Case Study for Hypothesis Testing Notes

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Objectives

1) Define and use properly in context all new terminology.

2) Conduct a hypothesis test using a permutation test to include all 4 steps.

Introduction

We now have the foundation to move onto statistical modeling. First we will begin with inference where we use the ideas of estimation and the variance of estimates to make decisions about the population. We will also briefly introduce the ideas of prediction. Then in the final block of material, we will examine some common linear models and use them both in prediction situations as well as inference.

Foundation for inference

Suppose a professor randomly splits the students in class into two groups: students on the left and students on the right. If $\hat{p}_{_L}$ and $\hat{p}_{_R}$ represent the proportion of students who own an Apple product on the left and right, respectively, would you be surprised if $\hat{p}_{_L}$ did not *exactly* equal $\hat{p}_{_R}$?

While the proportions would probably be close to each other, they are probably not exactly the same. We would probably observe a small difference due to *chance*.

> **Exercise**:
> If we don't think the side of the room a person sits on in class is related to whether the person owns an Apple product, what assumption are we making about the relationship between these two variables?[1]

Studying randomness of this form is a key focus of statistical modeling. In this block, we'll explore this type of randomness in the context of several applications, and we'll learn new tools and ideas that can be applied to help make decisions from data.

Randomization case study: gender discrimination

We consider a study investigating gender discrimination in the 1970s, which is set in the context of personnel decisions within a bank.[2] The research question we hope to answer is, "Are females discriminated against in promotion decisions made by male managers?"

[1] We would be assuming that these two variables are **independent**, meaning they are unrelated.

[2] Rosen B and Jerdee T. 1974. "Influence of sex role stereotypes on personnel decisions." Journal of Applied Psychology 59(1):9-14.

Variability within data

The participants in this study were 48 male bank supervisors attending a management institute at the University of North Carolina in 1972. They were asked to assume the role of the personnel director of a bank and were given a personnel file to judge whether the person should be promoted to a branch manager position. The files given to the participants were identical, except that half of them indicated the candidate was male and the other half indicated the candidate was female. These files were randomly assigned to the subjects.

> **Exercise**:
> Is this an observational study or an experiment? How does the type of study impact what can be inferred from the results?[3]

For each supervisor we recorded the gender associated with the assigned file and the promotion decision. Using the results of the study summarized in the table below, we would like to evaluate if females are unfairly discriminated against in promotion decisions. In this study, a smaller proportion of females are promoted than males (0.583 versus 0.875), but it is unclear whether the difference provides **convincing evidence** that females are unfairly discriminated against.

		Decision		
		promoted	not promoted	Total
Gender	male	21	3	24
	female	14	10	24
	Total	35	13	48

> *Thought Question*:
> Statisticians are sometimes called upon to evaluate the strength of evidence. When looking at the rates of promotion for males and females in this study, why might we be tempted to immediately conclude that females are being discriminated against?

The large difference in promotion rates (58.3% for females versus 87.5% for males) suggest there might be discrimination against women in promotion decisions. Most people come to this conclusion because they think these sample statistics are the actual population parameters. We cannot yet be sure if the observed difference represents discrimination or is just from random variability. Generally there is fluctuation in sample data; if we conducted the experiment again, we would get different values. We also wouldn't expect the sample proportions to be **exactly** equal, even if the truth was that the promotion decisions were independent of gender. To make a decision, we must understand the random variability and compare it with the observed difference.

This question is a reminder that the observed outcomes in the sample may not perfectly reflect the true relationships between variables in the underlying population. The table shows there were 7 fewer promotions in the female group than in the male group, a difference in promotion rates of 29.2% $\left(\frac{21}{24} - \frac{14}{24} = 0.292\right)$. This observed difference is what we call a *point estimate* of the true effect. The point estimate of the difference is large, but the sample size for the study is small, making it unclear if this observed difference represents discrimination or whether it is simply due to chance. We label these two competing claims, chance or discrimination, as H_0 and H_A:

H_0: **Null hypothesis.** The variables *gender* and *decision* are independent. They have no relationship, and the observed difference between the proportion of males and females who were promoted, 29.2%, was due to chance.

[3]The study is an experiment, as subjects were randomly assigned a male file or a female file. Since this is an experiment, the results can be used to evaluate a causal relationship between gender of a candidate and the promotion decision.

H_A: **Alternative hypothesis.** The variables *gender* and *decision* are *not* independent. The difference in promotion rates of 29.2% was not due to chance, and equally qualified females are less likely to be promoted than males.

