2

A Summary of Basic Determinant Theory

2.1 Introduction

This chapter consists entirely of a summary of basic determinant theory, a prerequisite for the understanding of later chapters. It is assumed that the reader is familiar with these relations, although not necessarily with the notation used to describe them, and few proofs are given. If further proofs are required, they can be found in numerous undergraduate textbooks.

Several of the relations, including Cramer's formula and the formula for the derivative of a determinant, are expressed in terms of column vectors, a notation which is invaluable in the description of several analytical processes.

2.2 Row and Column Vectors

Let row i (the ith row) and column j (the jth column) of the determinant $A_n = |a_{ij}|_n$ be denoted by the boldface symbols $\mathbf{R}_i$ and $\mathbf{C}_j$ respectively:

$$\begin{aligned}\mathbf{R}_i &= \begin{bmatrix} a_{i1}\ a_{i2}\ a_{i3} \cdots a_{in} \end{bmatrix}, \\ \mathbf{C}_j &= \begin{bmatrix} a_{1j}\ a_{2j}\ a_{3j} \cdots a_{nj} \end{bmatrix}^T \end{aligned} \tag{2.2.1}$$

where T denotes the transpose. We may now write

$$A_n = \begin{vmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \vdots \\ \mathbf{R}_n \end{vmatrix} = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|. \tag{2.2.2}$$

The column vector notation is clearly more economical in space and will be used exclusively in this and later chapters. However, many properties of particular determinants can be proved by performing a sequence of row and column operations and in these applications, the symbols $\mathbf{R}_i$ and $\mathbf{C}_j$ appear with equal frequency.

If every element in $\mathbf{C}_j$ is multiplied by the scalar k, the resulting vector is denoted by $k\mathbf{C}_j$:

$$k\mathbf{C}_j = \left[ka_{1j}\ ka_{2j}\ ka_{3j} \cdots ka_{nj}\right]^T.$$

If $k = 0$, this vector is said to be zero or null and is denoted by the boldface symbol $\mathbf{O}$.

If a_{ij} is a function of x, then the derivative of $\mathbf{C}_j$ with respect to x is denoted by $\mathbf{C}'_j$ and is given by the formula

$$\mathbf{C}'_j = \left[a'_{1j}\ a'_{2j}\ a'_{3j} \cdots a'_{nj}\right]^T.$$

2.3 Elementary Formulas

2.3.1 Basic Properties

The arbitrary determinant

$$A = |a_{ij}|_n = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|,$$

where the suffix n has been omitted from A_n, has the properties listed below. Any property stated for columns can be modified to apply to rows.

a. The value of a determinant is unaltered by transposing the elements across the principal diagonal. In symbols,

$$|a_{ji}|_n = |a_{ij}|_n.$$

b. The value of a determinant is unaltered by transposing the elements across the secondary diagonal. In symbols

$$|a_{n+1-j,n+1-i}|_n = |a_{ij}|_n.$$

c. If any two columns of A are interchanged and the resulting determinant is denoted by B, then $B = -A$.

Example.

$$\left|\mathbf{C}_1\ \mathbf{C}_3\ \mathbf{C}_4\ \mathbf{C}_2\right| = -\left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_4\ \mathbf{C}_3\right| = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3\ \mathbf{C}_4\right|.$$

Applying this property repeatedly,

i.

$$\left|\mathbf{C}_m\ \mathbf{C}_{m+1}\cdots\mathbf{C}_n\ \mathbf{C}_1\ \mathbf{C}_2\cdots\mathbf{C}_{m-1}\right| = (-1)^{(m-1)(n-1)}A,\quad 1 < m < n.$$

The columns in the determinant on the left are a cyclic permutation of those in A.

ii. $\left|\mathbf{C}_n\ \mathbf{C}_{n-1}\ \mathbf{C}_{n-2}\cdots\mathbf{C}_2\ \mathbf{C}_1\right| = (-1)^{n(n-1)/2}A.$

d. Any determinant which contains two or more identical columns is zero.

$$\left|\mathbf{C}_1\cdots\mathbf{C}_j\cdots\mathbf{C}_j\cdots\mathbf{C}_n\right| = 0.$$

e. If every element in any one column of A is multiplied by a scalar k and the resulting determinant is denoted by B, then $B = kA$.

$$B = \left|\mathbf{C}_1\ \mathbf{C}_2\cdots(k\mathbf{C}_j)\cdots\mathbf{C}_n\right| = kA.$$

Applying this property repeatedly,

$$\begin{aligned}|ka_{ij}|_n &= \left|(k\mathbf{C}_1)\ (k\mathbf{C}_2)\ (k\mathbf{C}_3)\cdots(k\mathbf{C}_n)\right|\\ &= k^n|a_{ij}|_n.\end{aligned}$$

