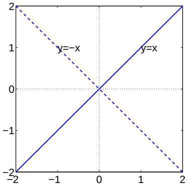


Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope $\frac{1}{2}$
because $-2 \cdot \frac{1}{2} = -1$

Perpendicular lines

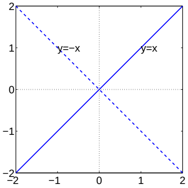
For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$



Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope $\frac{1}{2}$
because $-2 \cdot \frac{1}{2} = -1$
Thus $y = \frac{1}{2}x + b$,

For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$



Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope $\frac{1}{2}$
because $-2 \cdot \frac{1}{2} = -1$
Thus $y = \frac{1}{2}x + b$, now fill in $(1, 1)$:
$1 = \frac{1}{2} + b \implies b = \frac{1}{2}$ so the line $y = \frac{1}{2}x + \frac{1}{2}$. [Do A P.2 T49]

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

Distance between (x, y) and $(1, 2)$ is

Circle with center $(0,0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

Distance between (x, y) and $(1, 2)$ is $\sqrt{(x-1)^2 + (y-2)^2}$

If the distance is equal to r is then is holds

$\sqrt{(x-1)^2 + (y-2)^2} = r \iff (x-1)^2 + (y-2)^2 = r^2.$

General equation of a circle in $\mathbb{R}^2$ with center (a, b) and radius r:

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle with center (1,2) and radius 3:

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

Distance between (x, y) and $(1, 2)$ is $\sqrt{(x-1)^2 + (y-2)^2}$

If the distance is equal to r is then is holds

$\sqrt{(x-1)^2 + (y-2)^2} = r \iff (x-1)^2 + (y-2)^2 = r^2.$

General equation of a circle in $\mathbb{R}^2$ with center (a, b) and radius r:

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle with center (1,2) and radius 3:

$(x-1)^2 + (y-2)^2 = 3^2$

Disc, circle and “hole”

The circle with radius r and center (a, b) consists of all points (x, y) for which it holds

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \text{circle line}$$

The disk with radius r and center (a, b) consists of all points (x, y) inside the circle for which it holds

$$(x-a)^2 + (y-b)^2 < r^2 \qquad \text{circle interior} \rightarrow \text{open disk}$$
$$(x-a)^2 + (y-b)^2 \leq r^2 \qquad \text{circle interior} + \text{line} \rightarrow \text{closed disk}$$

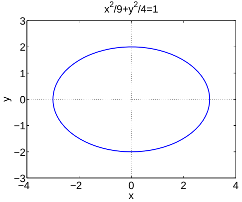
The “hole” with radius r and center (a, b) consists of alle points (x, y) outside the circle for which it holds

$$(x-a)^2 + (y-b)^2 > r^2 \qquad \text{circle exterior}$$

Ellipse with center $(0,0)$ and "radii" r_x and r_y

We saw:
Circle with radius r and center $(0,0)$:

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$



ellipse $$\boxed{\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 = 1}$$

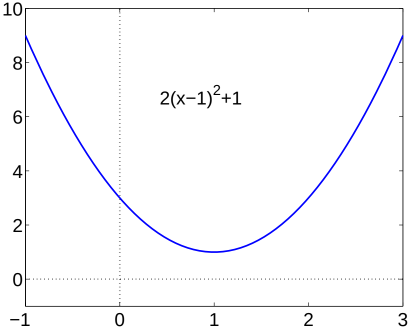
is thus of the form $cx^2 + dy^2 = e$

General ellipse with center (a,b) and semi-axes r_x and r_y

$$\left(\frac{x-a}{r_x}\right)^2 + \left(\frac{y-b}{r_y}\right)^2 = 1$$

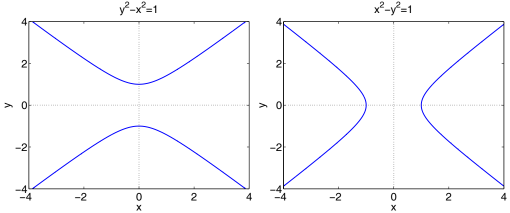
Parabolas

Parabolas with the origin as vertex and the y-axis as axis of symmetry:
$y = ax^2$ with $a \neq 0$.
Parabolas with (x_v, y_v) as vertex and the y-axis as axis of symmetry:

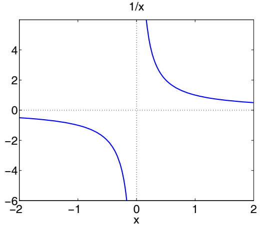


$$y = a\,(x - x_V)^2 + y_V$$

For example: $\left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 = 1$ and $\left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 = -1$