Hypothesis testing
These hypotheses are part of what is called a **hypothesis test**. A hypothesis test is a statistical technique used to evaluate competing claims using data. Often times, the null hypothesis takes a stance of **no difference** or **no effect** and thus is `skeptical` of the research claim. If the null hypothesis and the data notably disagree, then we will reject the null hypothesis in favor of the alternative hypothesis.

Don't worry if you aren't a master of hypothesis testing at the end of this lesson. We'll discuss these ideas and details many times in this block.

What would it mean if the null hypothesis, which says the variables *gender* and *decision* are unrelated, is true? It would mean each banker would decide whether to promote the candidate without regard to the gender indicated on the file. That is, the difference in the promotion percentages would be due to the way the files were randomly divided to the bankers, and the randomization just happened to give rise to a relatively large difference of 29.2%.

Consider the alternative hypothesis: bankers were influenced by which gender was listed on the personnel file. If this was true, and especially if this influence was substantial, we would expect to see some difference in the promotion rates of male and female candidates. If this gender bias was against females, we would expect a smaller fraction of promotion recommendations for female personnel files relative to the male files.

We will choose between these two competing claims by assessing if the data conflict so much with H_0 that the null hypothesis cannot be deemed reasonable. If this is the case, and the data support H_A, then we will reject the notion of independence and conclude that these data provide strong evidence of discrimination. Again, we will do this by determining how much difference in promotion rates would happen by random variation and compare this with the observed difference. We will make a decision based on probability considerations.

Simulating the study

The table of data shows that 35 bank supervisors recommended promotion and 13 did not. Now, suppose the bankers' decisions were independent of gender, that is the null hypothesis is true. Then, if we conducted the experiment again with a different random assignment of files, differences in promotion rates would be based only on random fluctuation. We can actually perform this **randomization**, which simulates what would have happened if the bankers' decisions had been independent of gender but we had distributed the files differently.[4] We will walk through the steps next.

First let's import the data.

```
discrim <- read_csv("data/discrimination_study.csv")
```

```
inspect(discrim)
```

```
##
## categorical variables:
##       name     class levels  n missing
## 1   gender character      2 48       0
## 2 decision character      2 48       0
##                                    distribution
## 1 female (50%), male (50%)
## 2 promoted (72.9%), not_promoted (27.1%)
```

[4]The test procedure we employ in this section is formally called a **permutation test**.

```
tally(~gender+decision,discrim,margins=TRUE)
```

```
##         decision
## gender   not_promoted promoted Total
##   female           10       14    24
##   male              3       21    24
##   Total            13       35    48
```

Let's do some categorical data cleaning. To get the `tally()` results to look like our table, we need to change to factors and reorder the levels.

We will use `mutate_if()` to convert characters to factors and `fct_relevel()` to change levels.

```
discrim <- discrim %>%
  mutate_if(is.character,as.factor) %>%
  mutate(gender=fct_relevel(gender,"male"),
         decision=fct_relevel(decision,"promoted"))
```

```
head(discrim)
```

```
## # A tibble: 6 x 2
##   gender decision
##   <fct>  <fct>
## 1 female not_promoted
## 2 female not_promoted
## 3 male   promoted
## 4 female promoted
## 5 female promoted
## 6 female promoted
```

```
tally(~gender+decision,discrim,margins = TRUE)
```

```
##         decision
## gender   promoted not_promoted Total
##   male         21            3    24
##   female       14           10    24
##   Total        35           13    48
```

Now that we have the data in form that we want, we are ready to conduct the *permutation test*. To think about this *simulation*, imagine we actually had the personnel files. We thoroughly shuffle 48 personnel files, 24 labeled *male* and 24 labeled *female*, and deal these files into two stacks. We will deal 35 files into the first stack, which will represent the 35 supervisors who recommended promotion. The second stack will have 13 files, and it will represent the 13 supervisors who recommended against promotion. Remember that the files are identical except for the listed gender. This simulation then assumes that gender is not important and thus we can randomly assign the files to any of the supervisors. Then, as we did with the original data, we tabulate the results and determine the fraction of *male* and *female* who were promoted. Since we don't actually physically have the files, we will do this shuffle via computer code.

Since the randomization of files in this simulation is independent of the promotion decisions, any difference in the two fractions is entirely due to chance. The following code shows the results of such a simulation.

```
set.seed(101)
tally(~shuffle(gender)+decision,discrim,margins = TRUE)
```

```
##                 decision
## shuffle(gender) promoted not_promoted Total
##          male         18            6    24
##          female       17            7    24
##          Total        35           13    48
```

The `shuffle()` function randomly rearranges the gender column while keeping the decision column the same. It is really a sampling without replacement.

