2WBB0 Calculus for BCS

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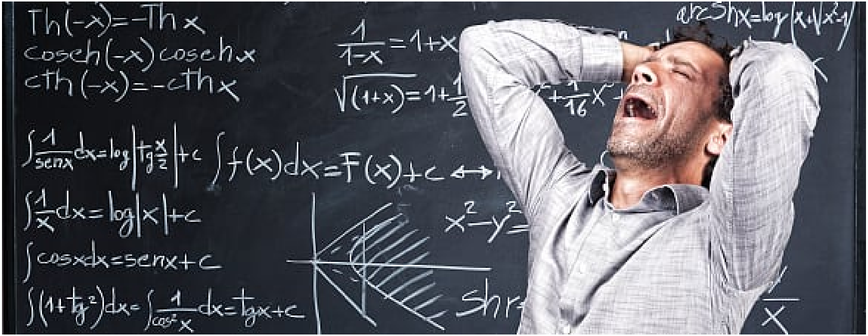
Why do we (=you) need Calculus?



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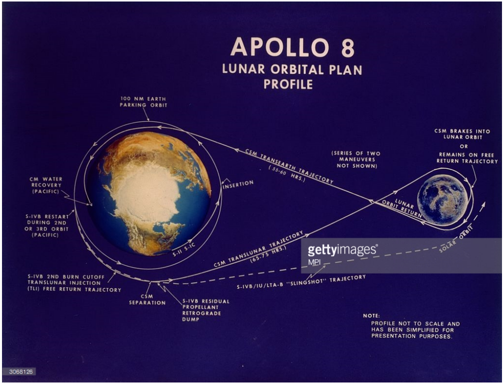

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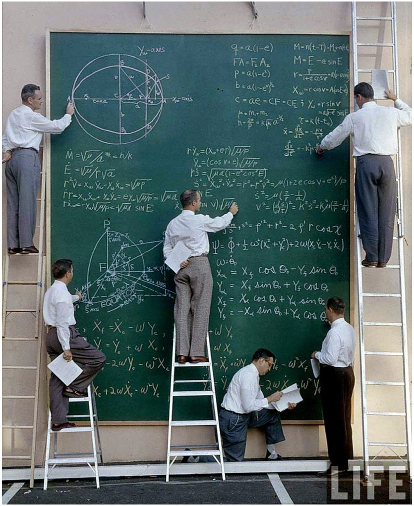

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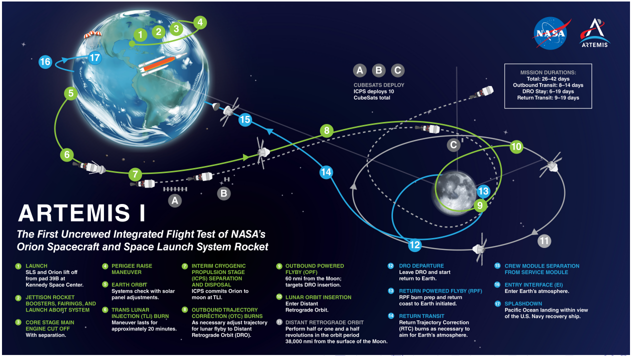
December 21, 1968?

November 16, 2022



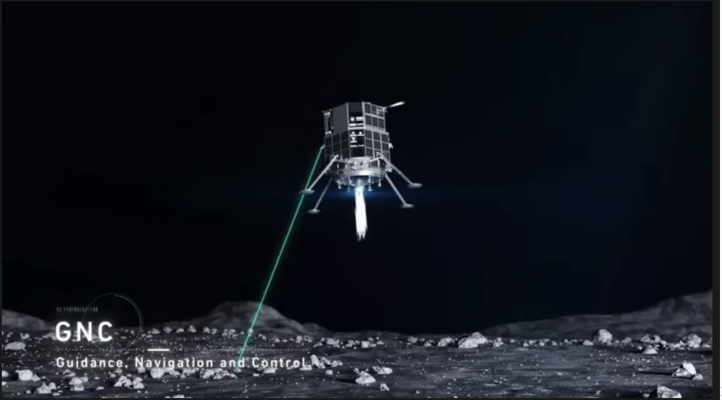
Why do we (=you) need Calculus?

April 26, 2023

THE GLITCH THAT BROUGHT DOWN JAPAN'S LUNAR LANDER

by: Matthew Carlson 37 Comments

June 3, 2023



Why do we (=you) need Calculus?

April 26, 2023

When a computer crashes, it usually doesn't leave debris. But when a computer happens to be descending towards the lunar surface and glitches out, that's a very different story. Turns out that's what happened on April 26th, as the Japanese Hakuto-R Lunar lander made its mark on the Moon...by crashing into it. [Scott Manley] dove in to try and understand the software bug that caused an otherwise flawless mission to go splat.

The lander began the descent sequence as expected at 100 km above the surface. However, as it descended, the altitude sensor reported the altitude as much lower than it was. It thought it was at zero altitude once it reached about 5 km above the surface. Confused by the fact it hadn't yet detected physical contact with the surface, the craft continued to slowly descend until it ran out of fuel and plunged to the surface.

Ultimately it all came down to sensor fusion. The lander merges several noisy sensors, such as accelerometers, gyroscopes, and radar, into one cohesive source of truth. The craft passed over a particularly large cliff that caused the radar altimeter to suddenly spike up 3 km. Like good filtering software, the craft reasons that the sensor must be getting spurious data and filters it out. It was now just estimating its altitude by looking at its acceleration. As anyone who has tried to track an object through space using just gyros and accelerometers alone can attest, errors accumulate, and suddenly you're not where you think you are.

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There are 20 people in the room. They all need to say good bye to each other, and shake hands. How many handshakes?

