Continuous Random Variables Applications Solutions

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Exercises

1. Let X be a continuous random variable on the domain $-k \leq X \leq k$. Also, let $f(x) = \frac{x^2}{18}$.

a) Assume that $f(x)$ is a valid pdf. Find the value of k.

Because f is a valid pdf, we know that $\int_{-k}^{k} \frac{x^2}{18} \mathrm{d}x = 1$. So,

$$\int_{-k}^{k} \frac{x^2}{18} \mathrm{d}x = \frac{x^3}{54}\bigg|_{-k}^{k} = \frac{k^3}{54} - \frac{-k^3}{54} = \frac{k^3}{27} = 1$$

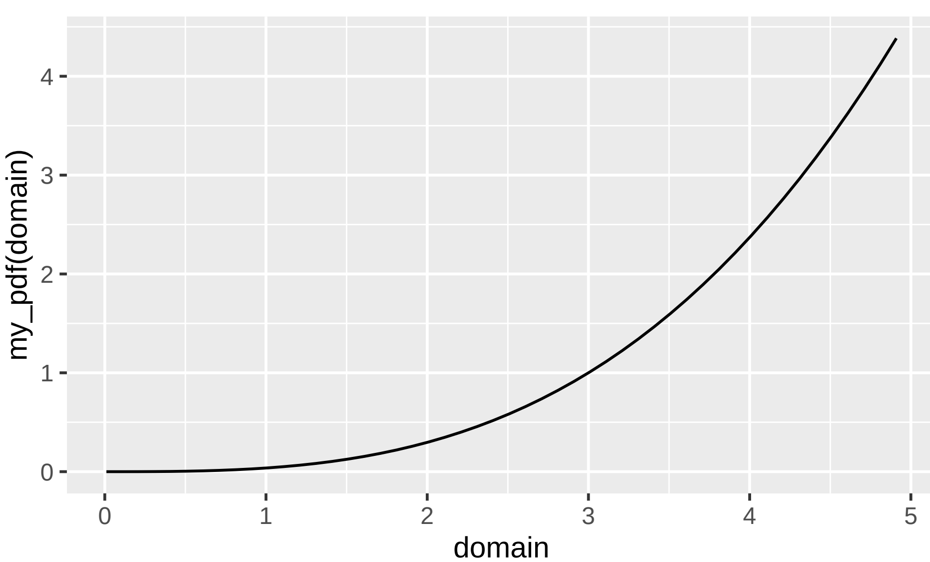
Thus, $k = 3$.

Using `R`, see if you can follow the code.

```
my_pdf <- function(x)integrate(function(y)y^2/18,-x,x)$value
```

```
my_pdf<-Vectorize(my_pdf)
```

```
domain <- seq(.01,5,.1)
gf_line(my_pdf(domain)~domain)
```

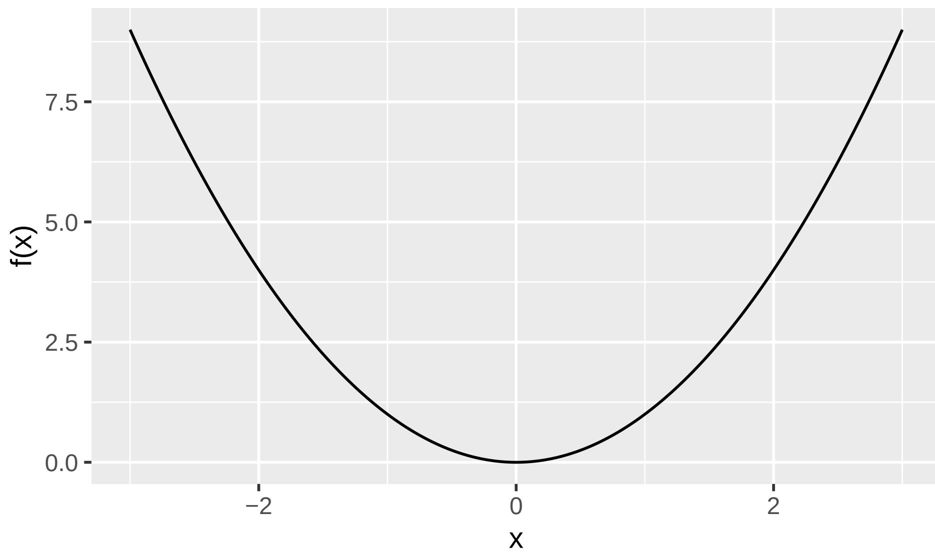


```r
uniroot(function(x)my_pdf(x)-1,c(-10,10))$root
```

