

# Multivariate Expectation Notes

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## Objectives

- 1) Given a joint pmf/pdf, obtain means and variances of random variables and functions of random variables.
- 2) Define the terms covariance and correlation, and given a joint pmf/pdf, obtain the covariance and correlation between two random variables.
- 3) Given a joint pmf/pdf, determine whether random variables are independent of one another.

## Expectation

Computing expected values of random variables in the joint context is similar to the univariate case. Let  $X$  and  $Y$  be discrete random variables with joint pmf  $f_{X,Y}(x,y)$ . Let  $g(X,Y)$  be some function of  $X$  and  $Y$ . Then:

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) f_{X,Y}(x,y)$$

(Note that  $\sum_x$  is shorthand for the sum across all possible values of  $x$ .)

In the case of continuous random variables with joint pdf  $f_{X,Y}$ , expectation becomes:

$$E[g(X,Y)] = \int_x \int_y g(x,y) f_{X,Y}(x,y) dy dx$$

## Expectation of a single variable

Given a joint pmf or pdf, one can find the mean of  $X$  by using the joint function or by finding the marginal pmf/pdf first and then using that to find  $E(X)$ . Demonstrating using the discrete case:

$$E(X) = \sum_x \sum_y x f_{X,Y}(x,y) = \sum_x x \sum_y f_{X,Y}(x,y)$$

The  $x$  can be moved outside the inner sum since the inner sum is with respect to variable  $y$  and  $x$  is a constant with respect to  $y$ . Now, note that the inner sum is the expression for the marginal pmf of  $X$ . So,

$$E(X) = \sum_x x \sum_y f_{X,Y}(x,y) = \sum_x x f_X(x)$$

*Example:*

Let  $X$  and  $Y$  be discrete random variables with joint pmf below.

		Y		
		0	1	2
X	0	0.10	0.08	0.11
	1	0.18	0.20	0.12
	2	0.07	0.05	0.09

Find  $E(X)$  first by using the joint pmf directly, then by finding the marginal pmf of  $X$  and using that.

$$E(X) = \sum_{x=0}^2 \sum_{y=0}^2 x f_{X,Y}(x, y) = 0 * 0.10 + 0 * 0.08 + 0 * 0.11 + 1 * 0.18 + \dots + 2 * 0.09 = 0.92$$

The marginal pmf of  $X$  is

$$f_X(x) = \begin{cases} 0.10 + 0.08 + 0.11, & x = 0 \\ 0.18 + 0.20 + 0.12, & x = 1 \\ 0.07 + 0.05 + 0.09, & x = 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 0.29, & x = 0 \\ 0.50, & x = 1 \\ 0.21, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

So,  $E(X) = 0 * 0.29 + 1 * 0.5 + 2 * 0.21 = 0.92$ .

**Exercise:** Let  $X$  and  $Y$  be defined above. Find  $E(Y)$ ,  $E(X + Y)$ ,  $E(XY)$ , and  $E\left(\frac{1}{2X+Y+1}\right)$ .

As with  $E(X)$ ,  $E(Y)$  can be found in two ways. Using the marginal pmf of  $Y$ , which we will not derive, we will use that:

$$E(Y) = \sum_{y=0}^2 f_Y(y) = 0 * 0.35 + 1 * 0.33 + 2 * 0.32 = 0.97$$

To find  $E(X + Y)$  we will use the joint pmf. In the discrete case, it helps to first identify all of the possible values of  $X + Y$  and then figure out what probabilities are associated with each value. This problem is really a transformation problem where we are finding the distribution of  $X + Y$ . In this example,  $X + Y$  can take on values 0, 1, 2, 3, and 4. The value 0 only happens when  $X = Y = 0$  and the probability of this outcome is 0.10. The value 1 occurs when  $X = 0$  and  $Y = 1$  or when  $X = 1$  and  $Y = 0$ . This occurs with probability  $0.08 + 0.18$ . We continue in this fashion:

$$\begin{aligned} E(X + Y) &= \sum_{x=0}^2 \sum_{y=0}^2 (x + y) f_{X,Y}(x, y) = 0 * 0.1 + 1 * (0.18 + 0.08) + 2 * (0.11 + 0.07 + 0.20) \\ &\quad + 3 * (0.12 + 0.05) + 4 * 0.09 = 1.89 \end{aligned}$$