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Without Calculus

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for i := 1 to X do
    for j := 1 to X do
        if j>i then inc(cnt);
Writeln(cnt);
```

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Readln('how many people?', X);
Writeln(X*(X-1)/2);
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- For 20 people, nobody can tell the difference.
- For one million people in the room, you get to do 10^{14} loops. You need a 64 bit machine for storing the counter. At 3 GHz, that takes 166 seconds (way more in real life). With Calculus: instant!

Calculus in Computer Science

For video game programming.
Calculus, when taught alongside Physics, gives you the background necessary to understand how objects can interact and move based on forces. Even if you don't ever write your own game engine, you'll be a better game programmer if you have a mastery of calculus.

For machine learning.
Basically machine learning is mathematics meets computer science meets data. This is one of the hottest topics in CS right now, but you have to know your mathematics to succeed with ML.

For scientific applications.
A lot of computer scientists work alongside scientists like biologists, physicists, and chemists. CS people help them solve their data-analysis problems using large clusters of computers, and anytime you have to understand how equations work, it is hard to escape calculus.

In research.
Researchers such as Professors and PhD students use mathematics when writing papers. If you want to one day pursue an advanced degree, you'll have to decipher these papers and then write some of your own. I would not recommend starting out a career in research without at least a basic command over calculus.

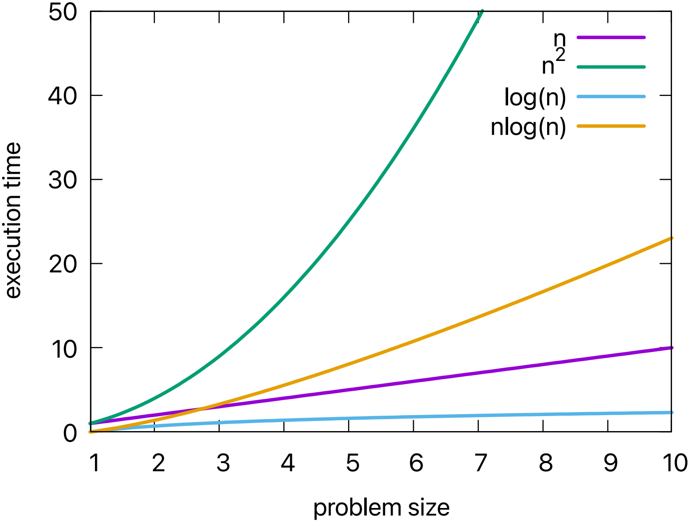
Analyze a Computer Algorithm

Imagine you are a programmer of an automatic, autonomous guidance system (think of cars, rockets). You have an algorithm that takes data collected from a number of input sources (sensors) and processes it to adjust the control of the vehicle. It takes 50ms to obtain the adjustment after the acquisition of data (reaction time).

Now, the engineers plan to double the number of sensors to improve the guidance. The maximal acceptable reaction time for the vehicle is 150ms.

Can your algorithm deal with double the amount of data within the 150ms?

Different scaling of (parts of complex) algorithms



Example: Bubble Sort

Sorting an unordered list of five numbers by swapping adjacent elements that are out of place:

17	**12**	8	3	7
12	**17**	**8**	3	7
12	8	**17**	**3**	7
12	8	3	**17**	**7**
12	8	3	7	**17**

How many comparisons? $n - 1$

Example: Bubble Sort

Now, do it again

12	**8**	3	7	17
8	**12**	**3**	7	17
8	3	**12**	**7**	17
8	3	7	**12**	**17**

How many comparisons? $n - 2$

And again:

8	**3**	7	12	17
3	**8**	**7**	12	17
3	7	**8**	**12**	**17**

How many comparisons? $n - 3$

Example: Bubble Sort

Finally:

3 7 8 12 17

How many comparisons? $n-4$
So all in all:

$$(n-1)+(n-2)+(n-3)+\ldots+1=\frac{n(n-1)}{2}$$

Ranking Different Algorithms

- How to compare different algorithms?
- Two algorithms

$$\text{Algorithm 1} \rightarrow f(n) = n$$
$$\text{Algorithm 2} \rightarrow g(n) = n^2$$

- Which one is "faster"?
- Algorithm 1, because it grows at a slower rate with n

Ranking Different Algorithms

- How to formulate this more generally?
  $\rightarrow$ Limits
-

$$\lim_{n\to\infty} \frac{f}{g} = 0 \qquad g \text{ grows faster}$$

$$\lim_{n\to\infty} \frac{f}{g} = c \qquad g \text{ grows same as } f$$

$$\lim_{n\to\infty} \frac{f}{g} = \infty \qquad g \text{ grows slower}$$

Limits not always straightforward

- $f(n) = n\log(n)$
- $g(n) = n^{3/2}$
- $$\lim_{n\to\infty} \frac{n^{3/2}}{n\log(n)} = ?$$
- To evaluate we will need:
  - derivatives
  - Taylor series/polynomials
  - Exponential functions and Logarithms
  - rational functions
  - ...

Overview

- W1: Numbers, Functions, (In-)equalities, Polynomials, Rational functions
- W2: Trigonometric functions/Vectors in 2 and 3 dimensions
- W3: Limits, continuity, differentiation I
- W4: Differentiation II and Inverse functions
- W5: Exponential function and logarithm/Taylor series/Limits II
- W6: Integration/Integration techniques I
- W7: Integration techniques II/1st order differential equations
- W8: Exam and W1 – W7 summary (no new material)

Shall we begin?

Topics in Week 1

- Real numbers and the real line
- Graphs of quadratic equations
- Polynomials and rational functions
- Powers and roots
- Cartesian coordinates in the plane
- More functions and their graphs
- Combining functions to make new functions

Real numbers

$$\mathbb{N} \underset{\substack{\uparrow \\ \text{is a subset of}}}{\subset} \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$\mathbb{N}$ natural numbers: $0, 1, 2, \ldots$ (sometimes without 0)
$\mathbb{Z}$ integer numbers: $\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$ rational numbers = fractions: $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$
$\mathbb{R}$ real numbers: $\mathbb{Q}$ and π, $\sqrt{2}$, etc
$\mathbb{C}$ complex numbers: $2 + 3\mathbf{i}$, with $\mathbf{i} \cdot \mathbf{i} = \mathbf{i}^2 = -1$ (not in curriculum)

Decimal Expansion

For rational numbers, the "decimal expansion" has a repeating pattern, for irrational numbers, this is not the case:

$$
\begin{aligned}
\frac{1}{2} &= 0.50000|0|\cdots \\
\frac{1}{3} &= 0.33333|3|\cdots \\
\frac{1}{11} &=
\end{aligned}
$$

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\tfrac{1}{7} &=
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\pi &=
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\tfrac{1}{7} &= 0.142857|142857|\cdots \\
\pi &= 3.141592653589793\cdots
\end{aligned}$$

Intervals

Notation for intervals:

- [,]: end points are part of the interval
- (,): end points are NOT part of the interval

interval	alternative notation	type	type
$(0,1)$	$0 < x < 1$	open	finite
$[0,1)$	$0 \leq x < 1$	half open	finite
$(0,1]$	$0 < x \leq 1$	half open	finite
$[0,1]$	$0 \leq x \leq 1$	closed	finite
$[1,\infty)$	$1 \leq x < \infty$		infinite
$(-\infty,3)$	$-\infty < x < 3$		infinite
$(-\infty,\infty)$	$-\infty < x < \infty$		infinite

Symbols

$\cup$	union
$\cap$	intersection
$\in$	element of
$\vee$	or
$\wedge$	and
(A) $\Longrightarrow$ (B)	from (A) follows (B)
(A) $\Leftrightarrow$ (B)	from (A) follows (B) and from (B) follows (A)
	in other words: they are equivalent
	(A) "if and only if" (B) (iff)

For two sets A and B:

- $A - B$
- $A \backslash B$

represents alle $x \in A$ for which $x \notin B$.

Polynomials

A polynomial is a "many term" construct

The degree (order) of a polynomial is the highest ocurring power

Ex: $p(x) = x^3 - 2x^2 + 1$ is of degree 3

Ex: $p(x) = 0 \cdot x^3 - 2x^2 + 1$ is of degree 2

Ex: $p(x) = (x+1)^2 - (x-1)^2$ is of degree 1

Ex: $p(x) = 3 = 3 \cdot x^0$ is of degree 0

Computers can only do

- additions/subtractions
- multiplications

All other operations, such as division or calculation of a root have be performed using only these basis operations – and then cost 3 to 5 times more calculational time

Frequently appearing polynomials

Degree	Name	Example
0	constant	$p(x) = 2$
1	linear	$p(x) = 2x + 3$
2	quadratic	$p(x) = 5x^2 + 4x + 5$
3	cubic	$p(x) = 2x^3 + 1$

Equations with polynomials of degree 1 always have a solution:

Ex: $4x - 3 = 0$

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Equations with polynomials of degree 1 always have a solution:

Ex: $4x - 3 = 0 \implies 4x = 3 \implies x = \frac{3}{4}$

Theorem:

> Equations of polynomials of degree n have a maximum of n real solutions

Equations with a polynomial of degree 2

Quadratic functions are more than algebraic curiosities – they are widely used in science, business, and engineering:

- trajectories of water jets in a fountain or of a bouncing ball
- parabolic reflectors that form the base of satellite dishes and car headlights
- forecast business profit and loss
- data interpolation

Equations of quadratic polynomials can always be written into the general form:

$$ax^2 + bx + c = 0$$

Equations with a polynomial of degree 2

Equations of quadratic polynomials can always be written into the general form:

$$ax^2 + bx + c = 0$$

Our task is to find the solutions (or zeros) of this equation.

3 different ways to find the solution:

- completing the square
- abc-formula (quadratic formula)
- "Sum-product"-formula

Use the way that you like best **for this purpose** but learn the concept of **completing the square**! It will be useful later.

Idea behind Completing the Square

Ex: : $x^2 = 3$

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Ex: : $x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$

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This is easy (and also an equation with a quadratic polynomial)!

So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.

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Ex: $(x-1)^2 = 9$

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This is easy (and also an equation with a quadratic polynomial)!

So, in general, if we have an equation

$$(\text{something})^2 = \text{something else}$$

we can simply take the square root on both sides.

Ex: $(x-1)^2 = 9 \Leftrightarrow x - 1 = \pm 3 \Leftrightarrow x = 1 \pm 3$

> Rewrite a quadratic equation to a form with a complete square on the left-hand side
>
> $$(x \pm R)^2 = T$$
>
> and take the square root on both sides, if possible.

The ONE basic trick for completing the square

Remember the binomial formulas:

$$\boxed{\begin{aligned}(n+m)^2 &= n^2+2nm+m^2\\(n-m)^2 &= n^2-2nm+m^2\end{aligned}}$$

and that one can also use them from right to left!

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Ex: $x^2-4x+4=(x-2)^2$

Ex: $4x^2+12x+9=(\underbrace{2x}_{=n})^2+\underbrace{2}_{\text{binom. form.}}\cdot\underbrace{2x}_{=n}\cdot\underbrace{3}_{=m}+(\underbrace{3}_{=m})^2=(2x+3)^2$

Those examples are complete squares. They can be rewritten directly into one of the two binomial formulas.

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But what about x^2-4x+3? We are missing $+1$ for a complete square!

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But what about x^2-4x+3? We are missing $+1$ for a complete square!

$$x^2-4x+3=x^2-4x+3+0=x^2-4x+3+1-1=x^2-4x+4-1=(x-2)^2-1$$

Completing the square to solve quadratic equations

Solve:

$$ax^2 + bx + c = 0$$

1. Identify n and m in the expression.
   - n is the square root of the term with x^2
     $ax^2 \to n = \sqrt{a}x$
   - m is the linear term in x, divided by $2n$
     $m = bx/(2 \cdot \sqrt{a}x) = b/(2\sqrt{a})$
2. Use binomial formula
   $(n+m)^2 = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = ax^2 + bx + \frac{b^2}{4a}$
3. what is missing/too much
   $ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 - \frac{b^2}{4a} + c = 0$
4. isolate the square, take roots, simplify
   $(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = \frac{b^2}{4a} - c \Leftrightarrow \sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$
   $$\Leftrightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = 0$ via abc-formula (I)

2nd degree polynomial $ax^2 + bx + c = 0$ with $a \neq 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

How many solutions are possible?

Discriminant: $D = b^2 - 4ac$

$D > 0$

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$D = 0$ 1 double solution (2 solutions $x_1 = x_2$)

$D < 0$ no (real) solution

The zeros determine the factoring of the polynomial:

$$ax^2 + bx + c = (x - x_1)(x - x_2).$$

Pay attention to the minus sign!

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions $(x_1 \neq x_2)$
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0$

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions ($x_1 \neq x_2$)
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
$D < 0$	no (real) solution

Ex: $x^2 + 1 = 0 \implies x^2 = -1 \implies$ no solution

indeed: $D = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0 \implies$ no solution

Ex: $x^2 + 2x + 1 = 0$

$ax^2 + bx + c = 0$ via abc-formula (II)

Discriminant: $D = b^2 - 4ac$

$D > 0$	2 different solutions ($x_1 \neq x_2$)
$D = 0$	1 double solution (2 solutions $x_1 = x_2$)
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Ex: $x^2 + 1 = 0 \implies x^2 = -1 \implies$ no solution
indeed: $D = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0 \implies$ no solution

Ex: $x^2 + 2x + 1 = 0$: $D = 2^2 - 4 \cdot 1 \cdot 1 = 0 \implies$ double solution
indeed: $x^2 + 2x + 1 = 0 \implies (x+1)^2 = 0$
$\implies x + 1 = 0$ or $x + 1 = 0 \implies$ double solution $x = -1$

$ax^2 + bx + c = 0$ via "Sum-product"

Set $a = 1$:

$$x^2 + \underbrace{(r+s)}_{=b(\text{sum})} x + \underbrace{rs}_{=c(\text{product})} = (x+r)(x+s)$$

Ex: $x^2 + 5x + 6 =$

$ax^2 + bx + c = 0$ via "Sum-product"

Set $a = 1$:

$$x^2 + \underbrace{(r+s)}_{=b(\text{sum})} x + \underbrace{rs}_{=c(\text{product})} = (x+r)(x+s)$$

Ex: $x^2 + 5x + 6 = (x+2)(x+3)$

Because $b = 5 = 2 + 3$ and $c = 6 = 2 \cdot 3$

Ex: $x^2 - 9 = (x-3)(x+3)$

Attention! $x^2 = 9 \not\Longrightarrow x = 3$, but :

$$x^2 = 9 \implies x^2 - 9 = 0 \implies (x-3)(x+3) = 0 \implies$$

$$x - 3 = 0 \text{ or } x + 3 = 0 \implies x = 3 \quad \text{or} \quad x = -3$$

Examples: Quadratic Equations

1. Solve: $x^2 - 3x + 2 = 0$
2. Factorize: $x^2 = 2$
3. Solve: $x^2 + 2 = 0$
4. Complete the square: $x^2 + 3x + \frac{1}{4}$
5. Complete the square: $4x^2 + 12x + 1$

Equations with a 3rd degree polynomial

1. Guess first zero. Try out first $a = 0, 1, -1, 2, -2, \ldots$
2. Divide by factor $x - a$
3. The rest must be 0 (if not, you made a mistake)
4. Find solutions of the remaining 2nd degree polynomial equation, if they exist

Ex: $x^3 + 2x^2 - x - 2 = 0$
Try:

Equations with a 3rd degree polynomial

1. Guess first zero. Try out first $a = 0, 1, -1, 2, -2, \ldots$
2. Divide by factor $x - a$
3. The rest must be 0 (if not, you made a mistake)
4. Find solutions of the remaining 2nd degree polynomial equation, if they exist

Ex: $x^3 + 2x^2 - x - 2 = 0$
Try: $x = 1$ is a solution
Then is $x^3 + 2x^2 - x - 2 = (x - 1) \cdot (\text{2nd degree polynomial})$

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To determine the 2nd order polynomial, divide out (see later):

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

And then $x^2 + 3x + 2 =$

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And then $x^2 + 3x + 2 = (x + 2)(x + 1)$ via abc-formula
Conclusion $x^3 + 2x^2 - x - 2 = (x - 1)(x + 2)(x + 1)$
Thus solution of $x^3 + 2x^2 - x - 2 = 0$:

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Conclusion $x^3 + 2x^2 - x - 2 = (x-1)(x+2)(x+1)$
Thus solution of $x^3 + 2x^2 - x - 2 = 0$: $x = 1$ or $x = -1$ or $x = -2$

Equations with 3rd, 4th, 5th, ...-degree polynomials

Principle: guess, divide, guess, etc, until 2nd-degree remains

1. First, have a look if there is not a special easier case, as $p(x) = 0$ with $p(x) = x^3 - 8$ or $p(x) = x^4 + 2x^2 + 1$
2. If not, guess a zero, say $x = a$, $a = 0, 1, -1, 2, -2, \ldots$
3. Divide out by Long Division: $p(x)/(x - a)$
   (here **must** have rest 0),
   this yields a new polynomial $q(x)$
4. If the degree of $q(x)$ is still larger than 2, repeat the above
5. If the degree of $q(x)$ is equal to 2, try with sum-product technique or the abc-formula

Degree 4, 5, . . .: Special cases

Ex: $x^4 - 50x^2 + 49 = 0$

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4th degree but 2nd degree in x^2! Rename $p = x^2$
It holds that $(x^2)^2 = x^2 \cdot x^2 = x^{2+2} = x^4$

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$$p^2 - 50p + 49 = 0 \implies (p-49)(p-1) = 0 \implies p = 49 \text{ of } p = 1$$
$$\underset{p=x^2}{\implies} \implies x^2 = 49 \text{ or } x^2 = 1 \implies x = \pm 7 \text{ or } x = \pm 1$$

Higher degree but only produts of lower degrees

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$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$
$$\implies x = 0 \text{ or } x - 2 = 0$$

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Higher degree but only produts of lower degrees

$$x^4(x-2)^2 = 0 \implies x^4 = 0 \text{ or } (x-2)^2 = 0$$
$$\implies x = 0 \text{ or } x - 2 = 0 \implies x = 0 \text{ or } x = 2$$

(respectively 4-fold and 2-fold)

Factoring and roots (zeros) of polynomials

Writing a polynomial as a product of two or more polynomials is called factoring.

Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2)$
The polynomials $x - 3$ and $x^2 - 2$ are factors with degree 1 and 2.

Factoring breaks up a complicated polynomial into easier, lower degree pieces.

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In the example, we can do more:
Ex: $x^3 - 3x^2 - 2x + 6 = (x-3)(x^2-2) = (x-3)(x-\sqrt{2})(x+\sqrt{2})$

This polynomial is factored into three **linear** polynomials. We can't do ant better, the polynomial is **factored completely**.

> Knowing all roots of a polynomial $\Leftrightarrow$ Knowing the factoring of a polynomial

Polynomials: When to let factors stand as they are?

When a polynomial consists of one term of factors:
$p(x) = (x+1)^3$ or $p(x) = (x-1)(x+2)$

Most of the times, it is useful to let these stand, because:

- it is easy, no extra effort
- you see the zeros at once
- sometimes this is easier to differentiate/integrate (later in Calculus) (eg derivative or primitive of $(x+1)^{40}$)
- useful for limits (eg $\lim_{x \to 1^+} (x-1)^3$ etc)

Polynomials: When to write factors out?

When a polynomial consists of several terms (of factors), eg:
$p(x) = (x+1)^3 - (x-1)^3$

Most of the times, it is useful to write these out, because:

- Terms can cancel each other out, eg:
  $(x+1)^3 - (x-1)^3 = 6x^2 + 2$ shows to be of degree 2 (not 3)

Examples: Higher order polynomial equations

1. Solve $3x^4 = 9x^2$
2. Solve $x^3 + 9x^2 + 26x + 24 = 0$
3. Solve $x^4 - 26x^2 + 25 = 0$
4. What are all solutions to $(x^2 - 1)^4(x + 3)^7 = 0$
5. Which of these polynomials has roots $x = 1 \vee x = 2 \vee x = 3/4$
   a) $p(x) = (x + 1)(x + 2)(x + 3/4)$
   b) $p(x) = (x - 1)(x - 2)(x - 3/4)$

Inequalities with polynomials

Finding solutions to inequalities of the kind

$$p(x) > d \qquad p(x) \geq d \qquad p(x) < d \qquad p(x) \leq d$$

is done in the following steps:

1. Replace $<$, $\leq$, $>$ or $\geq$ by $=$
2. Get a 0 to the right hand side
3. Mark the solutions (zeros) on the x-axis
4. Determine the signs for each interval (e.g., by filling in an arbitrary point
5. Determine the solution of the inequality

Simple example: solve $x^2 > 9$

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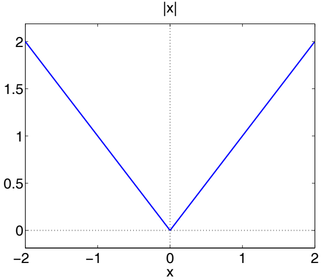
5. Determine the solution of the inequality

Simple example: solve $x^2 > 9 \iff x \in (-\infty, -3) \cup (3, \infty)$

Absolute value

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



"Distance to 0". Properties:

- $|x| = |-x|$
- $|xy| = |x| \cdot |y|$
- $|x + y| \leq |x| + |y|$
- $|x - a|$ "distance to a"

For example:

- $|(-2)(-4)| = |-2| \cdot |-4|$
- $|2 + 4| = |2| + |4|, \qquad |2 - 4| < |2| + |-4|$

Absolute values: Examples

Attention!

$$(\sqrt{a})^2 = a \text{ but } \sqrt{a^2} = |a|$$

Ex: $\sqrt{(-3)^2} = 3$

Only if you know that $a \geq 0$, then $\sqrt{a^2} = |a| = a$, eg

$$\sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2+1)^2} = \left|x^2 + 1\right| = x^2 + 1$$

The last step only because $x^2 + 1$ always ≥ 0 (even ≥ 1)

Equations and inequalities with absolute values

1. Equations
   split into the two parts and solve separately
2. Inequalities

- Just as for polynomial inequalities: replace $<$ or $\leq$ by $=$
- Solve and mark solutions on the x-axis with 0s
- Determine sign by filling in arbitrary points within the intervals
- Determine the final solution

Fractions

Addition: make denominators same, then you can add

$$\frac{1}{p}+\frac{1}{q}=\frac{q}{pq}+\frac{p}{pq}=\frac{p+q}{pq}$$

Up to "smallest common multiple":

$$\tfrac{1}{4}+\tfrac{1}{12}=\tfrac{3}{12}+\tfrac{1}{12}=\tfrac{4}{12}=\tfrac{1}{3} \qquad \tfrac{1}{6}+\tfrac{1}{8}=\tfrac{4}{24}+\tfrac{3}{24}=\tfrac{7}{24}$$

Multiplication: Multiply numerators and denominators

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

Simplify: divide numerator and denominator by the same factor

$$\tfrac{144}{30}=\tfrac{72}{15}=\tfrac{24}{5}$$

$$\tfrac{2x+6}{4}=\tfrac{x+3}{2}=\tfrac{1}{2}x+\tfrac{3}{2} \quad \text{but} \quad \tfrac{4}{2x+6}=\tfrac{2}{x+3}\neq\tfrac{2}{x}+\tfrac{2}{3}\ !$$

Division by a fraction is multiplication with the inverse

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\Big/\frac{c}{d}=\tfrac{a}{b}\cdot\tfrac{d}{c}=\tfrac{ad}{bc}$$

Equations with rational functions

Rational functions $= \frac{\text{polynomial}}{\text{polynomial}}$ $\qquad R(x) = \frac{P(x)}{Q(x)}$

Two strategies:

(1) Crosswise multiplication

(2) Bring everything on the same denominator

$$(1):\quad \frac{3}{x+1} = \frac{2}{x+2}$$

Equations with rational functions

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Two strategies:

(1) Crosswise multiplication (2) Bring everything on the same denominator

$$(1): \quad \frac{3}{x+1} = \frac{2}{x+2} \implies 3\,(x+2) = 2\,(x+1) \text{ and } x \neq -1 \text{ and } x \neq -2$$

Equations with rational functions

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Two strategies:

$$\boxed{\begin{array}{ll} (1) & \text{Crosswise multiplication} \\ (2) & \text{Bring everything on the same denominator} \end{array}}$$

$$(1): \quad \frac{3}{x+1} = \frac{2}{x+2} \implies 3\,(x+2) = 2\,(x+1) \text{ and } x \neq -1 \text{ and } x \neq -2$$

$$\implies x = -4 \quad (\text{ and } x \neq -1 \text{ and } x \neq -2)$$

(however the condition $x \neq -1$ and $x \neq -2$ is thus not relevant)

$$(2): \frac{3\,(x+2)}{(x+1)(x+2)} - \frac{2\,(x+1)}{(x+1)(x+2)} = 0 \implies \frac{x+4}{(x+1)(x+2)} = 0$$

Thus numerator $= 0$: $\quad x = -4$
and denominator $\neq 0$: $\quad x \neq -1$ and $x \neq -2$, but that does not apply here

Division with rest

Rewrite a fraction into an integer + "something with a numerator smaller than the denominator": $\frac{7}{4} = 1 + \frac{3}{4}$:

This also works with rational functions:

$$\boxed{\frac{\text{polynomial}}{\text{polynomial}} = \text{polynomial} + \frac{\text{polynomial P}}{\text{polynomial Q}}}$$

so that the degree of P $<$ degree of Q.

Previous example: $x = 1$ was zero of $x^3 + 2x^2 - x - 2$ and

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

with rest-term $\frac{\text{polynomial P}}{\text{polynomial Q}}$ equal to zero.

Division with rest, different formulation

Target: Write "rational function = polynomial + rational function" so that the degree of numerator < degree of denominator

Ex: $$\frac{x^3 + 2}{x^2 + x}$$

Procedure: The same as for long division for finding zeros.
However, the rest does not have to be 0.

Long division with rest

Target: rational function = polynomial+other rational function while the degree of the numerator < degree of the denominator

$$x^2 + x \quad / \quad x^3 \qquad\qquad +2 \quad \backslash$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x - 1 + \frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

integer number + "something with numerator smaller than denominator"

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\begin{array}{rcccc}
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x \cdot & (x^2 + x) & \to & x^3 + x^2 &
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 & & - \; \overline{} & & \\
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 & & - & \overline{\qquad\qquad\qquad} & \\
 & & & \quad -x^2 \quad +2 & \\
-1\cdot & (x^2+x) & \to & \quad -x^2-x &
\end{array}
$$

Conclusion

$$\frac{x^3+2}{x^2+x} = x-1+\frac{x+2}{x^2+x}$$

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-1\cdot & (x^2+x)\rightarrow & \quad -x^2-x \\
 & - & \qquad \overline{\qquad\quad} \\
 & & \qquad\qquad x+2
\end{array}
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Conclusion

$$\frac{x^3+2}{x^2+x} = x-1+\frac{x+2}{x^2+x}$$

Indeed: here degree of numerator is smaller than the degree of the denominator

Somewhat analog to $\frac{7}{4} = 1 + \frac{3}{4}$:

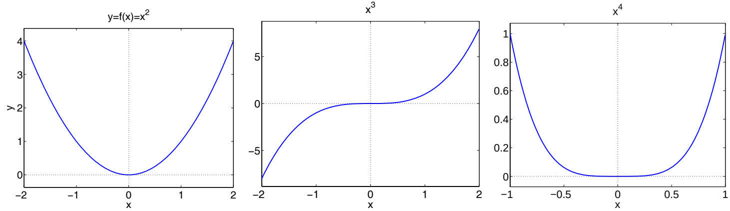
integer number + "something with numerator smaller than denominator"

1. $x^2 - 3x \geq -2$
2. $\frac{x+1}{x} \geq 2$
3. Simplify $\sqrt{x^4 + 6x^2 + 9}$
4. $|2x - 5| < 6$

Roots $\sqrt[n]{a}$

$x^2 = a \implies x = \pm\sqrt{a}$ here must be $a \geq 0$
$x^3 = a \implies x = \sqrt[3]{a}$ here a can be everything ($a \in \mathbb{R}$)
$x^4 = a \implies x = \pm\sqrt[4]{a}$ here must be $a \geq 0$
$x^5 = a \implies x = \sqrt[5]{a}$ here a can be everything ($a \in \mathbb{R}$)

etc; these are standard results that you can use in exams



Ex: $x^3 = 8$ thus $x^3 - 8 = 0$: you could say right away $x = 2$
but you could also solve this using the recipe for higher order polynomials

- guess solution: $x = 2$
- long division: $\frac{x^3-8}{x-2} = x^2 + 2x + 4$
- the discriminant of $x^2 + 2x^2 + 4 = -12 < 0$
  thus no further solutions

n-th power roots: examples

$$x^3 = \quad 27$$

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$$x^3 = \quad 27 \qquad\qquad x = \sqrt[3]{27} =$$

n-th power roots: examples

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\begin{aligned}
x^3 &= 27 \qquad & x &= \sqrt[3]{27} = 3 \\
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x^3 &= 27 & \qquad x &= \sqrt[3]{27} = 3 \\
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x^4 &= 4
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 & & \qquad \text{different notation: } \pm 4^{\frac{1}{4}} = \pm(4^{\frac{1}{2}})^{\frac{1}{2}} = \pm 2^{\frac{1}{2}} \\
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x^4 & = & -4 & \qquad \text{no solutions}
\end{array}
$$

Working with roots

Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
Simplification in steps: decompose into factors:

$$\sqrt{1125} =$$

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$$\sqrt{1125} = \sqrt{5 \cdot 225} = \sqrt{5 \cdot 5 \cdot 45} = 5\sqrt{45} = 5\sqrt{9 \cdot 5} = 5 \cdot 3\sqrt{5} = 15\sqrt{5}$$

Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} =$$

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$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} =$$

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"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} = \frac{1}{\sqrt{a} - 2} \cdot \frac{\sqrt{a} + 2}{\sqrt{a} + 2} = \frac{\sqrt{a} + 2}{(\sqrt{a})^2 - 2^2} = \frac{\sqrt{a} + 2}{a - 4}$$

$$\frac{1}{\sqrt{a} + \sqrt{b}} =$$

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Basic property (definition): $\sqrt{x} \cdot \sqrt{x} = x \quad (x \geq 0)$
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Remove the root from the denominator:

$$\frac{1}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{2}}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} = \tfrac{1}{2}\sqrt{2}$$

"Root trick": Make use of $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$:

$$\frac{1}{\sqrt{a} - 2} = \frac{1}{\sqrt{a} - 2} \cdot \frac{\sqrt{a} + 2}{\sqrt{a} + 2} = \frac{\sqrt{a} + 2}{(\sqrt{a})^2 - 2^2} = \frac{\sqrt{a} + 2}{a - 4}$$

$$\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{1}{\sqrt{a} + \sqrt{b}} \cdot \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{(\sqrt{a})^2 - (\sqrt{b})^2} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$$

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\Longrightarrow$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x - 1$$

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x-1 \implies 4x+1 = (x-1)^2 \implies x^2-6x=0$$
$$\implies x\,(x-6)=0 \implies x=0 \text{ or } x=6$$

Check:

You have to pay attention to one thing: by taking a square you can come to solutions that cannot exist $\implies$ $\boxed{\text{extra check}}$

$$\sqrt{4x+1} = x-1 \implies 4x+1 = (x-1)^2 \implies x^2 - 6x = 0$$
$$\implies x\,(x-6) = 0 \implies x = 0 \text{ or } x = 6$$

Check: $x = 0$ is no solution, $x = 6$ is

- Equations: solve by taking squares (evt several times)
  check answers always at the end
- Inequalities: solve the equality
  also pay attention that for the solutions (intervals) all arguments under the roots have to be ≥ 0

Algebraic skills: frequently made mistakes

FAIL: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

GOOD: $\sqrt{xy} = \sqrt{x}\sqrt{y} \qquad (\text{if } x \geq 0, \ \ y \geq 0)$

FAIL: $x\,(x-2) = 1 \implies x = 1 \ \text{ or } \ x - 2 = 1$

GOOD: $x\,(x-2) = 0 \implies x = 0 \ \text{ or } \ x - 2 = 0$

FAIL: $x^2 = 3x \implies x = 3$

GOOD: $x^2 = 3x \implies x = 3 \text{ or } x = 0 \quad$ because $x\,(x-3) = 0$

FAIL: $\sqrt{x^2} = x$

GOOD: $\sqrt{x^2} = |x|$

FAIL: $x^2 = a \implies x = \sqrt{a}$

GOOD: $x^2 = a \implies x = \pm\sqrt{a}$

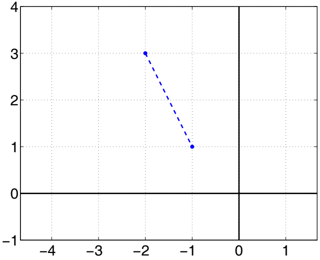
FAIL: $\frac{2}{x+3} = \frac{2}{x} + \frac{2}{3}$

GOOD: $\frac{x+3}{2} = \frac{x}{2} + \frac{3}{2}$

The plane $\mathbb{R}^2$, distance, lines

$\mathbb{R}^2$: notation for a 2-dimensional space

Distance d between 2 points $(x_1, y_1) = (-1, 1)$ adn $(x_2, y_2) = (-2, 3)$



- $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2$
- $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1+4} = \sqrt{5}$ (Pythagoras! [A B12])

Slope m of the line between the points $(-1, 1)$ and $(-2, 3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{2}{-1} = -2 \qquad \text{[A P.2 E7]}$$

Lines (I)

The line through 2 points (x_1, y_1) and (x_2, y_2)

two-point equation: $(y - y_1)\Delta x = (x - x_1)\Delta y$

(Remember: $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$).

point-slope equation: $y = m(x - x_1) + y_1$

Formula is correct because filling in yields:

- (x_1, y_1): $0 \cdot \Delta x = 0 \cdot \Delta y$
- (x_2, y_2): $(y_2 - y_1)\Delta x = (x_2 - x_1)\Delta y \Longrightarrow \Delta y \Delta x = \Delta x \Delta y$

Lines (II)

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$(y - y_1)\underbrace{\Delta x}_{=-1} = (x - x_1)\underbrace{\Delta y}_{=2} \implies (y-1)\cdot -1 = (x+1)\cdot 2$$
$$\implies (y-1) = -2(x+1)$$
$$\implies y = -2x - 1$$

because $\Delta x = x_2 - x_1 = -2 - (-1) = -1, \quad \Delta y = y_2 - y_1 = 3 - 1 = 2.$
Straight lines thus have the form $ax + by = c$. Special cases:

- $a = 0$: horizontal line
- $b = 0$: vertical line

Ex: Set point $(-2, 3)$
Horizontal line through the point has the eq.:

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Vertical line through this point has the eq.:

Lines (II)

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$
\begin{aligned}
(y - y_1)\underbrace{\Delta x}_{=-1} = (x - x_1)\underbrace{\Delta y}_{=2} \quad &\Longrightarrow \quad (y-1)\cdot -1 = (x+1)\cdot 2 \\
&\Longrightarrow \quad (y-1) = -2(x+1) \\
&\Longrightarrow \quad y = -2x - 1
\end{aligned}
$$

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Ex: Set point $(-2, 3)$
Horizontal line through the point has the eq.: $y = 3$ $(b = 1)$
Vertical line through this point has the eq.: $x = -2$ $(a = 1)$

Lines (III)

Intersections with the x- and y-axes are called "intercepts"

Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$y = -2x - 1$$

Where does this line intersect the x-axis?

Lines (III)

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Ex: The line through $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (-2, 3)$ is

$$y = -2x - 1$$

Where does this line intersect the x-axis? Then $y = 0$ thus

$0 = -2x - 1 \implies -2x = 1 \implies x = -\frac{1}{2}$ (thus “x-intercept” $= -\frac{1}{2}$)

Where does this line intersect the y-axis?

Lines (III)

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Where does this line intersect the y-axis?

Then $x = 0$ thus $y = -2 \cdot 0 - 1 = -1$ (thus "y-intercept" $= -1$)

Do [A P.2 T38]

Vertical and general lines (I)

A line is vertical if $x_1 = x_2$, then it follows

$$(y - y_1)\Delta x = (x - x_1)\Delta y \implies 0 = (x - x_1)\Delta y \implies 0 = x - x_1$$

A vertical line in $\mathbb{R}^2$ is given by $x = x_1$.

Other lines: $x_1 \neq x_2 \implies \Delta x \neq 0$ thus

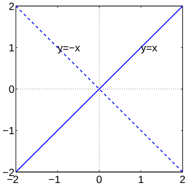
$$(y - y_1)\Delta x = (x - x_1)\Delta y \implies (y - y_1) = \frac{\Delta y}{\Delta x}(x - x_1) \implies$$
$$(y - y_1) = m(x - x_1)$$

A general line in $\mathbb{R}^2$ is described by the equation $y = mx + b$

with $b = m \cdot x_1 + y_1$ the y-intercept and $m = \Delta y / \Delta x$ the slope.

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:

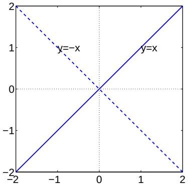


Ex: Which line is perpendicular to $y = -2x - 1$ and goes through $(1, 1)$?

The given line has the slope -2
Thus the sought line has the slope $\frac{1}{2}$
because $-2 \cdot \frac{1}{2} = -1$
Thus $y = \frac{1}{2}x + b$, now fill in $(1, 1)$:
$1 = \frac{1}{2} + b \implies b = \frac{1}{2}$ so the line $y = \frac{1}{2}x + \frac{1}{2}$. [Do A P.2 T49]

Perpendicular lines

For lines with slopes m_1 and m_2 that are perpendicular to each other:
$m_1 \cdot m_2 = -1$



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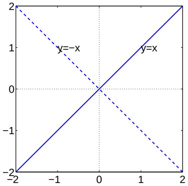

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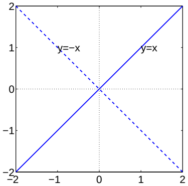

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The given line has the slope -2
Thus the sought line has the slope

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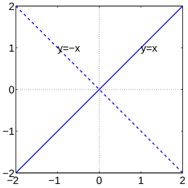

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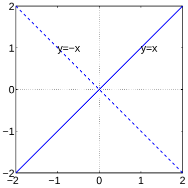

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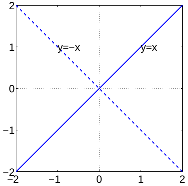

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Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

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Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or

Circle with radius r, center (0,0) has equation

$$\boxed{x^2 + y^2 = r^2} \text{ or } \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

Distance between (x, y) and $(1, 2)$ is

Circle with center $(0, 0)$ and radius r

Circle with radius r, center (0,0) has equation

$$\boxed{x^2 + y^2 = r^2} \text{ or } \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

Distance between (x, y) and $(1, 2)$ is $\sqrt{(x-1)^2 + (y-2)^2}$

If the distance is equal to r is then is holds

$\sqrt{(x-1)^2 + (y-2)^2} = r \iff (x-1)^2 + (y-2)^2 = r^2$.

> General equation of a circle in $\mathbb{R}^2$ with center (a, b) and radius r:
> $$(x-a)^2 + (y-b)^2 = r^2$$

Circle with center (1,2) and radius 3:

Circle with center $(0,0)$ and radius r

Circle with radius r, center (0,0) has equation

$\boxed{x^2 + y^2 = r^2}$ or $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ or $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

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> $$(x-a)^2 + (y-b)^2 = r^2$$

Circle with center (1,2) and radius 3:

$(x-1)^2 + (y-2)^2 = 3^2$

Disc, circle and "hole"

The circle with radius r and center (a, b) consists of all points (x, y) for which it holds

$(x-a)^2 + (y-b)^2 = r^2$	circle line

The disk with radius r and center (a, b) consists of all points (x, y) inside the circle for which it holds

$(x-a)^2 + (y-b)^2 < r^2$	circle interior $\rightarrow$ open disk
$(x-a)^2 + (y-b)^2 \leq r^2$	circle interior + line $\rightarrow$ closed disk

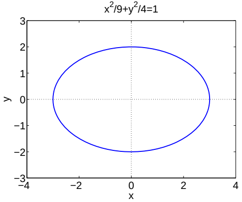
The "hole" with radius r and center (a, b) consists of alle points (x, y) outside the circle for which it holds

$(x-a)^2 + (y-b)^2 > r^2$	circle exterior

Ellipse with center $(0,0)$ and "radii" r_x and r_y

We saw:
Circle with radius r and center $(0,0)$:

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$



ellipse $\boxed{\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 = 1}$

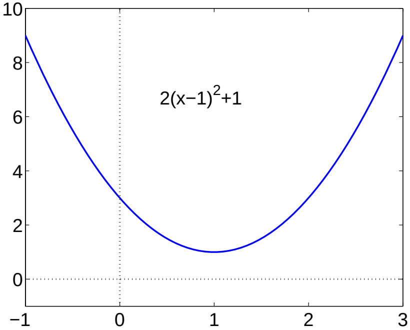
is thus of the form $cx^2 + dy^2 = e$

General ellipse with center (a,b) and semi-axes r_x and r_y

$$\left(\frac{x-a}{r_x}\right)^2 + \left(\frac{y-b}{r_y}\right)^2 = 1$$

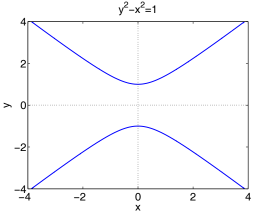
Parabolas

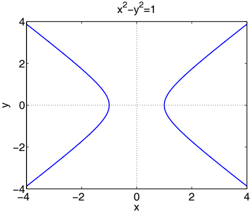
Parabolas with the origin as vertex and the y-axis as axis of symmetry:
$y = ax^2$ with $a \neq 0$.
Parabolas with (x_v, y_v) as vertex and the y-axis as axis of symmetry:



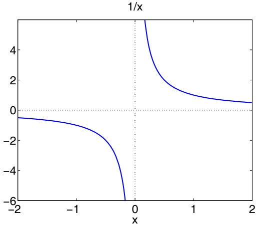
$$y = a\,(x - x_V)^2 + y_V$$

For example: $\left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 = 1$ and $\left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 = -1$





Also: $xy = c$ thus $y = \frac{c}{x}$



Overview of quadratic equations

Standard form has "$= 1$" on the right hand side

Circle ex: $$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1 \quad \text{or} \quad x^2 + y^2 = r^2$$

Ellipse ex: $$\left(\frac{x-a}{r_x}\right)^2 + \left(\frac{y-b}{r_y}\right)^2 = 1 \quad \text{or} \quad cx^2 + dy^2 = e$$

Parabola ex: $$\left(\frac{x-a}{r_x}\right)^2 + \frac{y-b}{r_y} = 1 \quad \text{or} \quad y = cx^2 + d$$

Hyperbola ex: $$\left(\frac{x-a}{r_x}\right)^2 - \left(\frac{y-b}{r_y}\right)^2 = \pm 1 \quad \text{or} \quad xy = c$$

Definition of Curve

A curve is an equation of the form $f(x, y) = 0$.

Ex:

- $f(x, y) = x^2 - (2x - y^2) = 0$
  $\rightarrow x^2 = 2x - y^2$
- $f(x, y) = 2y - 3x + 2 = 0$
  $\rightarrow 2y = 3x - 2$ (straight line)
- $f(x, y) = y - (3x + 2)$
  $\rightarrow y = 3x + 2$ (straight line)

What kind of curve is $x^2 = 2x - y^2$?
ellipse, parabola, ...?

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$x^2 - 2x + y^2 = 0$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$\begin{aligned} & x^2 - 2x + y^2 = 0 \\ \implies & x^2 - 2x + 1 - 1 + y^2 = 0 \end{aligned}$$

Determining the type of curve

Now what does $x^2 = 2x - y^2$ represent?
Complete the square

$$\begin{aligned} & x^2 - 2x + y^2 = 0 \\ \implies & x^2 - 2x + 1 - 1 + y^2 = 0 \\ \implies & (x-1)^2 + y^2 = 1 \end{aligned}$$

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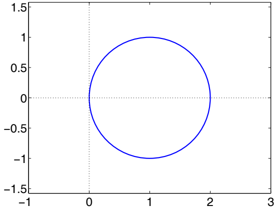
Circle with radius 1 and center (1,0)



See [A page 18], do [A, P.2 T9,11] and [A P.3 T7,15]

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$

$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

Complete the square

$$x^2 - 3x + y^2 = 0$$

$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$

$$\implies (x - \tfrac{3}{2})^2 + y^2 = (\tfrac{3}{2})^2$$

Determining the type of curve

What does $x^2 = 3x - y^2$ represent?

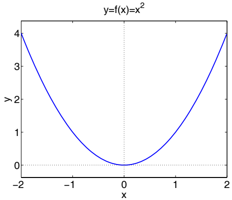
Complete the square

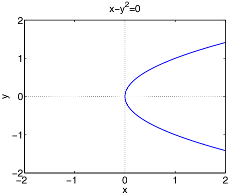
$$x^2 - 3x + y^2 = 0$$
$$\implies x^2 - (\tfrac{3}{2} + \tfrac{3}{2})x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + y^2 = 0$$
$$\implies (x - \tfrac{3}{2})^2 + y^2 = (\tfrac{3}{2})^2$$

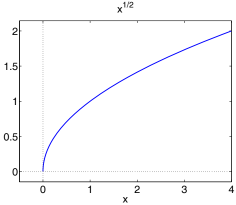
Circle with radius 3/2 and center (3/2,0)

Functions and graphs

Function f is a rule that:
prescribes for each $x \in D$ (domain) exactly one $f(x)$
Notation: $y = f(x)$
x is the independent, y is the dependent variable
Function or not?







$y = f(x) = x^2$
function

Do [A P.4 T7]

$y = \pm\sqrt{x}$
no function
but curve

$y = \sqrt{x}$
function on $[0, \infty)$
no function on $\mathbb{R}$

Functions and curves

Function $y = g(x)$ is described by the curve

$$y - g(x) = 0$$

that means by $f(x, y) = 0$ with $f(x, y) = y - g(x)$. Each function "is" therefore a curve.

Domain and range

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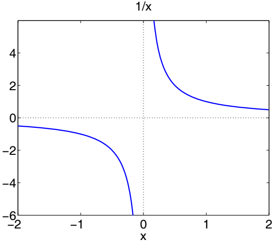

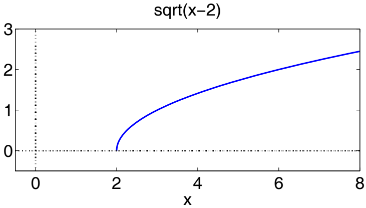

Do [A P.4 T6] and [A P.6 T16]

Even/odd function I

Even function

- $f(x) = f(-x)$ for all x
- means: graph of f is the same as if it is mirrored at the y-axis
- ex: $f(x) = x^2$, $f(x) = x^6 + x^2$, $f(x) = 1$, $f(x) = \cos(x)$
- sums of even powers in x are even

Odd function

- $f(x) = -f(-x)$ for all x
- means: graph of f is the same as first being mirrored at the y-axis, then at x-axis
- ex: $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^7 + x^3$, $f(x) = \sin(x)$
- sums of odd powers in x are odd

Ex: $f(x) = x^2 + x \Rightarrow f(-x) = (-x)^2 + (-x) = x^2 - x$
Is function even? Is $x^2 - x = x^2 + x$ for all x?

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Is function even? Is $x^2 - x = x^2 + x$ for all x? No
Is function odd? Is $x^2 - x = -(x^2 + x)$ for all x?

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Is function odd? Is $x^2 - x = -(x^2 + x)$ for all x? No

A function does NOT have to be even or odd!

Symmetries

Even and odd are special cases of (anti-)symmetry:
namely around $x = 0$
One can also have symmetries around $x = 1$:

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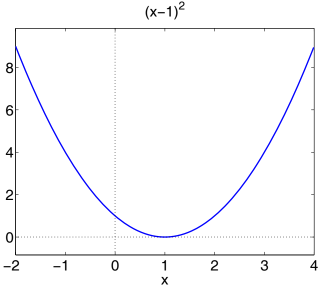

This can easily be seen by subsituting $w = x - 1$: $y = w^2$, even function: symmetric around $w = 0$,
this coincides with $x = 1$
Ex: function which is anti-symmetric around $x = 2$?

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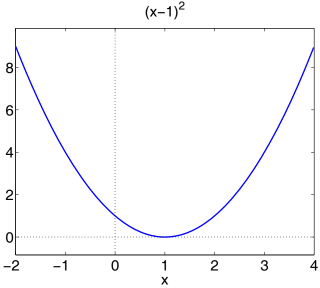

This can easily be seen by subsituting $w = x - 1$: $y = w^2$, even function:
symmetric around $w = 0$,
this coincides with $x = 1$
Ex: function which is anti-symmetric around $x = 2$?
$f(x) = x - 2$ or $f(x) = (x-2)^3$ or ...

Functions and their graphs

- Give domain and range of $f(x) = -x^2 + 2x - 7$
- Give domain and range of for $f(x) = \sqrt{9 - x^2}$
- Draw the graph of $y = -x^2$. How does the equation change for shifting by 7 left, right, up, down?
- Give function and domain for shifting $f(x) = \sqrt{x}$ 1 to the left
- Even, odd or nothing?
  $\sqrt{1 - x^2}, \quad x + 1, \quad x^3 + x$

Addition and Multiplication

If f and g functions are, then also sums and products of them are functions:

operation			example
$(fg)(x)$	$=$	$f(x) \cdot g(x)$	$x^2 \sin(x)$
$(f+g)(x)$	$=$	$f(x) + g(x)$	$x^2 + \sin(x)$
$(f-g)(x)$	$=$	$f(x) - g(x)$	$x^2 - \sin(x)$
$\left(\frac{f}{g}\right)(x)$	$=$	$\frac{f(x)}{g(x)}$	$x^2/\sin(x)$

Do [A P.5 T2]

Composite functions – one after the other

Composite functions: $(f \circ g)(x) = f(g(x))$ "f after g"

- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$
  $(f \circ g)(x) =$

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  $(g \circ f)(x) =$

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  $$(f \circ g)(x) = 4\,(x^2 - 1) - 1$$

  $$(g \circ f)(x) = (4x - 1)^2 - 1$$

- $\sqrt{\sin(x^2)} = f(g(h(x)))$ with:

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- $f(x) = 4x - 1, \;\; g(x) = x^2 - 1$
  $$(f \circ g)(x) = 4\,(x^2 - 1) - 1$$
  $$(g \circ f)(x) = (4x - 1)^2 - 1$$
- $\sqrt{\sin(x^2)} = f(g(h(x)))$ with:
  $f(x) = \sqrt{x}, \;\; g(x) = \sin(x), \;\; h(x) = x^2$

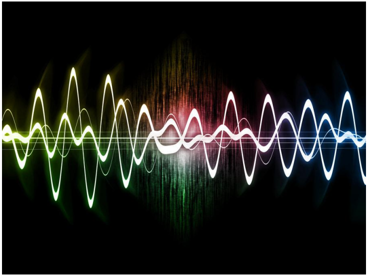
$\mathrm{D}_{f \circ g}$ consists of all $x \in D_g$ for which $g(x) \in D_f$

Ex: $f(x) = 1/x$, $g(x) = 1/x$. Then is $(f \circ g)(x) = x$ and $\mathrm{D}_{f \circ g} = \mathbb{R} - \{0\}$.
Do [A P.5 T6,7,9,13,15]

Week 2: Trigonometrics, powers, and vectors

- Trigonometric functions (sin, . . .)
- Powers
- Vectors in 2 and 3 dimensions
- Lines and planes
- Distances, angles and the inner product
- The Cross Product

Sound



Sound, water, electro-magnetic and other waves are a sum of elementary waves: $A \sin(f \cdot 2\pi \cdot t)$

- $\sum_{k=1}^{n} A_k \sin(2\pi \cdot f_k \cdot t)$

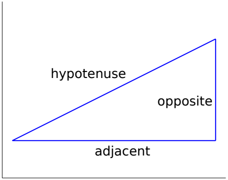
where the frequency f determines the pitch and
the amplitude A the intensity of the sound. sin is the sine function

Trigonometric functions: sin, cos, tan

$\sin = \frac{\text{opp}}{\text{hyp}}$, $\cos = \frac{\text{adj}}{\text{hyp}}$, $\tan = \frac{\text{opp}}{\text{adj}}$

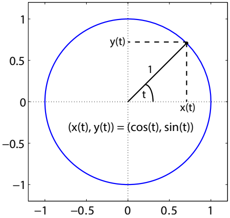
Relation degrees – radians:

$30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, $90^\circ = \frac{\pi}{2}$



Radians = the length of a piece of the circle

Complete circle $= 2\pi = 360^\circ$



Point on a circle with radius 1 has the coordinates $(\cos(t), \sin(t))$

This agrees with $\sin^2(t) + \cos^2(t) = (\sin(t))^2 + (\cos(t))^2 = 1$

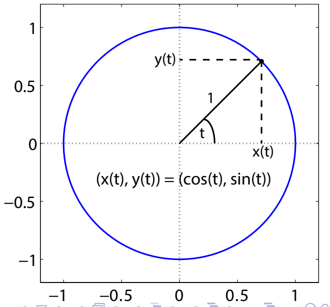
Similar: point on the circle with radius a has the coordinates $(a\cos(t), a\sin(t)) = a(\cos(t), \sin(t))$

The 4 quadrants

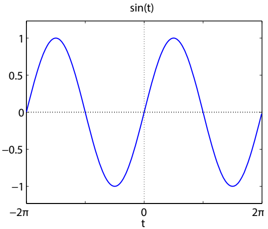
1. $t \in [0, \pi/2] \Longrightarrow (cos(t), sin(t))$ is in the 1st quadrant
2. $t \in [\pi/2, \pi] \Longrightarrow (cos(t), sin(t))$ is in the 2nd quadrant
3. $t \in [\pi, 3/2 \cdot \pi] \Longrightarrow (cos(t), sin(t))$ is in the 3rd quadrant
4. $t \in [3/2 \cdot \pi, 2\pi] \Longrightarrow (cos(t), sin(t))$ is in the 4th quadrant

so that we define

1. the 1st quadrant: $t \in [0, \pi/2]$
2. the 2nd quadrant: $t \in [\pi/2, \pi]$
3. the 3rd quadrant: $t \in [\pi, 3/2 \cdot \pi]$
4. the 4th quadrant: $t \in [3/2 \cdot \pi, 2\pi]$



Sine, Cosine, and Tangens functions



$\sin(t + 2\pi) = \sin(t)$

odd function:

$\sin(-t) = -\sin(t)$

$\sin(t) = 0$

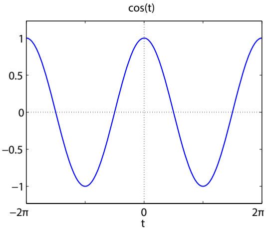
$\implies \quad t = k\pi \qquad (k \in \mathbb{Z})$

$\sin(t) = 1$

$\implies \quad t = \frac{\pi}{2} + 2k\pi$

$\sin(t) = -1$

$\implies \quad t = \frac{3\pi}{2} + 2k\pi$



$\cos(t + 2\pi) = \cos(t)$

even function:

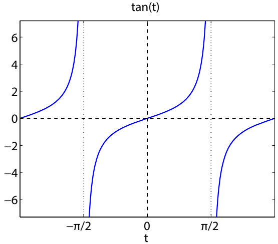
$\cos(-t) = \cos(t)$

$\cos(t) = 0$

$\implies \quad t = \frac{\pi}{2} + k\pi$

$\sin(t) = \cos(\frac{\pi}{2} - t)$

$\cos(t) = \sin(\frac{\pi}{2} - t)$



$\tan(t) = \frac{\sin(t)}{\cos t}$

odd function:

$\tan(-t) = -\tan(t)$

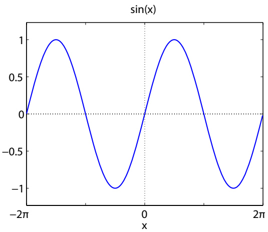
$\tan(t) = 0$

$\implies \quad t = k\pi \qquad (k \in \mathbb{Z})$

vertical asymptote in

$\frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

Sine, Cosine, and Tangens functions



$\sin(x + 2\pi) = \sin(x)$

odd function:

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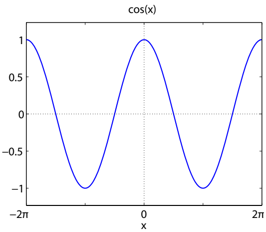


$\cos(x + 2\pi) = \cos(x)$

even function:

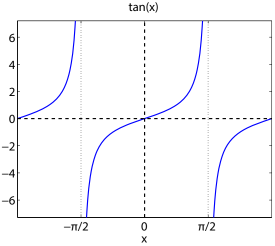
$\cos(-x) = \cos(x)$

$\cos(x) = 0$

$\implies \quad x = \frac{\pi}{2} + k\pi$

$\sin(x) = \cos(\frac{\pi}{2} - x)$

$\cos(x) = \sin(\frac{\pi}{2} - x)$



$\tan(x) = \frac{\sin(x)}{\cos x}$

odd function:

$\tan(-x) = -\tan(x)$

$\tan(x) = 0$

$\implies \quad x = k\pi \qquad (k \in \mathbb{Z})$

vertical asymptote in

$\frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

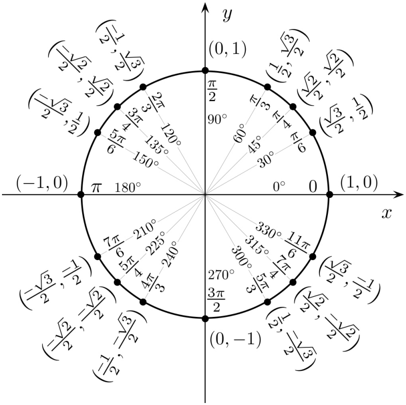
Trigonometric identities

Know at least the red ones by heart!

$$
\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x)
\end{aligned}
$$

Furthermore, we can use the last two lines to rewrite $\sin^2(x)$ and $\cos^2(x)$

$$\boxed{\sin^2(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \qquad \cos^2(x) = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)}$$



Do we need to remember all of those?!?
No, just those in the 1st quandrant:

x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$\sin(x)$	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
$\cos(x)$	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$
$\tan(x)$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	ng

$$
\begin{aligned}
\sin(\pi - x) &= \sin(\pi + (-x)) \\
&\underset{\sin(x+y)=\dots}{=} \sin(\pi)\cos(-x) + \cos(\pi)\sin(-x) \\
&= 0 \cdot \cos(-x) + -1 \cdot \sin(-x) \\
&= \sin(x)
\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [\frac{\pi}{2}, 2\pi]$ (I)

Use the formulas (that you know by heart)

- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

one can relate angles in Q2-4 to one in Q1!

Very practical: use π and 2π!

$$\begin{array}{lll}
\sin(x) = \sin(\pi - x) & \cos(x) = -\cos(\pi - x) & \tan(x) = -\tan(\pi - x) \\
\sin(x) = -\sin(x - \pi) & \cos(x) = -\cos(x - \pi) & \tan(x) = \tan(x - \pi) \\
\sin(x) = -\sin(-x) & \cos(x) = \cos(-x) & \tan(x) = -\tan(-x)
\end{array}$$

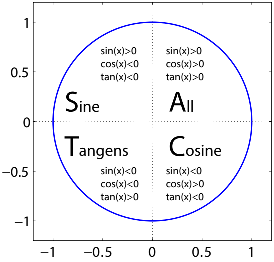
Ex: Calculate $\sin(x)$ for $x = \frac{5}{6}\pi$.
$\frac{5}{6}\pi$ is in the 2nd quadrant and
$\pi - \frac{5}{6}\pi$ in the 1st: $\sin(\frac{5}{6}\pi) = \sin(\pi - \frac{5}{6}\pi) = \sin(\frac{1}{6}\pi) = \frac{1}{2}$.

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [\frac{\pi}{2}, 2\pi]$ (II)

Who can (and wants to) remember all 9 relations?
Instead, remember to change the argument in the trigonometric function according to:

- if x is in 2nd quadrant $[\frac{1}{2}\pi, \pi]$: $x \to \pi - x$
- if x is in 3rd quadrant $[\pi, \frac{3}{2}\pi]$: $x \to x - \pi$
- if x is in 4th quadrant $[\frac{3}{2}\pi, 2\pi]$: $x \to 2\pi - x$

and the CAST rule to check for a changing sign.



Ex: Calculate $\sin(x)$ for $x = \frac{7}{6}\pi$.
$\frac{7}{6}\pi$ is in the 3rd quadrant.
$\frac{7}{6}\pi \to \frac{7}{6}\pi - \pi = \frac{1}{6}\pi$
$sin(x)$ changes sign from Q3 to Q1
so that $\sin(\frac{7}{6}\pi) = -\sin(\frac{7}{6}\pi - \pi) = -\sin(\frac{1}{6}\pi) = -\frac{1}{2}$.

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$\sin(\frac{\pi}{6}) \quad =$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\begin{aligned} \sin\left(\tfrac{\pi}{6}\right) &= \tfrac{1}{2} \\ \cos\left(\tfrac{2}{3}\pi\right) &= \end{aligned}$$

$$\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &=
\end{aligned}$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &&= -\tfrac{1}{2} \qquad \text{qu 2, } \cos(\pi - x) \\
\tan(-\tfrac{\pi}{4}) &=
\end{aligned}$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &&= -\tfrac{1}{2} && \text{qu 2, } \cos(\pi - x) \\
\tan(-\tfrac{\pi}{4}) &= -\tan(\tfrac{\pi}{4}) &&=
\end{aligned}$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\sin(\tfrac{\pi}{6}) = \tfrac{1}{2}$$

$$\cos(\tfrac{2}{3}\pi) = -\cos(\tfrac{1}{3}\pi) = -\tfrac{1}{2} \qquad \text{qu 2, } \cos(\pi - x)$$

$$\tan(-\tfrac{\pi}{4}) = -\tan(\tfrac{\pi}{4}) = -1 \qquad \text{qu 1, tan is odd}$$

$$\sin(\tfrac{4\pi}{3}) =$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$
\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &&= -\tfrac{1}{2} && \color{blue}{\text{qu 2, } \cos(\pi - x)} \\
\tan(-\tfrac{\pi}{4}) &= -\tan(\tfrac{\pi}{4}) &&= -1 && \color{blue}{\text{qu 1, tan is odd}} \\
\sin(\tfrac{4\pi}{3}) &= -\sin(\tfrac{\pi}{3}) &&=
\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$
\begin{array}{llll}
\sin(\frac{\pi}{6}) & = \frac{1}{2} & & \\
\cos(\frac{2}{3}\pi) & = -\cos(\frac{1}{3}\pi) & = -\frac{1}{2} & \text{qu 2, } \cos(\pi - x) \\
\tan(-\frac{\pi}{4}) & = -\tan(\frac{\pi}{4}) & = -1 & \text{qu 1, tan is odd} \\
\sin(\frac{4\pi}{3}) & = -\sin(\frac{\pi}{3}) & = -\frac{1}{2}\sqrt{3} & \text{qu 3, } \sin(x - \pi) \\
\cos(\frac{7}{6}\pi) & = & &
\end{array}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

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\begin{array}{llll}
\sin(\frac{\pi}{6}) & = \frac{1}{2} & & \\
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\cos(\frac{7}{6}\pi) & = -\cos(\frac{1}{6}\pi) & = &
\end{array}
$$

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$$
\begin{aligned}
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\tan(-\tfrac{\pi}{4}) &= -\tan(\tfrac{\pi}{4}) &&= -1 && \text{qu 1, tan is odd} \\
\sin(\tfrac{4\pi}{3}) &= -\sin(\tfrac{\pi}{3}) &&= -\tfrac{1}{2}\sqrt{3} && \text{qu 3, } \sin(x - \pi) \\
\cos(\tfrac{7}{6}\pi) &= -\cos(\tfrac{1}{6}\pi) &&= -\tfrac{1}{2}\sqrt{3} && \text{qu 3, } \cos(x - \pi) \\
\tan(\tfrac{11}{6}\pi) &=
\end{aligned}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$
\begin{array}{llll}
\sin(\frac{\pi}{6}) & = \frac{1}{2} & & \\
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\tan(\frac{11}{6}\pi) & = -\tan(\frac{1}{6}\pi) & = &
\end{array}
$$

Determining $\sin(x)$, $\cos(x)$, $\tan(x)$, $x \in [0, 2\pi]$ (III)

$$\begin{aligned}
\sin(\tfrac{\pi}{6}) &= \tfrac{1}{2} && && \\
\cos(\tfrac{2}{3}\pi) &= -\cos(\tfrac{1}{3}\pi) &&= -\tfrac{1}{2} && \text{qu 2, } \cos(\pi - x) \\
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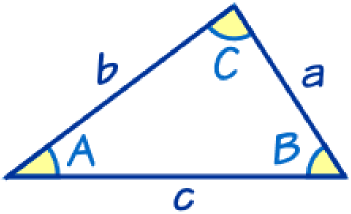
\cos(\tfrac{7}{6}\pi) &= -\cos(\tfrac{1}{6}\pi) &&= -\tfrac{1}{2}\sqrt{3} && \text{qu 3, } \cos(x - \pi) \\
\tan(\tfrac{11}{6}\pi) &= -\tan(\tfrac{1}{6}\pi) &&= -\tfrac{1}{3}\sqrt{3} && \text{qu 4, } \tan(x - 2\pi)
\end{aligned}$$

Do [A P.7 T5]

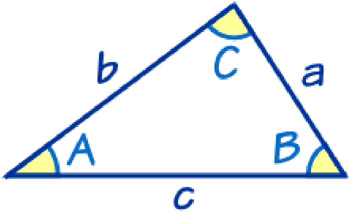
Sine and Cosine Rules



Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Sine and Cosine Rules



Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine rule

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

Do [A P.7 T31,37].

Trigonometric Equations and Inequalities

$\sin(x) = \sin(a)$

Trigonometric Equations and Inequalities

$$\begin{aligned} \sin(x) = \sin(a) \quad &\Longrightarrow \quad x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\ \cos(x) = \cos(a) \end{aligned}$$

Trigonometric Equations and Inequalities

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\sin(x) = \sin(a) \quad &\Longrightarrow \quad x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\
\cos(x) = \cos(a) \quad &\Longrightarrow \quad x = \pm a + 2k\pi \\
\tan(x) = \tan(a) \quad &
\end{aligned}$$

Trigonometric Equations and Inequalities

$$\begin{aligned}
\sin(x) = \sin(a) &\implies x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\
\cos(x) = \cos(a) &\implies x = \pm a + 2k\pi \\
\tan(x) = \tan(a) &\implies x = a + k\pi \\
\sin(x) = \cos(a) &
\end{aligned}$$

$$
\begin{array}{rcl}
\sin(x) = \sin(a) & \Longrightarrow & x = a + 2k\pi \text{ or } x = \pi - a + 2k\pi \\
\cos(x) = \cos(a) & \Longrightarrow & x = \pm a + 2k\pi \\
\tan(x) = \tan(a) & \Longrightarrow & x = a + k\pi \\
\sin(x) = \cos(a) & \Longrightarrow & \text{rewrite cos to sin :} \\
 & & \cos(a) \underset{\sin(x+y)=\ldots}{=} \sin(\frac{\pi}{2} - a)
\end{array}
$$

Recipe: Trigonometric Equations/Inequalities

1. Use substitution to rewrite to an equation with (powers of) only 1 unknown, eg $\sin(x)$
   and with only 1 argument (so no mix of $\sin(x)$ and $\sin(2x)$)
   or products of different functions (avoid roots as much as possible).
2. Maybe use substitution, eg $y = \sin(x)$, and solve the equation / inequality
3. Rewrite y back to eg $\sin(x)$, and solve, eg
   $\sin(x) = \frac{1}{2} \Longrightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$
4. Pay attention to "$+k\,2\pi$" (sin and cos) or "$+k\,\pi$" for tan

Exercises

1. For $\sin(x) = -\frac{5}{13}$ and $x \in [3/2\pi, 2\pi]$, determine $\cos(x)$ and $\tan(x)$.

2. For $\tan(x) = -\frac{7}{24}$, and x in the 4th quadrant, determine $\sin(x)$ and $\cos(x)$.

3. Calculate $\sin(\frac{5\pi}{12})$. Hint: make use of the $\sin(x+y)$ identity.

4. Solve $3\cos^2(x) - \sin^2(x) = 0$ for $x \in (0, 2\pi)$.

5. Solve $\cos^2(x) + \cos(x) = \sin^2(x)$.

Exercises

1. For $\sin(x) = -\frac{5}{13}$ and $x \in [3/2\pi, 2\pi]$, determine $\cos(x)$ and $\tan(x)$.
   $\cos(x) = \frac{12}{13}$ and $\tan(x) = -\frac{5}{12}$
2. For $\tan(x) = -\frac{7}{24}$, and x in the 4th quadrant, determine $\sin(x)$ and $\cos(x)$.
   $\cos(x) = \frac{24}{25}$ and $\sin(x) = -\frac{7}{25}$
3. Calculate $\sin(\frac{5\pi}{12})$. Hint: make use of the $\sin(x+y)$ identity.
   $\frac{1}{4}(\sqrt{6}+\sqrt{2})$
4. Solve $3\cos^2(x) - \sin^2(x) = 0$ for $x \in (0, 2\pi)$.
   $x = \frac{\pi}{3} \vee x = \frac{4\pi}{3} \vee x = \frac{2\pi}{3} \vee x = \frac{5\pi}{3}$
5. Solve $\cos^2(x) + \cos(x) = \sin^2(x)$.
   $x = \pi + 2n\pi \vee x = \frac{\pi}{3} + 2n\pi \vee x = \frac{5\pi}{3} + 2n\pi$ with $n \in \mathbb{Z}$

Powers and roots

Reminder: Rules for calculating powers and roots

$$a^p a^q = a^{p+q} \qquad (ab)^p = a^p b^p \qquad a^{-p} = \frac{1}{a^p}$$

$$a^{1/2} = \sqrt{a} \qquad a^{1/n} = \sqrt[n]{a} \qquad a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

$$(a^p)^q = a^{pq} \qquad a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1}{\sqrt[q]{a^p}} \qquad \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

Useful to rewrite expressions into more simplified forms:

Ex: $$\frac{(3a^{-\frac{2}{3}}b)^2}{2a^2b^{-\frac{1}{3}}} = \frac{3^2(a^{-\frac{2}{3}})^2b^2}{2a^2b^{-\frac{1}{3}}} = \frac{9a^{-\frac{4}{3}}b^2}{2a^2b^{-\frac{1}{3}}} = \frac{9}{2}a^{-\frac{10}{3}}b^{\frac{7}{3}}$$

Equations with powers

Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

Equations with powers

Rewrite to

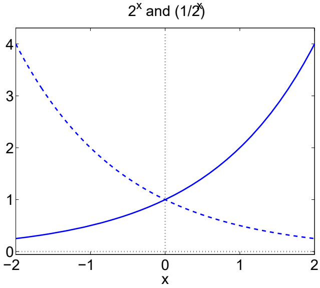
$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

- $f(x) = a^x$ everywhere increasing function if $a > 1$
- $f(x) = a^x$ everywhere decreasing function if $0 < a < 1$

Ex:

$2^{x+1} + 2^{x+3} = 320$



Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

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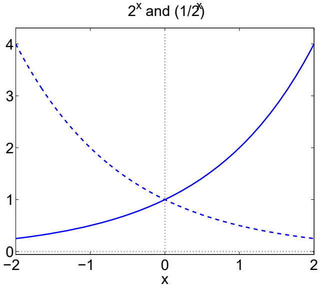

Ex:

$$2^{x+1} + 2^{x+3} = 320 \implies 2^x(2+2^3) = 320 \implies 2^x = 32 = 2^5 \implies x = 5$$

Ex: $16^{3x+3} = 8^{x^2+4}$

Rewrite to

$$a^{\text{expression}_1} = a^{\text{expression}_2}$$

conclude that the exponents must be equal because:

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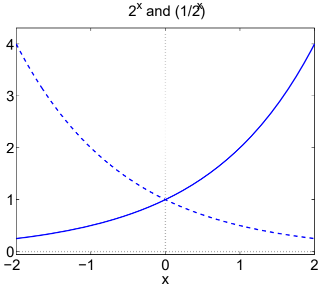


Ex:

$$2^{x+1} + 2^{x+3} = 320 \implies 2^x(2+2^3) = 320 \implies 2^x = 32 = 2^5 \implies x = 5$$

Ex: $16^{3x+3} = 8^{x^2+4} \implies (2^4)^{3x+3} = (2^3)^{x^2+4} \implies 2^{12x+12} = 2^{3x^2+12}$
$\implies 12x + 12 = 3x^2 + 12 \implies 3x(x-4) = 0 \implies x = 0$ or $x = 4$

Inequalities with powers

Again: get the same base number, then compare the powers
Remember:

- $f(x) = a^x$ is increasing everywhere if $a > 1$
- $f(x) = a^x$ is decreasing everywhere if $0 < a < 1$

Ex: the base number: is it < 1 or > 1?

- $f(x) = 2^x$ is increasing everywhere because $2 > 1$
- $f(x) = 2^{-x}$

Inequalities with powers

Again: get the same base number, then compare the powers
Remember:

- $f(x) = a^x$ is increasing everywhere if $a > 1$
- $f(x) = a^x$ is decreasing everywhere if $0 < a < 1$

Ex: the base number: is it < 1 or > 1?

- $f(x) = 2^x$ is increasing everywhere because $2 > 1$
- $f(x) = 2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$ is decreasing everywhere because $0 < 1/2 < 1$

Ex: Inequalities

- $2^x > 2^4 \implies x > 4$ because 2^x is increasing
- $(\frac{1}{2})^x > (\frac{1}{2})^4 \implies x < 4$ because $(\frac{1}{2})^x$ is decreasing

Quiz: Powers

1. Solve $e^{x^2} = 5e^{2x-1}$.

2. Solve $\frac{2^{2x}-8}{2^x-4} \leq 0$

3. $f(x) = 5^x$ is
   (a) increasing everywhere
   (b) decreasing everywhere
   (c) increasing for $x > 0$, decreasing for $x < 0$
   (d) increasing for $x < 0$, decreasing for $x > 0$

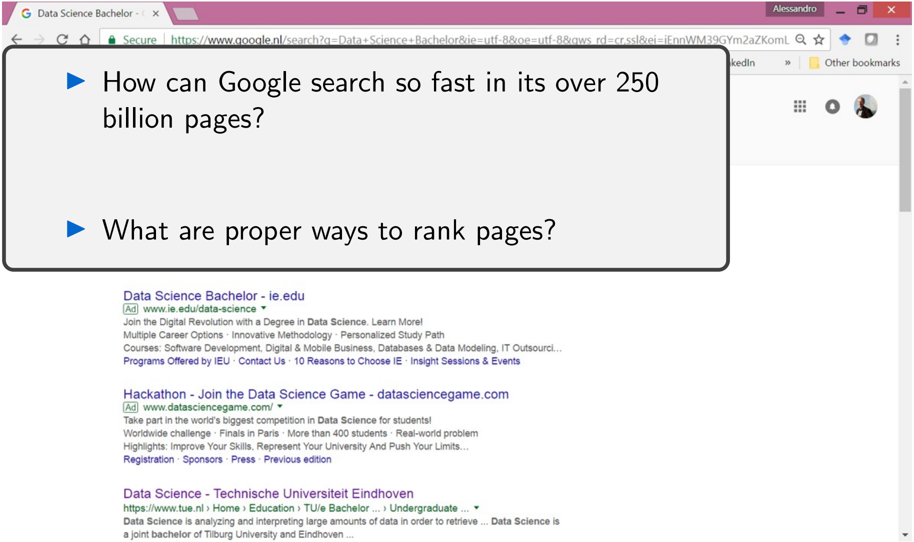
4. If $x < 5$, then
   (a) $5^5 > 5^x$ (b) $5^x \leq 5^5$ (c) $5^5 < 5^x$ (d) $5^x \geq 5^5$

Quiz: Powers

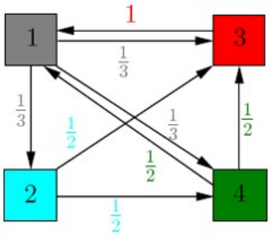
1. Solve $e^{x^2} = 5e^{2x-1}$.
   $x = 1 + \pm\sqrt{\ln(5)}$
2. Solve $\frac{2^{2x}-8}{2^x-4} \le 0$
   $\frac{3}{2} \le x < 2$
3. $f(x) = 5^x$ is
   (a) increasing everywhere
   (b) decreasing everywhere
   (c) increasing for $x > 0$, decreasing for $x < 0$
   (d) increasing for $x < 0$, decreasing for $x > 0$
4. If $x < 5$, then
   (a) $5^5 > 5^x$ (b) $5^x \le 5^5$ (c) $5^5 < 5^x$ (d) $5^x \ge 5^5$

Google Search

- How can Google search so fast in its over 250 billion pages?
- What are proper ways to rank pages?



Mathematical Model for Weblinks



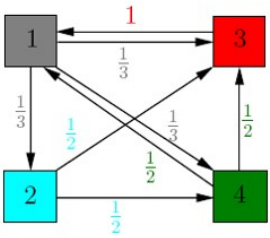
Model the WWW as a **graph**:

- Squares (nodes, vertices) denote web pages
- Arrows (edges) denote links from one page to another

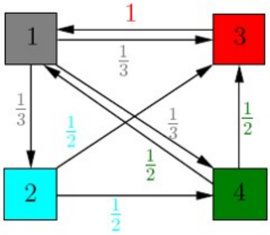
> Relevance is translated into number of incoming links weighted by the importance of referring pages.

(What comes first?)

System of Linear Equations



$$
\begin{aligned}
x_1 &= 1 \cdot x_3 + \frac{1}{2} \cdot x_4 \\
x_2 &= \frac{1}{3} \cdot x_1 \\
x_3 &= \frac{1}{3} \cdot x_1 + \frac{1}{2} \cdot x_2 + \frac{1}{2} \cdot x_4 \\
x_4 &= \frac{1}{3} \cdot x_1 + \frac{1}{2} \cdot x_2
\end{aligned}
$$



$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} = A$$

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{31} \cdot \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

Information about page ranks is determined as a **vector** of page relevance (solutions of linear systems).

More Matrix-Vector problems

Alice	4			4
Bob		5	4	
Joe		5	4	
Sam	5			5

- User preferences/histories are encoded as vector data.
- How similar is your vector to those of others?
- completely similar = “parallel”
- completely different = “perpendicular”
- Trigonometric functions (just done) used with vector calculus (coming up)

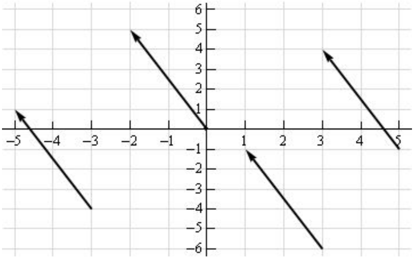
Introduction: Vectors in Analytic Geometry

We will introduce vectors and their calculus in a specific application: Analytic Geometry

- describe directions in two or three space dimensions
- addition and scaling of vectors
- linear combinations
- lines and line segments in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Planes in $\mathbb{R}^3$
- Orthogonality, angle between vectors
- Intersections of lines, lines and planes, planes

A vector is represented in plane or space by an arrow with direction and length (magnitude).

Vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (2 to the left, 5 up)



These vectors are equivalent, even though they have different starting points.

Vectors in 2 and 3 dimensions: Notation

In a plane (2D): 2 to the left and 5 up is a vector:

- Notation: Write $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ or $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (lectures: latter notation)
- Sometimes "also": $<-2,5>$: eg in books to save space...
- $\mathbf{v}$, $\underline{v}$, $\vec{v}$, or simply v (on the slides $\mathbf{v}$, on paper, I write v of $\underline{v}$)
- Sketch: Arrow (a combination of a line segment and directions)

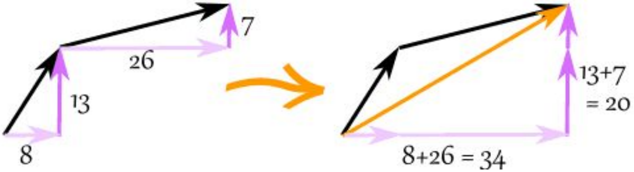
A vector has a direction and length (here $\sqrt{(-2)^2+5^2}$).

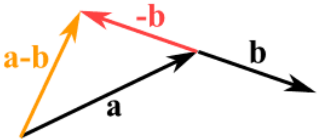
Vectors: Addition and subtraction

- add or subtract component-wise:

$$\begin{bmatrix} 8 \\ 13 \end{bmatrix} + \begin{bmatrix} 26 \\ 7 \end{bmatrix} = \begin{bmatrix} 34 \\ 20 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

- graphically: parallellogram





Vectors from point A to point B

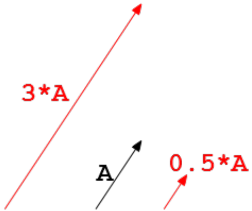
Because $A = (x_1, y_1)$ and $B = (x_2, y_2)$ then the vector from A to B is:

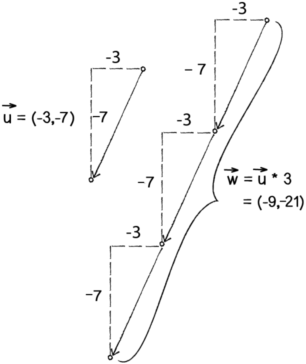
$$B - A = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Notation: AB or $\vec{AB}$.
Do [V 5 T1]

Scaling of vectors

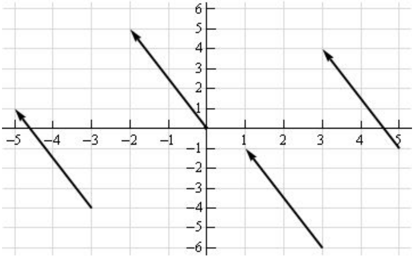
Scaling (scalar multiplication) changes the length and evt the direction:





- scalar means using (real) number
- multiply component-wise: $2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Scaling of vectors



- multiplication by 2:
  vector becomes 2 times as long, keeps the same direction
- multiplication by -2:
  vector becomes 2 times as long, in opposite direction

(Later for more about length)

Subtraction of vectors

Convention: Subtraction is addition of the vector with opposite direction:

$$\mathbf{b} - \mathbf{a} = \mathbf{b} + ((-1) \cdot \mathbf{a})$$

Ex: $\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Length of a vector

Notation:

- $\|\mathbf{x}\|$ (chosen)
- or sometimes also $|\mathbf{x}|$

Calculate: if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then

$$\boxed{\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} \quad \text{thus} \quad \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\boxed{\text{Distance between two points } \mathbf{x} \text{ and } \mathbf{y}: \quad \|\mathbf{x} - \mathbf{y}\|}$$

Ex: Distance between $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \sqrt{5}$

is also the length of the difference vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\left(\text{and } \begin{bmatrix} -1 \\ -2 \end{bmatrix}\right)$

Length of a vector

Ex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length

Length of a vector

Ex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $\sqrt{2}$

Sometimes we seek a vector in the same direction, but with length 1

Ex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sqrt{2} \cdot \underbrace{\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}}_{\text{length } 1}$$

Angle between vectors

Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$

Inner product in 2D: $\boxed{\mathbf{a} \cdot \mathbf{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2}$

Inner product nD (so also for $n = 3$):

$$\boxed{\mathbf{a} \cdot \mathbf{b} = (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \cdots + a_n b_n}$$

Useful for:

- length: $\boxed{\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2}}$
- angle: $\boxed{\cos(\mathbf{a}, \mathbf{b}) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}} \Leftarrow \boxed{\underbrace{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\mathbf{a}, \mathbf{b})}_{\text{inner product rule}}}$

Thus also $\boxed{\mathbf{a} \cdot \mathbf{b} = \cos(\alpha) \, \|\mathbf{a}\| \, \|\mathbf{b}\|, \quad \alpha = \angle(\mathbf{a}, \mathbf{b})}$

Different notation for inner product: $\boxed{\langle \mathbf{a}, \mathbf{b} \rangle}$ and $\boxed{\mathbf{a}^T \mathbf{b}}$

Angle between vectors

Ex: Calculate the angle θ between $\underbrace{\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}}_{\mathbf{v}}$ and $\underbrace{\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}}_{\mathbf{w}}$: Inner product rule:

$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} = \left| \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right| \cdot \left| \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \right| \cdot \cos\theta$$

or (length of both vectors is $\sqrt{(\pm 1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$):

$$\underbrace{1 \cdot -1 + \sqrt{3} \cdot \sqrt{3}}_{\mathbf{v} \cdot \mathbf{w}} = \underbrace{2}_{|\mathbf{v}|} \cdot \underbrace{2}_{|\mathbf{w}|} \cdot \cos\theta$$

and $2 = 4 \cdot \cos\theta \implies \cos\theta = \frac{1}{2} \implies \theta = \pm\pi/3$ (solution with minus sign goes away)
(the angle between two vectors in maximal π!)

Distances, angles and the inner product

- $\mathbf{x} \cdot \mathbf{x} \geq 0$

  logical, is same as $\|\mathbf{x}\|^2$, thus ≥ 0
- $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$

  only the $\mathbf{0}$-vector has length 0
- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

  just as for the multiplication of numbers, the order does not matter
- $(\lambda \mathbf{x}) \cdot \mathbf{y} = \lambda\,(\mathbf{x} \cdot \mathbf{y})$

  if $\mathbf{x}$ eg 2 times as long, then the inner product is 2 times as big

  logical, since $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos(\alpha)$

Orthogonal vectors

Vectors $\mathbf{a}$ and $\mathbf{b}$ are called $\boxed{\text{orthogonal (or perpendicular) if } \mathbf{a} \cdot \mathbf{b} = 0}$

Notation: $\mathbf{a} \perp \mathbf{b}$

Works with: $\cos(\mathbf{a}, \mathbf{b}) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}$

because then the numerator is 0, so $\cos(\mathbf{a}, \mathbf{b}) = 0$, thus $\angle(\mathbf{a}, \mathbf{b}) = \frac{\pi}{2}$

Ex: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are indeed orthogonal

but also $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Orthogonal vectors

In $\mathbb{R}^2$ the vectors that are orthogonal to a given vector can be calculated

Ex: Which vectors are $\perp (1,2)$?
We need to have: $(a_1, a_2) \cdot (1,2) = a_1 + 2a_2 = 0$

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switch the components, and give one an extra minus sign, thus:

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- $(2, -1)$, because $(2, -1) \cdot (1, 2) = 0$
- $(-2, 1)$, because $(-2, 1) \cdot (1, 2) = 0$
- and also all multiples of $(-2, 1)$ and $(2, -1)$:
  $(-2\lambda, \lambda) = \lambda(-2, 1)$ for $\lambda \in \mathbb{R}$

(In $\mathbb{R}^2$ a line of vectors is perpendicular on a given vector!)

Quiz: Vector calculus

1. Given vectors $\mathbf{v} =< -2, 3 >$ and $\mathbf{u} =< 4, 6 >$, find
   $\mathbf{v} + 2\mathbf{u}$ and $\mathbf{u} - 4\mathbf{u}$

2. Given vectors $\mathbf{v} =< 1, -2 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $\mathbf{v} + 3\mathbf{u} = 0$

3. Given vectors $\mathbf{v} =< 4, 1 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $2\mathbf{v} - 3\mathbf{u} = 0$

4. Calculate the distance between the points $A(3, 1, 7)$ and $B(5, -1, 3)$

5. Calculate the angle between the vectors $\mathbf{v} =< 1, 1, 1 >$ and $\mathbf{w} =< 4, 0, -3 >$

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   $\mathbf{v} + 2\mathbf{u} =< 6, 15 >$ and $\mathbf{u} - 4\mathbf{u} =< -12, -18 >$
2. Given vectors $\mathbf{v} =< 1, -2 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $\mathbf{v} + 3\mathbf{u} = 0$
   $\mathbf{u} =< -\frac{1}{3}, \frac{2}{3}$
3. Given vectors $\mathbf{v} =< 4, 1 >$ and $\mathbf{u} =< u_1, u_2 >$, find components of vector $\mathbf{u}$ so that $2\mathbf{v} - 3\mathbf{u} = 0$
   $\mathbf{u} =< \frac{8}{3}, \frac{2}{3} >$
4. Calculate the distance between the points $A(3, 1, 7)$ and $B(5, -1, 3)$
   $d = 2\sqrt{5}$
5. Calculate the angle between the vectors $\mathbf{v} =< 1, 1, 1 >$ and $\mathbf{w} =< 4, 0, -3 >$
   $\cos(\phi) = \frac{1}{5\sqrt{2}}$

Linear combination of vectors

In linear algebra, we say for vectors $\mathbf{v}_1, \dots \mathbf{v}_n$ and real numbers $a_1, \dots, a_n$ that the sum (-vector) $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$
is a linear combination.

Let's get to work with that.

Linear functions and combinations

A function is called linear if:

- in 1 variable: $f(x) = ax$ (or sometimes also with "$+b$")
- in 2 variables: $f(x, y) = ax + by$
- in n variables: $f(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$
- in n vectors: $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$

Thus linear is always:

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Thus linear is always: sum of constants times variables

What is not-linear?

Linear functions and combinations

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- in n vectors: $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n$

Thus linear is always: sum of constants times variables
What is not-linear? Everything else, eq:

- everything containing multiplications of variables, eg x^2, xy, etc
- sin, cos, tan, e^x, $\ln(x)$, etc etc

Read examples [V 1.11]

Lines: An alternative to $ax + by = c$ in $\mathbb{R}^2$ and also $\mathbb{R}^3$

We have seen: Every line in $\mathbb{R}^2$ is described by:

All (x, y) for which: $ax + by = c$
Different a, b, c represent possibly different lines.
But it must not: $2x - 3y = 1$ is the same line as $3y - 2x = -1$
(multiply left/right with -1).

In $\mathbb{R}^3$

$$ax + by + cz = d$$

does not represent a line.

Lines: An alternative to $ax + by = c$ in $\mathbb{R}^2$ and also $\mathbb{R}^3$

Ex: What does $ax + by + cz = d$ describe in $\mathbb{R}^3$? Let's take an example:

$$0 \cdot x + 0 \cdot y + 1 \cdot z = 0$$

($a = b = d = 0$ and $c = 1$) – or $z = 0$. This is the xy-plane!: Table:

x	y	z	$0 \cdot x + 0 \cdot y + z$
0	0	0	$0 + 0 + 0 = 0$
1	0	0	$0 \cdot 1 + 0 + 0 = 0$
0	1	0	$0 + 0 \cdot 1 + 0 = 0$
1	1	0	$0 \cdot 1 + 0 \cdot 1 + 0 = 0$
$\vdots$	$\vdots$	0	$0 = 0$

The 4 points $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$ are the corners of a square in the xy-plane. It cannot be a straight line!
We need: **An expression for a line that works in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$ ...**

Parameter representation of a Line

Let $\mathbf{a} = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} x_B \\ y_B \end{bmatrix}$ be two coordinate vectors of the points A and B. (the same for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$). It holds that

$$\mathbf{x} = \mathbf{x}(\lambda) = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) \text{ for all } \lambda \in \mathbb{R}$$

describes all points on the line through A and B. Specifically

- $\lambda = 0$: $\mathbf{x} = \mathbf{a} + 0 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{a}$
- $\lambda = 1$: $\mathbf{x} = \mathbf{a} + 1 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{b}$

The representation $\mathbf{x} = \mathbf{x}(\lambda) = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$ with the parameter λ is called parameter representation or vector representation. Definitions:

- $\mathbf{a}$ is the support vector or the start point
- $\mathbf{b} - \mathbf{a}$ is the direction vector

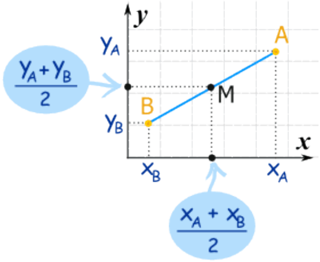
Parameter representation of a Line, now in $\mathbb{R}^3$

Ex: Parameter representation of a line through points $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$:

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2-1 \\ 3-1 \\ 1-1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\text{direction vector}} \\
&= \begin{bmatrix} 1+\lambda \\ 1+2\lambda \\ 1+0\cdot\lambda \end{bmatrix} = \begin{bmatrix} 1+\lambda \\ 1+2\lambda \\ 1 \end{bmatrix}
\end{aligned}$$

Lines, line segments, straight lines

For all $\lambda \in \mathbb{R}$ is $\mathbf{x} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$ on the line through A en B:



- $\lambda = 0$: $\mathbf{x} = \mathbf{a} + 0 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{a}$ (point A)
- $\lambda = 1$: $\mathbf{x} = \mathbf{a} + 1 \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{b}$ (point B)
- $\lambda = \frac{1}{2}$: $\mathbf{x} = \mathbf{a} + \frac{1}{2} \cdot (\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ middle between $\mathbf{a}$ en $\mathbf{b}$
- $\lambda = \frac{1}{4}$: $\mathbf{x} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$ middle between M and A.

Complete lines, half lines and line segments

Parameter representation $\boxed{\mathbf{x} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})}$

yields for each $\lambda \in \mathbb{R}$

one point $\mathbf{x}$ on the line through points B and A

Depending on what λ can be, this is a line segment, line etc:

- $\lambda \in [0,1]$: line segment between A and B including these points
- $\lambda \geq 0$: half line from A through B and further
- $\lambda \in \mathbb{R}$: (complete) line through points A and B

Note that $\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$ is a linear combination because $\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ is of the form: scalar times vector plus scalar times vector.

Ex: A line has several parameter representations:

Parameter representations are not unique! $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} \qquad \text{just take } \mu = -\tfrac{1}{2}\lambda$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} \qquad \text{just take } \mu = -\tfrac{1}{2}\lambda - \tfrac{1}{2}$$

If your neighbor gets a different answer, it does not mean anything! ...
But: All direction vectors are “the same” except for a minus sign.

Lines in $\mathbb{R}^2$: Function vs. parameter representation

Change from function to parameter representation:

Ex: $y = 2x + 3$ goes through $(0,3)$ and $(1,5)$, thus

parameter representation $\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \lambda \\ 3 + 2\lambda \end{bmatrix}$

Ex: Let a parameter representation be $\begin{bmatrix} 1 + 2\lambda \\ 2 + 3\lambda \end{bmatrix}$

then the line goes through $(1,2)$ (for $\lambda = 0$) and $(3,5)$ (for $\lambda = 1$)

thus it's the equation $y = \frac{3}{2}x + \frac{1}{2}$

Lines in $\mathbb{R}^3$

Can be given as in $\mathbb{R}^2$ by parametrisation:
starting point and direction vector

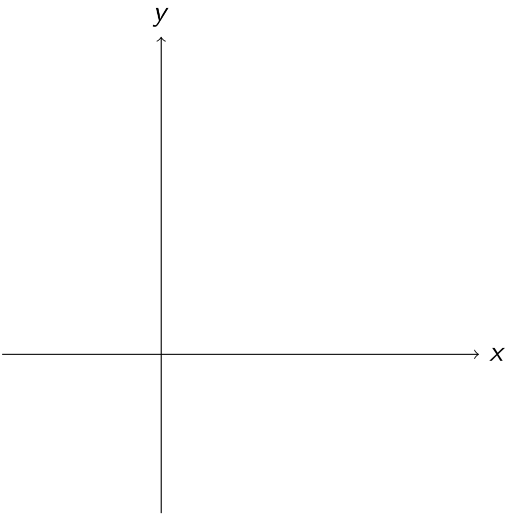
$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

A line can also be given by the intersection of two planes

Lines

- Determine a parametric representation of the lines through:
  - $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$
  - $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- Give an equation for the line:
  $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
- Find a parametric representation for the following lines in $\mathbb{R}^2$
  $3x - 4y + 7 = 0$

Vectors in $\mathbb{R}^2$



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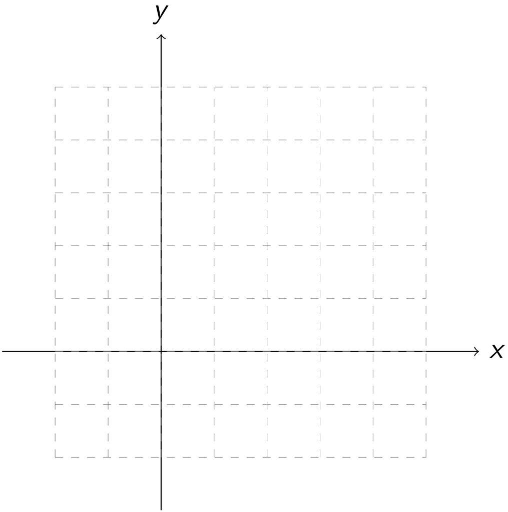

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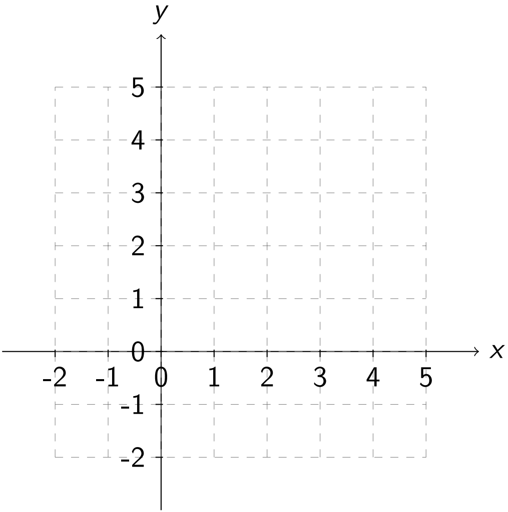

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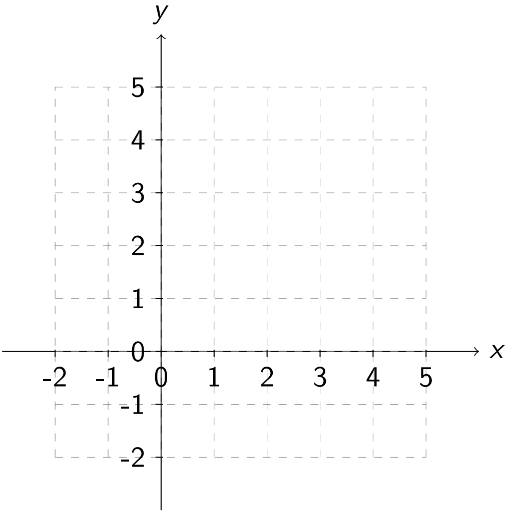

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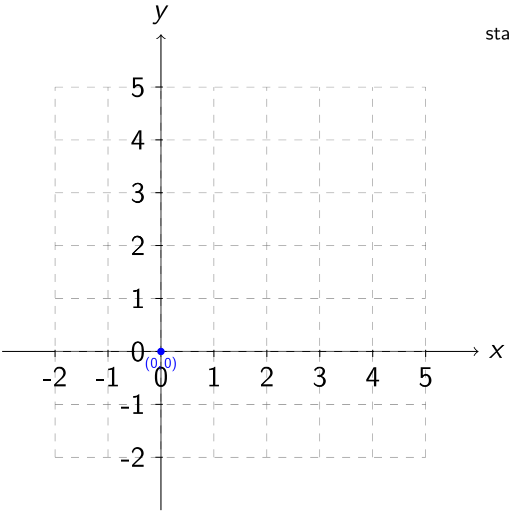
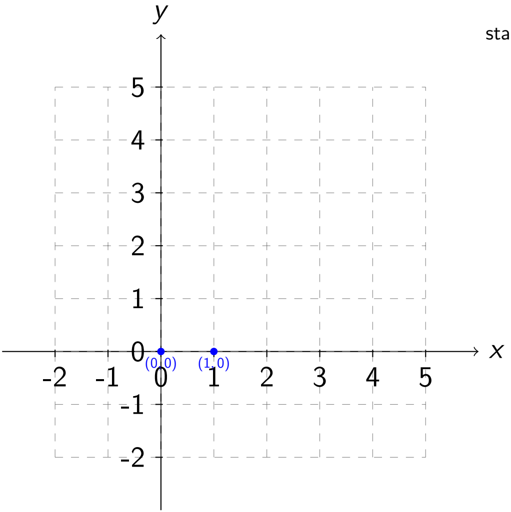


start point $\mathbf{p} = (0, 0)$



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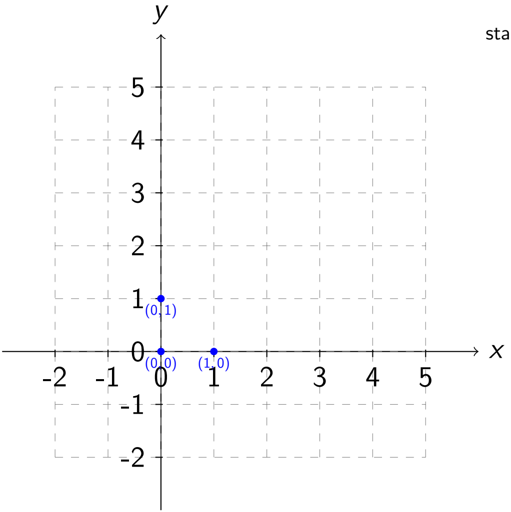

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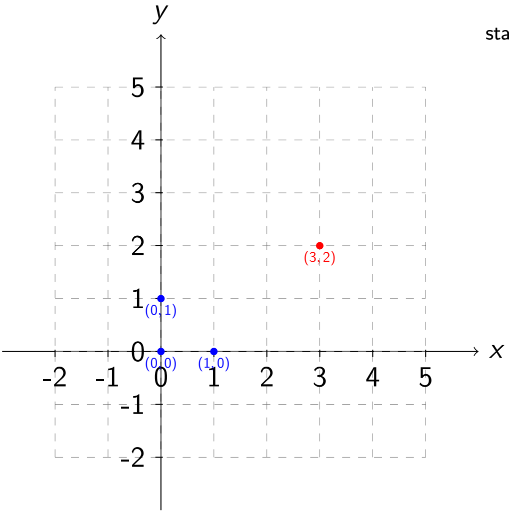
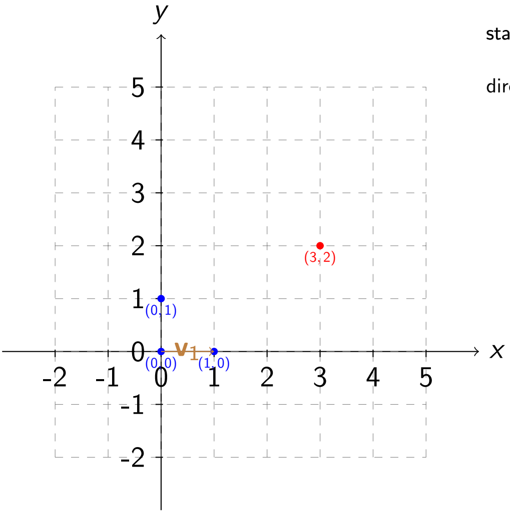


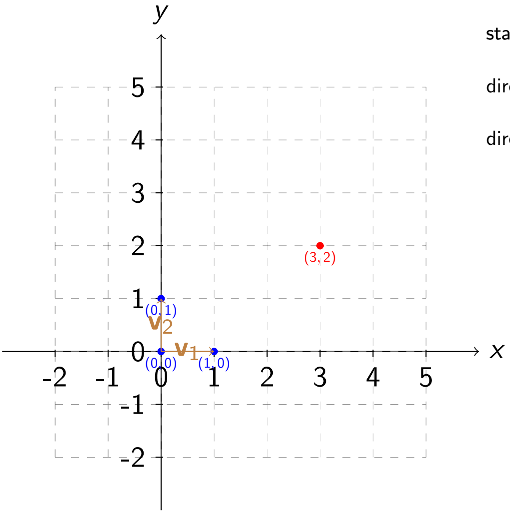
start point $\mathbf{p} = (0, 0)$, to reach $(3, 2)$

Vectors in $\mathbb{R}^2$



start point $\mathbf{p} = (0,0)$, to reach $(3,2)$

direction $\mathbf{v}_1 = (1,0) - (0,0) = (1,0)$



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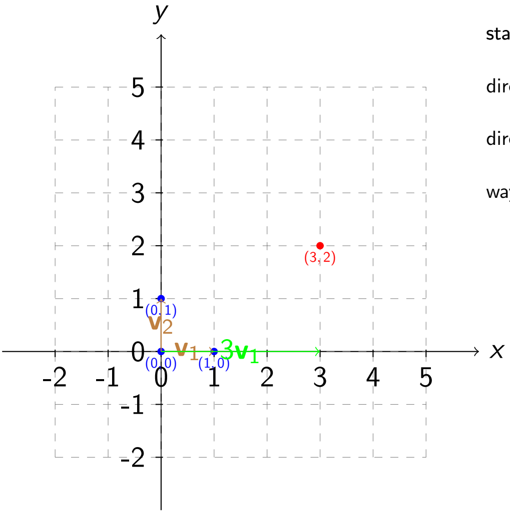
direction $\mathbf{v}_1 = (1, 0) - (0, 0) = (1, 0)$

direction $\mathbf{v}_2 = (0, 1) - (0, 0) = (0, 1)$

Vectors in $\mathbb{R}^2$



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direction $\mathbf{v}_1 = (1, 0) - (0, 0) = (1, 0)$

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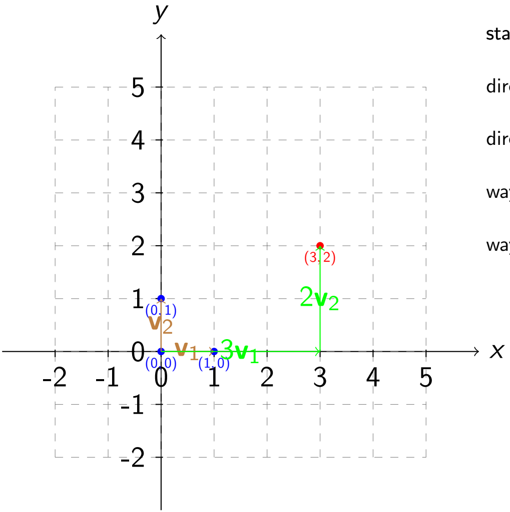
way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

Vectors in $\mathbb{R}^2$



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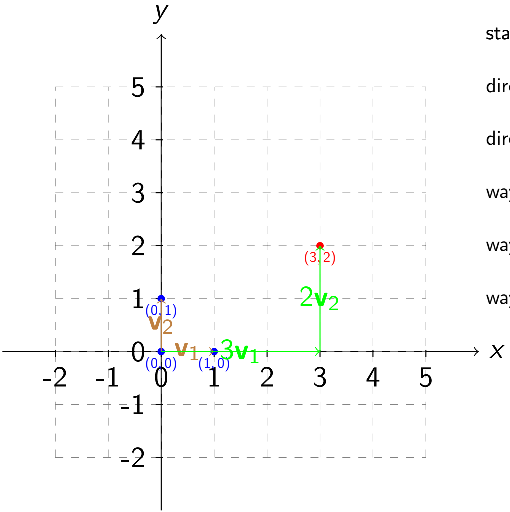
Vectors in $\mathbb{R}^2$



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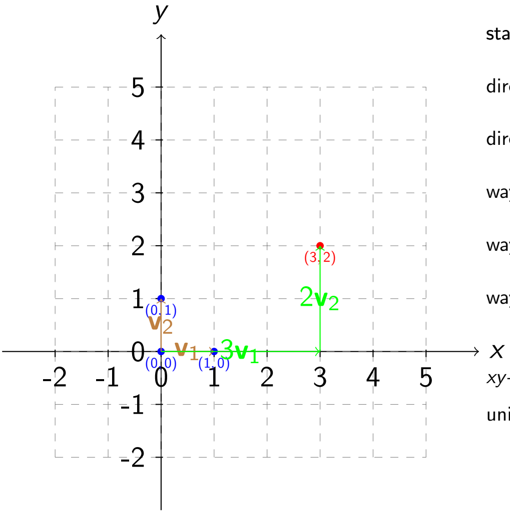
way to $\begin{pmatrix}3\\2\end{pmatrix} = \mathbf{p} + 3\mathbf{v}_1 + 2\mathbf{v}_2$



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direction $\mathbf{v}_2 = (0, 1) - (0, 0) = (0, 1)$

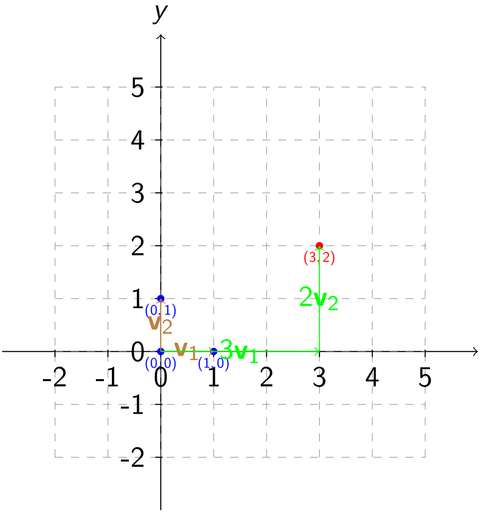
way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{p} + 3\mathbf{v}_1 + 2\mathbf{v}_2$

xy-plane: Each point (x, y) determined by unique λ, μ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Vectors in $\mathbb{R}^2$



start point $\mathbf{p} = (0, 0)$, to reach $(3, 2)$

direction $\mathbf{v}_1 = (1, 0) - (0, 0) = (1, 0)$

direction $\mathbf{v}_2 = (0, 1) - (0, 0) = (0, 1)$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dots$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

way to $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \mathbf{p} + 3\mathbf{v}_1 + 2\mathbf{v}_2$

xy-plane: Each point (x, y) determined by unique λ, μ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

xy-plane: $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{p} + \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$

Why the parameter representation?

The 2D notation $y = ax + b$ does not work for equations in the 3D space!

Ex: What object is $x - y + z = 1$ in $\mathbb{R}^3$?
Answer: That depends on the set of points (x, y, z) that satisfies the equation.
Let's try to find a description for that:

- 1 equation with 3 unknowns. Chose 2 arbitrarily:
- $z = \lambda$ and $y = \mu$ and calculate x:
- $x = 1 - \lambda + \mu$

So for all $\lambda, \mu \in \mathbb{R}$ the resulting $\mathbf{x}$ satisfies $x - y + z = 1$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - \lambda + \mu \\ \mu \\ \lambda \end{bmatrix}$$

Why the parameter representation?

Different notation:

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1-\lambda+\mu \\ \mu \\ \lambda \end{bmatrix} \\
&= \begin{bmatrix} 1+(-1)\cdot\lambda+1\cdot\mu \\ 0+0\cdot\lambda+1\cdot\mu \\ 0+1\cdot\lambda+0\cdot\mu \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

represents all linear combinations of 2 directions from 1 support point.
Such an object is always a plane!
Summarized:

- in 2D: $ax + by = c$ is the equation of a line
- in 3D: $ax + by = c \iff ax + by + 0 \cdot z = c$ is the equation of a plane!

Planes in $\mathbb{R}^3$

Planes can be given by a parametrisation:
start point and 2 direction vectors:

$$\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) + \mu\,(\mathbf{c} - \mathbf{a}) \text{ for all } \lambda, \mu \in \mathbb{R}$$

λ and μ are the parameters

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} + \mu \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

Ex: the xy-plane: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The parameter representation of the xy-plane

Addition yields $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix}$ that the points $(\lambda, \mu, 0)$ are described for $\lambda, \mu \in \mathbb{R}$.

Ex: The equation $z = 0$ describes the same xy-plane:

- 1 equation with 3 unknowns: $0 \cdot x + 0 \cdot y + z = 0$. Choose 2 arbitrarily:
- $x = \lambda$ and $y = \mu$ and calculate z:
- $z = 0$.

So that for all $\lambda, \mu \in \mathbb{R}$ the resulting $\mathbf{x}$ satisfy $z = 0$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The parameter representation is not unique

Take V the xy-plane:

$$V: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}} + \mu \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{w}}$$

All points $P = (a, b, 0)$ are in V:

$$V: \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We obtain the point P by choosing $\lambda = a$ and $\mu = b$

