Transformations Application Solutions

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Exercises

1. Let X be a random variable and let g be a function. By this point, it should be clear that $\mathrm{E}[g(X)]$ is not necessarily equal to $g(\mathrm{E}[X])$. Give one example (excluding examples from the Lesson 13 narrative) of X and g where $\mathrm{E}[g(X)] \neq g(\mathrm{E}[X])$. What are $\mathrm{E}[g(X)]$ and $g(\mathrm{E}[X])$? Use `R` to find $\mathrm{E}[g(X)]$.

Let $X \sim \textsf{Expon}(\lambda = 0.5)$. Let $g(X) = X^2$. We know that $\mathrm{E}(X) = \frac{1}{0.5} = 2$ so $g(\mathrm{E}(X)) = \mathrm{E}(X)^2 = 4$.

However, let $Y = X^2$.

```
sims<-rexp(10000,0.5)
mean(sims^2)
```

```
## [1] 8.04536
```

So, $g(\mathrm{E}(X)) = 4$ and $\mathrm{E}(g(X)) \approx 7.84$.

2. (From Pruim 2011, Section 2.5) Let $X \sim \textsf{Binom}(n, \pi)$. What is the pmf for $X + 3$? Make sure you specify the domain of Y. [Note, in Lesson 8, we used p for the probability of success in a binomial distribution. In the Pruim text, π was used.]

Let $Y = X + 3$:

$$f_Y(y) = \mathrm{P}(Y = y) = \mathrm{P}(X + 3 = y) = \mathrm{P}(X = y - 3) = f_X(y - 3) = \binom{n}{y-3}\pi^{y-3}(1-\pi)^{n-y+3}$$

where $3 \leq Y \leq n + 3$.

3. Let $X \sim \textsf{Expon}(\lambda)$. Let $Y = X^2$. Find the pdf of Y.

CDF method:

$$F_Y(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(X \leq \sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}$$

So,

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = -e^{-\lambda\sqrt{y}} \times \frac{-\lambda}{2\sqrt{y}} = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}}$$

for $y > 0$.

PDF method:

$$f_Y(y) = \lambda e^{-\lambda\sqrt{y}}\frac{1}{2\sqrt{y}} = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}}$$

for $y > 0$.

4. ADVANCED: In exercise 3, you found the pdf of $Y = X^2$ when $X \sim \mathsf{Expon}(\lambda)$. Rearrange the pdf to show that $Y \sim \mathsf{Weibull}$ and find the parameters of that distribution.

$$f_Y(y) = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}} = \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}} = \frac{\lambda^2}{2}\frac{1}{\lambda\sqrt{y}} e^{-\sqrt{\lambda^2 y}} = \frac{1/2}{1/\lambda^2}\left(\frac{y}{1/\lambda^2}\right)^{\frac{1}{2}-1} e^{-\left(\frac{y}{1/\lambda^2}\right)^{\frac{1}{2}}}$$

So, $Y \sim \mathsf{Weibull}\left(\alpha = \frac{1}{2}, \beta = \frac{1}{\lambda^2}\right)$.