CONIC

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***Problem Statement* - A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+2y^2=6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2+2y^2=6$ are at right angles.**

1 Solution

Given,

$$\mathbf{x^2}+4\mathbf{y^2}=4 \tag{1}$$

The standard equation of the conics is given as

$$\mathbf{x}^\top\mathbf{V}\mathbf{x}+2\mathbf{u}^\top\mathbf{x}+f=0 \tag{2}$$

By Comparing (1) and (2) we get,

$$\mathbf{V_1}=\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix},\mathbf{u_1}=\begin{pmatrix}0\\0\end{pmatrix},f_1=-4 \tag{3}$$

Equation of the ellipse is given as,

$$\mathbf{x^2}+2\mathbf{y^2}=6 \tag{4}$$

By comparing (4) with (2), we get

$$\mathbf{V_2}=\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix},\mathbf{u_2}=\begin{pmatrix}0\\0\end{pmatrix},f_2=-6 \tag{5}$$

On substituting these values in the standard equation of conics, we get

$$\mathbf{x}^\top\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\mathbf{x}-6=0 \tag{6}$$

If $\mathbf{q}$ is the point of contact of the tangent to the ellipse then standard equation of the tangent to the conic is given as

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^\top\mathbf{x}+\mathbf{u}^\top\mathbf{q}+f=0 \tag{7}$$

1.1 Equation of tangent at $\mathbf{q_1}$

$$Let\ \mathbf{q_1}=\begin{pmatrix}0\\1\end{pmatrix} \tag{8}$$

By substituting (8) and (3) in (7) we get the equation of tangent as

$$\begin{pmatrix}0 & 4\end{pmatrix}\mathbf{x}=4 \tag{9}$$

By comparing (9) with the standard vector equation, we get

$$\mathbf{n}=\begin{pmatrix}0\\4\end{pmatrix},\mathbf{m}=\begin{pmatrix}-4\\0\end{pmatrix} \tag{10}$$

Points of intersection of the line ((9)) with the curve ((6)) are given by

$$\mathbf{x}_i=\mathbf{q_1}+\mu_i\mathbf{m} \tag{11}$$

On solving we get the points of intersection as

$$\mathbf{P}=\begin{pmatrix}-2\\1\end{pmatrix},\mathbf{Q}=\begin{pmatrix}2\\1\end{pmatrix} \tag{12}$$

By substituting (12) in (7) we get the equations of tangents at points of intersection **P** and **Q**

Equation of tangent at **P** is given as

$$\begin{pmatrix}-2 & 2\end{pmatrix}\mathbf{x}=6 \tag{13}$$

Equation of tangent at **Q** is given as

$$\begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x}=6 \tag{14}$$

By comparing (13) and (14)

$$\mathbf{P_1}=\begin{pmatrix}-2\\2\end{pmatrix},\mathbf{Q_1}=\begin{pmatrix}2\\2\end{pmatrix} \tag{15}$$

$$\mathbf{P_1}^\top\mathbf{Q_1}=\begin{pmatrix}-2 & 2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}=-4+4=0 \tag{16}$$

We know that if two vectors are orthogonal (perpendicular) then

$$\mathbf{A_1}^\top\mathbf{B_1}=0 \tag{17}$$

Hence, from (16), the tangents at **P** and **Q** of the ellipse $\mathbf{x}^\top\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\mathbf{x}-6=0$ are at right angle.

1.2 Equation of tangent at $\mathbf{q_2}$

$$Let\ \mathbf{q_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{18}$$

By substituting (17) and (3) in (7) we get the equation of tangent as

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{19}$$

By comparing (19) with the standard vector equation, we get

$$\mathbf{n} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{20}$$

Points of intersection of the line ((18)) with the curve ((6)) are given by

$$\mathbf{x_i} = \mathbf{q_2} + \mu_i \mathbf{m} \tag{21}$$

On solving we get the points of intersection as

$$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{22}$$

By substituting (22) in (7) we get the equations of tangents at points of intersection **A** and **B**

Equation of tangent at **A** is given as

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{23}$$

Equation of tangent at **B** is given as

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{24}$$

By comparing (23) and (24)

$$\mathbf{A_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{B_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{25}$$

$$\mathbf{A_1}^\top \mathbf{B_1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - 1 = 0 \tag{26}$$

We know that if two vectors are orthogonal (perpendicular) then

$$\mathbf{A_1}^\top \mathbf{B_1} = 0 \tag{27}$$

Hence, from (26), the tangents at **A** and **B** of the ellipse $\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} - 6 = 0$ are at right angle.

2 Construction

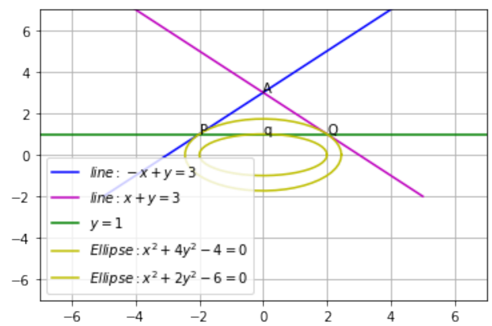

Figure 1: Tangents at $\mathbf{q_1}$

Symbol	Value	Description
A	(0,3)	point **A**
P	(-2,1)	point **P**
Q	(2,1)	point **Q**
$\mathbf{q_1}$	(0,1)	point $\mathbf{q_1}$

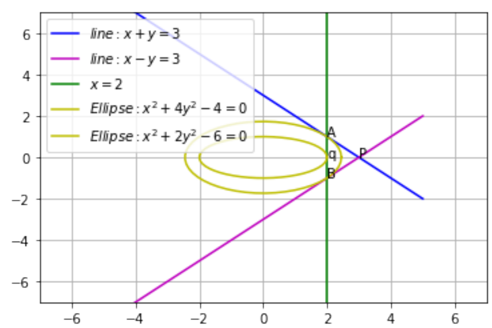

Figure 2: Tangents at $\mathbf{q_2}$

Symbol	Value	Description
A	(2,1)	point **A**
B	(2,-1)	point **B**
P	(3,0)	point **P**
$\mathbf{q_2}$	(2,0)	point $\mathbf{q_2}$

The above constructions are realized by executing the following codes.

https://github.com/BavyaVemulapalli/FWC-IITH/tree/main/Conic/Codes

3 Conclusion

Hence we proved that the tangents at **A** and **B** of the ellipse $\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} - 6 = 0$ are at right angle.