# 4

# Particular Determinants

## 4.1 Alternants

### *4.1.1 Introduction*

Any function of $n$ variables which changes sign when any two of the variables are interchanged is known as an alternating function. It follows that an alternating function vanishes if any two of the variables are equal. Any determinant function which possess these properties is known as an alternant.

The simplest form of alternant is

$$|f_j(x_i)|_n = \begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \dots & \dots & \dots & \dots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{vmatrix}_n . \tag{4.1.1}$$

The interchange of any two $x$'s is equivalent to the interchange of two rows which gives rise to a change of sign. If any two of the $x$'s are equal, the determinant has two identical rows and therefore vanishes.

The double or two-way alternant is

$$|f(x_i, y_j)|_n = \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) & \cdots & f(x_1, y_n) \\ f(x_2, y_1) & f(x_2, y_2) & \cdots & f(x_2, y_n) \\ \dots & \dots & \dots & \dots \\ f(x_n, y_1) & f(x_n, y_2) & \cdots & f(x_n, y_n) \end{vmatrix}_n . \tag{4.1.2}$$

If the $x$'s are not distinct, the determinant has two or more identical rows. If the $y$'s are not distinct, the determinant has two or more identical columns. In both cases, the determinant vanishes.

**Illustration.** The Wronskian $|D_x^{j-1}(f_i)|_n$ is an alternant. The double Wronskian $|D_x^{j-1}D_y^{i-1}(f)|_n$ is a double alternant, $D_x = \partial/\partial x$, etc.

**Exercise.** Define two third-order alternants $\phi$ and $\psi$ in column vector notation as follows:

$$\begin{aligned}\phi &= |\mathbf{c}(x_1)\ \mathbf{c}(x_2)\ \mathbf{c}(x_3)|,\\ \psi &= |\mathbf{C}(x_1)\ \mathbf{C}(x_2)\ \mathbf{C}(x_3)|.\end{aligned}$$

Apply l'Hopital's formula to prove that

$$\lim\left(\frac{\phi}{\psi}\right) = \frac{|\mathbf{c}(x)\ \mathbf{c}'(x)\ \mathbf{c}''(x)|}{|\mathbf{C}(x)\ \mathbf{C}'(x)\ \mathbf{C}''(x)|},$$

where the limit is carried out as $x_i \to x$, $1 \le i \le 3$, provided the numerator and denominator are not both zero.

### *4.1.2 Vandermondians*

The determinant

$$\begin{aligned}X_n &= |x_i^{j-1}|_n\\ &= \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1}\\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1}\end{vmatrix}_n\\ &= V(x_1, x_2, \ldots, x_n) \qquad (4.1.3)\end{aligned}$$

is known as the alternant of Vandermonde or simply a Vandermondian.

**Theorem.**

$$X_n = \prod_{1\le r<s\le n}(x_s - x_r).$$

*The expression on the right is known as a difference–product and contains* $(n/2) = \frac{1}{2}n(n-1)$ *factors.*

*First Proof.* The expansion of the determinant consists of the sum of $n!$ terms, each of which is the product of $n$ elements, one from each row and one from each column. Hence, $X_n$ is a polynomial in the $x_r$ of degree

$$0 + 1 + 2 + 3 + \cdots + (n-1) = \tfrac{1}{2}n(n-1).$$

One of the terms in this polynomial is the product of the elements in the leading diagonal, namely

$$+\ x_2 x_3^2 x_4^3 \cdots x_n^{n-1}. \qquad (4.1.4)$$

When any two of the $x_r$ are equal, $X_n$ has two identical rows and therefore vanishes. Hence, very possible difference of the form $(x_s - x_r)$ is a factor of $X_n$, that is,

$$\begin{aligned} X_n = \quad & K(x_2 - x_1)(x_3 - x_1)(x_4 - x_1)\cdots(x_n - x_1) \\ & \qquad\qquad (x_3 - x_2)(x_4 - x_2)\cdots(x_n - x_2) \\ & \qquad\qquad\qquad\qquad (x_4 - x_3)\cdots(x_n - x_3) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \cdots\cdots \\ & \qquad\qquad\qquad\qquad\qquad\qquad (x_n - x_{n-1}) \\ = K & \prod_{1\le r<s\le n} (x_s - x_r), \end{aligned}$$

which is the product of $K$ and $\frac{1}{2}n(n-1)$ factors. One of the terms in the expansion of this polynomial is the product of $K$ and the first term in each factor, namely

$$Kx_2x_3^2x_4^3\cdots x_n^{n-1}.$$

Comparing this term with (4.1.4), it is seen that $K = 1$ and the theorem is proved.

*Second Proof.* Perform the column operations

$$\mathbf{C}_j' = \mathbf{C}_j - x_n\mathbf{C}_{j-1}$$

in the order $j = n, n-1, n-2, \ldots, 3, 2$. The result is a determinant in which the only nonzero element in the last row is a 1 in position $(n, 1)$. Hence,

$$X_n = (-1)^{n-1}V_{n-1},$$

where $V_{n-1}$ is a determinant of order $(n-1)$. The elements in row $s$ of $V_{n-1}$ have a common factor $(x_s - x_n)$. When all such factors are removed from $V_{n-1}$, the result is

$$X_n = X_{n-1}\prod_{r=1}^{n-1}(x_n - x_r),$$

which is a reduction formula for $X_n$. The proof is completed by reducing the value of $n$ by 1 repeatedly and noting that $X_2 = x_2 - x_1$. □

## Exercises

**1.** Let

$$A_n = \left|\binom{j-1}{i-1}(-x_i)^{j-i}\right|_n = 1.$$

Postmultiply the Vandermondian $V_n(\mathbf{x})$ or $V_n(x_1, x_2, \ldots, x_n)$ by $A_n$, prove the reduction formula

$$V_n(x_1, x_2, \ldots, x_n) = V_{n-1}(x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1) \prod_{p=2}^{n} (x_p - x_1),$$

and hence evaluate $V_n(\mathbf{x})$.

**2.** Prove that

$$|x_i^{j-1} y_i^{n-j}|_n = \prod_{1 \le r < s \le n} \begin{vmatrix} y_r & x_r \\ y_s & x_s \end{vmatrix}.$$

**3.** If

$$x_i = \frac{z + c_i}{\rho},$$

prove that

$$|x_i^{j-1}|_n = \rho^{-n(n-1)/2} |c_i^{j-1}|_n,$$

which is independent of $z$. This relation is applied in Section 6.10.3 on the Einstein and Ernst equations.

### *4.1.3 Cofactors of the Vandermondian*

**Theorem 4.1.** *The scaled cofactors of the Vandermonian $X_n = |x_{ij}|_n$, where $x_{ij} = x_i^{j-1}$ are given by the quotient formula*

$$X_n^{ij} = \frac{(-1)^{n-j} \sigma_{i,n-j}^{(n)}}{g_{ni}(x_i)},$$

*where*

$$g_{nr}(x) = \sum_{s=0}^{n-1} (-1)^s \sigma_{rs}^{(n)} x^{n-1-s}.$$

*Notes on the symmetric polynomials $\sigma_{rs}^{(n)}$ and the function $g_{nr}(x)$ are given in Appendix A.7.*

PROOF. Denote the quotient by $F_{ij}$. Then,

$$\begin{aligned} \sum_{k=1}^{n} x_{ik} F_{jk} &= \frac{1}{g_{nj}(x_j)} \sum_{k=1}^{n} (-1)^{n-k} \sigma_{j,n-k}^{(n)} x_i^{k-1} \qquad \text{(Put } k = n - s\text{)} \\ &= \frac{1}{g_{nj}(x_j)} \sum_{s=0}^{n-1} (-1)^s \sigma_{js}^{(n)} x_i^{n-s-1} \\ &= \frac{g_{nj}(x_i)}{g_{nj}(x_j)} \\ &= \delta_{ij}. \end{aligned}$$

Hence,

$$\begin{aligned}
[x_{ij}]_n [F_{ji}]_n &= \mathbf{I}, \\
[F_{ji}]_n &= [x_{ij}]^{-1} \\
&= [X_n^{ji}]_n.
\end{aligned}$$

The theorem follows. □

**Theorem 4.2.**

$$X_{nj}^{(n)} = (-1)^{n-j} X_{n-1} \sigma_{n-j}^{(n-1)}.$$

Proof. Referring to equations (A.7.1) and (A.7.3) in Appendix A.7,

$$\begin{aligned}
X_n &= X_{n-1} \prod_{r=1}^{n-1} (x_n - x_r) \\
&= X_{n-1} f_{n-1}(x_n) \\
&= X_{n-1} g_{nn}(x_n).
\end{aligned}$$

From Theorem 4.1,

$$\begin{aligned}
X_{nj}^{(n)} &= \frac{(-1)^{n-j} X_n \sigma_{n,n-j}^{(n)}}{g_{nn}(x_n)} \\
&= (-1)^{n-j} X_{n-1} \sigma_{n,n-j}^{(n)}.
\end{aligned}$$

The proof is completed using equation (A.7.4) in Appendix A.7. □

### 4.1.4 A Hybrid Determinant

Let $Y_n$ be a second Vandermondian defined as

$$Y_n = |y_i^{j-1}|_n$$

and let $H_{rs}$ denote the hybrid determinant formed by replacing the $r$th row of $X_n$ by the $s$th row of $Y_n$.

**Theorem 4.3.**

$$\frac{H_{rs}}{X_n} = \frac{g_{nr}(y_s)}{g_{nr}(x_r)}.$$

Proof.

$$\begin{aligned}
\frac{H_{rs}}{X_n} &= \sum_{j=1}^{n} y_s^{j-1} X_n^{rj} \\
&= \frac{1}{g_{nr}(x_r)} \sum_{j=1}^{n} (-1)^{n-j} \sigma_{r,n-j}^{(n)} y_s^{j-1} \qquad (\text{Put } j = n-k) \\
&= \frac{1}{g_{nr}(x_r)} \sum_{k=0}^{n-1} (-1)^k \sigma_{rk}^{(n)} y_s^{n-1-k}.
\end{aligned}$$

This completes the proof of Theorem 4.3 which can be expressed in the form

$$\frac{H_{rs}}{X_n} = \frac{\prod_{i=1}^{n}(y_s - x_i)}{(y_s - x_r)\prod_{\substack{i=1\\i\neq r}}^{n}(x_r - x_i)}. \qquad \square$$

Let

$$\begin{aligned} A_n &= |\sigma_{i,j-1}^{(m)}|_n \\ &= \begin{vmatrix} \sigma_{10}^{(m)} & \sigma_{11}^{(m)} & \cdots & \sigma_{1,n-1}^{(m)} \\ \sigma_{20}^{(m)} & \sigma_{21}^{(m)} & \cdots & \sigma_{2,n-1}^{(m)} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n0}^{(m)} & \sigma_{n1}^{(m)} & \cdots & \sigma_{n,n-1}^{(m)} \end{vmatrix}_n, \quad m \geq n. \end{aligned}$$

**Theorem 4.4.**

$$A_n = (-1)^{n(n-1)/2} X_n.$$

PROOF.

$$A_n = \left|\mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \ldots \mathbf{C}_{n-1},\right|_n$$

where, from the lemma in Appendix A.7,

$$\begin{aligned} \mathbf{C}_j &= \left[\sigma_{1j}^{(m)}\ \sigma_{2j}^{(m)}\ \sigma_{3j}^{(m)}\ \ldots\ \sigma_{nj}^{(m)}\right]^T \\ &= \sum_{p=0}^{j} \sigma_p^{(m)} \left[v_1^{j-p}\ v_2^{j-p}\ v_3^{j-p}\ \ldots\ v_n^{j-p}\right]^T, \quad v_r = -x_r,\ \sigma_0^{(m)} = 1. \end{aligned}$$

Applying the column operations

$$\mathbf{C}_j' = \mathbf{C}_j - \sum_{k=1}^{j} \sigma_k^{(m)} \mathbf{C}_{j-k}$$

in the order $j = 1, 2, 3, \ldots$ so that each new column created by one operation is applied in the next operation, it is found that

$$\mathbf{C}_j' = \left[v_1^j\ v_2^j\ v_3^j\ \ldots\ v_n^j\right]^T, \quad j = 0, 1, 2, \ldots.$$

Hence

$$\begin{aligned} A_n &= |v_i^{j-1}|_n \\ &= (-1)^{n(n-1)/2} |x_i^{j-1}|_n. \end{aligned}$$

Theorem 4.4 follows. $\square$

### 4.1.5 The Cauchy Double Alternant

The Cauchy double alternant is the determinant

$$A_n = \left| \frac{1}{x_i - y_j} \right|_n,$$

which can be evaluated in terms of the Vandermondians $X_n$ and $Y_n$ as follows.

Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_n, \quad 1 \le j \le n-1,$$

and then remove all common factors from the elements of rows and columns. The result is

$$A_n = \frac{\prod_{r=1}^{n-1} (y_r - y_n)}{\prod_{r=1}^{n} (x_r - y_n)} B_n, \tag{4.1.5}$$

where $B_n$ is a determinant in which the last column is

$$\begin{bmatrix} 1 \; 1 \; 1 \ldots 1 \end{bmatrix}_n^T$$

and all the other columns are identical with the corresponding columns of $A_n$.

Perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_n, \quad 1 \le i \le n-1,$$

on $B_n$, which then degenerates into a determinant of order $(n-1)$. After removing all common factors from the elements of rows and columns, the result is

$$B_n = \frac{\prod_{r=1}^{n-1} (x_n - x_r)}{\prod_{r=1}^{n-1} (x_n - y_r)} A_{n-1}. \tag{4.1.6}$$

Eliminating $B_n$ from (4.1.5) and (4.1.6) yields a reduction formula for $A_n$, which, when applied, gives the formula

$$A_n = \frac{(-1)^{n(n-1)/2} X_n Y_n}{\prod_{r,s=1}^{n} (x_r - y_s)}.$$

*Exercises*

**1.** Prove the reduction formula

$$A_{ij}^{(n)} = A_{ij}^{(n-1)} \prod_{\substack{r=1\\ r\neq i}}^{n-1} \left(\frac{x_n - x_r}{x_r - y_n}\right) \prod_{\substack{s=1\\ s\neq j}}^{n-1} \left(\frac{y_s - y_n}{x_n - y_s}\right).$$

Hence, or otherwise, prove that

$$A_n^{ij} = \frac{1}{x_i - y_j} \frac{f(y_j)g(x_i)}{f'(x_i)g'(y_j)},$$

where

$$f(t) = \prod_{r=1}^{n}(t - x_r),$$

$$g(t) = \prod_{s=1}^{n}(t - y_s).$$

**2.** Let

$$V_n = \begin{vmatrix} & & & & & f(x_1) \\ & & & & & f(x_2) \\ & & [a_{ij}]_n & & & \vdots \\ & & & & & \vdots \\ & & & & & f(x_n) \\ 1 & 1 & \dots & \dots & 1 & 1 \end{vmatrix}_{n+1},$$

$$W_n = \begin{vmatrix} & & & & & f(x_1) \\ & & & & & f(x_2) \\ & & [a_{ij}]_n & & & \vdots \\ & & & & & \vdots \\ & & & & & f(x_n) \\ -1 & -1 & \dots & \dots & -1 & 1 \end{vmatrix}_{n+1},$$

where

$$a_{ij} = \left(\frac{1 - x_i y_j}{x_i - y_j}\right) f(x_i),$$

$$f(x) = \prod_{i=1}^{n}(x - y_i).$$

Show that

$$V_n = (-1)^{n(n+1)/2} X_n Y_n \prod_{i=1}^{n}(x_i - 1)(y_i + 1),$$

$$W_n = (-1)^{n(n+1)/2} X_n Y_n \prod_{i=1}^{n} (x_i + 1)(y_i - 1).$$

Removing $f(x_1), f(x_2), \ldots, f(x_n)$, from the first $n$ rows in $V_n$ and $W_n$, and expanding each determinant by the last row and column, deduce that

$$\left| \frac{1 - x_i y_j}{x_i - y_j} \right|_n = \frac{1}{2} \left| \frac{1}{x_i - y_j} \right|_n \left\{ \prod_{i=1}^{n} (x_i + 1)(y_i - 1) + \prod_{i=1}^{n} (x_i - 1)(y_i + 1) \right\}.$$

### 4.1.6 A Determinant Related to a Vandermondian

Let $P_r(x)$ be a polynomial defined as

$$P_r(x) = \sum_{s=1}^{r} a_{sr} x^{s-1}, \quad r \geq 1.$$

Note that the coefficient is $a_{sr}$, not the usual $a_{rs}$.

Let

$$X_n = |x_j^{i-1}|_n.$$

**Theorem.**

$$|P_i(x_j)|_n = (a_{11}\ a_{22} \cdots a_{nn}) X_n.$$

PROOF. Define an upper triangular determinant $U_n$ as follows:

$$\begin{aligned} U_n &= |a_{ij}|_n, \qquad a_{ij} = 0, \quad i > j, \\ &= a_{11}\ a_{22} \cdots a_{nn}. \end{aligned} \tag{4.1.7}$$

Some of the cofactors of $U_i$ are given by

$$U_{ij}^{(i)} = \begin{cases} 0, & j > i, \\ U_{i-1}, & j = i,\ U_0 = 1. \end{cases}$$

Those cofactors for which $j < i$ are not required in the analysis which follows. Hence, $|U_{ij}^{(i)}|_n$ is also upper triangular and

$$|U_{ij}^{(i)}|_n = \begin{cases} U_{11}^{(1)} U_{22}^{(2)} \cdots U_{nn}^{(n)}, & U_{11}^{(1)} = 1, \\ U_1\ U_2 \cdots U_{n-1}. \end{cases} \tag{4.1.8}$$

Applying the formula for the product of two determinants in Section 1.4,

$$|U_{ij}^{(j)}|_n |P_i(x_j)|_n = |q_{ij}|_n, \tag{4.1.9}$$

where

$$\begin{aligned}
q_{ij} &= \sum_{r=1}^{i} U_{ir}^{(i)} P_r(x_j) \\
&= \sum_{r=1}^{i} U_{ir}^{(i)} \sum_{s=1}^{r\to i} a_{sr} x_j^{s-1} \quad (a_{sr} = 0,\ s > r) \\
&= \sum_{s=1}^{i} x_j^{s-1} \sum_{r=1}^{i} a_{sr} U_{ir}^{(i)} \\
&= U_i \sum_{s=1}^{i} x_j^{s-1} \delta_{si} \\
&= U_i x_j^{i-1}.
\end{aligned}$$

Hence, referring to (4.1.8),

$$\begin{aligned}
|q_{ij}|_n &= \big(U_1\ U_2 \cdots U_n\big)|x_j^{i-1}| \\
&= U_n |U_{ij}^{(i)}|_n X_n.
\end{aligned}$$

The theorem follows from (4.1.7) and (4.1.9). □

## 4.1.7 *A Generalized Vandermondian*

**Lemma.**

$$\left|\sum_{k=1}^{N} y_k x_k^{i+j-2}\right|_n = \sum_{k_1 \ldots k_n = 1}^{N} \left(\prod_{r=1}^{n} y_{k_r}\right)\left(\prod_{s=2}^{n} x_{k_s}^{s-1}\right) \left|x_{k_j}^{i-1}\right|_n.$$

Proof. Denote the determinant on the left by $A_n$ and put

$$a_{ij}^{(k)} = y_k x_k^{i+j-2}$$

in the last identity in Property (g) in Section 2.3.1. Then,

$$A_n = \sum_{k_1 \ldots k_n = 1}^{N} \left|y_{k_j} x_{k_j}^{i+j-2}\right|_n.$$

Now remove the factor $y_{k_j} x_{k_j}^{j-1}$ from column $j$ of the determinant, $1 \le j \le n$. The lemma then appears and is applied in Section 6.10.4 on the Einstein and Ernst equations. □

## 4.1.8 *Simple Vandermondian Identities*

**Lemmas.**

**a.** $V_n = V_{n-1} \displaystyle\prod_{r=1}^{n-1} (x_n - x_r), \quad n > 1, \quad V(x_1) = 1$

**b.** $V_n = V(x_2, x_3, \ldots, x_n) \prod_{r=2}^{n} (x_r - x_1)$.

**c.** $V(x_t, x_{t+1}, \ldots, x_n) = V(x_{t+1}, x_{t+2}, \ldots, x_n) \prod_{r=t+1}^{n} (x_r - x_t)$.

**d.** $V_{1n}^{(n)} = (-1)^{n+1} V(x_2, x_3, \ldots, x_n) = \dfrac{(-1)^{n+1} V_n}{\prod_{r=2}^{n} (x_r - x_1)}$.

**e.** 
$$\begin{aligned} V_{in}^{(n)} &= (-1)^{n+i} V(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &= \frac{(-1)^{n+i} V_n}{\prod_{r=1}^{i-1} (x_i - x_r) \prod_{r=i+1}^{n} (x_r - x_i)}, \quad i > 1 \end{aligned}$$

**f.** If $\{j_1\ j_2 \cdots j_n\}$ is a permutation of $\{1\ 2 \ldots n\}$, then

$$V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_n}\big) = \operatorname{sgn} \begin{Bmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{Bmatrix} V(x_1, x_2, \ldots, x_n).$$

The proofs of (a) and (b) follow from the difference–product formula in Section 4.1.2 and are elementary. A proof of (c) can be constructed as follows. In (b), put $n = m - t + 1$, then put $x_r = y_{r+t-1}$, $r = 1, 2, 3, \ldots$, and change the dummy variable in the product from $r$ to $s$ using the formula $s = r + t - 1$. The resut is (c) expressed in different symbols. When $t = 1$, (c) reverts to (b). The proofs of (d) and (e) are elementary. The proof of (f) follows from Property (c) in Section 2.3.1 and Appendix A.2 on permutations and their signs.

Let the minors of $V_n$ be denoted by $M_{ij}$. Then,

$$\begin{aligned} M_i &= M_{in} = V(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\ M_n &= M_{nn} = V_{n-1}. \end{aligned}$$

**Theorems.**

**a.** $\displaystyle\prod_{r=1}^{m} M_r = \frac{V(x_{m+1}, x_{m+2}, \ldots, x_n) V_n^{m-1}}{V(x_1, x_2, \ldots, x_m)}, \quad 1 \le m \le n-1$

**b.** $\displaystyle\prod_{r=1}^{n} M_r = V_n^{n-2}$

**c.** $\displaystyle\prod_{r=1}^{m} M_{k_r} = \frac{V(x_{k_{m+1}}, x_{k_{m+2}}, \ldots, x_{k_n}) V_n^{m-1}}{V(x_{k_1}, x_{k_2}, \ldots, x_{k_m})}$

PROOF. Use the method of induction to prove (a), which is clearly valid when $m = 1$. Assume it is valid when $m = s$. Then, from Lemma (e) and referring to Lemma (a) with $n \to s+1$ and Lemma (c) with $m \to s+1$,

$$\begin{aligned}\prod_{r=1}^{s+1} M_r &= \left[\frac{V_n}{\prod_{r=1}^{s}(x_{s+1}-x_r)\prod_{r=s+2}^{n}(x_r-x_{s+1})}\right]\frac{V(x_{s+1},x_{s+2},\ldots,x_n)V_n^{s-1}}{V(x_1,x_2,\ldots,x_s)}\\ &= \frac{V_n^s}{\left[V(x_1,x_2,\ldots,x_s)\prod_{r=1}^{s}(x_{s+1}-x_r)\right]}\left[\frac{V(x_{s+1},x_{s+2},\ldots,x_n)}{\prod_{r=s+2}^{n}(x_r-x_{s+1})}\right]\\ &= \frac{V(x_{s+2},x_{s+3},\ldots,x_n)V_n^s}{V(x_1,x_2,\ldots,x_{s+1})}.\end{aligned}$$

Hence, (a) is valid when $m = s+1$, which proves (a). To prove (b), put $m = n-1$ in (a) and use $M_n = V_{n-1}$. The details are elementary.

The proof of (c) is obtained by applying the permutation

$$\left\{\begin{matrix} 1 & 2 & 3 & \cdots & n \\ k_1 & k_2 & k_3 & \cdots & k_n \end{matrix}\right\}$$

to (a). The only complication which arises is the determination of the sign of the expression on the right of (c). It is left as an exercise for the reader to prove that the sign is positive. □

**Exercise.** Let $A_6$ denote the determinant of order 6 defined in column vector notation as follows:

$$\mathbf{C}_j = \left[a_j \; a_jx_j \; a_jx_j^2 \; b_j \; b_jy_j \; b_jy_j^2\right]^T, \quad 1 \le j \le 6.$$

Apply the Laplace expansion theorem to prove that

$$A_6 = \sum_{\substack{i<j<k\\ p<q<r}} \sigma a_ia_ja_kb_pb_qb_r V(x_i,x_j,x_k)V(y_p,y_q,y_r),$$

where

$$\sigma = \operatorname{sgn}\left\{\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & p & q & r \end{matrix}\right\}$$

and where the lower set of parameters is a permutation of the upper set. The number of terms in the sum is $\binom{6}{3} = 20$.

Prove also that

$$A_6 = 0 \quad \text{when} \quad a_j = b_j, \quad 1 \le j \le 6.$$

Generalize this result by giving an expansion formula for $A_{2n}$ from the first $m$ rows and the remaining $(2n-m)$ rows using the dummy variables $k_r$, $1 \le r \le 2n$. The generalized formula and Theorem (c) are applied in Section 6.10.4 on the Einstein and Ernst equations.

### *4.1.9 Further Vandermondian Identities*

The notation

$$\begin{aligned} N_m &= \{1\ 2 \cdots m\}, \\ J_m &= \{j_1\ j_2 \cdots j_m\}, \\ K_m &= \{k_1\ k_2 \cdots k_m\}, \end{aligned}$$

where $J_m$ and $K_m$ are permutations of $N_m$, is used to simplify the following lemmas.

**Lemma 4.5.**

$$V(x_1, x_2, \ldots, x_m) = \sum_{J_m}^{N_m} \operatorname{sgn} \begin{Bmatrix} N_m \\ J_m \end{Bmatrix} \sum_{r=1}^{m} x_{j_r}^{r-1}.$$

PROOF. The proof follows from the definition of a determinant in Section 1.2 with $a_{ij} \to x_i^{j-1}$. □

**Lemma 4.6.**

$$V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = \operatorname{sgn} \begin{Bmatrix} N_m \\ J_m \end{Bmatrix} V(x_1, x_2, \ldots, x_m).$$

This is Lemma (f) in Section 4.1.8 expressed in the present notation with $n \to m$.

**Lemma 4.7.**

$$\sum_{J_m}^{K_m} F\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = \begin{Bmatrix} N_m \\ J_m \end{Bmatrix} \sum_{J_m}^{N_m} F\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big).$$

In this lemma, the permutation symbol is used as a substitution operator. The number of terms on each side is $m^2$.

**Illustration.** Put $m = 2$, $F(x_{j_1}, x_{j_2}) = x_{j_1} + x_{j_2}^2$ and denote the left and right sides of the lemma by $P$ and $Q$ respectively. Then,

$$\begin{aligned} P &= x_{k_1} + x_{k_1}^2 + x_{k_2} + x_{k_2}^2 \\ Q &= \begin{Bmatrix} 1 & 2 \\ k_1 & k_2 \end{Bmatrix} (x_1 + x_1^2 + x_2 + x_2^2) \\ &= P. \end{aligned}$$

**Theorem.**

**a.** $$\sum_{J_m}^{N_m} \left( \prod_{r=1}^{m} x_{j_r}^{r-1} \right) V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = [V(x_1, x_2, \ldots, x_m)]^2,$$

**b.** $$\sum_{J_m}^{K_m} \left( \prod_{r=1}^{m} x_{j_r}^{r-1} \right) V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = \big[V\big(x_{k_1}, x_{k_2}, \ldots, x_{k_m}\big)\big]^2.$$

PROOF. Denote the left side of (a) by $S_m$. Then, applying Lemma 4.6,

$$\begin{aligned} S_m &= \sum_{J_m}^{N_m} \left( \prod_{r=1}^{m} x_{j_r}^{r-1} \right) \operatorname{sgn} \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} V(x_1, x_2, \ldots, x_m) \\ &= V(x_1, x_2, \ldots, x_m) \sum_{J_m}^{N_m} \operatorname{sgn} \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} \prod_{r=1}^{m} x_{j_r}^{r-1}. \end{aligned}$$

The proof of (a) follows from Lemma 4.5. The proof of (b) follows by applying the substitution operation $\left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\}$ to both sides of (a). □

This theorem is applied in Section 6.10.4 on the Einstein and Ernst equations.