This formula contrasts with the corresponding matrix formula, namely

$$[ka_{ij}]_n = k[a_{ij}]_n.$$

Other formulas of a similar nature include the following:

i. $|(-1)^{i+j}a_{ij}|_n = |a_{ij}|_n,$
ii. $|ia_{ij}|_n = |ja_{ij}|_n = n!|a_{ij}|_n,$
iii. $|x^{i+j-r}a_{ij}|_n = x^{n(n+1-r)}|a_{ij}|_n.$

f. Any determinant in which one column is a scalar multiple of another column is zero.

$$\left|\mathbf{C}_1\cdots\mathbf{C}_j\cdots(k\mathbf{C}_j)\cdots\mathbf{C}_n\right| = 0.$$

g. If any one column of a determinant consists of a sum of m subcolumns, then the determinant can be expressed as the sum of m determinants, each of which contains one of the subcolumns.

$$\left|\mathbf{C}_1\cdots\left(\sum_{s=1}^{m}\mathbf{C}_{js}\right)\cdots\mathbf{C}_n\right| = \sum_{s=1}^{m}\left|\mathbf{C}_1\cdots\mathbf{C}_{js}\cdots\mathbf{C}_n\right|.$$

Applying this property repeatedly,

$$\left|\left(\sum_{s=1}^{m}\mathbf{C}_{1s}\right)\cdots\left(\sum_{s=1}^{m}\mathbf{C}_{js}\right)\cdots\left(\sum_{s=1}^{m}\mathbf{C}_{ns}\right)\right|$$

$$= \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_n=1}^{m} \left| \mathbf{C}_{1k_1} \cdots \mathbf{C}_{jk_j} \cdots \mathbf{C}_{nk_n} \right|_n.$$

The function on the right is the sum of m^n determinants. This identity can be expressed in the form

$$\left| \sum_{k=1}^{m} a_{ij}^{(k)} \right|_n = \sum_{k_1,k_2,\ldots,k_n=1}^{m} \left| a_{ij}^{(k_j)} \right|_n.$$

h. *Column Operations.* The value of a determinant is unaltered by adding to any one column a linear combination of all the other columns. Thus, if

$$\begin{aligned} \mathbf{C}'_j &= \mathbf{C}_j + \sum_{r=1}^{n} k_r \; \mathbf{C}_r \qquad && k_j = 0, \\ &= \sum_{r=1}^{n} k_r \; \mathbf{C}_r, && k_j = 1, \end{aligned}$$

then

$$\left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}'_j \cdots \mathbf{C}_n \right| = \left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_j \cdots \mathbf{C}_n \right|.$$

$\mathbf{C}'_j$ should be regarded as a new column j and will not be confused with the derivative of $\mathbf{C}_j$. The process of replacing $\mathbf{C}_j$ by $\mathbf{C}'_j$ is called a column operation and is extensively applied to transform and evaluate determinants. Row and column operations are of particular importance in reducing the order of a determinant.

Exercise. If the determinant $A_n = |a_{ij}|_n$ is rotated through $90°$ in the clockwise direction so that a_{11} is displaced to the position $(1, n)$, a_{1n} is displaced to the position (n, n), etc., and the resulting determinant is denoted by $B_n = |b_{ij}|_n$, prove that

$$\begin{aligned} b_{ij} &= a_{j,n-i} \\ B_n &= (-1)^{n(n-1)/2} A_n. \end{aligned}$$

2.3.2 Matrix-Type Products Related to Row and Column Operations

The row operations

$$\mathbf{R}'_i = \sum_{j=i}^{3} u_{ij} \mathbf{R}_j, \quad u_{ii} = 1, \quad 1 \le i \le 3; \quad u_{ij} = 0, \quad i > j, \qquad (2.3.1)$$

namely

$$\begin{aligned}
\mathbf{R}_1' &= \mathbf{R}_1 + u_{12}\mathbf{R}_2 + u_{13}\mathbf{R}_3\\
\mathbf{R}_2' &= \qquad\quad\ \ \mathbf{R}_2 + u_{23}\mathbf{R}_3\\
\mathbf{R}_3' &= \qquad\qquad\qquad\quad\ \ \mathbf{R}_3,
\end{aligned}$$

can be expressed in the form

$$\begin{bmatrix} \mathbf{R}_1' \\ \mathbf{R}_2' \\ \mathbf{R}_3' \end{bmatrix} = \begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix}.$$

