Multivariate Expectation Application Solutions

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Exercises

1. Let X and Y be continuous random variables with joint pmf:

$$f_{X,Y}(x,y) = x + y$$

where $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

a) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$.

$$\mathrm{E}(X) = \int_0^1 x\left(x + \frac{1}{2}\right)\mathrm{d}x = \frac{x^3}{3} + \frac{x^2}{4}\bigg|_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} = 0.583$$

$$\mathrm{E}(Y) = \int_0^1 y\left(y + \frac{1}{2}\right)\mathrm{d}y = 0.583$$

b) Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$$
$$\mathrm{E}(X^2) = \int_0^1 x^2\left(x + \frac{1}{2}\right)\mathrm{d}x = \frac{x^4}{4} + \frac{x^3}{6}\bigg|_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12} = 0.417$$

So, $\mathrm{Var}(X) = 0.417 - 0.583^2 = 0.076$.

Similarly, $\mathrm{Var}(Y) = 0.076$.

c) Find $\mathrm{Cov}(X,Y)$ and ρ. Are X and Y independent?

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$$

$$\mathrm{E}(XY) = \int_0^1\int_0^1 xy(x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3}\bigg|_0^1 \mathrm{d}x = \int_0^1 \frac{x^2}{2} + \frac{x}{3}\,\mathrm{d}x$$
$$= \frac{x^3}{6} + \frac{x^2}{6}\bigg|_0^1 = \frac{1}{3} = 0.333$$

So,

$$\mathrm{Cov}(X,Y) = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -0.007$$

$$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{-0.007}{\sqrt{0.076 \times 0.076}} = -0.909$$

With a non-zero covariance, X and Y are not independent.

d) Find $\text{Var}(3X + 2Y)$.

$$\text{Var}(3X + 2Y) = \text{Var}(3X) + \text{Var}(2Y) + 2\text{Cov}(3X, 2Y) = 9\text{Var}(X) + 4\text{Var}(Y) + 12\text{Cov}(X,Y)$$
$$= 9 * 0.076 + 4 * 0.076 + 12 * -0.007 = 0.910$$

2. Let X and Y be continuous random variables with joint pmf:

$$f_{X,Y}(x,y) = 1$$

where $0 \leq x \leq 1$ and $0 \leq y \leq 2x$.

a) Find $\text{E}(X)$ and $\text{E}(Y)$.

$$\text{E}(X) = \int_0^1 x \cdot 2x \,\text{d}x = \left.\frac{2x^3}{3}\right|_0^1 = 0.667$$

$$\text{E}(Y) = \int_0^2 y\left(1 - \frac{y}{2}\right) \text{d}y = \left.\frac{y^2}{2} - \frac{y^3}{6}\right|_0^2 = 2 - \frac{8}{6} = 0.667$$

b) Find $\text{Var}(X)$ and $\text{Var}(Y)$.

$$\text{E}(X^2) = \int_0^1 x^2 \cdot 2x \,\text{d}x = \left.\frac{x^4}{2}\right|_0^1 = 0.5$$

So, $\text{Var}(X) = 0.5 - \left(\frac{2}{3}\right)^2 = \frac{1}{18} = 0.056$

$$\text{E}(Y^2) = \int_0^2 y^2\left(1 - \frac{y}{2}\right) \text{d}y = \left.\frac{y^3}{3} - \frac{y^4}{8}\right|_0^2 = \frac{8}{3} - 2 = 0.667$$

So, $\text{Var}(Y) = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9} = 0.222$

c) Find $\text{Cov}(X,Y)$ and ρ. Are X and Y independent?

$$\text{E}(XY) = \int_0^1 \int_0^{2x} xy \,\text{d}y \,\text{d}x = \int_0^1 \left.\frac{xy^2}{2}\right|_0^{2x} \text{d}x = \int_0^1 2x^3 \,\text{d}x = \left.\frac{x^4}{2}\right|_0^1 = \frac{1}{2}$$

So,

$$\text{Cov}(X,Y) = \frac{1}{2} - \frac{2}{3}\frac{2}{3} = \frac{1}{18} = 0.056$$

$$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\frac{1}{18}}{\sqrt{\frac{1}{18}\frac{2}{9}}} = 0.5$$

X and Y appear to be positively correlated (thus not independent).

d) Find $\text{Var}\left(\frac{X}{2} + 2Y\right)$.

$$\text{Var}\left(\frac{X}{2} + 2Y\right) = \frac{1}{4}\text{Var}(X) + 4\text{Var}(Y) + 2\text{Cov}(X, Y) = \frac{1}{72} + \frac{8}{9} + \frac{1}{9} = 1.014$$

3. Suppose X and Y are *independent* random variables. Show that $\text{E}(XY) = \text{E}(X)\text{E}(Y)$.

If X and Y are independent, then $\text{Cov}(X, Y) = 0$. So,

$$\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X)\text{E}(Y) = 0$$

Thus,

$$\text{E}(XY) = \text{E}(X)\text{E}(Y)$$

4. ADVANCED: Let $X_1, X_2, ..., X_n$ be independent, identically distributed random variables. (This is often abbreviated as "iid"). Each X_i has mean μ and variance σ^2 (i.e., for all i, $\text{E}(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$).

Let $S = X_1 + X_2 + ... + X_n = \sum_{i=1}^{n} X_i$. And let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$.

Find $\text{E}(S)$, $\text{Var}(S)$, $\text{E}(\bar{X})$ and $\text{Var}(\bar{X})$.

$$\text{E}(S) = \text{E}(X_1 + X_2 + ... + X_n) = \text{E}(X_1) + \text{E}(X_2) + ... + \text{E}(X_n) = \mu + \mu + ... + \mu = n\mu$$

Since the X_is are all independent:

$$\text{Var}(S) = \text{Var}(X_1 + X_2 + ... + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + ... + \text{Var}(X_n) = n\sigma^2$$

$$\text{E}(\bar{X}) = \frac{1}{n}\text{E}(X_1 + X_2 + ... + X_n) = \frac{1}{n}n\mu = \mu$$
$$\text{Var}(\bar{X}) = \frac{1}{n^2}\text{Var}(X_1 + X_2 + ... + X_n) = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}$$