## 4.2 Symmetric Determinants

If $A = |a_{ij}|_n$, where $a_{ji} = a_{ij}$, then $A$ is symmetric about its principal diagonal. By simple reasoning,

$$\begin{aligned} A_{ji} &= A_{ij}, \\ A_{js,ir} &= A_{ir,js}, \end{aligned}$$

etc. If $a_{n+1-j,n+1-i} = a_{ij}$, then $A$ is symmetric about its secondary diagonal. Only the first type of determinant is normally referred to as symmetric, but the second type can be transformed into the first type by rotation through 90° in either the clockwise or anticlockwise directions. This operation introduces the factor $(-1)^{n(n-1)/2}$, that is, there is a change of sign if $n = 4m+2$ and $4m+3$, $m = 0, 1, 2, \ldots$.

**Theorem.** *If $A$ is symmetric,*

$$\sum_{\mathrm{ep}\{p,q,r\}} A_{pq,rs} = 0,$$

*where the symbol* $\mathrm{ep}\{p,q,r\}$ *denotes that the sum is carried out over all even permutations of* $\{p,q,r\}$*, including the identity permutation.*

In this simple case the even permutations are also the cyclic permutations [Appendix A.2].

PROOF. Denote the sum by $S$. Then, applying the Jacobi identity (Section 3.6.1),

$$\begin{aligned} AS &= AA_{pq,rs} + AA_{qr,ps} + AA_{rp,qs} \\ &= \begin{vmatrix} A_{pr} & A_{ps} \\ A_{qr} & A_{qs} \end{vmatrix} + \begin{vmatrix} A_{qp} & A_{qs} \\ A_{rp} & A_{rs} \end{vmatrix} + \begin{vmatrix} A_{rq} & A_{rs} \\ A_{pq} & A_{ps} \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} A_{pr} & A_{ps} \\ A_{qr} & A_{qs} \end{vmatrix} + \begin{vmatrix} A_{pq} & A_{qs} \\ A_{pr} & A_{rs} \end{vmatrix} + \begin{vmatrix} A_{qr} & A_{rs} \\ A_{pq} & A_{ps} \end{vmatrix}$$
$$= 0.$$

The theorem follows immediately if $A \neq 0$. However, since the identity is purely algebraic, all the terms in the expansion of $S$ as sums of products of elements must cancel out in pairs. The identity must therefore be valid for all values of its elements, including those values for which $A = 0$. The theorem is clearly valid if the sum is carried out over even permutations of any three of the four parameters. □

Notes on skew-symmetric, circulant, centrosymmetric, skew-centrosymmetric, persymmetric (Hankel) determinants, and symmetric Toeplitz determinants are given under separate headings.

## 4.3 Skew-Symmetric Determinants

### *4.3.1 Introduction*

The determinant $A_n = |a_{ij}|_n$ in which $a_{ji} = -a_{ij}$, which implies $a_{ii} = 0$, is said to be skew-symmetric. In detail,

$$A_n = \begin{vmatrix} \bullet & a_{12} & a_{13} & a_{14} & \cdots \\ -a_{12} & \bullet & a_{23} & a_{24} & \cdots \\ -a_{13} & -a_{23} & \bullet & a_{34} & \cdots \\ -a_{14} & -a_{24} & -a_{34} & \bullet & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n . \tag{4.3.1}$$

**Theorem 4.8.** *The square of an arbitrary determinant of order $n$ can be expressed as a symmetric determinant of order $n$ if $n$ is odd or a skew-symmetric determinant of order $n$ if $n$ is even.*

PROOF. Let

$$A = |a_{ij}|_n.$$

Reversing the order of the rows,

$$A = (-1)^N |a_{n+1-i,j}|_n, \qquad N = \left[\frac{n}{2}\right]. \tag{4.3.2}$$

Transposing the elements of the original determinant across the secondary diagonal and changing the signs of the elements in the new rows $2, 4, 6, \ldots$,

$$A = (-1)^N |(-1)^{i+1} a_{n+1-j,n+1-i}|_n. \tag{4.3.3}$$

Hence, applying the formula for the product of two determinants in Section 1.4,

$$A^2 = |a_{n+1-i,j}|_n |(-1)^{i+1} a_{n+1-j,n+1-i}|_n$$

$$= |c_{ij}|_n,$$

where

$$\begin{aligned}
c_{ij} &= \sum_{r=1}^{n} (-1)^{r+1} a_{n+1-i,r} a_{n+1-j,n+1-r} \qquad (\text{put } r = n+1-s) \\
&= (-1)^{n+1} \sum_{s=1}^{n} (-1)^{s+1} a_{n+1-j,s} a_{n+1-i,n+1-s} \\
&= (-1)^{n+1} c_{ji}. \qquad (4.3.4)
\end{aligned}$$

The theorem follows. □

**Theorem 4.9.** *A skew-symmetric determinant of odd order is identically zero.*

Proof. Let $A^*_{2n-1}$ denote the determinant obtained from $A_{2n-1}$ by changing the sign of every element. Then, since the number of rows and columns is odd,

$$A^*_{2n-1} = -A_{2n-1}.$$

But,

$$A^*_{2n-1} = A^T_{2n-1} = A_{2n-1}.$$

Hence,

$$A_{2n-1} = 0,$$

which proves the theorem. □

The cofactor $A^{(2n)}_{ii}$ is also skew-symmetric of odd order. Hence,

$$A^{(2n)}_{ii} = 0. \qquad (4.3.5)$$

By similar arguments,

$$\begin{aligned}
A^{(2n)}_{ji} &= -A^{(2n)}_{ij}, \\
A^{(2n-1)}_{ji} &= A^{(2n-1)}_{ij}. \qquad (4.3.6)
\end{aligned}$$

It may be verified by elementary methods that

$$\begin{aligned}
A_2 &= a_{12}^2, \qquad (4.3.7) \\
A_4 &= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2. \qquad (4.3.8)
\end{aligned}$$

**Theorem 4.10.** *$A_{2n}$ is the square of a polynomial function of its elements.*

Proof. Use the method of induction. Applying the Jacobi identity (Section 3.6.1) to the zero determinant $A_{2n-1}$,

$$\begin{vmatrix} A^{(2n-1)}_{ii} & A^{(2n-1)}_{ij} \\ A^{(2n-1)}_{ji} & A^{(2n-1)}_{jj} \end{vmatrix} = 0,$$

$$\left[A_{ij}^{(2n-1)}\right]^2 = A_{ii}^{(2n-1)} A_{jj}^{(2n-1)}. \tag{4.3.9}$$

It follows from the section on bordered determinants (Section 3.7.1) that

$$\left| \begin{array}{cc} & x_1 \\ A_{2n-1} & \vdots \\ & x_{2n-1} \\ \cdots\cdots\cdots & \\ y_1 \cdots y_{2n-1} & \bullet \end{array} \right|_{2n} = -\sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} A_{ij}^{(2n-1)} x_i y_j. \tag{4.3.10}$$

Put $x_i = a_{i,2n}$ and $y_j = -a_{j,2n}$. Then, the identity becomes

$$\begin{aligned} A_{2n} &= \sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} A_{ij}^{(2n-1)} a_{i,2n} a_{j,2n} &(4.3.11)\\ &= \sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} \left[A_{ii}^{(2n-1)} A_{jj}^{(2n-1)}\right]^{1/2} a_{i,2n} a_{j,2n} \\ &= \left[\sum_{i=1}^{2n-1} \left[A_{ii}^{(2n-1)}\right]^{1/2} a_{i,2n}\right] \left[\sum_{j=1}^{2n-1} \left[A_{jj}^{(2n-1)}\right]^{1/2} a_{j,2n}\right] \\ &= \left[\sum_{i=1}^{2n-1} \left[A_{ii}^{(2n-1)}\right]^{1/2} a_{i,2n}\right]^2. &(4.3.12) \end{aligned}$$

However, each $A_{ii}^{(2n-1)}$, $1 \le i \le (2n-1)$, is a skew-symmetric determinant of even order $(2n-2)$. Hence, if each of these determinants is the square of a polynomial function of its elements, then $A_{2n}$ is also the square of a polynomial function of its elements. But, from (4.3.7), it is known that $A_2$ is the square of a polynomial function of its elements. The theorem follows by induction. □

This proves the theorem, but it is clear that the above analysis does not yield a unique formula for the polynomial since not only is each square root in the series in (4.3.12) ambiguous in sign but each square root in the series for each $A_{ii}^{(2n-1)}$, $1 \le i \le (2n-1)$, is ambiguous in sign.

A unique polynomial for $A_{2n}^{1/2}$, known as a Pfaffian, is defined in a later section. The present section ends with a few theorems and the next section is devoted to the solution of a number of preparatory lemmas.

**Theorem 4.11.** *If*

$$a_{ji} = -a_{ij},$$

*then*

**a.** $|a_{ij} + x|_{2n} = |a_{ij}|_{2n}$,

**b.** $|a_{ij} + x|_{2n-1} = x \times \left( \begin{array}{l} \textit{the square of a polyomial function} \\ \textit{of the elements } a_{ij} \end{array} \right)$

PROOF. Let $A_n = |a_{ij}|_n$ and let $E_{n+1}$ and $F_{n+1}$ denote determinants obtained by bordering $A_n$ in different ways:

$$E_{n+1} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ -x & \bullet & a_{12} & a_{13} & \cdots \\ -x & -a_{12} & \bullet & a_{23} & \cdots \\ -x & -a_{13} & -a_{23} & \bullet & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n+1}$$

and $F_{n+1}$ is obtained by replacing the first column of $E_{n+1}$ by the column

$$\begin{bmatrix} 0 & -1 & -1 & -1 & \cdots \end{bmatrix}_{n+1}^T.$$

Both $A_n$ and $F_{n+1}$ are skew-symmetric. Then,

$$E_{n+1} = A_n + xF_{n+1}.$$

Return to $E_{n+1}$ and perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_1, \quad 2 \le j \le n+1,$$

which reduces every element to zero except the first in the first row and increases every other element in columns 2 to $(n+1)$ by $x$. The result is

$$E_{n+1} = |a_{ij} + x|_n.$$

Hence, applying Theorems 4.9 and 4.10,

$$\begin{aligned} |a_{ij} + x|_{2n} &= A_{2n} + xF_{2n+1} \\ &= A_{2n}, \\ |a_{ij} + x|_{2n-1} &= A_{2n-1} + xF_{2n} \\ &= xF_{2n}. \end{aligned}$$

The theorem follows. □

**Corollary.** *The determinant*

$$A = |a_{ij}|_{2n}, \quad \textit{where} \quad a_{ij} + a_{ji} = 2x,$$

*can be expressed as a skew-symmetric determinant of the same order.*

PROOF. The proof begins by expressing $A$ in the form

$$A = \begin{vmatrix} x & a_{12} & a_{13} & a_{14} & \cdots \\ 2x - a_{12} & x & a_{23} & a_{24} & \cdots \\ 2x - a_{13} & 2x - a_{23} & x & a_{34} & \cdots \\ 2x - a_{14} & 2x - a_{24} & 2x - a_{34} & x & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{2n}$$

and is completed by subtracting $x$ from each element. □

Let

$$A_n = |a_{ij}|_n, \quad a_{ji} = -a_{ij},$$

and let $B_{n+1}$ denote the skew-symmetric determinant obtained by bordering $A_n$ by the row

$$\begin{bmatrix}-1 & -1 & -1 \cdots -1 & 0\end{bmatrix}_{n+1}$$

below and by the column

$$\begin{bmatrix}1 & 1 & 1 \cdots 1 & 0\end{bmatrix}_{n+1}^T$$

on the right.

**Theorem 4.12** (Muir and Metzler). *$B_{n+1}$ is expressible as a skew-symmetric determinant of order $(n-1)$.*

PROOF. The row and column operations

$$\begin{aligned}\mathbf{R}'_i &= \mathbf{R}_i + a_{in}\mathbf{R}_{n+1}, \quad 1 \le i \le n-1,\\ \mathbf{C}'_j &= \mathbf{C}_j + a_{jn}\mathbf{C}_{n+1}, \quad 1 \le j \le n-1,\end{aligned}$$

when performed on $B_{n+1}$, result in the elements $a_{ij}$ and $a_{ji}$ being transformed into $a^*_{ij}$ and $a^*_{ji}$, where

$$\begin{aligned}a^*_{ij} &= a_{ij} - a_{in} + a_{jn}, \quad 1 \le i \le n-1,\\ a^*_{ji} &= a_{ji} - a_{jn} + a_{in}, \quad 1 \le j \le n-1,\\ &= -a^*_{ij}.\end{aligned}$$

In particular, $a^*_{in} = 0$, so that all the elements except the last in both column $n$ and row $n$ are reduced to zero. Hence, when a Laplace expansion from the last two rows or columns is performed, only one term survives and the formula

$$B_{n+1} = |a^*_{ij}|_{n-1}$$

emerges, which proves the theorem. When $n$ is even, both sides of this formula are identically zero. □

### *4.3.2 Preparatory Lemmas*

Let

$$B_n = |b_{ij}|_n$$

where

$$b_{ij} = \begin{cases} 1, & i < j-1 \\ 0, & i = j-1 \\ -1, & i > j-1. \end{cases}$$

In detail,

$$B_n = \begin{vmatrix} -1 & \bullet & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & \bullet & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \bullet & \cdots & 1 & 1 \\ \multicolumn{7}{c}{\dotfill} \\ -1 & -1 & -1 & -1 & \cdots & -1 & \bullet \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}_n .$$

**Lemma 4.13.**

$$B_n = (-1)^n.$$

PROOF. Perform the column operation

$$\mathbf{C}_2' = \mathbf{C}_2 - \mathbf{C}_1$$

and then expand the resulting determinant by elements from the new $\mathbf{C}_2$. The result is

$$B_n = -B_{n-1} = B_{n-2} = \cdots = (-1)^{n-1} B_1.$$

But $B_1 = -1$. The result follows. □

**Lemma 4.14.**

**a.** $\displaystyle\sum_{k=1}^{2n} (-1)^{j+k+1} = 0,$

**b.** $\displaystyle\sum_{k=1}^{i-1} (-1)^{j+k+1} = (-1)^j \delta_{i,\text{even}},$

**c.** $\displaystyle\sum_{k=i}^{2n} (-1)^{j+k+1} = (-1)^{j+1} \delta_{i,\text{even}},$

*where the $\delta$ functions are defined in Appendix A.1. All three identities follow from the elementary identity*

$$\sum_{k=p}^{q} (-1)^k = (-1)^p \delta_{q-p,\text{even}}. \qquad \square$$

*Define the function $E_{ij}$ as follows:*

$$E_{ij} = \begin{cases} (-1)^{i+j+1}, & i < j \\ 0, & i = j \\ -(-1)^{i+j+1}, & i > j. \end{cases}$$

**Lemma 4.15.**

**a.** $\displaystyle\sum_{k=1}^{2n} E_{jk} = (-1)^{j+1},$

**b.** $$\sum_{k=i}^{2n} E_{jk} = (-1)^{j+1}\delta_{i,\text{odd}}, \quad i \le j$$
$$= (-1)^{j+1}\delta_{i,\text{even}}, \quad i > j$$

**c.** $$\sum_{k=1}^{i-1} E_{jk} = (-1)^{j+1}\delta_{i,\text{even}}, \quad i \le j$$
$$= (-1)^{j+1}\delta_{i,\text{odd}}, \quad i > j.$$

PROOF. Referring to Lemma 4.14(b,c),

$$\begin{aligned}\sum_{k=1}^{2n} E_{jk} &= \sum_{k=1}^{j-1} E_{jk} + E_{jj} + \sum_{k=j+1}^{2n} E_{jk}\\ &= -\sum_{k=1}^{j-1}(-1)^{j+k+1} + 0 + \sum_{k=j+1}^{2n}(-1)^{j+k+1}\\ &= (-1)^{j+1}(\delta_{j,\text{even}} + \delta_{j,\text{odd}})\\ &= (-1)^{j+1},\end{aligned}$$

which proves (a).

If $i \le j$,

$$\begin{aligned}\sum_{k=i}^{2n} E_{jk} &= \left[\sum_{k=1}^{2n} - \sum_{k=1}^{i-1}\right] E_{jk}\\ &= (-1)^{j+1} + \sum_{k=1}^{i-1}(-1)^{j+k+1}\\ &= (-1)^{j+1}(1 - \delta_{j,\text{even}})\\ &= (-1)^{j+1}\delta_{i,\text{odd}}.\end{aligned}$$

If $i > j$,

$$\begin{aligned}\sum_{k=i}^{2n} E_{jk} &= \sum_{k=i}^{2n}(-1)^{j+k+1}\\ &= (-1)^{j+1}\delta_{i,\text{even}},\end{aligned}$$

which proves (b).

$$\sum_{k=1}^{i-1} E_{jk} = \left[\sum_{k=1}^{2n} - \sum_{k=i}^{2n}\right] E_{jk}.$$

Part (c) now follows from (a) and (b). □

Let $E_n$ be a skew-symmetric determinant defined as follows:

$$E_n = |\varepsilon_{ij}|_n,$$

where $\varepsilon_{ij} = 1$, $i < j$, and $\varepsilon_{ji} = -\varepsilon_{ij}$, which implies $\varepsilon_{ii} = 0$.

**Lemma 4.16.**

$$E_n = \delta_{n,\text{even}}.$$

PROOF. Perform the column operation

$$\mathbf{C}'_n = \mathbf{C}_n + \mathbf{C}_1,$$

expand the result by elements from the new $\mathbf{C}_n$, and apply Lemma 4.13

$$\begin{aligned} E_n &= (-1)^{n-1} B_{n-1} - E_{n-1} \\ &= 1 - E_{n-1} \\ &= 1 - (1 - E_{n-2}) \\ &= E_{n-2} = E_{n-4} = E_{n-6}, \text{ etc.} \end{aligned}$$

Hence, if $n$ is even,

$$E_n = E_2 = 1$$

and if $n$ is odd,

$$E_n = E_1 = 0,$$

which proves the result. □

**Lemma 4.17.** *The function $E_{ij}$ defined in Lemma 4.15 is the cofactor of the element $\varepsilon_{ij}$ in $E_{2n}$.*

PROOF. Let

$$\lambda_{ij} = \sum_{k=1}^{2n} \varepsilon_{ik} E_{jk}.$$

It is required to prove that $\lambda_{ij} = \delta_{ij}$.

$$\begin{aligned} \lambda_{ij} &= \sum_{k=1}^{i-1} \varepsilon_{ik} E_{jk} + 0 + \sum_{k=i+1}^{2n} \varepsilon_{ik} E_{jk} \\ &= -\sum_{k=1}^{i-1} E_{jk} + \sum_{k=i+1}^{2n} E_{jk} \\ &= \left[ \sum_{k=i}^{2n} - \sum_{k=1}^{i-1} \right] E_{jk} - E_{ji}. \end{aligned}$$

If $i < j$,

$$\begin{aligned} \lambda_{ij} &= (-1)^{j+1} \left[ \delta_{i,\text{odd}} - \delta_{i,\text{even}} + (-1)^i \right] \\ &= 0. \end{aligned}$$

If $i > j$,

$$\lambda_{ij} = (-1)^{j+1} \left[ \delta_{i,\text{even}} - \delta_{i,\text{odd}} - (-1)^i \right]$$

$$
\begin{aligned}
&= 0 \\
\lambda_{ii} &= (-1)^{i+1}\left[\delta_{i,\text{odd}} - \delta_{i,\text{even}}\right] \\
&= 1.
\end{aligned}
$$

This completes the proofs of the preparatory lemmas. The definition of a Pfaffian follows. The above lemmas will be applied to prove the theorem which relates it to a skew-symmetric determinant. □

### *4.3.3 Pfaffians*

The $n$th-order Pfaffian $\text{Pf}_n$ is defined by the following formula, which is similar in nature to the formula which defines the determinant $A_n$ in Section 1.2:

$$
\text{Pf}_n = \sum \text{sgn} \left\{ \begin{matrix} 1 & 2 & 3 & 4 & \cdots & (2n-1) & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n \end{matrix} \right\}_{2n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}, \tag{4.3.13}
$$

where the sum extends over all possible distinct terms subject to the restriction

$$
1 \le i_s < j_s \le n, \quad 1 \le s \le n.. \tag{4.3.14}
$$

Notes on the permutations associated with Pfaffians are given in Appendix A.2. The number of terms in the sum is

$$
\prod_{s=1}^{n} (2s-1) = \frac{(2n)!}{2^n n!}. \tag{4.3.15}
$$

### *Illustrations*

$$
\begin{aligned}
\text{Pf}_1 &= \sum \text{sgn} \left\{ \begin{matrix} 1 & 2 \\ i & j \end{matrix} \right\} a_{ij} \quad \text{(1 term)} \\
&= a_{12}, \\
A_2 &= [\text{Pf}_1]^2 \\
\text{Pf}_2 &= \sum \text{sgn} \left\{ \begin{matrix} 1 & 2 & 3 & 4 \\ i_1 & j_1 & i_2 & j_2 \end{matrix} \right\} a_{i_1 j_1} a_{i_2 j_2} \quad \text{(3 terms)}.
\end{aligned} \tag{4.3.16}
$$

Omitting the upper parameters,

$$
\begin{aligned}
\text{Pf}_2 &= \text{sgn}\{1\ 2\ 3\ 4\} a_{12} a_{34} + \text{sgn}\{1\ 3\ 2\ 4\} a_{13} a_{24} + \text{sgn}\{1\ 4\ 2\ 3\} a_{14} a_{23} \\
&= a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} \\
A_4 &= [\text{Pf}_2]^2.
\end{aligned} \tag{4.3.17}
$$

These results agree with (4.3.7) and (4.3.8).

The coefficient of $a_{r,2n}$, $1 \le r \le (2n-1)$, in $\mathrm{Pf}_n$ is found by putting $(i_s, j_s) = (r, 2n)$ for any value of $s$. Choose $s = 1$. Then, the coefficient is

$$\sum \sigma_r a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n},$$

where

$$\begin{aligned}
\sigma_r &= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) & 2n \\ r & 2n & i_2 & j_2 & \ldots & i_n & j_n \end{Bmatrix}_{2n} \\
&= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) & 2n \\ r & i_2 & j_2 & i_3 & \ldots & j_n & 2n \end{Bmatrix}_{2n} \\
&= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) \\ r & i_2 & j_2 & i_3 & \ldots & j_n \end{Bmatrix}_{2n-1} \qquad (4.3.18) \\
&= (-1)^{r+1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (r-1)r(r+1) & \ldots & (2n-1) \\ i_2 & j_2 & i_3 & j_3 & \ldots & r & \ldots & j_n \end{Bmatrix}_{2n-1}, \quad r > 1 \\
&= (-1)^{r+1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (r-1)(r+1) & \ldots & (2n-1) \\ i_2 & j_2 & i_3 & j_3 & \ldots & \ldots & \ldots & j_n \end{Bmatrix}_{2n-2}, \quad r > 1.
\end{aligned}$$

From (4.3.18),

$$\begin{aligned}
\sigma_1 &= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) \\ 1 & i_2 & j_2 & i_3 & \ldots & j_n \end{Bmatrix}_{2n-1} \\
&= \operatorname{sgn} \begin{Bmatrix} 2 & 3 & 4 & \ldots & (2n-1) \\ i_2 & j_2 & i_3 & \ldots & j_n \end{Bmatrix}_{2n-2}.
\end{aligned}$$

Hence,

$$\mathrm{Pf}_n = \sum_{r=1}^{2n-1} (-1)^{r+1} a_{r,2n} Pf_r^{(n)}, \qquad (4.3.19)$$

where

$$\begin{aligned}
\mathrm{Pf}_r^{(n)} = \sum \operatorname{sgn} &\begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & (r-1)(r+1) & \cdots & (2n-2) & (2n-1) \\ i_2 & j_2 & i_3 & j_3 & \cdots & \cdots & \cdots & i_n & j_n \end{Bmatrix}_{2n-2} \\
&\times a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n}, \qquad 1 < r \le 2n-1, \qquad (4.3.20)
\end{aligned}$$

which is a Pfaffian of order $(n-1)$ in which no element contains the row parameter $r$ or the column parameter $2n$. In particular,

$$\begin{aligned}
\mathrm{Pf}_{2n-1}^{(n)} &= \sum \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & (2n-3) & (2n-2) \\ i_2 & j_2 & i_3 & j_3 & \cdots & i_n & j_n \end{Bmatrix}_{2n-2} \\
&= \mathrm{Pf}_{n-1}. \qquad (4.3.21)
\end{aligned}$$

Thus, a Pfaffian of order $n$ can be expressed as a linear combination of $(2n-1)$ Pfaffians of order $(n-1)$.

In the particular case in which $a_{ij} = 1$, $i < j$, denote $\mathrm{Pf}_n$ by $\mathrm{pf}_n$ and denote $\mathrm{Pf}_r^{(n)}$ by $\mathrm{pf}_r^{(n)}$.

**Lemma.**

$$\mathrm{pf}_n = 1.$$

The proof is by induction. Assume $\mathrm{pf}_m = 1$, $m < n$, which implies $\mathrm{pf}_r^{(n)} = 1$. Then, from (4.3.19),

$$\mathrm{pf}_n = \sum_{r=1}^{2n-1} (-1)^{r+1} = 1.$$

Thus, if every Pfaffian of order $m < n$ is equal to 1, then every Pfaffian of order $n$ is also equal to 1. But from (4.3.16), $\mathrm{pf}_1 = 1$, hence $\mathrm{pf}_2 = 1$, which is confirmed by (4.3.17), $\mathrm{pf}_3 = 1$, and so on.

The following important theorem relates Pfaffians to skew-symmetric determinants.

**Theorem.**

$$A_{2n} = [\mathrm{Pf}_n]^2.$$

The proof is again by induction. Assume

$$A_{2m} = [\mathrm{Pf}_m]^2, \quad m < n,$$

which implies

$$A_{ii}^{(2n-1)} = \left[\mathrm{Pf}_i^{(n)}\right]^2.$$

Hence, referring to (4.3.9),

$$\begin{aligned} \left[A_{ij}^{(2n-1)}\right]^2 &= A_{ii}^{(2n-1)} A_{jj}^{(2n-1)} \\ &= \left[\mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)}\right]^2 \end{aligned}$$

$$\frac{A_{ij}^{(2n-1)}}{\mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)}} = \pm 1 \tag{4.3.22}$$

for all elements $a_{ij}$ for which $a_{ji} = -a_{ij}$. Let $a_{ij} = 1$, $i < j$. Then

$$\begin{aligned} A_{ij}^{(2n-1)} &\to E_{ij}^{(2n-1)} = (-1)^{i+j}, \\ \mathrm{Pf}_i^{(n)} &\to \mathrm{pf}_i^{(n)} = 1. \end{aligned}$$

Hence,

$$\frac{A_{ij}^{(2n-1)}}{\mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)}} = \frac{E_{ij}^{(2n-1)}}{\mathrm{pf}_i^{(n)} \mathrm{pf}_j^{(n)}} = (-1)^{i+j}, \tag{4.3.23}$$

which is consistent with (4.3.22). Hence,

$$A_{ij}^{(2n-1)} = (-1)^{i+j}\mathrm{Pf}_i^{(n)}\mathrm{Pf}_j^{(n)}. \tag{4.3.24}$$

Returning to (4.3.11) and referring to (4.3.19),

$$\begin{aligned} A_{2n} &= \left[\sum_{i=1}^{2n-1}(-1)^{i+1}\mathrm{Pf}_i^{(n)}a_{i,2n}\right]\left[\sum_{j=1}^{2n-1}(-1)^{j+1}\mathrm{Pf}_j^{(n)}a_{j,2n}\right] \\ &= \left[\sum_{i=1}^{2n-1}(-1)^{i+1}\mathrm{Pf}_i^{(n)}a_{i,2n}\right]^2 \\ &= [\mathrm{Pf}_n]^2, \end{aligned}$$

which completes the proof of the theorem.

The notation for Pfaffians consists of a triangular array of the elements $a_{ij}$ for which $i < j$:

$$\mathrm{Pf}_n = \left| \begin{matrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,2n} \\ & a_{23} & a_{24} & \cdots & a_{2,2n} \\ & & a_{34} & \cdots & a_{3,2n} \\ & & & \cdots\cdots\cdots & \\ & & & a_{2n-1,2n} & \end{matrix} \right|_{2n-1}. \tag{4.3.25}$$

$\mathrm{Pf}_n$ is a polynomial function of the $n(2n-1)$ elements in the array.