Note that  $E(X + Y) = E(X) + E(Y)$ . (The proof of this is left to the reader.)

$$\begin{aligned} E(XY) &= \sum_{x=0}^2 \sum_{y=0}^2 xy f_{X,Y}(x, y) = 0 * (0.1 + 0.08 + 0.11 + 0.18 + 0.07) + 1 * 0.20 \\ &\quad + 2 * (0.12 + 0.05) + 4 * 0.09 = 0.9 \end{aligned}$$

Note that  $E(XY)$  is not necessarily equal to  $E(X)E(Y)$ .

$$E\left(\frac{1}{2X+Y+1}\right) = \sum_{x=0}^2 \sum_{y=0}^2 \frac{1}{2x+y+1} f_{X,Y}(x,y) = 1 * 0.1 + \frac{1}{2} * 0.08 + \frac{1}{3} * (0.11 + 0.18) \\ + \frac{1}{4} * 0.20 + \frac{1}{5} * (0.12 + 0.07) + \frac{1}{6} * 0.05 + \frac{1}{7} * 0.09 = 0.3125$$

Let's consider an example with continuous random variables where summation is replaced with integration:

*Example:*

Let  $X$  and  $Y$  be continuous random variables with joint pdf:

$$f_{X,Y}(x,y) = xy$$

for  $0 \leq x \leq 2$  and  $0 \leq y \leq 1$ .

Find  $E(X)$ ,  $E(X+Y)$ ,  $E(XY)$ , and  $\text{Var}(XY)$ .

We found the marginal pdf of  $X$  in a previous lesson, so we should use that:

$$E(X) = \int_0^2 x \frac{x}{2} dx = \frac{x^3}{6} \Big|_0^2 = \frac{4}{3}$$

To find  $E(X+Y)$ , we could use the joint pdf directly, or use the marginal pdf of  $Y$  to find  $E(Y)$  and then add the result to  $E(X)$ . The reason for this is because when we integrate  $x$  with the joint pdf, integrating with respect to  $y$  we can treat  $x$  as a constant and bring it out side the integral of  $y$ .

We'll use the joint pdf:

$$E(X+Y) = \int_0^2 \int_0^1 (x+y)xy dy dx = \int_0^2 \int_0^1 (x^2y + xy^2) dy dx = \int_0^2 \frac{x^2y^2}{2} + \frac{xy^3}{3} \Big|_0^1 dx \\ = \int_0^2 \frac{x^2}{2} + \frac{x}{3} dx = \frac{x^3}{6} + \frac{x^2}{6} \Big|_0^2 = \frac{8}{6} + \frac{4}{6} = 2$$

If we wanted to simulate, we could simulate variables from the marginal of  $X$  and  $Y$  and then add them together to create a new variable.

The cdf for  $X$  is  $\frac{x}{2}$  so we simulation by sampling from a random uniform and then taking the inverse of the cdf. For  $Y$  the cdf is  $2y$ .

```
set.seed(1820)
new_data <- data.frame(x=2*sqrt(runif(10000)),y=sqrt(runif(10000)))
new_data %>%
  mutate(z=x+y) %>%
  summarize(Ex=mean(x),Ey=mean(y),Explusy = mean(z))
```

```
##           Ex           Ey  Explusy
## 1  1.338196 0.6695514  2.007748
```

We can see that  $E(X + Y) = E(X) + E(Y)$ .

Next, we have

$$\begin{aligned} E(XY) &= \int_0^2 \int_0^1 xy * xy \, dy \, dx = \int_0^2 \frac{x^2 y^3}{3} \Big|_0^1 \, dx = \int_0^2 \frac{x^2}{3} \, dx \\ &= \frac{x^3}{9} \Big|_0^2 = \frac{8}{9} \end{aligned}$$

```
set.seed(191)
new_data <- data.frame(x=2*sqrt(runif(10000)),y=sqrt(runif(10000)))
new_data %>%
  mutate(z=x*y) %>%
  summarize(Ex=mean(x),Ey=mean(y),Extimesy = mean(z))
```

```
##           Ex           Ey  Extimesy
## 1  1.33096  0.6640436  0.8837552
```