> **Exercise**: What is the difference in promotion rates between the two simulated groups? How does this compare to the observed difference 29.2% from the actual study?[5]

Calculating by hand will not help in a simulation, so we must write a function or use an existing one. We will use `diffprop` from the `mosiac` package. The code to find the difference for the original data is:

```
(obs<-diffprop(decision~gender,data=discrim))
```

```
##   diffprop
## -0.2916667
```

Notice that this is subtracting proportion of males promoted from the proportion of females. This does not impact our results as this is an arbitrary decision. We just need to be consistent in our analysis. If we prefer to use positive values we can adjust the order easily.

```
diffprop(decision~fct_relevel(gender,"female"),data=discrim)
```

```
##  diffprop
## 0.2916667
```

Notice that what we have done here, we developed a single number metric to measure the relationship between *gender* and *decision*. This single value metric is called the **test statistic**. We could have used a number of different metrics to include just the difference in males and females. The key idea in hypothesis testing is that once you decide on a test statistic, you need to find the distribution of that test statistic assuming the null hypothesis is true.

Checking for independence

We computed one possible difference under the null hypothesis in the exercise above, which represents one difference due to chance. Repeating the simulation, we get another difference due to chance: -0.042. And another: 0.208. And so on until we repeat the simulation enough times that we have a good idea of what represents the **distribution of differences from chance alone**. That is the difference if there really is no relationship between gender and the promotion decision. We are using a simulation when there is actually a finite number of permutations of the `gender` label. From our lesson on counting, we have 48 labels of which 24 are `male` and 24 are `female`. Thus the total number of ways to arrange the labels differently is:

$$\frac{48!}{24! \cdot 24!} \approx 3.2 \cdot 10^{13}$$

[5] $18/24 - 17/24 = 0.042$ or about 4.2% in favor of the men. This difference due to chance is much smaller than the difference observed in the actual groups.

```
factorial(48)/(factorial(24)*factorial(24))
```

```
## [1] 3.22476e+13
```

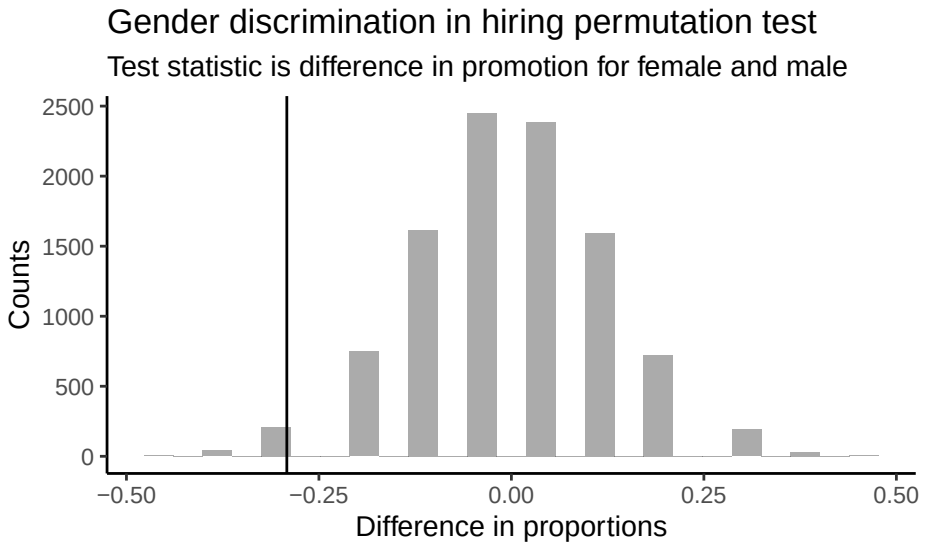
This number of permutations is too large to find by hand or even via code and thus we will use a simulation.