### *Illustrations*

From (4.3.16), (4.3.17), and (4.3.25),

$$\begin{aligned} \mathrm{Pf}_1 &= |a_{12}| = a_{12}, \\ \mathrm{Pf}_2 &= \left| \begin{matrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{matrix} \right| \\ &= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}. \end{aligned}$$

It is left as an exercise for the reader to evaluate $\mathrm{Pf}_3$ directly from the definition (4.3.13) with the aid of the notes given in the section on permutations associated with Pfaffians in Appendix A.2 and to show that

$$\begin{aligned} \mathrm{Pf}_3 &= \left| \begin{matrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{34} & a_{35} & a_{36} \\ & & & a_{45} & a_{46} \\ & & & & a_{56} \end{matrix} \right| \\ &= a_{16}\left| \begin{matrix} a_{23} & a_{24} & a_{25} \\ & a_{34} & a_{35} \\ & & a_{45} \end{matrix} \right| - a_{26}\left| \begin{matrix} a_{13} & a_{14} & a_{15} \\ & a_{34} & a_{35} \\ & & a_{45} \end{matrix} \right| + a_{36}\left| \begin{matrix} a_{12} & a_{14} & a_{15} \\ & a_{24} & a_{25} \\ & & a_{45} \end{matrix} \right| \end{aligned}$$

$$
- \; a_{46}\begin{vmatrix} a_{12} & a_{13} & a_{15} \\ & a_{23} & a_{25} \\ & & a_{35} \end{vmatrix} + a_{56}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{vmatrix}
$$

$$
= \sum_{r=1}^{5} (-1)^{r+1} a_{r6} \mathrm{Pf}_r^{(3)}, \tag{4.3.26}
$$

which illustrates (4.3.19). This formula can be regarded as an expansion of $\mathrm{Pf}_3$ by the five elements from the fifth column and their associated second-order Pfaffians. Note that the second of these five Pfaffians, which is multiplied by $a_{26}$, is *not* obtained from $\mathrm{Pf}_3$ by deleting a particular row and a particular column. It is obtained from $\mathrm{Pf}_3$ by deleting *all* elements whose suffixes include either 2 or 6 whether they be row parameters or column parameters. The other four second-order Pfaffians are obtained in a similar manner.

It follows from the definition of $\mathrm{Pf}_n$ that one of the terms in its expansion is

$$
+ \, a_{12} a_{34} a_{56} \cdots a_{2n-1,2n} \tag{4.3.27}
$$

in which the parameters are in ascending order of magnitude. This term is known as the principal term. Hence, there is no ambiguity in signs in the relations

$$
\begin{aligned}
\mathrm{Pf}_n &= A_{2n}^{1/2} \\
\mathrm{Pf}_i^{(n)} &= \left[A_{ii}^{(2n-1)}\right]^{1/2}.
\end{aligned} \tag{4.3.28}
$$

Skew-symmetric determinants and Pfaffians appear in Section 5.2 on the generalized Cusick identities.

## *Exercises*

1. **Theorem (Muir and Metzler)** *An arbitrary determinant $A_n = |a_{ij}|_n$ can be expressed as a Pfaffian of the same order.*

   Prove this theorem in the particular case in which $n = 3$ as follows: Let

$$
\begin{aligned}
b_{ij} &= \tfrac{1}{2}(a_{ij} + a_{ji}) = b_{ji}, \\
c_{ij} &= \tfrac{1}{2}(a_{ij} - a_{ji}) = -c_{ji}.
\end{aligned}
$$

   Then

$$
\begin{aligned}
b_{ii} &= a_{ii}, \\
c_{ii} &= 0, \\
a_{ij} - b_{ij} &= c_{ij}, \\
a_{ij} + c_{ji} &= b_{ij}.
\end{aligned}
$$

Applying the Laplace expansion formula (Section 3.3) in reverse,

$$A_3^2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & & & \\ a_{21} & a_{22} & a_{23} & & & \\ a_{31} & a_{32} & a_{33} & & & \\ -b_{31} & -b_{32} & -b_{33} & a_{33} & a_{23} & a_{13} \\ -b_{21} & -b_{22} & -b_{23} & a_{32} & a_{22} & a_{12} \\ -b_{11} & -b_{12} & -b_{13} & a_{31} & a_{21} & a_{11} \end{vmatrix}.$$

Now, perform the column and row operations

$$\begin{aligned} \mathbf{C}'_j &= \mathbf{C}_j + \mathbf{C}_{7-j}, \quad 4 \le j \le 6, \\ \mathbf{R}'_i &= \mathbf{R}_i + \mathbf{R}_{7-i}, \quad 1 \le i \le 3, \end{aligned}$$

and show that the resulting determinant is skew-symmetric. Hence, show that

$$A_3 = \left| \begin{matrix} c_{12} & c_{13} & b_{13} & b_{12} & b_{11} \\ & c_{23} & b_{23} & b_{22} & b_{21} \\ & & b_{33} & b_{32} & b_{31} \\ & & & c_{23} & c_{13} \\ & & & & c_{12} \end{matrix} \right|.$$

2. **Theorem (Muir and Metzler)** *An arbitrary determinant of order $2n$ can be expressed as a Pfaffian of order $n$.*
Prove this theorem in the particular case in which $n = 2$ as follows: Denote the determinant by $A_4$, transpose it and interchange first rows 1 and 2 and then rows 3 and 4. Change the signs of the elements in the (new) rows 2 and 4. These operations leave the value of the determinant unaltered. Multiply the initial and final determinants together, prove that the product is skew-symmetric, and, hence, prove that

$$A_4 = \left| \begin{matrix} (N_{12,12} + N_{12,34}) & (N_{13,12} + N_{13,34}) & (N_{14,12} + N_{14,34}) \\ & (N_{23,12} + N_{23,34}) & (N_{24,12} + N_{24,34}) \\ & & (N_{34,12} + N_{34,34}) \end{matrix} \right|.$$

where $N_{ij,rs}$ is a retainer minor (Section 3.2.1).

3. Expand $\mathrm{Pf}_3$ by the five elements from the first row and their associated second-order Pfaffians.
4. A skew-symmetric determinant $A_{2n}$ is defined as follows:

$$A_{2n} = |a_{ij}|_{2n},$$

where

$$a_{ij} = \frac{x_i - x_j}{x_i + x_j}.$$

Prove that the corresponding Pfaffian is given by the formula

$$\mathrm{Pf}_{2n-1} = \prod_{1 \le i < j \le 2n} a_{ij},$$

that is, the Pfaffian is equal to the product of its elements.

## 4.4 Circulants

### 4.4.1 Definition and Notation

A circulant $A_n$ is denoted by the symbol $A(a_1, a_2, a_3, \dots, a_n)$ and is defined as follows:

$$\begin{aligned} A_n &= A(a_1, a_2, a_3, \dots, a_n) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{vmatrix}_n . \end{aligned} \tag{4.4.1}$$

Each row is obtained from the previous row by displacing each element, except the last, one position to the right, the last element being displaced to the first position. The name circulant is derived from the circular nature of the displacements.

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} a_{j+1-i}, & j \geq i, \\ a_{n+j+1-i}, & j < i. \end{cases} \tag{4.4.2}$$

### 4.4.2 Factors

After performing the column operation

$$\mathbf{C}'_1 = \sum_{j=1}^{n} \mathbf{C}_j, \tag{4.4.3}$$

it is easily seen that $A_n$ has the factor $\sum_{r=1}^{n} a_r$ but $A_n$ has other factors. When all the $a_r$ are real, the first factor is real but some of the other factors are complex.

Let $\omega_r$ denote the complex number defined as follows and let $\bar{\omega}_r$ denote its conjugate:

$$\begin{aligned} \omega_r &= \exp(2ri\pi/n) \quad 0 \leq r \leq n-1, \\ &= \omega_1^r, \\ \omega_r^n &= 1, \\ \omega_r \bar{\omega}_r &= 1, \\ \omega_0 &= 1. \end{aligned} \tag{4.4.4}$$

$\omega_r$ is also a function of $n$, but the $n$ is suppressed to simplify the notation. The $n$ numbers

$$1, \omega_r, \omega_r^2, \ldots, \omega_r^{n-1} \tag{4.4.5}$$

are the $n$th roots of unity for any value of $r$. Two different choices of $r$ give rise to the same set of roots but in a different order. It follows from the third line in (4.4.4) that

$$\sum_{s=0}^{n-1} \omega_r^s = 0, \quad 0 \le r \le n-1. \tag{4.4.6}$$

**Theorem.**

$$A_n = \prod_{r=0}^{n-1} \sum_{s=1}^{n} \omega_r^{s-1} a_s.$$

Proof. Let

$$\begin{aligned} z_r &= \sum_{s=1}^{n} \omega_r^{s-1} a_s \\ &= a_1 + \omega_r a_2 + \omega_r^2 a_3 + \cdots + \omega_r^{n-1} a_n, \quad \omega_r^n = 1. \end{aligned} \tag{4.4.7}$$

Then,

$$\left.\begin{aligned} \omega_r z_r &= a_n + \omega_r a_1 + \omega_r^2 a_2 + \cdots + \omega_r^{n-1} a_{n-1} \\ \omega_r^2 z_r &= a_{n-1} + \omega_r a_n + \omega_r^2 a_1 + \cdots + \omega_r^{n-1} a_{n-2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \omega_r^{n-1} z_r &= a_2 + \omega_r a_3 + \omega_r^2 a_4 + \cdots + \omega_r^{n-1} a_1 \end{aligned}\right\}. \tag{4.4.8}$$

Express $A_n$ in column vector notation and perform a column operation:

$$\begin{aligned} A_n &= \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right| \\ &= \left|\mathbf{C}_1'\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|, \end{aligned}$$

where

$$\begin{aligned} \mathbf{C}_1' &= \sum_{j=1}^{n} \omega_r^{j-1} \mathbf{C}_j \\ &= \begin{bmatrix} a_1 \\ a_n \\ a_{n-1} \\ \vdots \\ a_2 \end{bmatrix} + \omega_r \begin{bmatrix} a_2 \\ a_1 \\ a_n \\ \vdots \\ a_3 \end{bmatrix} + \omega_r^2 \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ \vdots \\ a_4 \end{bmatrix} + \cdots + \omega_r^{n-1} \begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \end{bmatrix} \\ &= z_r \mathbf{W}_r, \end{aligned}$$

where

$$\mathbf{W}_r = \left[1\ \omega_r\ \omega_r^2 \cdots \omega_r^{n-1}\right]^T. \tag{4.4.9}$$

Hence,

$$A = z_r \big| \mathbf{W}_r \ \mathbf{C}_2 \ \mathbf{C}_3 \cdots \mathbf{C}_n \big|. \tag{4.4.10}$$

It follows that each $z_r$, $0 \le r \le n-1$, is a factor of $A_n$. Hence,

$$A_n = K \prod_{r=0}^{n-1} z_r, \tag{4.4.11}$$

but since $A_n$ and the product are homogeneous polynomials of degree $n$ in the $a_r$, the factor $K$ must be purely numerical. It is clear by comparing the coefficients of $a_1^n$ on each side that $K = 1$. The theorem follows from (4.4.7). □

**Illustration.** When $n = 3$, $\omega_r = \exp(2ri\pi/3)$, $\omega_r^3 = 1$.

$$\begin{aligned} \omega_0 &= 1, \\ \omega = \omega_1 &= \exp(2i\pi/3), \\ \omega_2 &= \exp(4i\pi/3) = \omega_1^2 = \omega^2 = \bar{\omega}, \\ \omega_2^2 &= \omega_1 = \omega. \end{aligned}$$

Hence,

$$\begin{aligned} A_3 &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} \\ &= (a_1 + a_2 + a_3)(a_1 + \omega_1 a_2 + \omega_1^2 a_3)(a_1 + \omega_2 a_2 + \omega_2^2 a_3) \\ &= (a_1 + a_2 + a_3)(a_1 + \omega a_2 + \omega^2 a_3)(a_1 + \omega^2 a_2 + \omega a_3). \end{aligned} \tag{4.4.12}$$

**Exercise.** Factorize $A_4$.

### *4.4.3 The Generalized Hyperbolic Functions*

Define a matrix $\mathbf{W}$ as follows:

$$\begin{aligned} \mathbf{W} &= \left[\omega^{(r-1)(s-1)}\right]_n \quad (\omega = \omega_1) \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2n-2} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{n-1} & \omega^{2n-2} & \omega^{3n-3} & \cdots & \omega^{(n-1)^2} \end{bmatrix}_n . \end{aligned} \tag{4.4.13}$$

**Lemma 4.18.**

$$\mathbf{W}^{-1} = \frac{1}{n}\overline{\mathbf{W}}.$$

PROOF.

$$\overline{\mathbf{W}} = \left[\omega^{-(r-1)(s-1)}\right]_n.$$

Hence,

$$\mathbf{W}\overline{\mathbf{W}} = [\alpha_{rs}]_n,$$

where

$$\begin{aligned}\alpha_{rs} &= \sum_{t=1}^{n} \omega^{(r-1)(t-1)-(t-1)(s-1)}\\ &= \sum_{t=1}^{n} \omega^{(t-1)(r-s)},\\ \alpha_{rr} &= n.\end{aligned} \tag{4.4.14}$$

Put $k = r - s$, $s \neq r$. Then, referring to (4.4.6),

$$\begin{aligned}\alpha_{rs} &= \sum_{t=1}^{n} \omega^{(t-1)k} \quad (\omega^k = \omega_1^k = \omega_k)\\ &= \sum_{t=1}^{n} \omega_k^{t-1}\\ &= 0, \qquad s \neq r.\end{aligned} \tag{4.4.15}$$

Hence,

$$\begin{aligned}[\alpha_{rs}] &= n\mathbf{I},\\ \mathbf{W}\overline{\mathbf{W}} &= n\mathbf{I}.\end{aligned}$$

The lemma follows. □

The $n$ generalized hyperbolic functions $H_r$, $1 \leq r \leq n$, of the $(n-1)$ independent variables $x_r$, $1 \leq r \leq n-1$, are defined by the matrix equation

$$\mathbf{H} = \frac{1}{n}\mathbf{W}\mathbf{E}, \tag{4.4.16}$$

where $\mathbf{H}$ and $\mathbf{E}$ are column vectors defined as follows:

$$\begin{aligned}\mathbf{H} &= \big[H_1\ H_2\ H_3 \ldots H_n\big]^T,\\ \mathbf{E} &= \big[E_1\ E_2\ E_3 \ldots E_n\big]^T,\\ E_r &= \exp\left[\sum_{t=1}^{n-1} \omega^{(r-1)t} x_t\right], \quad 1 \leq r \leq n.\end{aligned} \tag{4.4.17}$$

**Lemma 4.19.**

$$\prod_{r=1}^{n} E_r = 1.$$

Proof. Referring to (4.4.15),

$$\prod_{r=1}^{n} E_r = \prod_{r=1}^{n} \exp\left[\sum_{t=1}^{n-1} \omega^{(r-1)t} x_t\right]$$

$$
\begin{aligned}
&= \exp\left[\sum_{r=1}^{n}\sum_{t=1}^{n-1}\omega^{(r-1)t}x_t\right]\\
&= \exp\left[\sum_{t=1}^{n-1}x_t\sum_{r=1}^{n}\omega^{(r-1)t}\right]\\
&= \exp(0).
\end{aligned}
$$

The lemma follows. □

**Theorem.**

$$A = A(H_1, H_2, H_3, \ldots, H_n) = 1.$$

PROOF. The definition (4.4.16) implies that

$$
\begin{aligned}
\mathbf{A}(H_1, H_2, H_3, \ldots, H_n) &= \begin{bmatrix} H_1 & H_2 & H_3 & \cdots & H_n \\ H_n & H_1 & H_2 & \cdots & H_{n-1} \\ H_{n-1} & H_n & H_1 & \cdots & H_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_2 & H_3 & H_4 & \cdots & H_1 \end{bmatrix}_n \\
&= \mathbf{W}^{-1}\mathbf{diag}\big(E_1\ E_2\ E_3 \ldots E_n\big)\mathbf{W}. \qquad (4.4.18)
\end{aligned}
$$

Taking determinants,

$$A(H_1, H_2, H_3, \ldots, H_n) = \big|\mathbf{W}^{-1}\mathbf{W}\big|\prod_{r=1}^{n}E_r.$$

The theorem follows from Lemma 4.19. □

## *Illustrations*

When $n = 2$, $\omega = \exp(i\pi) = -1$.

$$
\begin{aligned}
\mathbf{W} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\\
\mathbf{W}^{-1} &= \tfrac{1}{2}\mathbf{W},\\
E_r &= \exp[(-1)^{r-1}x_1], \qquad r = 1, 2.
\end{aligned}
$$

Let $x_1 \to x$; then,

$$
\begin{aligned}
E_1 &= e^x,\\
E_2 &= e^{-x},\\
\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^x \\ e^{-x} \end{bmatrix},\\
H_1 &= \operatorname{ch} x,\\
H_2 &= \operatorname{sh} x,
\end{aligned}
$$

the simple hyperbolic functions;

$$A(H_1, H_2) = \begin{vmatrix} H_1 & H_2 \\ H_2 & H_1 \end{vmatrix} = 1. \tag{4.4.19}$$

When $n = 3$, $\omega_r = \exp(2ri\pi/3)$,

$$\begin{aligned}
\omega_r^3 &= 1, \\
\omega = \omega_1 &= \exp(2i\pi/3), \\
\omega^2 &= \bar{\omega}, \\
\omega\bar{\omega} &= 1. \\
\mathbf{W} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \\
\mathbf{W}^{-1} &= \tfrac{1}{3}\mathbf{W}, \\
E_r &= \exp\left[\sum_{t=0}^{2} \omega^{(r-1)t} x_t\right] \\
&= \exp\left[\omega^{r-1}x_1 + \omega^{2r-2}x_2\right].
\end{aligned}$$

Let $(x_1, x_2) \to (x, y)$. Then,

$$\begin{aligned}
E_1 &= \exp(x + y), \\
E_2 &= \exp(\omega x + \bar{\omega} y), \\
E_3 &= \exp(\bar{\omega} x + \omega y) = \bar{E}_2.
\end{aligned} \tag{4.4.20}$$

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \tag{4.4.21}$$

$$\begin{aligned}
H_1 &= \tfrac{1}{3}\left[e^{x+y} + e^{\omega x + \bar{\omega} y} + e^{\bar{\omega} x + \omega y}\right], \\
H_2 &= \tfrac{1}{3}\left[e^{x+y} + \omega e^{\omega x + \bar{\omega} y} + \bar{\omega} e^{\bar{\omega} x + \omega y}\right], \\
H_3 &= \tfrac{1}{3}\left[e^{x+y} + \bar{\omega} e^{\omega x + \bar{\omega} y} + \omega e^{\bar{\omega} x + \omega y}\right].
\end{aligned} \tag{4.4.22}$$

Since the complex terms appear in conjugate pairs, all three functions are real:

$$A(H_1, H_2, H_3) = \begin{vmatrix} H_1 & H_2 & H_3 \\ H_3 & H_1 & H_2 \\ H_2 & H_3 & H_1 \end{vmatrix} = 1. \tag{4.4.23}$$

A bibliography covering the years 1757–1955 on higher-order sine functions, which are closely related to higher-order or generalized hyperbolic functions, is given by Kaufman. Further notes on the subject are given by Schmidt and Pipes, who refer to the generalized hyperbolic functions as cyclical functions and by Izvercianu and Vein who refer to the generalized hyperbolic functions as Appell functions.

*Exercises*

**1.** Prove that when $n = 3$ and $(x_1, x_2) \to (x, y)$,

$$\frac{\partial}{\partial x}[H_1, H_2, H_3] = [H_2, H_3, H_1],$$
$$\frac{\partial}{\partial y}[H_1, H_2, H_3] = [H_3, H_1, H_2]$$

and apply these formulas to give an alternative proof of the particular circulant identity

$$A(H_1, H_2, H_3) = 1.$$

If $y = 0$, prove that

$$H_1 = \sum_{r=0}^{\infty} \frac{x^{3r}}{(3r)!},$$
$$H_2 = \sum_{r=0}^{\infty} \frac{x^{3r+2}}{(3r+2)!},$$
$$H_3 = \sum_{r=0}^{\infty} \frac{x^{3r+1}}{(3r+1)!}.$$

**2.** Apply the partial derivative method to give an alternative proof of the general circulant identity as stated in the theorem.

## 4.5 Centrosymmetric Determinants

### *4.5.1 Definition and Factorization*

The determinant $A_n = |a_{ij}|_n$, in which

$$a_{n+1-i,n+1-j} = a_{ij} \tag{4.5.1}$$

is said to be centrosymmetric. The elements in row $(n+1-i)$ are identical with those in row $i$ but in reverse order; that is, if

$$\mathbf{R}_i = \left[a_{i1}\ a_{i2} \ldots a_{i,n-1}\ a_{in}\right],$$

then

$$\mathbf{R}_{n+1-i} = \left[a_{in}\ a_{i,n-1} \ldots a_{i2}\ a_{i1}\right].$$

A similar remark applies to columns. $A_n$ is unaltered in form if it is transposed first across one diagonal and then across the other, an operation which is equivalent to rotating $A_n$ in its plane through 180° in either direction. $A_n$ is not necessarily symmetric across either of its diagonals. The most general centrosymmetric determinant of order 5 is of the form

$$A_5 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_2 & c_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix}. \tag{4.5.2}$$

**Theorem.** *Every centrosymmetric determinant can be factorized into two determinants of lower order. $A_{2n}$ has factors each of order $n$, whereas $A_{2n+1}$ has factors of orders $n$ and $n+1$.*

PROOF. In the row vector

$$\mathbf{R}_i + \mathbf{R}_{n+1-i} = \left[(a_{i1}+a_{in})(a_{i2}+a_{i,n-1})\cdots(a_{i,n-1}+a_{i2})(a_{in}+a_{i1})\right],$$

the $(n+1-j)$th element is identical to the $j$th element. This suggests performing the row and column operations

$$\begin{aligned} \mathbf{R}'_i &= \mathbf{R}_i + \mathbf{R}_{n+1-i}, \quad 1 \le i \le \left[\tfrac{1}{2}n\right], \\ \mathbf{C}'_j &= \mathbf{C}_j - \mathbf{C}_{n+1-j}, \quad \left[\tfrac{1}{2}(n+1)\right] + 1 \le j \le n, \end{aligned}$$

where $\left[\frac{1}{2}n\right]$ is the integer part of $\frac{1}{2}n$. The result of these operations is that an array of zero elements appears in the top right-hand corner of $A_n$, which then factorizes by applying a Laplace expansion (Section 3.3). The dimensions of the various arrays which appear can be shown clearly using the notation $\mathbf{M}_{rs}$, etc., for a matrix with $r$ rows and $s$ columns. $\mathbf{0}_{rs}$ is an array of zero elements.

$$\begin{aligned} A_{2n} &= \begin{vmatrix} \mathbf{R}_{nn} & \mathbf{0}_{nn} \\ \mathbf{S}_{nn} & \mathbf{T}_{nn} \end{vmatrix}_{2n} \\ &= |\mathbf{R}_{nn}|\,|\mathbf{T}_{nn}|, \end{aligned} \tag{4.5.3}$$

$$\begin{aligned} A_{2n+1} &= \begin{vmatrix} \mathbf{R}^*_{n+1,n+1} & \mathbf{0}_{n+1,n} \\ \mathbf{S}^*_{n,n+1} & \mathbf{T}^*_{nn} \end{vmatrix}_{2n+1} \\ &= |\mathbf{R}^*_{n+1,n+1}|\,|\mathbf{T}^*_{nn}|. \end{aligned} \tag{4.5.4}$$

□

The method of factorization can be illustrated adequately by factorizing the fifth-order determinant $A_5$ defined in (4.5.2).

$$A_5 = \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 & a_4+a_2 & a_5+a_1 \\ b_1+b_5 & b_2+b_4 & 2b_3 & b_4+b_2 & b_5+b_1 \\ c_1 & c_2 & c_3 & c_2 & c_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 & \bullet & \bullet \\ b_1+b_5 & b_2+b_4 & 2b_3 & \bullet & \bullet \\ c_1 & c_2 & c_3 & \bullet & \bullet \\ b_5 & b_4 & b_3 & b_2-b_4 & b_1-b_5 \\ a_5 & a_4 & a_3 & a_2-a_4 & a_1-a_5 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 \\ b_1+b_5 & b_2+b_4 & 2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} b_2-b_4 & b_1-b_5 \\ a_2-a_4 & a_1-a_5 \end{vmatrix}$$

$$= \tfrac{1}{2}|\mathbf{E}|\,|\mathbf{F}|, \tag{4.5.5}$$

where

$$\mathbf{E} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} + \begin{bmatrix} a_5 & a_4 & a_3 \\ b_5 & b_4 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} b_2 & b_1 \\ a_2 & a_1 \end{bmatrix} - \begin{bmatrix} b_4 & b_5 \\ a_4 & a_5 \end{bmatrix}. \tag{4.5.6}$$

Two of these matrices are submatrices of $A_5$. The other two are submatrices with their rows or columns arranged in reverse order.

**Exercise.** If a determinant $A_n$ is symmetric about its principal diagonal and persymmetric (Hankel, Section 4.8) about its secondary diagonal, prove analytically that $A_n$ is centrosymmetric.

### *4.5.2 Symmetric Toeplitz Determinants*

The classical Toeplitz determinant $A_n$ is defined as follows:

$$A_n = |a_{i-j}|_n$$

$$= \begin{vmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & \cdots & \cdots & \cdots & \cdots & a_0 \end{vmatrix}_n.$$

The symmetric Toeplitz determinant $T_n$ is defined as follows:

$$T_n = |t_{|i-j|}|_n$$

$$= \begin{vmatrix} t_0 & t_1 & t_2 & t_3 & \cdots & t_{n-1} \\ t_1 & t_0 & t_1 & t_2 & \cdots & \cdots \\ t_2 & t_1 & t_0 & t_1 & \cdots & \cdots \\ t_3 & t_2 & t_1 & t_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n-1} & \cdots & \cdots & \cdots & \cdots & t_0 \end{vmatrix}_n, \tag{4.5.7}$$

which is centrosymmetric and can therefore be expressed as the product of two determinants of lower order. $T_n$ is also persymmetric about its secondary diagonal.

Let $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{E}_n$ denote Hankel matrices defined as follows:

$$\begin{aligned}\mathbf{A}_n &= \left[t_{i+j-2}\right]_n,\\ \mathbf{B}_n &= \left[t_{i+j-1}\right]_n,\\ \mathbf{E}_n &= \left[t_{i+j}\right]_n.\end{aligned} \tag{4.5.8}$$

Then, the factors of $T_n$ can be expressed as follows:

$$\begin{aligned}T_{2n-1} &= \tfrac{1}{2}|\mathbf{T}_{n-1} - \mathbf{E}_{n-1}|\,|\mathbf{T}_n + \mathbf{A}_n|,\\ T_{2n} &= |\mathbf{T}_n + \mathbf{B}_n|\,|\mathbf{T}_n - \mathbf{B}_n|.\end{aligned} \tag{4.5.9}$$

Let

$$\begin{aligned}P_n &= \tfrac{1}{2}|\mathbf{T}_n - \mathbf{E}_n| = \tfrac{1}{2}|t_{|i-j|} - t_{i+j}|_n,\\ Q_n &= \tfrac{1}{2}|\mathbf{T}_n + \mathbf{A}_n| = \tfrac{1}{2}|t_{|i-j|} + t_{i+j-2}|_n,\\ R_n &= \tfrac{1}{2}|\mathbf{T}_n + \mathbf{B}_n| = \tfrac{1}{2}|t_{|i-j|} + t_{i+j-1}|_n,\\ S_n &= \tfrac{1}{2}|\mathbf{T}_n - \mathbf{B}_n| = \tfrac{1}{2}|t_{|i-j|} - t_{i+j-1}|_n,\end{aligned} \tag{4.5.10}$$

$$\begin{aligned}U_n &= R_n + S_n,\\ V_n &= R_n - S_n.\end{aligned} \tag{4.5.11}$$

Then,

$$\begin{aligned}T_{2n-1} &= 2P_{n-1}Q_n,\\ T_{2n} &= 4R_nS_n\\ &= U_n^2 - V_n^2.\end{aligned} \tag{4.5.12}$$

**Theorem.**

**a.** $T_{2n-1} = U_{n-1}U_n - V_{n-1}V_n$,
**b.** $T_{2n} = P_nQ_n + P_{n-1}Q_{n+1}$.

PROOF. Applying the Jacobi identity (Section 3.6),

$$\begin{vmatrix} T_{11}^{(n)} & T_{1n}^{(n)} \\ T_{n1}^{(n)} & T_{nn}^{(n)} \end{vmatrix} = T_n T_{1n,1n}^{(n)}.$$

But

$$\begin{aligned}T_{11}^{(n)} &= T_{nn}^{(n)} = T_{n-1},\\ T_{n1}^{(n)} &= T_{1n}^{(n)},\\ T_{1n,1n}^{(n)} &= T_{n-2}.\end{aligned}$$

Hence,

$$T_{n-1}^2 = T_nT_{n-2} + \left(T_{1n}^{(n)}\right)^2. \tag{4.5.13}$$

The element $t_{2n-1}$ does not appear in $T_n$ but appears in the bottom right-hand corner of $B_n$. Hence,

$$\begin{aligned}\frac{\partial R_n}{\partial t_{2n-1}} &= R_{n-1},\\ \frac{\partial S_n}{\partial t_{2n-1}} &= -S_{n-1}.\end{aligned} \tag{4.5.14}$$

The same element appears in positions $(1, 2n)$ and $(2n, 1)$ in $T_{2n}$. Hence, referring to the second line of (4.5.12),

$$\begin{aligned}T_{1,2n}^{(2n)} &= \frac{1}{2}\,\frac{\partial T^{(2n)}}{\partial t_{2n-1}}\\ &= 2\frac{\partial}{\partial t_{2n-1}}(R_n S_n)\\ &= 2(R_{n-1}S_n - R_n S_{n-1}).\end{aligned} \tag{4.5.15}$$

Replacing $n$ by $2n$ in (4.5.13),

$$\begin{aligned}T_{2n-1}^2 &= T_{2n}T_{2n-2} + \bigl(T_{1,2n}^{(2n)}\bigr)^2\\ &= 4\bigl[4R_nS_nR_{n-1}S_{n-1} + (R_{n-1}S_n - R_nS_{n-1})^2\bigr]\\ &= 4(R_{n-1}S_n + R_nS_{n-1})^2.\end{aligned}$$

The sign of $T_{2n-1}$ is decided by putting $t_0 = 1$ and $t_r = 0$, $r > 0$. In that case, $\mathbf{T}_n = \mathbf{I}_n$, $\mathbf{B}_n = \mathbf{O}_n$, $R_n = S_n = \frac{1}{2}$. Hence, the sign is positive:

$$T_{2n-1} = 2(R_{n-1}S_n + R_nS_{n-1}). \tag{4.5.16}$$

Part (a) of the theorem follows from (4.5.11).