The parameter representation is not unique

Take V the xy-plane:

$$V: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{r}} + \mu \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{s}}$$

All points $P = (a, b, 0)$ are in V:

$$V: \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}a \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + (b + \frac{1}{2}a) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We obtain the point P by choosing $\lambda = \frac{1}{2}a$ and $\mu = b + \frac{1}{2}a$
A plane can be described by many different directions
Here $\mathbf{r} = 2\mathbf{v} - \mathbf{w}$ and $\mathbf{s} = \mathbf{w}$

Orthogonal vectors

In $\mathbb{R}^3$

Orthogonal vectors

In $\mathbb{R}^3$ a plane of vectors is orthogonal to a given vector.

Ex: Which vectors $\mathbf{v} \in \mathbb{R}^3$ are perpendicular on $\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$? For such a vector

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \text{ it must hold: } \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} = 0 \iff -3v_x + 4v_y + v_z = 0$$

- 1 equation with 3 unknown. Choose 2 arbitrarily:
- $v_x = \lambda$ and $v_y = \mu$ and calculate v_z:
- $v_z = 3\lambda - 4\mu$

which is a plane through the origin:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ 3\lambda - 4\mu \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$$

Orthogonal vectors

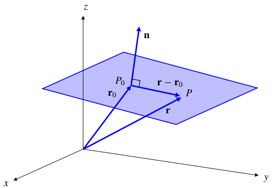
Ex: For which numbers $a \in \mathbb{R}$ is $\begin{bmatrix} a \\ 4 \\ 1 \end{bmatrix}$ orthogonal to $\begin{bmatrix} a \\ a \\ 3 \end{bmatrix}$?

We must have:

$$\begin{aligned} (a, 4, 1) \cdot (a, a, 3) &= 0 \\ \iff a^2 + 4a + 3 &= 0 \\ \iff (a+1)(a+3) &= 0 \\ \iff a = -1 \vee a = -3 & \end{aligned}$$

Planes and normals (see also [A 10.4])

Let $P_0 = (x_0, y_0, z_0)$ be an arbitrary point in the plane
and $\mathbf{n} = (n_1, n_2, n_3)$ a normal(vector) on this plane (length does not matter):



normal vector = vector that is orthogonal to the plane

The plane consists of all points $P = (x, y, z)$ for which it holds:
$(x - x_0,\ y - y_0,\ z - z_0)$ is orthogonal to $\mathbf{n}$
Thus for (x, y, z) in the plane:
$0 = \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0)$
So $n_1 x + n_2 y + n_3 z = \underbrace{n_1 x_0 + n_2 y_0 + n_3 z_0}_{d}$

Normal vector form of a plane: $\mathbf{n} \cdot \mathbf{x} = d$

Planes and normals

Earlier, we saw that the plane V with equation $x - y + z = 1$ has the vector representation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and normal } \mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The normal **n** must be orthogonal to both directions V!

Check:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot -1 + -1 \cdot 0 + 1 \cdot 1 = 0 \text{ works}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot 1 + -1 \cdot 1 + 1 \cdot 0 = 0 \text{ works too}$$

Planes and normals

The plane V through point $P = (1, 2, 3)$ with normal $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$?

has the equation $n_1 x + n_2 y + n_3 z = d$

Thus $-x + 0 \cdot y + z = d \Longrightarrow -x + z = d$

Point P lies in V: Filling in yields $-1 + 3 = 2 = d$.

Planes and normals

The plane V through point $P = (1, 2, 3)$ with normal $\mathbf{n} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$?

has the equation $-x + z = 2$ (previous slide)

But also the equation $-2x + 2z = 4$ with normal $\mathbf{n} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

and equation $x - z = -2$ with normal $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

The normals: All parallel!

Equations for a plane and normal on plane

$$n_1 x + n_2 y + n_3 z = \underbrace{n_1 x_0 + n_2 y_0 + n_3 z_0}_{d}$$

Formula for d unimportant, but:

- From the equation of the plane $n_1 x + n_2 y + n_3 z = d$ you can see:
  normal $= (n_1, n_2, n_3)$
- And vice versa: from the normal $= (n_1, n_2, n_3)$ you know the equation of the plane: $n_1 x + n_2 y + n_3 z = d$ where you still need to determine d via

Equations for a plane and normal on plane

$$n_1x + n_2y + n_3z = \underbrace{n_1x_0 + n_2y_0 + n_3z_0}_{d}$$

Formula for d unimportant, but:

- From the equation of the plane $n_1x + n_2y + n_3z = d$ you can see: normal $= (n_1, n_2, n_3)$
- And vice versa: from the normal $= (n_1, n_2, n_3)$ you know the equation of the plane: $n_1x + n_2y + n_3z = d$ where you still need to determine d via filling in 1 point P in the plane

Lines in $\mathbb{R}^3$

Reminder: 2 options: given by

- parameter representation
- or 2 equations: intersection of 2 planes

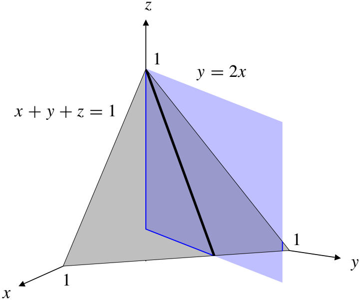
Parameter representation: $(x, y, z) = (x_1, y_1, z_1) + t\,(a, b, c), \quad t \in \mathbb{R}$
support point $+\ t \cdot$ direction vector

Ex: $\begin{aligned} x &= 2+t \\ y &= 3 \\ z &= -4t \end{aligned}$ is rewritten as $(x, y, z) = (2, 3, 0) + t\,(1, 0, -4)$

Ex: Line through $(1, -2, 3)$ orthogonal to plane $x - 2y + 4z = 5$ has parameter representation
$(x, y, z) = (1, -2, 3) + t\,(1, -2, 4) = (1 + t,\ -2 - 2t,\ 3 + 4t)$

Lines in $\mathbb{R}^3$

Also possible by 2 equations: intersection of 2 planes



One can make a parameter representation out of this by finding 2 points, eg:

choose $x = 0$ then $y = 0$ and $z = 1$

choose $x = 1$ then $y = 2$ and $z = -2$

A parameter representation is then

$(0, 0, 1) + t\,((1, 2, -2) - (0, 0, 1)) = (0, 0, 1) + t\,(1, 2, -3)$

Planes

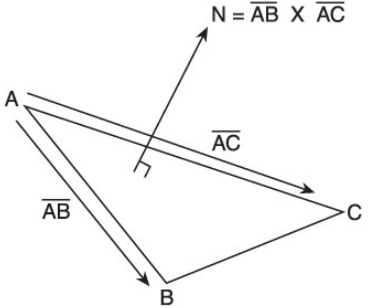
- Give a parameter representation of the line through $(5, -2, -5)$ parallel to a vector $(7, 7, 10)$
- The same for the line through $(-8, -7, 2)$ orthogonal to the plane $3x + 3y + 3z = 8$
- Give the equation for the plane through $(3, 5, 5)$ and normal to the vector $(8, 6, 9)$
- Where does the line $x = -8 + 3t, \;\; y = 4 + 7t, \;\; z = -9 - 8t$ intersect the plane $9x - 6y + 3z = -11$

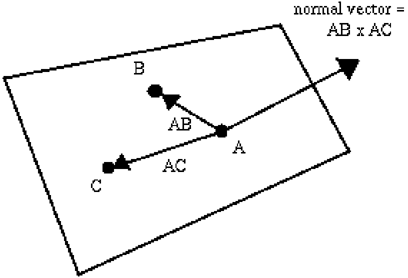
Line in $\mathbb{R}^2$ and plane in $\mathbb{R}^3$

Note the similarities and differences:
Line in $\mathbb{R}^2$: $ax + by = c$
(a, b, c fixed, x, y variable)
Plane in $\mathbb{R}^3$: $ax + by + cz = d$
(a, b, c, d fixed, x, y, z variable)

The cross product vector

Cross product vector: A vector orthogonal to two different vectors.
Three points A, B and C define a start point (eg A) and two directions from A ($\vec{AB} = \mathbf{b} - \mathbf{a}$ and $\vec{AC} = \mathbf{c} - \mathbf{a}$) and with that a plane with vector representation: $\mathbf{x} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$.
The normal N (vector $\mathbf{n}$) is orthogonal to both vectors.





and can be calculated with the so called cross product.

Definition of the cross product

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

Ex: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Ex: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Ex: $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ Ex: $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

What is noteworthy about these examples for $\mathbf{v} \times \mathbf{w}$?

Definition of the cross product

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

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What is noteworthy about these examples for $\mathbf{v} \times \mathbf{w}$?

- Cross product is orthogonal to the 2 vectors **v** and **w**
- Length is the same as the surface of a parallellogram formed by **v** and **w**

This always turns out to be true.

Cross product and normal vector

The cross product is useful for calculating the normal vector on the plane

Ex: Plane with parametrisation $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

this is the

Cross product and normal vector

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Ex: Plane with parametrisation $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

this is the (x, y)-plane

Normal is

Cross product and normal vector

The cross product is useful for calculating the normal vector on the plane

Ex: Plane with parametrisation $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

this is the (x, y)-plane

Normal is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Cross product: properties

- $\mathbf{x} \times \mathbf{x} = \mathbf{0}$

  cross product of a vector with itself is the **0**-vector

  logical, because then the parallellogram has the surface 0
- $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{x} = 0$

  cross product of one vector with another is perpendicular to the other

  this is 1 of the 2 basic properties of the cross product
- $\mathbf{y} \times \mathbf{x} = -(\mathbf{x} \times \mathbf{y})$

  when you swap the order, you get the opposite direction

  "corkscrew-rule"
- $(\lambda \mathbf{x}) \times \mathbf{y} = \lambda (\mathbf{x} \times \mathbf{y})$

  if **x** is eg 2 times longer, then the cross product is 2 times as long

  logical, because the length of the cross product is surface of a parallellogram

Distance of a point to a plane

Distance of point (p, q, r) to a plane $ax + by + cz = d$

1. Set up the eq. for the line through point orthogonal to plane:
   $(x, y, z) = (p, q, r) + \lambda (a, b, c)$
2. Calculate the intersection point of the line with the plane: fill in $x = p + \lambda a, \quad y = q + \lambda b, \quad z = r + \lambda c$ into the plane equation
3. Calculate distance between (p, q, r) and intersection point

Distance of a point to a line

Distance of the point to a line $\mathbb{R}^3$

1. Set up the eq. of a plane through point orthogonal to the line;
   $ax + by + cz = d$, with (a, b, c) the direction vector of the line
2. Calculate d by filling in the point
3. calculate intersection of line with plane
4. calculate the distance between point and intersection

Surface of a triangle in plane and space

Surface triangle (a, b, c), (d, e, f), (g, h, i) in $\mathbb{R}^3$

1. calculate cross product of $(d, e, f) - (a, b, c)$ and $(g, h, i) - (a, b, c)$
2. surface $= \frac{1}{2} \cdot$ length of this vector

Surface triangle (a, b), (c, d), (e, f) in $\mathbb{R}^2$

1. Calculate cross product of $(c, d, 0) - (a, b, 0)$ and $(e, f, 0) - (a, b, 0)$
2. surface $= \frac{1}{2} \cdot$ length of this vector

Note: there are also other methods you can use.

When inner product/when cross product

Let $\mathbf{u}, \mathbf{v}$ be two vectors.

Inner product:

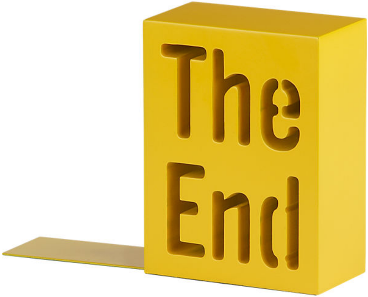
- To see if $\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0$
- Calculation of the angle θ between $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$

Cross product:

- Calculation of a vector $\mathbf{n}$ orthogonal to $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{n} = \mathbf{u} \times \mathbf{v}$
- To see if $\mathbf{u} \parallel \mathbf{v} \iff \mathbf{u} \times \mathbf{v} = 0$
- Calculation of the surface O of the parallellogram spanned by $\mathbf{u}$ and $\mathbf{v}$: $O = |\mathbf{u} \times \mathbf{v}|$

Week 2: We have seen

- Trigonometric functions sin, ...
- Powers e^x 2^x, ...
- Vectors in 2 and 3 dimensions
- Lines and planes
- Distance, angle, inner product
- Cross product



Week 3: Limits, Continuity and Differentiation I

- Limits of functions
- Limits at Infinity and Infinite limits
- Continuity, maxima and minima
- The Intermediate-Value Theorem
- The Derivative
- Derivatives of trigonometris functions
- Rules for Calculating Derivatives
- The Chain Rule
- Tangent lines and their direction

Limits in Computer Science and Data Science

- Foundation to understand complexity analysis. Without this, you'll never understand algorithms, theory of computation.
- Concepts and definitions allow to go from *continuous* to discrete, numerical analysis, leading to numerical algorithms.
- Needed in statistics (law of large numbers) $\rightarrow$ data science
- software testing and deployment

Ranking Different Algorithms

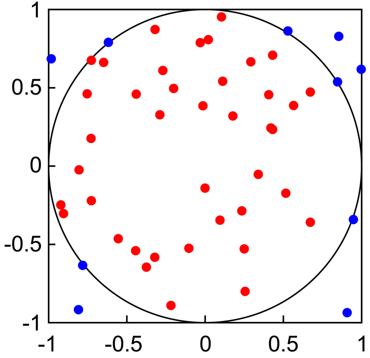
- How to formulate this more generally?
  $\rightarrow$ Limits
- $$\lim_{n\to\infty} \frac{f}{g} = 0 \qquad g \text{ grows faster}$$
  $$\lim_{n\to\infty} \frac{f}{g} = c \qquad g \text{ grows same as } f$$
  $$\lim_{n\to\infty} \frac{f}{g} = \infty \qquad g \text{ grows slower}$$

Limits not always straightforward

- $f(n) = n \log(n)$
- $g(n) = n^{3/2}$
- $$\lim_{n\to\infty} \frac{n^{3/2}}{n\log(n)} = ?$$
- To evaluate we will need:
  - derivatives
  - Taylor series/polynomials
  - Exponential functions and Logarithms
  - rational functions
  - ...

Statistical estimate of π

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\pi \cdot r^2}{(2r)^2} = \frac{\pi}{4}$$



Statistical estimate of π

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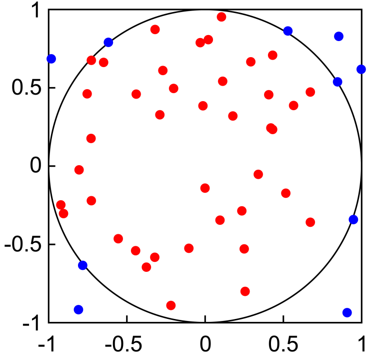

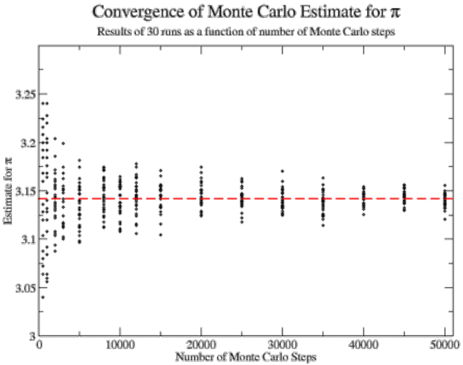

Relative Error $\sim \frac{1}{\sqrt{N}}$

N larger, error smaller

- randomly select a large number of points within the square
- ratio of number of points within the circle to the number points within the square provides an estimate for $\pi/4$

The limit of a series of numbers

Consider the sequence of numbers:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$$

Can we describe the sequence with a simple expression?
Let x_k be the k-th entry:
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The limit of a series of numbers

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$$x_k = \frac{k}{k+1} =$$

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Which value does x_k take for $k = 1, 2, 3, \ldots$?

$$x_k = \frac{k}{k+1} = \frac{(k+1) - 1}{k+1} =$$

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$$x_k = \frac{k}{k+1} = \frac{(k+1)-1}{k+1} = \frac{k+1}{k+1} - \frac{1}{k+1} =$$

The limit of a series of numbers

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The limit of a series of numbers

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$$x_k = \frac{k}{k+1} = \frac{(k+1)-1}{k+1} = \ldots = 1 - \frac{1}{k+1}$$

Table

k	x_k
1	0.500000000000000
2	0.666666666666667
3	0.750000000000000
4	0.800000000000000
5	0.833333333333333
$\vdots$	$\vdots$
98	0.989898989898990
99	0.990000000000000
100	0.990099009900990

Clearly for $k = 1, 2, 3, \ldots \to \infty$: $x_k \to 1$

The limit of a series of numbers

$$x_k = \frac{k}{k+1} = \frac{(k+1)-1}{k+1} = \ldots =$$

The limit of a series of numbers

$$x_k = \frac{k}{k+1} = \frac{(k+1)-1}{k+1} = \ldots = 1 - \frac{1}{k+1}$$

Clearly for $k = 1, 2, 3, \ldots \to \infty$: $x_k \to 1$

For $k = 1, 2, \ldots \to \infty$, x_k approaches the limit 1, notation:

$$\lim_{k\to\infty} x_k = 1$$

Not every series of numbers has a limit: $x_k = k \to \infty$ for $k = 1, 2, \ldots, \infty$

Limits: $x \to a$

There are THREE possible directions to approach a:

$x \to a^-$ for "x approaches a" from left
$x \to a^+$ for "x approaches a" from right
$x \to a$ for "x approaches a", from both sides

Some examples:

$x \to 0^-$: $x = -1, -0.1, -0.01, -0.001, \ldots$ approaches 0 "quickly"
$x \to 0^-$: $x = -1, -1/2, -1/3, -1/4, \ldots$ approaches 0 "slowly"

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$x \to 0^+$: $x = 1, 0.1, 0.01, 0.001, \ldots$
$x \to 2^-$: $x = 1.9, 1.99, 1.999, 1.9999, \ldots$
$x \to 2^+$: $x = 2.1, 2.01, 2.001, 2.0001, \ldots$

Limits: $x \to a$

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$x \to 2^-$: $x = 1.9, 1.99, 1.999, 1.9999, \ldots$
$x \to 2^+$: $x = 2.1, 2.01, 2.001, 2.0001, \ldots$
$x \to -2^-$: $x = -2.1, -2.01, -2.001, -2.0001, \ldots$
$x \to -2^+$: $x = -1.9, -1.99, -1.999, -1.9999, \ldots$
$x \to 0$: $x = 1, 0.1, -0.1, 0.01, -0.01, 0.001, -0.001, \ldots$
$x \to 0$: $x = 1, 0.1, 0.05, 0.01, 0.005, 0.001, 0.0005, \ldots$

If $x \to a$ then $f(x) \to ?$

Ex: If $x \to 0^-$ then $f(x) = x^2 \to^+ 0$ because:

$x \to 0^-$	$f(x) = x^2$
-1	1
-0.1	0.01
-0.01	0.001
-0.001	0.00001
-0.0001	0.0000001

Without filling 0 into f: $x \to 0^-$ then $f(x) \to f(0)$

Ex: If $x \to 0^+$ then $f(x) = x^2 \to 0$ because:

$x \to 0^+$	$f(x) = x^2$
1	1
0.1	0.01
0.01	0.001
0.001	0.00001
0.0001	0.0000001

Without filling 0 into f:

$$\left.\begin{array}{ccc} x \to 0^+ & \Longrightarrow & f(x) \to f(0) \\ x \to 0^- & \underset{\text{previous slide}}{\Longrightarrow} & f(x) \to f(0) \end{array}\right. \Longrightarrow x \to 0 \Longrightarrow f(x) \to 0$$

Three types limits of functions

The left $\lim\limits_{x\to a^-} f(x)$, right limit $\lim\limits_{x\to a^+} f(x)$ and "THE" limit $\lim\limits_{x\to a} f(x)$

Notation: The limit reached by f for $x \to a$ is:

$$
\begin{array}{lllll}
(x \to a^- & \Longrightarrow & f(x) \to L) & \underset{\text{notation}}{\Longleftrightarrow} & \lim\limits_{x\to a^-} f(x) = L \\
(x \to a^+ & \Longrightarrow & f(x) \to R) & \underset{\text{notation}}{\Longleftrightarrow} & \lim\limits_{x\to a^+} f(x) = R \\
(x \to a & \Longrightarrow & f(x) \to M) & \underset{\text{definition}}{\Longleftrightarrow} & \lim\limits_{x\to a^-} f(x) = M = \lim\limits_{x\to a^+} f(x)
\end{array}
$$

Limits L and R can exist even though $f(a)$ does not exist/is not defined!

If $f(a)$ is a real number, then this is the limit!

Attention:

If $L = R = \infty$ or $L = R = -\infty$, the limit exists and is ∞ resp. $-\infty$.

Ex: If $x \to 1^-$ then $f(x) = x + 1 \to 2$ because:

$x \to 1^-$	$f(x) = x + 1$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999
0.99999	1.99999

So without filling 1 into f: $\lim_{x \to 1^-} x + 1 = 2 = f(1)$,

If $x \to a$ then $f(x) \to ?$

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0.99	1.99
0.999	1.999
0.9999	1.9999
0.99999	1.99999

So without filling 1 into f: $\lim_{x \to 1^-} x + 1 = 2 = f(1)$,
because $1.999999\ldots = 1.\overline{9} = 2$.

Limit: Functions with jumps

Ex: The sign-function (signum) jumps at $x = 0$ from -1 to 1:

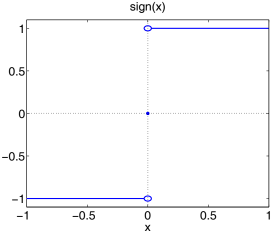
$$\operatorname{sign}(x) = \begin{cases} 1 & \textit{for} \quad x > 0 \\ 0 & \textit{for} \quad x = 0 \\ -1 & \textit{for} \quad x < 0 \end{cases}$$

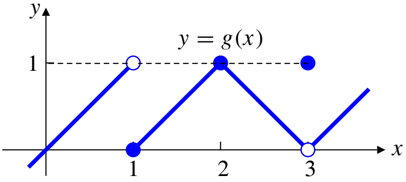
$$\lim_{x \to 0^-} \operatorname{sign}(x) = -1$$

$$\lim_{x \to 0^+} \operatorname{sign}(x) = 1$$

$$\lim_{x \to 0} \operatorname{sign}(x) \quad \text{does not exist}$$

Now: $\operatorname{sign}(0)$ exists but is different to the left and right limits!





$\lim_{x \to 1^-} g(x) = 1$

$\lim_{x \to 1^+} g(x) = 0$ $\qquad \lim_{x \to 1} g(x)$ does not exist; $\quad g(1) = 0$

$\lim_{x \to 2^-} g(x) = 1$

$\lim_{x \to 2^+} g(x) = 1$ $\qquad \lim_{x \to 2} g(x) = 1;$ $\quad$ is equal to $g(2) = 1$

$\lim_{x \to 3^-} g(x) = 0$

$\lim_{x \to 3^+} g(x) = 0$ $\qquad \lim_{x \to 3} g(x) = 0;$ $\quad$ but $g(3) = 1$

The property that for a a function $\lim_{x \to a} f(x) = f(a)$ is special. Such functions are called continuous at/in a (see [A 1.4]).

Calculation rules limits

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ exist, then

- $\lim_{x\to a} [f(x) \pm g(x)] = L \pm M$
- $\lim_{x\to a} f(x)\, g(x) = LM$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ als $M \neq 0$
- If $f(x) \leq g(x)$ in an interval around a then $L \leq M$

Ex: $\lim_{x\to a} x + x^2 = \lim_{x\to a} x + \lim_{x\to a} x^2 = a + a^2$

It is necessary that both $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ exist:

$$1 = \lim_{x\to 0} 1 \neq \frac{\lim_{x\to 0} x^2}{\lim_{x\to 0} x^2} = \frac{0}{0} \quad \dots$$

Limits: The difficult cases

$\frac{\neq 0}{0}$ (coming soon!)

$\frac{0}{0}$ (divide factor out, l'Hôpital, Taylor)

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Limits: The difficult cases

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$\infty - \infty$	(Week 5)

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Horizontal, vertical asymptote

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Horizontal, vertical asymptote	
$0 \cdot \infty$	(Week 5, ex $\lim\limits_{x\to 0^+} x\ln(x)$)

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1^∞	(ex $\lim\limits_{x\to\infty} (1+\dfrac{1}{x})^x$)

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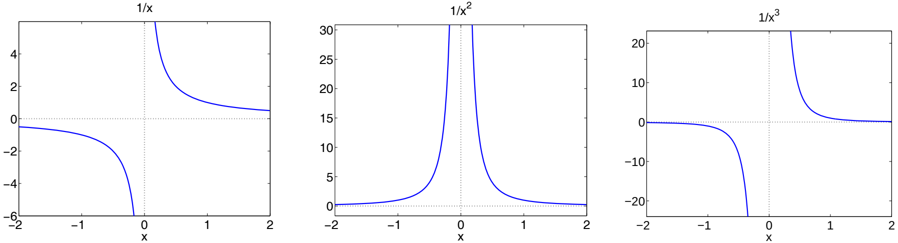
0^0	(ex $\lim_{x\to 0^+} x^x$)

Limits: $f(x) = t(x)/n(x)$, type $\frac{\neq 0}{0}$

Typical (simple) cases:



$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^+} \frac{1}{x^2} = \infty \qquad \lim_{x\to 0^+} \frac{1}{x^3} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^-} \frac{1}{x^2} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x^3} = -\infty$$

$\lim_{x\to 0} \frac{1}{x}$ does not exist. $\quad \lim_{x\to 0} \frac{1}{x^2} = \infty \quad$ $\lim_{x\to 0} \frac{1}{x^3}$ does not exist

Limits: $f(x) = t(x)/n(x)$, type $\frac{\neq 0}{0}$

Ex: $\lim_{x \to -2} \dfrac{x}{x^2 + 8x + 12}$. Filling $x = -2$ in yields $\frac{-2}{0}$

1. Factorize the denominator!
2. Analyze both $\lim_{x \to -2^+}$ and $\lim_{x \to -2^-}$

Because $x^2 + 8x + 12 = (x+2)(x+6)$:

$$\lim_{x \to -2^+}$$

Limits: $f(x) = t(x)/n(x)$, type $\frac{\neq 0}{0}$

Ex: $\displaystyle\lim_{x\to -2} \frac{x}{x^2+8x+12}$. Filling $x = -2$ in yields $\frac{-2}{0}$

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Because $x^2 + 8x + 12 = (x+2)(x+6)$:

$$\lim_{x\to -2^+} \frac{x}{(x+2)(x+6)} =$$

Limits: $f(x) = t(x)/n(x)$, type $\frac{\neq 0}{0}$

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Because $x^2 + 8x + 12 = (x+2)(x+6)$:

$$\lim_{x \to -2^+} \frac{x}{(x+2)(x+6)} = \frac{-2^+}{0^+ \cdot 4^+} =$$

Limits: $f(x) = t(x)/n(x)$, type $\frac{\neq 0}{0}$

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$$\lim_{x \to -2^+} \frac{x}{(x+2)(x+6)} = \frac{-2^+}{0^+ \cdot 4^+} = \frac{-2}{0^+ \cdot 4} = \frac{-2}{0^+} \quad =$$

Ex: $\lim\limits_{x\to -2} \dfrac{x}{x^2+8x+12}$. Filling $x=-2$ in yields $\frac{-2}{0}$

1. Factorize the denominator!
2. Analyze both $\lim\limits_{x\to -2^+}$ and $\lim\limits_{x\to -2^-}$

Because $x^2+8x+12 = (x+2)(x+6)$:

$$\lim_{x\to -2^+} \frac{x}{(x+2)(x+6)} = \frac{-2^+}{0^+\cdot 4^+} = \frac{-2}{0^+\cdot 4} = \frac{-2}{0^+} \quad = \quad -\infty$$

$(-2^+ + 2 = (-2+2)^+ = 0^+)$

Ex: $\lim_{x \to -2} \dfrac{x}{x^2 + 8x + 12}$. Filling $x = -2$ in yields $\frac{-2}{0}$

1. Factorize the denominator!
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Because $x^2 + 8x + 12 = (x + 2)(x + 6)$:

$$\lim_{x \to -2^+} \frac{x}{(x + 2)(x + 6)} = \frac{-2^+}{0^+ \cdot 4^+} = \frac{-2}{0^+ \cdot 4} = \frac{-2}{0^+} = -\infty$$

$(-2^+ + 2 = (-2 + 2)^+ = 0^+)$

Same: $\lim_{x \to -2^-} \dfrac{x}{(x + 2)(x + 6)} = \infty$

thus $\lim_{x \to -2} \dfrac{x}{x^2 + 8x + 12}$ does not exist

Why is $3 - 3^+ = 0^-$?

$x \to 3^+$	$3 - 3^+$	0^-
3.1	-0.1	-0.1
3.01	-0.01	-0.01
3.001	-0.001	-0.001
3.0001	-0.0001	-0.0001
3.00001	-0.00001	-0.00001

Or: Sequence of numbers $x \to 3^+$ of the form $x_k = 3 + h_k$ with $h_k > 0$. Then

$$3 - x_k = 3 - (3 + h_k) = -h_k \to 0^-.$$

Limits: $f(x) = t(x)/n(x)$, generals

$$\lim_{x\to a} \frac{f(x)}{g(x)} = ? \quad \text{if} \quad \lim_{x\to a} g(x) = g(a) = 0$$

- If numerator does **not** approach 0, then

  $$\lim_{x\to a^-} \frac{f(x)}{g(x)} = \pm\infty \quad \text{and} \quad \lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$$

  Check sign for both cases. Are left and right limits the same?

  3 possibilities for $\lim_{x\to a}$: ∞, $-\infty$ or does not exist

  Useful to factorize the denominator

- If numerator **also** approaches 0:

  (a) try to divide a factor out

  Ex: $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} x+3 = 6$

  (b) or with l'Hôpital (later in Calculus)

  (c) or with Taylor series (later in Calculus)

Limits of functions

- $\lim_{x \to 0} \frac{1}{x^5} = ?$

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  $\lim\limits_{x \to 0^+} \dfrac{1}{x^5} =$

Limits of functions

- $\displaystyle\lim_{x\to 0} \frac{1}{x^5} = ?$

  $\displaystyle\lim_{x\to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$

  $\displaystyle\lim_{x\to 0^-} \frac{1}{x^5} =$

Limits of functions

- $\lim\limits_{x \to 0} \dfrac{1}{x^5} = ?$

  $\lim\limits_{x \to 0^+} \dfrac{1}{x^5} = \dfrac{1}{0^+} = \infty$

  $\lim\limits_{x \to 0^-} \dfrac{1}{x^5} = \dfrac{1}{0^-} = -\infty$ thus $\lim\limits_{x \to 0} \dfrac{1}{x^5}$

Limits of functions

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  $\lim\limits_{x \to 0^-} \dfrac{1}{x^5} = \dfrac{1}{0^-} = -\infty$ thus $\lim\limits_{x \to 0} \dfrac{1}{x^5}$ does not exist

- $\lim\limits_{x \to 0} \dfrac{x-2}{x^4} = ?$

Limits of functions

- $\lim_{x\to 0} \frac{1}{x^5} = ?$

  $\lim_{x\to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$

  $\lim_{x\to 0^-} \frac{1}{x^5} = \frac{1}{0^-} = -\infty$ thus $\lim_{x\to 0} \frac{1}{x^5}$ does not exist

- $\lim_{x\to 0} \frac{x-2}{x^4} = ?$

  $\lim_{x\to 0^+} \frac{x-2}{x^4} =$

- $\lim_{x \to 0} \frac{1}{x^5} = ?$

  $$\lim_{x \to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$$

  $$\lim_{x \to 0^-} \frac{1}{x^5} = \frac{1}{0^-} = -\infty \qquad \text{thus} \qquad \lim_{x \to 0} \frac{1}{x^5} \text{ does not exist}$$

- $\lim_{x \to 0} \frac{x-2}{x^4} = ?$

  $$\lim_{x \to 0^+} \frac{x-2}{x^4} = \frac{-2^+}{0^+} = -\infty$$

  $$\lim_{x \to 0^-} \frac{x-2}{x^4} =$$

- $\displaystyle\lim_{x\to 0}\frac{1}{x^5} = ?$

  $$\lim_{x\to 0^+}\frac{1}{x^5} = \frac{1}{0^+} = \infty$$

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- $\lim\limits_{x \to 2} \dfrac{x^2}{2-x} = ?$

Limits of functions

- $\lim_{x\to 0} \frac{1}{x^5} = ?$

  $\lim_{x\to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$

  $\lim_{x\to 0^-} \frac{1}{x^5} = \frac{1}{0^-} = -\infty$ thus $\lim_{x\to 0} \frac{1}{x^5}$ does not exist

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- $\lim_{x\to 2} \frac{x^2}{2-x} = ?$

  $\lim_{x\to 2^+} \frac{x^2}{2-x} =$

Limits of functions

- $\lim_{x\to 0} \frac{1}{x^5} = ?$

  $$\lim_{x\to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$$

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- $\lim_{x\to 2} \frac{x^2}{2-x} = ?$

  $$\lim_{x\to 2^+} \frac{x^2}{2-x} = \frac{4}{0^-} = -\infty$$

Limits of functions

- $\lim\limits_{x\to 0} \dfrac{1}{x^5} = ?$

  $\lim\limits_{x\to 0^+} \dfrac{1}{x^5} = \dfrac{1}{0^+} = \infty$

  $\lim\limits_{x\to 0^-} \dfrac{1}{x^5} = \dfrac{1}{0^-} = -\infty$ thus $\lim\limits_{x\to 0} \dfrac{1}{x^5}$ does not exist

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- $\lim\limits_{x\to 2} \dfrac{x^2}{2-x} = ?$

  $\lim\limits_{x\to 2^+} \dfrac{x^2}{2-x} = \dfrac{4}{0^-} = -\infty$

  $\lim\limits_{x\to 2^-} \dfrac{x^2}{2-x} =$

- $\lim_{x\to 0} \frac{1}{x^5} = ?$

  $\lim_{x\to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$

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Limits of functions

- $\lim_{x\to 0} \frac{1}{x^5} = ?$

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  $\lim_{x\to 2^+} \frac{x^2}{2-x} = \frac{4}{0^-} = -\infty$

  $\lim_{x\to 2^-} \frac{x^2}{2-x} = \frac{4}{0^+} = \infty$ thus $\lim_{x\to 2} \frac{x^2}{2-x}$ does not exist

- $\lim_{x\to 2} \frac{x}{x^2-4x+4} =$

Limits of functions

- $\displaystyle\lim_{x\to 0}\frac{1}{x^5} = ?$

$$\lim_{x\to 0^+}\frac{1}{x^5} = \frac{1}{0^+} = \infty$$

$$\lim_{x\to 0^-}\frac{1}{x^5} = \frac{1}{0^-} = -\infty \qquad \text{thus} \qquad \lim_{x\to 0}\frac{1}{x^5} \text{ does not exist}$$

- $\displaystyle\lim_{x\to 0}\frac{x-2}{x^4} = ?$

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- $\displaystyle\lim_{x\to 2}\frac{x^2}{2-x} = ?$

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- $\displaystyle\lim_{x\to 2}\frac{x}{x^2-4x+4} = \lim_{x\to 2}\frac{x}{(x-2)^2} \qquad \text{thus}$

Limits of functions

- $\lim\limits_{x\to 0} \dfrac{1}{x^5} = ?$

$$\lim_{x\to 0^+} \frac{1}{x^5} = \frac{1}{0^+} = \infty$$

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- $\lim\limits_{x\to 2} \dfrac{x}{x^2-4x+4} = \lim\limits_{x\to 2} \dfrac{x}{(x-2)^2}$ thus $= \infty$!

Because the limit from both sides is $\frac{2}{0^+} = \infty$

Limits: $f(x) = t(x)/n(x)$, type $\frac{0}{0}$

What results for $\frac{0}{0}$ depends on whether numerator goes faster towards 0 than the denominator:

Ex: $\lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$ (numerator faster)

Ex: $\lim\limits_{x\to 0} \dfrac{x}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist (denominator faster)

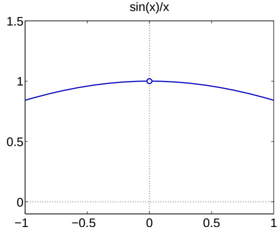
Ex: $\lim\limits_{x\to 0} \dfrac{x}{x} = \lim\limits_{x\to 0} 1 = 1$ (equally fast)

Ex: $\lim\limits_{x\to 0} \dfrac{3x}{x} = \lim\limits_{x\to 0} 3 = 3$ (both same powers, but different constants)

Limits: Type $\frac{0}{0}$ with variable substitution and $\cdot \frac{g(x)}{g(x)}$

Accept now: $\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1}$

Determine now:



$$\lim_{x\to 0^+} \frac{\sin(\sqrt{4x})}{\sqrt{4x}} = \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1$$

$$\lim_{x\to 0} \frac{\sin(x)}{\sin(3x)} = \lim_{x\to 0} \frac{1}{3} \frac{\sin(x)\,3x}{x\,\sin(3x)}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{3x}{\sin(3x)} = \frac{1}{3}$$

$$\lim_{x\to 0} \frac{\tan(11x)}{x} = \lim_{x\to 0} \frac{\sin(11x)}{\cos(11x)\,x} = \lim_{x\to 0} \frac{11\sin(11x)}{\cos(11x)\,11x}$$

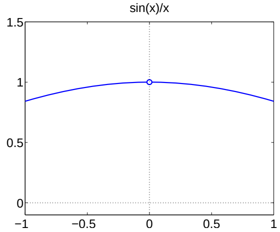
$$= 11 \cdot \lim_{x\to 0} \frac{\sin(11x)}{11x} \cdot \lim_{x\to 0} \frac{1}{\cos(11x)} = 11$$

Accepting the fact above, you can calculate other limits, with substitution.

Limits: Type $\frac{0}{0}$ with variable substitution and $\cdot\frac{g(x)}{g(x)}$

Accept now: $\boxed{\lim_{x\to 0}\frac{\sin(x)}{x}=1}$

Determine now:



$$\lim_{x\to 0^+}\frac{\sin(\sqrt{4x})}{\sqrt{4x}}=$$

$$\lim_{x\to 0}\frac{1}{3}\,\frac{\sin(x)\,3x}{x\,\sin(3x)}$$

$$=\frac{1}{3}\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{3x}{\sin(3x)}=\frac{1}{3}$$

$$\lim_{x\to 0}\frac{\tan(11x)}{x}=\lim_{x\to 0}\frac{\sin(11x)}{\cos(11x)\,x}=\lim_{x\to 0}\frac{11\sin(11x)}{\cos(11x)\,11x}$$

$$=11\cdot\lim_{x\to 0}\frac{\sin(11x)}{11x}\cdot\lim_{x\to 0}\frac{1}{\cos(11x)}=11$$

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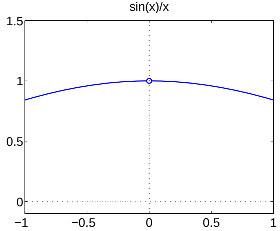

$$\lim_{x\to 0^+} \frac{\sin(\sqrt{4x})}{\sqrt{4x}} =$$

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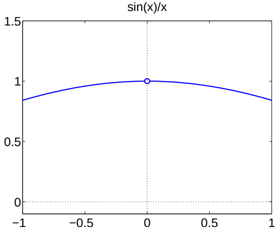

$$\lim_{x\to 0^+}\frac{\sin(\sqrt{4x})}{\sqrt{4x}}=$$

$$\frac{1}{3}$$

$$\lim_{x\to 0}\frac{\tan(11x)}{x}=\lim_{x\to 0}\frac{\sin(11x)}{\cos(11x)\,x}=\lim_{x\to 0}\frac{11\sin(11x)}{\cos(11x)\,11x}$$
$$=11\cdot\lim_{x\to 0}\frac{\sin(11x)}{11x}\cdot\lim_{x\to 0}\frac{1}{\cos(11x)}=11$$

Accepting the fact above, you can calculate other limits, with substitution.

Limits: Type $\frac{0}{0}$ with variable substitution and $\cdot\frac{g(x)}{g(x)}$

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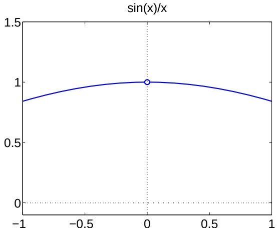


$$\lim_{x\to 0^+}\frac{\sin(\sqrt{4x})}{\sqrt{4x}} =$$

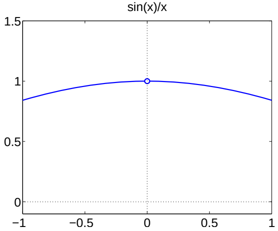
$$\lim_{x\to 0}\frac{\sin(11x)}{\cos(11x)\,x} = \lim_{x\to 0}\frac{11\sin(11x)}{\cos(11x)\,11x}$$

$$= 11\cdot\lim_{x\to 0}\frac{\sin(11x)}{11x}\cdot\lim_{x\to 0}\frac{1}{\cos(11x)} = 11$$

Accepting the fact above, you can calculate other limits, with substitution.

Limits: Type $\frac{0}{0}$ with variable substitution and $\cdot\frac{g(x)}{g(x)}$

Accept now: $\boxed{\lim_{x\to 0}\frac{\sin(x)}{x} = 1}$



Determine now:

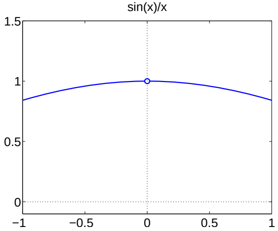
$$\lim_{x\to 0^+}\frac{\sin(\sqrt{4x})}{\sqrt{4x}} =$$

$$\lim_{x\to 0}\frac{11\sin(11x)}{\cos(11x)\,11x}$$

$$= 11\cdot\lim_{x\to 0}\frac{\sin(11x)}{11x}\cdot\lim_{x\to 0}\frac{1}{\cos(11x)} = 11$$

Accepting the fact above, you can calculate other limits, with substitution.

Accept now: $\boxed{\lim_{x\to 0}\frac{\sin(x)}{x}=1}$



Determine now:

$$\lim_{x\to 0^+}\frac{\sin(\sqrt{4x})}{\sqrt{4x}} = \lim_{y\to 0^+}\frac{\sin(y)}{y} = 1$$

$$\lim_{x\to 0}\frac{\sin(x)}{\sin(3x)} = \lim_{x\to 0}\frac{1}{3}\,\frac{\sin(x)\,3x}{x\,\sin(3x)}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{3x}{\sin(3x)} = \frac{1}{3}$$

$$\lim_{x\to 0}\frac{\tan(11x)}{x} = \lim_{x\to 0}\frac{\sin(11x)}{\cos(11x)\,x} = \lim_{x\to 0}\frac{11\sin(11x)}{\cos(11x)\,11x}$$

$$= 11\cdot\lim_{x\to 0}\frac{\sin(11x)}{11x}\cdot\lim_{x\to 0}\frac{1}{\cos(11x)} = 11$$

Accepting the fact above, you can calculate other limits, with substitution.

- Already seen: limits themselves can be $\pm\infty$ ($\notin \mathbb{R}$)
- New (now!): Calculate limits for $x \to \pm\infty$

For $\lim\limits_{x\to\pm\infty}$ of polynomials: largest power is important

$$
\begin{array}{llll}
\lim\limits_{x\to\infty} x = \infty & & \lim\limits_{x\to-\infty} x = -\infty \\
\lim\limits_{x\to\infty} x^2 = \infty & & \lim\limits_{x\to-\infty} x^2 = \infty \\
\lim\limits_{x\to\infty} x^2 - x = \infty & & \lim\limits_{x\to-\infty} x^2 - x = \infty
\end{array}
$$

The limit can be a number:

Ex: $\lim\limits_{x\to-\infty} e^x = 0$

Ex: $\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$

Limits: $\lim_{x\to\pm\infty} t(x)/n(x)$, type $\frac{\infty}{\infty}$ or $\frac{0}{0}$

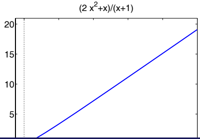
Rational function: $\frac{p(x)}{q(x)}$, with $p(x)$ and $q(x)$ polynomials

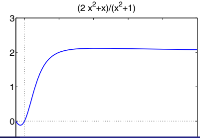
Divide numerator and denominator by largest power of x in the denominator

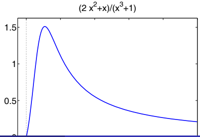
$$\lim_{x\to\infty} \frac{2x^2+x}{x+1} = \lim_{x\to\infty} \frac{2x+1}{1+\frac{1}{x}} = \frac{\infty}{1} = \infty$$

$$\lim_{x\to\infty} \frac{2x^2+x}{x^2+1} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{2}{1} = 2$$

$$\lim_{x\to\infty} \frac{2x^2+x}{x^3+1} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x^3}} = \frac{0}{1} = 0$$







$\frac{\infty}{\infty}$ can have any answer

It depends on whether the numerator or denominator approaches ∞ faster, equally fast fast etc

Ex: $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$ (numerator faster)

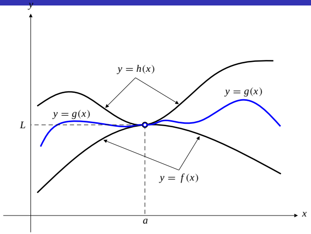
Ex: $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$ (denominator faster)

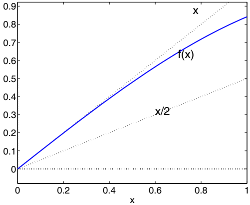
Ex: $\lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$ (equally fast)

Ex: $\lim_{x\to\infty} \frac{3x}{x} = \lim_{x\to\infty} 3 = 3$ (same power, different constant)

(See also examples on previous slide)

Squeeze theorem





If $f(x) \leq g(x) \leq h(x)$ around $x = a$, then

$$\boxed{\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x)}$$

If also $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$ then

$$\boxed{\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)}$$

Ex: Let $f(x) = sin(x)$, assume $\frac{x}{2} \leq f(x) \leq x$ for $x > 0$. Then

$$0 = \lim_{x \to 0^+} \frac{x}{2} \leq \lim_{x \to 0^+} \sin(x) \leq \lim_{x \to 0^+} x = 0 \text{ thus } \lim_{x \to 0^+} \sin(x) = 0$$

$\lim_{x \to \pm\infty} t(x)/n(x)$

Squeeze Theorem is the last resort:

Ex: $\lim_{x \to \infty} \frac{\sin(x)}{x}$? Use $-1 \leq \sin(x) \leq 1$ for all x:

Squeeze Theorem is the last resort:

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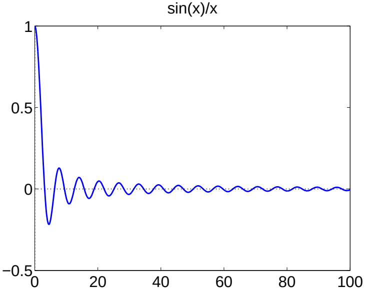

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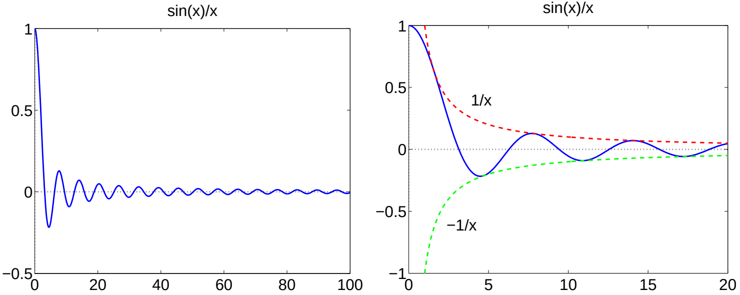
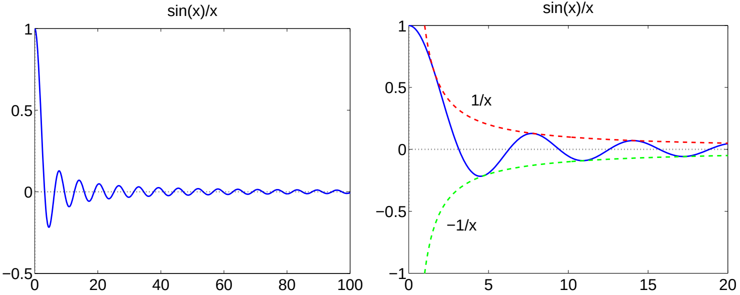


Solve these with

$\lim_{x \to \pm\infty} t(x)/n(x)$

Squeeze Theorem is the last resort:

Ex: $\lim_{x \to \infty} \frac{\sin(x)}{x}$? Use $-1 \leq \sin(x) \leq 1$ for all x:



Solve these with squeeze theorem:

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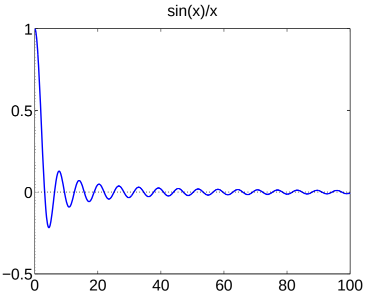
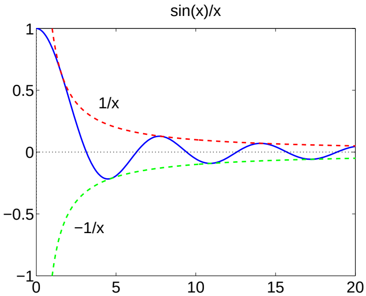




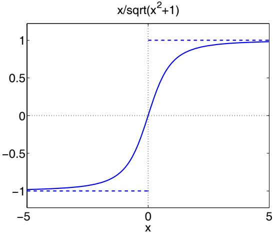
Solve these with squeeze theorem: $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$ thus

$$\underbrace{\lim_{x\to\infty} -\frac{1}{x}}_{=0} \leq \underbrace{\lim_{x\to\infty} \frac{\sin(x)}{x}}_{\text{must therefore also be 0!}} \leq \underbrace{\lim_{x\to\infty} \frac{1}{x}}_{=0}$$

Do [A 1.2 T75]

Horizontal asymptote

If $\lim_{x\to\pm\infty} f(x) = a \quad (a \in \mathbb{R}, \text{ number}, \neq \pm\infty)$,
then f has a horizontal asymptote $y = a$ for $x \to \pm\infty$



Ex: $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} =$

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} =$$

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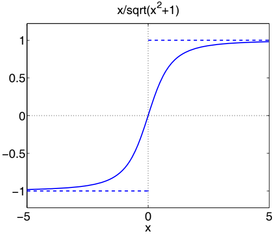

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$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$

$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$ because then $|x| = -x$

Thus horizontal asymptote $y = 1$ to the right and $y = -1$ to the left

Vertical asymptote

f has a vertical asymptote $x = a$
if $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim\limits_{x \to a^-} f(x) = \pm\infty$

Here it is allowed that the limit does not exist, thus eg $\lim\limits_{x \to a^+} f(x) = \infty$ and $\lim\limits_{x \to a^-} f(x) = -\infty$

Ex: $\frac{-6}{x+4}$ has a vertical asymptote in $x = -4$ because

$$\lim_{x \to -4^+} \frac{-6}{x+4} = -\infty \quad \text{and} \quad \lim_{x \to -4^-} \frac{-6}{x+4} = \infty$$

Ex: $\ln(x)$ has a vertical asymptote in $x = 0$ because

$$\lim_{x \to 0^+} \ln(x) = -\infty \quad \left(\lim_{x \to 0^-} \ln(x) \text{ does not exist, domain} \right)$$

Some standard limits

If $a > 0$, then:

$$\boxed{\begin{aligned} \lim_{x\to\infty} \frac{x^a}{e^x} &= 0 \\ \lim_{x\to\infty} \frac{\ln(x)}{x^a} &= 0 \\ \lim_{x\to-\infty} |x|\,e^x &= 0 \\ \lim_{x\to 0^+} x^a \ln(x) &= 0 \end{aligned}}$$

This also holds for b^x $(b > 1)$ in particular for e^x

For $x \to \infty$, b^x $(b > 1)$ approaches ∞ faster than any polynomial (or any positive power), and any polynomial faster than $\ln(x)$

For $x \to -\infty$, b^x $(b > 1)$ approaches 0 faster than any polynomial approaches ∞

For $x \to 0^+$ approaches any positive power 0 faster than $\ln(x)$ approaches $-\infty$

Limits of functions

- $\lim\limits_{x\to\pm\infty} \dfrac{-8+\frac{5}{x}}{7-\frac{4}{x^2}}$
- $\lim\limits_{x\to\infty} \dfrac{3x^7+4x^6+6}{9x^8}$
- $\lim\limits_{x\to\infty} \dfrac{7\sqrt{x}+x^{-4}}{4x-5}$
- $\lim\limits_{x\to-\infty} \dfrac{8-8x+\sin(8x)}{8x-\cos(8x)}$
- $\lim\limits_{x\to\infty} \dfrac{2^{3x}+3^{2x}}{7^{x+9}+9^x}$
- $\lim\limits_{x\to\infty} \dfrac{7x^3+8x^5+2^{2x}}{1+4^{x-1}}$

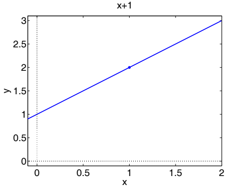
Continuity of function f in point a

f is right continuous in a if $\lim_{x \to a^+} f(x) = f(a)$ [A 1.4 D6]

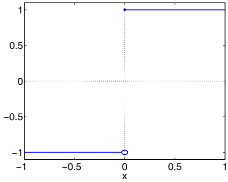
f is left continuous in a if $\lim_{x \to a^-} f(x) = f(a)$ [A 1.4 D6]

f is continuous in a if $\lim_{x \to a} f(x) = f(a)$ [A 1.4 D4]

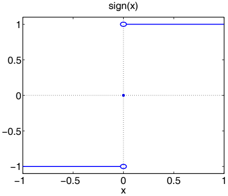
Thus f is continuous in a $\iff$ left and right continuous in a



left continuous in $x = 1$
right continuous in $x = 1$
continuous in $x = 1$



not left continuous in $x = 0$
right continuous in $x = 0$
no continuous in $x = 0$



not left continuous in $x = 0$
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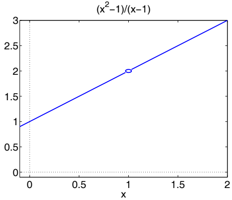
not continuous in $x = 0$



not continuous in $x = 1$
but make continuous by defining $f(1) = 2$
"removable discontinuity"

Continuity of function f in point a

Repeat: In mathematical terms f is continuous in a

(I) $f(a)$ exists
(II) $\lim_{x \to a^+} f(x) = f(a)$ exists
(III) $\lim_{x \to a^-} f(x) = f(a)$ exists;

It means (graphically): There is no hole/jump in the graph of f at a

Continuity of function f in $x \in [a, b]$

The definition:

f is continuous on an interval $[a, b]$ (or on $\mathbb{R}$)
if f is continuous in every $x \in (a, b)$ (or in every $x \in \mathbb{R}$)
If an endpoint as a is involved:

- $f(a)$ must exist and
- the right limit must be $R = f(a)$

Similar for right endpoint.

Ex: $f(x) = 1/x$ is continuous on $(0, \infty)$

If f is continuous on a CLOSED interval $[a, b]$ then there are no holes/jumps in the graph f over $[a, b]$, so then f obviously has a minimal and an maximal value!

Intermediate-Value Theorem

If f is continuous on $[a, b]$ then f takes every value between $f(a)$ and $f(b)$ because f does not have any holes or jumps in $[a, b]$:
If f is continuous on $[a, b]$ then there is for every $s \in [f(a), f(b)]$ a c with $f(c) = s$:

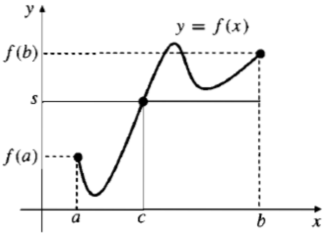
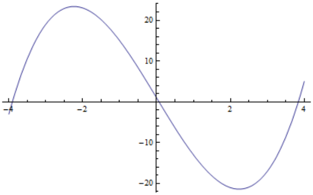


Figure 1.32 The continuous function f takes on the value s at some point c between a and b

Ex: $f(x) = x^3 - 15x + 1$ has three zeros on $[-4, 4]$:



x	$f(x)$	sign
-4	-3	$-$
-3	19	$+$
-2	23	$+$
-1	15	$+$
-0	1	$+$
1	-13	$-$
2	-21	$-$
3	-17	$-$
4	5	$+$

At $x = -4$ is $f(-4) < 0$ and $x = -3$ $f(-3) > 0$. f is continuous thus it has at least one zero on $[-3, -4]$. Same for zeros in $(-1, 0)$ and in $(3, 4)$

Continuity: Examples

Ex: Continuous on its complete domain D are:

- polynomials $D = \mathbb{R}$
- powers a^x on $D = \mathbb{R}$ for $a \geq 0$, $\sqrt{x}$ for $D = [0, \infty)$
- sin, cos on $D = \mathbb{R}$
- tan on $\mathbb{R} - \{\frac{\pi}{2} + k\pi\}$ for all integers k
- sums, differences, products of continuous functions
- quotient of functions: $\frac{f(x)}{g(x)}$ on

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- compositions of continuous functions

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- $\sin(y)$ continuous, $\sin(\sqrt{x^2-1})$ continuous on $(-\infty,-1] \cup [1,\infty)$

If $\lim_{x \to a} g(x)$ exists and f is continuous at $g(a)$ then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(g(a))$

Holds too for left and right limits. Ex: Calculate:

$$\lim_{x \to 0} \cos\left(\frac{\pi - \pi \cos^2 x}{x^2}\right).$$

The cosine is a continuous function (for all inputs), thus first:

$$\begin{aligned} \lim_{x \to 0} \frac{\pi - \pi \cos^2 x}{x^2} &= \lim_{x \to 0} \frac{\pi \sin^2 x}{x^2} = \pi \lim_{x \to 0} \frac{\sin^2 x}{x^2} \\ &= \pi \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = \pi \cdot 1^2 = \pi \end{aligned}$$

so that

$$\lim_{x \to 0} \cos\left(\frac{\pi - \pi \cos^2 x}{x^2}\right) = \cos\left(\lim_{x \to 0} \frac{\pi - \pi \cos^2 x}{x^2}\right) = \cos(\pi) = -1$$

- For which x is $f(x) = \frac{8}{x-4} - 3x$ continuous?
- For which x is $f(x) = |9x - 4| - 4\sin(5x)$ continuous?
- Let $f(x) = \frac{6x^2-150}{6x-30}$
  Determine $f(5)$ so that f is continuous for $x = 5$

Do [A 1.4 T9,15]

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  $f(5)$ is not defined because $0/0$
  $$\lim_{x\to 5} \frac{6x^2 - 150}{6x - 30} = \lim_{x\to 5} \frac{x^2 - 25}{x - 5}$$

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  $$\lim_{x\to5} \frac{6x^2-150}{6x-30} = \lim_{x\to5} \frac{x^2-25}{x-5} = \lim_{x\to5} \frac{(x-5)(x+5)}{x-5} = 10$$
  So we define $f(5) = 10$, then is f continuous everywhere

Do [A 1.4 T9,15]

Secure Key Exchange: Elliptic Diffie Hellman

- based on addition of points on an *elliptic curve*
- mathematically, *groups* consist of elements (points on elliptic curve) and operations (addition of points) that fulfil the four fundamental properties of closure, associativity, the identity property, and the inverse property.
- in other words: the operation "addition" (and so multiplication) can be a very strange looking operation compared to the common addition of real numbers
- $P \oplus Q$ is the addition of two points on an ellipic curve
- $nP = P \oplus P \oplus P \oplus \ldots \oplus P$ is the multiplication of a point

Secure Key Exchange: Elliptic Diffie Hellman

Elliptic curve: $y^2 = x^3 + ax + b$

1. User1 initiates a connection with User2, and selects a generator P (a point on the curve) and the parameters a and b of the elliptic curve equation, and sends them across the wire as plain text.

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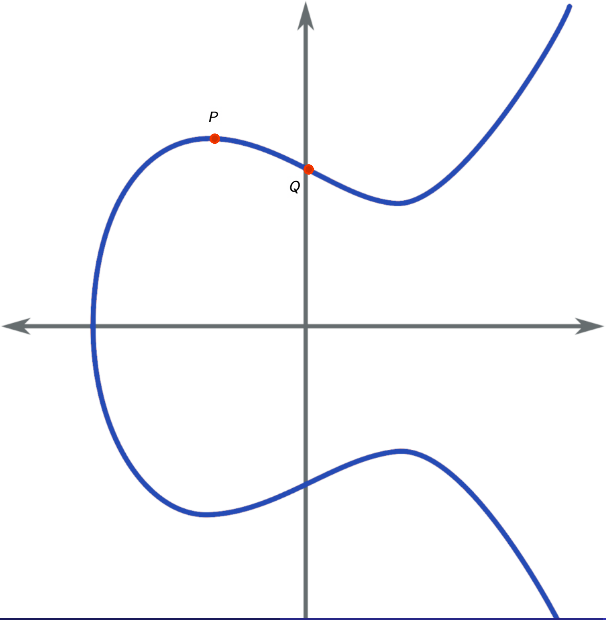
6. The point is used as the symmetrical encryption key for all data transfers.

Secure Key Exchange: Elliptic Diffie Hellman



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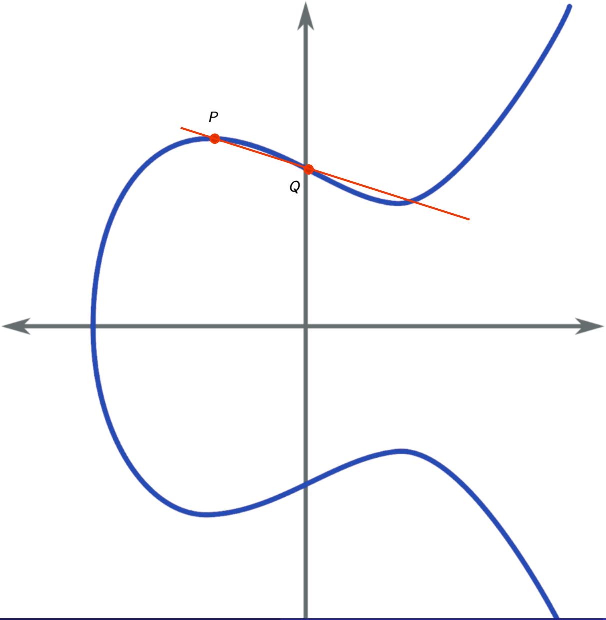

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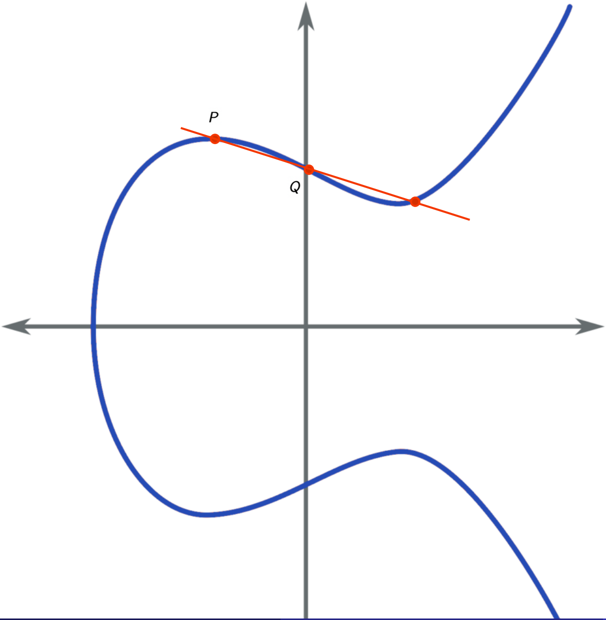

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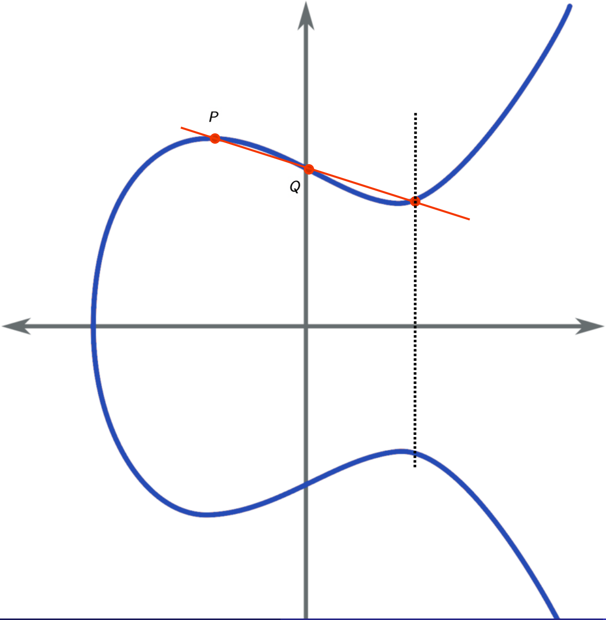

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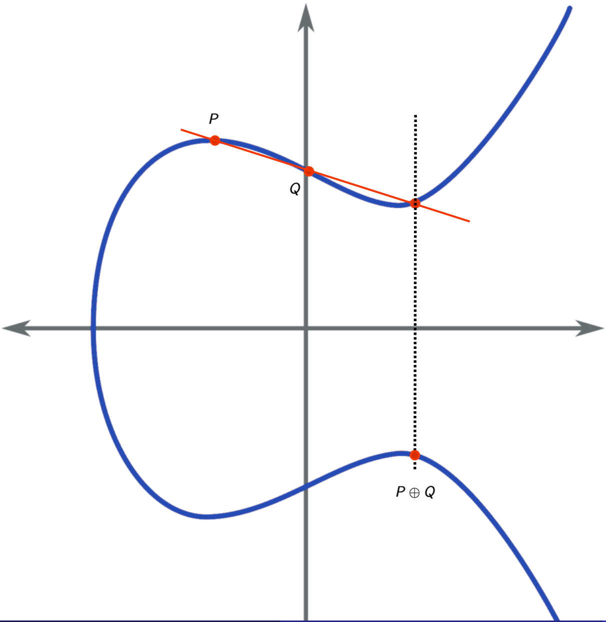

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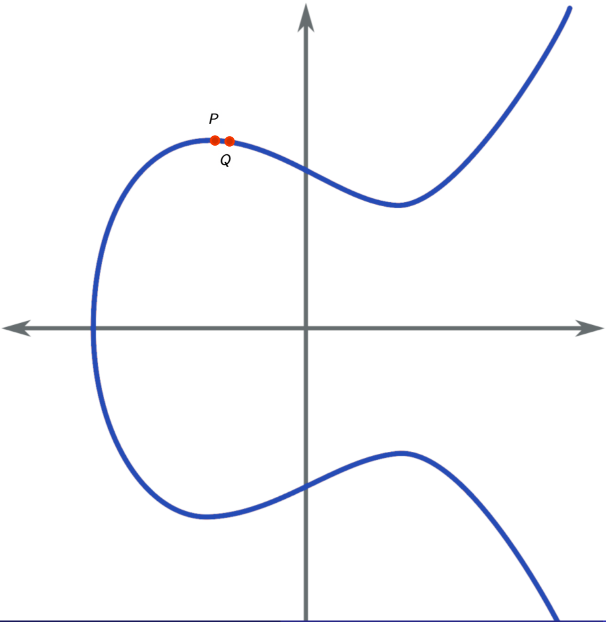

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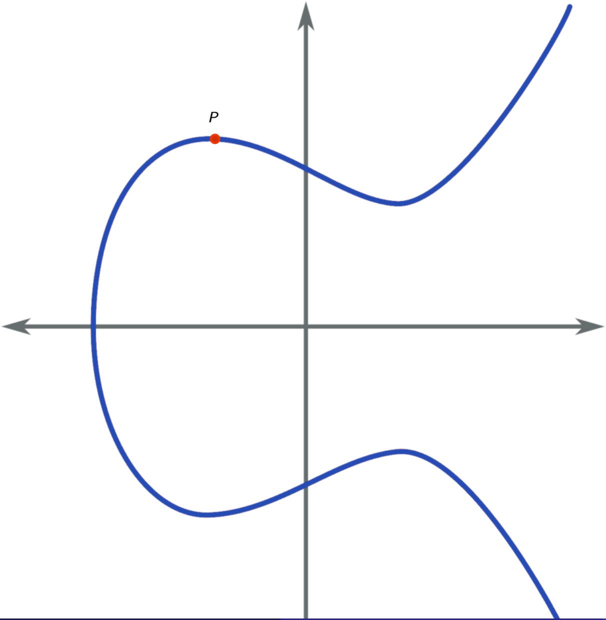

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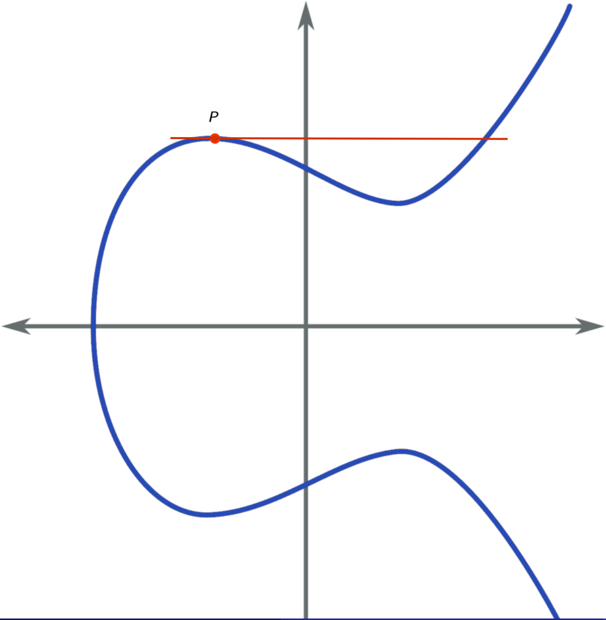

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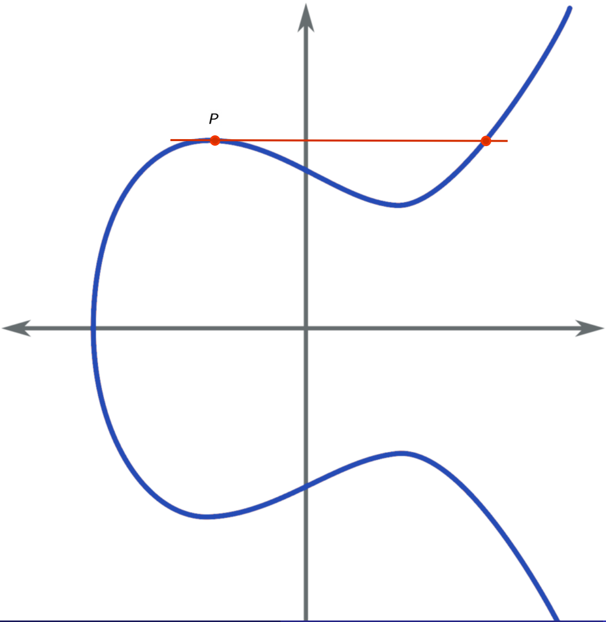

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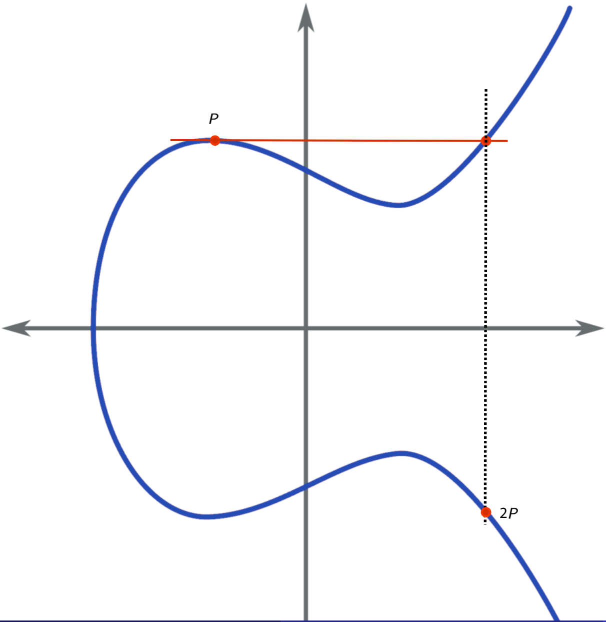

The Derivative

You might have learned at school :

- The derivative of $x^n = n \cdot x^{n-1}$
- The derivative of $\cos(x) = -\sin(x)$

How is that determined/calculated?

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How is that determined/calculated? With limits from calculus ...

Derivative of a function f in point x

If both

Left derivative $\displaystyle\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \mathrm{L}$

Right derivative $\displaystyle\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \mathrm{R}$

exist and $L = R \neq \pm\infty$ then the derivative of f in x exists:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

This is the **slope of the tangent line** in $(x, f(x))$.

If this limit is a number, then the function f is called differentiable in x

Derivative of function f in point x

Derivative f' indicates: f increasing/rising, decreasing/falling, or horizontal.

If $f'(x) > 0$ then the function rises at x
If $f'(x) < 0$ then the function falls at x
If $f'(x) = 0$ then the function f has a maximum, minimum or inflection point at x
If $f''(x) = 0$ and f'' changes sign at x then f has an inflection point at x

Convention: If the derivative is $\pm\infty$, it does not exist.

For limits, we learned that they CAN exist if they are $\pm\infty$ ....
A convention must not be logical. So, just learn it.

Derivative: Notation

Different notation for the derivative f' is:

$$f' = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f$$

If we call the function y, it's the same (eg $y(x) = x^2$):
Different notation for y' is:

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} y$$

Derivative: Examples

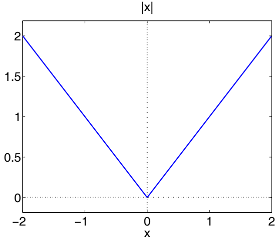
Ex: Function $f(x) = x^2$:

- at $x = 0$: $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 - 0}{h} = \lim_{h\to 0} h = 0$
- at $x = 1$:
  $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = 2$
- For general x:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = 2x \end{aligned}$$

Derivative: Example $f(x) = |x|$

If $x > 0$: $f(x) = \quad x$, thus $f'(x) = \quad 1$
If $x < 0$: $f(x) = -x$, thus $f'(x) = -1$
If $x = 0$: Left and right derivative:



- $$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|0+h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 \ (h > 0!)$$
- $$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|0+h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \ (h < 0!)$$

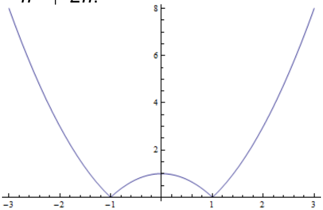
are different thus $\displaystyle\lim_{h\to 0} \frac{|0+h|-|0|}{h}$ does not exist

Conclusion: $f(x) = |x|$ is not differentiable in $x = 0$

Equivalent: There is no clear tangent to the graph at $(0, 0)$

$f(x) = \left|x^2 - 1\right|/x$ for $x = 1$

Right derivative: $h \to 0^+ \implies h > 0 \implies \left|(1+h)^2 - 1\right| = h^2 + 2h.$

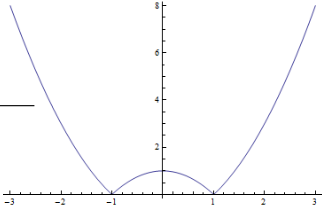


$$
\begin{aligned}
\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} &\overset{x=1}{=} \lim_{h\to 0^+} \left(\frac{\left|(1+h)^2 - 1\right|}{1} - \underbrace{\frac{\left|1^2 - 1\right|}{1}}_{=0} \right) / h \\
&= \lim_{h\to 0^+} \frac{h^2 + 2h}{h} \\
&= \lim_{h\to 0^+} h + 2 \\
&= 2 =: R
\end{aligned}
$$

$f(x) = \left|x^2 - 1\right|/x$ for $x = 1$

$f(x)$ for $x \to 1^-$: $(1+h)^2 - 1 = h^2 + 2h$ for $h < 0$:

$h \to 0^-$	$h^2 + 2h$
-3	$9 - 6 = 3$
-2	$4 - 4 = 0$
-1	$1 - 2 = -1$
-0.1	$0.01 - 0.2 = -0.19$
-0.01	$0.0001 - 0.02 = -0.0199$
-0.001	$0.000001 - 0.002 = -0.001999$
-0.0001	$0.00000001 - 0.0001 = -0.00019999$

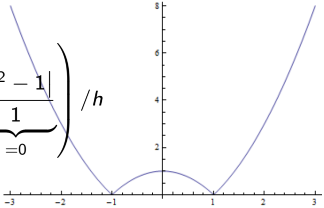


thus $(1+h)^2 - 1 < 0$ for $h \to^- 0$ and with that
$\left|(1+h)^2 - 1\right| = -(h^2 + 2h)$ for $h \to 0^-$

$f(x) = \left|x^2 - 1\right|/x$ for $x = 1$

Left derivative: $\left|(1+h)^2 - 1\right| = -(h^2 + 2h)$ for $h \to 0^-$:

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} &\overset{x=1}{=} \lim_{h\to 0^-} \left(\frac{\left|(1+h)^2 - 1\right|}{1} - \underbrace{\frac{\left|1^2 - 1\right|}{1}}_{=0} \right) / h \\
&= \lim_{h\to 0^-} \frac{-h^2 - 2h}{h} \\
&= \lim_{h\to 0^-} -(h+2) \\
&= -2 =: L
\end{aligned}$$



Left and right derivatives are different: $-2 = L \neq R = 2$ thus the derivative does not exist.

Derivatives: Differentiable on $[a, b]$

For endpoints a and b only a right or left derivative can exist:

Left derivative $\quad f'(b) = \lim_{h \to 0^-} \dfrac{f(b+h) - f(b)}{h}$

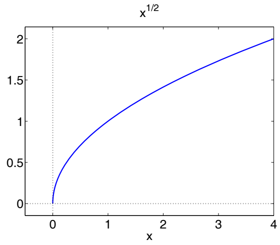
Right derivative $\quad f'(a) = \lim_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$

For all other points in $[a, b]$ it must hold that for the left derivative L and right derivative R exist and that $L = R \neq \pm\infty$.

Ex: $\sqrt{x}$: domain $[0, \infty)$.
Thus for $x = 0$ only $\lim\limits_{h \to 0^+}$ needs to exist:
$\sqrt{x}$ is not right differentiable at $x = 0$ since

$$\lim_{h \to 0^+} \frac{\sqrt{h} - \sqrt{0}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty$$

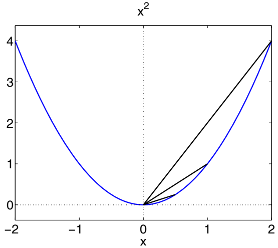
This is a vertical tangent.



Tangents to graph of a function

The tangent to the graph of f at point $x = a$

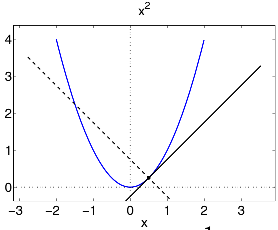
1. includes the point of the graph, thus $(a, f(a))$
2. has direction coefficient $f'(a)$



and has the equation: $y - f(a) = f'(a)(x - a)$.
Thus $y = f(a) + f'(a)(x - a)$ – see Taylor series later!

Ex: Function $f(x) = x^2$.
Tangent to the graph at $a = 0$ is x-axis:
$f(x) = x^2$, $a = 0 \Longrightarrow f(a) = a^2 = 0$
$f'(x) = 2x$, $a = 0 \Longrightarrow f'(a) = 0$.
Thus tangent: $y - 0 = 0 \cdot (x - 0)$

Normal in a point
= line orthogonal to the graph in a point
= line orthogonal to the tangent in that point



Tangent in $x = \frac{1}{2}$ has slope $m = 1$, goes through $(\frac{1}{2}, \frac{1}{4})$,
this it has the equation $\quad y = x - \frac{1}{4}$

Normal has slope $m_2 = -1$, also goes through $(\frac{1}{2}, \frac{1}{4})$,
thus it has the equation $\quad y = -x + \frac{3}{4}$

Quiz: Derivatives with Definition

Determine:

- $f'(0)$ and $f'(7)$ for $f(x) = 5 + x^2$
- $\frac{dy}{dx}$ if $y = -8x^3 + 6x$
- $\frac{dy}{dx}$ if $y = \frac{5}{\sqrt{x+7}}$
- Derivative of $f(x) = 4x + \frac{5}{x}$ and slope of the tangent at $x = 1$
- Derivative of $f(x) = 4 + \sqrt{3 - x}$ and equation for the tangent at $(x, y) = (-6, 7)$

The derivative with help of the definitions

Just seen: Lots of work. Often, we can do that easier: Recipe:

1. Learn a number of derivatives: eg $sin'(x) = cos(x)$, $ln'(x) = 1/x$, etc
2. Use calculation rules: $(f+g)' = f' + g'$, $(f \cdot g)' = f' \cdot g + f \cdot g'$, etc

Attention: You must be able to calculate the derivative of a (simple?) function using the definition! That could happen in an exam!

Learn these derivatives!

$f(x)$	$f'(x)$
c	0
x^r	$r\,x^{r-1}$
a^x	$a^x \ln(a)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$1+\tan^2(x) = \frac{1}{\cos^2(x)}$
$f(x)^{g(x)}$	via $e^{g(x)\ln(f(x))}$

Differentiation rules

f' via definition is a lot of work $\Longrightarrow$ therefore rules, eg $(x^r)' = rx^{r-1}$

$$
\begin{aligned}
(f+g)'(x) &= f'(x) + g'(x) \\
(f-g)'(x) &= f'(x) - g'(x) \\
(fg)'(x) &= f'(x)g(x) + f(x)g'(x) \\
\left(\frac{f}{g}\right)'(x) &= \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2} \\
(cg)'(x) &= 0 \cdot g(x) + cg'(x) = cg'(x) \\
\left(\frac{1}{g}\right)'(x) &= \frac{0\cdot g(x)-1\cdot g'(x)}{(g(x))^2} = -\frac{g'(x)}{(g(x))^2}
\end{aligned}
$$

Proof: with the definition ..., eg:

$$
\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \frac{(f(x+h) - f(x))g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

Derivatives of sin, cos, tan

We have already calculated the derivative of sine.
The one of the cosine works just the same way.
For the one of the tan function:

$$\begin{aligned}
\tan'(x) &= \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = 1 + \tan^2(x)
\end{aligned}$$

Chain Rule: Examples

Chain rule for $(f \circ g)(x) = f(g(x))$:

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Ex: $y = (7x - 3)^{10} = f(g(x))$, with

Chain Rule: Examples

Chain rule for $(f \circ g)(x) = f(g(x))$:

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Ex: $y = (7x-3)^{10} = f(g(x))$, with $f(x) = x^{10}$ and $g(x) = 7x - 3$

$f'(x) = 10x^9, \;\; g'(x) = 7$
$y' = f'(g(x)) \cdot g'(x) = 10\,(7x-3)^9 \cdot 7 = 70\,(7x-3)^9$

Ex: $\frac{d}{dx}\left(\frac{1}{f}\right)(x) = \frac{d}{dx}(f(x))^{-1} = (-1) \cdot (f(x))^{-2} \cdot f'(x) = -\frac{f'(x)}{(f(x))^2}$
Also seen already with quotient rule!

Ex: Determine $f'(x)$ if $f(x) = (x^2 + 1)^2$

Here we can first rewrite f: $f(x) = x^4 + 2x^2 + 1$

and then calculate the derivative: $f(x) = 4x^3 + 4x$

or with chain rule: $f'(x) = 2(x^2 + 1) \cdot 2x = 4x(x^2 + 1)$ same!

(because $f(x) = g(h(x))$ with $g(x) = x^2$ and $h(x) = x^2 + 1$)

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Ex: But now: determine $f'(x)$ if $f(x) = (x^2+1)^{40}$

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Ex: But now: determine $f'(x)$ if $f(x) = (x^2+1)^{40}$

Rewriting f first is not very nice!

But with chain rule:

$f'(x) = 40(x^2+1)^{39} \cdot 2x = 80x(x^2+1)^{39}$ very quick!

(because $f(x) = g(h(x))$ with $g(x) = x^{40}$ and $h(x) = x^2 + 1$)

Quiz: Sum, product, quotient and chain rule

Calculate $y'(x)$ for:

- $y = \left(-4 - \frac{5x}{7}\right)^{-7}$
- $y = \sqrt{-4 + 3x}$
- $y = \sqrt{-9 + \sqrt{3x}}$
- $y = \frac{\sin(x)}{1+\sin(x)}$
- $y = \cos(\sin(x))$
- $y = x\,(7x^2 + 1)^{1/2}$
- $y = \sqrt{\dfrac{1}{3x^5}}$
- $y = (5x + 2)^4(2x + 5)^{-3}$
- $y = \sin^2(3x - \pi)$
- $y = \sin\left((8x - 3)^{-9/2}\right)$
- Let $f(x) = x^2 - 1$ and $g(x) = x^2 + 1$
  Calculate $(f \circ g)'$ in 2 ways:
  (a) "usual" way: first $(f \circ g)(x)$ and then differentiate
  (b) with the chain rule $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- Tangent and normal to $y = \sqrt{x}$ in $(4, 2)$
- Slope and equation for the tangent to $y = 2 - 9x^2$ in $(3, -79)$
- Slope and equation for the tangent to $y = 6\sqrt{x}$ in $(1, 6)$

- Limits $\lim_{x\to a} f(x)$
- Limits $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to a} f(x) = \pm\infty$ (horizontal/vertical asymptotes)
- Definition continuous (without holes and jumps)
- Intermediate-Value Theorem
- Tangent to $(a, f(a))$: $y - f(a) = f'(a)(x - a)$
- Definition derivative $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$
- Calculation rules: derivative/chain rule $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- Derivatives of trigonometric functions
  $\tan'(x) = \dfrac{1}{\cos^2(x)} = 1 + \tan^2(x)$

Week 4: Differentiation II/Inverse functions

- Mean Value Theorem
- Higher order derivatives
- "Implicit" differentiation
- Extreme values (maxima/minima/inflection points)
- Extreme value problems
- Inverse functions
- Inverse trigonometric functions

Logarithm

We know that 2 raised to the 4th power equals 16. This is expressed by the exponential equation $2^4 = 16$.
If we want to know "2 raised to which power equals 16?". This is expressed by the logarithmic equation $\log_2(16) = 4$.

$$2^4 = 16 \Leftrightarrow log_2(16) = 4$$

The logarithm is defined as (with $a > 0$ and $a \neq 1$):

$$a^L = x \Leftrightarrow L = \log_a x$$

$\log_{10} x$ is commonly used in scientific texts, written as $\log(x)$.
$\log_e x$ is called the natural logarithm and is usually written as $\ln x$.

Logarithm

${}^a\log(x)$, with domain $(0,\infty)$ and range $(-\infty,\infty)$

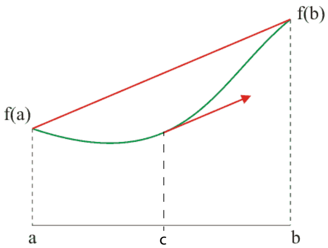
$$\begin{aligned}
{}^a\log(xy) &= {}^a\log(x) + {}^a\log(y)\\
{}^a\log(\tfrac{x}{y}) &= {}^a\log(x) - {}^a\log(y)\\
{}^a\log(x^r) &= r \cdot {}^a\log(x)\\
{}^a\log(\tfrac{1}{x}) &= -\, {}^a\log(x)\\
{}^a\log(1) &= 0\\
{}^a\log(x) &= {}^a\log(b) \cdot {}^b\log(x)
\end{aligned}$$

$$\begin{aligned}
\ln(xy) &= \ln(x) + \ln(y)\\
\ln(\tfrac{x}{y}) &= \ln(x) - \ln(y)\\
\ln(x^r) &= r \cdot \ln(x)\\
\ln(\tfrac{1}{x}) &= -\ln(x)\\
\ln(1) &= 0
\end{aligned}$$

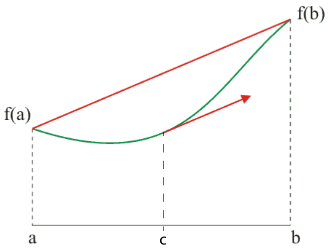
Notation in book: $\log_a(x)$ instead of ${}^a\log(x)$

More next week!

Mean Value Theorem



Mean Value Theorem

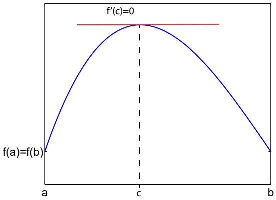


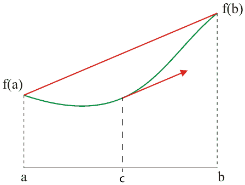
If f is differentiable on $[a, b]$ (this implies that f is continuous on $[a, b]$) then there is *at least* one c between a and b with

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

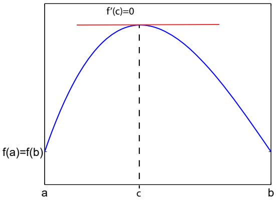
In words: There is one point c at which the slope is equal to the mean slope between a and b

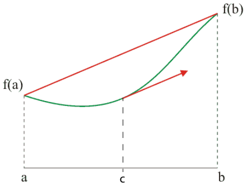
Mean Value Theorem: Rolle's Theorem





Mean Value Theorem: Rolle's Theorem





If f is differentiable on $[a, b]$ (this implies that f is continuous on $[a, b]$) then there is *at least* one c between a and b with

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Thus: If $f(a) = f(b)$ then there is one $c \in (a, b)$ with

$$0 = \frac{0}{b - a} = \frac{f(b) - f(a)}{b - a} = f'(c)$$

derivative 0, thus one extremum (maximum or minimum). This statement (existence of c with $f'(c) = 0$) is Rolle's Theorem.

Mean Value Theorem: Application

Show that $\sin(x) < x$ for all $0 < x < 2\pi$

With Mean-Value Theorem: Let $0 < x < 2\pi$. Chose $a = 0$, $b = x$, and $f = \sin$.

Now enter this into the MVT:

$$\frac{f(b) - f(a)}{b - a} = \frac{\sin(x) - \sin(0)}{x - 0} = \cos(c)$$

This holds for a c with $0 < c < x$. We also know that $\cos(c) < 1$, so that:

$$\frac{\sin(x)}{x} = cos(c) < 1 \Rightarrow \sin(x) < x$$

Mean Value Theorem: Application Examples

1. Show that $\tan(x) > x$ for all $0 < x < \frac{\pi}{2}$
2. Show that $e^x \leq 1$ for all $x \in \mathbb{R}$
3. Show for all $x > 0$ that:

$$\frac{1}{x+1} < \ln(x+1) - \ln(x) < \frac{1}{x}.$$

Higher order derivatives: Second derivative

Ex: Take the derivative of the derivative:

$$(\sin(x))' = \cos(x), \quad (\cos(x))' = -\sin(x) \implies (\sin(x))'' = -\sin(x)$$

Notation: For function with name f – fully with argument x:

$$\boxed{(f')'(x) = f''(x) = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = \frac{d^2 f}{dx^2}(x)}$$

Notation: For function with name y – shorthand without x:

$$\boxed{(y')' = y'' = \frac{d}{dx}\frac{d}{dx}y = \frac{d^2}{dx^2}y = \frac{d^2 y}{dx^2}}$$

One can repet $\boxed{\text{Notation: } f''' = f^{(3)}, \quad f'''' = f^{(4)}, \quad \ldots, \quad f = f^{(0)}}$

Ex: $(fg)' = f'g + fg'$ thus
$(fg)'' = (f'g + fg')' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$

Don't try to remember this! Calculate $(fg)''$ by taking the derivative of fg two times

Examples Higher order derivatives

Determine the second derivative of:

- $y = \cos(x^2)$

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Determine the second derivative of:

- $y = \cos(x^2)$

  $y' \underset{(f \circ g)'}{=} -\sin(x^2) \cdot 2x = -2x\sin(x^2)$

Examples Higher order derivatives

Determine the second derivative of:

- $y = \cos(x^2)$

  $y' \underset{(f\circ g)'}{=} -\sin(x^2)\cdot 2x = -2x\sin(x^2)$

  $y'' \underset{(f\cdot g)'}{=} -2\sin(x^2) - 2x\cos(x^2)\cdot 2x = -2\sin(x^2) - 4x^2\cos(x^2)$

- $y = (3-2x)^7$

Examples Higher order derivatives

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- $y = (3 - 2x)^7$

  $y' = 7(3 - 2x)^6 \cdot (-2) = -14(3 - 2x)^6$

Examples Higher order derivatives

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- $y = \cos(x^2)$

  $y' \underset{(f\circ g)'}{=} -\sin(x^2)\cdot 2x = -2x\sin(x^2)$

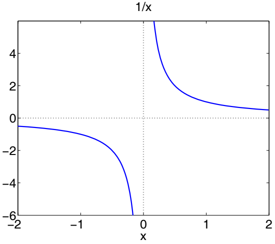
  $y'' \underset{(f\cdot g)'}{=} -2\sin(x^2) - 2x\cos(x^2)\cdot 2x = -2\sin(x^2) - 4x^2\cos(x^2)$

- $y = (3-2x)^7$

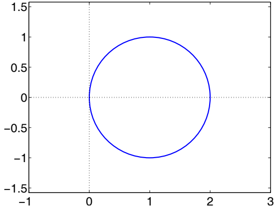
  $y' = 7(3-2x)^6\cdot(-2) = -14(3-2x)^6$

  $y'' = -14\cdot 6(3-2x)^5\cdot(-2) = 168(3-2x)^5$

A function is of the form $y = f(x)$.
A curve is of the form $F(x, y) = 0$.



Function $y = 1/x$



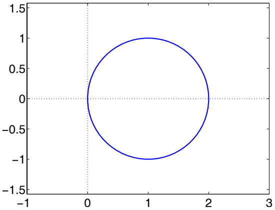
Curve $(x-1)^2 + y^2 = 1$

Remember:

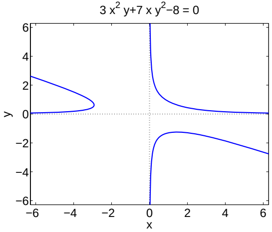
- Every function is a curve (with just one y-value per input x)
- Curves describe more phenomena than functions
- Curves also have tangents in (x, y) on their graphs

Aim of implicit differentiation: determine tangents to curves.

Curves have tangents for nearly all points (x, y) on their graphs:



$(x-1)^2 + y^2 = 1$

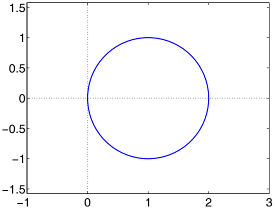


$3x^2y + 7xy^2 = 8$

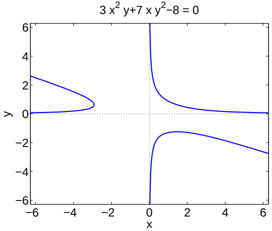
- $(x-1)^2 + y^2 = 1$ has an tangent for all points on its graph (also in $(0,0)$ and $(2,0)$ on the graph where $\dfrac{dy}{dx}$ does not exist)
- $3x^2y + 7xy^2 = 8$ has a tangent for all points on its graph ($x = 0$ and $y = 0$ are asymptotes but no tangents because points with $x = 0$ or $y = 0$ do not lie on the graph of the curve

For nearly all (x, y) on the graph $F(x, y) = 0$, we realize that $y = y(x)$ but often

- the **explicit form** is valid only close to (x, y)
- the **explicit form** cannot be calculated at all



$(x-1)^2 + y^2 = 1$

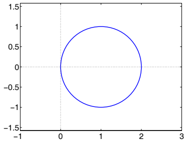


$3x^2y + 7xy^2 = 8$

Without knowing $y(x)$ explicitly, how can we determine the tangent to a curve? $\Rightarrow$ IMPLICIT differentiation!

Tangent to curve: Example

We know that around $(1,1)$ on the graph of $(x-1)^2+y^2=1$, $x\in[0,2]$: $y=y(x)$. Determine the tangent to $(1,1)$:



$(x-1)^2+y^2=1$

Explicit differentiation: Determine equation for $y=y(x)$ and then y':

$$y(x) = +\sqrt{1-(x-1)^2} \Longrightarrow y'(x) = -\frac{x-1}{\sqrt{1-(x-1)^2}}$$

This the tangent to $(1,1)$: $y-1=y'(1)(x-1) \Longrightarrow y=1.$

Tangent to curve: Example

Implicit differentiation:

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \iff (x-1)^2 + y^2(x) = 1$$

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \Longleftrightarrow (x-1)^2 + y^2(x) = 1$$

Attention: Left and right sides are functions in x: differentiate on left and right:

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \iff (x-1)^2 + y^2(x) = 1$$

Attention: Left and right sides are functions in x: differentiate on left and right:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x-1)^2 + y^2\right) \quad = \quad \frac{\mathrm{d}}{\mathrm{d}x}1 = 0 \underset{(f+g)'=f'+g'}{\Longrightarrow}$$

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \Longleftrightarrow (x-1)^2 + y^2(x) = 1$$

Attention: Left and right sides are functions in x: differentiate on left and right:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((x-1)^2 + y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}1 = 0 \underset{(f+g)'=f'+g'}{\Longrightarrow} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left((x-1)^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) &= 0 \underset{\text{chain rule}}{\Longrightarrow}
\end{aligned}$$

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

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$$2\cdot(x-1) + 2\cdot y\cdot y' = 0 \Longrightarrow 2y\cdot y' = 2(1-x)$$

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \Longleftrightarrow (x-1)^2 + y^2(x) = 1$$

Attention: Left and right sides are functions in x: differentiate on left and right:

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$$2\cdot(x-1) + 2\cdot y\cdot y' = 0 \Longrightarrow 2y\cdot y' = 2(1-x)$$

for $y \neq 0$

$$y' = \frac{2(1-x)}{2y}.$$

now to $(1,1)$, enter into y' thus:

Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \Longleftrightarrow (x-1)^2 + y^2(x) = 1$$

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Tangent to curve: Example

Implicit differentiation: Determine y' without knowing $y(x)$:

$$(x-1)^2 + y^2 = 1 \Longleftrightarrow (x-1)^2 + y^2(x) = 1$$

Attention: Left and right sides are functions in x: differentiate on left and right:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((x-1)^2 + y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}1 = 0 \underset{(f+g)'=f'+g'}{\Longrightarrow} \\
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\end{aligned}$$

$$2\cdot(x-1) + 2\cdot y\cdot y' = 0 \Longrightarrow 2y\cdot y' = 2(1-x)$$

for $y \neq 0$

$$y' = \frac{2(1-x)}{2y}.$$

now to $(1, 1)$, enter into y' thus: $y' = \frac{2(1-1)}{2\cdot 1} = 0$

Tangent to $(0, 0)$: $2\cdot 0\cdot y' = 2(0-1)$ so it cannot be determined like this.

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2		

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
y^2		

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
y^2	$2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2yy'$	chain
$x \cdot y$		

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
y^2	$2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2yy'$	chain
$x \cdot y$	$1 \cdot y + x \cdot y' = y + xy'$	product
xy^2		

Implicit differentiation: examples

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x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
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$x \cdot y$	$1 \cdot y + x \cdot y' = y + xy'$	product
xy^2	$1 \cdot y^2 + x \cdot (2yy') = y^2 + 2xyy'$	product/chain
$\sin(y)$		

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
y^2	$2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2yy'$	chain
$x \cdot y$	$1 \cdot y + x \cdot y' = y + xy'$	product
xy^2	$1 \cdot y^2 + x \cdot (2yy') = y^2 + 2xyy'$	product/chain
$\sin(y)$	$\cos(y) \cdot y'$	chain
$\sin(y^2)$		

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
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$\sin(y)$	$\cos(y) \cdot y'$	chain
$\sin(y^2)$	$\cos(y^2) \cdot 2y \cdot y'$	2 x chain
$x + y^2$		

Implicit differentiation: examples

Expression	Derivative to x	Rule
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xy^2	$1 \cdot y^2 + x \cdot (2yy') = y^2 + 2xyy'$	product/chain
$\sin(y)$	$\cos(y) \cdot y'$	chain
$\sin(y^2)$	$\cos(y^2) \cdot 2y \cdot y'$	2 x chain
$x + y^2$	$1 + 2yy'$	sum
$(x+y)^2$		

Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\dfrac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
y^2	$2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2yy'$	chain
$x \cdot y$	$1 \cdot y + x \cdot y' = y + xy'$	product
xy^2	$1 \cdot y^2 + x \cdot (2yy') = y^2 + 2xyy'$	product/chain
$\sin(y)$	$\cos(y) \cdot y'$	chain
$\sin(y^2)$	$\cos(y^2) \cdot 2y \cdot y'$	2 x chain
$x + y^2$	$1 + 2yy'$	sum
$(x+y)^2$	$2(1 + y')$	sum/chain
x^3y^2		

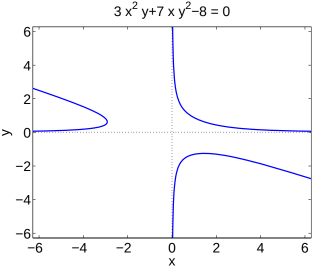
Implicit differentiation: examples

Expression	Derivative to x	Rule
y	y'	
x^2	$2x\frac{\mathrm{d}x}{\mathrm{d}x} = 2x \cdot 1 = 2x$	chain
y^2	$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2yy'$	chain
$x \cdot y$	$1 \cdot y + x \cdot y' = y + xy'$	product
xy^2	$1 \cdot y^2 + x \cdot (2yy') = y^2 + 2xyy'$	product/chain
$\sin(y)$	$\cos(y) \cdot y'$	chain
$\sin(y^2)$	$\cos(y^2) \cdot 2y \cdot y'$	2 x chain
$x + y^2$	$1 + 2yy'$	sum
$(x + y)^2$	$2(1 + y')$	sum/chain
x^3y^2	$3x^2y^2 + 2x^3yy'$	product/chain

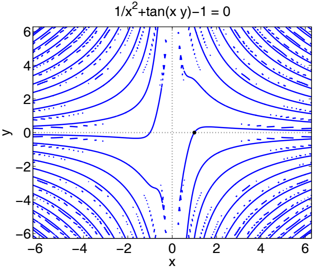
Here we consider $y = y(x)$ always as a function of x.

Implicit differentiation: More examples

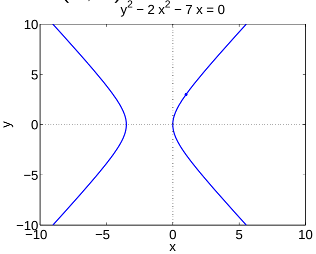
Ex: Determine $y' = \frac{dy}{dx}$ for the curve $3x^2y + 7xy^2 = 8$



Ex: Determine the equation of the tangent in the point $(1, 0)$ to the curve $\frac{1}{x^2} + \tan(xy) = 1$ (exam 14.11.07)



Determine y' and y'' for $y^2 = 2x^2 + 7x$ in the point $(1, 3)$



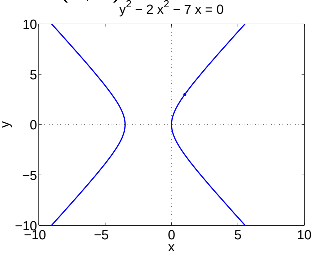
Implicit differentiation to determine y':

$$y^2 = 2x^2 + 7x \Longleftrightarrow (y^2)' = (2x^2 + 7x)' \Longleftrightarrow 2yy' = 4x + 7$$

and once more to determine y'':

Implicit differentiaion and higher order derivatives

Determine y' and y'' for $y^2 = 2x^2 + 7x$ in the point $(1,3)$



Implicit differentiation to determine y':

$$y^2 = 2x^2 + 7x \iff (y^2)' = (2x^2+7x)' \iff 2yy' = 4x+7$$

and once more to determine y'':

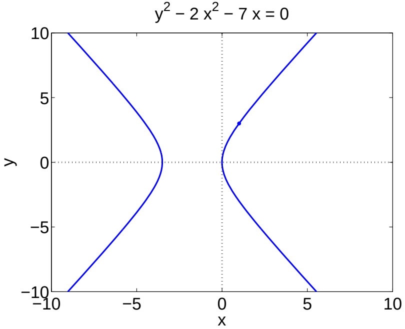
$$2yy' = 4x + 7 \iff (2yy')' = (4x+7)' \iff 2y'y' + 2yy'' = 4 \iff (y')^2 + yy'' = 2$$

Thus yields $y'(1) = \frac{11}{6}$ and $y'' = \frac{2-\left(\frac{11}{6}\right)^2}{3} = -\frac{49}{108}$

Reminder: Determine y'' for $y^2 = 2x^2 + 7x$ in the point $(1, 3)$

Here: $y = \sqrt{2x^2 + 7x}$ not $-\sqrt{2x^2 + 7x}$, we take the "positive sign" because of $(1, 3)$)

and differentiate this 2 times



We can:

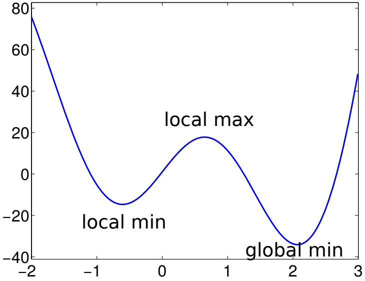
- Determine derivative of composite functions $f \circ g$ without first calculating the function itself (with help of chain rule)
  (seen before)
- Determine the derivative of implicitly defined functions without first calculating the function itself
  (in an explicit form $y = \cdots$)
  (seen today)
- Determine the derivative of an inverse function without first calculating the inverse function itself (in the form $x = f^{-1}(y)$)
  (to be seen on Wednesday)

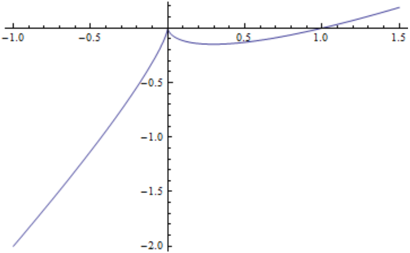
Exercises on Implicit Differentiation

- Determine y' for $3y^2 = \frac{2x-5}{2x+5}$, $x = \cos(y)$, $x^{-3/4} + y^{-3/4} = -5$
- Determine the tangent to $y = 4\sin(\pi x + y)$ in $(-1, 0)$
- What is the slope of the curve $2y^9 + 9x^4 = 7y + 4x$ in $(1, 1)$
- Same for $x^3 + y^3 = 63$ in point $(-1, 4)$
- What is the tangent and normal to $x^2y^2 = 100$ in $(5, -2)$?
- What is the slope of $y^4 = y^2 - x^2$ in $(\frac{\sqrt{3}}{4}, \frac{1}{2})$
- Determine an equation for the tangent in the point $(3, -1)$ to the curve $x^2y^3 + 2x^3y^2 - 8x + y = 20$
- Same for the point $(2, 0)$ to the curve $\sin(x^2y^2) + x^3y = x^2 - 4$
- Same for the point $(2, 0)$ to the curve $\arctan(xy) + y^2 = x^2 - 4$

Extreme values: Optimization

Find local and global min/max of function f on interval D:

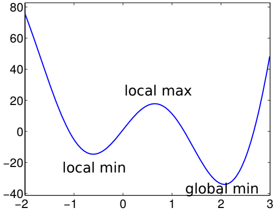




- f has absolute/global min at a if $f(a) \leq f(x)$ for all x
- f has absolute/global max at a if $f(a) \geq f(x)$ for all x
- f has local min at a if $f(a) \leq f(x)$ $\forall x$ near a
- f has local max at a if $f(a) \geq f(x)$ $\forall x$ near a

Extreme values: Optimization

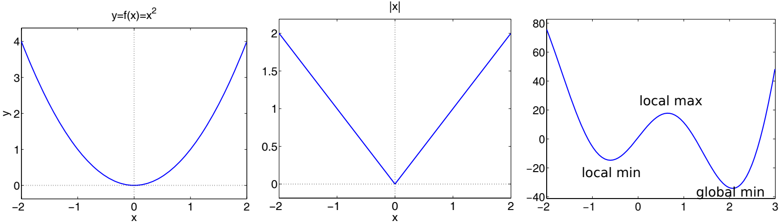
Find local and global min/max of function f on interval D:



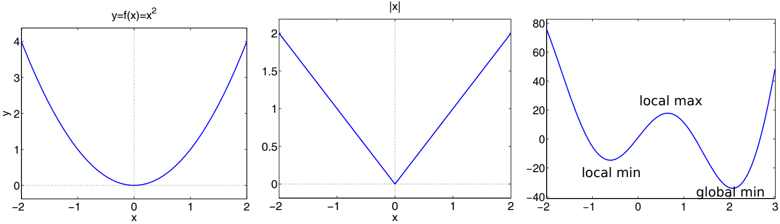
Ex: Function f on interval $[-2, 3]$ has:

- Global max at -2 with value $f(-2)$ (-2 fill in, is end point)
- Local max at 3 with value $f(3)$ (3 fill in, is end point)
- Local min between -1 and 0
- Local max between 0 and 1
- Global min between 1 and 3

Find local/global min/max of function f over interval D:



Find local/global min/max of function f over interval D:



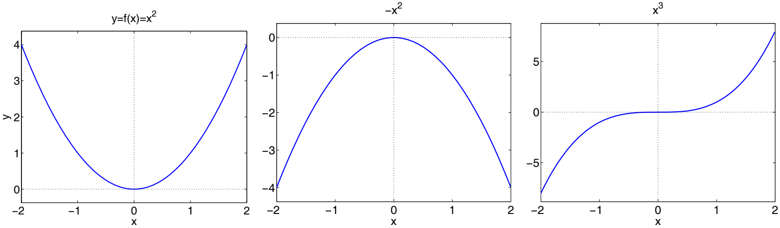
You calculate all candidates with the recipe:

1. Points x where $f'(x) = 0$ (critical points)
2. Points x where f' does not exist (singular points)
3. End points of D ($x = a, b$ if $I = [a, b]$; $x = a$ if $I = [a, b)$)

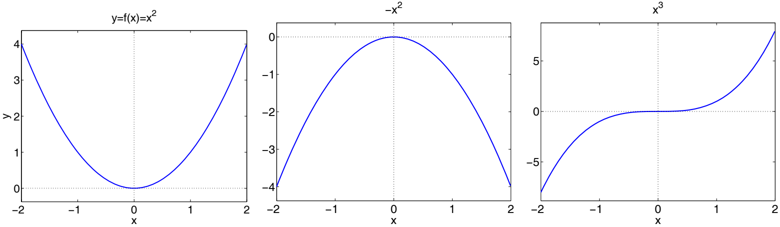
Determine local/global min/max of $f(x)$ on D

1. Solve $f'(x) = 0$ and point at which $f'(x)$ does not exist
2. Make a sign sketch for f' and with that
3. decide: is a local max/min at critical point? ($f'(x) = 0$)
4. decide: is a local max/min at a singular point? (f' does not exist)
5. calculate: f for the end points of D that contribute
6. calculate the function values in all these points
7. Check again if the points $x \in D$!
8. Compare the local max/min with each other
   to determine the global min/max
9. Evt: Calculate inflection points with $f''(x) = 0$ etc. Later

Min/max/stationary points: $f'(x) = 0$



Min/max/stationary points: $f'(x) = 0$

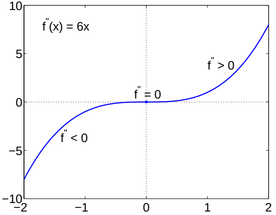


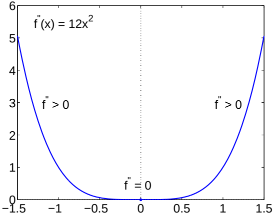
This is possible: $\boxed{\text{min, max, or just } \textit{stationary}}$

Type	$f(x)$	$f'(x)$	$f'(-1)$	$f'(0)$	$f'(1)$
min	x^2	$2x$	$-$	0	$+$
max	$-x^2$	$-2x$	$+$	0	$-$
stat	x^3	$3x^2$	$+$	0	$+$
stat	$-x^3$	$-3x^2$	$-$	0	$-$

Inflection points: Points with $f''(x) = 0$

$f''(x) = 0$ is an inflection point if f'' changes sign. Else not!





Type	$f(x)$	$f''(x)$	x	$f''(-1)$	$f''(0)$	$f''(1)$
infl	x^3	$6x$	0	$-$	0	$+$
nope	x^4	$12x$	0	$+$	0	$+$

Existence of maximum / minimum

$\boxed{\text{Continuous } f \text{ on limited and closed interval has min and max}}$

Functions do not have to have a max/min on not-closed/not-limited intervals:

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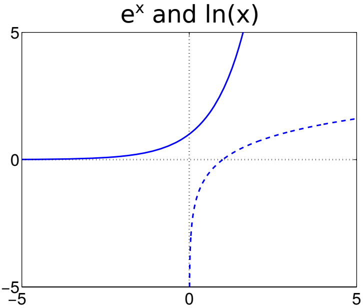

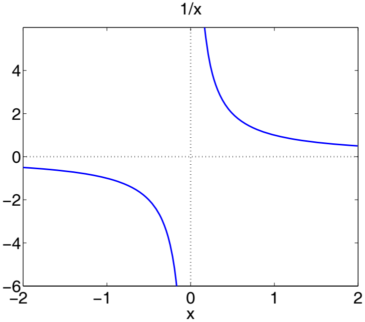

- e^x has no max on interval $D = [0, \infty)$

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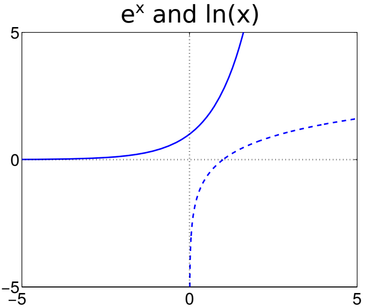

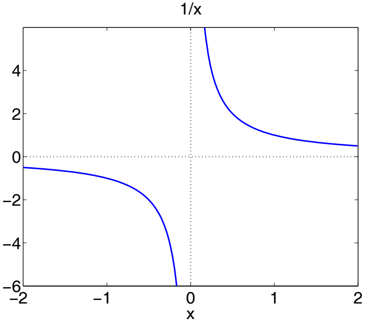

- e^x has no max on interval $D = [0, \infty)$ (D not limited)
- $\frac{1}{x}$ has no max on interval $D = (0, 1]$

Continuous f on limited and closed interval has min and max

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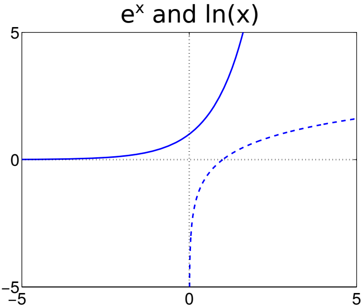
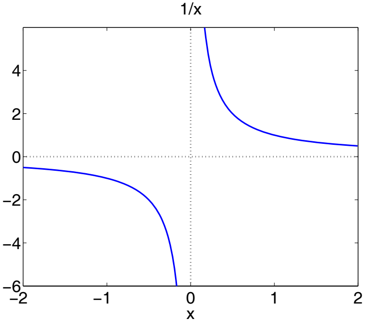
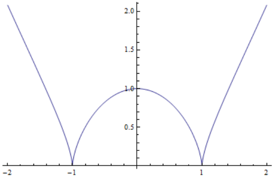




- e^x has no max on interval $D = [0, \infty)$ (D not limited)
- $\frac{1}{x}$ has no max on interval $D = (0, 1]$ (D not closed)

Extreme values: example

Find loc/glob min/max of $f(x) = (x^2 - 1)^{2/3}$ on interval $I = [a, b) = [-2, 2)$:



- End points: $f(a) = f(-2) = 3^{2/3}$. End point b does not work!
- Local: $f'(x) = \frac{4}{3}\frac{x}{(x^2-1)^{1/3}}$ max at 0 with $f(0) = 1$
- Local min: Where $f'(x)$ n.b.: $x = \pm 1$ with value $f(\pm 1) = 0$

Thus global max at $x = -2$ because $3^{2/3} > 1$. Local max at $x = 0$ and double global min at $x = \pm 1$

Exercises: Extreme values

Find all extrema for

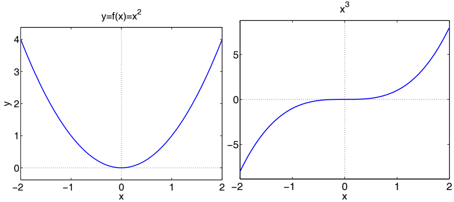
- $x^{8/3} - 8x^{2/3}$ on $[-3, 4]$
- $f(x) = e^{x^3+x^2-x-1}$ on $(-\infty, \frac{5}{4}]$
- $\frac{1}{\sin(x)}$ on $[\frac{1}{4}\pi, \frac{5}{6}\pi]$
- x^2e^{-x} on $\mathbb{R}$
- $\cos(x) + x\sin(x)$ on $[-2\pi, 2\pi]$
- $f(x) = e^{2\cos^2(x)}\sin(x)$ on $[0, \pi]$

1-to-1 (injective) functions

A function f assignes to each x ($\in D_f$) precisely one y
Does this also hold the other way around?

1-to-1 (injective) functions

A function f assignes to each x ($\in D_f$) precisely one y
Does this also hold the other way around? Not always:



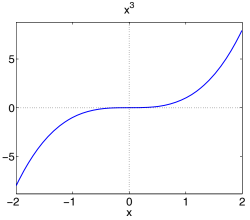
Functions for which this is the case are called injective / 1-to-1 / one-to-one: $f(x_1) = f(x_2) \implies x_1 = x_2$

Ex: $f(x) = x^3$ is indeed 1-to-1 on $D = \mathbb{R}$ because
$f(x) = f(y) \iff x^3 = y^3 \iff x = y$

Ex: $f(x) = x^2$ is not 1-to-1 on $D = \mathbb{R}$ because
$f(x) = f(y) \iff x^2 = y^2 \iff x = \pm y$

Ex: $f(x) = x^2$ is indeed 1-to-1 on $D = [0, \infty)$ because
$f(x) = f(y) \iff x^2 = y^2 \iff x = y$
because $x, y \in D_f = [0, \infty)$ is valid in $x, y > 0$

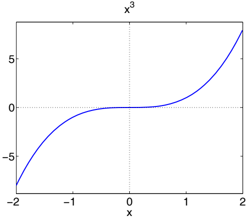
Increasing and decreasing functions



Function f is a increasing/decreasing function on domain $[a, b]$ if and only if $f'(x) > 0$/$f'(x) < 0$ for all $x \in (a, b)$, except for a finite number of points where $f'(x) = 0$

Theorem If f is increasing/decreasing on domain $D = [a, b]$ then f is 1-to-1 on $[a, b]$ for all $x \in [a, b]$

Ex: $f(x) = x^3$ is increasing on $D = [0, \infty)$; $f'(x) = 3x^2 > 0$ for all $x > 0$
Ex: $f(x) = x^3$ is increasing on $D = (-\infty, \infty)$; $f'(x) = 3x^2 > 0$ for all x except for $x = 0$
Ex: $f(x) = x^2$ is increasing on $D = [0, \infty)$; $f'(x) = 2x > 0$ for all $x > 0$



> **Theorem** If f is 1-to-1 on domain $D = [a, b]$ then f has an inverse function on $[a, b]$ for all $x \in [a, b]$

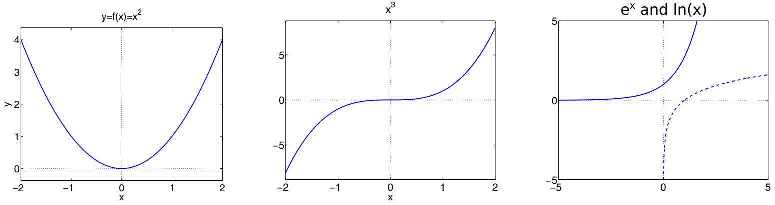
Ex: $f(x) := x^3$ is 1-to-1 on $D = \mathbb{R}$ and $g(x) = \sqrt[3]{x}$ then is g the inverse of f on $D = \mathbb{R}$ because

$$(g \circ f)(x) = g(x^3) \underset{g(x)=\sqrt[3]{x}}{=} \sqrt[3]{x^3} = x$$

$$(f \circ g)(x) = f(\sqrt[3]{x}) \underset{f(x)=x^3}{=} (\sqrt[3]{x})^3 = x$$

Function g is called the inverse of f and f the inverse of g on D

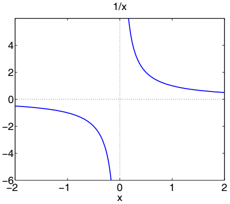
Inverse functions



Examples:

- $f(x) = x^3$ is 1-to-1 on $D = \mathbb{R}$ because
  $f(x) = f(y) \iff x^3 = y^3 \iff x = y$
  thus the inverse of f is $g(x) = \sqrt[3]{x}$
- $f(x) = x^2$ is 1-to-1 on $D = [0, \infty)$ because
  $f(x) = f(y) \iff x^2 = y^2 \iff x = y$
  thus the inverse of f on D is $g(x) = \sqrt{x}$
- $f(x) = e^x$ is 1-to-1 on $D = \mathbb{R}$ because
  $f(x) = f(y) \iff e^x = e^y \iff x = y$
  and the inverse of f on D is $g(x) = \ln x$

Inverse functions



Ex: $f(x) = 1/x$ is 1-to-1 on $D = \mathbb{R} - \{0\}$ because
$f(x) = f(y) \iff 1/x = 1/y \iff x = y$
and the inverse of f on D is $g(x) = 1/x$

Existence of inverse functions of trigonometric functions

- $f(x) = \sin(x)$ is increasing on $D = [-\pi/2, \pi/2]$ thus f is 1-to-1 on D and has an inverse function on D! Attention: the sine function on $D = \mathbb{R}$ is not 1-to-1 and thus has no inverse function
- $f(x) = \cos(x)$ is decreasing on $D = [0, \pi]$ thus f is 1-to-1 on D and has an inverse function on D!
- $f(x) = \tan(x)$ is increasing on $D = [-\pi/2, \pi/2]$ thus f is 1-to-1 on D and has an inverse function on D!

What do we call these inverse functions?

Existence of inverse functions of trigonometric functions

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What do we call these inverse functions? Later ...

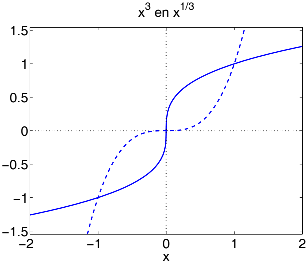
Inverse functions

Our calculus course focuses on the inverse functions of:

- polynomials
- trigonometric functions
- $f(x) = e^x$
- compositions of the above

Inverse functions: notation drama

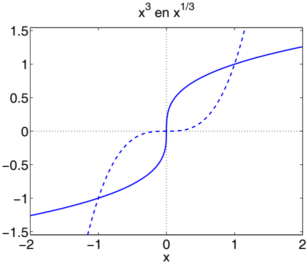
The inverse function of f on domain D is written as
f^{-1} or $arcf(x)$
The first often used notation is not consistent with another mathematical notation ...
if x is a number, then $x^{-1} = \dfrac{1}{x}$ but if f is a function, then $f^{-1} \neq \dfrac{1}{f}$
Ex: $f(x) = x^2$ on $D = [0, \infty)$ has inverse $f^{-1}(x) = \sqrt{x}$. Clearly
$(x^2)^{-1} \neq \sqrt{x}$



Ex: $f(x) = x^3$ has inverse $f^{-1}(x) = \sqrt[3]{x}$

- $f^{-1}(f(x)) = x$ or written differently $(f^{-1} \circ f)(x) = x$
  $f(f^{-1}(y)) = y$ of written differently $(f \circ f^{-1})(y) = y$
  both $f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity
- Graph of f^{-1} = mirroring of graph of f at the line $y = x$
- Domain of f^{-1} =

Properties of inverse functions



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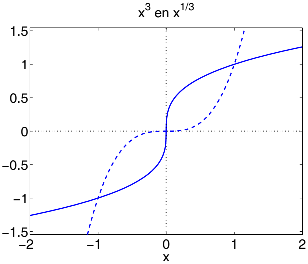

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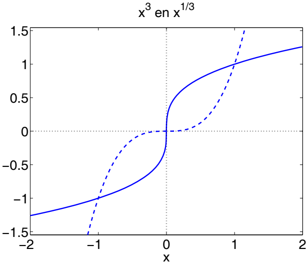

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- Range of f^{-1} = domain of f
- $(f^{-1})^{-1} = f$ (on domain of f)

Findinf derivatives of inverse functions

Starting point: $(f^{-1} \circ f)(x) = x$, with chain rule:

$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \implies (f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

or with $y = f(x)$:

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Ex: $y = x^3, \quad f^{-1}(y) = y^{1/3}, \quad (f^{-1})'(y) = \frac{1}{3}y^{-2/3}$

With formula: $(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}$ because $x = y^{1/3}$

Advantage: we can calculate $(f^{-1})'(y) = \frac{1}{f'(x)}$

for a given x without calculating f^{-1} itself!

- Is $f(x) = |x|$ 1-to-1?
- $f(x) = x^7 + 9$, give f^{-1}
- Is $f(x) = x^3 - 2$ 1-to-1? If so, what is f^{-1}? Sketch the graph. What is the range and domain of f and f^{-1}?
- Same for $f(x) = \frac{4}{x^2}, \quad x > 0$
- Show that $f(x) = \frac{e^x}{e^x+1}$ is 1-to-1, calculate the inverse function
- Give f^{-1} for $f(x) = -\frac{6}{7}x + 6$. Sketch $f(x)$ and $f^{-1}(x)$ in 1 graph. Give $\frac{df}{dx}$ in $x = 1$ and $\frac{df^{-1}}{dy}$ at $y = f(1)$
- $f(x) = 4x^3 - 7x^2 - 3, \quad x \geq 1.5$.
  Find the value of $\frac{df^{-1}}{dy}$ at $y = 42 = f(3)$
- Calculate the inverse function $f^{-1}(y)$ and give the domain and range of f^{-1}, and explain your answer:
  - $f(x) = \dfrac{x-1}{x+1}$
  - $f(x) = \dfrac{e^{2x}}{e^{2x}+1}$
  - $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
  - $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

Inverse functions

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- Same for $f(x) = \frac{4}{x^2}$, $x > 0$

  $y = \frac{4}{x^2} \implies x^2 = \frac{4}{y} \implies x = \sqrt{\frac{4}{y}} = \frac{2}{\sqrt{y}}$
- Show that $f(x) = \frac{e^x}{e^x+1}$ is 1-to-1,
  and calculate the inverse function $x = g(y)$

Inverse functions

- Give f^{-1} for $f(x) = -\frac{6}{7}x + 6$. Sketch $f(x)$ and $f^{-1}(x)$ in 1 graph.
  Give $\frac{df}{dx}$ in $x = 1$ and $\frac{df^{-1}}{dy}$ at $y = f(1)$

  $y = -\frac{6}{7}x + 6 \implies x = -\frac{7}{6}(y - 6)$

  thus $f^{-1}(y) = -\frac{7}{6}y + 6$ and $(f^{-1})'(y) = -\frac{7}{6}$

  Also possible with formula $(f^{-1})'(y) = \frac{1}{f'(x)}$:

  $x = 1$, $y = f(1) = \frac{36}{7}$ and $f'(1) = -\frac{6}{7}$

  thus $(f^{-1})'(\frac{36}{7}) = \frac{1}{-6/7} = -\frac{7}{6}$

- $f(x) = 4x^3 - 7x^2 - 3, \;\; x \geq 1.5$.