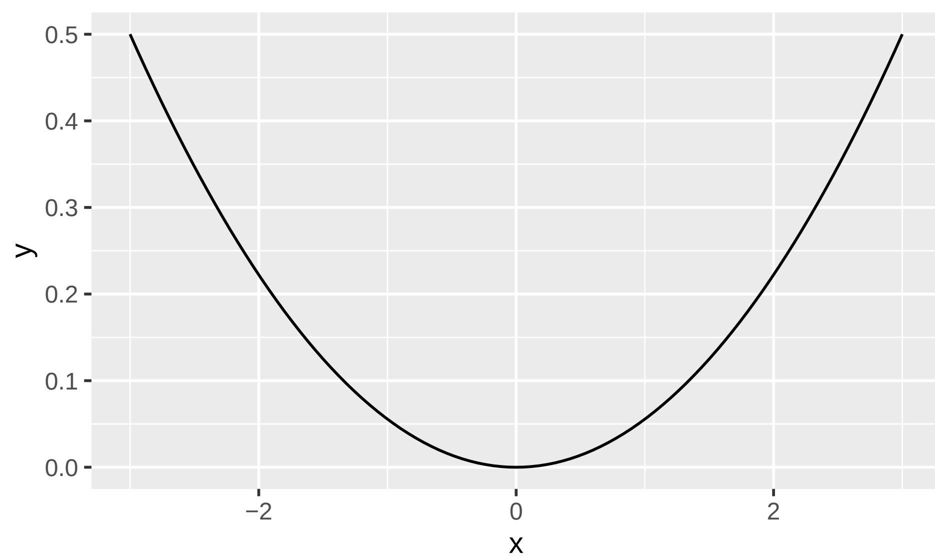
```
## [1] 2.999997
```

b) Plot the pdf of X.

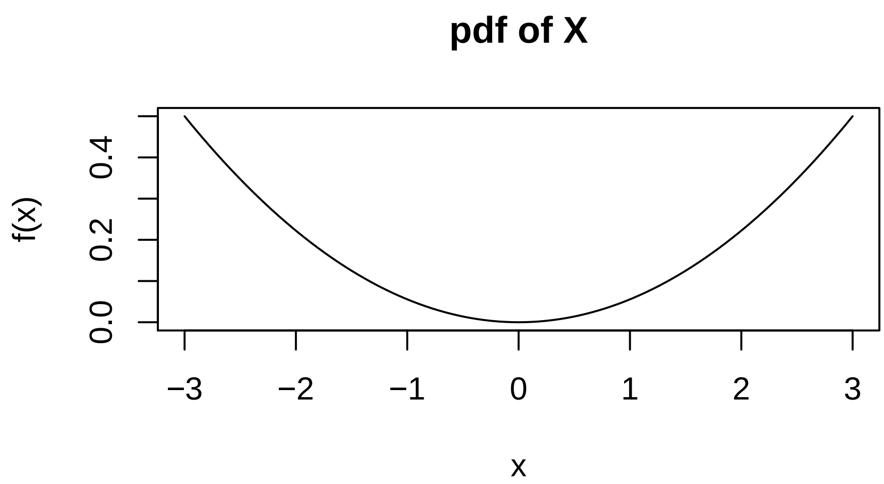
```r
x<-seq(-3,3,0.001)
fx<-x^2
gf_line(fx~x,ylab="f(x)",main="pdf of X")
```




```r
ggplot(data.frame(x=c(-3, 3)), aes(x)) +
 stat_function(fun=function(x) x^2/18)
```



```r
curve(x^2/18,from=-3,to=3,ylab="f(x)",main="pdf of X")
```