Also: $xy = c$ thus $y = \frac{c}{x}$



Overview of quadratic equations

Standard form has "$= 1$" on the right hand side

Circle ex: $$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1 \quad \text{or} \quad x^2 + y^2 = r^2$$

Ellipse ex: $$\left(\frac{x-a}{r_x}\right)^2 + \left(\frac{y-b}{r_y}\right)^2 = 1 \quad \text{or} \quad cx^2 + dy^2 = e$$

Parabola ex: $$\left(\frac{x-a}{r_x}\right)^2 + \frac{y-b}{r_y} = 1 \quad \text{or} \quad y = cx^2 + d$$

Hyperbola ex: $$\left(\frac{x-a}{r_x}\right)^2 - \left(\frac{y-b}{r_y}\right)^2 = \pm 1 \quad \text{or} \quad xy = c$$

Definition of Curve

A curve is an equation of the form $f(x, y) = 0$.

Ex:

- $f(x, y) = x^2 - (2x - y^2) = 0$
  $\rightarrow x^2 = 2x - y^2$
- $f(x, y) = 2y - 3x + 2 = 0$
  $\rightarrow 2y = 3x - 2$ (straight line)
- $f(x, y) = y - (3x + 2)$
  $\rightarrow y = 3x + 2$ (straight line)

What kind of curve is $x^2 = 2x - y^2$?
ellipse, parabola, ...?

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$x^2 - 2x + y^2 = 0$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$\begin{aligned} & x^2 - 2x + y^2 = 0 \\ \implies & x^2 - 2x + 1 - 1 + y^2 = 0 \end{aligned}$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

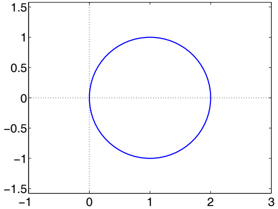
$$
\begin{aligned}
& x^2 - 2x + y^2 = 0 \\
\implies & x^2 - 2x + 1 - 1 + y^2 = 0 \\
\implies & (x-1)^2 + y^2 = 1
\end{aligned}
$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$\begin{aligned} & x^2 - 2x + y^2 = 0 \\ \implies & x^2 - 2x + 1 - 1 + y^2 = 0 \\ \implies & (x-1)^2 + y^2 = 1 \end{aligned}$$

Circle with radius 1 and center (1,0)



See [A page 18], do [A, P.2 T9,11] and [A P.3 T7,15]

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

What does $x^2 = 3x - y^2$ represent?

Complete the square
$$x^2 - 3x + y^2 = 0$$

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$

$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$\begin{aligned} & x^2 - 3x + y^2 = 0 \\ \implies\ & x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0 \\ \implies\ & (x - \tfrac{3}{2})^2 + y^2 = (\tfrac{3}{2})^2 \end{aligned}$$

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$

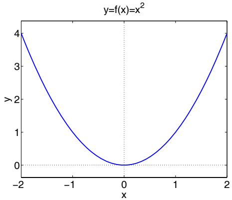
$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$

$$\implies (x - \tfrac{3}{2})^2 + y^2 = (\tfrac{3}{2})^2$$

Circle with radius 3/2 and center (3/2,0)

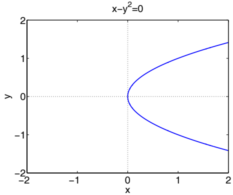
Functions and graphs

Function f is a rule that:
prescribes for each $x \in D$ (domain) exactly one $f(x)$
Notation: $y = f(x)$
x is the independent, y is the dependent variable
Function or not?

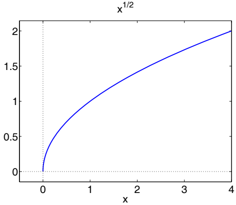


$y = f(x) = x^2$
function

Do [A P.4 T7]



$y = \pm\sqrt{x}$
no function
but curve



$y = \sqrt{x}$
function on $[0, \infty)$
no function on $\mathbb{R}$

Function $y = g(x)$ is described by the curve

$$y - g(x) = 0$$

that means by $f(x, y) = 0$ with $f(x, y) = y - g(x)$. Each function "is" therefore a curve.