Recall that the variance of a random variable is the expected value of the squared difference from its mean. So,

$$\begin{aligned} \text{Var}(XY) &= E[(XY - E(XY))^2] = E\left[\left(XY - \frac{8}{9}\right)^2\right] \\ &= \int_0^2 \int_0^1 \left(xy - \frac{8}{9}\right)^2 xy \, dy \, dx = \int_0^2 \int_0^1 \left(x^2 y^2 - \frac{16xy}{9} + \frac{64}{81}\right) xy \, dy \, dx \end{aligned}$$

Yuck!! But we can continue because we are determined to integrate.

$$\begin{aligned} &= \int_0^2 \int_0^1 \left(x^3 y^3 - \frac{16x^2 y^2}{9} + \frac{64xy}{81}\right) \, dy \, dx = \int_0^2 \frac{x^3 y^4}{4} - \frac{16x^2 y^3}{27} + \frac{32xy^2}{81} \Big|_0^1 \, dx = \int_0^2 \frac{x^3}{4} - \frac{16x^2}{27} + \frac{32x}{81} \, dx \\ &= \frac{x^4}{16} - \frac{16x^3}{81} + \frac{16x^2}{81} \Big|_0^2 \\ &= \frac{16}{16} - \frac{128}{81} + \frac{64}{81} = 0.2098765 \end{aligned}$$

```
set.seed(816)
new_data <- data.frame(x=2*sqrt(runif(10000)),y=sqrt(runif(10000)))
new_data %>%
  mutate(z=(x*y-8/9)^2) %>%
  summarize(Var = mean(z))
```

```
##           Var
## 1  0.2098769
```

That was much easier. Notice that we are really just estimating these expectations with the simulations. The mathematical answers are the true population values while simulations are sample estimate. In a few lessons we will discuss estimators in more detail.

## Covariance/Correlation

We have discussed expected values of random variables and functions of random variables in a joint context. It would be helpful to have some kind of consistent measure to describe *how* two random variables are related to one another. *Covariance* and *correlation* do just that. It is important to understand that these are measures of a linear relationship between variables.

Consider two random variables  $X$  and  $Y$ . (We could certainly consider more than two, but for demonstration, let's consider only two for now). The covariance between  $X$  and  $Y$  is denoted as  $\text{Cov}(X, Y)$  and is found by:

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$$

We can simplify this expression to make it a little more usable:

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E[XY - YE(X) - XE(Y) + E(X)E(Y)]$$

$$E(XY) - E(YE(X)) - E(XE(Y)) + E(X)E(Y) = E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$$

Thus,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

This expression is a little easier to use, since it's typically straightforward to find each of these quantities.

It is important to note that while variance is a positive quantity, covariance can be positive or negative. A positive covariance implies that as the value of one variable increases, the other tends to increase. This is a statement about a linear relationship. Likewise, a negative covariance implies that as the value of one variable increases, the other tends to decrease.

*Example:*

An example of positive covariance is human height and weight. As height increase, weight tends to increase. An example of negative covariance is gas mileage and car weight. As car weight increases, gas mileage decreases.

Remember that if  $a$  and  $b$  are constants,  $E(aX + b) = aE(X) + b$  and  $\text{Var}(aX + b) = a^2\text{Var}(X)$ . Similarly, if  $a$ ,  $b$ ,  $c$ , and  $d$  are all constants,

$$\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y)$$

One disadvantage of covariance is its dependence on the scales of the random variables involved. This makes it difficult to compare covariances of multiple sets of variables. *Correlation* avoids this problem. Correlation is a scaled version of covariance. It is denoted by  $\rho$  and found by:

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

While covariance could take on any real number, correlation is bounded by -1 and 1. Two random variables with a correlation of 1 are said to be perfectly positively correlated, while a correlation of -1 implies perfect negative correlation. Two random variables with a correlation (and thus covariance) of 0 are said to be uncorrelated, that is they do not have a linear relationship but could have a non-linear relationship.

Let's look at some plots of different correlations. Remember that the correlation we are calculating in this section is for the population. The plots are showing sample points from a population.