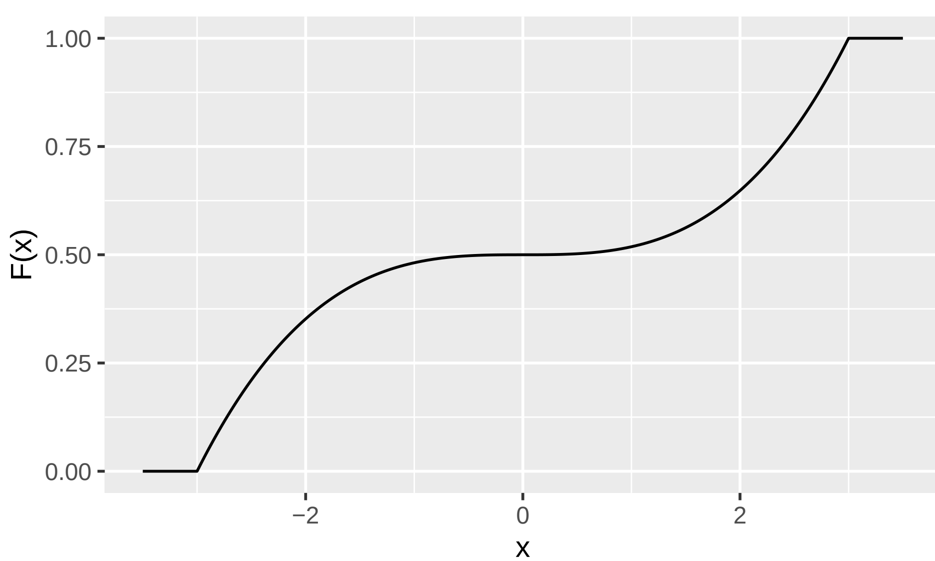


c) Find and plot the cdf of X.

$$F_X(x) = \mathrm{P}(X \le x) = \int_{-3}^{x} \frac{t^2}{18}\,\mathrm{d}t = \left.\frac{t^3}{54}\right|_{-3}^{x} = \frac{x^3}{54} + \frac{1}{2}$$

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \frac{x^3}{54} + \frac{1}{2}, & -3 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

```r
x<-seq(-3.5,3.5,0.001)
fx<-pmin(1,(1*(x>=-3)*(x^3/54+1/2)))
gf_line(fx~x,ylab="F(x)",main="cdf of X")
```



d) Find $\mathrm{P}(X < 1)$.

$$\mathrm{P}(X < 1) = F(1) = \frac{1}{54} + \frac{1}{2} = 0.519$$

```
integrate(function(x)x^2/18,-3,1)
```

```
## 0.5185185 with absolute error < 5.8e-15
```

e) Find $\mathrm{P}(1.5 < X \leq 2.5)$.

$$\mathrm{P}(1.5 < X \leq 2.5) = F(2.5) - F(1.5) = \frac{2.5^3}{54} + \frac{1}{2} - \frac{1.5^3}{54} - \frac{1}{2} = 0.227$$

```
integrate(function(x)x^2/18,1.5,2.5)
```

```
## 0.2268519 with absolute error < 2.5e-15
```

f) Find the 80th percentile of X (the value x for which 80% of the distribution is to the left of that value).

Need x such that $F(x) = 0.8$. Solving $\frac{x^3}{54} + \frac{1}{2} = 0.8$ for x yields $x = 2.530$.

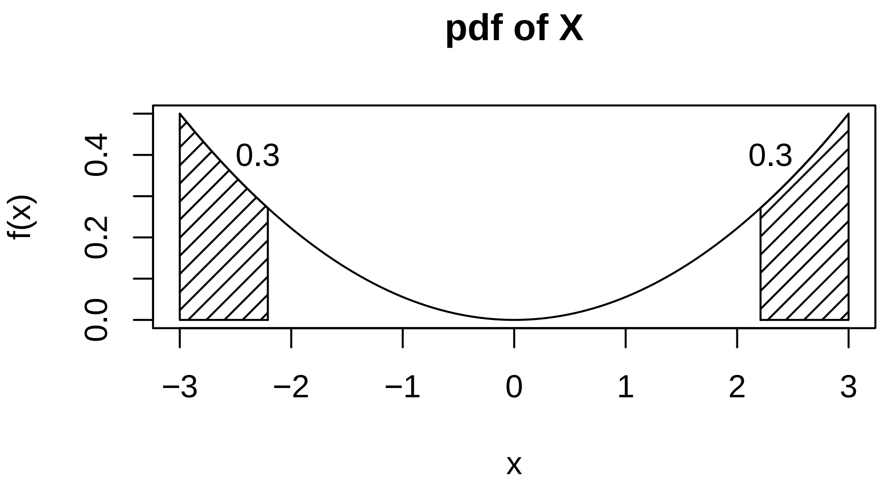
```
uniroot(function(x)x^3/54+.5-.8,c(-3,3))
```

```
## $root
## [1] 2.530293
##
## $f.root
## [1] -1.854422e-06
##
## $iter
## [1] 6
##
## $init.it
## [1] NA
##
## $estim.prec
## [1] 6.103516e-05
```

g) Find the value x such that $\mathrm{P}(-x \leq X \leq x) = 0.4$.