Denote the upper triangular matrix by $\mathbf{U}_3$. These operations, when performed in the given order on an arbitrary determinant $A_3 = |a_{ij}|_3$, have the same effect as *pre*multiplication of A_3 by the unit determinant U_3. In each case, the result is

$$A_3 = \begin{vmatrix} a_{11} + u_{12}a_{21} + u_{13}a_{31} & a_{12} + u_{12}a_{22} + u_{13}a_{32} & a_{13} + u_{12}a_{23} + u_{13}a_{33} \\ a_{21} + u_{23}a_{31} & a_{22} + u_{23}a_{32} & a_{23} + u_{23}a_{33} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{2.3.2}$$

Similarly, the column operations

$$\mathbf{C}_i' = \sum_{j=i}^{3} u_{ij}\mathbf{C}_j, \quad u_{ii} = 1, \quad 1 \le i \le 3; \quad u_{ij} = 0, \quad i > j, \tag{2.3.3}$$

when performed in the given order on A_3, have the same effect as *post*multiplication of A_3 by U_3^T. In each case, the result is

$$A_3 = \begin{vmatrix} a_{11} + u_{12}a_{12} + u_{13}a_{13} & a_{12} + u_{23}a_{13} & a_{13} \\ a_{21} + u_{12}a_{22} + u_{13}a_{23} & a_{22} + u_{23}a_{23} & a_{23} \\ a_{31} + u_{12}a_{32} + u_{13}a_{33} & a_{32} + u_{23}a_{33} & a_{33} \end{vmatrix}. \tag{2.3.4}$$

The row operations

$$\mathbf{R}_i' = \sum_{j=1}^{i} v_{ij}\mathbf{R}_j, \quad v_{ii} = 1, \quad 1 \le i \le 3; \quad v_{ij} = 0, \quad i < j, \tag{2.3.5}$$

can be expressed in the form

$$\begin{bmatrix} \mathbf{R}_1' \\ \mathbf{R}_2' \\ \mathbf{R}_3' \end{bmatrix} = \begin{bmatrix} 1 & & \\ v_{21} & 1 & \\ v_{31} & v_{32} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix}.$$

Denote the lower triangular matrix by $\mathbf{V}_3$. These operations, when performed *in reverse order* on A_3, have the same effect as *pre*multiplication of A_3 by the unit determinant V_3.

Similarly, the column operations

$$\mathbf{C}'_i = \sum_{j=1}^{i} v_{ij}\mathbf{C}_j, \quad v_{ii} = 1, \quad 1 \le i \le 3, \quad v_{ij} = 0, \quad i > j, \tag{2.3.6}$$

when performed on A_3 in *reverse order*, have the same effect as *post*multiplication of A_3 by V_3^T.

2.3.3 First Minors and Cofactors; Row and Column Expansions

To each element a_{ij} in the determinant $A = |a_{ij}|_n$, there is associated a subdeterminant of order $(n-1)$ which is obtained from A by deleting row i and column j. This subdeterminant is known as a first minor of A and is denoted by M_{ij}. The first cofactor A_{ij} is then defined as a signed first minor:

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{2.3.7}$$

It is customary to omit the adjective *first* and to refer simply to minors and cofactors and it is convenient to regard M_{ij} and A_{ij} as quantities which belong to a_{ij} in order to give meaning to the phrase "an element and its cofactor."

The expansion of A by elements from row i and their cofactors is

$$A = \sum_{j=1}^{n} a_{ij}A_{ij}, \quad 1 \le i \le n. \tag{2.3.8}$$

The expansion of A by elements from column j and their cofactors is obtained by summing over i instead of j:

$$A = \sum_{i=1}^{n} a_{ij}A_{ij}, \quad 1 \le j \le n. \tag{2.3.9}$$

Since A_{ij} belongs to but is independent of a_{ij}, an alternative definition of A_{ij} is

$$A_{ij} = \frac{\partial A}{\partial a_{ij}}. \tag{2.3.10}$$

Partial derivatives of this type are applied in Section 4.5.2 on symmetric Toeplitz determinants.

2.3.4 Alien Cofactors; The Sum Formula

The theorem on alien cofactors states that

$$\sum_{j=1}^{n} a_{ij}A_{kj} = 0, \quad 1 \le i \le n, \quad 1 \le k \le n, \quad k \ne i. \tag{2.3.11}$$

The elements come from row i of A, but the cofactors belong to the elements in row k and are said to be alien to the elements. The identity is merely an expansion by elements from row k of the determinant in which row $k =$ row i and which is therefore zero.

The identity can be combined with the expansion formula for A with the aid of the Kronecker delta function δ_{ik} (Appendix A.1) to form a single identity which may be called the sum formula for elements and cofactors:

$$\sum_{j=1}^{n} a_{ij}A_{kj} = \delta_{ik}A, \quad 1 \le i \le n, \quad 1 \le k \le n. \tag{2.3.12}$$

It follows that

$$\sum_{j=1}^{n} A_{ij}\mathbf{C}_j = [0 \ldots 0 \ A \ 0 \ldots 0]^T, \quad 1 \le i \le n,$$

where the element A is in row i of the column vector and all the other elements are zero. If $A = 0$, then

$$\sum_{j=1}^{n} A_{ij}\mathbf{C}_j = 0, \quad 1 \le i \le n, \tag{2.3.13}$$

that is, the columns are linearly dependent. Conversely, if the columns are linearly dependent, then $A = 0$.