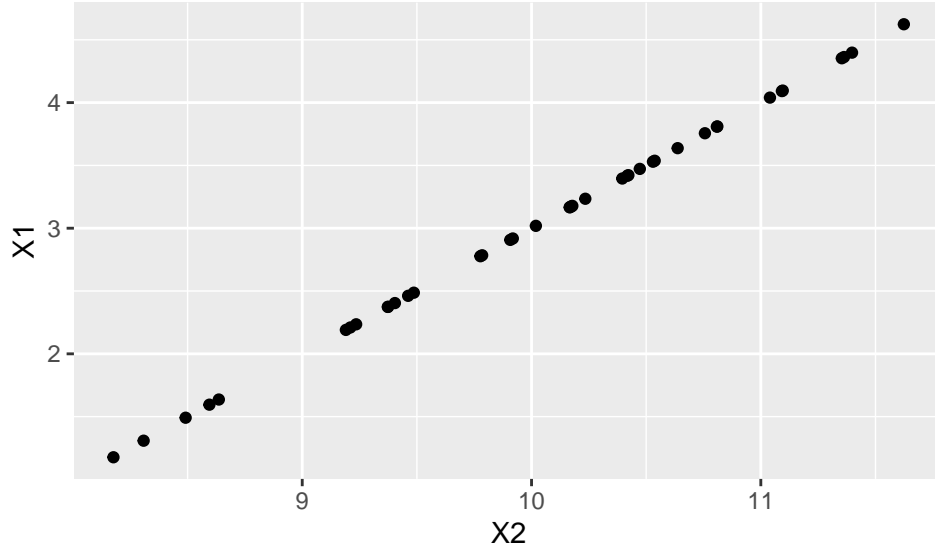


Figure 1: Correlation of 1

### Variance of sums

Suppose  $X$  and  $Y$  are two random variables. Then,

$$\text{Var}(X + Y) = E[(X + Y - E(X + Y))^2] = E[(X + Y)^2] - [E(X + Y)]^2$$

In the last step, we are using the alternative expression for variance ( $\text{Var}(X) = E(X^2) - E(X)^2$ ). Evaluating:

$$\text{Var}(X + Y) = E(X^2) + E(Y^2) + 2E(XY) - E(X)^2 - E(Y)^2 - 2E(X)E(Y)$$

Regrouping the terms:

$$= E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2 + 2(E(XY) - E(X)E(Y)) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

*Example:*

Let  $X$  and  $Y$  be defined as above. Find  $\text{Cov}(X, Y)$ ,  $\rho$ , and  $\text{Var}(X + Y)$ .

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.9 - 0.92 * 0.97 = 0.0076$$

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Quickly,  $\text{Var}(X) = E(X^2) - E(X)^2 = 1.34 - 0.92^2 = 0.4936$  and  $\text{Var}(Y) = 0.6691$ . So,

$$\rho = \frac{0.0076}{\sqrt{0.4936 * 0.6691}} = 0.013$$

With such a low  $\rho$ , we would say that  $X$  and  $Y$  are only slightly positively correlated.

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = 0.4936 + 0.6691 + 2 * 0.0076 = 1.178$$

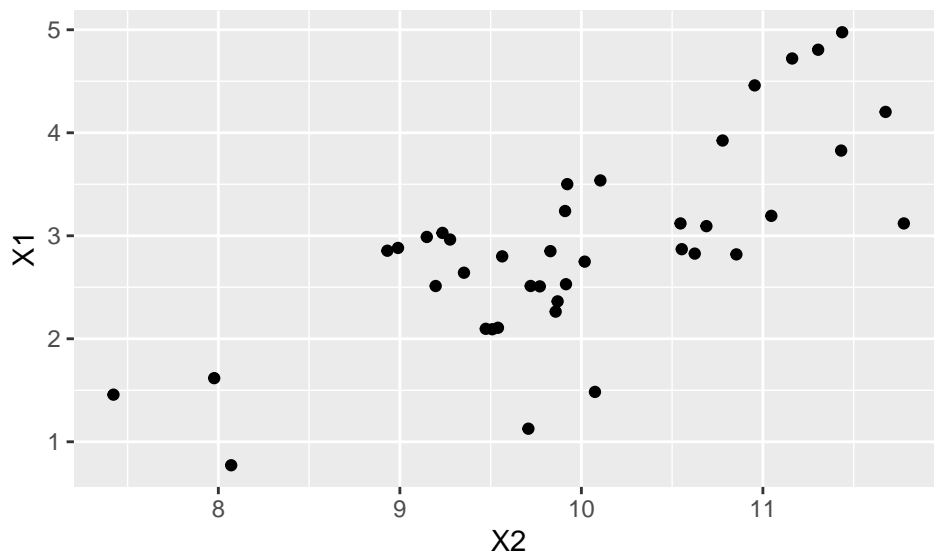


Figure 2: Correlation of .8

## Independence

Two random variables  $X$  and  $Y$  are said to be *independent* if their joint pmf/pdf is the product of their marginal pmfs/pdfs:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

If  $X$  and  $Y$  are independent, then  $\text{Cov}(X,Y) = 0$ . The converse is not necessarily true, however because they could have a non-linear relationship.