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

- number under root must be ≥ 0
- do not divide by 0
- $\ln(x)$: x must be > 0 (also for ${}^{10}\log$)

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

- number under root must be ≥ 0
- do not divide by 0
- $\ln(x)$: x must be > 0 (also for ${}^{10}\log$)

Range of f: set of all possible outputs $y = f(x)$

Ex: $f(x) = \frac{1}{x}$: domain $=$

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

- number under root must be ≥ 0
- do not divide by 0
- $\ln(x)$: x must be > 0 (also for ${}^{10}\log$)

Range of f: set of all possible outputs $y = f(x)$

Ex: $f(x) = \frac{1}{x}$: domain $= (-\infty, 0) \cup (0, \infty)$, (or $\mathbb{R}\backslash\{0\}$)

range $=$

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

- number under root must be ≥ 0
- do not divide by 0
- $\ln(x)$: x must be > 0 (also for $^{10}\log$)

Range of f: set of all possible outputs $y = f(x)$

Ex: $f(x) = \frac{1}{x}$: domain $= (-\infty, 0) \cup (0, \infty)$, (or $\mathbb{R}\backslash\{0\}$)

range $= (-\infty, 0) \cup (0, \infty)$

Ex: $f(x) = \sqrt{x-2}$: domain $=$

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

- number under root must be ≥ 0
- do not divide by 0
- $\ln(x)$: x must be > 0 (also for ${}^{10}\log$)

Range of f: set of all possible outputs $y = f(x)$

Ex: $f(x) = \frac{1}{x}$: domain $= (-\infty, 0) \cup (0, \infty)$, (or $\mathbb{R}\backslash\{0\}$)

range $= (-\infty, 0) \cup (0, \infty)$

Ex: $f(x) = \sqrt{x-2}$: domain $= [2, \infty)$, range $=$

Domain and range

Domain of f: the set of all x for which $f(x)$ is defined ("all possible inputs x"). Pay attention:

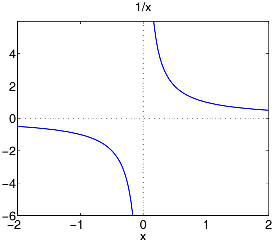
- number under root must be ≥ 0
- do not divide by 0
- $\ln(x)$: x must be > 0 (also for ${}^{10}\log$)

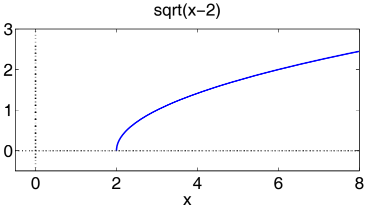
Range of f: set of all possible outputs $y = f(x)$

Ex: $f(x) = \frac{1}{x}$: domain $= (-\infty, 0) \cup (0, \infty)$, (or $\mathbb{R}\backslash\{0\}$)

range $= (-\infty, 0) \cup (0, \infty)$

Ex: $f(x) = \sqrt{x-2}$: domain $= [2, \infty)$, range $= [0, \infty)$





Do [A P.4 T6] and [A P.6 T16]

Even/odd function I

Even function

- $f(x) = f(-x)$ for all x
- means: graph of f is the same as if it is mirrored at the y-axis
- ex: $f(x) = x^2$, $f(x) = x^6 + x^2$, $f(x) = 1$, $f(x) = \cos(x)$
- sums of even powers in x are even

Odd function

- $f(x) = -f(-x)$ for all x
- means: graph of f is the same as first being mirrored at the y-axis, then at x-axis
- ex: $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^7 + x^3$, $f(x) = \sin(x)$
- sums of odd powers in x are odd

Ex: $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 + (-x) = x^2 - x$
Is function even? Is $x^2 - x = x^2 + x$ for all x?

Even/odd function I

Even function

- $f(x) = f(-x)$ for all x
- means: graph of f is the same as if it is mirrored at the y-axis
- ex: $f(x) = x^2$, $f(x) = x^6 + x^2$, $f(x) = 1$, $f(x) = \cos(x)$
- sums of even powers in x are even

Odd function

- $f(x) = -f(-x)$ for all x
- means: graph of f is the same as first being mirrored at the y-axis, then at x-axis
- ex: $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^7 + x^3$, $f(x) = \sin(x)$
- sums of odd powers in x are odd

Ex: $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 + (-x) = x^2 - x$
Is function even? Is $x^2 - x = x^2 + x$ for all x? No
Is function odd? Is $x^2 - x = -(x^2 + x)$ for all x?