Let's simulate the experiment and plot the simulated values of the difference in the proportions of male and female files recommended for promotion.

```
set.seed(2022)
results <- do(10000)*diffprop(decision~shuffle(gender),data=discrim)
```

In our plot, we will insert a vertical line at the value of our observed difference.

```
results %>%
  gf_histogram(~diffprop) %>%
  gf_vline(xintercept =-0.2916667 ) %>%
  gf_theme(theme_classic()) %>%
  gf_labs(x="Difference in proportions",y="Counts",
          title="Gender discrimination in hiring permutation test",
          subtitle="Test statistic is difference in promotion for female and male")
```



Note that the distribution of these simulated differences is centered around 0 and is roughly symmetrical. It is centered on zero because we simulated differences in a way that made no distinction between men and women. This makes sense: we should expect differences from chance alone to fall around zero with some random fluctuation for each simulation under the assumption of the null hypothesis. The histogram also looks like a normal distribution; this is not a coincidence, it is a result of what is called the **Central Limit Theorem** which we will learn about in this block.

Example:
How often would you observe a difference of at least -29.2% (-0.292) according to the figure? (Often, sometimes, rarely, or never?)

It appears that a difference of at least -29.2% due to chance alone would only happen rarely. We can estimate the probability using the `results` object.

```
results %>%
  summarise(p_value = mean(diffprop<=obs))
```

```
##   p_value
## 1  0.0257
```

In our simulations, only 2.8% of the simulated test statistics were less than or equal to the observed test statistic, more extreme relative to the null hypothesis. Such a low probability indicates that observing such a large difference in proportions from chance alone is rare. This probability is known as a **p-value**. The p-value is a conditional probability, the probability of the observed value or more extreme given that the null hypothesis is true.

The observed difference of -29.2% is a rare event if there really is no impact from listing gender in the candidates' files, which provides us with two possible interpretations of the study results:

H_0: **Null hypothesis.** Gender has no effect on promotion decision, and we observed a difference that is so large that it would only happen rarely.

H_A: **Alternative hypothesis.** Gender has an effect on promotion decision, and what we observed was actually due to equally qualified women being discriminated against in promotion decisions, which explains the large difference of -29.2%.

When we conduct formal studies, we reject a skeptical position if the data strongly conflict with that position.[6]

In our analysis, we determined that there was only a $\sim 2\%$ probability of obtaining a test statistic where the difference between female and male promotion proportions was 29.2% or larger assuming gender had no impact. So we conclude the data provide evidence of gender discrimination against women by the supervisors. In this case, we reject the null hypothesis in favor of the alternative.

Statistical inference is the practice of making decisions and conclusions from data in the context of uncertainty. Errors do occur, just like rare events, and the data set at hand might lead us to the wrong conclusion. While a given data set may not always lead us to a correct conclusion, statistical inference gives us tools to control and evaluate how often these errors occur.

Let's summarize what we did in this case study. We had a research question and some data to test the question. We then performed 4 steps:

1. State the null and alternative hypotheses.

2. Compute a test statistic.

3. Determine the p-value.

[6]This reasoning does not generally extend to anecdotal observations. Each of us observes incredibly rare events every day, events we could not possibly hope to predict. However, in the non-rigorous setting of anecdotal evidence, almost anything may appear to be a rare event, so the idea of looking for rare events in day-to-day activities is treacherous. For example, we might look at the lottery: there was only a 1 in 176 million chance that the Mega Millions numbers for the largest jackpot in history (March 30, 2012) would be (2, 4, 23, 38, 46) with a Mega ball of (23), but nonetheless those numbers came up! However, no matter what numbers had turned up, they would have had the same incredibly rare odds. That is, *any set of numbers we could have observed would ultimately be incredibly rare*. This type of situation is typical of our daily lives: each possible event in itself seems incredibly rare, but if we consider every alternative, those outcomes are also incredibly rare. We should be cautious not to misinterpret such anecdotal evidence.

4. Draw a conclusion.

We decided to use a randomization, a permutation test, to answer the question. When creating a randomization distribution, we attempted to satisfy 3 guiding principles.

1. Be consistent with the null hypothesis.
   We need to simulate a world in which the null hypothesis is true. If we don't do this, we won't be testing our null hypothesis. In our problem, we assumed gender and promotion were independent.

2. Use the data in the original sample.
   The original data should shed light on some aspects of the distribution that are not determined by null hypothesis. For our problem we used the difference in promotion rates. The data does not give us the distribution direction, but it gives us an idea that there is a large difference.

3. Reflect the way the original data were collected.
   There were 48 files and 48 supervisors. A total of 35 files indicated promote. We keep this the same in our simulation.

The remainder of this block expands on the ideas of this case study.

File Creation Information

- File creation date: 2020-10-13
- Windows version: Windows 10 x64 (build 18362)
- R version 3.6.3 (2020-02-29)
- `mosaic` package version: 1.7.0
- `tidyverse` package version: 1.3.0