  Find the value of $\frac{df^{-1}}{dy}$ at $y = 42 = f(3)$

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  Find the value of $\frac{df^{-1}}{dy}$ at $y = 42 = f(3)$

  $y = 4x^3 - 7x^2 - 3, \quad x = f^{-1}(y) =$??? does not work!
  Now we have to use the formula $(f^{-1})'(y) = \frac{1}{f'(x)}$:
  $x = 3$, $y = f(3)$ and $f'(x) = 12x^2 - 14x$ thus $f'(3) = 66$:
  thus $(f^{-1})'(42) = \frac{1}{66}$

For the following functions $y = f(x)$:
(a) Calculate the inverse function $f^{-1}(y)$
(b) Give the domain and range of f^{-1}, and explain your answer.

- $f(x) = \dfrac{x-1}{x+1}$ (exam 04.11.10)
- $f(x) = \dfrac{e^{2x}}{e^{2x}+1}$ (exam 25.01.11)
- $f(x) = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$ (exam 11.11.11)
- $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ (exam 24.01.12)

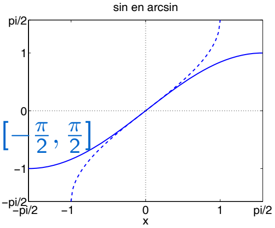
See also AS 14. arcsin $= \sin^{-1}$: the inverse of sin

sin with domain $[0, 2\pi]$ is not 1-to-1

$\implies$ focus on sin on domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

then range $[-1, 1]$

Notation [A]: $\mathrm{Sin}(x) = \sin(x)$ restricted to domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

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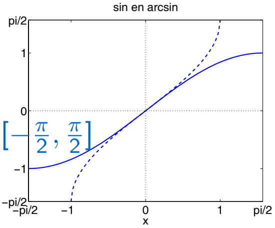

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$$
\begin{aligned}
(f^{-1})'(f(x)) &= \frac{1}{f'(x)} \\
\arcsin'(\sin(x)) &= \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2(x)}}
\end{aligned}
$$

Inverse functions of sin, cos, tan

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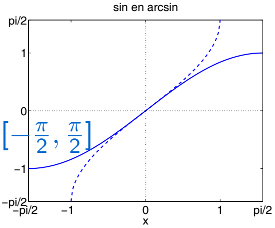


$$
\begin{aligned}
(f^{-1})'(f(x)) &= \frac{1}{f'(x)} \\
\arcsin'(\sin(x)) &= \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2(x)}} \\
y=\sin(x) \implies \arcsin'(y) &= \frac{1}{\sqrt{1-y^2}}
\end{aligned}
$$

Inverse functions of sin, cos, tan

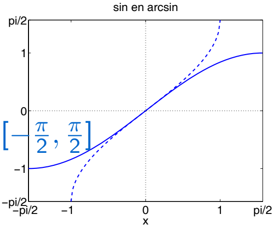
See also AS 14. arcsin $= \sin^{-1}$: the inverse of sin

sin with domain $[0, 2\pi]$ is not 1-to-1

$\implies$ focus on sin on domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

then range $[-1, 1]$

Notation [A]: $\mathrm{Sin}(x) = \sin(x)$ restricted to domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$



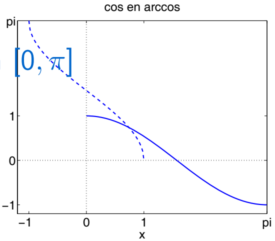
$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$\arcsin'(\sin(x)) = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2(x)}}$$

$$y = \sin(x) \implies \arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$$

$$\implies \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

cos with domain $[0, \pi]$ is 1-to-1
with range $[-1, 1]$
Notation [A]: $\mathrm{Cos}(x) = \cos(x)$ restricted to domain $[0, \pi]$



$$(f^{-1})'(y) = \frac{1}{f'(x)}, \qquad y = f(x)$$

cos with domain $[0, \pi]$ is 1-to-1
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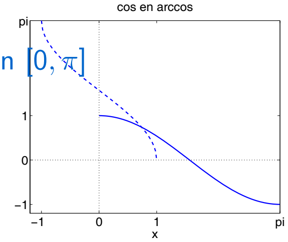

$$
\begin{aligned}
(f^{-1})'(y) &= \frac{1}{f'(x)}, \qquad y = f(x) \\
\arccos'(y) &= \frac{1}{-\sin(x)}, \qquad y = \cos(x)
\end{aligned}
$$

arccos = $\cos^{-1}$: the inverse of cos

cos with domain $[0, \pi]$ is 1-to-1
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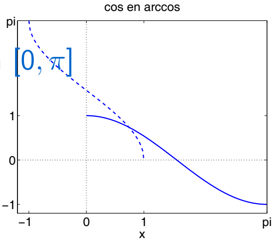

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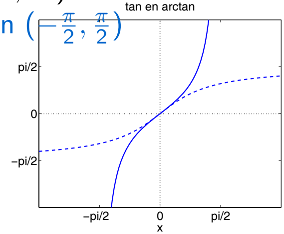
\end{aligned}
$$

Now: $\quad \sin(x) = \sqrt{1 - \cos^2(x)} = \sqrt{1 - y^2}, \quad$ thus

$$
\boxed{\arccos'(y) = -\frac{1}{\sqrt{1 - y^2}}}
$$

tan with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ is 1-to-1 with range $(-\infty, \infty)$
Notation [A]: $\mathrm{Tan}(x) = \tan(x)$ restricted to domain $(-\frac{\pi}{2}, \frac{\pi}{2})$



$$(f^{-1})'(y) = \frac{1}{f'(x)}, \qquad y = f(x)$$

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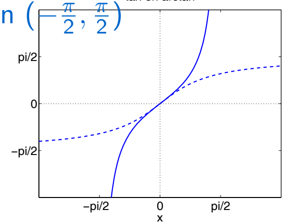
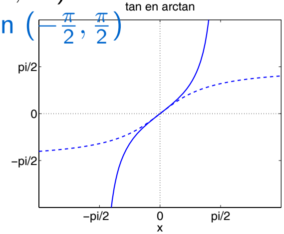


$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(x)}, & y = f(x) \\ \arctan'(y) &= \frac{1}{1+\tan^2(x)}, & y = \tan(x) \end{aligned}$$

$\arctan = \tan^{-1}$: the inverse of tan

tan with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ is 1-to-1 with range $(-\infty, \infty)$
Notation [A]: $\mathrm{Tan}(x) = \tan(x)$ restricted to domain $(-\frac{\pi}{2}, \frac{\pi}{2})$



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(f^{-1})'(y) &= \frac{1}{f'(x)}, \qquad y = f(x) \\
\arctan'(y) &= \frac{1}{1+\tan^2(x)}, \qquad y = \tan(x)
\end{aligned}
$$

Thus $\boxed{\arctan'(y) = \dfrac{1}{1+y^2}}$

arcsin, arccos, arctan

Function	Domain	Range
Sin	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
Cos	$[0, \pi]$	$[-1, 1]$
Tan	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$

Function	Domain	Range
arcsin	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
arccos	$[-1, 1]$	$[0, \pi]$
arctan	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

If $y \in [-1, 1]$ then $\sin(\arcsin(y)) = y$

If $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ then $\arcsin(\sin(x)) = x$

But what is $\arcsin(\sin(x))$ if $x \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$?
Calculate with period
Ex: $\arctan(\tan(\frac{2}{3}\pi))$

arcsin, arccos, arctan

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arcsin, arccos, arctan

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But what is $\arcsin(\sin(x))$ if $x \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$?
Calculate with period
Ex: $\arctan(\tan(\frac{2}{3}\pi))$
In principle are tan an arctan each others inverse thus then $\frac{2}{3}\pi$??
But: arctan is inverse of "Tan" with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$
Thus range arctan is also $(-\frac{\pi}{2}, \frac{\pi}{2})$
tan is π-periodic, thus $\tan(\frac{2}{3}\pi) = \tan(-\frac{1}{3}\pi)$ and $-\frac{1}{3}\pi \in (-\frac{\pi}{2}, \frac{\pi}{2})$
Thus $\arctan(\tan(\frac{2}{3}\pi)) = -\frac{1}{3}\pi$

Exercises for inverses of trgonometric functions

- $\arctan(-1)$?
- Given that $\alpha = \arcsin(\frac{21}{29})$, find $\cos(\alpha)$
- $\cos(\arcsin(\frac{1}{7}))$
- $\tan(\arcsin(-\frac{1}{2}\sqrt{2}))$
- Derivative of $y = \arccos(2x^9)$
- Calculate derivative of $\arccos(2^x)$
- Same for $\arcsin(e^{(x^3)})$
- $\arcsin(-\frac{\sqrt{3}}{2})$
- $\arccos(-\frac{\sqrt{3}}{2})$

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  Take $x = \arctan(-1)$ then $\tan(x) = -1$ and $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
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  $\sin(\alpha) = \frac{21}{29}$, sketch now the triangle

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  $\sin(\alpha) = \frac{21}{29}$, sketch now the triangle with sides 21, 29 en 20
- $\cos(\arcsin(\frac{1}{7}))$ (exam 12.10.06)
- $\tan(\arcsin(-\frac{1}{2}\sqrt{2}))$
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  $y' = -\frac{1}{\sqrt{1-(2x^9)^2}}$

Exercises for inverses of trgonometric functions

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- $\tan(\arcsin(-\frac{1}{2}\sqrt{2}))$
- Derivative of $y = \arccos(2x^9)$
  $y' = -\frac{1}{\sqrt{1-(2x^9)^2}} \cdot 18x^8 = -\frac{18x^8}{\sqrt{1-4x^{18}}}$
- Calculate derivative of $\arccos(2^x)$ (exam 14.11.07)
- Same for $\arcsin(e^{(x^3)})$ (exam 10.1.07)

- $\arcsin(-\frac{\sqrt{3}}{2})$ (exam 16.1.07)

  If $x = \arcsin(-\frac{\sqrt{3}}{2})$ then is $\sin(x) = -\frac{\sqrt{3}}{2}$

  $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, thus $x = -\frac{\pi}{3} + k2\pi$ or $x = \frac{4\pi}{3} + k2\pi$

  Convention: range of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

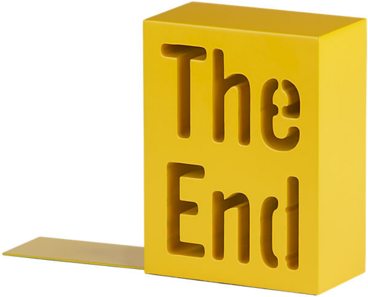
  x must be in here, thus $x = -\frac{\pi}{3}$

Inverse trigonometric functions

- $\arcsin(-\frac{\sqrt{3}}{2})$ (exam 16.1.07)
  If $x = \arcsin(-\frac{\sqrt{3}}{2})$ then is $\sin(x) = -\frac{\sqrt{3}}{2}$
  $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, thus $x = -\frac{\pi}{3} + k2\pi$ or $x = \frac{4\pi}{3} + k2\pi$
  Convention: range of arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  x must be in here, thus $x = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2})$ (exam 16.1.07)
  If $x = \arccos(-\frac{\sqrt{3}}{2})$ then is $\cos(x) = -\frac{\sqrt{3}}{2}$
  $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, thus $x = \pi - \frac{\pi}{6} + k2\pi$ or $x = \pi + \frac{\pi}{6} + k2\pi$
  Convention: range of arccos is $[0, \pi]$
  x must be in here, thus $x = \frac{5}{6}\pi$

Week 4: We have seen

- The Mean Value Theorem
- Higher order derivatives $f^{(k)}$
- "Implicit" differentiation
- Extrema: minima/maxima/stationary points
- Inverse functions
- Inverse trigonometric functions



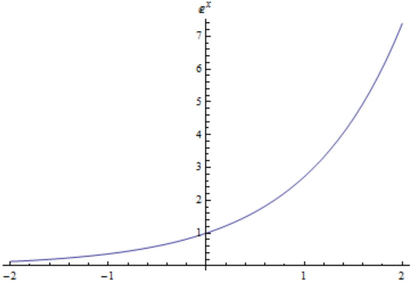
- The natural logarithm and exponent
- Exponential and logarithmic functions
- Growth and Decay
- Linear approximations
- Taylor polynomials
- Indeterminate form

The factorial of an integer number

For numbers $n = 0, 1, 2, \ldots$ define $0! = 1$ and n! ("n-factorial'):

$$\boxed{n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots 2 \cdot 1}$$

- $1! = 1$
- $3! = 3 \cdot 2 \cdot 1 = 6$
- $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
- $100! = 100 \cdot 99 \cdot \ldots \cdot 2 \cdot 1 \approx 9.332621544394410 \cdot 10^{157}$ has 158 digits!



For all $x \in \mathbb{R}$ e^x is defined as:

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

Although all terms are powers of x is e^x not a polynomial!

A sum that does not stop (such as the one for e^x) is called a series

Is this really *the* definition of e^x? Yes!

Domain of $f(x) = e^x$ is $D_f = \mathbb{R}$ and range is $(0, \infty)$

Again: for all $x \in \mathbb{R}$ e^x is defined as:

$$\boxed{e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}_{\text{series}} = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

This means $e = e^1 =$

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This means $e = e^1 = 1 + 1 + 1/2 + 1/6 + 1/24 + \approx$

The function e^x

Again: for all $x \in \mathbb{R}$ e^x is defined as:

$$\boxed{e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}_{\text{series}} = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

This means $e = e^1 = 1 + 1 + 1/2 + 1/6 + 1/24 + \approx 2.718281828459046$

Again: for all $x \in \mathbb{R}$ e^x is defined as:

$$\boxed{e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}_{\text{series}} = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

where the infinite sum is defined with the help of limits:

$$\boxed{e^x = \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{\text{no polynomial in } x} = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \frac{x^k}{k!}}_{\text{polynomial of degree } n \text{ in } x}}$$

The fact that e^x is this series for all $x \in \mathbb{R}$ is really special:
Often the limit only exists for x in a small interval!

The derivative of the function e^x

For the derivative it holds that $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$ because:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} e^x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x} 1 + \frac{\mathrm{d}}{\mathrm{d}x} x + \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{2} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^4}{4!} + \dots\right) \\
&= \left(0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots\right) \\
&= e^x
\end{aligned}$$

with that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^{k+1}}{(k+1)!} = (k+1) \cdot \frac{x^k}{(k+1)!} = \frac{x^k}{k!}$$

The inverse of e^x is the function $\ln(x)$

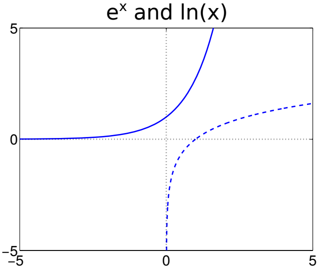
Function e^x is increasing, thus 1-to-1, thus it has an inverse, with name $\ln(x)$
Inverse function $\ln(x)$ is thus also 1-to-1, has domain $(0, \infty)$ and range $(-\infty, \infty)$

$$\boxed{y = e^x \Longleftrightarrow x = \ln(y)}$$

$$\boxed{e^{\ln(x)} = x, \;\; \ln(e^x) = x}$$

The standard power-properties hold:

- $e^x e^y = e^{x+y}$
- $e^{-x} = \dfrac{1}{e^x}$, etc

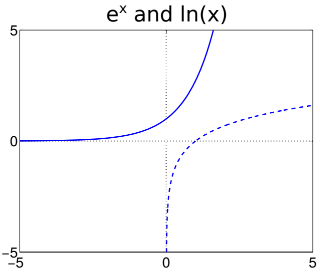


For the derivative of the inverse $\ln(x)$ of e^x it holds (set $f(x) = e^x$, inverse $g(x) = \ln(x)$) $(g \circ f)(x) = x$ thus with use of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(g \circ f)(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \iff g'(f(x)) \cdot f'(x) = 1 \implies$$

$$(g)'(f(x)) \underset{\text{chain rule}}{=} \frac{1}{f'(x)} \iff (g)'(e^x) \underset{\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x}{=} \frac{1}{e^x}$$

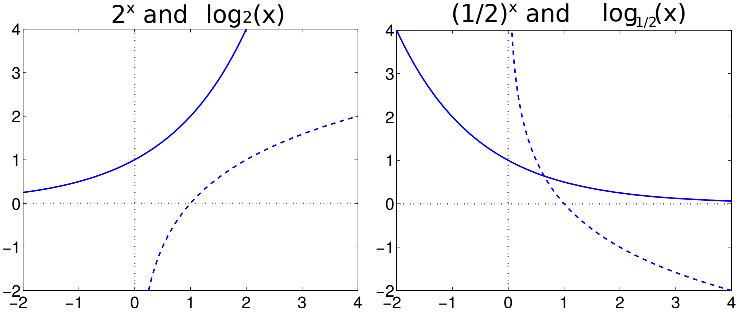
$$\iff (g)'(y) \underset{y:=a^x}{=} \frac{1}{y} \iff (\ln)'(y) \underset{g=\ln}{=} \frac{1}{y}$$



so

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$$

a^x with domain $(-\infty, \infty)$ and range $(0, \infty)$ is increasing thus 1-to-1



$\Longrightarrow$ has inverse, we call $\log_a(x)$,

with domain $(0, \infty)$ and range $(-\infty, \infty)$

this one you can easily sketch by mirroring at the line $y = x$

$\boxed{y = a^x \Longleftrightarrow x = \log_a(y)}$ $\quad$ $\boxed{a^{\log_a(x)} = x, \;\; \log_a(a^x) = x}$

Remember:

$$\begin{aligned}(e^x)' &= e^x \Longrightarrow \\ (a^x)' &= \left(e^{\ln(a^x)}\right)' = \left(e^{x\ln(a)}\right)' = e^{x\ln(a)} \cdot \ln(a) = a^x \cdot \ln(a)\end{aligned}$$

Remember:

$$(\ln(x))' = 1/x.$$

With function $f(x) = a^x$ and inverse of this $g(x) = {}^a\log(x)$:

$$\begin{aligned}(g)'(y) &\underset{\text{chain rule}}{=} \frac{1}{f'(x)} = \frac{1}{a^x \cdot \ln(a)} \\ &\underset{y:=e^x}{=} \frac{1}{y\ln(a)} \Longrightarrow \\ \frac{d}{dx}\log_a(x) &= \frac{1}{a^x \ln(a)}\end{aligned}$$

for all $x > 0$.

$\log_{10}$	used very often
$\log_e = \ln$	used often
$\log_2$	only used often in computer science
$\log_a$	little used

$$
\begin{aligned}
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a\left(\tfrac{x}{y}\right) &= \log_a(x) - \log_a(y) \\
\log_a(x^r) &= r \cdot \log_a(x) \\
\log_a\left(\tfrac{1}{x}\right) &= -\log_a(x) \\
\log_a(1) &= 0 \\
\log_a(x) &= \log_a(b) \cdot \log_b(x)
\end{aligned}
$$

Logarithmic equations

Basic principle

- to get rid of e-power, take ln
- to get rid of ln, take e-power

Alternatively: see that you get:

$$\ln(\text{something}) = \ln(\text{something else}),$$

then you can conclude

$$\text{something} = \text{something else}$$

because ln is an increasing function (1-to-1)

Check always the obtained solutions! (as with equations inequalities with roots) $\Longrightarrow$ argument of ln must always be > 0

Logarithmic equations

Examples:

- $\ln(x) + \ln(x+2) = 1$

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  $x = -1 + \sqrt{1+e}$
- $\ln^2(x) + \ln(x^2) - 3 = 0$

Examples:

- $\ln(x) + \ln(x+2) = 1$
  $x = -1 + \sqrt{1+e}$
- $\ln^2(x) + \ln(x^2) - 3 = 0$
  $x = e^{-3}$ or $x = e$
- Solve y from $e^{\sqrt{y}} = x^6 \quad$ for $x \geq 1$
  $y = 36 \ln^2(x)$

Logarithmic inequalities

Basic princple: $\ln(x)$ is an increasing function, thus:

$\ln(x) > \ln(4) \implies$

Logarithmic inequalities

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$\ln(x) < \ln(4) \implies 0 < x < 4$

Pay attention to the domain!

- $\ln(2x - 3) < 3 - \ln(x)$

Logarithmic inequalities

Basic princple: $\ln(x)$ is an increasing function, thus:

$\ln(x) > \ln(4) \implies x > 4$

$\ln(x) < \ln(4) \implies 0 < x < 4$

Pay attention to the domain!

- $\ln(2x-3) < 3 - \ln(x)$

  $\implies e^{\ln(2x-3)} < e^{3-\ln(x)}$ because e^x is an increasing function

  $\implies 2x-3 < e^3 \cdot e^{-\ln(x)}$

  $\implies 2x-3 < e^3 \cdot e^{\ln(x^{-1})}$

  $2x-3 < \frac{e^3}{x} \implies$ solve ...

  (solve first for the equation, or $\frac{2x^2-3x-e^3}{x} < 0$ etc)

  But extra requirement for domain:

Logarithmic inequalities

Basic princple: $\ln(x)$ is an increasing function, thus:

$\ln(x) > \ln(4) \implies x > 4$

$\ln(x) < \ln(4) \implies 0 < x < 4$

Pay attention to the domain!

- $\ln(2x-3) < 3 - \ln(x)$

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  (solve first for the equation, or $\frac{2x^2-3x-e^3}{x} < 0$ etc)

  But extra requirement for domain:

  $2x - 3 > 0$ and $x > 0$ thus $x > \frac{3}{2}$

Exercises: Exponential and logarithmic equations

- Write $\ln\left(\frac{8}{9}\right)$ in terms of $\ln(2)$ and $\ln(3)$
- Simplify $\ln(\sin(x)) - \ln\left(\frac{\sin(x)}{4}\right)$
- Simplify $e^{-\ln(5x^6)}$
- Simplify $e^{\ln(5x)-\ln(2y)}$
- Solve y from $\ln(y) = 2x + 7$
- Solve y from $\ln(y-3) - \ln(5) = 3x + \ln(x)$
- Solve $e^{2x} = 9$
- Derivative of $y = \cos(e^{-4x^3})$
- Derivative of $y = \ln(x^8)$
- Derivative of $y = \ln\left(\frac{9}{x}\right)$
- $\frac{dy}{dx}$ for $x^{3y} = y^{5x}$
- Solve $\ln(2x) + 3\ln((x+\frac{3}{2})^{1/3}) = 1$

Computer algorithms for Mathematical Operations

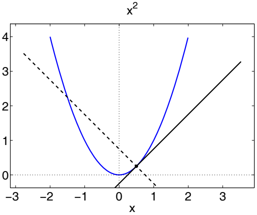
1. Arithmetic operations
   $\rightarrow$ addition, multiplication as in schoolbook

$$\begin{array}{r} {\scriptstyle 1} \\ 47 \\ +\,89 \\ \hline 136 \\ \hline \end{array}$$

2. Algebraic functions
   $\rightarrow$ polynomials $ax^n + bx^{n-1} + \ldots + c = 0$
3. Special functions
   $\rightarrow$ $\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, $\ldots$???

Approximate $f(x)$ with 1st degree polynomial $p_1(x)$

First degree polynomial p approximates f at $x = a$: $p(x) = c_0 + c_1(x - a)$ has two degrees of freedom.



Thus we need 2 conditions: same function value and same derivative:

- $p(a) = f(a)$ so it goes through the point; and
- $p'(a) = f'(a)$ thus slope is the same

because from $\underbrace{p(x)}_{=:y} = f(a) + f'(a)(x - a)$ it follows that

$$\begin{aligned} p(x) &= f(a) + f'(a)(x-a) \qquad & p(a) &= f(a) + 0 \\ p'(x) &= 0 + f'(a) & p'(a) &= f'(a) \end{aligned}$$

gives the tangent line!!

Linearization examples

- Give the linearization of $f(x) = 3x^3 + 5x + 4$ in $x = -1$

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  $f(-1) = -4, \quad f'(x) = 9x^2 + 5, \quad f'(-1) = 14$
  thus $p_1(x) = -4 + 14(x + 1)$. p_1 approximates f around -1 thus
  Check close to $x = -1$
  $f(-1.01) = -4.140903 \approx p(-1.01) = -4 - 0.14 = -4.14$
  $f(-0.99) = -3.860897 \approx p_1(-0.99) = -4 + 0.14 = -3.86$
- $f(x) = 5x^2 - 3x$. Give an estimate for $f(-1.9)$ with a linearization in $x = -2$

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  True value: $f(-1.9) = 23.75$

Approximate $f(x)$ with 2nd degree polynomial $p_2(x)$

Suppose we want to use more derivatives of f and p identically:

- $p(a) = f(a)$ goes through the point; and
- $p'(a) = f'(a)$ has the same slope
- $p''(a) = f''(a)$ has the same curvature

This is polynomial of degree 2 (check the degree!)

$$p_2(x) = \underbrace{f(a) + (x-a)f'(a)}_{\text{tangent line } p_1(x)} + \frac{(x-a)^2}{2} f''(a)$$

fulfills the 3 requirements:

$$\begin{aligned} p(x) &= \ldots(\text{above})\ldots & p(a) &= f(a) + 0 + 0 \\ p'(x) &= 0 + f'(a) + (x-a)f''(a) & p'(a) &= f'(a) + 0 \\ p''(x) &= f''(a) & p''(a) &= f''(a) \end{aligned}$$

This polynomial is unique.

Approximate $f(x)$ with 3rd, 4th, ... degree polynomial

Suppose we want to use more derivatives of f and p identical:

- $p(a) = f(a)$ goes through the point; and
- $p'(a) = f'(a)$ has the same slope
- $p''(a) = f''(a)$ has same curvature
- $p'''(a) = f'''(a)$ etc

This polynomial of degree 3 (check the degree!)

$$p_3(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

fulfills the 4 requirements:

$p(x) = \dots$ (see above)

$p'(x) = 0 + f'(a) + (x-a)f''(a) + \dfrac{(x-a)^2}{2}f'''(a)$

$p''(x) = f''(a) + (x-a)f'''(a)$

$p'''(x) = f'''(a)$

$p(a) = f(a) + 0 + 0 + 0$

$p'(a) = f'(a) + 0 + 0$

$p''(a) = f''(a) + 0$

$p'''(a) = f'''(a)$

This polynomial is unique. The same for degree 4, 5, etc.

Approximate $f(x)$ with 3rd, 4th, ... degree polynomial

Ex: Approximate the function $f(x) = \sin(x)$ at $a = 0$. That yields the derivatives

$$\begin{array}{ll} f(x) = \sin(x) & f(0) = 0 \\ f'(x) = \cos(x) & f'(0) = 1 \\ f''(x) = -\sin(x) & f''(0) = 0 \\ f'''(x) = -\cos(x) & f'''(0) = -1 \end{array}$$

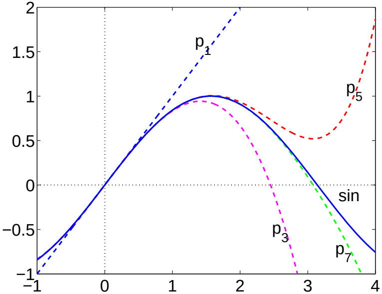
and approximating polynomial of degree 3

$$\begin{aligned} p_3(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \\ &= f(a) + (x-0)f'(a) + \frac{(x-0)^2}{2} f''(0) + \frac{(x-0)^3}{3!} f'''(0) \\ &= 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot -1 = x - \frac{x^3}{6} \end{aligned}$$

Does p_3 approximate the sine better than p_1 (p_1 is the tangent line)?

Approximate $f(x)$ with 3rd, 4th, ... degree polynomial

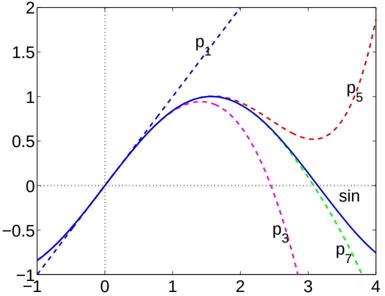
Ex: Remember: Take function $f(x) = \sin(x)$ and approximate f at $a = 0$



$f(x) = \sin(x)$	$f(0.03) = 0.029995500202496$
$p_1(x) = x$	$p_1(0.03) = 0.030000000000000$
$p_3(x) = x - \frac{1}{6}x^3$	$p_3(0.03) = 0.029995500000000$
$p_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$	$p_5(0.03) = 0.029995500202500$
$p_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$	$p_7(0.03) = 0.029995500202496$

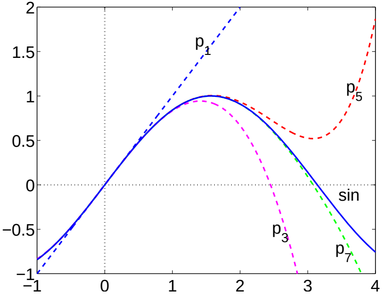
Approximate $f(x)$ with 3rd, 4th, ... degree polynomial

Ex: Remember: Take function $f(x) = \sin(x)$ and approximate f at $a = 0$



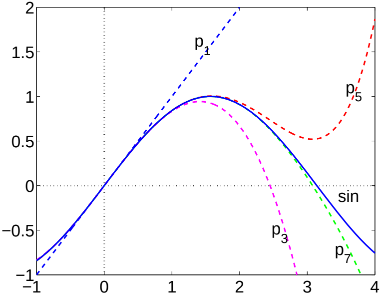
When we approximate $f(0.01)$ with $p_n(0.01)$ the difference $|f(x) - p_n(x)|$ is called the approximation error

Ex: Remember: Take function $f(x) = \sin(x)$ and approximate f at $a = 0$



Although p_1, p_3, p_5, p_7 only approximate sin at $a = 0$
the approximations at $a \neq 0$ always get better!
The approximations of a higher degree also become better close to $a = 0$ (later) but we can't see that here.

Ex: Remember: Take function $f(x) = \sin(x)$ and approximate f at $a = 0$



Why are we doing such approximations with p_1, p_3, p_5, p_7, ...?
Answer: Because computers can only add/subtract and multiply!
so computers use these kind of approximations to calculate $\sin(a)$ for a given a!

Taylor polynomial p_n approximates f around point a

The polynomial approximation p_n of degree n of the function f in point a:

$$\boxed{\begin{aligned} p_n(x) = f(a) \quad &+ \quad f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 \\ &+ \quad \frac{1}{3!}f'''(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n \end{aligned}}$$

is "the" Taylor polynomial – and *is* a polynomial of degree $\leq n$
If $a = 0$ the Taylor polynomial is also called Maclaurin polynomial

Taylor series: Matching all derivatives ...

The function p_∞ that approximates the function f in a and which matches all derivatives $f^{(0)}(a) = f(a)$, $f'(a)$, $f''(a)$, $f^{(3)}(a)$, $f^{(4)}(a)$, ... in a is:

$$\begin{aligned} p_\infty(x) = f(a) \quad &+ \quad f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 \\ &+ \quad \frac{1}{3!}f'''(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \cdots \end{aligned}$$

is called Taylor series of f in a.

Attention: $p_\infty(x)$ is no polynomial!

It can be shown for all the good functions that: $p_\infty(x) = f(x)$ (really identical!) for x close to a.

This means only for x so that $|x-a| < r$ and r is really small.

Only for $e^x, \sin(x)$ and $\cos(x)$ it holds that $p_\infty(x) = f(x)$ for all $x \in \mathbb{R}$!

Match only $n+1$ derivatives ...

Seen: $p_\infty = f$ around x and also $p_n \neq f$ for most f and n.
Generalization of the Mean Value Theorem:
For x around a there is a number c so that

$$\boxed{\underbrace{p_\infty(x)}_{f(x)} = p_n(x) + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}}$$

Written differently: Approximation error

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

can be estimated. Computers approximate all functions e^x, $\sin(x)$, $\cos(x)$ etc by polynomials p_n and do this so that the approximation error cannot be seen.

But there are rounding errors

Taylor series in $a = 0$ for known functions

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + \cdots
\end{aligned}
$$

Taylor series in $a = 0$ for known functions

$p(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots$

- of e^x:

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- of e^x: all derivatives of e^x are also e^x

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  thus $f(0) = f'(0) = f''(0) = \cdots = 1$

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  thus $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$ (odd powers: odd function)

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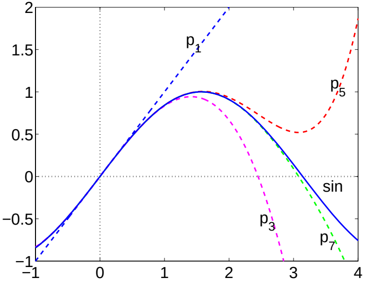
  thus $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$ (even powers: even function)

Ex: Taylor polynomial in $x = 0$ of $\sin(x)$

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$



$\sin(x)$ with approximations:

$p_1(x) = x$

$p_3(x) = x - \frac{1}{6}x^3$

$p_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$p_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

Some notation for every Taylor polynomial

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n = \sum_{k=0}^{n} \frac{1}{k!}x^k$$

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

Taylor polynomial/series of polynomials

Ex: $f(x) = x^2$
Preparation: $f'(x) = 2x, \;\; f''(x) = 2, \;\; f'''(x) = 0,$ etc.
$p(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots$

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$$\begin{aligned} p(x) &= f(0) + f'(0)x + \tfrac{1}{2!}f''(0)x^2 + \tfrac{1}{3!}f'''(0)x^3 + \cdots \\ &= 0 + 0 \cdot x + \tfrac{1}{2!} \cdot 2x^2 + \tfrac{1}{3!} \cdot 0 \cdot x^3 + 0 + 0 + \cdots = x^2! \end{aligned}$$

We find that for all polynomials it holds that:

$\boxed{\text{Taylor series of polynomial} = \text{polynomial itself}}$

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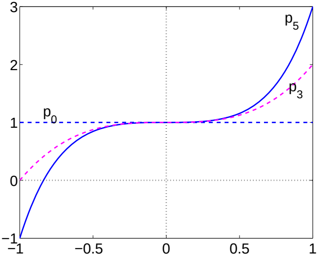
Evt Taylor polynomial can also exist $\neq$ polynomial itself!:

Ex: Taylor polynomial of $x^5 + 3x^3 + 1$:

0th-degree:



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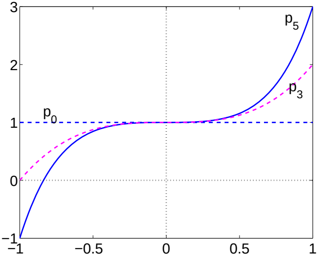
1st-degree:



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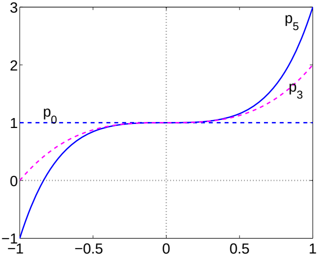
2nd-degree:



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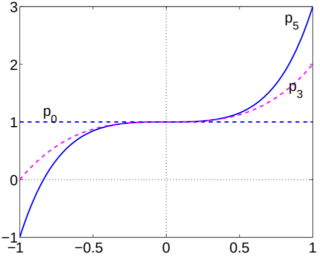

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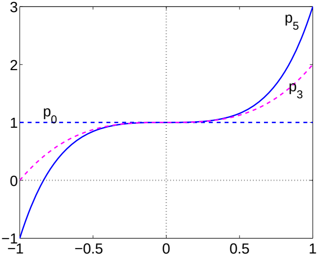
3rd-degree: $p_3(x) = 3x^3 + 1$

4th-degree:



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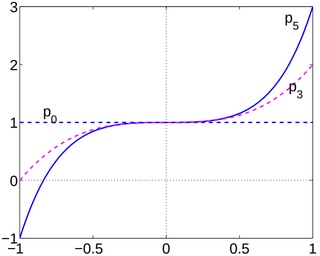
3rd-degree: $p_3(x) = 3x^3 + 1$

4th-degree: $p_4(x) = 3x^3 + 1$

5th-degree:



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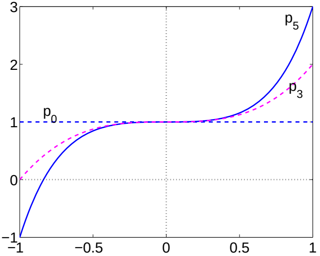
4th-degree: $p_4(x) = 3x^3 + 1$

5th-degree: $p_5(x) = x^5 + 3x^3 + 1$

6th-and higher degrees:



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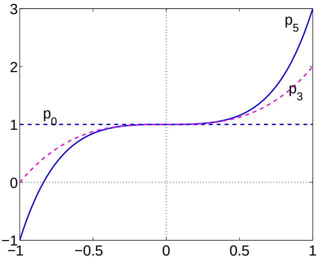
4th-degree: $p_4(x) = 3x^3 + 1$

5th-degree: $p_5(x) = x^5 + 3x^3 + 1$

6th-and higher degrees: also $x^5 + 3x^3 + 1$



Taylor series cut off and $\mathcal{O}$-notation

Often a certain number of terms is more than enough in a Taylor series

Ex: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

For small x ("$x \ll 1$", "x much smaller than 1"),
is x^5 is really very small $\implies$ we often neglect

Then "cut off", with notation:
$\sin(x) = x - \frac{x^3}{3!} + \mathcal{O}(x^5)$ if $x \to 0$
$\mathcal{O}$: pronounce "big-O", or "order"
$\mathcal{O}(x^5)$ means: depends on x^5,

Ex: $100x^2 + 5x^3 = \mathcal{O}(x^2)$ if $x \to 0$:

x	$100x^2 + 5x^3$	$100x^2$	$5x^3$
1	105	100	5
0.1	1.005	1	0.005
0.01	0.010005	0.01	0.000005
0.001	0.000100005	0.0001	0.000000005

because $100x^2$ and $100x^2 + 5x^3$ most resemble each other as $x \to 0$.
Ex: $100x^2 + 5x^3 = 100x^2 + \mathcal{O}(x^3)$ if $x \to 0$
Ex: $-\frac{1}{2}x^4 = \mathcal{O}(x^4)$
a minus sign you can integrate in $\mathcal{O}$

Ex: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \mathcal{O}(x^5) \quad$ als $x \to 0$

But you can also say:

$\sin(x) = x + \mathcal{O}(x^3) \quad$ if $x \to 0$

then you cut the series off earlier, and you neglect more terms

Ex: $(x + \mathcal{O}(x^3))^2 =$

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Ex: $(x + \mathcal{O}(x^3))^2 = x^2 + \mathcal{O}(x^4)$ because

$(x + \mathcal{O}(x^3))(x + \mathcal{O}(x^3)) = x^2 + \mathcal{O}(x^4) + \mathcal{O}(x^4) + \mathcal{O}(x^6) = x^2 + \mathcal{O}(x^4)$

Ex: $\frac{x^2 + \mathcal{O}(x^3)}{x} =$

Calculation with $\mathcal{O}$-notation

Ex: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

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Ex: $\frac{x^2 + \mathcal{O}(x^3)}{x} = x + \mathcal{O}(x^2)$

Determination of Taylor series

- With definition (determine derivatives), or easier:
- Try using known Taylor series!

Ex: Taylor series of $\frac{1}{1+2x^2}$ around $x = 0$

Resembles

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Resembles $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} = 1 + (-2x^2) + (-2x^2)^2 + (-2x^2)^3 + \cdots$

Ex: Taylor polynomial of degree 3 for $f(x) = \frac{1}{8-x}$ in $x = 0$

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Ex: Taylor polynomial of degree 3 for $f(x) = \frac{1}{8-x}$ in $x = 0$

Resembles $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, therefore:

$f(x) = \frac{1}{8(1-\frac{x}{8})} = \frac{1}{8} \cdot \frac{1}{1-\frac{x}{8}} = \frac{1}{8} \cdot (1 + \frac{x}{8} + (\frac{x}{8})^2 + (\frac{x}{8})^3 + \mathcal{O}(x^4))$

Ex: Taylor series of e^{3x} around $x = 0$

Determination of Taylor series

Ex: Taylor series of e^{3x} around $x = 0$
$e^{3x} = 1 + (3x) + \frac{1}{2!}(3x)^2 + \frac{1}{3!}(3x)^3 + \cdots = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots$
Ex: Taylor series of e^{x+3} around $x = 0$

Determination of Taylor series

Ex: Taylor series of e^{3x} around $x = 0$

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Ex: Taylor series of e^{x+3} around $x = 0$

WRONG: $e^{x+3} = 1 + (x+3) + \frac{1}{2}(x+3)^2 + \frac{1}{6}(x+3)^3 + \cdots$ because here this is not easily possible to rewrite into powers of x

But this works:

Determination of Taylor series

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But this works: $e^{x+3} = e^3 e^x = e^3(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots)$

$= e^3 + e^3 x + \frac{1}{2}e^3 x^2 + \frac{1}{6}e^3 x^3 + \cdots$

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But this works: $e^{x+3} = e^3 e^x = e^3(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots)$

$= e^3 + e^3 x + \frac{1}{2}e^3x^2 + \frac{1}{6}e^3x^3 + \cdots$

This also works well with definition:

$f(x) = e^{x+3}, \quad f'(x) = e^{x+3} \quad f''(x) = e^{x+3}, \ldots$

thus $f(0) = e^3, \; f'(0) = e^3, \; f''(0) = e^3, \ldots$

So Taylor polynomial:

$f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3$

$= e^3 + e^3 x + \frac{1}{2}e^3x^2 + \frac{1}{6}e^3x^3 + \cdots$

Examples Taylor series

Ex: Taylor series of $\frac{x}{e^x}$ around $x = 0$

WRONG: $\dfrac{x}{1 + x + \frac{1}{2}x^2 + \cdots}$

because

Examples Taylor series

Ex: Taylor series of $\frac{x}{e^x}$ around $x = 0$

WRONG: $\dfrac{x}{1 + x + \frac{1}{2}x^2 + \cdots}$

because this is not a polynomial in x, but a rational function

GOOD: $x\, e^{-x} = x\,(1 - x + \frac{1}{2}x^2 - \cdots)$

Ex: Taylor series of e^{x^3} with 3rd degree around $x = 0$

$1 + x^3 + \frac{1}{2}x^6 + \cdots = 1 + x^3 + \mathcal{O}(x^6)$

here only 2 terms remain

Ex: Taylor series of $\cos(x^4)$ with 3rd degree around $x = 0$

$1 - \frac{1}{2}x^8 + \cdots = 1 + \mathcal{O}(x^8)$

here only 1 term remains

Step-by-step: Taylor series in $a \neq 0$

1. You need powers $y = x - a$
2. write $x = y + a$ and fill that in into $f(x)$
3. Try to find a Taylor series with powers of y
   (and not eg $y - 1$)
4. Replace y again by $x - a$

Linear, quadratic approximations: examples

Taylor polynomial of degree 0, 1, 2 for $f(x) = \sqrt{1+x}$ in $x = 0$

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$\Longrightarrow$ use the definition

Linear, quadratic approximations: examples

Taylor polynomial of degree 0, 1, 2 for $f(x) = \sqrt{1+x}$ in $x = 0$

Of $\sqrt{1+x}$ we do not know a standard Taylor series

$\Longrightarrow$ use the definition

$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \quad f''(x) = -\frac{1}{4}(1+x)^{-3/2}$

Degree 0: $p(x) = f(0) = 1$

Degree 1: $p(x) = 1 + f'(0)x = 1 + \frac{1}{2}x$

Degree 2: $p(x) = 1 + \frac{1}{2}x + \frac{1}{2} \cdot (-\frac{1}{4})x^2$

Useful: $\boxed{\sqrt{1+x} \approx 1 + \dfrac{1}{2}x}$

Assume we want to approximate $\sqrt{1.1}$

Linear approximation: $\sqrt{1.1} \approx 1 + \frac{1}{2} \cdot 0.1 = 1.05$

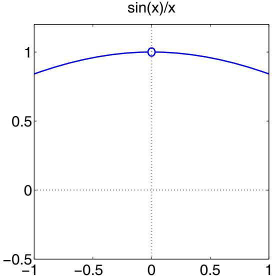
Quadratic approximation: $\sqrt{1.1} \approx 1 + \frac{1}{2}0.1 - \frac{1}{8} \cdot (0.1)^2 = 1.04875$

True value: $\sqrt{1.1} = 1.04880 \cdots$

Exercises: Taylor polynomial and Taylor series

- Find the Taylor polynomial of degree 0, 1, 2 and 3 of $f(x) = \ln(x)$ in $x = 1$
- Taylor series of $6 + e^{4x^3}$
- Find the 3rd-order Taylor polynomial of $f(x) = \ln((1+2x)^2) + e^{-x+1}$ in $x = 0$
- Find the 4th-order Taylor polynomial of $f(x) = \ln(x) + \sin(\pi x) + \frac{1}{1+(x-1)^2}$ in $x = 1$

Reminder: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ why??



This sort of limit we can determine with l'Hôpital

Limits with l'Hôpital

Assume $f(a) = 0$ and $g(a) = 0$; the question is: $\lim_{x \to a} \frac{f(x)}{g(x)} = ?$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

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Limits with l'Hôpital

Assume $f(a) = 0$ and $g(a) = 0$; the question is: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = ?$

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Therefore rule of l'Hôpital:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

only in the following 2 situations:

- if limit is $0/0$ after filling in
- if limit is ∞/∞ after filling in (or $-\infty$ instead of ∞; proof is more difficult)

Ex: $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = \lim\limits_{x \to 0} \dfrac{\cos(x)}{1} = 1$

Attention! Do not mix up l'Hôpital with the quotient rule $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Limits with l'Hôpital

With l'Hôpital this also works: $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3} \dfrac{2x}{1} = 6$

l'Hôpital can also be repeated if after taking the derivatives, there is still 0/0 (or ∞/∞):
$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to a} \dfrac{f''(x)}{g''(x)}$ etc (until no more $\frac{0}{0}$ or $\frac{\infty}{\infty}$ occurs)

Ex: $\lim\limits_{x\to 0} \dfrac{\sin(x)-x}{\cos(x)-1}$

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Ex: $\displaystyle \lim_{x\to 0} \frac{\sin(x)-x}{\cos(x)-1} = \lim_{x\to 0} \frac{\cos(x)-1}{-\sin(x)}$

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Ex: $\lim_{x\to\infty} \frac{e^x}{x^2}$ (of type $\frac{\infty}{\infty}$)

Limits with l'Hôpital

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$= \lim\limits_{x\to\infty} \dfrac{e^x}{2x}$

Limits with l'Hôpital

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$= \lim\limits_{x\to\infty} \dfrac{e^x}{2x} = \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \infty$

You can also use Taylor for the limit, it's an alternative to l'Hôpital

Taylor makes it hereby possible to divide by a factor

Ex: $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = \lim\limits_{x\to 0} \dfrac{x - \frac{1}{6}x^3 + \cdots}{x} = \lim\limits_{x\to 0} \dfrac{1 - \frac{1}{6}x^2 + \cdots}{1} = 1$

With $\mathcal{O}$-notation: $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = \lim\limits_{x\to 0} \dfrac{x + \mathcal{O}(x^3)}{x} = \lim\limits_{x\to 0} \dfrac{1 + \mathcal{O}(x^2)}{1} = 1$

One disadvantage: it is often not immediately clear to which order you have to use Taylor, (Exam 24.10.05)

$$\lim_{x\to 0} \frac{\frac{1}{2}\ln(1+2x^2) + \frac{1}{1+x^2} - 1}{\sin(x^2) - x^2\cos(x^2)} =$$

Limit with Taylor

One disadvantage: it is often not immediately clear to which order you have to use Taylor, (Exam 24.10.05)

$$\lim_{x\to 0} \frac{\frac{1}{2}\ln(1+2x^2)+\frac{1}{1+x^2}-1}{\sin(x^2)-x^2\cos(x^2)} = \lim_{x\to 0} \frac{\frac{1}{2}(2x^2-\frac{4x^4}{2}+\mathcal{O}(x^6))-x^2+x^4+\mathcal{O}(x^6)}{x^2+\mathcal{O}(x^6)-x^2(1-\frac{1}{2}x^4+\mathcal{O}(x^8))}$$

$$= \lim_{x\to 0} \frac{\mathcal{O}(x^6)}{\mathcal{O}(x^6)+\mathcal{O}(x^{10})} =$$

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$$= \lim_{x\to 0} \frac{\mathcal{O}(x^6)}{\mathcal{O}(x^6)+\mathcal{O}(x^{10})} = \;?? \quad \text{no luck! “constants are in the } \mathcal{O}\text{”}$$

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Therefore now Taylor with 6th powers:

$$\lim_{x\to 0} \frac{\frac{1}{2}(2x^2 - \frac{4x^4}{2} + \frac{8x^6}{3} + \mathcal{O}(x^8)) - x^2 + x^4 - x^6 + \mathcal{O}(x^8)}{x^2 - \frac{x^6}{6} + \mathcal{O}(x^{10}) - x^2(1 - \frac{1}{2}x^4 + \mathcal{O}(x^8))}$$

$$= \lim_{x\to 0} \frac{(\frac{4}{3} - 1)\,x^6 + \mathcal{O}(x^8)}{(-\frac{1}{6} + \frac{1}{2})\,x^6 + \mathcal{O}(x^{10})} = \lim_{x\to 0} \frac{\frac{1}{3} + \mathcal{O}(x^2)}{\frac{1}{3} + \mathcal{O}(x^4)} = \frac{1}{3}\Big/\frac{1}{3} = 1$$

Works also with l'Hôpital, but then you have to use it $6\times$ ...

Problem cases for limits ...

What are the problem cases and why exactly?
We have seen earlier:

- $\frac{\neq 0}{0}$: can be ∞, $-\infty$ or can not exist;
  consider left and right limits
- $\frac{0}{0}$: l'Hôpital, factor division or Taylor
- $\frac{\infty}{\infty}$: divide by largest in denominator or l'Hôpital

We will now also consider:

- $0 \cdot \infty$: (which is stronger?)
- $\infty - \infty$: (which is stronger?)
- 0^0: $0^x = 0$ if $x > 0$, but $x^0 = 1$ for all x, thus problem
- ∞^0: $\infty^x = \infty$ if $x > 0$, but $x^0 = 1$ if $x < \infty$: problem
- 1^∞: $1^x = 1$ if $x \neq \infty$
  but $(1 + 10^{-10})^\infty = \infty$ and $(1 - 10^{-10})^\infty = 0$

Case $0 \cdot \infty$ or $0 \cdot (-\infty)$

$\boxed{0 \cdot \infty: \quad \text{rewrite to } \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$

Ex: $\lim\limits_{x \to 0^+} x \ln(x)$ is of type $0 \cdot (-\infty)$:

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$$= \lim_{x \to 0^+} \frac{\ln(x)}{1/x} \quad \text{(now type } -\infty/\infty\text{, thus l'Hôpital)}$$

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$$= \lim_{x \to 0^+} \frac{\ln(x)}{1/x} \quad \text{(now type } -\infty/\infty\text{, thus l'Hôpital)}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

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$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$$

(but in this case you may also say that it is a standard limit)

Case $\infty - \infty$

$\boxed{\infty - \infty: \quad \text{Try to combine to 1 term}}$

Ex: $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin(x)} \right)$

Case $\infty - \infty$

$$\boxed{\infty - \infty: \text{ Try to combine to 1 term}}$$

Ex: $\displaystyle \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \lim_{x\to 0^+} \left(\frac{\sin(x) - x}{x\sin(x)}\right)$

Case $\infty - \infty$

$$\boxed{\infty - \infty\text{: Try to combine to 1 term}}$$

Ex: $\lim\limits_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \lim\limits_{x\to 0^+}\left(\frac{\sin(x)-x}{x\sin(x)}\right)$

$$= \lim_{x\to 0^+}\left(\frac{\cos(x)-1}{\sin(x)+x\cos(x)}\right)$$

Case $\infty - \infty$

$\boxed{\infty - \infty: \quad \text{Try to combine to 1 term}}$

Ex: $\lim\limits_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) = \lim\limits_{x \to 0^+} \left(\frac{\sin(x) - x}{x \sin(x)} \right)$

$$= \lim_{x \to 0^+} \left(\frac{\cos(x) - 1}{\sin(x) + x \cos(x)} \right) = \lim_{x \to 0^+} \left(\frac{-\sin(x)}{\cos(x) + \cos(x) - x \sin(x)} \right)$$

$\boxed{\infty - \infty: \quad \text{Try to combine to 1 term}}$

Ex: $\lim\limits_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) = \lim\limits_{x \to 0^+} \left(\frac{\sin(x) - x}{x \sin(x)} \right)$

$= \lim\limits_{x \to 0^+} \left(\frac{\cos(x) - 1}{\sin(x) + x\cos(x)} \right) = \lim\limits_{x \to 0^+} \left(\frac{-\sin(x)}{\cos(x) + \cos(x) - x\sin(x)} \right) =$

$\frac{0}{2} = 0$

(this rewrite $\infty - \infty$ to $\frac{0}{0}$ and use l'Hôpital)

Ex: $\lim\limits_{x \to \infty} \sqrt{x^2 - x} - x =$

$\boxed{\infty - \infty: \quad \text{Try to combine to 1 term}}$

Ex: $\lim\limits_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \lim\limits_{x\to 0^+} \left(\frac{\sin(x) - x}{x\sin(x)}\right)$

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$\frac{0}{2} = 0$

(this rewrite $\infty - \infty$ to $\frac{0}{0}$ and use l'Hôpital)

Ex: $\lim\limits_{x\to\infty} \sqrt{x^2 - x} - x = \lim\limits_{x\to\infty} (\sqrt{x^2 - x} - x) \cdot \frac{\sqrt{x^2 - x} + x}{\sqrt{x^2 - x} + x}$

$\boxed{\infty - \infty: \quad \text{Try to combine to 1 term}}$