The element $t_{2n}$ appears in the bottom right-hand corner of $\mathbf{E}_n$ but does not appear in either $\mathbf{T}_n$ or $\mathbf{A}_n$. Hence, referring to (4.5.10),

$$\begin{aligned}\frac{\partial P_n}{\partial t_{2n}} &= -P_{n-1},\\ \frac{\partial Q_n}{\partial t_{2n}} &= Q_{n-1}.\end{aligned} \tag{4.5.17}$$

$$\begin{aligned}T_{1,2n+1}^{(2n+1)} &= \frac{1}{2}\,\frac{\partial T_{2n+1}}{\partial t_{2n}}\\ &= \frac{\partial}{\partial t_{2n}}(P_nQ_{n+1})\\ &= P_nQ_n - P_{n-1}Q_{n+1}.\end{aligned} \tag{4.5.18}$$

Return to (4.5.13), replace $n$ by $2n+1$, and refer to (4.5.12):

$$\begin{aligned}T_{2n}^2 &= T_{2n+1}T_{2n-1} + \bigl(T_{1,2n+1}^{(2n+1)}\bigr)^2\\ &= 4P_nQ_{n+1}P_{n-1}Q_n + (P_nQ_n - P_{n-1}Q_{n+1})^2\\ &= (P_nQ_n + P_{n-1}Q_{n+1})^2.\end{aligned} \tag{4.5.19}$$

When $t_0 = 1$, $t_r = 0$, $r > 0$, $T_{2n} = 1$, $\mathbf{E}_n = \mathbf{O}_n$, $\mathbf{A}_n = \text{diag}[1\ 0\ 0 \ldots 0]$. Hence, $P_n = \frac{1}{2}$, $Q_n = 1$, and the sign of $T_{2n}$ is positive, which proves part (b) of the theorem. $\square$

The above theorem is applied in Section 6.10 on the Einstein and Ernst equations.

**Exercise.** Prove that

$$T_{12}^{(n)} = T_{n-1,n}^{(n)} = T_{1n;1,n+1}^{(n+1)}.$$

### *4.5.3 Skew-Centrosymmetric Determinants*

The determinant $A_n = |a_{ij}|_n$ is said to be skew-centrosymmetric if

$$a_{n+1-i,n+1-j} = -a_{ij}.$$

In $A_{2n+1}$, the element at the center, that is, in position $(n+1, n+1)$, is necessarily zero, but in $A_{2n}$, no element is necessarily zero.

#### *Exercises*

1. Prove that $A_{2n}$ can be expressed as the product of two determinants of order $n$ which can be written in the form $(P+Q)(P-Q)$ and hence as the difference between two squares.
2. Prove that $A_{2n+1}$ can be expressed as a determinant containing an $(n+1) \times (n+1)$ block of zero elements and is therefore zero.
3. Prove that if the zero element at the center of $A_{2n+1}$ is replaced by $x$, then $A_{2n+1}$ can be expressed in the form $x(p+q)(p-q)$.

## 4.6 Hessenbergians

### *4.6.1 Definition and Recurrence Relation*

The determinant

$$H_n = |a_{ij}|_n,$$

where $a_{ij} = 0$ when $i - j > 1$ or when $j - i > 1$ is known as a Hessenberg determinant or simply a Hessenbergian. If $a_{ij} = 0$ when $i - j > 1$, the Hessenbergian takes the form

$$H_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ & a_{32} & a_{33} & \cdots & \cdots & \cdots \\ & & a_{43} & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots \\ & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & a_{n,n-1} & a_{nn} \end{vmatrix}_n. \tag{4.6.1}$$

If $a_{ij} = 0$ when $j - i > 1$, the triangular array of zero elements appears in the top right-hand corner. $H_n$ can be expressed neatly in column vector notation.

Let

$$\mathbf{C}_{jr} = \left[a_{1j}\ a_{2j}\ a_{3j} \ldots a_{rj}\ \mathbf{O}_{n-r}\right]_n^T, \tag{4.6.2}$$

where $\mathbf{O}_i$ represents an unbroken sequence of $i$ zero elements. Then

$$H_n = \left|\mathbf{C}_{12}\ \mathbf{C}_{23}\ \mathbf{C}_{34} \ldots \mathbf{C}_{n-1,n}\ \mathbf{C}_{nn}\right|_n. \tag{4.6.3}$$

Hessenbergians satisfy a simple recurrence relation.

**Theorem 4.20.**

$$H_n = (-1)^{n-1} \sum_{r=0}^{n-1} (-1)^r p_{r+1,n} H_r, \quad H_0 = 1,$$

*where*

$$p_{ij} = \begin{cases} a_{ij} a_{j,j-1} a_{j-1,j-2} \cdots a_{i+2,i+1} a_{i+1,i}, & j > i \\ a_{ii}, & j = i. \end{cases}$$

PROOF. Expanding $H_n$ by the two nonzero elements in the last row,

$$H_n = a_{nn} H_{n-1} - a_{n,n-1} K_{n-1},$$

where $K_{n-1}$ is a determinant of order $(n-1)$ whose last row also contains two nonzero elements. Expanding $K_{n-1}$ in a similar manner,

$$K_{n-1} = a_{n-1,n} H_{n-2} - a_{n-1,n-2} K_{n-2},$$

where $K_{n-2}$ is a determinant of order $(n-2)$ whose last row also contains two nonzero elements. The theorem appears after these expansions are repeated a sufficient number of times. $\square$

**Illustration.**

$$\begin{aligned} H_5 = \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{34}\mathbf{C}_{45}\mathbf{C}_{55}\right| &= a_{55} H_4 - a_{54} \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{34}\mathbf{C}_{54}\right|, \\ \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{34}\mathbf{C}_{54}\right| &= a_{45} H_3 - a_{43} \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{53}\right|, \\ \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{53}\right| &= a_{35} H_2 - a_{32} \left|\mathbf{C}_{12}\mathbf{C}_{52}\right|, \\ \left|\mathbf{C}_{12}\mathbf{C}_{52}\right| &= a_{25} H_1 - a_{21} a_{15} H_0. \end{aligned}$$

Hence,

$$\begin{aligned} H_5 &= a_{55} H_4 - (a_{45} a_{54}) H_3 + (a_{35} a_{54} a_{43}) H_2 \\ &\quad - (a_{25} a_{54} a_{43} a_{32}) H_1 + (a_{15} a_{54} a_{43} a_{32} a_{21}) H_0 \\ &= p_{55} H_4 - p_{45} H_3 + p_{35} H_2 - p_{25} H_1 + p_{15} H_0. \end{aligned}$$

Muir and Metzler use the term *recurrent* without giving a definition of the term. A *recurrent* is any determinant which satisfies a recurrence relation.

### *4.6.2 A Reciprocal Power Series*

**Theorem 4.21.** *If*

$$\sum_{r=0}^{\infty}(-1)^r\psi_n t^r = \left[\sum_{r=0}^{\infty}\phi_r t^r\right]^{-1}, \quad \phi_0 = \psi_0 = 1,$$

*then*

$$\psi_r = \begin{vmatrix} \phi_1 & \phi_0 & & & & \\ \phi_2 & \phi_1 & \phi_0 & & & \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \phi_{n-1} & \phi_{n-2} & \cdots & \cdots & \phi_1 & \phi_0 \\ \phi_n & \phi_{n-1} & \cdots & \cdots & \phi_2 & \phi_1 \end{vmatrix}_n,$$

*which is a Hessenbergian.*

PROOF. The given equation can be expressed in the form

$$(\phi_0 + \phi_1 t + \phi_2 t^2 + \phi_3 t^3 + \cdots)(\psi_0 - \psi_1 t + \psi_2 t^2 - \psi_3 t^3 + \cdots) = 1.$$

Equating coefficients of powers of $t$,

$$\sum_{i=0}^{n}(-1)^{i+1}\phi_i\psi_{n-i} = 0 \tag{4.6.4}$$

from which it follows that

$$\phi_n = \sum_{i=1}^{n}(-1)^{i+1}\phi_{n-i}\psi_i. \tag{4.6.5}$$

In some detail,

$$\begin{aligned}
&\phi_0\psi_1 &&= \phi_1\\
&\phi_1\psi_1 - \phi_0\psi_2 &&= \phi_2\\
&\phi_2\psi_1 - \phi_1\psi_2 + \phi_0\psi_3 &&= \phi_3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && \\
&\phi_{n-1}\psi_1 - \phi_{n-2}\psi_2 + \cdots + (-1)^{n+1}\phi_0\psi_n &&= \phi_n.
\end{aligned}$$

These are $n$ equations in the $n$ variables $(-1)^{r-1}\psi_r$, $1 \le r \le n$, in which the determinant of the coefficients is triangular and equal to 1. Hence,

$$(-1)^{n-1}\psi_n = \begin{vmatrix} \phi_0 & & & & & & \phi_1 \\ \phi_1 & \phi_0 & & & & & \phi_2 \\ \phi_2 & \phi_1 & \phi_0 & & & & \phi_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{n-2} & \phi_{n-3} & \phi_{n-4} & \cdots & \phi_1 & \phi_0 & \phi_{n-1} \\ \phi_{n-1} & \phi_{n-2} & \phi_{n-3} & \cdots & \phi_2 & \phi_1 & \phi_n \end{vmatrix}_n.$$

The proof is completed by transferring the last column to the first position, an operation which introduces the factor $(-1)^{n-1}$. □

In the next theorem, $\phi_m$ and $\psi_m$ are functions of $x$.

**Theorem 4.22.** *If*

$$\phi'_m = (m+a)F\phi_{m-1}, \quad F = F(x),$$

*then*

$$\psi'_m = (a+2-m)F\psi_{m-1}.$$

Proof. It follows from (4.6.4) that

$$\psi_n = \sum_{i=1}^{n} (-1)^{i+1}\phi_i\psi_{n-i}. \tag{4.6.6}$$

It may be verified by elementary methods that

$$\begin{aligned}
\psi'_1 &= (a+1)F\psi_0, \\
\psi'_2 &= aF\psi_1, \\
\psi'_3 &= (a-1)F\psi_2,
\end{aligned}$$

etc., so that the theorem is known to be true for small values of $m$. Assume it to be true for $1 \leq m \leq n-1$ and apply the method of induction. Differentiating (4.6.6),

$$\begin{aligned}
\psi'_n &= \sum_{i=1}^{n} (-1)^{i+1}(\phi'_i\psi_{n-i} + \phi_i\psi'_{n-i}) \\
&= F\sum_{i=1}^{n} (-1)^{i+1}[(i+a)\phi_{i-1}\psi_{n-i} + (a+2-n+i)\phi_i\psi_{n-1-i}] \\
&= F(S_1 + S_2 + S_3),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{i=1}^{n} (-1)^{i+1}(i+a)\phi_{i-1}\psi_{n-i}, \\
S_2 &= (a+2-n)\sum_{i=1}^{n} (-1)^{i+1}\phi_i\psi_{n-1-i}, \\
S_3 &= \sum_{i=1}^{n} (-1)^{i+1} i\phi_i\psi_{n-1-i}.
\end{aligned}$$

Since the $i = n$ terms in $S_2$ and $S_3$ are zero, the upper limits in these sums can be reduced to $(n-1)$. It follows that

$$S_2 = (a+2-n)\psi_{n-1}.$$

Also, adjusting the dummy variable in $S_1$ and referring to (4.6.4) with $n \to n-1$,

$$\begin{aligned}
S_1 &= \sum_{i=0}^{n-1} (-1)^i (i+1+a)\phi_i \psi_{n-1-i} \\
&= \sum_{i=1}^{n-1} (-1)^i i\phi_i \psi_{n-1-i} + (1+a) \sum_{i=0}^{n-1} (-1)^i \phi_i \psi_{n-1-i} \\
&= -S_3.
\end{aligned}$$

Hence, $\psi'_n = (a+2-n)F\psi_{n-1}$, which is equivalent to the stated result. Note that if $\phi'_m = (m-1)\phi_{m-1}$, then $\psi'_m = -(m-1)\psi_{m-1}$. □

### *4.6.3 A Hessenberg–Appell Characteristic Polynomial*

Let

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} a_{j-i+1}, & j \geq i, \\ -j, & j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

In some detail,

$$A_n = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\ -1 & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ & -2 & a_1 & a_2 & \cdots & \cdots & \cdots \\ & & -3 & a_1 & \cdots & \cdots & \cdots \\ & & & & \cdots & \cdots & \cdots \\ & & & & & a_1 & a_2 \\ & & & & & -(n-1) & a_1 \end{vmatrix}_n . \tag{4.6.7}$$

Applying the recurrence relation in Theorem 4.20,

$$A_n = (n-1)! \sum_{r=0}^{n-1} \frac{a_{n-r} A_r}{r!}, \quad n \geq 1, \quad A_0 = 1. \tag{4.6.8}$$

Let $B_n(x)$ denote the characteristic polynomial of the matrix $\mathbf{A}_n$:

$$B_n = \left| \mathbf{A}_n - x\mathbf{I} \right|. \tag{4.6.9}$$

This determinant satisfies the recurrence relation

$$B_n = (n-1)! \sum_{r=0}^{n-1} \frac{b_{n-r} B_r}{r!}, \quad n \geq 1, \quad B_0 = 1, \tag{4.6.10}$$

where

$$b_1 = a_1 - x,$$

$$b_r = a_r, \quad r > 1.$$

$$\begin{aligned} B_n(0) &= A_n, \\ B_{ij}^{(n)}(0) &= A_{ij}^{(n)}. \end{aligned} \tag{4.6.11}$$

**Theorem 4.23.**

**a.** $B'_n = -nB_{n-1}$.

**b.** $\displaystyle\sum_{r=1}^{n} A_{rr}^{(n)} = nA_{n-1}$.

**c.** $\displaystyle B_n = \sum_{r=0}^{n} \binom{n}{r} A_r(-x)^{n-r}$.

PROOF.

$$\begin{aligned} B_1 &= -x + A_1, \\ B_2 &= x^2 - 2A_1x + A_2, \\ B_3 &= -x^3 + 3A_1x^2 - 3A_2x + A_3, \end{aligned} \tag{4.6.12}$$

etc., which are Appell polynomials (Appendix A.4) so that (a) is valid for small values of $n$. Assume that

$$B'_r = -rB_{r-1}, \quad 2 \le r \le n-1,$$

and apply the method of induction.

From (4.6.10),

$$\begin{aligned} B_n &= (n-1)!\sum_{r=0}^{n-2} \frac{a_{n-r}B_r}{r!} + (a_1 - x)B_{n-1}, \\ B'_n &= -(n-1)!\sum_{r=1}^{n-2} \frac{a_{n-r}rB_{r-1}}{r!} - (n-1)(a_1 - x)B_{n-2} - B_{n-1} \\ &= -(n-1)!\sum_{r=1}^{n-2} \frac{a_{n-r}B_{r-1}}{(r-1)!} - (n-1)(a_1 - x)B_{n-2} - B_{n-1} \\ &= -(n-1)!\sum_{r=0}^{n-3} \frac{a_{n-1-r}B_r}{r!} - (n-1)(a_1 - x)B_{n-2} - B_{n-1} \\ &= -(n-1)!\sum_{r=0}^{n-2} \frac{b_{n-1-r}B_r}{r!} - B_{n-1} \\ &= -(n-1)B_{n-1} - B_{n-1} \\ &= -nB_{n-1}, \end{aligned}$$

which proves (a).

The proof of (b) follows as a corollary since, differentiating $B_n$ by columns,

$$B_n' = -\sum_{r=1}^{n} B_{rr}^{(n)}.$$

The given result follows from (4.6.11).

To prove (c), differentiate (a) repeatedly, apply the Maclaurin formula, and refer to (4.6.11) again:

$$\begin{aligned} B_n^{(r)} &= \frac{(-1)^r n! B_{n-r}}{(n-r)!}, \\ B_n &= \sum_{r=0}^{n} \frac{B_n^{(r)}(0)}{r!} x^r \\ &= \sum_{r=0}^{n} \binom{n}{r} A_{n-r}(-x)^r. \end{aligned}$$

Put $r = n - s$ and the given formula appears. It follows that $B_n$ is an Appell polynomial for all values of $n$. □

## *Exercises*

**1.** Let

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} \psi_{j-i+1}, & j \geq i, \\ j, & j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that if $A_n$ satisfies the Appell equation $A_n' = nA_{n-1}$ for small values of $n$, then $A_n$ satisfies the Appell equation for all values of $n$ and that the elements must be of the form

$$\begin{aligned} \psi_1 &= x + \alpha_1, \\ \psi_m &= \alpha_m, \qquad m > 1, \end{aligned}$$

where the $\alpha$'s are constants.

**2.** If

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} \phi_{j-i}, & j \geq i, \\ -j, & j = i-1, \\ 0, & \text{otherwise,} \end{cases}$$

and where

$$\phi'_m = (m+1)\phi_{m-1}, \quad \phi_0 = \text{constant},$$

prove that

$$A'_n = n(n-1)A_{n-1}.$$

**3.** Prove that

$$\prod_{r=1}^{n} \begin{vmatrix} 1 & a_r x \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & b_{12}x & b_{13}x^2 & \cdots & \cdots & b_{1,n+1}x^n \\ -1 & 1 & b_{23}x & \cdots & \cdots & b_{2,n+1}x^{n-1} \\ & -1 & 1 & \cdots & \cdots & b_{3,n+1}x^{n-2} \\ & & \cdots & \cdots & \cdots & \cdots \\ & & & & -1 & 1 \end{vmatrix}_{n+1},$$

where

$$b_{ij} = \prod_{r=i}^{j-1} a_r.$$

**4.** If

$$U_n = \begin{vmatrix} u' & u''/2! & u'''/3! & u^{(4)}/4! & \cdots \\ u & u' & u''/2! & u'''/3! & \cdots \\ & u & u' & u''/2! & \cdots \\ & & u & u' & \cdots \\ & & & \cdots & \cdots \end{vmatrix}_n,$$

prove that

$$U_{n+1} = u'U_n - \frac{uU'_n}{n+1}. \qquad \text{(Burgmeier)}$$

## 4.7 Wronskians

### *4.7.1 Introduction*

Let $y_r = y_r(x)$, $1 \le r \le n$, denote $n$ functions each with derivatives of orders up to $(n-1)$. These functions are said to be linearly dependent if there exist coefficients $\lambda_r$, independent of $x$ and not all zero, such that

$$\sum_{r=1}^{n} \lambda_r y_r = 0 \qquad (4.7.1)$$

for all values of $x$.

**Theorem 4.24.** *The necessary condition that the functions $y_r$ be linearly dependent is that*

$$\left|y_j^{(i-1)}\right|_n = 0$$

*identically.*

PROOF. Equation (4.7.1) together with its first $(n-1)$ derivatives form a set of $n$ homogeneous equations in the $n$ coefficients $\lambda_r$. The condition that not all the $\lambda_r$ be zero is that the determinant of the coefficients of the $\lambda_r$ be zero, that is,

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0$$

for all values of $x$, which proves the theorem. □

This determinant is known as the Wronskian of the $n$ functions $y_r$ and is denoted by $W(y_1, y_2, \ldots, y_n)$, which can be abbreviated to $W_n$ or $W$ where there is no risk of confusion. After transposition, $W_n$ can be expressed in column vector notation as follows:

$$W_n = W(y_1, y_2, \ldots, y_n) = \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-1)}\right|$$

where

$$\mathbf{C} = \left[y_1\ y_2 \cdots y_n\right]^T. \tag{4.7.2}$$

If $W_n \neq 0$, identically the $n$ functions are linearly independent.

**Theorem 4.25.** *If $t = t(x)$,*

$$W(ty_1, ty_2, \ldots, ty_n) = t^n W(y_1, y_2, \ldots, y_n).$$

PROOF.

$$\begin{aligned} W(ty_1, ty_2, \ldots, ty_n) &= \left|(t\mathbf{C})\ (t\mathbf{C})'\ (t\mathbf{C})'' \cdots (t\mathbf{C})^{(n-1)}\right| \\ &= \left|\mathbf{K}_1\ \mathbf{K}_2\ \mathbf{K}_3 \cdots \mathbf{K}_n\right|, \end{aligned}$$

where

$$\mathbf{K}_j = (t\mathbf{C})^{(j-1)} = D^{j-1}(t\mathbf{C}), \quad D = \frac{d}{dx}.$$

Recall the Leibnitz formula for the $(j-1)$th derivative of a product and perform the following column operations:

$$\begin{aligned} \mathbf{K}_j' &= \mathbf{K}_j + t\sum_{s=1}^{j-1} \binom{j-1}{s} D^s\left(\frac{1}{t}\right)\mathbf{K}_{j-s}, \quad j = n, n-1, \ldots, 3.2. \\ &= t\sum_{s=0}^{j-1} \binom{j-1}{s} D^s\left(\frac{1}{t}\right)\mathbf{K}_{j-s} \\ &= t\sum_{s=0}^{j-1} \binom{j-1}{s} D^s\left(\frac{1}{t}\right)D^{j-1-s}(t\mathbf{C}) \\ &= tD^{(j-1)}(\mathbf{C}) \\ &= t\mathbf{C}^{(j-1)}. \end{aligned}$$

Hence,

$$\begin{aligned} W(ty_1, ty_2, \ldots, ty_n) &= \left|(t\mathbf{C})\ (t\mathbf{C}')\ (t\mathbf{C}'') \cdots (t\mathbf{C}^{(n-1)})\right| \\ &= t^n \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-1)}\right|. \end{aligned}$$

The theorem follows. □

**Exercise.** Prove that

$$\frac{d^n x}{dy^n} = \frac{(-1)^{n+1} W\{y'', (y^2)'', (y^3)'' \ldots (y^{n-1})''\}}{1!2!3! \cdots (n-1)!(y')^{n(n+1)/2}},$$

where $y' = dy/dx$, $n \geq 2$. (Mina)

### 4.7.2 The Derivatives of a Wronskian

The derivative of $W_n$ with respect to $x$, when evaluated in column vector notation, consists of the sum of $n$ determinants, only one of which has distinct columns and is therefore nonzero. That determinant is the one obtained by differentiating the last column:

$$W_n' = \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-3)}\ \mathbf{C}^{(n-2)}\ \mathbf{C}^{(n)}\right|.$$

Differentiating again,

$$\begin{aligned} W_n'' = &\left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-3)}\ \mathbf{C}^{(n-1)}\ \mathbf{C}^{(n)}\right| \\ &+ \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-3)}\ \mathbf{C}^{(n-2)}\ \mathbf{C}^{(n+1)}\right|, \end{aligned} \tag{4.7.3}$$

etc. There is no simple formula for $W_n^{(r)}$. In some detail,

$$W_n' = \begin{vmatrix} y_1 & y_1' & \cdots & y_1^{(n-2)} & y_1^{(n)} \\ y_2 & y_2' & \cdots & y_2^{(n-2)} & y_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_n & y_n' & \cdots & y_n^{(n-2)} & y_n^{(n)} \end{vmatrix}_n. \tag{4.7.4}$$

The first $(n-1)$ columns of $W_n'$ are identical with the corresponding columns of $W_n$. Hence, expanding $W_n'$ by elements from its last column,

$$W_n' = \sum_{r=1}^{n} y_r^{(n)} W_{rn}^{(n)}. \tag{4.7.5}$$

Each of the cofactors in the sum is itself a Wronskian of order $(n-1)$:

$$W_{rn}^{(n)} = (-1)^{r+n} W(y_1, y_2, \ldots, y_{r-1}, y_{r+1}, \ldots, y_n). \tag{4.7.6}$$

$W_n'$ is a cofactor of $W_{n+1}$:

$$W_n' = -W_{n+1,n}^{(n+1)}. \tag{4.7.7}$$

Repeated differentiation of a Wronskian of order $n$ is facilitated by adopting the notation

$$W_{ijk\ldots r} = \left|\mathbf{C}^{(i)}\ \mathbf{C}^{(j)}\ \mathbf{C}^{(k)} \cdots \mathbf{C}^{(r)}\right|$$

$$\begin{aligned} &= 0 \quad \text{if the parameters are not distinct} \\ W'_{ijk\ldots r} &= \text{the sum of the determinants obtained by increasing} \\ &\quad \text{the parameters one at a time by 1 and discarding} \\ &\quad \text{those determinants with two identical parameters.} \end{aligned} \tag{4.7.8}$$

**Illustration.** Let

$$W = \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}''\right| = W_{012}.$$

Then

$$\begin{aligned} W' &= W_{013}, \\ W'' &= W_{014} + W_{023}, \\ W''' &= W_{015} + 2W_{024} + W_{123}, \\ W^{(4)} &= W_{016} + 3W_{025} + 2W_{034} + 3W_{124}, \\ W^{(5)} &= W_{017} + 4W_{026} + 5W_{035} + 6W_{125} + 5W_{134}, \end{aligned} \tag{4.7.9}$$

etc. Formulas of this type appear in Sections 6.7 and 6.8 on the K dV and KP equations.

### *4.7.3 The Derivative of a Cofactor*

In order to determine formulas for $(W_{ij}^{(n)})'$, it is convenient to change the notation used in the previous section.

Let

$$W = |w_{ij}|_n,$$

where

$$w_{ij} = y_i^{(j-1)} = D^{j-1}(y_i), \quad D = \frac{d}{dx},$$

and where the $y_i$ are arbitrary $(n-1)$ differentiable functions.

Clearly,

$$w'_{ij} = w_{i,j+1}.$$

In column vector notation,

$$W_n = \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_n\right|,$$

where

$$\begin{aligned} \mathbf{C}_j &= \left[y_1^{(j-1)}\ y_2^{(j-1)} \cdots y_n^{(j-1)}\right]^T, \\ \mathbf{C}'_j &= \mathbf{C}_{j+1}. \end{aligned}$$

**Theorem 4.26.**

**a.** $\left(W_{ij}^{(n)}\right)' = -W_{i,j-1}^{(n)} - W_{i,n+1;jn}^{(n+1)}$.

**b.** $\left(W_{i1}^{(n)}\right)' = -W_{i,n+1;1n}^{(n+1)}$.

**c.** $\left(W_{in}^{(n)}\right)' = -W_{i,n-1}^{(n)}$.

PROOF. Let $\mathbf{Z}_i$ denote the $n$-rowed column vector in which the element in row $i$ is 1 and all the other elements are zero.

Then

$$W_{ij}^{(n)} = \left|\mathbf{C}_1 \cdots \mathbf{C}_{j-2}\ \mathbf{C}_{j-1}\ \mathbf{Z}_i\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n, \tag{4.7.10}$$

$$\begin{aligned}\left(W_{ij}^{(n)}\right)' &= \left|\mathbf{C}_1 \cdots \mathbf{C}_{j-2}\ \mathbf{C}_{j}\ \mathbf{Z}_i\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n \\ &\quad + \left|\mathbf{C}_1 \cdots \mathbf{C}_{j-2}\ \mathbf{C}_{j-1}\ \mathbf{Z}_i\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_{n+1}\right|_n.\end{aligned} \tag{4.7.11}$$

Formula (a) follows after $\mathbf{C}_j$ and $\mathbf{Z}_i$ in the first determinant are interchanged. Formulas (b) and (c) are special cases of (a) which can be proved by a similar method but may also be obtained from (a) by referring to the definition of first and second cofactors. $W_{i0} = 0$; $W_{rs,tt} = 0$. □

**Lemma.** *When* $1 \leq j, s \leq n$,

$$\sum_{r=0}^{n} w_{r,s+1} W_{rj}^{(n)} = \begin{cases} W_n, & s = j-1,\ j \neq 1, \\ -W_{n+1,j}^{(n+1)}, & s = n, \\ 0, & \text{otherwise.} \end{cases}$$

The first and third relations are statements of the sum formula for elements and cofactors (Section 2.3.4):

$$\begin{aligned}\sum_{r=1}^{n} w_{r,n+1} W_{rj}^{(n)} &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{j-1}\ \mathbf{C}_{n+1}\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n \\ &= (-1)^{n-j}\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{j-1}\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\ \mathbf{C}_{n+1}\right|_n.\end{aligned}$$

The second relation follows.