2.3.5 Cramer's Formula

The set of equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad 1 \le i \le n,$$

can be expressed in column vector notation as follows:

$$\sum_{j=1}^{n} \mathbf{C}_j x_j = \mathbf{B},$$

where

$$\mathbf{B} = \begin{bmatrix} b_1 \ b_2 \ b_3 \cdots b_n \end{bmatrix}^T.$$

If $A = |a_{ij}|_n \ne 0$, then the unique solution of the equations can also be expressed in column vector notation. Let

$$A = \left|\mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_j \cdots \mathbf{C}_n\right|.$$

Then

$$x_j = \frac{1}{A}\left|\mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_{j-1} \ \mathbf{B} \ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|$$

$$= \frac{1}{A} \sum_{i=1}^{n} b_i A_{ij}. \tag{2.3.14}$$

The solution of the triangular set of equations

$$\sum_{j=1}^{i} a_{ij} x_j = b_i, \quad i = 1, 2, 3, \dots$$

(the upper limit in the sum is i, not n as in the previous set) is given by the formula

$$x_i = \frac{(-1)^{i+1}}{a_{11} a_{22} \cdots a_{ii}} \begin{vmatrix} b_1 & a_{11} \\ b_2 & a_{21} & a_{22} \\ b_3 & a_{31} & a_{32} & a_{33} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i-1} & a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,i-1} \\ b_i & a_{i1} & a_{i2} & a_{i3} & \cdots & a_{i,i-1} \end{vmatrix}_i . \tag{2.3.15}$$

The determinant is a Hessenbergian (Section 4.6).

Cramer's formula is of great theoretical interest and importance in solving sets of equations with algebraic coefficients but is unsuitable for reasons of economy for the solution of large sets of equations with numerical coefficients. It demands far more computation than the unavoidable minimum. Some matrix methods are far more efficient. Analytical applications of Cramer's formula appear in Section 5.1.2 on the generalized geometric series, Section 5.5.1 on a continued fraction, and Section 5.7.2 on the Hirota operator.

Exercise. If

$$f_i^{(n)} = \sum_{j=1}^{n} a_{ij} x_j + a_{in}, \quad 1 \le i \le n,$$

and

$$f_i^{(n)} = 0, \quad 1 \le i \le n, \quad i \ne r,$$

prove that

$$f_r^{(n)} = \frac{A_n x_r}{A_{rn}^{(n)}}, \quad 1 \le r < n,$$

$$f_n^{(n)} = \frac{A_n (x_n + 1)}{A_{n-1}},$$

where

$$A_n = |a_{ij}|_n,$$

provided

$$A_{rn}^{(n)} \ne 0, \quad 1 \le i \le n.$$

2.3.6 *The Cofactors of a Zero Determinant*

If $A = 0$, then

$$A_{p_1q_1}A_{p_2q_2} = A_{p_2q_1}A_{p_1q_2}, \tag{2.3.16}$$

that is,

$$\begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} \\ A_{p_2q_1} & A_{p_2q_2} \end{vmatrix} = 0, \quad 1 \le p_1, p_2, q_1, q_2 \le n.$$

It follows that

$$\begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} & A_{p_1q_3} \\ A_{p_2q_1} & A_{p_2q_2} & A_{p_2q_3} \\ A_{p_3q_1} & A_{p_3q_2} & A_{p_3q_2} \end{vmatrix} = 0$$

since the second-order cofactors of the elements in the last (or any) row are all zero. Continuing in this way,

$$\begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} & \cdots & A_{p_1q_r} \\ A_{p_2q_1} & A_{p_2q_2} & \cdots & A_{p_2q_r} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p_rq_1} & A_{p_rq_2} & \cdots & A_{p_rq_r} \end{vmatrix}_r = 0, \quad 2 \le r \le n. \tag{2.3.17}$$

This identity is applied in Section 3.6.1 on the Jacobi identity.

2.3.7 *The Derivative of a Determinant*

If the elements of A are functions of x, then the derivative of A with respect to x is equal to the sum of the n determinants obtained by differentiating the columns of A one at a time:

$$\begin{aligned} A' &= \sum_{j=1}^{n} \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}'_j \cdots \mathbf{C}_n \right| \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a'_{ij} A_{ij}. \end{aligned} \tag{2.3.18}$$