An easy way to determine if continuous variables are independent is to check the the domain only contains constants, it is rectangular, and the the joint pdf can be written as a product of a function of  $X$  only and a function of  $Y$  only.

Thus for our examples  $f(x,y) = xy$  were independent while  $f(x,y) = x + y$  were not.

## Conditional Expectation

An important idea in graph theory, network analysis, Bayesian networks, and queuing theory is conditional expectation. We will only briefly introduce the ideas here so that you have a basic understanding. This does not imply it is not an important topic.

Let's start with a simple example to illustrate the ideas.

*Example:*

Sam will read either one chapter of his history book or one chapter of his philosophy book. If the number of misprints in a chapter of his history book is Poisson with mean 2 and the if the number of misprints in a chapter of his philosophy book is Poisson with mean 5, then assuming Sam is equally likely to choose either book, what is the expected number of misprints that Same will find?

Note: in the next lesson we are working with transformations and could attack the problem using that method.

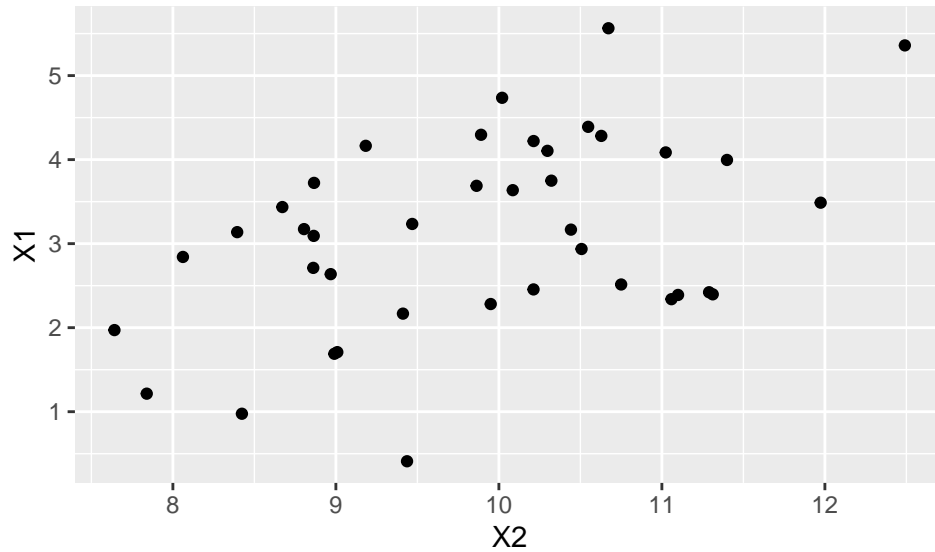


Figure 3: Correlation of .5

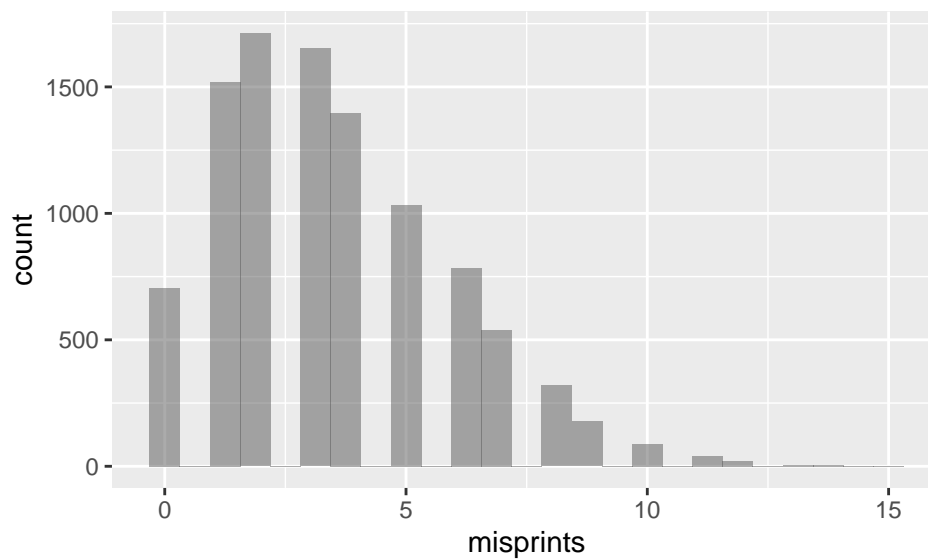
First let's use simulation to get an idea and then use mathematics.