Because this distribution is symmetric, finding x is equivalent to finding x such that $\mathrm{P}(X > x) = 0.3$. (It helps to draw a picture). Thus, we need x such that $F(x) = 0.7$. Solving $\frac{x^3}{54} + \frac{1}{2} = 0.7$ for x yields $x = 2.210$.

```
curve(x^2/18,from=-3,to=3,ylab="f(x)",main="pdf of X")
t<-seq(2.21,3,0.001)
polygon(c(2.21,t,3),c(0,t^2/18,0),density=15)
polygon(c(-2.21,-t,-3),c(0,t^2/18,0),density=15)
text(-2.3,0.4,"0.3")
text(2.3,0.4,"0.3")
```



h) Find the mean and variance of X.

$$\mathrm{E}(X) = \int_{-3}^{3} x \cdot \frac{x^2}{18}\,\mathrm{d}x = \left.\frac{x^4}{72}\right|_{-3}^{3} = \frac{81}{72} - \frac{81}{72} = 0$$

$$\mathrm{E}(X^2) = \int_{-3}^{3} x^2 \cdot \frac{x^2}{18}\,\mathrm{d}x = \left.\frac{x^5}{90}\right|_{-3}^{3} = \frac{243}{90} - \frac{-243}{90} = 5.4$$

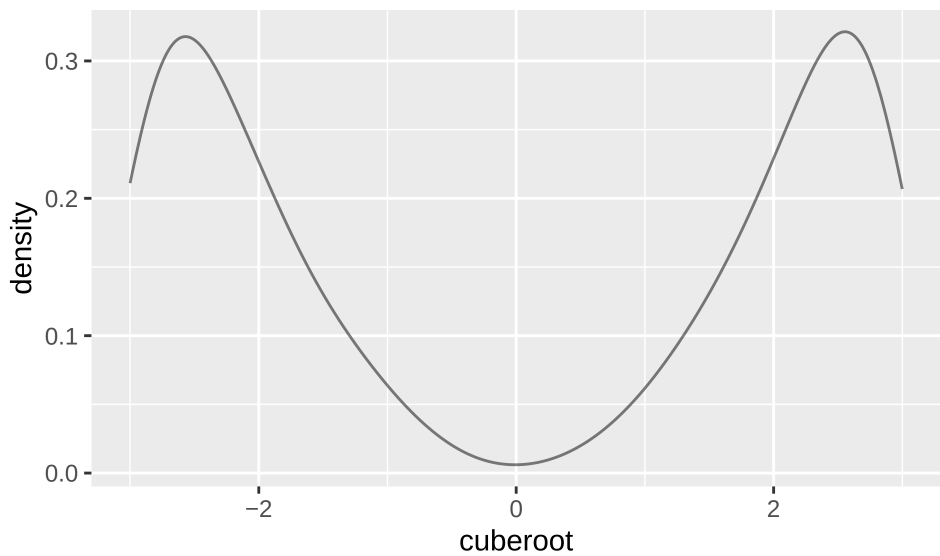
$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 5.4 - 0^2 = 5.4$$

i) This is tricky since we need a cube root function. Just raising to the one-third power won't work. Let's write our own function.

```
cuberoot <- function(x) {
  sign(x) * abs(x)^(1/3)}
```

```
set.seed(4)
results <- do(10000)*cuberoot((runif(1)-.5)*54)
```

```
results %>%
  gf_dens(~cuberoot)
```




```
inspect(results)
```

```
##
## quantitative variables:
##       name   class       min        Q1     median       Q3      max
## 1 cuberoot numeric -2.999981 -2.382864 -0.1574198 2.376346 2.999347
##           mean       sd     n missing
## 1 -0.002416475 2.322639 10000       0
```

2. Let X be a continuous random variable. Prove that the cdf of X, $F_X(x)$ is a non-decreasing function. (Hint: show that for any $a < b$, $F_X(a) \leq F_X(b)$.)

Let $a < b$, where a and b are both in the domain of X. Note that $F_X(a) = \mathrm{P}(X \leq a)$ and $F_X(b) = \mathrm{P}(X \leq b)$. Since $a < b$, we can partition $\mathrm{P}(X \leq b)$ as $\mathrm{P}(X \leq a) + \mathrm{P}(a < X \leq b)$. One of the axioms of probability is that it must be non-negative, so I know that $\mathrm{P}(a < X \leq b) \geq 0$. Thus,

$$\mathrm{P}(X \leq b) = \mathrm{P}(X \leq a) + \mathrm{P}(a < X \leq b) \geq \mathrm{P}(X \leq a)$$

So, we have shown that $F_X(a) \leq F_X(b)$. Thus, $F_X(x)$ is a non-decreasing function.