Even/odd function I

Even function

- $f(x) = f(-x)$ for all x
- means: graph of f is the same as if it is mirrored at the y-axis
- ex: $f(x) = x^2$, $f(x) = x^6 + x^2$, $f(x) = 1$, $f(x) = \cos(x)$
- sums of even powers in x are even

Odd function

- $f(x) = -f(-x)$ for all x
- means: graph of f is the same as first being mirrored at the y-axis, then at x-axis
- ex: $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^7 + x^3$, $f(x) = \sin(x)$
- sums of odd powers in x are odd

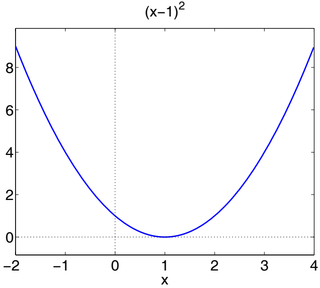
Ex: $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 + (-x) = x^2 - x$
Is function even? Is $x^2 - x = x^2 + x$ for all x? No
Is function odd? Is $x^2 - x = -(x^2 + x)$ for all x? No
A function does NOT have to be even or odd!

Symmetries

Even and odd are special cases of (anti-)symmetry:
namely around $x = 0$
One can also have symmetries around $x = 1$:

Symmetries

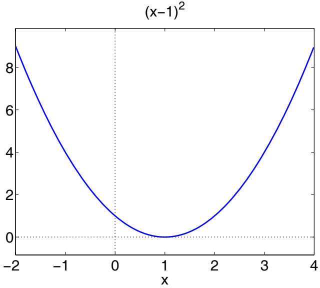
Even and odd are special cases of (anti-)symmetry:
namely around $x = 0$
One can also have symmetries around $x = 1$: $f(x) = (x-1)^2$



This can easily be seen by subsituting $w = x - 1$: $y = w^2$, even function:
symmetric around $w = 0$,
this coincides with $x = 1$
Ex: function which is anti-symmetric around $x = 2$?

Symmetries

Even and odd are special cases of (anti-)symmetry:
namely around $x = 0$
One can also have symmetries around $x = 1$: $f(x) = (x-1)^2$



This can easily be seen by subsituting $w = x - 1$: $y = w^2$, even function:
symmetric around $w = 0$,
this coincides with $x = 1$
Ex: function which is anti-symmetric around $x = 2$?
$f(x) = x - 2$ or $f(x) = (x-2)^3$ or ...

Functions and their graphs

- Give domain and range of $f(x) = -x^2 + 2x - 7$
- Give domain and range of for $f(x) = \sqrt{9 - x^2}$
- Draw the graph of $y = -x^2$. How does the equation change for shifting by 7 left, right, up, down?
- Give function and domain for shifting $f(x) = \sqrt{x}$ 1 to the left
- Even, odd or nothing?
  $\sqrt{1 - x^2}, \quad x + 1, \quad x^3 + x$

Addition and Multiplication

If f and g functions are, then also sums and products of them are functions:

operation			example
$(fg)(x)$	$=$	$f(x) \cdot g(x)$	$x^2 \sin(x)$
$(f+g)(x)$	$=$	$f(x) + g(x)$	$x^2 + \sin(x)$
$(f-g)(x)$	$=$	$f(x) - g(x)$	$x^2 - \sin(x)$
$\left(\frac{f}{g}\right)(x)$	$=$	$\frac{f(x)}{g(x)}$	$x^2/\sin(x)$

Do [A P.5 T2]

Composite functions – one after the other

Composite functions: $(f \circ g)(x) = f(g(x))$ "f after g"

- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$
  $(f \circ g)(x) =$

Composite functions – one after the other

Composite functions: $(f \circ g)(x) = f(g(x))$ "f after g"

- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$

  $(f \circ g)(x) = 4\,(x^2 - 1) - 1$

  $(g \circ f)(x) =$

Composite functions: $(f \circ g)(x) = f(g(x))$ "f after g"

- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$
  $$(f \circ g)(x) = 4\,(x^2 - 1) - 1$$
  $$(g \circ f)(x) = (4x - 1)^2 - 1$$
- $\sqrt{\sin(x^2)} = f(g(h(x)))$ with:

Composite functions – one after the other

Composite functions: $(f \circ g)(x) = f(g(x))$ "f after g"

- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$
  $(f \circ g)(x) = 4\,(x^2 - 1) - 1$
  $(g \circ f)(x) = (4x - 1)^2 - 1$
- $\sqrt{\sin(x^2)} = f(g(h(x)))$ with:
  $f(x) = \sqrt{x}, \;\; g(x) = \sin(x), \;\; h(x) = x^2$

$\mathrm{D}_{f \circ g}$ consists of all $x \in D_g$ for which $g(x) \in D_f$

Ex: $f(x) = 1/x$, $g(x) = 1/x$. Then is $(f \circ g)(x) = x$ and $\mathrm{D}_{f \circ g} = \mathbb{R} - \{0\}$.
Do [A P.5 T6,7,9,13,15]