**Theorem 4.27.**

$$\begin{vmatrix} W_{ij}^{(n)} & W_{in}^{(n)} \\ W_{n+1,j}^{(n+1)} & W_{n+1,n}^{(n+1)} \end{vmatrix} = W_n W_{i,n+1;jn}^{(n+1)}.$$

This identity is a particular case of Jacobi variant (B) (Section 3.6.3) with $(p, q) \to (j, n)$, but the proof which follows is independent of the variant.

PROOF. Applying double-sum relation (B) (Section 3.4),

$$\left(W_n^{ij}\right)' = -\sum_{r=1}^{n}\sum_{s=1}^{n} w'_{rs} W_n^{is} W_n^{rj}.$$

Reverting to simple cofactors and applying the above lemma,

$$\left(\frac{W_{ij}^{(n)}}{W_n}\right)' = -\frac{1}{W_n^2}\sum_r \sum_s w'_{rs} W_{is}^{(n)} W_{rj}^{(n)}$$

$$= -\frac{1}{W_n^2} \sum_{s=j-1,n} W_{is}^{(n)} \sum_r w_{r,s+1} W_{rj}^{(n)},$$

$$W_n \big(W_{ij}^{(n)}\big)' - W_{ij}^{(n)} W_n' = -W_n W_{i,j-1}^{(n)} + W_{in}^{(n)} W_{n+1,j}^{(n+1)}.$$

Hence, referring to (4.7.7) and Theorem 4.26(a),

$$\begin{aligned} W_{ij}^{(n)} W_{n+1,n}^{(n+1)} - W_{in}^{(n)} W_{n+1,j}^{(n+1)} &= -W_n \big[(W_{ij}^{(n)})' + W_{i,j-1}^{(n)}\big] \\ &= W_n W_{i,n+1;jn}^{(n+1)}, \end{aligned}$$

which proves Theorem 4.27. □

### 4.7.4 An Arbitrary Determinant

Since the functions $y_i$ are arbitrary, we may let $y_i$ be a polynomial of degree $(n-1)$. Let

$$y_i = \sum_{r=1}^{n} \frac{a_{ir} x^{r-1}}{(r-1)!}, \tag{4.7.12}$$

where the coefficients $a_{ir}$ are arbitrary. Furthermore, since $x$ is arbitrary, we may let $x = 0$ in algebraic identities. Then,

$$\begin{aligned} w_{ij} &= y_i^{(j-1)}(0) \\ &= a_{ij}. \end{aligned} \tag{4.7.13}$$

Hence, an arbitrary determinant $A_n = |a_{ij}|_n$ can be expressed in the form $(W_n)_{x=0}$ and any algebraic identity which is satisfied by an arbitrary Wronskian is valid for $A_n$.

### 4.7.5 Adjunct Functions

**Theorem.**

$$W(y_1, y_2, \ldots, y_n) W(W^{1n}, W^{2n}, \ldots, W^{nn}) = 1.$$

PROOF. Since

$$\big|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-2)}\ \mathbf{C}^{(r)}\big| = \begin{cases} 0, & 0 \le r \le n-2 \\ W, & r = n-1, \end{cases}$$

it follows by expanding the determinant by elements from its last column and scaling the cofactors that

$$\sum_{i=1}^{n} y_i^{(r)} W^{in} = \delta_{r,n-1}.$$

Let

$$\varepsilon_{rs} = \sum_{i=1}^{n} y_i^{(r)} (W^{in})^{(s)}. \tag{4.7.14}$$

Then,

$$\varepsilon'_{rs} = \varepsilon_{r+1,s} + \varepsilon_{r,s+1} \tag{4.7.15}$$

and

$$\varepsilon_{r0} = \delta_{r,n-1}. \tag{4.7.16}$$

Differentiating (4.7.16) repeatedly and applying (4.7.15), it is found that

$$\varepsilon_{rs} = \begin{cases} 0, & r+s < n-1 \\ (-1)^s, & r+s = n-1. \end{cases} \tag{4.7.17}$$

Hence,

$$W(y_1, y_2, \ldots, y_n) W(W^{1n}, W^{2n}, \ldots, W^{nn})$$
$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_n \begin{vmatrix} W^{1n} & (W^{1n})' & \cdots & (W^{1n})^{(n-1)} \\ W^{2n} & (W^{2n})' & \cdots & (W^{2n})^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ W^{nn} & (W^{nn})' & \cdots & (W^{nn})^{(n-1)} \end{vmatrix}_n$$
$$= \begin{vmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \cdots & \cdots & \varepsilon_{0,n-2} & \varepsilon_{0,n-1} \\ \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1,n-3} & \varepsilon_{1,n-2} & \star \\ \varepsilon_{20} & \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2,n-3} & \star & \star \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon_{n-2,0} & \varepsilon_{n-2,1} & \star & \cdots & \star & \star & \star \\ \varepsilon_{n-1,0} & \star & \star & \cdots & \star & \star & \star \end{vmatrix}_n . \tag{4.7.18}$$

From (4.7.17), it follows that those elements which lie above the secondary diagonal are zero: those on the secondary diagonal from bottom left to top right are

$$1, -1, 1, \ldots, (-1)^{n+1}$$

and the elements represented by the symbol $\star$ are irrelevant to the value of the determinant, which is 1 for all values of $n$. The theorem follows. □

The set of functions $\{W^{1n}, W^{2n}, \ldots, W^{nn}\}$ are said to be adjunct to the set $\{y_1, y_2, \ldots, y_n\}$.

**Exercise.** Prove that

$$W(y_1, y_2, \ldots, y_n) W(W^{r+1,n}, W^{r+2,n}, \ldots, W^{nn}) = W(y_1, y_2, \ldots, y_r),$$
$$1 \le r \le n-1,$$

by raising the order of the second Wronskian from $(n-r)$ to $n$ in a manner similar to that employed in the section of the Jacobi identity.

### *4.7.6 Two-Way Wronskians*

Let

$$W_n = |f^{(i+j-2)}|_n = |D^{i+j-2} f|_n, \quad D = \frac{d}{dx},$$

$$= \begin{vmatrix} f & f' & f'' & \cdots & f^{(n-1)} \\ f' & f'' & f''' & \cdots & \cdots \\ f'' & f''' & f^{(4)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f^{(n-1)} & \cdots & \cdots & \cdots & f^{(2n-2)} \end{vmatrix}_n . \tag{4.7.19}$$

Then, the rows and columns satisfy the relation

$$\begin{aligned} \mathbf{R}'_i &= \mathbf{R}_{i+1}, \\ \mathbf{C}'_j &= \mathbf{C}_{j+1}, \end{aligned} \tag{4.7.20}$$

which contrasts with the simple Wronskian defined above in which only one of these relations is valid. Determinants of this form are known as two-way or double Wronskians. They are also Hankelians. A more general two-way Wronskian is the determinant

$$W_n = \left| D_x^{i-1} D_y^{j-1}(f) \right|_n \tag{4.7.21}$$

in which

$$\begin{aligned} D_x(\mathbf{R}_i) &= \mathbf{R}_{i+1}, \\ D_y(\mathbf{C}_j) &= \mathbf{C}_{j+1}. \end{aligned} \tag{4.7.22}$$

Two-way Wronskians appear in Section 6.5 on Toda equations.

**Exercise.** Let $A$ and $B$ denote Wronskians of order $n$ whose columns are defined as follows:

In $A$,

$$\mathbf{C}_1 = \begin{bmatrix} 1 \; x \; x^2 \cdots x^{n-1} \end{bmatrix}, \quad \mathbf{C}_j = D_x(\mathbf{C}_{j-1}).$$

In $B$,

$$\mathbf{C}_1 = \begin{bmatrix} 1 \; y \; y^2 \cdots y^{n-1} \end{bmatrix}, \quad \mathbf{C}_j = D_y(\mathbf{C}_{j-1}).$$

Now, let $E$ denote the hybrid determinant of order $n$ whose first $r$ columns are identical with the first $r$ columns of $A$ and whose last $s$ columns are identical with the first $s$ columns of $B$, where $r + s = n$. Prove that

$$E = \begin{bmatrix} 0! \; 1! \; 2! \cdots (r-1)! \end{bmatrix} \begin{bmatrix} 0! \; 1! \; 2! \cdots (s-1)! \end{bmatrix} (y-x)^{rs}. \quad \text{(Corduneanu)}$$

## 4.8 Hankelians 1

### *4.8.1 Definition and the $\phi_m$ Notation*

A Hankel determinant $A_n$ is defined as

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = f(i+j). \tag{4.8.1}$$

It follows that

$$a_{ji} = a_{ij},$$

so that Hankel determinants are symmetric, but it also follows that

$$a_{i+k,j-k} = a_{ij}, \qquad k = \pm 1, \pm 2, \dots . \tag{4.8.2}$$

In view of this additional property, Hankel determinants are described as persymmetric. They may also be called Hankelians.

A single-suffix notation has an advantage over the usual double-suffix notation in some applications.

Put

$$a_{ij} = \phi_{i+j-2}. \tag{4.8.3}$$

Then,

$$A_n = \begin{vmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_{n-1} \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{n-1} & \phi_n & \phi_{n+1} & \cdots & \phi_{2n-2} \end{vmatrix}_n, \tag{4.8.4}$$

which may be abbreviated to

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2. \tag{4.8.5}$$

In column vector notation,

$$A_n = \left|\mathbf{C}_0 \ \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_n,$$

where

$$\mathbf{C}_j = \left[\phi_j \ \phi_{j+1} \ \phi_{j+2} \cdots \phi_{j+n-1}\right]^T, \quad 0 \le j \le n-1. \tag{4.8.6}$$

The cofactors satisfy $A_{ji} = A_{ij}$, but $A_{ij} \ne F(i+j)$ in general, that is, $\operatorname{adj} A$ is symmetric but not Hankelian except possibly in special cases.

The elements $\phi_2$ and $\phi_{2n-4}$ each appear in three positions in $A_n$. Hence, the cofactor

$$\begin{vmatrix} \phi_2 & \cdots & \phi_{n-1} \\ \vdots & & \vdots \\ \phi_{n-1} & \cdots & \phi_{2n-4} \end{vmatrix} \tag{4.8.7}$$

also appears in three positions in $A_n$, which yields the identities

$$A^{(n)}_{12;n-1,n} = A^{(n)}_{1n,1n} = A^{(n)}_{n-1,n;12}.$$

Similarly

$$A^{(n)}_{123;n-2,n-1,n} = A^{(n)}_{12n;1,n,n-1} = A^{(n)}_{1n,n-1;12n} = A^{(n)}_{n-2,n-1,n;123}. \tag{4.8.8}$$

Let

$$\begin{aligned} A_n &= |\phi_{i+j-2}|_n = |\phi_m|_n, & 0 \le m \le 2n-2, \\ B_n &= |x^{i+j-2}\phi_{i+j-2}|_n = |x^m\phi_m|_n, & 0 \le m \le 2n-2. \end{aligned} \tag{4.8.9}$$

**Lemma.**

**a.** $B_n = x^{n(n-1)}A_n$.
**b.** $B_{ij}^{(n)} = x^{n(n-1)-(i+j-2)}A_{ij}^{(n)}$.
**c.** $B_n^{ij} = x^{-(i+j-2)}A_n^{ij}$.

PROOF OF (A). Perform the following operations on $B_n$: Remove the factor $x^{i-1}$ from row $i$, $1 \le i \le n$, and the factor $x^{j-1}$ from column $j$, $1 \le j \le n$. The effect of these operations is to remove the factor $x^{i+j-2}$ from the element in position $(i, j)$.

The result is

$$B_n = x^{2(1+2+3+\cdots+\overline{n-1})}A_n,$$

which yields the stated result. Part (b) is proved in a similar manner, and (c), which contains scaled cofactors, follows by division. □

### *4.8.2 Hankelians Whose Elements are Differences*

The $h$ difference operator $\Delta_h$ is defined in Appendix A.8.

**Theorem.**

$$|\phi_m|_n = |\Delta_h^m\phi_0|_n;$$

*that is, a Hankelian remains unaltered in value if each $\phi_m$ is replaced by $\Delta_h^m\phi_0$.*

PROOF. *First Proof.* Denote the determinant on the left by $A$ and perform the row operations

$$\mathbf{R}_i' = \sum_{r=0}^{i-1}(-h)^r\binom{i-1}{r}\mathbf{R}_{i-r}, \quad i = n, n-1, n-2, \ldots, 2, \tag{4.8.10}$$

on $A$. The result is

$$A = \left|\Delta_h^{i-1}\phi_{j-1}\right|_n. \tag{4.8.11}$$

Now, restore symmetry by performing the same operations on the columns, that is,

$$\mathbf{C}_j' = \sum_{r=0}^{j-1}(-h)^r\binom{j-1}{r}\mathbf{C}_{j-r}, \quad j = n, n-1, n-2, \ldots, 2. \tag{4.8.12}$$

The theorem appears. Note that the values of $i$ and $j$ are taken in descending order of magnitude.

The second proof illustrates the equivalence of row and column operations on the one hand and matrix-type products on the other (Section 2.3.2).

*Second Proof.* Define a triangular matrix $\mathbf{P}(x)$ as follows:

$$\begin{aligned}\mathbf{P}(x) &= \left[\binom{i-1}{j-1} x^{i-j}\right]_n \\ &= \begin{bmatrix} 1 & & & \\ x & 1 & & \\ x^2 & 2x & 1 & \\ x^3 & 3x^2 & 3x & 1 \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots} \end{bmatrix}_n . \end{aligned} \tag{4.8.13}$$

Since $|\mathbf{P}(x)| = |\mathbf{P}^T(x)| = 1$ for all values of $x$.

$$\begin{aligned} A &= |\mathbf{P}(-h)\mathbf{A}\mathbf{P}^T(-h)|_n \\ &= \left|(-h)^{i-j}\binom{i-1}{j-1}\right|_n |\phi_{i+j-2}|_n \left|(-h)^{j-i}\binom{j-1}{i-1}\right|_n \\ &= |\alpha_{ij}|_n \end{aligned} \tag{4.8.14}$$

where, applying the formula for the product of three determinants at the end of Section 3.3.5,

$$\begin{aligned} \alpha_{ij} &= \sum_{r=1}^{i}\sum_{s=1}^{j}(-h)^{i-r}\binom{i-1}{r-1}\phi_{r+s-2}(-h)^{j-s}\binom{j-1}{s-1} \\ &= \sum_{r=0}^{i-1}\binom{i-1}{r}(-h)^{i-1-r}\sum_{s=0}^{j-1}\binom{j-1}{s}(-h)^{j-1-s}\phi_{r+s} \\ &= \sum_{r=0}^{i-1}\binom{i-1}{r}(-h)^{i-1-r}\Delta_h^{j-1}\phi_r \\ &= \Delta_h^{j-1}\sum_{r=0}^{i-1}\binom{i-1}{r}(-h)^{i-1-r}\phi_r \\ &= \Delta_h^{j-1}\Delta_h^{i-1}\phi_0 \\ &= \Delta_n^{i+j-2}\phi_0. \end{aligned} \tag{4.8.15}$$

The theorem follows. Simple differences are obtained by putting $h = 1$. □

**Exercise.** Prove that

$$\sum_{r=1}^{n}\sum_{s=1}^{n} h^{r+s-2} A_{rs}(x) = A_{11}(x-h).$$

### *4.8.3 Two Kinds of Homogeneity*

The definitions of a function which is homogeneous in its variables and of a function which is homogeneous in the suffixes of its variables are given in Appendix A.9.

**Lemma.** *The determinant $A_n = |\phi_m|_n$ is*

**a.** *homogeneous of degree $n$ in its elements and*
**b.** *homogeneous of degree $n(n-1)$ in the suffixes of its elements.*

PROOF. Each of the $n!$ terms in the expansion of $A_n$ is of the form

$$\pm\phi_{1+k_1-2}\phi_{2+k_2-2}\cdots\phi_{n+k_n-2},$$

where $\{k_r\}_1^n$ is a permutation of $\{r\}_1^n$. The number of factors in each term is $n$, which proves (a). The sum of the suffixes in each term is

$$\begin{aligned}\sum_{r=1}^{n}(r+k_r-2) &= 2\sum_{r=1}^{n} r - 2n\\ &= n(n-1),\end{aligned}$$

which is independent of the choice of $\{k_r\}_1^n$, that is, the sum is the same for each term, which proves (b). $\square$

**Exercise.** Prove that $A_{ij}^{(n)}$ is homogeneous of degree $(n-1)$ in its elements and homogeneous of degree $(n^2-n+2-i-j)$ in the suffixes of its elements. Prove also that the scaled cofactor $A_n^{ij}$ is homogeneous of degree $(-1)$ in its elements and homogeneous of degree $(2-i-j)$ in the suffixes of its elements.

### *4.8.4 The Sum Formula*

The sum formula for general determinants is given in Section 3.2.4. The sum formula for Hankelians can be expressed in the form

$$\sum_{m=1}^{n} \phi_{m+r-2}A_n^{ms} = \delta_{rs}, \quad 1 \le r, s \le n. \tag{4.8.16}$$

**Exercise.** Prove that, in addition to the sum formula,

**a.** $\displaystyle\sum_{m=1}^{n} \phi_{m+n-1}A_{im}^{(n)} = -A_{i,n+1}^{(n+1)}, \quad 1 \le i \le n,$

**b.** $\displaystyle\sum_{m=1}^{n} \phi_{m+n}A_{im}^{(n)} = A_{1n}^{(n+1)},$

where the cofactors are unscaled. Show also that there exist further sums of a similar nature which can be expressed as cofactors of determinants of orders $(n+2)$ and above.

### 4.8.5 Turanians

A Hankelian in which $a_{ij} = \phi_{i+j-2+r}$ is called a Turanian by Karlin and Szegö and others.

Let

$$T^{(n,r)} = \begin{cases} |\phi_{m+r}|_n, & 0 \le m \le 2n-2, \\ |\phi_m|_n, & r \le m \le 2n-2+r, \\ \begin{vmatrix} \phi_r & \cdots & \phi_{n-1+r} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \phi_{n-1+r} & \cdots & \phi_{2n-2+r} \end{vmatrix}_n \\ \left| \mathbf{C}_r \; \mathbf{C}_{r+1} \; \mathbf{C}_{r+2} \cdots \mathbf{C}_{n-1+r} \right|. \end{cases} \tag{4.8.17}$$

**Theorem 4.28.**

$$\begin{vmatrix} T^{(n,r+1)} & T^{(n,r)} \\ T^{(n,r)} & T^{(n,r-1)} \end{vmatrix} = T^{(n+1,r-1)} T^{(n-1,r+1)}.$$

PROOF. Denote the determinant by $T$. Then, each of the Turanian elements in $T$ is of order $n$ and is a minor of one of the corner elements in $T^{(n+1,r-1)}$. Applying the Jacobi identity (Section 3.6),

$$\begin{aligned} T &= \begin{vmatrix} T_{11}^{(n+1,r-1)} & T_{1,n+1}^{(n+1,r-1)} \\ T_{n+1,1}^{(n+1,r-1)} & T_{n+1,n+1}^{(n+1,r-1)} \end{vmatrix} \\ &= T^{(n+1,r-1)} T_{1,n+1;1,n+1}^{(n+1,r-1)} \\ &= T^{(n+1,r-1)} T^{(n-1,r+1)}, \end{aligned}$$

which proves the theorem. □

Let

$$\begin{aligned} A_n &= T^{(n,0)} = |\phi_{i+j-2}|_n, \\ F_n &= T^{(n,1)} = |\phi_{i+j-1}|_n, \\ G_n &= T^{(n,2)} = |\phi_{i+j}|_n. \end{aligned} \tag{4.8.18}$$

Then, the particular case of the theorem in which $r = 1$ can be expressed in the form

$$A_n G_n - A_{n+1} G_{n-1} = F_n^2. \tag{4.8.19}$$

This identity is applied in Section 4.12.2 on generalized geometric series.

Omit the parameter $r$ in $T^{(n,r)}$ and write $T_n$.

**Theorem 4.29.** *For all values of $r$,*

$$\begin{vmatrix} T_{11}^{(n)} & T_{1,n+1}^{(n+1)} \\ T_{n1}^{(n)} & T_{n,n+1}^{(n+1)} \end{vmatrix} - T_n T_{1n;1,n+1}^{(n+1)} = 0.$$

Proof. The identity is a particular case of Jacobi variant (A) (Section 3.6.3),

$$\begin{vmatrix} T_{ip}^{(n)} & T_{i,n+1}^{(n+1)} \\ T_{jp}^{(n)} & T_{j,n+1}^{(n+1)} \end{vmatrix} - T_n T_{ij;p,n+1}^{(n+1)} = 0, \tag{4.8.20}$$

where $(i, j, p) = (1, n, 1)$.

Let

$$\begin{aligned} A_n &= T^{(n,r)}, \\ B_n &= T^{(n,r+1)}. \end{aligned}$$

Then Theorem 4.29 is satisfied by both $A_n$ and $B_n$. □

**Theorem 4.30.** *For all values of r,*

**a.** $A_n B_{n+1,n}^{(n+1)} - B_n A_{n+1,n}^{(n+1)} + A_{n+1} B_{n-1} = 0.$
**b.** $B_{n-1} A_{n+1,n}^{(n+1)} - A_n B_{n,n-1}^{(n)} + A_{n-1} B_n = 0.$

Proof.

$$\begin{aligned} B_n &= (-1)^n A_{1,n+1}^{(n+1)}, \\ B_{n+1,n}^{(n+1)} &= (-1)^n A_{n1}^{(n+1)}, \\ B_{n-1} &= (-1)^{n-1} A_{1n}^{(n)} \\ &= (-1)^n A_{n,n+1;1,n+1}^{(n+1)}, \\ A_{1n;n,n+1}^{(n+1)} &= A_{1,n-1}^{(n)} \\ &= (-1)^{n-1} B_{n,n-1}^{(n)}. \end{aligned} \tag{4.8.21}$$

Denote the left-hand side of (a) by $Y_n$. Then, applying the Jacobi identity to $A_{n+1}$,

$$\begin{aligned} (-1)^n Y_n &= \begin{vmatrix} A_{n1}^{(n+1)} & A_{n,n+1}^{(n+1)} \\ A_{n+1,1}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} - A_{n+1} A_{n,n+1;1,n+1}^{(n+1)} \\ &= 0, \end{aligned}$$

which proves (a).

The particular case of (4.8.20) in which $(i, j, p) = (n, 1, n)$ and $T$ is replaced by $A$ is

$$\begin{vmatrix} A_{n-1} & A_{n,n+1}^{(n+1)} \\ A_{1n}^{(n)} & A_{1,n+1}^{(n+1)} \end{vmatrix} - A_n A_{n1;n,n+1}^{(n+1)} = 0. \tag{4.8.22}$$

The application of (4.8.21) yields (b). □

This theorem is applied in Section 6.5.1 on Toda equations.

### 4.8.6 Partial Derivatives with Respect to $\phi_m$

In $A_n$, the elements $\phi_m$, $\phi_{2n-2-m}$, $0 \le m \le n-2$, each appear in $(m+1)$ positions. The element $\phi_{n-1}$ appears in $n$ positions, all in the secondary diagonal. Hence, $\partial A_n/\partial\phi_m$ is the sum of a number of cofactors, one for each appearance of $\phi_m$. Discarding the suffix $n$,

$$\frac{\partial A}{\partial \phi_m} = \sum_{p+q=m+2} A_{pq}. \tag{4.8.23}$$

For example, when $n \ge 4$,

$$\begin{aligned}\frac{\partial A}{\partial \phi_3} &= \sum_{p+q=5} A_{pq} \\ &= A_{41} + A_{32} + A_{23} + A_{14}.\end{aligned}$$

By a similar argument,

$$\frac{\partial A_{ij}}{\partial \phi_m} = \sum_{p+q=m+2} A_{ip,jq}, \tag{4.8.24}$$

$$\frac{\partial A_{ir,js}}{\partial \phi_m} = \sum_{p+q=m+2} A_{irp,jsq}. \tag{4.8.25}$$

Partial derivatives of the scaled cofactors $A^{ij}$ and $A^{ir,js}$ can be obtained from (4.8.23)–(4.8.25) with the aid of the Jacobi identity:

$$\frac{\partial A^{ij}}{\partial \phi_m} = -\sum_{p+q=m+2} A^{iq}A^{pj} \tag{4.8.26}$$

$$= \sum_{p+q=m+2} \begin{vmatrix} A^{ij} & A^{iq} \\ A^{pj} & \bullet \end{vmatrix}. \tag{4.8.27}$$

The proof is simple.

**Lemma.**

$$\frac{\partial A^{ir,js}}{\partial \phi_m} = \sum_{p+q=m+2} \begin{vmatrix} A^{ij} & A^{is} & A^{iq} \\ A^{rj} & A^{rs} & A^{rq} \\ A^{pj} & A^{ps} & \bullet \end{vmatrix}, \tag{4.8.28}$$

*which is a development of (4.8.27).*

PROOF.

$$\begin{aligned}\frac{\partial A^{ir,js}}{\partial \phi_m} &= \frac{1}{A^2}\left[A\frac{\partial A_{ir,js}}{\partial \phi_m} - A_{ir,js}\frac{\partial A}{\partial \phi_m}\right] \\ &= \frac{1}{A^2}\sum_{p,q}\left[AA_{irp,jsq} - A_{ir,js}A_{pq}\right] \\ &= \sum_{p,q}\left[A^{irp,jsq} - A^{ir,js}A^{pq}\right].\end{aligned} \tag{4.8.29}$$

The lemma follows from the second-order and third-order Jacobi identities. □

### *4.8.7 Double-Sum Relations*

When $A_n$ is a Hankelian, the double-sum relations (A)–(D) in Section 3.4 with $f_r = g_r = \frac{1}{2}$ can be expressed as follows. Discarding the suffix $n$,

$$\frac{A'}{A} = D(\log A) = \sum_{m=0}^{2n-2} \phi'_m \sum_{p+q=m+2} A^{pq}, \tag{A$_1$}$$

$$(A^{ij})' = -\sum_{m=0}^{2n-2} \phi'_m \sum_{p+q=m+2} A^{ip}A^{jq}, \tag{B$_1$}$$

$$\sum_{m=0}^{2n-2} \phi_m \sum_{p+q=m+2} A^{pq} = n, \tag{C$_1$}$$

$$\sum_{m=0}^{2n-2} \phi_m \sum_{p+q=m+2} A^{ip}A^{jq} = A^{ij}. \tag{D$_1$}$$

Equations (C$_1$) and (D$_1$) can be proved by putting $a_{ij} = \phi_{i+j-2}$ in (C) and (D), respectively, and rearranging the double sum, but they can also be proved directly by taking advantage of the first kind of homogeneity of Hankelians and applying the Euler theorem in Appendix A.9.

$A_n$ and $A_{ij}^{(n)}$ are homogeneous polynomial functions of their elements of degrees $n$ and $n-1$, respectively, so that $A_n^{ij}$ is a homogeneous function of degree $(-1)$. Hence, denoting the sums in (C$_1$) and (D$_1$) by $S_1$ and $S_2$,

$$\begin{aligned} AS_1 &= \sum_{m=0}^{2n-2} \phi_m \frac{\partial A}{\partial \phi_m} \\ &= nA, \\ S_2 &= -\sum_{m=0}^{2n-2} \phi_m \frac{\partial A^{ij}}{\partial \phi_m} \\ &= A^{ij}. \end{aligned}$$

which prove (C$_1$) and (D$_1$).

**Theorem 4.31.**

$$\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+2} A^{pq} = n(n-1), \tag{C$_2$}$$

$$\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+1} A^{ip}A^{jq} = (i+j-2)A^{ij}. \tag{D$_2$}$$

These can be proved by putting $a_{ij} = \phi_{i+j-2}$ and $f_r = g_r = r-1$ in (C) and (D), respectively, and rearranging the double sum, but they can also be proved directly by taking advantage of the second kind of homogeneity of Hankelians and applying the modified Euler theorem in Appendix A.9.