Ex: $\lim\limits_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) = \lim\limits_{x \to 0^+} \left(\frac{\sin(x) - x}{x \sin(x)} \right)$

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Ex: $\lim\limits_{x \to \infty} \sqrt{x^2 - x} - x = \lim\limits_{x \to \infty} (\sqrt{x^2 - x} - x) \cdot \frac{\sqrt{x^2 - x} + x}{\sqrt{x^2 - x} + x}$

$= \lim\limits_{x \to \infty} \frac{x^2 - x - x^2}{\sqrt{x^2 - x} + x} = \lim\limits_{x \to \infty} \frac{-x}{x\left(\sqrt{1 - \frac{1}{x}} + 1\right)} = -\frac{1}{2}$

(thus rewrite $\infty - \infty$ with root trick to $\frac{\infty}{\infty}$ and divide through the biggest power in denominator)

$\lim_{x\to\infty}(1+\frac{3}{x})^x$:
Is type "1^∞"; remember these examples with all tricks

Ex: $\lim_{x\to\infty}(1+\frac{3}{x})^x = \lim_{x\to\infty} e^{\ln(1+\frac{3}{x})^x} = \lim_{x\to\infty} e^{x\ln(1+\frac{3}{x})}$
Consider exponent, it is of form $\infty \cdot 0$:

$\lim_{x\to\infty} x\ln(1+\frac{3}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{3}{x})}{1/x}$ is thus in the form $\frac{0}{0}$

l'Hôpital: $\lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot(-\frac{3}{x^2})}{-1/x^2} = \lim_{x\to\infty}\frac{3}{1+\frac{3}{x}} = 3$ (etc)

- $\lim_{x\to\infty} x^{1/\sqrt{x}}$ is of the form ∞^0 thus
  $= \lim_{x\to\infty} e^{\ln (x)^{1/\sqrt{x}}} = \lim_{x\to\infty} e^{\frac{\ln(x)}{\sqrt{x}}}$
  Consider exponent with l'Hôpital:
  $$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$
  thus answer to the question: $\lim_{x\to\infty} x^{1/\sqrt{x}} = 1$

- $\lim_{x\to\infty} (\sqrt{x})^{1/x}$ is of the form ∞^0 thus
  $= \lim_{x\to\infty} e^{\ln(x^{1/(2x)})} = \lim_{x\to\infty} e^{\frac{\ln(x)}{2x}}$
  Consider exponent with l'Hôpital: $\lim_{x\to\infty} \frac{\ln(x)}{2x} = \lim_{x\to\infty} \frac{1/x}{2} = 0$
  thus answer to the question: $\lim_{x\to\infty} (\sqrt{x})^{1/x} = 1$

Step-by-step plan limits

1. Always first

Step-by-step plan limits

1. Always first try filling in
   if it is no problem case: done!
2. $\frac{\neq 0}{0}$:

Step-by-step plan limits

1. Always first try filling in
   if it is no problem case: done!
2. $\frac{\neq 0}{0}$: consider $\lim\limits_{x \to a^+}$ and $\lim\limits_{x \to a^-}$
   if eg ∞ and ∞ then the limit is also ∞
   if eg ∞ and $-\infty$ the limit does not exist
3. $\frac{0}{0}$:

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   if it is no problem case: done!
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3. $\frac{0}{0}$: l'Hôpital (several times), or Taylor,
   or dividing factor
4. $\frac{\infty}{\infty}$:

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4. $\frac{\infty}{\infty}$: divide through biggest power in denominator or l'Hôpital)

5. $0 \cdot \infty$:

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5. $0 \cdot \infty$: rewrite to $\frac{0}{0}$ or $\frac{\infty}{\infty}$
6. $\infty - \infty$:

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2. $\frac{\neq 0}{0}$: consider $\lim\limits_{x\to a^+}$ and $\lim\limits_{x\to a^-}$
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4. $\frac{\infty}{\infty}$: divide through biggest power in denominator or l'Hôpital)
5. $0\cdot\infty$: rewrite to $\frac{0}{0}$ or $\frac{\infty}{\infty}$
6. $\infty-\infty$: rewrite as 1 term
7. ∞^0, 1^∞, 0^0:

Step-by-step plan limits

1. Always first try filling in
   if it is no problem case: done!
2. $\frac{\neq 0}{0}$: consider $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$
   if eg ∞ and ∞ then the limit is also ∞
   if eg ∞ and $-\infty$ the limit does not exist
3. $\frac{0}{0}$: l'Hôpital (several times), or Taylor, or dividing factor
4. $\frac{\infty}{\infty}$: divide through biggest power in denominator or l'Hôpital)
5. $0 \cdot \infty$: rewrite to $\frac{0}{0}$ or $\frac{\infty}{\infty}$
6. $\infty - \infty$: rewrite as 1 term
7. ∞^0, 1^∞, 0^0: trick with $e^{\ln}$

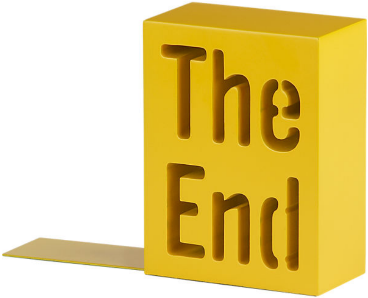
Note: $\frac{0}{\infty}$, 0^∞ and ∞^∞ are no problem cases

Exercises: Limits Indeterminate Form

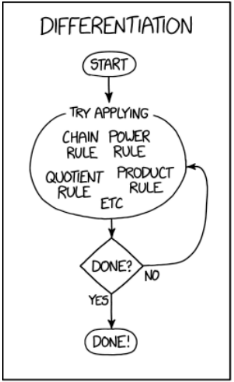
- $\displaystyle\lim_{x\to 8} \frac{2x-16}{6x^2-384}$
- $\displaystyle\lim_{x\to 0} \frac{3x^2}{\cos(x)-1}$ (a) with l'Hôpital; (b) with Taylor
- $\displaystyle\lim_{x\to 0} \frac{2x(\cos(2x)-1)}{\sin(3x)-3x}$ (a) with l'Hôpital; (b) with Taylor
- $\displaystyle\lim_{x\to 0} \frac{e^{x^3}-\cos(x^3)-\frac{1}{4}\ln(1+4x^3)}{\sin(x^2)+\arctan(x^2)-2x^2}$
- $\displaystyle\lim_{x\to -3} \frac{x^3-4x+15}{x^2-x-12}$ (a) with l'Hôpital; (b) with dividing factor
- determine $\displaystyle\lim_{x\to\infty} \frac{e^x+x^2}{2e^x-x}$ with l'Hôpital
- $\displaystyle\lim_{x\to 1^+} x^{9/(1-x)}$
- $\displaystyle\lim_{x\to\infty} (1+2x)^{19/(2\ln(x))}$

Week 5: We have seen

- Exponential and logarithmic functions
- Taylor polynomials and series
- Limits: L'Hôpital



This week and next



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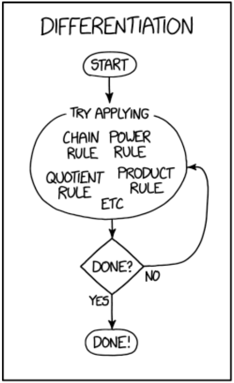

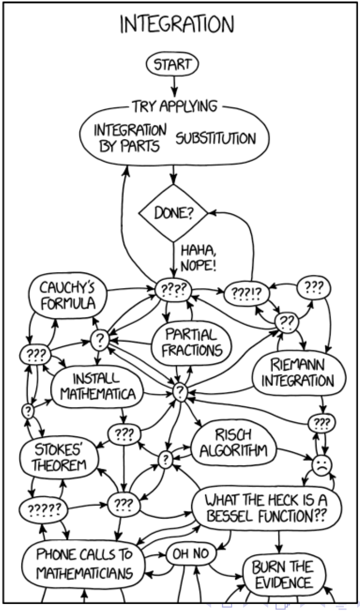

Week 6: Integration I

- Sum and Sigma notation
- Area as limit of sums
- The definite integral $\int_a^b$
- Properties of the integral
- Fundamental Theorem of calculus $\int_a^b f'(x)\,dx = f(b) - f(a)$
- The substitution method
- Partial integration

Forget calculation rules for now ...

Forget the calculation rules:

$$\int x\,dx = \frac{1}{2}x^2 + C.$$

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How do we then determine the area between $f(x) = x$ and the x-axis:

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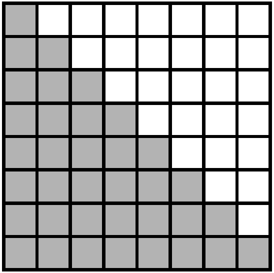
without these rules?

Answer: By approximating the area with a lot of rectangles!

Sum notation Σ for $1 + 2 + \cdots + n$

$1 + 2 + \cdots + n = ?$



$$1 + 2 + \ldots + n = \sum_{k=1}^{n} k =$$

Ex: $1 + 2 + \cdots + 99 + 100 = \sum_{k=1}^{100} = \frac{1}{2} \cdot 100 \cdot 101 = 5050$

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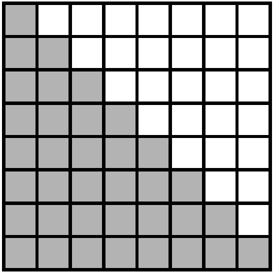

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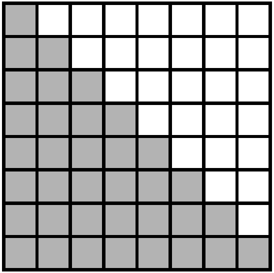

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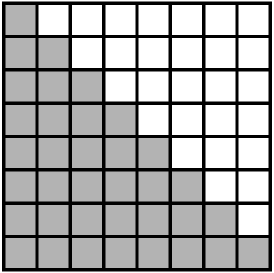

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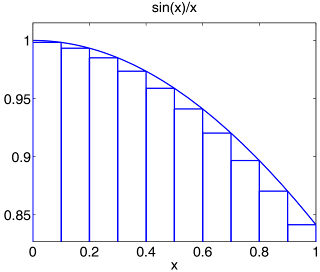
Integral notation $\int$ areas

Definite integral: $\int_a^b f(x)\,dx =$ "area between graph an x-axis"



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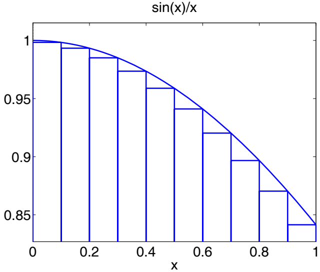

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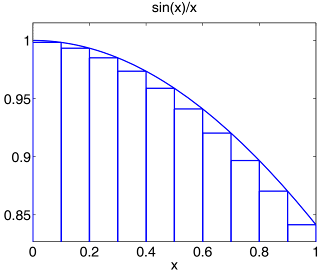

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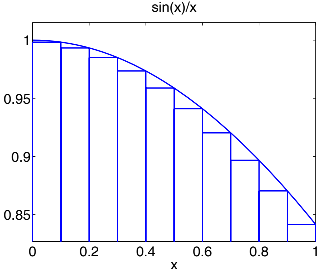

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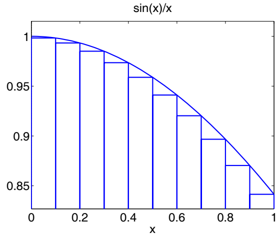
Approximate integral $\int$ (area) with $\sum$



The areas of the rectangles $R_1, \ldots, R_{10}$ approximate the area below the graph:

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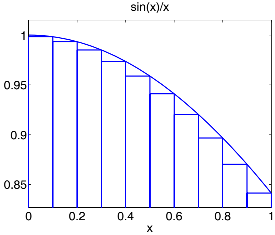

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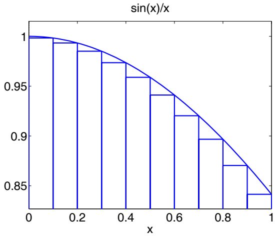

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$$\int_0^1 f(x)\,dx \approx \underbrace{0.1 \cdot f(0.1)}_{A(R_1)} + \underbrace{0.1 \cdot f(0.2)}_{A(R_2)} + \cdots + \underbrace{0.1 \cdot f(1)}_{A(R_{10})}$$

$$=$$

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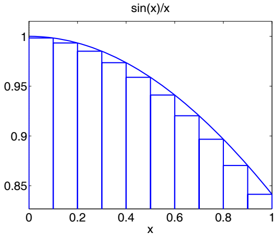

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$$= \underbrace{\sum_{k=1}^{10} 0.1 \cdot f\left(\frac{k}{10}\right)}_{\text{Riemann sum}}$$

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$$\int_a^b f(x)\,dx \underset{\text{definition}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\frac{b-a}{n}}_{\text{width}} \cdot \underbrace{f\left(a + k \cdot \frac{b-a}{n}\right)}_{\text{height}}$$

Integral calculation with limit

Question: For $f(x) = x$ determine with the limit definition

$$\int_0^1 x\,dx = \int_0^1 f(x)\,dx \underset{a=0,b=1}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1-0}{n} \cdot f(0 + k \cdot \frac{1-0}{n})$$

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&= \quad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot f(\frac{k}{n}) \\
&= \quad \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} f(\frac{k}{n}) \\
&\underset{f(x)=x}{=} \quad \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{k}{n} \\
&= \quad \lim_{n\to\infty} \frac{1}{n^2} \cdot \sum_{k=1}^{n} k
\end{aligned}$$

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&= \frac{1}{2} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}
\end{aligned}$$

so indeed:

Integral calculation with limit

Remember: For $f(x) = x$ determine with the limit definition

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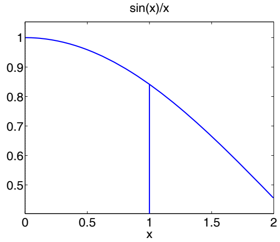
so indeed: $\int_0^1 x\,dx = [\frac{1}{2}x^2]_0^1 = \frac{1}{2}1^2 - \frac{1}{2}0^2 = \frac{1}{2}$.

Properties of the integral

The limit definition for $\displaystyle\int_a^b f(x)\,dx$ leads to

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



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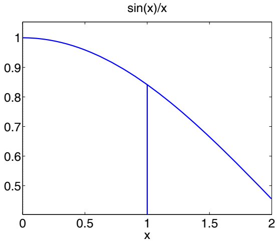
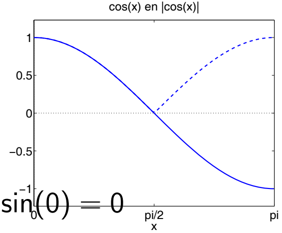
Ex: $$\int_0^2 \frac{\sin(x)}{x}\,dx = \int_0^1 \frac{\sin(x)}{x}\,dx + \int_1^2 \frac{\sin(x)}{x}\,dx$$



Properties of the integral

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Ex: $\underbrace{\left|\int_0^\pi \cos(x)\,dx\right|}_{=0} \leq \underbrace{\int_0^\pi |\cos(x)|}_{=2}\,dx$ because



$$\int_0^\pi \cos(x)\,dx = [\sin(x)]_0^\pi = \sin(\pi) - \sin(0) = 0$$

and

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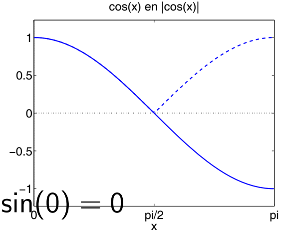

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$$\int_0^\pi |\cos(x)|\,dx = \underbrace{\int_0^{\frac{1}{2}\pi} |\cos(x)|\,dx}_{\cos(x)\ge 0,\, x\in[0,\pi/2]} + \underbrace{\int_{\frac{1}{2}\pi}^{\pi} |\cos(x)|\,dx}_{\cos(x)\le 0,\, x\in[\pi/2,\pi]}$$

$$=$$

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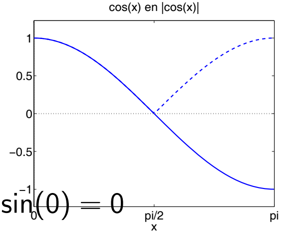

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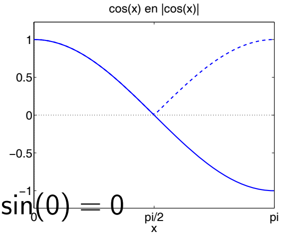

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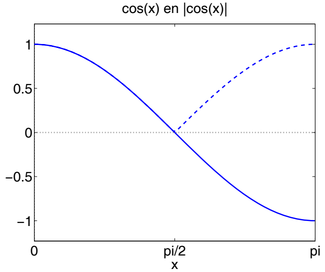
&= \int_0^{\frac{1}{2}\pi} \cos(x)\,dx + \int_{\frac{1}{2}\pi}^{\pi} -\cos(x)\,dx \\
&= 1 - -1 = 2
\end{aligned}$$

Ex: Integral with absolute value: Determine where $f(x) = 0$:



$$\int_0^{3\pi/2} |\cos(x)| \, dx =$$

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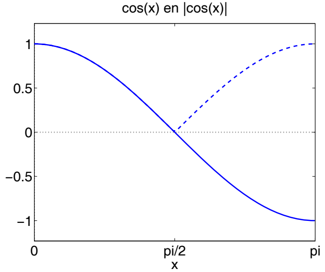
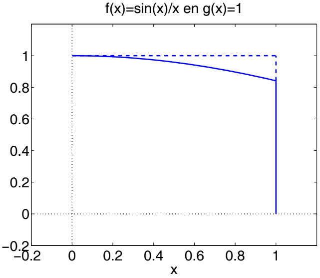


$$\int_0^{3\pi/2} |\cos(x)|\, dx = \int_0^{\pi/2} \cos(x)\, dx + \int_{\pi/2}^{3\pi/2} (-\cos(x))\, dx$$

Properties of the integral

Area under 2 graphs:
If $a \leq b$ and $f(x) \leq g(x)$ for $x \in [a, b]$ then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$



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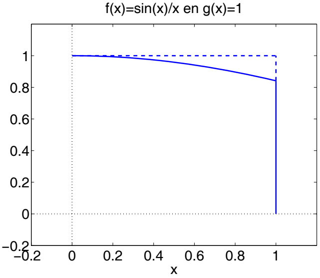
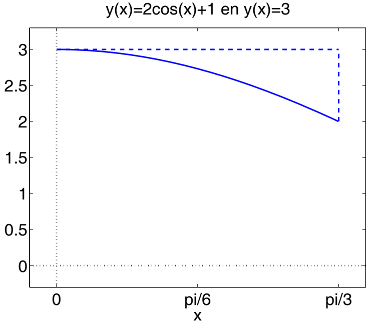


Ex: $\displaystyle\int_0^1 \frac{\sin(x)}{x}\,dx \le \int_0^1 1\,dx = 1$

Area between 2 graphs: If $f(x) \geq g(x)$ on $[a, b]$,

then the area between the graphs of f and g is: $\boxed{\int_a^b (f(x) - g(x))\, dx}$

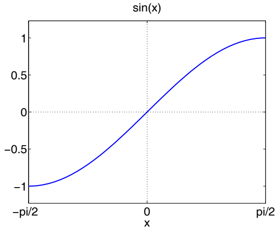


area $= \int_0^{\pi/3} (3 - (2\cos(x) + 1))\, dx$
One is calculating the difference of two areas.

If f is an odd function

then $\int_{-a}^{a} f(x)\,dx = 0$

Ex: $\int_{-\pi/2}^{\pi/2} \sin(x)\,dx = 0$



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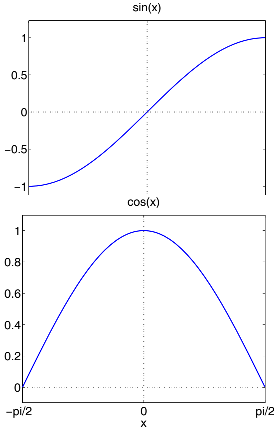

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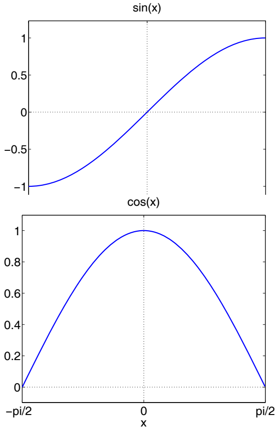
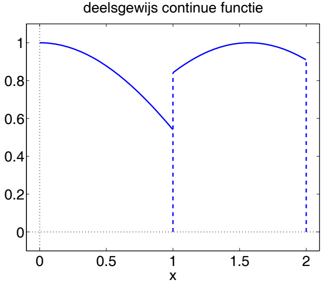


Properties of the integral (VI)

$\int$ for piecewise continuous function: divide in pieces and add:



Ex: $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{b} f(x)\,dx$

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- If f is an even function, than $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
- $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

Average value of a series of numbers and functions

Average value of a series of numbers:

$$\boxed{\tfrac{1}{n}(a_1 + a_2 + \cdots + a_n) = \tfrac{1}{n}\sum_{k=1}^{n} a_k}$$

"Integral = infinite sum"

Average value of a function on interval:

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

Primitive functions

Assume $F(x)$ is a function with

$$F'(x) = f(x)$$

as derivative. Then $F(x)$ is called a primitive (function) of $f(x)$

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If $F(x)$ is a primitive of $f(x)$, then is $F(x) + C$ and also:

$$\frac{d}{dx}(F(x) + C) = \underbrace{\frac{d}{dx}F(x)}_{f(x)} + \underbrace{\frac{d}{dx}C}_{0} = f(x)$$

so a primitive is only determined up to a constant $+ C$!

So do not say the primitive of f but a primitive of f. Ex: $f(x) = x$ has as primitive $\frac{1}{2}x^2 + C$, C: integration constant

Assume F is a primitive function of f (thus $F' = f$):

Fundamental theorem of calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Primitives

f	$F = \int f + C$
x^r	

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$-\frac{1}{\sqrt{1-x^2}}$	$\arccos(x) + C$
$\tan(x)$	$-\ln\lvert\cos(x)\rvert + C \quad$ (see[S 5.6])

Quiz: Integration and primitives

- $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$
- $\int_{0}^{\ln(3)} e^{3x}\, dx$
- $\int_{0}^{3} (\sqrt{2} + 1) x^{\sqrt{2}}\, dx$
- $\int_{-3}^{-2} \frac{dx}{x} = \int_{-3}^{-2} \frac{1}{x}\, dx$
- $\int_{-2}^{2} (3x + 2)^2\, dx$
- $\int_{-\pi/5}^{\pi/5} (x^2 + \sin(x))\, dx$
- $\int_{1}^{8} x^{-5/3}\, dx$
- average value of $f(x) = (x - 7)^2$ on $[0, 12]$?

Substitution method

Consider the example

$$\int \cos(x^2)2x\,dx$$

We can't see a primitive of $f(x) = \cos(x^2)2x$.
But we see that $2x$ is the derivative of $g(x) = x^2$.

$$\int \cos(x^2)2x\,dx = \int \cos(g(x))g'(x)\,dx \overset{u=g(x)}{=} \int \cos(u)\,du$$
$$= \sin(u) + C = \sin(x^2) + C$$

Why is $g'(x)\,dx = du$?

$$g'(x) = \frac{d}{dx}g(x) \overset{u=g(x)}{=} \frac{d}{dx}u = \frac{du}{dx} \overset{\cdot dx}{\Leftrightarrow} g'(x)\,dx = du$$

Substitution rule

$$\int_a^b f(g(x))g'(x)\,dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)\,du$$

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How to determine $\int_0^2 f(x)\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx$

The function in the integral is not immediately obvious to be of type $f(g(x))g'(x)$... but it doesn't have to be!

To determine $\int_0^2 f(x)\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}} \quad dx$

We can't see a primitive of f – and try substitution:

$\boxed{x^2 = u} \implies x = \pm\sqrt{(u)}$ and thus also $\frac{x}{\sqrt{1+x^2}} = \frac{\pm\sqrt{u}}{\sqrt{1+u}}$ Further

$$\frac{\mathrm{d}}{\mathrm{d}x}u \quad = \quad \frac{\mathrm{d}}{\mathrm{d}x}x^2 \implies \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \implies \mathrm{d}u = 2x\mathrm{d}x \implies$$

Integrals with substitution

To determine $\int_0^2 f(x)\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\quad dx$

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$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}u &= \frac{\mathrm{d}}{\mathrm{d}x}x^2 \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Longrightarrow \mathrm{d}u = 2x\mathrm{d}x \Longrightarrow \\
\mathrm{d}x &= \frac{1}{2x}\mathrm{d}u = \frac{1}{2}\frac{1}{\pm\sqrt{u}}\mathrm{d}u
\end{aligned}$$

Also

To determine $\int_0^2 f(x)\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\;dx$

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Also $x = 0 \Longrightarrow u = 0^2 = 0$ and $x = 2 \Longrightarrow u = 4$

Together

$$\int_0^2 \frac{x}{\sqrt{1+x^2}}\;dx \quad =$$

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Integrals with substitution

To determine $\int_0^2 f(x)\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx$

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$$\begin{aligned} \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx &= \int_0^4 \frac{\pm\sqrt{u}}{\sqrt{1+u}}\frac{1}{2}\frac{1}{\pm\sqrt{u}}\mathrm{d}u = \frac{1}{2}\int_0^4 \frac{1}{\sqrt{1+u}}\mathrm{d}u \\ &= \end{aligned}$$

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To determine $\int_0^2 f(x)\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^2 \frac{x}{\sqrt{1+x^2}} \quad dx$

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Together

$$\begin{aligned} \int_0^2 \frac{x}{\sqrt{1+x^2}} \quad dx &= \int_0^4 \frac{\pm\sqrt{u}}{\sqrt{1+u}}\frac{1}{2}\frac{1}{\pm\sqrt{u}}\mathrm{d}u = \frac{1}{2}\int_0^4 \frac{1}{\sqrt{1+u}}\mathrm{d}u \\ &= \frac{1}{2}\left[2\sqrt{1+u}\right]_0^4 = \sqrt{5} - \sqrt{1} \end{aligned}$$

Substitution

Substitution rule

$$\int_a^b f(g(x))g'(x)\,dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)\,du$$

Step-by-step guide integration with substitution:

1. Replace $g(x)$ by u (substitute $u = g(x)$)
2. Replace $g'(x)\,dx$ by du
3. See to it that you also replace all remaining x by u
4. Evt: boundaries $x \to$ boundaries for u
   is boundaries are given, and this is possible

Substitution tips

- Fractions $\int \frac{t(x)}{n(x)}$: Try $u = n(x)$
- Fractions $\int \frac{t(x)}{\sqrt{n(x)}}$: Try $u = n(x)$ or $u = \sqrt{n(x)}$
- Powers $\int \cos(x)e^{\sin(x)}$: Try exponent $u = \sin(x)$
- Chain rule $\int h(x) \cdot f'(x) \cdot f(x)$: Try $u = f(x)$. Ex: If $f(x) = \arccos(x)$ then is $f'(x) = -\frac{1}{\sqrt{1-x^2}}$ in
$$\int -\frac{(\arccos(x))^4}{\sqrt{1-x^2}}\, dx$$
- Practice!

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- Practice! ...Practice more!

Primitives with substitution

Difference between calculating primitives and integrals:

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1. skip integration boundaries

Primitives with substitution

Difference between calculating primitives and integrals:

1. skip integration boundaries
2. add integration constant $+C$!

Primitives with substitution

Difference between calculating primitives and integrals:

1. skip integration boundaries
2. add integration constant $+\, C$!
3. back substitution

Primitives with substitution

To determine $\displaystyle\int \frac{x^5}{2-x^6}\,dx = \int \frac{1}{2-x^6}\;\; x^5\,dx$

We can't see a primitive of f – and try substitution:

$\boxed{2-x^6=u}$ Further

$$\frac{\mathrm{d}}{\mathrm{d}x}u \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\left(2-x^6\right) \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -6x^5 \Longrightarrow$$

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Integration boundaries: not needed. Together

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Primitives with substitution

To determine $\int x^3\sqrt{x^2+2}\,dx \left(= \int x^2\sqrt{x^2+2}\cdot x\,dx \right)$

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$$\frac{\mathrm{d}}{\mathrm{d}x}u = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+2\right) \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Longrightarrow$$

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Integration boundaries: not needed. Together

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$$\begin{aligned} \int x^3\sqrt{x^2+2}\,dx &= \int x^2\sqrt{x^2+2}\;x\,dx = \int (u-2)\sqrt{u}\cdot\frac{1}{2}\mathrm{d}u \\ &= \end{aligned}$$

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\int x^3\sqrt{x^2+2}\,dx &= \int x^2\sqrt{x^2+2}\;x\,dx = \int (u-2)\sqrt{u}\cdot\frac{1}{2}\mathrm{d}u \\
&= \frac{1}{2}\int u^{3/2} - 2u^{1/2}\mathrm{d}u \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right) + C \\
&= \frac{1}{2}\left(\frac{2}{5}(x^2+2)^{5/2} - \frac{4}{3}(x^2+2)^{3/2}\right) + C
\end{aligned}$$

Quiz: The substitution rule

- $\int \frac{x^5}{\sqrt{2-x^6}}\,dx$
- $\int x\sqrt[5]{49+x^2}\,dx$
- $\int \sin(3x-2)\,dx$ "direct" and with substitution
- $\int x^2\left(\frac{x^3}{27}+3\right)^8 dx$
- $\int \frac{3}{x^4}\cos\left(\frac{1}{x^3}-2\right)dx$
- $\int \frac{1}{(x+3)\ln(x+3)}\,dx$
- $\int \frac{21}{4+25x^2}\,dx$
- $\int_0^{3\sqrt{3}} \frac{13}{\sqrt{36-x^2}}\,dx$
- $\int_0^{\pi/4}(1-\cos(2x))\sin(2x)\,dx$

Integration/primitives with cos and sin

To determine $\int f(x)\,dx = \int \frac{6\sin(6x)}{5-\cos(6x)}\,dx$

We can't see a primitive of f – and try substitution:

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$$\frac{\mathrm{d}}{\mathrm{d}x}u = \frac{\mathrm{d}}{\mathrm{d}x}(5-\cos(6x)) \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 6\sin(6x) \Longrightarrow$$

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$$\begin{aligned}\int \frac{6\sin(6x)}{5-\cos(6x)}\,dx &= \int \frac{1}{5-\cos(6x)}\;6\sin(6x)\,dx = \int \frac{1}{u}\cdot \mathrm{d}u \\ &=\end{aligned}$$

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$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx$

Substitution:

Integration/primitives with cos and sin

$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx$
Substitution: $u = \cos(x)$, then $du = -\sin(x)dx$, thus:
$\int \frac{\sin(x)}{\cos(x)}\, dx = \int \frac{-1}{u}\, du = -\ln|u| = -\ln|\cos(x)|$

$\int \tan^2(x)\, dx$

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$\int \tan^2(x)\,dx = \int \frac{\sin^2(x)}{\cos^2(x)}\,dx = \int \frac{1-\cos^2(x)}{\cos^2(x)}\,dx$

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$\int \tan^2(x)\, dx = \int \frac{\sin^2(x)}{\cos^2(x)}\, dx = \int \frac{1-\cos^2(x)}{\cos^2(x)}\, dx$

$= \int \left(\frac{1}{\cos^2(x)} - 1\right) dx$

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$\int \tan^2(x)\, dx = \int \frac{\sin^2(x)}{\cos^2(x)}\, dx = \int \frac{1-\cos^2(x)}{\cos^2(x)}\, dx$
$= \int \left(\frac{1}{\cos^2(x)} - 1\right) dx = \tan(x) - x$

Integrals with $\sin^m(x)$, $\cos^n(x)$

$\int \sin^4(x)\cos(x)\,dx =$

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$\int \sin^4(x)\cos(x)\,dx = \frac{1}{5}\sin^5(x)$

Too fast? $u = \sin(x)$ then $du = \cos(x)\,dx$ and

$\int \sin^4(x)\cos(x)\,dx = \int u^4 du = \frac{1}{5}u^5 = \frac{1}{5}\sin^5(x)$

$\int \sin^4(x)\cos^3(x)\,dx =$

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$\int \sin^4(x)\cos^3(x)\,dx = \int \sin^4(x)(1-\sin^2(x))\cos(x)\,dx$

$= \int(\sin^4(x) - \sin^6(x))\cos(x)\,dx =$

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$\int \sin^4(x)\cos^3(x)\,dx = \int \sin^4(x)(1-\sin^2(x))\cos(x)\,dx$

$= \int(\sin^4(x) - \sin^6(x))\cos(x)\,dx = \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x)$

With these techniques, all cases with $\sin^m(x)\cos^n(x)$ are covered if m or n (or both) is odd

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$\int \sin^2(x)\,dx = \int (\frac{1}{2} - \frac{1}{2}\cos(2x))\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x)$

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$\int \cos^2(x)\,dx = \int(\frac{1}{2} + \frac{1}{2}\cos(2x))\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x)$

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With these techniques, all cases with $\sin^m(x)\cos^n(x)$ are covered if m or n (or both) is odd

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$\int \cos^2(x)\,dx = \int(\frac{1}{2}+\frac{1}{2}\cos(2x))\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x)$

$\int \sin^4(x)\,dx = \int(\sin^2(x))^2\,dx = \int(\frac{1}{2}-\frac{1}{2}\cos(2x))^2\,dx$

$= \int(\frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x))\,dx$

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$\int \sin^4(x)\,dx = \int (\sin^2(x))^2\,dx = \int (\frac{1}{2} - \frac{1}{2}\cos(2x))^2\,dx$

$= \int (\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x))\,dx$

and use $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$

Integrals in $\sin(x)$ and $\cos(x)$

The substitution $\tan(\theta/2) = x$ transforms sin and cos to polynomials in x:

$$\begin{aligned} \cos\theta &= \frac{1-x^2}{1+x^2} \\ \sin\theta &= \frac{2x}{1+x^2} \\ d\theta &= \frac{2}{1+x^2}\,dx \end{aligned}$$

Quiz: Integration

- $\int_0^{\pi/6} 2^{\cos(3x)} \sin(3x)\, dx$
- $\int \frac{1}{\sqrt{23x}\,(\sqrt{23x}+3)}\, dx$
- $\int \frac{169}{25+169x^2}\, dx$
- $\int \frac{1}{64e^{-6x}+e^{6x}}\, dx$
- $\int \frac{1}{\sqrt{1-36x^2}}\, dx$
- $\int \frac{1}{\sqrt{36-x^2}}\, dx$
- $\int 8\,(13^{8x+9})\, dx$
- $\int \frac{4x^7}{x^4-5}\, dx$

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- $\int \frac{1}{\sqrt{36-x^2}}\, dx$
- $\int 8\,(13^{8x+9})\, dx$
- $\int \frac{4x^7}{x^4-5}\, dx = \int \frac{4x^3 \cdot x^4}{x^4-5}\, dx$
- $\int \frac{5-50x}{\sqrt{81-25x^2}}\, dx$
- $\int \cos^3(\frac{x}{2})\, dx$
- $\int_0^{\pi/4} 3x\, \sqrt{1-\cos(2x)}\, dx$

Partial integration

$$\int_a^b f\,G\,dx = [FG]_a^b - \int_a^b F\,g\,dx$$

with F primitive of f and G primitive of g (integration by parts)
Why?

$$\begin{aligned}
(FG)' &= Fg + fG \\
\int_a^b (FG)'\,dx &= \int_a^b Fg\,dx + \int_a^b fG\,dx \\
[FG]_a^b &= \int_a^b Fg\,dx + \int_a^b fG\,dx
\end{aligned}$$

Partial integration

$$\boxed{\int_a^b f\; G\; dx = [FG]_a^b - \int_a^b F\; g\; dx}$$

What to chose for f and G?

- Rule of thumb: chose as G the function you want to simplify
- Experience
- Trial: if it does not work, try a different choice for f and G
- Chose $f = 1$ is there is only one factor G

Of course, the book uses a different notation that can be confused with substitution

So with or without integration boundaries:

$$\int_a^b f\,G\,dx = [FG]_a^b - \int_a^b F\,g\,dx$$
$$\int f\,G\,dx = F\,G - \int F\,g\,dx$$

Ex:

$$\int_0^1 x\,e^x\,dx = \int_0^1 e^x x\,dx$$
$$=$$

Partial integration

So with or without integration boundaries:

$$\int_a^b f\,G\,dx = [FG]_a^b - \int_a^b F\,g\,dx$$

$$\int f\,G\,dx = F\,G - \int F\,g\,dx$$

Ex:

$$\int_0^1 x\,e^x\,dx = \int_0^1 e^x x\,dx$$

$$= \int_0^1 \underbrace{x}_{G}\underbrace{e^x}_{f}\,dx$$

So with or without integration boundaries:

$$\int_a^b f\,G\,dx = [FG]_a^b - \int_a^b F\,g\,dx$$
$$\int f\,G\,dx = F\,G - \int F\,g\,dx$$

Ex:

$$\begin{aligned}\int_0^1 x\,e^x\,dx &= \int_0^1 e^x x\,dx \\ &= \int_0^1 \underbrace{x}_{G}\underbrace{e^x}_{f}\,dx = [\underbrace{e^x}_{F}\underbrace{x}_{G}]_0^1 - \int_0^1 \underbrace{e^x}_{F}\underbrace{1}_{g}\,dx \\ &= (e-0) - [e^x]_0^1 = e - (e-1) = 1\end{aligned}$$

Partial integration with $f = 1$

Ex: Not a product of 2 functions!

$$\int \ln(x)\, dx \quad =$$

Partial integration with $f = 1$

Ex: Not a product of 2 functions!

$$\int \ln(x)\,dx = \int \underbrace{1}_{f}\underbrace{\ln(x)}_{G}\,dx = \underbrace{x}_{F}\underbrace{\ln(x)}_{G} - \int \underbrace{x}_{F}\underbrace{\frac{1}{x}}_{g}\,dx$$

$$= x\ln(x) - x$$

The same way also to get: ("p.i." is partial integration)

- $$\int \arcsin(x)\,dx = \int 1\cdot\arcsin(x)\,dx$$
  $$\underset{\text{p.i.}}{=} x\cdot\arcsin(x) - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$
  $$\underset{u=1-x^2}{=} x\ \arcsin(x) + \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \cdots$$
- $\int \arccos(x)\,dx$ and $\int \arctan(x)\,dx$

Partial integration (P.I.): Repeat

$$\begin{aligned}
\int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\
&= x^2 e^x - \left(2x e^x - \int 2 e^x \, dx \right) \\
&= x^2 e^x - (2x e^x - 2 e^x) \\
&= (x^2 - 2x + 2) e^x
\end{aligned}$$

P.I.: Repeating restores integral!

Sometimes you will get back to the start after 2 times partial integration, but this can be useful:

$$
\begin{aligned}
\int e^{ax}\sin(x)\,dx &= e^{ax}(-\cos(x)) - \int ae^{ax}(-\cos(x))\,dx \\
&\underset{\text{p.i.}}{=} -e^{ax}\cos(x) + a\int e^{ax}\cos(x)\,dx \\
&\underset{\text{p.i.}}{=} -e^{ax}\cos(x) + a\left(e^{ax}\sin(x) - \int ae^{ax}\sin(x)\,dx\right)
\end{aligned}
$$

Thus $\int e^{ax}\sin(x)\,dx = -e^{ax}\cos(x) + ae^{ax}\sin(x) - a^2\int e^{ax}\sin(x)\,dx$

and $(1+a^2)\int e^{ax}\sin(x)\,dx = -e^{ax}\cos(x) + ae^{ax}\sin(x)$

so that $\displaystyle\int e^{ax}\sin(x)\,dx = \frac{1}{1+a^2}e^{ax}(-\cos(x) + a\sin(x))$

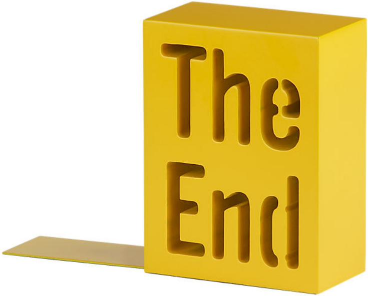
Quiz: Partial Integration

- $\int x \sin(5x)\, dx$
- $\int_{1/7}^{5} 16x \ln(7x)\, dx$
- $\int (6x^2 - 5x) e^{5x}\, dx$
- $\int_0^{2\pi} x^2 \sin(\frac{x}{4})\, dx$
- $\int \sin(\ln(5x))\, dx =$

Quiz: Partial Integration

- $\int x \sin(5x)\, dx$
- $\int_{1/7}^{5} 16x \ln(7x)\, dx$
- $\int (6x^2 - 5x)e^{5x}\, dx$
- $\int_0^{2\pi} x^2 \sin(\frac{x}{4})\, dx$
- $\int \sin(\ln(5x))\, dx = \int 1 \cdot \sin(\ln(5x))\, dx$
- $\int e^{3x} \cos(4x)\, dx$
- $\int x \tan^2(x)\, dx$

- Sums, sigma notation, area as limit of sums
- The definite integral $\int_a^b$
- The Fundamental Theorem of calculus
- Variable substitution
- Partial integration



Integration II/1st-order differential equations

- Integrals of rational functions $\displaystyle\int_a^b \frac{p(x)}{n(x)}\,dx$ 2 (L13)
- $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,dt$
- Improper integrals such as $\displaystyle\int_a^\infty f(x)\,dx$
- estimation of integrals
- 1st order differential equations $y' = p(x)y + q(x)$

1. First "divide-out" until the degree of numerator $<$ denominator
2. Factorize denominator by linear/quadratic factors
3. Partial fraction decomposition
4. Find primitives for the individual terms

Step 1: Division so that degree numerator $<$ denominator

Ex:

$$\int \frac{x^3+2}{x^3-x}\,dx =$$

Integration of rational function $\frac{p(x)}{n(x)}$

1. First "divide-out" until the degree of numerator $<$ denominator
2. Factorize denominator by linear/quadratic factors
3. Partial fraction decomposition
4. Find primitives for the individual terms

Step 1: Division so that degree numerator $<$ denominator

Ex:

$$\int \frac{x^3+2}{x^3-x}\,dx = \int \left(1 + \frac{x+2}{x^3-x}\right)\,dx$$

Sometimes you can do this by eye

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int \left(1 - \frac{1}{1+x^2}\right)dx = x - \arctan(x)$$

Factorize denominator

Step 2: Factorize denominator:
$x^2 - 1 =$

Factorize denominator

Step 2: Factorize denominator:
$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$$x^2 + 1 \underset{\substack{\text{Discriminant} \\ D<0}}{=} x^2 + 1$$

Factorize denominator

Step 2: Factorize denominator:
$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$x^2 + 1 \underset{\substack{\text{Discriminant} \\ D<0}}{=} x^2 + 1$ nothing to do: 1 quadratic factor (single)

$x^2 - 2x + 1 =$

Factorize denominator

Step 2: Factorize denominator:
$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$x^2 + 1 \underset{\substack{\text{Discriminant} \\ D<0}}{=} x^2 + 1$ nothing to do: 1 quadratic factor (single)

$x^2 - 2x + 1 = (x-1)^2$: 1 linear factor (double)

$x^2 - 2x + 2 \underset{\substack{\text{Discriminant} \\ D<0 \\ \text{split off square}}}{=}$

Factorize denominator

Step 2: Factorize denominator:
$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$x^2 + 1 \underset{\substack{\text{Discriminant} \\ D<0}}{=} x^2 + 1$ nothing to do: 1 quadratic factor (single)

$x^2 - 2x + 1 = (x-1)^2$: 1 linear factor (double)

$x^2 - 2x + 2 \underset{\substack{\text{Discriminant} \\ D<0 \\ \text{split off square}}}{=} (x-1)^2 + 1$: 1 quadratic factor (single)

$x^3 - x =$

Step 2: Factorize denominator:

$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$x^2 + 1 \underset{\substack{\text{Discriminant} \\ D<0}}{=} x^2 + 1$ nothing to do: 1 quadratic factor (single)

$x^2 - 2x + 1 = (x-1)^2$: 1 linear factor (double)

$x^2 - 2x + 2 \underset{\substack{\text{Discriminant} \\ D<0 \\ \text{split off square}}}{=} (x-1)^2 + 1$: 1 quadratic factor (single)

$x^3 - x = x\,(x^2 - 1) =$

Factorize denominator

Step 2: Factorize denominator:

$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$x^2 + 1 \underset{\substack{\text{Discriminant} \\ D<0}}{=} x^2 + 1$ nothing to do: 1 quadratic factor (single)

$x^2 - 2x + 1 = (x-1)^2$: 1 linear factor (double)

$x^2 - 2x + 2 \underset{\substack{\text{Discriminant} \\ D<0 \\ \text{split off square}}}{=} (x-1)^2 + 1$: 1 quadratic factor (single)

$x^3 - x = x\,(x^2 - 1) = x(x+1)(x-1)$: 3 linear factors (single)

$x^3 + x =$

Factorize denominator

Step 2: Factorize denominator:

$x^2 - 1 = (x+1)(x-1)$: 2 different linear factors (single)

$x^2 + 1 \underset{\substack{\text{Discriminant}\\ D<0}}{=} x^2 + 1$ nothing to do: 1 quadratic factor (single)

$x^2 - 2x + 1 = (x-1)^2$: 1 linear factor (double)

$x^2 - 2x + 2 \underset{\substack{\text{Discriminant}\\ D<0\\ \text{split off square}}}{=} (x-1)^2 + 1$: 1 quadratic factor (single)

$x^3 - x = x\,(x^2 - 1) = x(x+1)(x-1)$: 3 linear factors (single)

$x^3 + x = x\,(x^2 + 1)$: 1 linear (single) and 1 quadratic factor (single)

Quadratic factors: form $x^2 + a^2$

Quadratic factors: always of form $x^2 + a^2$: Split off square

$$\begin{aligned} x^2 - 2x + 2 &= (x-1)^2 + 1 \quad , D = 4 - 8 < 0 \\ x^2 + 2x + 4 &= (x+1)^2 + 3 \quad , D = 4 - 16 < 0 \end{aligned}$$

Substitution $y = (x \pm 1)$ leads to form $y^2 + a^2$:

$$\begin{aligned} x^2 - 2x + 2 &= (x-1)^2 + 1 \underset{y=x-1}{=} y^2 + 1^2 \\ x^2 + 2x + 4 &= (x+1)^2 + 3 \underset{y=x+1}{=} y^2 + \sqrt{3}^2 \end{aligned}$$

Polynomial $(x-\alpha)^2 - \beta^2$ has 2 factors:

$$(x-\alpha)^2 - \beta = x^2 - 2\alpha x + \alpha^2 - \beta^2 \Longrightarrow D = (2\alpha)^2 - 4 \cdot 1 \cdot (\alpha^2 - \beta^2) = \underbrace{4\beta^2}_{\geq 0}$$

Always check if $D < 0$!

Partial fractions: Linear factors

Step 3: Partial fraction decomposition: (inverse of bringing on one denominator):

$$\text{Linear factor single:}\quad \frac{t(x)}{x-a} = \frac{A}{x-a}$$

$$\text{Linear factor double:}\quad \frac{t(x)}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\text{Linear factor triple:}\quad \frac{t(x)}{(x-a)^3} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

Determine then the constants $A, \;\; B, \;\; \ldots$ and integrate

Partial fractions: Quadratic factors

Step 3: Partial fraction decomposition: (inverse of bringing on one denominator):

- Quadratic factor single yields 1 linear numerators eg

$$\frac{t(x)}{x^2+a^2} = \frac{Bx+C}{x^2+a^2}$$

- Quadratic factors double yields 2 linear numerators eg

$$\frac{t(x)}{(x^2+a^2)^2} = \frac{Bx+C}{x^2+a^2} + \frac{Dx+E}{(x^2+a^2)^2}$$

- Quadratic factors triple yields 3 linear numerators eg

$$\frac{t(x)}{(x^2+a^2)^3} = \frac{Bx+C}{x^2+a^2} + \frac{Dx+E}{(x^2+a^2)^2} + \frac{Fx+G}{(x^2+a^2)^3}$$

Determine then the constants $B, \ C, \ \ldots$ and integrate

Partial fractions: Mix of different factors

Step 3: Partial fraction decomposition: (inverse of bringing on one denominator):

- Mix linear factors eg
$$\frac{t(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \qquad \text{if } a \neq b$$
- Mix linear and quadratic factors eg
$$\frac{t(x)}{x\,(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$
- Mix linear and linear power 2 factor eg
$$\frac{t(x)}{x\,(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
- Mix linear and quadratic factor power 2 eg
$$\frac{t(x)}{x\,(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$

Partial fractions: Example

Ex: Split $\dfrac{p(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$. $\deg(p) < \deg(n)$. Steps:

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1. Division:

Partial fractions: Example

Ex: Split $\dfrac{p(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$. $\deg(p) < \deg(n)$. Steps:

1. Division: not needed. Thus $\dfrac{t(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$

Partial fractions: Example

Ex: Split $\dfrac{p(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$. $\deg(p) < \deg(n)$. Steps:

1. Division: not needed. Thus $\dfrac{t(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$
2. Factorize denominator: $n(x) = (x+8)(x+6)$
   (thus 2 linear single factors)

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   (thus 2 linear single factors)
3. Partial fractions:

$$\frac{t(x)}{n(x)} \quad = \quad \frac{13x+86}{(x+8)(x+6)} \underset{\text{partial fractions}}{=}$$

Partial fractions: Example

Ex: Split $\dfrac{p(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$. $\deg(p) < \deg(n)$. Steps:

1. Division: not needed. Thus $\dfrac{t(x)}{n(x)} = \dfrac{13x+86}{(x+8)(x+6)}$
2. Factorize denominator: $n(x) = (x+8)(x+6)$
   (thus 2 linear single factors)
3. Partial fractions:

$$\begin{aligned}
\frac{t(x)}{n(x)} &= \frac{13x+86}{(x+8)(x+6)} \underset{\text{partial fractions}}{=} \frac{A}{x+8} + \frac{B}{x+6} \\
&\underset{\text{bring on one denominator}}{=} \frac{A(x+6)+B(x+8)}{(x+8)(x+6)} \\
&= \frac{(A+B)x + 6A + 8B}{(x+8)(x+6)}
\end{aligned}$$

thus $A+B=13$ and $6A+8B=86 \Longrightarrow A=9$ and $B=4$

thus result: $\dfrac{13x+86}{(x+8)(x+6)} = \dfrac{9}{x+8} + \dfrac{4}{x+6}$

Step 4: Find primitives: Linear factors

Linear factors always have a constant numerator (type A)!

Is always of linear type $1 - 3$:

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Is always of linear type 1 – 3:

1. Single linear factor:

$$\int \frac{1}{x-a}\,dx =$$

Step 4: Find primitives: Linear factors

Linear factors always have a constant numerator (type A)!

Is always of linear type 1 – 3:

1. Single linear factor:

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C$$

Step 4: Find primitives: Linear factors

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2. Double linear factor:

$$\int \frac{1}{(x-a)^2}\,dx =$$

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3. Multiple linear factor ($k > 1$):

$$\int \frac{1}{(x-a)^k}\,dx =$$

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3. Multiple linear factor ($k > 1$):

$$\int \frac{1}{(x-a)^k}\,dx = \int (x-a)^{-k}\,dx = \frac{1}{1-k}\frac{1}{(x-a)^{k-1}} + C$$

Step 4: Find primitives: Quadratic factors (I)

Quadratic factors have a linear numerator (type $Ax + B$)!

Is always of quadratic type $1 - 7$:

1. $\displaystyle\int \frac{1}{a^2 + x^2}\, dx \underset{\substack{\text{evt subst} \\ au=x}}{=}$

Step 4: Find primitives: Quadratic factors (I)

Quadratic factors have a linear numerator (type $Ax + B$)!

Is always of quadratic type 1 – 7:

1. $$\int \frac{1}{a^2 + x^2}\, dx \underset{\substack{\text{evt subst} \\ au=x}}{=} \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$
2. $$\int \frac{x}{a^2 + x^2}\, dx \underset{\substack{\text{evt subst} \\ u=a^2+x^2}}{=}$$

Step 4: Find primitives: Quadratic factors (I)

Quadratic factors have a linear numerator (type $Ax + B$)!

Is always of quadratic type 1 – 7:

1. $$\int \frac{1}{a^2 + x^2}\, dx \underset{\substack{\text{evt subst} \\ au=x}}{=} \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$
2. $$\int \frac{x}{a^2 + x^2}\, dx \underset{\substack{\text{evt subst} \\ u=a^2+x^2}}{=} \frac{1}{2} \ln(a^2 + x^2) + C$$
3. $$\begin{aligned} \int \frac{A + Bx}{a^2 + x^2}\, dx \quad & \underset{\text{split}}{=} \quad A \int \frac{1}{a^2 + x^2}\, dx + B \int \frac{x}{a^2 + x^2}\, dx \\ & \underset{\substack{\text{use} \\ \text{1. and 2.}}}{=} \quad \frac{A}{a} \arctan\left(\frac{x}{a}\right) + \frac{B}{2} \ln(a^2 + x^2) + C \end{aligned}$$

Quadratic types 4 – 6: Ex:

4. $$\int \frac{1}{x^2+2x+4}\,dx \underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{1}{(\sqrt{3})^2+(x+1)^2}\,dx \underset{\substack{\text{evt subst}\\ \sqrt{3}u=x+1}}{=}$$

Step 4: Find primitives: Quadratic factors (II)

Quadratic types 4 – 6: Ex:

4. $$\int \frac{1}{x^2+2x+4}\,dx \underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{1}{(\sqrt{3})^2+(x+1)^2}\,dx \underset{\substack{\text{evt subst}\\ \sqrt{3}u=x+1}}{=} \frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right)+C$$

5. $$\int \frac{x}{x^2+2x+4}\,dx \underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx \underset{\text{later}}{=} \dots$$

6. $$\int \frac{A+Bx}{x^2+2x+4}\,dx \underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{A+Bx}{(\sqrt{3})^2+(x+1)^2}\,dx \underset{\text{split}}{=}$$
$$A\int \frac{1}{(\sqrt{3})^2+(x+1)^2}\,dx + B\int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx \underset{\substack{\text{combine}\\ \text{4. and 5.}}}{=} \dots$$

Step 4: Find primitives: Quadratic factors (III)

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$\underset{\substack{\text{split off}\\ \text{square}}}{=}$$

Step 4: Find primitives: Quadratic factors (III)

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$\underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx$$

$$\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=}$$

Step 4: Find primitives: Quadratic factors (III)

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$\underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx$$

$$\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=} \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2+(\sqrt{3})^2u^2}\,\sqrt{3}du =$$

Step 4: Find primitives: Quadratic factors (III)

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$\underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx$$