Simulate 5000 reads from the history book and 5000 from philosophy and combine

```
my_data<-data.frame(misprints=c(rpois(5000,2),rpois(5000,5)))
```

Let's plot.

```
gf_histogram(~misprints,data=my_data)
```



And now find the average.



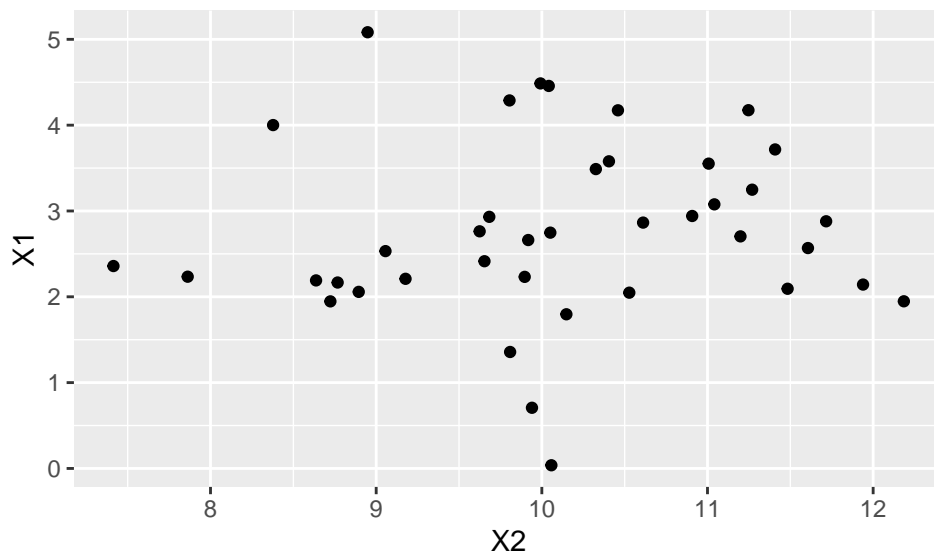


Figure 4: Correlation of 0

```
mean(~misprints,data=my_data)
```

```
## [1] 3.5017
```

Now for a mathematical solution. Let  $X$  stand for the number of misprints and  $Y$  for the book, to make it a random variable let's call 0 history and 1 philosophy. Then  $E[X|Y]$  is the expected number of misprints given the book. This is a conditional expectation. For example  $E[X|Y = 0]$  is the expected number of misprints in the history book.

Now here is the tricky part, without specifying  $Y$  this expectation function is a random variable that depends on  $Y$ . If we take the expected value of this random variable we get

$$E[X] = E[E[X|Y]]$$

The inner expectation is for the conditional distribution and the outer is for the marginal with respect to  $Y$ . This seems confusing, so let's go back to our example.

$$\begin{aligned} E[X] &= E[E[X|Y]] \\ &= E[X|Y = 0]P(Y = 0) + E[X|Y = 1]P(Y = 1) \\ &= 2 * \frac{1}{2} + 5 * \frac{1}{2} = \frac{7}{2} = 3.5 \end{aligned}$$

These ideas are going to be similar for continuous random variables. Note that we can also use conditional expectations to find probabilities but that is beyond the scope of this course.

## File Creation Information

- File creation date: 2020-06-24
- Windows version: Windows 10 x64 (build 17763)
- R version 3.6.3 (2020-02-29)

- `mosaic` package version: 1.6.0
- `tidyverse` package version: 1.3.0
- `cubature` package version: 2.0.4
- `mosaicCalc` package version: 0.5.1