PROOF. $A_n$ and $A_n^{ij}$ are homogeneous functions of degree $n(n-1)$ and $(2-i-j)$, respectively, in the suffixes of their elements. Hence, denoting the sums by $S_1$ and $S_2$. respectively,

$$\begin{aligned} AS_1 &= \sum_{m=1}^{2n-2} m\phi_m \frac{\partial A}{\partial \phi_m} \\ &= n(n-1)A, \\ S_2 &= -\sum_{m=1}^{2n-2} m\phi_m \frac{\partial A^{ij}}{\partial \phi_m} \\ &= -(2-i-j)A^{ij}. \end{aligned}$$

The theorem follows. □

**Theorem 4.32.**

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(r+s-2)\phi_{r+s-3}A^{rs} = 0, \tag{E}$$

*which can be rearranged in the form*

$$\sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{pq} = 0 \tag{E$_1$}$$

*and*

$$\begin{aligned} \sum_{r=1}^{n}\sum_{s=1}^{n}(r+s-2)\phi_{r+s-3}A^{ir}A^{sj} &= iA^{i+1,j} + jA^{i,j+1} \\ &= 0, \qquad (i,j) = (n,n). \end{aligned} \tag{F}$$

*which can be rearranged in the form*

$$\begin{aligned} \sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{ip}A^{jq} &= iA^{i+1,j} + jA^{i,j+1} \\ &= 0, \qquad (i,j) = (n,n). \end{aligned} \tag{F$_1$}$$

Proof of (F). Denote the sum by $S$ and apply the Hankelian relation $\phi_{r+s-3} = a_{r,s-1} = a_{r-1,s}$.

$$\begin{aligned} S &= \sum_{s=1}^{n}(s-1)A^{sj}\sum_{r=1}^{n} a_{r,s-1}A^{ir} + \sum_{r=1}^{n}(r-1)A^{ir}\sum_{s=1}^{n} a_{r-1,s}A^{sj} \\ &= \sum_{s=1}^{n}(s-1)A^{sj}\delta_{s-1,i} + \sum_{r=1}^{n}(r-1)A^{ir}\delta_{r-1,j}. \end{aligned}$$

The proof of (F) follows. Equation (E) is proved in a similar manner. □

## *Exercises*

Prove the following:

**1.** $\displaystyle\sum_{p+q=m+2} A^{ij,pq} = 0.$

**2.** $\displaystyle\sum_{m=0}^{2n-2} \phi_m \sum_{q+q=m+2} A_{ip,jq} = (n-1)A_{ij}.$

**3.** $\displaystyle\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+2} A_{ip,jq} = (n^2-n-i-j+2)A_{ij}.$

**4.** $\displaystyle\sum_{m=0}^{2n-2} \phi_m \sum_{p+q=m+2} A^{ijp,hkq} = nA^{ij,hk}.$

**5.** $\displaystyle\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+2} A^{ijp,hkq} = (n^2-n-i-j-h-k-4)A^{ij,hk}.$

**6.** $\displaystyle\sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{ijp,hkq}$
$= iA^{i+1,j;hk} + jA^{i,j+1;hk} + hA^{ij;h+1,k} + kA^{ij;h,k+1}.$

**7.** $\displaystyle\sum_{m=0}^{2n-2} \sum_{p+q=m+2} \phi_{p+r-1}\phi_{q+r-1}A^{pq} = \phi_{2r}, \quad 0 \le r \le n-1.$

**8.** $\displaystyle\sum_{m=1}^{2n-2} m \sum_{p+q=m+2} \phi_{p+r-1}\phi_{q+r-1}A^{pq} = 2r\phi_{2r}, \quad 0 \le r \le n-1.$

**9.** Prove that

$$\sum_{r=1}^{n-1} rA^{r+1,j} \sum_{m=1}^{n} \phi_{m+r-2}A^{im} = iA^{i+1,j}$$

by applying the sum formula for Hankelians and, hence, prove ($F_1$) directly. Use a similar method to prove ($E_1$) directly.

# 4.9 Hankelians 2

## *4.9.1 The Derivatives of Hankelians with Appell Elements*

The Appell polynomial

$$\phi_m = \sum_{r=0}^{m} \binom{m}{r} \alpha_r x^{m-r} \tag{4.9.1}$$

and other functions which satisfy the Appell equation

$$\phi'_m = m\phi_{m-1}, \quad m = 1, 2, 3, \dots, \tag{4.9.2}$$

play an important part in the theory of Hankelians. Extensive notes on these functions are given in Appendix A.4.

**Theorem 4.33.** *If*

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

*where $\phi_m$ satisfies the Appell equation, then*

$$A'_n = \phi'_0 A_{11}^{(n)}.$$

PROOF. Split off the $m = 0$ term from the double sum in relation $(\mathrm{A}_1)$ in Section 4.8.7:

$$\begin{aligned}\frac{A'}{A} &= \phi'_0 \sum_{p+q=2} A^{pq} + \sum_{m=1}^{2n-2} \phi'_m \sum_{p+q=m+2} A^{pq} \\ &= \phi'_0 A^{11} + \sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{pq}.\end{aligned}$$

The theorem follows from $(\mathrm{E}_1)$ and remains true if the Appell equation is generalized to

$$\phi'_m = mF\phi_{m-1}, \quad F = F(x). \tag{4.9.3}$$

$\square$

**Corollary.** *If $\phi_m$ is an Appell polynomial, then $\phi_0 = \alpha_0 =$ constant, $A' =$ 0, and, hence, $A$ is independent of $x$, that is,*

$$|\phi_m(x)|_n = |\phi_m(0)|_n = |\alpha_m|_n, \quad 0 \le m \le 2n-2. \tag{4.9.4}$$

*This identity is one of a family of identities which appear in Section 5.6.2 on distinct matrices with nondistinct determinants.*

If $\phi_m$ satisfies (4.9.3) and $\phi_0 =$ constant, it does not follows that $\phi_m$ is an Appell polynomial. For example, if

$$\phi_m = (1-x^2)^{-m/2} P_m,$$

where $P_m$ is the Legendre polynomial, then $\phi_m$ satisfies (4.9.3) with

$$F = (1 - x^2)^{-3/2}$$

and $\phi_0 = P_0 = 1$, but $\phi_m$ is not a polynomial. These relations are applied in Section 4.12.1 to evaluate $|P_m|_n$.

## *Examples*

**1.** If

$$\phi_m = \frac{1}{m+1}\sum_{r=1}^{k} b_r\{f(x) + c_r\}^{m+1},$$

where $\sum_{r=1}^{k} b_r = 0$, $b_r$ and $c_r$ are independent of $x$, and $k$ is arbitrary, then

$$\phi_m' = mf'(x)\phi_{m-1},$$
$$\phi_0 = \sum_{r=1}^{k} b_r c_r = \text{constant}.$$

Hence, $A = |\phi_m|_n$ is independent of $x$.

**2.** If

$$\phi_m(x, \xi) = \frac{1}{m+1}\left[(\xi + x)^{m+1} - c(\xi - 1)^{m+1} + (c-1)\xi^{m+1}\right],$$

then

$$\frac{\partial \phi_m}{\partial \xi} = m\phi_{m-1},$$
$$\phi_0 = x + c.$$

Hence, $A$ is independent of $\xi$. This relation is applied in Section 4.11.4 on a nonlinear differential equation.

## *Exercises*

**1.** Denote the three cube roots of unity by 1, $\omega$, and $\omega^2$, and let$A = |\phi_m|_n$, $0 \le m \le 2n-2$, where

**a.** $\phi_m = \dfrac{1}{3(m+1)}\left[(x+b+c)^{m+1} + \omega(x+\omega c)^{m+1} + \omega^2(x+\omega^2 c)^{m+1}\right],$

**b.** $\phi_m = \dfrac{1}{3(m+1)}\left[(x+b+c)^{m+1} + \omega^2(x+\omega c)^{m+1} + \omega(x+\omega^2 c)^{m+1}\right],$

**c.** $\phi_m = \dfrac{1}{3(m+1)(m+2)}\left[(x+c)^{m+2} + \omega^2(x+\omega c)^{m+2} + \omega(x+\omega^2 c)^{m+2}\right].$

Prove that $\phi_m$ and hence also $A$ is real in each case, and that in cases (a) and (b), $A$ is independent of $x$, but in case (c), $A' = cA_{11}$.

**2.** The Yamazaki–Hori determinant $A_n$ is defined as follows:

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \frac{1}{m+1}\left[p^2(x^2-1)^{m+1} + q^2(y^2-1)^{m+1}\right], \quad p^2+q^2=1.$$

Let

$$B_n = |\psi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\psi_m = \frac{\phi_m}{(x^2-y^2)^{m+1}}\,.$$

Prove that

$$\frac{\partial \psi_m}{\partial x} = mF\psi_{m-1},$$

where

$$F = -\frac{2x(y^2-1)}{(x^2-y^2)^2}\,.$$

Hence, prove that

$$\begin{aligned}\frac{\partial B_n}{\partial x} &= FB_{11}^{(n)},\\ (x^2-y^2)\frac{\partial A_n}{\partial x} &= 2x\left[n^2A_n - (y^2-1)A_{11}^{(n)}\right].\end{aligned}$$

Deduce the corresponding formulas for $\partial B_n/\partial y$ and $\partial A_n/\partial y$ and hence prove that $A_n$ satisfies the equation

$$\left(\frac{x^2-1}{x}\right)z_x + \left(\frac{y^2-1}{y}\right)z_y = 2n^2 z.$$

**3.** If $A_n = |\phi_m|_n$, $0 \le m \le 2n-2$, where $\phi_m$ satisfies the Appell equation, prove that

**a.** $(A_n^{ij})' = -\phi_0' A_n^{i1}A_n^{j1} - (iA_n^{i+1,j} + jA_n^{i,j+1}), \quad (i,j) \ne (n,n),$
**b.** $(A_n^{nn})' = -\phi_0'(A_n^{1n})^2.$

**4.** Apply Theorem 4.33 and the Jacobi identity to prove that

$$\left(\frac{A_n}{A_{n-1}}\right)' = \phi_0'\left(\frac{A_{1n}^{(n)}}{A_{n-1}}\right)^2.$$

Hence, prove (3b).

**5.** If

$$\begin{aligned}A_n &= |\phi_m|_n, \quad 0 \le m \le 2n-2,\\ F_n &= |\phi_m|_n, \quad 1 \le m \le 2n-1,\\ G_n &= |\phi_m|_n, \quad 2 \le m \le 2n,\end{aligned}$$

where $\phi_m$ is an Appell polynomial, apply Exercise 3a in which the cofactors are scaled to prove that

$$D(A_{ij}^{(n)}) = -\big(iA_{i+1,j}^{(n)} + jA_{i,j+1}^{(n)}\big)$$

in which the cofactors are unscaled. Hence, prove that

**a.** $D^r(F_n) = (-1)^{n+r} r! A_{r+1,n+1}^{(n+1)}, \quad 0 \le r \le n;$
**b.** $D^n(F_n) = n! A_n;$
**c.** $F_n$ is a polynomial of degree $n$;
**d.** $D^r(G_n) = (-1)^r r! \sum_{p+q=r+2} A_{pq}^{(n+1)}, \quad 0 \le r \le 2n;$
**e.** $D^{2n}(G_n) = (2n)! A_n;$
**f.** $G_n$ is a polynomial of degree $2n$.

**6.** Let $B_n$ denote the determinant of order $(n+1)$ obtained by bordering $A_n(0)$ by the row

$$\mathbf{R} = \big[1 \;\; -x \;\; x^2 \;\; -x^3 \cdots (-x)^{n-1} \;\; \bullet\big]_{n+1}$$

at the bottom and the column $\mathbf{R}^T$ on the right. Prove that

$$B_n = -\sum_{r=0}^{2n-2} (-x)^r \sum_{p+q=r+2} A_{pq}^{(n)}(0).$$

Hence, by applying a formula in the previous exercise and then the Maclaurin expansion formula, prove that

$$B_n = -G_{n-1}.$$

**7.** Prove that

$$D^r(A_{ij}) = \frac{(-1)^r r!}{(i-1)!(j-1)!} \sum_{s=0}^{r} \frac{(i+r-s-1)!(j+s-1)!}{s!(r-s)!} A_{i+r-s,j+s}.$$

**8.** Apply the double-sum relation ($\mathrm{A}_1$) in Section 4.8.7 to prove that $G_n$ satisfies the differential equation

$$\sum_{m=0}^{2n-1} \frac{(-1)^m \phi_m D^{m+1}(G_n)}{m!} = 0.$$

### 4.9.2 The Derivatives of Turanians with Appell and Other Elements

Let

$$T = T^{(n,r)} = \left|\mathbf{C}_r\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|_n, \tag{4.9.5}$$

where

$$\begin{aligned}\mathbf{C}_j &= \left[\phi_j\ \phi_{j+1}\ \phi_{j+2} \cdots \phi_{j+n-1}\right]^T,\\ \phi'_m &= mF\phi_{m-1}.\end{aligned}$$

**Theorem 4.34.**

$$T' = rF\left|\mathbf{C}_{r-1}\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|.$$

PROOF.

$$\mathbf{C}'_j = F\left(j\mathbf{C}_{j-1} + \mathbf{C}^*_j\right),$$

where

$$\mathbf{C}^*_j = \left[0\ \phi_j\ 2\phi_{j+1}\ 3\phi_{j+2} \cdots (n-1)\phi_{j+n-2}\right]^T.$$

Hence,

$$\begin{aligned}T' &= \sum_{j=r}^{r+n-1} \left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}_{j-1}\ \mathbf{C}'_j \cdots \mathbf{C}_{r+n-1}\right|\\ &= F\sum_{j=r}^{r+n-1} \left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}_{j-1}(j\mathbf{C}_{j-1} + \mathbf{C}^*_j) \cdots \mathbf{C}_{r+n-1}\right|\\ &= rF\left|\mathbf{C}_{r-1}\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|\\ &\quad + F\sum_{j=r}^{r+n-1} \left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}^*_j \cdots \mathbf{C}_{r+n-1}\right|\end{aligned}$$

after discarding determinants with two identical columns. The sum is zero by Theorem 3.1 in Section 3.1 on cyclic dislocations and generalizations. The theorem follows. □

The column parameters in the above definition of $T$ are consecutive. If they are not consecutive, the notation

$$T_{j_1 j_2 \ldots j_n} = \left|\mathbf{C}_{j_1}\ \mathbf{C}_{j_2} \cdots \mathbf{C}_{j_r} \cdots \mathbf{C}_{j_n}\right| \tag{4.9.6}$$

is convenient.

$$T'_{j_1 j_2 \ldots j_n} = F\sum_{r=1}^{n} j_r\left|\mathbf{C}_{j_1}\ \mathbf{C}_{j_2} \cdots \mathbf{C}_{(j_r-1)} \cdots \mathbf{C}_{j_n}\right|. \tag{4.9.7}$$

Higher derivatives may be found by repeated application of this formula, but no simple formula for $D^k(T_{j_1 j_2 \ldots j_n})$ has been found. However, the method can be illustrated adequately by taking the particular case in which $(n, r) = (4, 3)$ and $\phi_m$ is an Appell polynomial so that $F = 1$.

Let

$$T = \left|\mathbf{C}_3\ \mathbf{C}_4\ \mathbf{C}_5\ \mathbf{C}_6\right| = T_{3456}.$$

Then

$$\begin{aligned}
D(T) &= 3T_{2456},\\
D^2(T)/2! &= 3T_{1456} + 6T_{2356},\\
D^3(T)/3! &= T_{0456} + 8T_{1356} + 10T_{2346},\\
\cdots\cdots&\cdots\cdots\cdots\cdots\cdots\cdots\\
D^9(T)/9! &= T_{0126} + 8T_{0135} + 10T_{0234},\\
D^{10}(T)/10! &= 3T_{0125} + 6T_{0134},\\
D^{11}(T)/11! &= 3T_{0124},\\
D^{12}(T)/12! &= T_{0123},\\
&= |\phi_m|_4, \quad 0 \le m \le 6\\
&= \text{constant}.
\end{aligned} \tag{4.9.8}$$

The array of coefficients is symmetric about the sixth derivative. This result and several others of a similar nature suggest the following conjecture.

**Conjecture.**

$$\begin{aligned}
D^{nr}\{T^{(n,r)}\} &= (nr)!|\phi_m|_n, \quad 0 \le m \le 2n-2\\
&= \text{constant}.
\end{aligned}$$

Assuming this conjecture to be valid, $T^{(n,r)}$ is a polynomial of degree $nr$ and not $n(n+r-1)$ as may be expected by examining the product of the elements in the secondary diagonal. Hence, the loss of degree due to cancellations is $n(n-1)$.

Let

$$T = T^{(n,r)} = \left|\mathbf{C}_r\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2}\cdots\mathbf{C}_{r+n-1}\right|_n,$$

where

$$\begin{aligned}
\mathbf{C}_j &= \left[\psi_{r+j-1}\ \psi_{r+j}\ \psi_{r+j+1}\cdots\psi_{r+j+n-2}\right]_n^T\\
\psi_m &= \frac{f^{(m)}(x)}{m!}, \quad f(x) \text{ arbitrary}\\
\psi'_m &= (m+1)\psi_{m+1}.
\end{aligned} \tag{4.9.9}$$

**Theorem 4.35.**

$$\begin{aligned}
T' &= (2n-1+r)\left|\mathbf{C}_r\ \mathbf{C}_{r+1}\cdots\mathbf{C}_{r+n-2}\ \mathbf{C}_{r+n}\right|_n\\
&= -(2n-1+r)T^{(n+1,r)}_{n+1,n}.
\end{aligned}$$

Proof. The sum formula for $T$ can be expressed in the form

$$\sum_{j=1}^{n} \psi_{r+i+j-1} T_{ij}^{(n,r)} = -\delta_{in} T_{n+1,n}^{(n+1,r)}, \tag{4.9.10}$$

$$\mathbf{C}'_j = \left[(r+j)\psi_{r+j}\ (r+j+1)\psi_{r+j+1} \cdots (r+j+n-1)\psi_{r+j+n-1}\right]_n^T. \tag{4.9.11}$$

Let

$$\begin{aligned} \mathbf{C}^*_j &= \mathbf{C}'_j - (r+j)\mathbf{C}_{j+1} \\ &= \left[0\ \psi_{r+j+1}\ 2\psi_{r+j+2} \cdots (n-1)\psi_{r+j+n-1}\right]_n^T. \end{aligned} \tag{4.9.12}$$

Differentiating the columns of $T$,

$$T' = \sum_{j=1}^{n} U_j,$$

where

$$U_j = \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}'_j\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n, \quad 1 \le j \le n.$$

Let

$$\begin{aligned} V_j &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}^*_j\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n, \quad 1 \le j \le n \\ &= \sum_{i=2}^{n} (i-1)\psi_{r+i+j-1} T_{ij}. \end{aligned} \tag{4.9.13}$$

Then, performing an elementary column operation on $U_j$,

$$\begin{aligned} U_j &= V_j, \quad 1 \le j \le n-1 \\ U_n &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}'_n\right| \\ &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}^*_n\right| + (r+n)\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}_{n+1}\right| \\ &= V_n - (r+n) T_{n+1,n}^{(n+1,r)}. \end{aligned} \tag{4.9.14}$$

Hence,

$$\begin{aligned} T' + (r+n) T_{n+1,n}^{(n+1,r)} &= \sum_{j=1}^{n} V_j \\ &= \sum_{j=1}^{n} (i-1) \sum_{j=1}^{n} \psi_{r+i+j-1} T_{ij} \\ &= -T_{n+1,n}^{(n+1,r)} \sum_{i=2}^{n} (i-1)\delta_{in} \\ &= -(n-1) T_{n+1,n}^{(n+1,r)}. \end{aligned}$$

The theorem follows. □

**Theorem 4.36.**

$$D(T_{11}^{(n,r)}) = -(2n+r-1)T_{n,n-1}^{(n,r+2)}.$$

Proof.

$$T_{11}^{(n,r)} = T^{(n-1,r+2)}.$$

The theorem follows by adjusting the parameters in Theorem 4.35. Both these theorems are applied in Section 6.5.3 on the Milne–Thomson equation. □

### *4.9.3 Determinants with Simple Derivatives of All Orders*

Let $\mathbf{Z}_r$ denote the column vector with $(n+1)$ elements defined as

$$\mathbf{Z}_r = \left[0_r\ \phi_0\ \phi_1\ \phi_2 \cdots \phi_{n-r}\right]_{n+1}^T, \quad 1 \le r \le n, \tag{4.9.15}$$

where $0_r$ denotes an unbroken sequence of $r$ zero elements and $\phi_m$ is an Appell polynomial.

Let

$$B = \left|\mathbf{Z}_1\ \mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_{n+1}, \tag{4.9.16}$$

where $\mathbf{C}_j$ is defined in (4.9.5). Differentiating $B$ repeatedly, it is found that, apart from a constant factor, only the first column changes:

$$D^r(B) = (-1)^r r! \left|\mathbf{Z}_{r+1}\ \mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_{n+1}, \quad 0 \le r \le n-1.$$

Hence

$$\begin{aligned} D^{n-1}(B) &= (-1)^{n-1}(n-1)!\phi_0 \left|\mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_n \\ &= (-1)^{n-1}(n-1)!\phi_0 |\phi_m|_n, \quad 0 \le m \le 2n-2 \\ &= \text{constant}; \end{aligned}$$

that is, $B$ is a polynomial of degree $(n-1)$ and not $(n^2-1)$, as may be expected by examining the product of the elements in the secondary diagonal of $B$. Once again, the loss of degree due to cancellations is $n(n-1)$.

### *Exercise*

Let

$$S_m = \sum_{r+s=m} \phi_r \phi_s.$$

This function appears in Exercise 2 at the end of Appendix A.4 on Appell polynomials. Also, let

$$\begin{aligned} \mathbf{C}_j &= \left[S_{j-1}\ S_j\ S_{j+1} \cdots S_{j+n-2}\right]_n^T, \quad 1 \le j \le n, \\ \mathbf{K} &= \left[\bullet\ S_0\ S_1\ S_2 \cdots S_{n-2}\right]_n^T, \\ E &= |S_m|_n, \quad 0 \le m \le 2n-2. \end{aligned}$$

Prove that

$$D^r(E) = (-1)^{r+1} r! \sum_{i=2}^{n} S_{i-2} E_{ir}$$
$$= (-1)^{r+1} r! \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_{r-1} \ \mathbf{K} \ \mathbf{C}_{r+1} \cdots \mathbf{C}_n \right|_n.$$

## 4.10 Henkelians 3

### *4.10.1 The Generalized Hilbert Determinant*

The generalized Hilbert determinant $K_n$ is defined as

$$K_n = K_n(h) = |k_{ij}|_n,$$

where

$$k_{ij} = \frac{1}{h+i+j-1}, \quad h \neq 1-i-j, \quad 1 \le i,j \le n. \tag{4.10.1}$$

In some detail,

$$K_n = \begin{vmatrix} \frac{1}{h+1} & \frac{1}{h+2} & \cdots & \frac{1}{h+n} \\ \frac{1}{h+2} & \frac{1}{h+3} & \cdots & \frac{1}{h+n+1} \\ \dots & \dots & \dots & \dots \\ \frac{1}{h+n} & \frac{1}{h+n+1} & \cdots & \frac{1}{h+2n-1} \end{vmatrix}_n . \tag{4.10.2}$$

$K_n$ is of fundamental importance in the evaluation of a number of determinants, not necessarily Hankelians, whose elements are related to $k_{ij}$. The values of such determinants and their cofactors can, in some cases, be simplified by expressing them in terms of $K_n$ and its cofactors. The given restrictions on $h$ are the only restrictions on $h$ which may therefore be regarded as a continuous variable. All formulas in $h$ given below on the assumption that $h$ is zero, a positive integer, or a permitted negative integer can be modified to include other permitted values by replacing, for example, $(h+n)!$ by $\Gamma(h+n+1)$.

Let $V_{nr} = V_{nr}(h)$ denote a determinantal ratio (not a scaled cofactor) defined as

$$V_{nr} = \frac{1}{K_n} \begin{vmatrix} \frac{1}{h+1} & \frac{1}{h+2} & \cdots & \frac{1}{h+n} \\ \frac{1}{h+2} & \frac{1}{h+3} & \cdots & \frac{1}{h+n+1} \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \cdots & 1 \\ \dots & \dots & \dots & \dots \\ \frac{1}{h+n} & \frac{1}{h+n+1} & \cdots & \frac{1}{h+2n-1} \end{vmatrix}_n \quad \text{row } r, \tag{4.10.3}$$

where every element in row $r$ is 1 and all the other elements are identical with the corresponding elements in $K_n$. The following notes begin with the evaluation of $V_{nr}$ and end with the evaluation of $K_n$ and its scaled cofactor $K_n^{rs}$.

**Identities 1.**

$$V_{nr} = \sum_{j=1}^{n} K_n^{rj}, \quad 1 \le r \le n. \tag{4.10.4}$$

$$V_{nr} = \frac{(-1)^{n+r}(h+r+n-1)!}{(h+r-1)!(r-1)!(n-r)!}, \quad 1 \le r \le n. \tag{4.10.5}$$

$$V_{n1} = \frac{(-1)^{n+1}(h+n)!}{h!(n-1)!}. \tag{4.10.6}$$

$$V_{nn} = \frac{(h+2n-1)!}{(h+n-1)!(n-1)!}. \tag{4.10.7}$$

$$K_n^{rs} = \frac{V_{nr}V_{ns}}{h+r+s-1}, \quad 1 \le r,s \le n. \tag{4.10.8}$$

$$K_n^{r1} = \frac{V_{nr}V_{n1}}{h+r}. \tag{4.10.9}$$

$$K_n^{nn} = \frac{K_{n-1}}{K_n} = \frac{V_{nn}^2}{h+2n-1}. \tag{4.10.10}$$

$$K_n^{rs} = \frac{(h+r)(h+s)K_n^{r1}K_n^{s1}}{(h+r+s-1)V_{n1}^2}. \tag{4.10.11}$$

$$K_n = \frac{(n-1)!^2(h+n-1)!^2}{(h+2n-2)!(h+2n-1)!}K_{n-1}. \tag{4.10.12}$$

$$K_n = \frac{[1!\,2!\,3!\cdots(n-1)!]^2 h!(h+1)!\cdots(h+n-1)!}{(h+n)!(h+n+1)!\cdots(h+2n-1)!}. \tag{4.10.13}$$

$$(n-r)V_{nr} + (h+n+r-1)V_{n-1,r} = 0. \tag{4.10.14}$$

$$K_n \prod_{r=1}^{n} V_{nr} = (-1)^{n(n-1)/2}. \tag{4.10.15}$$

PROOF. Equation (4.10.4) is a simple expansion of $V_{nr}$ by elements from row $r$. The following proof of (4.10.5) is a development of one due to Lane.

Perform the row operations

$$\mathbf{R}_i' = \mathbf{R}_i - \mathbf{R}_r, \quad 1 \le i \le n, \quad i \ne r,$$

on $K_n$, that is, subtract row $r$ from each of the other rows. The result is

$$K_n = |k'_{ij}|_n,$$

where

$$\begin{aligned} k'_{rj} &= k_{rj}, \\ k'_{ij} &= k_{ij} - k_{rj} \\ &= \left(\frac{r-i}{h+r+j-1}\right)k_{ij}, \quad 1 \le i,j \le n, \quad i \ne r. \end{aligned}$$

After removing the factor $(r-i)$ from each row $i$, $i \ne r$, and the factor $(h+r+j-1)^{-1}$ from each column $j$ and then canceling $K_n$ the result can be expressed in the form

$$
\begin{aligned}
V_{nr} &= \prod_{j=1}^{n}(h+r+j-1)\left(\prod_{\substack{i=1\\i\neq r}}^{n}(r-i)\right)^{-1}\\
&= \frac{(h+r)(h+r-1)\cdots(h+r+n-1)}{[(r-1)(r-2)\cdots 1][(-1)(-2)\cdots(r-n)]},
\end{aligned}
$$

which leads to (4.10.5) and, hence, (4.10.6) and (4.10.7), which are particular cases.