$$\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=} \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2+(\sqrt{3})^2u^2}\sqrt{3}du = \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2(1+u^2)}\sqrt{3}du$$

$$=$$

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$
\begin{aligned}
&\underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx \\
&\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=} \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2+(\sqrt{3})^2u^2}\,\sqrt{3}du = \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2(1+u^2)}\,\sqrt{3}du \\
&= \frac{1}{(\sqrt{3})^2}\int \frac{\sqrt{3}u-1}{1+u^2}\sqrt{3}\,du \\
&=
\end{aligned}
$$

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$
\begin{aligned}
&\underset{\substack{\text{split off}\\ \text{square}}}{=} && \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx \\
&\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=} && \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2+(\sqrt{3})^2u^2}\sqrt{3}du = \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2(1+u^2)}\sqrt{3}du \\
&= && \frac{1}{(\sqrt{3})^2}\int \frac{\sqrt{3}u-1}{1+u^2}\sqrt{3}\,du \\
&= && \int \frac{u}{1+u^2}\,dx - \frac{1}{\sqrt{3}}\int \frac{1}{1+u^2}\,dx \\
&=
\end{aligned}
$$

Step 4: Find primitives: Quadratic factors (III)

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$
\begin{aligned}
&\underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx \\
&\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=} \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2+(\sqrt{3})^2u^2}\sqrt{3}du = \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2(1+u^2)}\sqrt{3}du \\
&= \frac{1}{(\sqrt{3})^2}\int \frac{\sqrt{3}u-1}{1+u^2}\sqrt{3}\,du \\
&= \int \frac{u}{1+u^2}\,dx - \frac{1}{\sqrt{3}}\int \frac{1}{1+u^2}\,dx \\
&= \frac{1}{2}\ln\left|1+u^2\right| - \frac{1}{\sqrt{3}}\arctan(u) + C \\
&\underset{\substack{\text{back subst}\\ u=(x+1)/\sqrt{3}\\ u^2>0}}{=}
\end{aligned}
$$

Step 4: Find primitives: Quadratic factors (III)

Previous slide, type 5: $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

$$
\begin{aligned}
&\underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x}{(\sqrt{3})^2+(x+1)^2}\,dx \\
&\underset{\substack{\text{substitution}\\ \sqrt{3}u=x+1\\ \sqrt{3}du=dx}}{=} \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2+(\sqrt{3})^2u^2}\,\sqrt{3}du = \int \frac{\sqrt{3}u-1}{(\sqrt{3})^2(1+u^2)}\,\sqrt{3}du \\
&= \frac{1}{(\sqrt{3})^2}\int \frac{\sqrt{3}u-1}{1+u^2}\sqrt{3}\,du \\
&= \int \frac{u}{1+u^2}\,dx - \frac{1}{\sqrt{3}}\int \frac{1}{1+u^2}\,dx \\
&= \frac{1}{2}\ln\left|1+u^2\right| - \frac{1}{\sqrt{3}}\arctan(u) + C \\
&\underset{\substack{\text{back subst}\\ u=(x+1)/\sqrt{3}\\ u^2>0}}{=} \frac{1}{2}\ln\left(1+\left(\frac{x+1}{\sqrt{3}}\right)^2\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C
\end{aligned}
$$

Step 4: Find primitives: Quadratic factors (III)

Type 7:

7.
$$\int \frac{1}{(a^2+x^2)^{2}}\,dx, \int \frac{1}{(a^2+x^2)^{3}}\,dx, \int \frac{1}{(a^2+x^2)^{4}}\,dx, \ldots$$

not in curriculum – if needed is on formula sheet)

Finding primitives $p(x)/q(x)$

Ex: Find primitive of $\int \frac{p(x)}{n(x)}\,dx = \int \frac{x+1}{x^2-2x+2}\,dx$. Steps:

Finding primitives $p(x)/q(x)$

Ex: Find primitive of $\int \frac{p(x)}{n(x)}\, dx = \int \frac{x+1}{x^2-2x+2}\, dx$. Steps:

1. Division:

Finding primitives $p(x)/q(x)$

Ex: Find primitive of $\displaystyle\int \frac{p(x)}{n(x)}\,dx = \int \frac{x+1}{x^2-2x+2}\,dx$. Steps:

1. Division: not needed: $\deg(p) < \deg(n)$. Thus $\displaystyle\frac{t(x)}{n(x)} = \frac{x+1}{x^2-2x+2}$

Finding primitives $p(x)/q(x)$

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2. Factorize denominator: split off square $n(x) = 1 + (x-1)^2$, (thus 1 quadratic single factor)

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3. Partial fractions $\dfrac{t(x)}{n(x)} \underset{\substack{\text{split off}\\ \text{square}}}{=} \displaystyle\int \frac{x+1}{1+(x-1)^2}\,dx$
4. Integration (type 6) $\displaystyle\int \frac{x+1}{1+(x-1)^2}\,dx$

$$\underset{\substack{\text{substitution}\\ u=x-1,x=u+1\\ du=dx}}{=}$$

Finding primitives $p(x)/q(x)$

Ex: Find primitive of $\displaystyle\int \frac{p(x)}{n(x)}\,dx = \int \frac{x+1}{x^2-2x+2}\,dx$. Steps:

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$$\underset{\substack{\text{substitution}\\ u=x-1,x=u+1\\ du=dx}}{=} \int \frac{u+2}{1+u^2}\,du \underset{\text{split}}{=}$$

Finding primitives $p(x)/q(x)$

Ex: Find primitive of $\displaystyle\int \frac{p(x)}{n(x)}\,dx = \int \frac{x+1}{x^2-2x+2}\,dx$. Steps:

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3. Partial fractions $\displaystyle\frac{t(x)}{n(x)} \underset{\substack{\text{split off}\\ \text{square}}}{=} \int \frac{x+1}{1+(x-1)^2}\,dx$
4. Integration (type 6) $\displaystyle\int \frac{x+1}{1+(x-1)^2}\,dx$

$$\underset{\substack{\text{substitution}\\ u=x-1,x=u+1\\ du=dx}}{=} \int \frac{u+2}{1+u^2}\,du \underset{\text{split}}{=} \int \frac{u}{1+u^2}\,du + \int \frac{2}{1+u^2}\,du$$

$$\underset{\text{integrate}}{=}$$

Ex: Find primitive of $\int \frac{p(x)}{n(x)}\,dx = \int \frac{x+1}{x^2-2x+2}\,dx$. Steps:

1. Division: not needed: $\deg(p) < \deg(n)$. Thus $\frac{t(x)}{n(x)} = \frac{x+1}{x^2-2x+2}$
2. Factorize denominator: split off square $n(x) = 1 + (x-1)^2$, (thus 1 quadratic single factor)
3. Partial fractions $\frac{t(x)}{n(x)} \underset{\text{split off square}}{=} \int \frac{x+1}{1+(x-1)^2}\,dx$
4. Integration (type 6) $\int \frac{x+1}{1+(x-1)^2}\,dx$

$$\underset{\substack{\text{substitution}\\ u=x-1,x=u+1\\ du=dx}}{=} \int \frac{u+2}{1+u^2}\,du \underset{\text{split}}{=} \int \frac{u}{1+u^2}\,du + \int \frac{2}{1+u^2}\,du$$

$$\underset{\text{integrate}}{=} \frac{1}{2}\ln\left|1+u^2\right| + 2\arctan(u) + C$$

$$\underset{\substack{\text{back subst}\\ u=x-1\\ u^2>0}}{=}$$

Finding primitives $p(x)/q(x)$

Ex: Find primitive of $\displaystyle\int \frac{p(x)}{n(x)}\,dx = \int \frac{x+1}{x^2-2x+2}\,dx$. Steps:

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$$\underset{\substack{\text{back subst}\\ u=x-1\\ u^2>0}}{=} \frac{1}{2}\ln\left(1+(x-1)^2\right) + 2\arctan(x-1) + C$$

Quiz: Partial fractions

- $\int \frac{dx}{169-49x^2}$
- $\int \frac{x^2+8x+16}{(x^2+16)^2}\, dx$
- $\int_8^{13} \frac{x}{x^2-2x-15}\, dx$
- $\int \frac{14x+14}{(x^2+1)(x-1)^3}\, dx$
- $\int \frac{6e^{6x}}{e^{12x}+15e^{6x}+54}\, dx$
- $\int \frac{9x^3-6x+2}{x^3-x^2}\, dx$

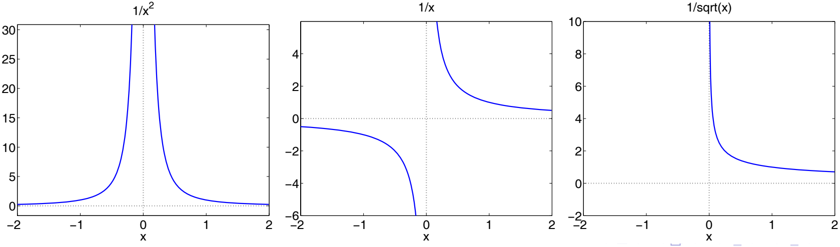
Improper integrals: The 2 types

Definition: An integral $\int_a^b f(x)\,dx$ is improper if

1. Type I: a or $b = \pm\infty$ or if
2. Type II: $f(x) \to \pm\infty$ if $x \to a$ or $x \to b$

Ex:

1. $\int_{1/2}^{\infty} \frac{1}{x^2}\,dx$ (left figure)
2. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (right figure)



Improper $\int$ Type I: a or b is $\pm\infty$

Improper integrals: Type I: a or b is $\pm\infty$. Ex: $\int_1^\infty 1/x^2\,dx$.
Definition: Assume F is the primitive of f so $F' = f$ then

$$\int_a^\infty f(x)\,dx =$$

Such an integral is either a number or does not exist:

1. $-\infty < \int_a^\infty f(x)\,dx < \infty$: The integral exists: "is convergent"
2. $\int_a^\infty f(x)\,dx = \pm\infty$: The integral does not exist: "is divergent"

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Definition: Assume F is the primitive of f so $F' = f$ then

$$\int_a^\infty f(x)\,dx = \left(\lim_{x\to\infty} F(x)\right) - F(a) \underset{\text{fill in}}{=}$$

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Ex: If $f(x) = x^{-2}$ then $F(x) = -x^{-1} + C = -\frac{1}{x} + C$ and

$$\int_1^\infty x^{-2}\,dx = \left(\lim_{x\to\infty} -\frac{1}{x}\right) - (-\frac{1}{-1}) = 0 - (-1) = 1 \qquad \text{(convergent).}$$

Also via filling in:

$$\int_1^\infty x^{-2}\,dx = \left[-x^{-1}\right]_1^\infty = 0 - (-1) = 1 \qquad \text{(convergent)}$$

Improper $\int$ Type I: a or b is $\pm\infty$

$$\int_1^\infty x^{-2}\,dx = [-x^{-1}]_1^\infty = \left[-\frac{1}{x}\right]_1^\infty = 0-(-1) = 1 \qquad \text{(convergent)}$$

$$\int_1^\infty x^{-3/2}\,dx = [-2x^{-1/2}]_1^\infty = \left[-2\frac{1}{x^{1/2}}\right]_1^\infty = 2 \qquad \text{(convergent)}$$

$$\int_1^\infty x^{-1}\,dx = [\ln|x|]_1^\infty = \infty - 0 = \infty \qquad \text{(divergent)}$$

$$\int_1^\infty x^{-1/2}\,dx = [2x^{1/2}]_1^\infty = [2\sqrt{x}]_1^\infty = \infty - 2 = \infty \qquad \text{(divergent)}$$

Lower boundary $1>0$ no influence on answer. "∞": If $a>0$ then

$$\boxed{\text{integral } \int_a^\infty x^r\,dx < \infty \text{ is convergent} \iff r < -1}$$

Different way of saying this: If $a > 0$ then

$$\boxed{\text{integral } \int_a^\infty x^r\,dx < \infty \text{ is convergent } \iff r < -1}$$

is the same as

$$\boxed{\text{integral } \int_a^\infty \frac{1}{x^r}\,dx < \infty \text{ is convergent } \iff r > 1}$$

Improper $\int$ Type II: f is $\pm\infty$ for a or b

Definition: Assume F is the primitive of f so $F' = f$ and assume $\lim_{x \to a^+} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx =$$

Definition: Assume F is the primitive of f so $F' = f$ and assume $\lim_{x \to a^+} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx = F(b) - \left(\lim_{x \to a^+} F(x)\right) \underset{\text{invullen}}{=}$$

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$$\boxed{\int_a^b f(x)\, dx = F(b) - \left(\lim_{x \to a^+} F(x)\right) \underset{\text{invullen}}{=} F(b) - F(a)}$$

Such an integral is a number or does not exist:

Definition: Assume F is the primitive of f so $F' = f$ and assume $\lim_{x \to a^+} f(x) = \pm\infty$ then

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Such an integral is a number or does not exist:

1. $-\infty < \int_a^b f(x)\,dx < \infty$: The integral exists: "is convergent"
2. $\int_a^b f(x)\,dx = \pm\infty$: The integral does not exist: is divergent

$$\int_0^1 x^{-3/2}\,dx = \left[-2\frac{1}{\sqrt{x}}\right]_{0^+}^{1} = -2-(-\infty) = \infty \qquad \text{(divergent)}$$

$$\int_0^1 x^{-1}\,dx = \ln(|1|) - \ln(\left|0^+\right|) = 0-(-\infty) = \infty \qquad \text{(divergent)}$$

$$\int_0^1 x^{-1/2}\,dx = \left[2\sqrt{x}\right]_{0^+}^{1} = 2-0 = 2 \qquad \text{(convergent)}$$

$$\int_0^1 x^{2}\,dx = \left[\frac{1}{3}x^3\right]_{0^+}^{1} = \frac{1}{3} - 0 \qquad \text{(convergent)}$$

Not different from type I! I and II together: If $a, b > 0$ then divergence if

$$\boxed{\begin{aligned} \int_0^b x^r\,dx &\underset{\text{Type II}}{=} \infty \iff r \leq -1 \text{ and} \\ \int_a^\infty x^r\,dx &\underset{\text{Type I}}{=} \infty \iff r \geq -1 \end{aligned}}$$

$0 \leq f(x) \leq g(x)$ then also $0 \leq \int_c^\infty f(x)\,dx \leq \int_c^\infty g(x)\,dx$ thus:

- $\int_c^\infty g(x)\,dx$ convergent $\implies \int_c^\infty f(x)\,dx$ convergent
- $\int_c^\infty f(x)\,dx$ divergent $\implies \int_c^\infty g(x)\,dx$ divergent

Ex: Convergent or divergent $\int_2^\infty f(x) = \int_2^\infty \frac{dx}{x^3+9}$?

$\int$ estimation: Convergence/divergence

$0 \leq f(x) \leq g(x)$ then also $0 \leq \int_c^\infty f(x)\,dx \leq \int_c^\infty g(x)\,dx$ thus:

- $\int_c^\infty g(x)\,dx$ convergent $\implies \int_c^\infty f(x)\,dx$ convergent
- $\int_c^\infty f(x)\,dx$ divergent $\implies \int_c^\infty g(x)\,dx$ divergent

Ex: Convergent or divergent $\int_2^\infty f(x) = \int_2^\infty \frac{dx}{x^3+9}$?

$$f(x) = \frac{1}{x^3+9} \underset{9\geq 0}{\leq} \frac{1}{x^3} = g(x) \text{ and}$$

$$\int_2^\infty g(x)\,dx = \int_2^\infty \frac{dx}{x^3}\,dx$$

$0 \le f(x) \le g(x)$ then also $0 \le \int_c^\infty f(x)\,dx \le \int_c^\infty g(x)\,dx$ thus:

- $\int_c^\infty g(x)\,dx$ convergent $\implies \int_c^\infty f(x)\,dx$ convergent
- $\int_c^\infty f(x)\,dx$ divergent $\implies \int_c^\infty g(x)\,dx$ divergent

Ex: Convergent or divergent $\int_2^\infty f(x) = \int_2^\infty \frac{dx}{x^3+9}$?

$f(x) = \frac{1}{x^3+9} \underset{9\ge 0}{\le} \frac{1}{x^3} = g(x)$ and

$\int_2^\infty g(x)\,dx = \int_2^\infty \frac{dx}{x^3}\,dx = \left[-\frac{1}{2}\frac{1}{x^2}\right]_2^\infty = 0 + \frac{1}{8} < \infty$ convergent thus

answer: The integral of f is convergent

$\int$ estimation: Accuracy

Reminder: an integral $\int_a^b f(x)\,dx$ is the area under the graph of f between a and b. Because for f continuous over a closed interval $[a, b]$

$$f_{\min} := \min_{x\in[a,b]}\{f(x)\} \leq f(x) \leq \max_{x\in[a,b]}\{f(x)\} =: f_{\max}$$

it holds (integration)

$$\begin{aligned}
\int_a^b f_{\min}\,dx &\leq \int_a^b f(x)\,dx \leq \int_a^b f_{\max}\,dx \iff \\
f_{\min}\cdot\int_a^b 1\,dx &\leq \int_a^b f(x)\,dx \leq f_{\max}\cdot\int_a^b 1\,dx \iff \\
f_{\min}\cdot(b-a) &\leq \int_a^b f(x)\,dx \leq f_{\max}\cdot(b-a)
\end{aligned}$$

$\int$ estimation: Accuracy

remember:

$$\underbrace{f_{\text{min}}}_{\min_{x\in[a,b]}\{f(x)\}} \cdot (b-a) \le \int_a^b f(x)\,dx \le \underbrace{f_{\text{max}}}_{\max_{x\in[a,b]}\{f(x)\}} \cdot (b-a)$$

Ex: Estimate $\int_0^2 f(x)\,dx$ for $f(x) = 10 + x^3$

Minimum and maximum of f on $[a,b] = [0,2]$:

Calculate critical points $f'(x) = 0$, etc., or

see that $f(x) = 10 + x^3$ increases so that

minimum $f(x)$ over $[0,2]$ is $f_{\text{min}} = f(0) = 10$

maximum $f(x)$ over $[0,2]$ is $f_{\text{max}} = f(2) = 10 + 2^3 = 18$, thus

$$\begin{aligned} f_{\text{min}} \cdot (b-a) &\le \int_a^b 10 + x^3\,dx \le f_{\text{max}} \cdot (b-a) \\ 10 \cdot (2-0) &\le \int_0^2 10 + x^3\,dx \le 18 \cdot (2-0) \\ 20 &\le \int_0^2 10 + x^3\,dx \le 36 \end{aligned}$$

$\int$ estimation: Accuracy

Remember:

$$\underbrace{f_{\min}}_{\min_{x\in[a,b]}\{f(x)\}} \cdot (b-a) \le \int_a^b f(x)\,dx \le \underbrace{f_{\max}}_{\max_{x\in[a,b]}\{f(x)\}} \cdot (b-a)$$

Ex: Estimate $\int_0^2 f(x)\,dx$ for $f(x) = \sin(x) + 1$

Minimum and maximum of f on $[a,b] = [0,2]$:

Calculate critical points $f'(x) = 0$, etc., or

see that $0 \le \sin(x) \le 1$ for all $x \in [0,2]$

so that $f_{\min} := 1 \le \sin(x) + 1 \le 2 =: f_{\max}$, thus

$$\begin{aligned} f_{\min} \cdot (b-a) &\le \int_a^b \sin(x) + 1\,dx \le f_{\max} \cdot (b-a) \\ 1 \cdot (2-0) &\le \int_0^2 \sin(x) + 1\,dx \le 2 \cdot (2-0) \\ 2 &\le \int_0^2 \sin(x) + 1\,dx \le 4 \end{aligned}$$

Quiz: Improper integrals

- $\int_5^\infty \frac{dx}{x^2+25}$
- $\int_0^1 \frac{dx}{x^{1/3}}$
- $\int_{14}^\infty \frac{9}{x^2-x}\,dx$
- $\int_{-\infty}^\infty \frac{2x}{(x^2+1)^5}\,dx$
- $\int_{-\infty}^0 e^{-|14x|}\,dx$
- $\int_0^3 \frac{2x+11}{\sqrt{x^2+11x}}\,dx$
- $\int_0^{1/3} x\ln(3x)\,dx$
- $\int_{-32}^2 \frac{dx}{\sqrt{|2x|}}$
- Convergent $\int_0^{\pi/8} \tan(4x)\,dx$?
- Convergent $\int_0^{\ln(5)} 9x^{-2}e^{-9/x}\,dx$?
- Convergent $\int_9^\infty \frac{e^{9x}}{x}\,dx$?
- Convergent $\int_{-\infty}^\infty \frac{dx}{x^{10}+5}$?

Master step plan integration

1. Standard (partially on formula sheet)
2. Substitution
   - Recognize derivative
   - $u =$ those where also the derivative shows up
   - If there is choice: substitute the hardest
3. Partial integration
   - If product of 2 functions; pick hardest $= G$
   - If 1 function, pick $f = 1$
   - Evt do several times
   - Evt 2 times and see restored integral
4. Rational functions
   - Divide out if degree numerator $\geq$ denominator
   - Factorize the denominator in linear or evt quadratic factors
   - Partial fractions
   - Integrate each term separately
     (linear factor gives ln, quadratic ln or arctan)
5. Improper integrals
6. Estimation of integrals

Old exam examples

$\int \frac{\ln(x)}{x^2}\,dx$

$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx$

$\int (1-\cos(3x))\sin(3x)\,dx$

$\int x^2\arctan(x)\,dx$

$\int \frac{e^x}{(e^x+1)^2}\,dx$

$\int 2x\arctan(x)\,dx$

$\int x^2(1+x)^{37}\,dx$

$\int \frac{x}{e^{2x}}\,dx$

$\int \sin(x)\cos(x)e^{\sin(x)}\,dx$

$\int \frac{x^4-13x-10}{x^3-2x-4}\,dx$

$\int (x+1)e^{5x}\,dx$

$\int \frac{1}{(1+x)\sqrt{x}}\,dx$

$\int \frac{\sin(\ln(x))}{x}\,dx$

$\int 2x^3e^{x^2}\,dx$

$\int \frac{1}{\sqrt{7+6x-x^2}}\,dx$

$\int e^{2x}\sin(e^x)\,dx$

$\int \frac{e^x}{\sqrt{10-e^x}}\,dx$

$\int x\cos(x)\,dx$

$\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

$\int \frac{\cos(\sqrt{x+2})}{\sqrt{x+2}}\,dx$

$\int \frac{1}{x^2+4x}\,dx$

$\int \frac{1}{x^3+9x}\,dx$

$\int (x^2+1)\ln(x)\,dx$

$\int 2x^3\sqrt{x^2+1}\,dx$

Differential equations (DEs)

Overview:

- Discrete growth
- 1st order separable
- 1st order linear

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T	Money
1	

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T	Money	
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T	Money
1	

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T		Money
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T		Money
1	$100 \cdot (1.03)$	$100 \cdot 1.0300$
2		

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T		Money
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T		Money
1	$100 \cdot (1.03)$	$100 \cdot 1.0300$
2	$100 \cdot (1.03) \cdot (1.03)$	$100 \cdot 1.0609$

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1		

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T		Money
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T		Money
1	$100 \cdot (1.03)$	$100 \cdot 1.0300$
2	$100 \cdot (1.03) \cdot (1.03)$	$100 \cdot 1.0609$

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2		

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T		Money
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T		Money
1	$100 \cdot (1.03)$	$100 \cdot 1.0300$
2	$100 \cdot (1.03) \cdot (1.03)$	$100 \cdot 1.0609$

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015) \cdot (1.015)$	$100 \cdot 1.0302$
3		

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T		Money
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T		Money
1	$100 \cdot (1.03)$	$100 \cdot 1.0300$
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Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015) \cdot (1.015)$	$100 \cdot 1.0302$
3	$100 \cdot (1.015) \cdot (1.015) \cdot (1.015)$	$100 \cdot 1.0457$
4		

Discrete growth

Assume 6% interest per $T = 1$ year, starting with 100 euro:

T		Money
1	$100 \cdot (1.06)$	$100 \cdot 1.06$

Assume $6/2 = 3\%$ interest per $T = 1$ semester, starting with 100 euro:

T		Money
1	$100 \cdot (1.03)$	$100 \cdot 1.0300$
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3	$100 \cdot (1.015) \cdot (1.015) \cdot (1.015)$	$100 \cdot 1.0457$
4	$100 \cdot (1.015)^4$	$100 \cdot 1.0614$

Interest distributed over shorter periods

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T	Money
1	

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2		

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3		

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3	$100 \cdot (1.015)^3$	$100 \cdot 1.0457$
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Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3	$100 \cdot (1.015)^3$	$100 \cdot 1.0457$
4	$100 \cdot (1.015)^4$	$100 \cdot 1.0614$

Assume $6/356 = 0.000164\%$ interest per $T = 1$ day, starting with 100:

T	Money
1	

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3	$100 \cdot (1.015)^3$	$100 \cdot 1.0457$
4	$100 \cdot (1.015)^4$	$100 \cdot 1.0614$

Assume $6/356 = 0.000164\%$ interest per $T = 1$ day, starting with 100:

T		Money
1	$100 \cdot (1.000164)$	$100 \cdot 1.0002$
2		

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3	$100 \cdot (1.015)^3$	$100 \cdot 1.0457$
4	$100 \cdot (1.015)^4$	$100 \cdot 1.0614$

Assume $6/356 = 0.000164\%$ interest per $T = 1$ day, starting with 100:

T		Money
1	$100 \cdot (1.000164)$	$100 \cdot 1.0002$
2	$100 \cdot (1.000164)^2$	$100 \cdot 1.0003$
3		

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3	$100 \cdot (1.015)^3$	$100 \cdot 1.0457$
4	$100 \cdot (1.015)^4$	$100 \cdot 1.0614$

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3	$100 \cdot (1.000164)^3$	$100 \cdot 1.0005$
4		

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

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Assume $6/356 = 0.000164\%$ interest per $T = 1$ day, starting with 100:

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2	$100 \cdot (1.000164)^2$	$100 \cdot 1.0003$
3	$100 \cdot (1.000164)^3$	$100 \cdot 1.0005$
4	$100 \cdot (1.000164)^4$	$100 \cdot 1.0007$
$\vdots$		
365		

Discrete growth

Assume $6/4 = 1.5\%$ interest per $T = 1$ quarter, starting with 100 euro:

T		Money
1	$100 \cdot (1.015)$	$100 \cdot 1.0150$
2	$100 \cdot (1.015)^2$	$100 \cdot 1.0302$
3	$100 \cdot (1.015)^3$	$100 \cdot 1.0457$
4	$100 \cdot (1.015)^4$	$100 \cdot 1.0614$

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T		Money
1	$100 \cdot (1.000164)$	$100 \cdot 1.0002$
2	$100 \cdot (1.000164)^2$	$100 \cdot 1.0003$
3	$100 \cdot (1.000164)^3$	$100 \cdot 1.0005$
4	$100 \cdot (1.000164)^4$	$100 \cdot 1.0007$
$\vdots$		
365	$100 \cdot (1.000164)^{365}$	$100 \cdot 1.0618$

Pay attention: End sum seems to stabilize factor relating start money with end amount is

$$\left(1+\frac{x}{n}\right)^n$$

for $x = 0.06$ $(= 6\%)$ and $n = 1, 2, 4, 365$. The limit exists

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

Our example with $x = 0.06$ limit is $e^{0.06} \approx 1.061836546545360$. Interest $x = 0.06$ distributed over $n = 31536000$ seconds:
$(1 + 0.06/31536000)^{31536000} = 1.061836543200352$
We will now consider continuous growth (continuous variation):

Differential Equations

1. Differential equations model the complex behavior of physical world.
2. Climate modelling, Big bang theory modelling. prediction of building failures all the natural phenomenon (heat equations, fluid flow equations, hooks law etc) can be modelled using differntial equations.
3. A large part of todays super computing powers is used to do modelling of differenetial equations and help us understand and predict the behavior of Nature.
4. even the heating and cooling problem of computer is modelled using the differential equation and solved by computer itself

First order differential equations

Definition

> Equation with function $y(t)$ and its derivative $y'(t)$

Attention: $y'(t)$: variation (growth/decay) at time t
Attention: $y(t)$ amount at time t

Abbreviation Differential Equation: DE

Ex: Interest: amount of money at t is $y(t)$ then

First order differential equations

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Ex: Interest: amount of money at t is $y(t)$ then
variation in amount of money at t is $y'(t)$
the variation is a factor a (eg $a = exp^{0.06}$) times the amount:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} y(t) = a \cdot y(t)}, \qquad \boxed{y'(t) = ay(t)}, \qquad \boxed{y' = ay}$$

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the variation is a factor a (eg $a = exp^{0.06}$) times the amount:

$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}y(t) = a \cdot y(t)}$, $\boxed{y'(t) = ay(t)}$, $\boxed{y' = ay}$

What is the mount of money $y(t)$ at t for all $t \in [0, \infty)$?

First order differential equations

Ex: Toilet seat: number of bacteria at t is $y(t)$ then

Ex: Toilet seat: number of bacteria at t is $y(t)$ then variation in number of bacteria at t is $y'(t)$

Ex: Toilet seat: number of bacteria at t is $y(t)$ then
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variation is a factor 3 time the number of bacteria:
$\boxed{y'(t) = 3 \cdot y(t)}$

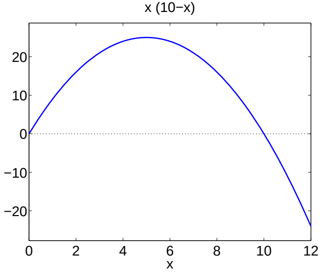
Ex: Toilet seat: number of bacteria at t is $y(t)$ then
variation in number of bacteria at t is $y'(t)$
variation is a factor 3 time the number of bacteria:
$\boxed{y'(t) = 3 \cdot y(t)}$
What is number of bacteria $y(t)$ at t for all $t \in [0, \infty)$?

Ex: Number of rabbits $y' = \alpha y(10 - y)$

("rabbits multiply")

("supply of grass only enough for 10 rabbits ")

Here: equilibrium situation for 0 and 10 rabbits, because then there is no growth/reduction because the variation $y'(t) = \alpha 0(10 - 0) = 0$, $y'(t) = \alpha 10(10 - 10) = 0$



Ex: Number of rabbits and foxes ("Predator-prey diff. eqns"):

$K' = \alpha K - \beta KV$

$V' = \gamma KV - \delta V$

$\alpha, \beta, \gamma, \delta > 0$

Lotka–Volterra equaiton

("rabbits multiply")

("foxes die out, unless there are rabbits")

Ex: Romeo and Julia's love resp. repulsion for each other. Positive values of J resp. R represent love for the other:

$$\begin{aligned} R' &= -\alpha J \\ J' &= \beta R \end{aligned}$$

$\alpha, \beta > 0$

harmonic oscillator equation

("neverending cycle of love and hate")

(see Steven H. Strogatz, Harvard, Love affairs and Differential equations, Mathematics Magazine, 61(1988), 35)

Ex: Toilet seat: number of bacteria at t is $y(t)$ then

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First order differential equations

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Ex: Rewriting: number of bacteria at time x is $y(x)$ then

First order differential equations

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Ex: Toilet seat: number of bacteria at t is $y(t)$ then
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What is number of bacteria $y(t)$ at t for all $t \in [0, \infty)$?

Ex: Rewriting: number of bacteria at time x is $y(x)$ then
variation in number of bacteria at time x is $y'(x)$
variation is a factor 3 time the number of bacteria:

$$\boxed{y'(x) = 3 \cdot y(x)}$$

What is number of bacteria $y(x)$ at x for all $t \in [0, \infty)$?

From now: $y = y(x)$ (mostly)

How can we solve this and other DEs?

Differential equations: Type "constant"

Type "constant DE": y' only depends on x :

$$\frac{dy}{dx} = f(x)$$

Solution: integrate:

$$y(x) = \int f(x)\,dx + C, \qquad C \in \mathbb{R}$$

because

$$\frac{\mathrm{d}}{\mathrm{d}x} y(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,dx + C\right) = f(x)$$

Ex:

$$\frac{dy}{dx} = x^2 \Longrightarrow y(x) = \int x^2\,dx + C = \frac{1}{3}x^3 + C$$

For each $C \in \mathbb{R}$ one solution, thus ∞ many solutions

Differential equations: Type "separable"

Type "separable DE": y' is product of a function of x and a function of y:

$$\frac{dy}{dx} = f(x)g(y)$$

Solve in steps:

1. Separate the variables: $\dfrac{dy}{g(y)} = f(x)dx$
2. Integrate both sides
   $$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
3. Solution: Express y in x and C
4. Fill in: $\dfrac{dy}{dx} = f(x)g(y)$ for all x and C?

Type "separable DE": $y' = f(x)g(y)$ is product of functions: Solution:

1. Separate the variables: $\dfrac{dy}{g(y)} = f(x)dx$
2. Integrate both sides
   $$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
3. Solution: Express y in x and C
4. Fill in: $\dfrac{dy}{dx} = f(x)g(y)$ for all x and C?

Ex: $y'(x) = x^2 y^3(x)$

$$\underset{\substack{\text{separate}\\\text{variables}}}{\Longrightarrow}$$

Differential equations: Type "separable"

Type "separable DE": $y' = f(x)g(y)$ is product of functions: Solution:

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4. Fill in: $\dfrac{dy}{dx} = f(x)g(y)$ for all x and C?

Ex: $y'(x) = x^2 y^3(x)$

$$\underset{\text{separate variables}}{\Longrightarrow} \quad \frac{dy}{y^3} = x^2\,dx \underset{\text{integrate}}{\Longrightarrow}$$

Differential equations: Type "separable"

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$$\underset{\text{separate variables}}{\Longrightarrow} \quad \frac{dy}{y^3} = x^2\,dx \underset{\text{integrate}}{\Longrightarrow} \int \frac{dy}{y^3} = \int x^2\,dx + C$$

$$\Longrightarrow$$

Differential equations: Type "separable"

Type "separable DE": $y' = f(x)g(y)$ is product of functions: Solution:

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   $$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
3. Solution: Express y in x and C
4. Fill in: $\dfrac{dy}{dx} = f(x)g(y)$ for all x and C?

Ex: $y'(x) = x^2 y^3(x)$

$$\underset{\substack{\text{separate}\\ \text{variables}}}{\Longrightarrow} \quad \frac{dy}{y^3} = x^2\,dx \underset{\text{integrate}}{\Longrightarrow} \int \frac{dy}{y^3} = \int x^2\,dx + C$$

$$\Longrightarrow \quad -\frac{1}{2y^2} = \frac{x^3}{3} + C \Longrightarrow$$

Differential equations: Type "separable"

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1. Separate the variables: $\dfrac{dy}{g(y)} = f(x)dx$
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$$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
3. Solution: Express y in x and C
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Ex: $y'(x) = x^2 y^3(x)$

$$\underset{\text{separate variables}}{\Longrightarrow} \quad \frac{dy}{y^3} = x^2\,dx \underset{\text{integrate}}{\Longrightarrow} \int \frac{dy}{y^3} = \int x^2\,dx + C$$

$$\Longrightarrow \quad -\frac{1}{2y^2} = \frac{x^3}{3} + C \Longrightarrow -\frac{1}{2y^2} = \frac{x^3 + 3C}{3}$$

$$\Longrightarrow$$

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$$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
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Ex: $y'(x) = x^2 y^3(x)$

$$\underset{\substack{\text{separate}\\\text{variables}}}{\Longrightarrow} \quad \frac{dy}{y^3} = x^2\,dx \underset{\text{integrate}}{\Longrightarrow} \int \frac{dy}{y^3} = \int x^2\,dx + C$$

$$\Longrightarrow \quad -\frac{1}{2y^2} = \frac{x^3}{3} + C \Longrightarrow -\frac{1}{2y^2} = \frac{x^3 + 3C}{3}$$

$$\Longrightarrow \quad -2y^2 = \frac{3}{x^3 + 3C} \underset{\substack{y \text{ expressed}\\ \text{in } x \text{ and } C}}{\Longrightarrow}$$

Differential equations: Type "separable"

Type "separable DE": $y' = f(x)g(y)$ is product of functions: Solution:

1. Separate the variables: $\dfrac{dy}{g(y)} = f(x)dx$
2. Integrate both sides
$$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
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Ex: $y'(x) = x^2y^3(x)$

$$\underset{\text{separate variables}}{\Longrightarrow} \quad \frac{dy}{y^3} = x^2\,dx \underset{\text{integrate}}{\Longrightarrow} \int \frac{dy}{y^3} = \int x^2\,dx + C$$

$$\Longrightarrow \quad -\frac{1}{2y^2} = \frac{x^3}{3} + C \Longrightarrow -\frac{1}{2y^2} = \frac{x^3 + 3C}{3}$$

$$\Longrightarrow \quad -2y^2 = \frac{3}{x^3 + 3C} \underset{y \text{ expressed in } x \text{ and } C}{\Longrightarrow} y = \pm\sqrt{\frac{3}{-2x^3 - 6C}}$$

Ex: Step 4: Fill in $y(x) = \pm\sqrt{\dfrac{3}{-2x^3 - 6C}}$ into DE $y'(x) = x^2y^3$ Because for the argument $a(x) = 3/(-2x^3 - 6C)$

$$\frac{\mathrm{d}}{\mathrm{d}x}a(x) \quad = \quad -3\frac{-6x^2}{(-2x^3 - 6C)^2} = 6\frac{3}{(-2x^3 - 6C)^2} \cdot x^2$$

we find

$$\begin{aligned}
y'(x) \quad &= \quad \frac{\mathrm{d}}{\mathrm{d}x}y = \pm \cdot \tfrac{1}{2}\left(\frac{3}{-2x^3 - 6C}\right)^{-1/2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}a(x) \\
&\underset{-\frac{1}{2}=\frac{1}{2}-1}{=} \quad y(x) \cdot \tfrac{1}{2} \cdot \left(\frac{3}{-2x^3 - 6C}\right)^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}x}a(x) \\
&= \quad y(x) \cdot \tfrac{1}{6}\left(-2x^3 - 6C\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}a(x) \\
&= \quad y(x) \cdot \tfrac{1}{6}\left(-2x^3 - 6C\right) \cdot 6\frac{3}{(-2x^3 - 6C)^2} \cdot x^2 \\
&= \quad y(x) \cdot \frac{3}{-2x^3 - 6C} \cdot x^2 = y(x) \cdot y^2(x) \cdot x^2 \\
&= \quad y^3(x)x^2 \qquad \text{for all } x \text{ and } C
\end{aligned}$$

Ex: Type "separable DE": $\dfrac{dy}{dx} = a \cdot y$

$\underset{\substack{\text{separate}\\ \text{variables}}}{\Longrightarrow}$

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Differential equations: Type "separable"

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$$\Longrightarrow \quad \ln|y| = ax + C \Longrightarrow e^{\ln|y|} = e^{ax+C} \Longrightarrow |y| = e^{ax} \cdot \underbrace{e^{C}}_{=:C_1>0}$$

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$$\Longrightarrow \quad y \underset{\substack{y\text{ expressed}\\\text{in } x \text{ and } C}}{=} e^{ax} \cdot \underbrace{\pm \cdot C_1}_{=:C_2\neq 0} \Longrightarrow$$

Differential equations: Type "separable"

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$$
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\end{aligned}
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with $C_2 \neq 0$. If $C_2 = 0$ then $y(x) = 0$ solves the DE, thus:

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with $C_2 \neq 0$. If $C_2 = 0$ then $y(x) = 0$ solves the DE, thus:

$$y(x) = C e^{ax}, \qquad C \in \mathbb{R}$$

Pay attention that the solution y is a function of x and C: $\boxed{y = y(x, C)}$
Too: for each $C \in \mathbb{R}$ one solution, thus ∞ many solutions

Ex: Step 4: Fill in $y(x) = Ce^{ax}$ into DE $\dfrac{dy}{dx} = a \cdot y$

$$y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(Ce^{ax}\right) = a \cdot Ce^{ax} = a \cdot y(x)$$

is true indeed for all x and C

In step 2: Two integration constants same as one constant

2. Integrate both sides $\int \frac{dy}{g(y)} + C_1 = \int f(x)dx + C_2, \quad C_1, C_2 \in \mathbb{R}$

is the same as

$$\int \frac{dy}{g(y)} = \int f(x)dx + C_2 - C_1, \quad C_1, C_2 \in \mathbb{R} \Longrightarrow$$

Differential equations: Type "separable"

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$$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$

Remember: the solution y is a function of x and C: $\boxed{y = y(x, C)}$!

Integration const. C follows from $y(a) = M$

Ex: DE $y'(x) = x^2 y^3(x)$ has solutions $y(x) = \sqrt{\dfrac{3}{-2x^3 - 6C}}$

If y at time a is equal to M meaning $y(a) = M$ then

$$M = y(a) = \sqrt{\frac{3}{-2a^3 - 6C}} \Longleftrightarrow$$

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6C &= -\frac{3}{M^2} - 2a^3 \iff
\end{aligned}$$

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6C &= -\frac{3}{M^2} - 2a^3 \iff \\
C &= -\frac{1}{2M^2} - \frac{1}{3}a^3
\end{aligned}$$

Thus:

- DE yields ∞ many solutions
- DE + condition $y(a) = M$ yields 1 solution

Condition $y(a) = M$ is called initial condition

Ex: Solve: $y' = x^2y^3$ with $y(1) = 3 \Longrightarrow y(x) = \sqrt{\frac{3}{-2x^3 - 6C}}$ thus

$$3 = y(1) = \sqrt{\frac{3}{-2 - 6C}} \Longleftrightarrow$$

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C &= -\frac{1}{18} - \frac{1}{3} = -\frac{7}{18}
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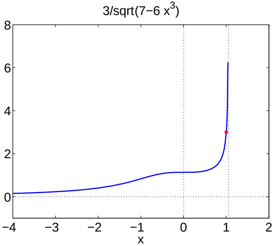
C &= -\frac{1}{18} - \frac{1}{3} = -\tfrac{7}{18}
\end{aligned}
$$

This solution holds for $x < \sqrt[3]{7/6}$ and "then blows up"



Differential equations: Type "separable"

Ex: $y' = \dfrac{x}{y}$

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$$\Longrightarrow \quad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$\underset{\substack{y \text{ expressed}\\ \text{in } x \text{ and } C}}{\Longrightarrow} \quad y = \pm\sqrt{x^2 + 2C}$$

Fill in:

$$y' = \frac{\mathrm{d}}{\mathrm{d}x} \pm \sqrt{x^2 + 2C} = \frac{1}{2}\frac{1}{\sqrt{x^2 + 2C}} \cdot 2x = \frac{x}{y}$$

works for all $x, C \in \mathbb{R}$

Solve:

- $\frac{dy}{dx} = \frac{1}{11}\sqrt{y}\cos^2(\sqrt{y})$
- $5\sqrt{xy}\,\frac{dy}{dx} = 3, \quad x, y > 0$
- $\frac{dy}{dx} = 4e^{x-y}$
- $\frac{dy}{dx} = 2x\sqrt{1-y^2}$
- $y' = 0.1(400-y)y, \quad y(0) = 1$
- $\frac{dy}{dx} + 4y = 7, \quad y(0) = 1$
- $e^x \frac{dy}{dx} = (x+1)(y+1)$
- $y' = 3 + y, \quad y(0) = 3$
- $\frac{dy}{dx} = 1 + x + y + xy$

DE: Type "separable": Summary

Solution of separable DE: $\dfrac{dy}{dx} = f(x)g(y)$:

1. Separation of variables: $\dfrac{dy}{g(y)} = f(x)dx$
2. Integrate both sides
$$\int \frac{dy}{g(y)} = \int f(x)dx + C, \quad C \in \mathbb{R}$$
3. Solution: y expressed in x and C
4. Fill in: $\dfrac{dy}{dx} = f(x)g(y)$ for all x and C?

The solution $y = y(x, C)$ depends on x and C!

We have also seen:

- DEs can be used to simulate all kinds of situations
- Integration constant is determined by initial values $y(a) = M$

First-order linear DE - Variation of parameter

Definition first-order linear DE: $\boxed{\frac{dy}{dx} = p(x)y + q(x)}$

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$\frac{dy}{dx} = p(x)y$: "homogeneous eqn" (is separable!)

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$\frac{dy}{dx} = p(x)y$: "homogeneous eqn" (is separable!)

$\frac{dy}{dx} = p(x)y + q(x)$: "inhomogeneous eqn" (new). Step by step:

- Solve $y'(x) \underset{\text{homogeneous equation}}{=} p(x)y$ (separable)!
- Give solution $y = y(x, C)$
- Variation of constants: $y = y(x, C(x))$
- Fill in $y'(x) \underset{\text{inhomogeneous equation}}{=} p(x)y + q(x) \Longrightarrow$ DE with $C'(x)$
- Solve equation for $C(x) = \ldots + D$

1st-order lin. DE $\frac{dy}{dx} = p(x)y + q(x)$

Ex: $y' + \dfrac{y}{x} = 1$. Std form: $y' = -\dfrac{y}{x} + 1$. 1: Solve $y' = -\dfrac{y}{x}$

$$\underset{\substack{\text{separate}\\\text{variables}}}{\Longrightarrow}$$

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$$\Longrightarrow \quad \ln|y| = -\ln|x| + C$$

$$\underset{\substack{y \text{ expressed}\\ \text{in } x \text{ and } C}}{\Longrightarrow} \quad e^{\ln|y|} = e^{\ln|x|^{-1}+C} \Longrightarrow |y| \underset{C_2 = e^C}{=} \frac{1}{|x|} \cdot C_2, \quad C_2 > 0$$

$$\Longrightarrow \quad y \underset{C_3 = \pm C_2}{=} C_3 \frac{1}{x}, \quad C_3 \neq 0$$

$$\Longrightarrow \quad y \underset{\substack{C=0\\ \text{also sol}}}{=} C\frac{1}{x}, \quad C \in \mathbb{R}$$

Fill in:

$$y' = \frac{\mathrm{d}}{\mathrm{d}x}\frac{C}{x} = C\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -C\frac{1}{x^2} = \left(C\frac{1}{x}\right)\cdot -\frac{1}{x} = -\frac{y}{x}$$

works for all $x, C \in \mathbb{R}$

2: Variation of constant: $y = C\dfrac{1}{x} \mapsto y = C(x)\dfrac{1}{x}$. Fill in:

$$\frac{dy}{dx} = -\frac{y}{x} + 1 \quad \underset{\text{fill in } y}{\Longrightarrow} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(C(x)\frac{1}{x}\right) = -\frac{C(x)\frac{1}{x}}{x} + 1$$

$$\underset{\text{work out}}{\Longrightarrow}$$

1st-order lin. DE $\frac{dy}{dx} = p(x)y + q(x)$

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$$\underset{\text{work out}}{\Longrightarrow} \quad C(x)\cdot -\frac{1}{x^2} + C'(x)\frac{1}{x} = -\frac{C(x)}{x^2} + 1$$

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$$\cancel{C(x)\cdot -\frac{1}{x^2}} + C'(x)\frac{1}{x} = \cancel{-\frac{C(x)}{x^2}} + 1$$

$$\underset{\substack{\text{terms must!}\\ \text{drop out}}}{\Longrightarrow}$$

2: Variation of constant: $y = C\dfrac{1}{x} \mapsto y = C(x)\dfrac{1}{x}$. Fill in:

$$\frac{dy}{dx} = -\frac{y}{x} + 1 \underset{\text{fill in } y}{\Longrightarrow} \frac{\mathrm{d}}{\mathrm{d}x}\left(C(x)\frac{1}{x}\right) = -\frac{C(x)\frac{1}{x}}{x} + 1$$

$$\underset{\text{work out}}{\Longrightarrow} C(x)\cdot -\frac{1}{x^2} + C'(x)\frac{1}{x} = -\frac{C(x)}{x^2} + 1$$

$$\cancel{C(x)\cdot -\frac{1}{x^2}} + C'(x)\frac{1}{x} = -\cancel{\frac{C(x)}{x^2}} + 1$$

$$\underset{\substack{\text{terms must!}\\ \text{drop out}}}{\Longrightarrow} C'(x)\frac{1}{x} = 1 \Longrightarrow$$

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$$\frac{dy}{dx} = -\frac{y}{x} + 1 \quad \underset{\text{fill in } y}{\Longrightarrow} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(C(x)\frac{1}{x}\right) = -\frac{C(x)\frac{1}{x}}{x} + 1$$

$$\underset{\text{work out}}{\Longrightarrow} \quad C(x)\cdot -\frac{1}{x^2} + C'(x)\frac{1}{x} = -\frac{C(x)}{x^2} + 1$$

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$$\underset{\substack{\text{terms must!}\\ \text{drop out}}}{\Longrightarrow} \quad C'(x)\frac{1}{x} = 1 \Longrightarrow C'(x) = x$$

$$\underset{\text{integrate}}{\Longrightarrow} \quad C(x) = \frac{1}{2}x^2 + D,\ D \in \mathbb{R}$$

3: Solution DE $y' + y/x = 1$:

$$y(x) = C(x)\frac{1}{x} = (\frac{1}{2}x^2 + D)\frac{1}{x} = \frac{1}{2}x + \frac{D}{x}$$

Quiz: 1st-order lin. diff. equations

- Determine the solution of the differential equation:
$$x\frac{dy}{dx} = 2x + 3y$$
with inital condition $y(1) = 5$
- Determine the solution of the differential equation:
$$\frac{dy}{dx} - (2x+1)y = e^{x^2+3x}$$
with inital condition $y(0) = 2$
- Determine the solution of the differential equation:
$$\frac{dy}{dx}\cos(x) + y\sin(x) = e^x\cos^2(x)$$
with inital condition $y(0) = 5$

Quiz: 1st-order lin. diff. equations

Determine the solution of:

- $(x-3)\frac{dy}{dx} + y = 5e^x$, $x > 0$
- $(x+4)y' + 4y = \frac{2\sin(x)}{(x+4)^3}, \quad x > 0$
- $xy' - 3y = 6x^3 \ln(x)$
- $x^2 \frac{dy}{dx} = 2x^3 e^{-\frac{1}{x}} + y, \;\; y(-1) = 2e$
- $e^{-x^2}\, \frac{dy}{dx} = 3x^2 + 2xye^{-x^2}$
- $x\frac{dy}{dx} + 3y = \frac{1}{x^2(1+x^2)}$
- $\frac{dy}{dx} = 2x^3 - 2xy$
- $\frac{dy}{dx} - \frac{2xy}{x^2+1} = 1$
- $x\,\frac{dy}{dx} = 2x + 3y$
- $\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos(x)}{x^2}$

Week 7: We have seen

- Integrals/primitives of rational functions
- Improper integrals such as $\displaystyle\int_a^{\infty} f(x)\,dx$
- Estimation of (convergence) of integrals
- Separable differential equations
- First order differential equations