Now, perform the column operations

$$
\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_s, \quad 1 \le j \le n, \quad j \neq s,
$$

on $V_{nr}$. The result is a multiple of a determinant in which the element in position $(r,s)$ is 1 and all the other elements in row $r$ are 0. The other elements in this determinant are given by

$$
\begin{aligned}
k''_{ij} &= k_{ij} - k_{is}\\
&= \left(\frac{s-j}{h+i+s-1}\right)k_{ij}, \quad 1 \le i,j \le n, \quad (i,j) \neq (r,s).
\end{aligned}
$$

After removing the factor $(s-j)$ from each column $j$, $j \neq s$, and the factor $(h+i+s-1)$ from each row $i$, the cofactor $K_{rs}$ appears and gives the result

$$
V_{nr} = K_n^{rs}\prod_{\substack{j=1\\j\neq s}}(s-j)\left(\prod_{\substack{i=1\\i\neq r}}^{n}(h+i+s-1)\right)^{-1},
$$

which leads to (4.10.8) and, hence, (4.10.9) and (4.10.10), which are particular cases. Equation (4.10.11) then follows easily. Equation (4.10.12) is a recurrence relation in $K_n$ which follows from (4.10.10) and (4.10.7) and which, when applied repeatedly, yields (4.10.13), an explicit formula for $K_n$. The proofs of (4.10.14) and (4.10.15) are elementary. □

## *Exercises*

Prove that

**1.** $K_n(-2n-h) = (-1)^n K_n(h), \quad h = 0,1,2,\ldots.$

**2.** $\dfrac{\partial}{\partial h}V_{nr} = V_{nr}\displaystyle\sum_{t=0}^{n-1}\frac{1}{h+r+t}.$

**3.** $\dfrac{\partial}{\partial h}K_n^{rs} = K_n^{rs}\left[\displaystyle\sum_{t=0}^{n-1}\left(\frac{1}{h+r+t}+\frac{1}{h+s+t}\right)-\frac{1}{h+r+s-1}\right].$

**4.** **a.** $K_n^{r1}(0) = \dfrac{(-1)^{r+1}n(r+n-1)!}{(r-1)!r!(n-r)!}$.

**b.** $K_n^{rs}(0) = \dfrac{(-1)^{r+s}rs}{r+s-1}\dbinom{n-1}{r-1}\dbinom{n-1}{s-1}\dbinom{r+n-1}{r}\dbinom{s+n-1}{s}$.

**c.** $K_n(0) = \dfrac{[1!2!3!\cdots(n-1)!]^3}{n!(n+1)!(n+2)!\cdots(2n-1)!}$.

**5.**

$$\begin{aligned}K_n\left(\frac{1}{2}\right) &= 2^n\left|\frac{1}{2i+2j-1}\right|_n\\ &= 2^{2n^2}[1!\,2!\,3!\cdots(n-1)!]^2\prod_{r=0}^{n-1}\frac{(2r+1)!(r+n)!}{r!(2r+2n+1)!}.\end{aligned}$$

[Apply the Legendre duplication formula in Appendix A.1].

**6.** By choosing $h$ suitably, evaluate $|1/(2i+2j-3)|_n$.

The next set of identities are of a different nature. The parameter $n$ is omitted from $V_{nr}$, $K_n^{ij}$, and so forth.

**Identities 2.**

$$\sum_j \frac{K^{sj}}{h+r+j-1} = \delta_{rs}, \quad 1 \le r \le n. \tag{4.10.16}$$

$$\sum_j \frac{V_j}{h+r+j-1} = 1, \quad 1 \le r \le n. \tag{4.10.17}$$

$$\sum_j \frac{V_j}{(h+r+j-1)(h+s+j-1)} = \frac{\delta_{rs}}{V_r}, \quad 1 \le r, s \le n. \tag{4.10.18}$$

$$\sum_j \frac{jK^{1j}}{h+r+j-1} = V_1 - h\delta_{r1}, \quad 1 \le r \le n. \tag{4.10.19}$$

$$\sum_j V_j = \sum_i\sum_j K^{ij} = n(n+h). \tag{4.10.20}$$

$$\sum_j jK^{1j} = (n^2+nh-h)V_1. \tag{4.10.21}$$

PROOF. Equation (4.10.16) is simply the identity

$$\sum_j k_{rj}K^{sj} = \delta_{rs}.$$

To prove (4.10.17), apply (4.10.9) with $r \to j$ and (4.10.4): and (4.10.12),

$$\begin{aligned}V_1\sum_j \frac{V_j}{h+r+j-1} &= \sum_j \frac{(h+j)K^{j1}}{h+r+j-1}\\ &= \sum_j\left(1 - \frac{r-1}{h+r+j-1}\right)K^{j1}\end{aligned}$$

$$= V_1 - (r-1)\sum_j \frac{K^{j1}}{h+r+j-1}$$
$$= V_1 - (r-1)\delta_{r1}, \quad 1 \le r \le n.$$

The second term is zero. The result follows.

The proof of (4.10.18) when $s \ne r$ follows from the identity

$$\frac{1}{(h+r+j-1)(h+s+j-1)} = \frac{1}{s-r}\left(\frac{1}{h+r+j-1} - \frac{1}{h+s+j-1}\right)$$

and (4.10.15). When $s = r$, the proof follows from (4.10.8) and (4.10.16):

$$V_r \sum_s \frac{V_s}{(h+r+s-1)^2} = \sum_s \frac{K^{rs}}{h+r+s-1} = 1.$$

To prove (4.10.19), apply (4.10.4) and (4.10.16):

$$\sum_j \frac{jK^{1j}}{h+r+j-1} = \sum_j \left(1 - \frac{h+r-1}{h+r+j-1}\right) K^{1j}$$
$$= V_1 - h\delta_{r1} - (r-1)\delta_{r1}, \quad 1 \le r \le n.$$

The third term is zero. The result follows.

Equation (4.10.20) follows from (4.10.4) and the double-sum identity (C) (Section 3.4) with $f_r = r$ and $g_s = s + h - 1$, and (4.10.21) follows from the identity (4.10.9) in the form

$$jK^{1j} = V_1 V_j - hK^{1j}$$

by summing over $j$ and applying (4.10.4) and (4.10.20). □

### *4.10.2 Three Formulas of the Rodrigues Type*

Let

$$R_n(x) = \sum_{j=1}^{n} K^{1j} x^{j-1}$$
$$= \frac{1}{K_n} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{vmatrix}_n .$$

**Theorem 4.37.**

$$R_n(x) = \frac{(h+n)!}{(n-1)!^2 h! x^{h+1}} D^{n-1}[x^{h+n}(1-x)^{n-1}].$$

PROOF. Referring to (4.10.9), (4.10.5), and (4.10.6),

$$D^{n-1}\left[x^{h+n}(1-x)^{n-1}\right] = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} D^{n-1}(x^{h+n+i})$$

$$= \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \frac{(h+n+i)!}{(h+i+1)!} x^{h+i+1}$$
$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{n-1}{j-1} \frac{(h+n+j-1)!}{(h+j)!} x^{h+j}$$
$$= \frac{(n-1)!^2 h! x^{h+1}}{(h+n)!} \sum_{j=1}^{n} K^{1j} x^{j-1}.$$

The theorem follows. □

Let

$$S_n(x,h) = \sum_{j=1}^{n} K_{nj}^{(n)} (-x)^{j-1}$$
$$= \begin{vmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n-1,1} & k_{n-1,2} & k_{n-1,3} & \cdots & k_{n-1,n} \\ 1 & -x & x^2 & \cdots & (-x)^{n-1} \end{vmatrix}_n .$$

The column operations

$$\mathbf{C}_j' = \mathbf{C}_j + x\mathbf{C}_{j-1}, \quad 2 \le j \le n,$$

remove the $x$'s from the last row and yield the formula

$$S_n(x,h) = (-1)^{n+1} \left| \frac{x}{h+i+j-1} + \frac{1}{h+i+j} \right|_{n-1} .$$

Let

$$T_n(x,h) = (-1)^{n+1} \left| \frac{1+x}{h+i+j-1} - \frac{1}{h+i+j} \right|_{n-1} .$$

**Theorem 4.38.**

**a.** $\dfrac{(h+n-1)!^2}{h!(n-1)!} \dfrac{S_n(x,h)}{S_n(0,h)} = D^{h+n-1}[x^{n-1}(1+x)^{h+n-1}].$

**b.** $\dfrac{(h+n-1)!}{h!(n-1)!} \dfrac{T_n(x,h)}{T_n(0,h)} = D^{h+n-1}[x^{h+n-1}(1+x)^{n-1}].$

PROOF.

$$S_n(0,h) = K_{n1}^{(n)}$$
$$= \frac{K_n(h) V_{nn} V_{n1}}{h+n}$$
$$= (-1)^{n+1} K_n(h) V_{nn} \binom{h+n-1}{h},$$

$$S_n(x,h) = K_n(h)V_{nn}\sum_{j=1}^{n}\frac{V_{nj}(-x)^{j-1}}{h+n+j-1}.$$

Hence,

$$\begin{aligned}
\binom{h+n-1}{h}\frac{S_n(x,h)}{S_n(0,h)} &= (-1)^{n+1}\sum_{j=1}^{n}\frac{V_{nj}(-x)^{j-1}}{h+n+j-1}\\
&= \sum_{j=1}^{n}\frac{1}{(n-j)!(h+j-1)!}\left[\frac{(h+n+j-2)!x^{j-1}}{(j-1)!}\right]\\
&= \frac{1}{(h+n-1)!}\sum_{j=1}^{n}\binom{h+n-1}{h+j-1}D^{h+n-1}(x^{h+n+j-2}),\\
\frac{(h+n-1)!^2}{h!(n-1)!}\,\frac{S_n(x,h)}{S_n(0,h)} &= D^{h+n-1}\left[x^{n-1}\sum_{j=1}^{n}\binom{h+n-1}{h+j-1}x^{h+j-1}\right]\\
&= D^{h+n-1}\left[x^{n-1}\sum_{r=h}^{h+n-1}\binom{h+n-1}{r}x^r\right]\\
&= D^{h+n-1}\big[x^{n-1}(1+x)^{h+n-1} - p_{h+n-2}(x)\big],
\end{aligned}$$

where $p_r(x)$ is a polynomial of degree $r$. Formula (a) follows. To prove (b), put $x=-1-t$. The details are elementary. □

Further formulas of the Rodrigues type appear in Section 4.11.4.

### *4.10.3 Bordered Yamazaki–Hori Determinants — 1*

Let

$$\begin{aligned}
A &= |a_{ij}|_n = |\theta_m|_n,\\
B &= |b_{ij}|_n = |\phi_m|_n, \quad 0\le m\le 2n-1,
\end{aligned}\tag{4.10.22}$$

denote two Hankelians, where

$$\begin{aligned}
a_{ij} &= \frac{1}{i+j-1}\big[p^2x^{2(i+j-1)} + q^2y^{2(i+j-1)} - 1\big],\\
\theta_m &= \frac{1}{m+1}\big[p^2x^{2m+2} + q^2y^{2m+2} - 1\big],\\
b_{ij} &= \frac{1}{i+j-1}\big[p^2X^{i+j-1} + q^2Y^{i+j-1}\big],\\
\phi_m &= \frac{1}{m+1}\big[p^2X^{m+1} + q^2Y^{m+1}\big],\\
p^2+q^2 &= 1,\\
X &= x^2-1,\\
Y &= y^2-1.
\end{aligned}\tag{4.10.23}$$

Referring to the section on differences in Appendix A.8,

$$\phi_m = \Delta^m \theta_0$$

so that

$$B = A.$$

The Hankelian $B$ arises in studies by M. Yamazaki and Hori of the Ernst equation of general relativity and $A$ arises in a related paper by Vein.

Define determinants $U(x)$, $V(x)$, and $W$, each of order $(n+1)$, by bordering $A$ in different ways. Since $a_{ij}$ is a function of $x$ and $y$, it follows that $U(x)$ and $V(x)$ are also functions of $y$. The argument $x$ in $U(x)$ and $V(x)$ refers to the variable which appears explicitly in the last row or column.

$$\begin{aligned} U(x) &= \begin{vmatrix} & & & & & x \\ & & & & & x^3/3 \\ & & [a_{ij}]_n & & & x^5/5 \\ & & & & & \cdots \\ & & & & & x^{2n-1}/(2n-1) \\ 1 & 1 & 1 & \cdots & 1 & \bullet \end{vmatrix}_{n+1} \\ &= -\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{A_{rs}x^{2r-1}}{2r-1}, \end{aligned} \tag{4.10.24}$$

$$\begin{aligned} V(x) &= \begin{vmatrix} & & & & & 1 \\ & & & & & 1/3 \\ & & [a_{ij}]_n & & & 1/5 \\ & & & & & \cdots \\ & & & & & 1/(2n-1) \\ x & x^3 & x^5 & \cdots & x^{2n-1} & \bullet \end{vmatrix}_{n+1} \\ &= -\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{A_{rs}x^{2s-1}}{2r-1}, \end{aligned} \tag{4.10.25}$$

$$W = U(1) = V(1). \tag{4.10.26}$$

**Theorem 4.39.**

$$p^2U^2(x) + q^2U^2(y) = W^2 - AW.$$

PROOF.

$$\begin{aligned} U^2(x) &= \sum_{i,s} \frac{A_{is}x^{2i-1}}{2i-1} \sum_{j,r}^{n} \frac{A_{jr}x^{2j-1}}{2j-1} \\ &= \sum_{i,j,r,s} \frac{A_{is}A_{jr}x^{2(i+j-1)}}{(2i-1)(2j-1)}. \end{aligned}$$

Hence,

$$p^2U^2(x) + q^2U^2(y) - W^2$$

$$
\begin{aligned}
&= \sum_{i,j,r,s} \frac{A_{is}A_{jr}}{(2i-1)(2j-1)}[p^2x^{2(i+j-1)} + q^2y^{2(i+j-1)} - 1] \\
&= \sum_{i,j,r,s} \frac{(i+j-1)a_{ij}A_{is}A_{rj}}{(2i-1)(2j-1)} \\
&= \tfrac{1}{2} \sum_{i,j,r,s} \left( \frac{1}{2i-1} + \frac{1}{2j-1} \right) a_{ij}A_{is}A_{rj} \\
&= \sum_{i,j,r,s} \frac{a_{ij}A_{is}A_{rj}}{2i-1} \\
&= \sum_{i,s} \frac{A_{is}}{2i-1} \sum_r \sum_j a_{ij}A_{rj} \\
&= A \sum_{i,s} \frac{A_{is}}{2i-1} \sum_r \delta_{ir} \\
&= -AW
\end{aligned}
$$

which proves the theorem. □

**Theorem 4.40.**

$$p^2V^2(x) + q^2V^2(y) = W^2 - AW.$$

This theorem resembles Theorem 4.39 closely, but the following proof bears little resemblance to the proof of Theorem 4.39. Applying double-sum identity (D) in Section 3.4 with $f_r = r$ and $g_s = s - 1$,

$$\sum_r \sum_s [p^2x^{2(r+s-1)} + q^2y^{2(r+s-1)} - 1]A^{is}A^{rj} = (i+j-1)A^{ij},$$

$$
\begin{aligned}
p^2 \left[ \sum_s A^{is}x^{2s-1} \right] \left[ \sum_r A^{rj}x^{2r-1} \right] + q^2 \left[ \sum_s A^{is}y^{2s-1} \right] \left[ \sum_r A^{rj}y^{2r-1} \right] \\
- \left[ \sum_s A^{is} \right] \left[ \sum_r A^{rj} \right] = (i+j-1)A^{ij}.
\end{aligned}
$$

Put

$$\lambda_i(x) = \sum_j A^{ij}x^{2j-1}.$$

Then,

$$p^2\lambda_i(x)\lambda_j(x) + q^2\lambda_i(y)\lambda_j(y) - \lambda_i(1)\lambda_j(1) = (i+j-1)A^{ij}.$$

Divide by $(2i-1)(2j-1)$, sum over $i$ and $j$ and note that

$$\sum_i \frac{\lambda_i(x)}{2i-1} = -\frac{V(x)}{A}.$$

The result is

$$\begin{aligned}\frac{1}{A^2}\left[p^2V^2(x) + q^2V^2(y) - W^2\right] &= \sum_i \sum_j \frac{i+j-1}{(2i-1)(2j-1)} A^{ij} \\ &= \tfrac{1}{2}\sum_i \sum_j \left(\frac{1}{2i-1} + \frac{1}{2j-1}\right) A^{ij} \\ &= -\frac{W}{A}.\end{aligned}$$

The theorem follows. The determinant $W$ appears in Section 5.8.6.

**Theorem 4.41.** *In the particular case in which* $(p, q) = (1, 0)$,

$$V(x) = (-1)^{n+1}U(x).$$

PROOF.

$$a_{ij} = \frac{x^{2(i+j-1)} - 1}{i+j-1} = a_{ji},$$

which is independent of $y$. Let

$$Z = \begin{vmatrix} & & & & & 1 \\ & & & & & 1 \\ & & [c_{ij}]_n & & & 1 \\ & & & & & \cdots \\ & & & & & 1 \\ x & x^3 & x^5 & \cdots & x^{2n-1} & \bullet \end{vmatrix}_{n+1},$$

where

$$\begin{aligned} c_{ij} &= (i-j)a_{ij} \\ &= -c_{ji}. \end{aligned}$$

The proof proceeds by showing that $U$ and $V$ are each simple multiples of $Z$. Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - x^{2j-1}\mathbf{C}_{n+1}, \quad 1 \le j \le n,$$

on $U$. This leaves the last column and the last row unaltered, but $[a_{ij}]_n$ is replaced by $[a'_{ij}]_n$, where

$$a'_{ij} = a_{ij} - \frac{x^{2(i+j-1)}}{2i-1}.$$

Now perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \frac{1}{2i-1}\mathbf{R}_{n+1}, \quad 1 \le i \le n.$$

The last column and the last row remain unaltered, but $[a'_{ij}]_n$ is replaced by $[a''_{ij}]_n$, where

$$a''_{ij} = a'_{ij} + \frac{1}{2i-1}$$

$$= \frac{c_{ij}}{2i-1}\,.$$

After removing the factor $(2i-1)^{-1}$ from row $i$, $1 \le i \le n$, the result is

$$U = \frac{2^n n!}{(2n)!}\begin{vmatrix} & & & & & x \\ & & & & & x^3 \\ & & [c_{ij}]_n & & & x^5 \\ & & & & & \cdots \\ & & & & & x^{2n-1} \\ 1 & 1 & 1 & \cdots & 1 & \bullet \end{vmatrix}_{n+1}.$$

Transposing,

$$U = \frac{2^n n!}{(2n)!}\begin{vmatrix} & & & & & 1 \\ & & & & & 1 \\ & & [-c_{ij}]_n & & & 1 \\ & & & & & \cdots \\ & & & & & 1 \\ x & x^3 & x^5 & \cdots & x^{2n-1} & \bullet \end{vmatrix}_{n+1}.$$

Now, change the signs of columns 1 to $n$ and row $(n+1)$. This introduces $(n+1)$ negative signs and gives the result

$$U = \frac{(-1)^{n+1}2^n n!}{(2n)!}\,Z. \tag{4.10.27}$$

Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j + \mathbf{C}_{n+1}, \quad 1 \le j \le n,$$

on $V$. The result is that $[a_{ij}]_n$ is replaced by $[a^*_{ij}]_n$, where

$$a^*_{ij} = a_{ij} + \frac{1}{2i-1}\,.$$

Perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \frac{x^{2i-1}}{2i-1}\mathbf{R}_{n+1}, \quad 1 \le i \le n,$$

which results in $[a^*_{ij}]_n$ being replaced by $[a^{**}_{ij}]_n$, where

$$\begin{aligned} a^{**}_{ij} &= a^*_{ij} - \frac{x^{2(i+j-1)}}{2i-1} \\ &= \frac{c_{ij}}{2i-1}\,. \end{aligned}$$

After removing the factor $(2i-1)^{-1}$ from row $i$, $1 \le i \le n$, the result is

$$V = \frac{2^n n!}{(2n)!}\,Z. \tag{4.10.28}$$

The theorem follows from (4.10.27) and (4.10.28). □

Let

$$A = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \frac{x^{2m+2}-1}{m+1}.$$

$A$ is identical to $|a_{ij}|_n$, where $a_{ij}$ is defined in Theorem 4.41. Let $Y$ denote the determinant of order $(n+1)$ obtained by bordering $A$ by the row

$$\begin{bmatrix}1 \; 1 \; 1 \ldots 1 \; \bullet\end{bmatrix}_{n+1}$$

below and the column

$$\begin{bmatrix}1 \; \frac{1}{3} \; \frac{1}{5} \cdots \frac{1}{2n-1} \; \bullet\end{bmatrix}_{n+1}^T$$

on the right.

**Theorem 4.42.**

$$Y = -nK_n\phi_0^{n(n-1)} \sum_{i=1}^{n} \frac{2^{2i-1}(n+i-1)!}{(n-i)!(2i)!}\phi_0^{n-i},$$

*where $K_n$ is the simple Hilbert determinant.*

PROOF. Perform the column operations

$$\mathbf{C}_j' = \mathbf{C}_j - \mathbf{C}_{j-1}$$

in the order $j = n, n-1, n-2, \ldots, 2$. The result is a determinant in which the only nonzero element in the last row is a 1 in position $(n+1, 1)$. Hence,

$$Y = (-1)^n \begin{vmatrix} \Delta\phi_0 & \Delta\phi_1 & \Delta\phi_2 & \cdots & \Delta\phi_{n-2} & 1 \\ \Delta\phi_1 & \Delta\phi_2 & \Delta\phi_3 & \cdots & \Delta\phi_{n-1} & \frac{1}{3} \\ \Delta\phi_2 & \Delta\phi_3 & \Delta\phi_4 & \cdots & \Delta\phi_n & \frac{1}{5} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \Delta\phi_{n-1} & \Delta\phi_n & \Delta\phi_{n+1} & \cdots & \Delta\phi_{2n-3} & \frac{1}{2n-1} \end{vmatrix}_n.$$

Perform the row operations

$$\mathbf{R}_i' = \mathbf{R}_i - \mathbf{R}_{i-1}$$

in the order $i = n, n-1, n-2, \ldots, 2$. The result is

$$Y = (-1)^n \begin{vmatrix} \Delta\phi_0 & \Delta\phi_1 & \Delta\phi_2 & \cdots & \Delta\phi_{n-2} & 1 \\ \Delta^2\phi_0 & \Delta^2\phi_1 & \Delta^2\phi_2 & \cdots & \Delta^2\phi_{n-1} & \Delta\alpha_0 \\ \Delta^2\phi_1 & \Delta^2\phi_2 & \Delta^2\phi_3 & \cdots & \Delta^2\phi_n & \Delta\alpha_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \Delta^2\phi_{n-2} & \Delta^2\phi_{n-1} & \Delta^2\phi_n & \cdots & \Delta^2\phi_{2n-4} & \Delta\alpha_{n-2} \end{vmatrix}_n,$$

where

$$\alpha_m = \frac{1}{2m+1}.$$

Now, perform the row and column operations

$$\mathbf{R}'_i = \sum_{r=0}^{i-2} (-1)^r \binom{i-2}{r} \mathbf{R}_{i-r}, \quad i = n, n-1, n-2, \ldots, 3,$$

$$\mathbf{C}'_j = \sum_{r=0}^{j-1} (-1)^r \binom{j-1}{r} \mathbf{C}_{j-r}, \quad j = n-1, n-2, \ldots, 2.$$

The result is

$$Y = (-1)^n \begin{vmatrix} \Delta\phi_0 & \Delta^2\phi_0 & \Delta^3\phi_0 & \cdots & \Delta^{n-1}\phi_0 & 1 \\ \Delta^2\phi_0 & \Delta^3\phi_0 & \Delta^4\phi_0 & \cdots & \Delta^n\phi_0 & \Delta\alpha_0 \\ \Delta^3\phi_0 & \Delta^4\phi_0 & \Delta^5\phi_0 & \cdots & \Delta^{n+1}\phi_0 & \Delta^2\alpha_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^n\phi_0 & \Delta^{n+1}\phi_0 & \Delta^{n+2}\phi_0 & \cdots & \Delta^{2n-2}\phi_0 & \Delta^{n-1}\alpha_0 \end{vmatrix}_n,$$

where

$$\Delta^m\phi_0 = \frac{\phi_0^{m+1}}{m+1}.$$

Transfer the last column to the first position, which introduces the sign $(-1)^{n+1}$, and then remove powers of $\phi_0$ from all rows and columns except the first column, which becomes

$$\left[1 \ \frac{\Delta\alpha_0}{\phi_0} \ \frac{\Delta^2\alpha_0}{\phi_0^2} \cdots \frac{\Delta^{n-1}\alpha_0}{\phi_0^{n-1}}\right]^T.$$

The other $(n-1)$ columns are identical with the corresponding columns of the Hilbert determinant $K_n$. Hence, expanding the determinant by elements from the first column,

$$Y = -\phi_0^{n(n-1)} \sum_{i=1}^{n} \left[K_{i1}^{(n)} \Delta^{i-1}\alpha_0\right] \phi_0^{n-i}.$$

The proof is completed with the aid of (4.10.5) and (4.10.8) and the formula for $\Delta^{i-1}\alpha_0$ in Appendix A.8. □

Further notes on the Yamazaki–Hori determinant appear in Section 5.8 on algebraic computing.

### *4.10.4 A Particular Case of the Yamazaki–Hori Determinant*

Let

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \frac{x^{2m+2}-1}{m+1}. \tag{4.10.29}$$

**Theorem.**

$$A_n = K_n(x^2-1)^{n^2}, \quad K_n = K_n(0).$$

PROOF.

$$\phi_0 = x^2 - 1.$$

Referring to Example A.3 (with $c = 1$) in the section on differences in Appendix A.8,

$$\Delta^m \phi_0 = \frac{\phi_0^{m+1}}{m+1}.$$

Hence, applying the theorem in Section 4.8.2 on Hankelians whose elements are differences,

$$\begin{aligned}
A_n &= |\Delta^m \phi_0|_n \\
&= \left|\frac{\phi_0^{m+1}}{m+1}\right|_n \\
&= \begin{vmatrix}
\phi_0 & \frac{1}{2}\phi_0^2 & \frac{1}{3}\phi_0^3 & \cdots & \frac{1}{n}\phi_0^n \\
\frac{1}{2}\phi_0^2 & \frac{1}{3}\phi_0^3 & \frac{1}{4}\phi_0^4 & \cdots & \cdots \\
\frac{1}{3}\phi_0^3 & \frac{1}{4}\phi_0^4 & \frac{1}{5}\phi_0^5 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{n}\phi_0^n & \cdots & \cdots & \cdots & \frac{1}{2n-1}\phi_0^{2n-1}
\end{vmatrix}_n.
\end{aligned}$$

Remove the factor $\phi_0^i$ from row $i$, $1 \le i \le n$, and then remove the factor $\phi_0^{j-1}$ from column $j$, $2 \le j \le n$. The simple Hilbert determinant $K_n$ appears and the result is

$$\begin{aligned}
A_n &= K_n \phi_0^{(1+2+3+\cdots+n)(1+2+3+\cdots+\overline{n-1})} \\
&= K_n \phi_0^{n^2},
\end{aligned}$$

which proves the theorem. $\square$

## *Exercises*

**1.** Define a triangular matrix $[a_{ij}]$, $1 \le i \le 2n-1$, $1 \le j \le 2n-i$, as follows:

$$\begin{aligned}
\text{column } 1 &= \left[1\ u\ u^2 \cdots u^{2n-2}\right]^T, \\
\text{row } 1 &= \left[1\ v\ v^2 \cdots v^{2n-2}\right].
\end{aligned}$$

The remaining elements are defined by the rule that the difference between consecutive elements in any one diagonal parallel to the secondary diagonal is constant. For example, one diagonal is

$$\left[u^3\ \frac{1}{3}(2u^3+v^3)\ \frac{1}{3}(u^3+2v^3)\ v^3\right]$$

in which the column difference is $\frac{1}{3}(v^3 - u^3)$.

Let the determinant of the elements in the first $n$ rows and the first $n$ columns of the matrix be denoted by $A_n$. Prove that

$$A_n = \frac{K_n n!^3}{(2n)!}(u - v)^{n(n+1)}.$$

**2.** Define a Hankelian $B_n$ as follows:

$$B_n = \left| \frac{\phi_m}{(m+1)(m+2)} \right|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \sum_{r=0}^{m} (m+1-r) u^{m-r} v^r.$$

Prove that

$$B_n = \frac{A_{n+1}}{n!(u-v)^{2n}},$$

where $A_n$ is defined in Exercise 1.

## 4.11 Hankelians 4

Throughout this section, $K_n = K_n(0)$, the simple Hilbert determinant.

### *4.11.1 v-Numbers*

The integers $v_{ni}$ defined by

$$v_{ni} = V_{ni}(0) = \frac{(-1)^{n+i}(n+i-1)!}{(i-1)!^2(n-i)!} \tag{4.11.1}$$

$$= (-1)^{n+i} i \binom{n-1}{i-1} \binom{n+i-1}{n-1}, \quad 1 \le i \le n, \tag{4.11.2}$$

are of particular interest and will be referred to as $v$-numbers.

A few values of the $v$-numbers $v_{ni}$ are given in the following table:

| $n$ \ $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | −2 | 6 | | | |
| 3 | 3 | −24 | 30 | | |
| 4 | −4 | 60 | −180 | 140 | |
| 5 | 5 | −120 | 630 | −1120 | 630 |

$v$-Numbers satisfy the identities

$$\sum_{k=1}^{n} \frac{v_{nk}}{i+k-1} = 1, \quad 1 \le i \le n, \tag{4.11.3}$$

$$v_{ni} \sum_{k=1}^{n} \frac{v_{nk}}{(i+k-1)(k+j-1)} = \delta_{ij}, \tag{4.11.4}$$

$$\frac{v_{ni}}{n+i-1} = -\frac{v_{n-1,i}}{n-i}, \tag{4.11.5}$$

$$\sum_{i=1}^{n} v_{ni} = n^2, \tag{4.11.6}$$

and are related to $K_n$ and its scaled cofactors by

$$K_n^{ij} = \frac{v_{ni} v_{nj}}{i+j-1}, \tag{4.11.7}$$

$$K_n \prod_{i=1}^{n} v_{ni} = (-1)^{n(n-1)/2}. \tag{4.11.8}$$

The proofs of these identities are left as exercises for the reader.

### *4.11.2 Some Determinants with Determinantal Factors*

This section is devoted to the factorization of the Hankelian

$$B_n = \det \mathbf{B}_n,$$

where

$$\begin{aligned} \mathbf{B}_n &= [b_{ij}]_n, \\ b_{ij} &= \frac{x^{2(i+j-1)} - t^2}{i+j-1}, \end{aligned} \tag{4.11.9}$$

and to the function

$$G_n = \sum_{j=1}^{n} (x^{2j-1} + t) B_{nj}, \tag{4.11.10}$$

which can be expressed as the determinant $|g_{ij}|_n$ whose first $(n-1)$ rows are identical to the first $(n-1)$ rows of $B_n$. The elements in the last row are given by

$$g_{nj} = x^{2j-1} + t, \quad 1 \le j \le n.$$

The analysis employs both matrix and determinantal methods.

Define five matrices $\mathbf{K}_n$, $\mathbf{Q}_n$, $\mathbf{S}_n$, $\mathbf{H}_n$, and $\overline{\mathbf{H}}_n$ as follows:

$$\mathbf{K}_n = \left[\frac{1}{i+j-1}\right]_n, \tag{4.11.11}$$

$$\mathbf{Q}_n = \mathbf{Q}_n(x) = \left[\frac{x^{2(i+j-1)}}{i+j-1}\right]_n. \tag{4.11.12}$$

Both $\mathbf{K}_n$ and $\mathbf{Q}_n$ are Hankelians and $\mathbf{Q}_n(1) = \mathbf{K}_n$, the simple Hilbert matrix.

$$\mathbf{S}_n = \mathbf{S}_n(x) = \left[\frac{v_{ni}x^{2j-1}}{i+j-1}\right]_n, \tag{4.11.13}$$

where the $v_{ni}$ are $v$-numbers.

$$\begin{aligned}\mathbf{H}_n = \mathbf{H}_n(x,t) &= \mathbf{S}_n(x) + t\mathbf{I}_n \\ &= \left[h_{ij}^{(n)}\right]_n,\end{aligned}$$

where

$$\begin{aligned}h_{ij}^{(n)} &= \frac{v_{ni}x^{2j-1}}{i+j-1} + \delta_{ij}t, \\ \overline{\mathbf{H}}_n &= \mathbf{H}_n(x,-t) = \mathbf{S}_n(x) - t\mathbf{I}_n \\ &= \left[\overline{h}_{ij}^{(n)}\right]_n,\end{aligned} \tag{4.11.14}$$

where

$$\begin{aligned}\overline{h}_{ij}^{(n)}(x,t) &= h_{ij}^{(n)}(x,-t), \\ \bar{H}_n(x,-t) &= (-1)^n H_n(-x,t).\end{aligned} \tag{4.11.15}$$

**Theorem 4.43.**

$$\mathbf{K}_n^{-1}\mathbf{Q}_n = \mathbf{S}_n^2.$$

PROOF. Referring to (4.11.7) and applying the formula for the product of two matrices,

$$\begin{aligned}\mathbf{K}_n^{-1}\mathbf{Q}_n &= \left[\frac{v_{ni}v_{nj}}{i+j-1}\right]_n \left[\frac{x^{2(i+j-1)}}{i+j-1}\right]_n \\ &= \left[\sum_{k=1}^n \frac{v_{ni}v_{nk}}{i+k-1}\,\frac{x^{2(k+j-1)}}{k+j-1}\right]_n \\ &= \left[\sum_{k=1}^n \left(\frac{v_{ni}x^{2k-1}}{i+k-1}\right)\left(\frac{v_{nk}x^{2j-1}}{k+j-1}\right)\right]_n \\ &= \mathbf{S}_n^2.\end{aligned}$$

□

**Theorem 4.44.**

$$\mathbf{B}_n = \mathbf{K}_n\mathbf{H}_n\overline{\mathbf{H}}_n,$$

*where the symbols can be interpreted as matrices or determinants.*

PROOF. Applying Theorem 4.43,

$$\mathbf{B}_n = \mathbf{Q}_n - t^2\mathbf{K}_n$$

$$\begin{aligned}
&= \mathbf{K}_n\big(\mathbf{K}_n^{-1}\mathbf{Q}_n - t^2\mathbf{I}_n\big)\\
&= \mathbf{K}_n\big(\mathbf{S}_n^2 - t^2\mathbf{I}_n\big)\\
&= \mathbf{K}_n(\mathbf{S}_n + t\mathbf{I}_n)(\mathbf{S}_n - t\mathbf{I}_n)\\
&= \mathbf{K}_n\mathbf{H}_n\overline{\mathbf{H}}_n.
\end{aligned}$$

□

**Corollary.**

$$\begin{aligned}
\mathbf{B}_n^{-1} &= \overline{\mathbf{H}}_n^{-1}\mathbf{H}_n^{-1}\mathbf{K}_n^{-1},\\
\big[\mathbf{B}_{ji}^{(n)}\big] &= \big[\overline{\mathbf{H}}_{ji}^{(n)}\big]\big[H_{ji}^{(n)}\big]\big[K_{ji}^{(n)}\big].
\end{aligned}$$

**Lemma.**

$$\sum_{i=1}^{n} h_{ij}^{(n)} = x^{2j-1} + t.$$

The proof applies (4.11.3) and is elementary.

Let $E_{n+1}$ denote the determinant of order $(n+1)$ obtained by bordering $H_n$ as follows:

$$\begin{aligned}
E_{n+1} &= \begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1n} & v_{n1}/n \\ h_{21} & h_{22} & \cdots & h_{2n} & v_{n2}/(n+1) \\ \dots & \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \cdots & h_{nn} & v_{nn}/(2n-1) \\ 1 & 1 & \cdots & 1 & \bullet \end{vmatrix}_{n+1}\\
&= -\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{v_{nr}H_{rs}}{n+r-1}. \qquad (4.11.16)
\end{aligned}$$

**Theorem 4.45.**

$$E_{n+1} = (-1)^n\overline{H}_{n-1}.$$

The proof consists of a sequence of row and column operations.

PROOF. Perform the column operation

$$\mathbf{C}_n' = \mathbf{C}_n - x^{2n-1}\mathbf{C}_{n+1} \qquad (4.11.17)$$

and apply (6b) with $j = n$. The result is

$$E_{n+1} = \begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1,n-1} & \bullet & v_{n1}/n \\ h_{21} & h_{22} & \cdots & h_{2,n-1} & \bullet & v_{n2}/(n+1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \cdots & h_{n,n-1} & t & v_{nn}/(2n-1) \\ 1 & 1 & \cdots & 1 & 1 & \bullet \end{vmatrix}_{n+1}. \qquad (4.11.18)$$

Remove the element in position $(n,n)$ by performing the row operation

$$\mathbf{R}_n' = \mathbf{R}_n - t\mathbf{R}_{n+1}. \qquad (4.11.19)$$

The only element which remains in column $n$ is a 1 in position $(n+1, n)$. Hence,

$$E_{n+1} = -\begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1,n-1} & v_{n1}/n \\ h_{21} & h_{22} & \cdots & h_{2,n-1} & v_{n2}/(n+1) \\ \dotfill & \dotfill & \dotfill & \dotfill & \dotfill \\ (h_{n1}-t) & (h_{n2}-t) & \cdots & (h_{n,n-1}-t) & v_{nn}/(2n-1) \end{vmatrix}_n . \tag{4.11.20}$$

It is seen from (4.11.3) (with $i = n$) that the sum of the elements in the last column is unity and it is seen from the lemma that the sum of the elements in column $j$ is $x^{2j-1}$, $1 \le j \le n-1$. Hence, after performing the row operation

$$\mathbf{R}'_n = \sum_{i=1}^{n} \mathbf{R}_i, \tag{4.11.21}$$

the result is

$$E_{n+1} = \begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1,n-1} & v_{n1}/n \\ h_{21} & h_{22} & \cdots & h_{2,n-1} & v_{n2}/(n+1) \\ \dotfill & \dotfill & \dotfill & \dotfill & \dotfill \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & v_{n,n-1}/(2n-2) \\ x & x^3 & \cdots & x^{2n-3} & 1 \end{vmatrix}_n . \tag{4.11.22}$$

The final set of column operations is

$$\mathbf{C}'_j = \mathbf{C}_j - x^{2j-1}\mathbf{C}_n, \quad 1 \le j \le n-1, \tag{4.11.23}$$

which removes the $x$'s from the last row. The result can then be expressed in the form

$$E_{n+1} = -\left|h_{ij}^{(n)^*}\right|_{n-1}, \tag{4.11.24}$$

where, referring to (4.11.5),

$$\begin{aligned} h_{ij}^{(n)^*} &= h_{ij}^{(n)} - \frac{v_{ni}x^{2j-1}}{n+i-1} \\ &= v_{ni}x^{2j-1}\left(\frac{1}{i+j-1} - \frac{1}{i+n-1}\right) + \delta_{ij}t \\ &= \left(\frac{v_{ni}}{i+n-1}\right)\left(\frac{(n-j)x^{2j-1}}{i+j-1}\right) + \delta_{ij}t \\ &= -\left(\frac{v_{n-1,i}}{n-i}\right)\left(\frac{(n-j)x^{2j-1}}{i+j-1}\right) + \delta_{ij}t \\ &= -\left(\frac{n-j}{n-i}\right)\left(\frac{v_{n-1,i}x^{2j-1}}{i+j-1} - \delta_{ij}t\right) \\ &= -\left(\frac{n-j}{n-i}\right)\bar{h}_{ij}^{(n-1)}, \\ \left|h_{ij}^{(n)^*}\right|_{n-1} &= -\left|-\bar{h}_{ij}^{(n-1)}\right|_{n-1}. \end{aligned} \tag{4.11.25}$$

Theorem 4.45 now follows from (4.11.24). □

**Theorem 4.46.**

$$G_n = (-1)^{n-1} v_{nn} K_n \overline{H}_n \overline{H}_{n-1},$$

*where $G_n$ is defined in (4.11.10).*

PROOF. Perform the row operation

$$\mathbf{R}'_i = \sum_{k=1}^{n} \mathbf{R}_k$$

on $\overline{H}_n$ and refer to the lemma. Row $i$ becomes

$$\left[(x+t), (x^3+t), (x^5+t), \ldots, (x^{2n-1}+t)\right].$$

Hence,

$$\overline{H}_n = \sum_{j=1}^{n} (x^{2j-1}+t)\overline{H}_{ij}^{(n)}, \quad 1 \le i \le n. \tag{4.11.26}$$

It follows from the corollary to Theorem 4.44 that

$$B_{ij}^{(n)} = B_{ji}^{(n)} = \sum_{r=1}^{n} \sum_{s=1}^{n} \overline{H}_{jr}^{(n)} H_{rs}^{(n)} K_{si}^{(n)}. \tag{4.11.27}$$

Hence, applying (4.11.7),

$$B_{ij}^{(n)} = K_n v_{ni} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{v_{ns} H_{rs}^{(n)} \overline{H}_{jr}^{(n)}}{i+s-1}. \tag{4.11.28}$$

Put $i = n$, substitute the result into (4.11.10), and apply (4.11.16) and (4.11.24):

$$\begin{aligned}
G_n &= K_n v_{nn} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{v_{ns} H_{rs}^{(n)}}{n+s-1} \sum_{j=1}^{n} (x^{2j-1}+t)\overline{H}_{jr}^{(n)} \\
&= K_n v_{nn} \overline{H}_n \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{v_{ns} H_{rs}^{(n)}}{n+s-1} \\
&= -K_n v_{nn} \overline{H}_n E_{n+1}.
\end{aligned} \tag{4.11.29}$$

The theorem follows from Theorem 4.45. □

### *4.11.3 Some Determinants with Binomial and Factorial Elements*

**Theorem 4.47.**

**a.** $\left| \binom{n+j-2}{n-i} \right|_n = (-1)^{n(n-1)/2},$

**b.** $\left|\dfrac{1}{(i+j-2)!}\right| = \dfrac{(-1)^{n(n-1)/2} 1!\,2!\,3!\cdots(n-2)!}{n!\,(n+1)!\cdots(2n-2)!}.$

*The second determinant is Hankelian.*

PROOF. Denote the first determinant by $A_n$. Every element in the last row of $A_n$ is equal to 1. Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_{j-1}, \quad j = n, n-1, n-2, \ldots, 2, \tag{4.11.30}$$

which remove all the elements in the last row except the one in position $(n,1)$. After applying the binomial identity

$$\binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1},$$

the result is

$$A_n = (-1)^{n+1}\left|\binom{n+j-2}{n-i-1}\right|_{n-1}. \tag{4.11.31}$$

Once again, every element in the last row is equal to 1. Repeat the column operations with $j = n-1, n-2, \ldots, 2$ and apply the binomial identity again. The result is

$$A_n = -\left|\binom{n+j-2}{n-i-2}\right|_{n-2}. \tag{4.11.32}$$

Continuing in this way,

$$\begin{aligned} A_n &= +\left|\binom{n+j-2}{n-i-4}\right|_{n-4} \\ &= -\left|\binom{n+j-2}{n-i-6}\right|_{n-6} \\ &= +\left|\binom{n+j-2}{n-i-8}\right|_{n-8} \\ &= \pm\left|\binom{n+j-2}{2-i}\right|_{2} \\ &= \pm 1, \end{aligned} \tag{4.11.33}$$

$$\operatorname{sign}(A_n) = \begin{cases} 1 & \text{when } n = 4m, 4m+1 \\ -1 & \text{when } n = 4m-2, 4m-1, \end{cases} \tag{4.11.34}$$

which proves (a).

Denote the second determinant by $B_n$. Divide $\mathbf{R}_i$ by $(n-i)!$, $1 \le i \le n-1$, and multiply $\mathbf{C}_j$ by $(n+j-2)!$, $1 \le j \le n$. The result is

$$\frac{(n-1)!\,n!\,(n+1)!\cdots(2n-2)!}{(n-1)!\,(n-2)!\,(n-3)!\cdots 1!} B_n = \left|\frac{(n+j-2)!}{(n-i)!\,(i+j-2)!}\right|_n$$

$$= \left| \binom{n+j-2}{n-i} \right|_n$$
$$= A_n,$$

which proves (b). □

*Exercises*

Apply similar methods to prove that

**1.** $\left| \binom{n+j-1}{n-i} \right|_n = (-1)^{n(n-1)/2},$

**2.** $\left| \dfrac{1}{(i+j-1)!} \right|_n = \dfrac{(-1)^{n(n-1)/2} K_n}{[1!\,2!\,3! \cdots (n-1)!]^2}.$

Define the number $\nu_i$ as follows:

$$(1+z)^{-1/2} = \sum_{i=0}^{\infty} \nu_i z^i. \tag{4.11.35}$$

Then

$$\nu_i = \frac{(-1)^i}{2^{2i}} \binom{2i}{i}. \tag{4.11.36}$$

Let

$$\begin{aligned} A_n &= |\nu_m|_n, \quad 0 \le m \le 2n-2, \\ &= \left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{n-1} \; \mathbf{C}_n \right|_n \end{aligned} \tag{4.11.37}$$

where

$$\mathbf{C}_j = \left[ \nu_{j-1} \; \nu_j \ldots \nu_{n+j-3} \; \nu_{n+j-2} \right]_n^T. \tag{4.11.38}$$

**Theorem 4.48.**

$$A_n = 2^{-(n-1)(2n-1)}.$$

Proof. Let

$$\lambda_{nr} = \frac{n}{n+r} \binom{n+r}{2r} 2^{2r}. \tag{4.11.39}$$

Then, it is shown in Appendix A.10 that

$$\sum_{j=1}^{n} \lambda_{n-1,j-1} \nu_{i+j-2} = \frac{\delta_{in}}{2^{2(n-1)}}, \quad 1 \le i \le n, \tag{4.11.40}$$

$$\begin{aligned} A_n &= 2^{-(2n-3)} \left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{n-1} \; (\lambda_{n-1,n-1} \mathbf{C}_n) \right|_n, \\ &= 2^{-(2n-3)} \left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{n-1} \; \mathbf{C}'_n \right|_n, \end{aligned} \tag{4.11.41}$$

where

$$\begin{aligned}\mathbf{C}'_n &= \lambda_{n-1,n-1}\mathbf{C}_n + \sum_{j=1}^{n-1} \lambda_{n-1,j-1}\mathbf{C}_j \\ &= \sum_{j=1}^{n-1} \lambda_{n-1,j-1}\left[\nu_{j-1}\ \nu_j \cdots \nu_{n+j-3}\ \nu_{n+j-2}\right]_n^T \\ &= 2^{-(4n-5)}\left[0\ 0 \cdots 0\ 1\right]_n^T. \end{aligned} \tag{4.11.42}$$

Hence,

$$\begin{aligned} A_n &= 2^{-4(n-1)+1}A_{n-1} \\ A_{n-1} &= 2^{-4(n-2)+1}A_{n-2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_2 &= 2^{-4(1)+1}A_1, \qquad (A_1 = \nu_0 = 1). \end{aligned} \tag{4.11.43}$$

The theorem follows by equating the product of the left-hand sides to the product of the right-hand sides. □

It is now required to evaluate the cofactors of $A_n$.

**Theorem 4.49.**

**a.** $A_{nj}^{(n)} = 2^{-(n-1)(2n-3)}\lambda_{n-1,j-1}$,
**b.** $A_{n1}^{(n)} = 2^{-(n-1)(2n-3)}$,
**c.** $A_n^{nj} = 2^{2(n-1)}\lambda_{n-1,j-1}$.

Proof. The $n$ equations in (4.11.40) can be expressed in matrix form as follows:

$$\mathbf{A}_n\mathbf{L}_n = \mathbf{C}'_n, \tag{4.11.44}$$

where

$$\mathbf{L}_n = \left[\lambda_{n0}\ \lambda_{n1} \cdots \lambda_{n,n-2}\ \lambda_{n,n-1}\right]_n^T. \tag{4.11.45}$$

Hence,

$$\begin{aligned}\mathbf{L}_n &= \mathbf{A}_n^{-1}\mathbf{C}'_n \\ &= A_n^{-1}\left[A_{ji}^{(n)}\right]_n \mathbf{C}'_n \\ &= 2^{(n-1)(2n-1)-2(n-1)}\left[A_{n1}\ A_{n2} \cdots A_{n,n-1}\ A_{nn}\right]_n^T, \end{aligned} \tag{4.11.46}$$

which yields part (a) of the theorem. Parts (b) and (c) then follow easily. □

**Theorem 4.50.**

$$A_{ij}^{(n)} = 2^{-n(2n-3)}\left[2^{2i-3}\lambda_{i-1,j-1} + \sum_{r=i+1}^{n-1} \lambda_{r-1,i-1}\lambda_{r-1,j-1}\right], \quad j \le i < n-1.$$

PROOF. Apply the Jacobi identity (Section 3.6.1) to $A_r$, where $r \geq i+1$:

$$\begin{aligned}
\begin{vmatrix} A_{ij}^{(r)} & A_{ir}^{(r)} \\ A_{rj}^{(r)} & A_{rr}^{(r)} \end{vmatrix} &= A_r A_{ir,jr}^{(r)}, \\
&= A_r A_{ij}^{(r-1)}, \\
A_{r-1}A_{ij}^{(r)} - A_r A_{ij}^{(r-1)} &= A_{ir}^{(r)} A_{jr}^{(r)}.
\end{aligned} \tag{4.11.47}$$

Scale the cofactors and refer to Theorems 4.48 and 4.49a:

$$\begin{aligned}
A_r^{ij} - A_{r-1}^{ij} &= \frac{A_r}{A_{r-1}}\left(A_r^{ri}\ A_r^{rj}\right) \\
&= 2^{-(4r-5)} A_r^{ri} A_r^{rj} \\
&= 2\lambda_{r-1,i-1}\lambda_{r-1,j-1}.
\end{aligned} \tag{4.11.48}$$

Hence,

$$\begin{aligned}
2\sum_{r=i+1}^{n} \lambda_{r-1,i-1}\lambda_{r-1,j-1} &= \sum_{r=i+1}^{n}\left(A_r^{ij} - A_{r-1}^{ij}\right) \\
&= A_n^{ij} - A_i^{ij} \\
&= A_n^{ij} - 2^{2(i-1)}\lambda_{i-1,j-1},
\end{aligned} \tag{4.11.49}$$

which yields a formula for the scaled cofactor $A_n^{ij}$. The stated formula for the simple cofactor $A_{ij}^{(n)}$ follows from Theorem 4.49a. □

Let

$$E_n = |P_m(0)|_n, \quad 0 \leq m \leq 2n-2, \tag{4.11.50}$$

where $P_m(x)$ is the Legendre polynomial [Appendix A.5]. Then,

$$\begin{aligned}
P_{2m+1}(0) &= 0, \\
P_{2m}(0) &= \nu_m.
\end{aligned} \tag{4.11.51}$$

Hence,

$$E_n = \begin{vmatrix} \nu_0 & \bullet & \nu_1 & \bullet & \nu_2 & \cdots \\ \bullet & \nu_1 & \bullet & \nu_2 & \bullet & \cdots \\ \nu_1 & \bullet & \nu_2 & \bullet & \nu_3 & \cdots \\ \bullet & \nu_2 & \bullet & \nu_3 & \bullet & \cdots \\ \nu_2 & \bullet & \nu_3 & \bullet & \nu_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n. \tag{4.11.52}$$

**Theorem 4.51.**

$$E_n = |P_m(0)|_n = (-1)^{n(n-1)/2} 2^{-(n-1)^2}.$$

PROOF. By interchanging first rows and then columns in a suitable manner it is easy to show that

$$E_n = \begin{vmatrix} \nu_0 & \nu_1 & \nu_2 & \cdots & & & \\ \nu_1 & \nu_2 & \nu_3 & \cdots & & & \\ \nu_2 & \nu_3 & \nu_4 & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ & & & & \nu_1 & \nu_2 & \cdots \\ & & & & \nu_2 & \nu_3 & \cdots \\ & & & & \cdots & \cdots & \cdots \end{vmatrix}_n . \tag{4.11.53}$$

Hence, referring to Theorems 4.11.5 and 4.11.6b,

$$\begin{aligned} E_{2n} &= (-1)^n A_n A_{n+1,1}^{(n+1)} \\ &= (-1)^n 2^{-(2n-1)^2}, \\ E_{2n+1} &= (-1)^n A_{n+1} A_{n+1,1}^{(n+1)} \\ &= (-1)^n 2^{-4n^2}. \end{aligned} \tag{4.11.54}$$

These two results can be combined into one as shown in the theorem which is applied in Section 4.12.1 to evaluate $|P_m(x)|_n$. □

**Exercise.** If

$$B_n = \left| \binom{2m}{m} \right|_n, \quad 0 \le m \le 2n-2,$$

prove that

$$\begin{aligned} B_n &= 2^{n-1}, \\ B_{ij}^{(n)} &= 2^{2[n(n-1)-(i+j-2)]} A_{ij}^{(n)}, \\ B_{n1}^{(n)} &= 2^{n-1}. \end{aligned}$$

### *4.11.4 A Nonlinear Differential Equation*

Let

$$G_n(x,h,k) = |g_{ij}|_n,$$

where

$$g_{ij} = \begin{cases} \frac{x^{h+i+k-1}}{h+i+k-1}, & j = k \\ \frac{1}{h+i+j-1}, & j \neq k. \end{cases} \tag{4.11.55}$$

Every column in $G_n$ except column $k$ is identical with the corresponding column in the generalized Hilbert determinant $K_n(h)$. Also, let

$$G_n(x,h) = \sum_{k=1}^{n} G_n(x,h,k). \tag{4.11.56}$$

**Theorem 4.52.**

$$(xG_n')' = K_n(h)x^h P_n^2(x,h),$$

*where*

$$P_n(x,h) = \frac{D^{h+n}[x^n(1+x)^{h+n-1}]}{(h+n-1)!}.$$

Proof. Referring to (4.10.8),

$$\begin{aligned}
G_n(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{K_{ij}^{(n)} x^{h+i+j-1}}{h+i+j-1} \\
&= K_n(h)\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{V_{ni}V_{nj}x^{h+i+j-1}}{(h+i+j-1)^2}. \qquad (4.11.57)
\end{aligned}$$

Hence,

$$\begin{aligned}
(xG_n')' &= K_n(h)x^h \sum_{i=1}^{n}\sum_{j=1}^{n} V_{ni}V_{nj}x^{i+j-2} \\
&= K_n(h)x^h P_n^2(x,h), \qquad (4.11.58)
\end{aligned}$$

where

$$\begin{aligned}
P_n(x,h) &= \sum_{i=1}^{n} (-1)^{n+i} V_{ni} x^{i-1} \\
&= \sum_{i=1}^{n} \frac{(h+n+i-1)!x^{i-1}}{(i-1)!\,(n-i)!\,(h+i-1)!} \\
&= \sum_{i-1}^{n} \frac{D^{h+n}(x^{h+n+i-1})}{(n-i)!\,(h+i-1)!}, \qquad (4.11.59)
\end{aligned}$$

$$\begin{aligned}
(h+n-1)!\,P_n(x,h) &= \sum_{i=1}^{n} \binom{h+n-1}{h+i-1} D^{h+n}(x^{h+n+i-1}) \\
&= D^{h+n}\left[x^n \sum_{i=1}^{n}\binom{h+n-1}{h+i-1} x^{h+i-1}\right] \\
&= D^{h+n}\left[x^n \sum_{r=h}^{h+n-1}\binom{h+n-1}{r} x^{r}\right] \\
&= D^{h+n}\big[x^n(1+x)^{h+n-1} - p_{h+n-1}(x)\big], \qquad (4.11.60)
\end{aligned}$$

where $p_r(x)$ is a polynomial of degree $r$. The theorem follows. □

Let

$$E(x) = |e_{ij}(x)|_{n-1},$$

where

$$e_{ij}(x) = \frac{(1+x)^{i+j+1} - x^{i+j+1}}{i+j+1}. \tag{4.11.61}$$

**Theorem 4.53.** *The polynomial determinant $E$ satisfies the nonlinear differential equation*

$$\left[\{x(1+x)E\}''\right]^2 = 4n^2(xE)'\{(1+x)E\}'.$$

PROOF. Let

$$A(x,\xi) = |\phi_m(x,\xi)|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m(x,\xi) = \frac{1}{m+1}\left[(\xi+x)^{m+1} - c(\xi-1)^{m+1} + (c-1)\xi^{m+1}\right]. \tag{4.11.62}$$

Then,

$$\begin{aligned} \frac{\partial}{\partial\xi}\phi_m(x,\xi) &= m\phi_{m-1}(x,\xi), \\ \phi_0(x,\xi) &= x+c. \end{aligned} \tag{4.11.63}$$

Hence, from Theorem 4.33 in Section 4.9.1, $A$ is independent of $\xi$. Put $\xi = 0$ and $-x$ in turn and denote the resulting determinants by $U$ and $V$, respectively. Then,

$$A = U = V, \tag{4.11.64}$$

where

$$\begin{aligned} U(x,c) &= |\phi_m(x,0)|_n \\ &= \left|\frac{x^{m+1} + (-1)^m c}{m+1}\right|_n, \quad 0 \le m \le 2n-2 \\ &= \left|\frac{x^{i+j-1} + (-1)^{i+j} c}{i+j-1}\right|_n. \end{aligned} \tag{4.11.65}$$

Put

$$\begin{aligned} \psi_m(x) &= \phi_m(x,-x) \\ &= \frac{(-1)^m}{m+1}[c(1+x)^{m+1} + (1-c)x^{m+1}] \end{aligned} \tag{4.11.66}$$

$$\begin{aligned} V(x,c) &= |\psi_m(x)|_n, \\ &= \left|\frac{c(1+x)^{m+1} + (1-c)x^{m+1}}{m+1}\right|_n \\ &= \left|\frac{c(1+x)^{i+j-1} + (1-c)x^{i+j-1}}{i+j-1}\right|_n. \end{aligned} \tag{4.11.67}$$

Note that $U_{ij} \neq V_{ij}$ in general. Since

$$\begin{aligned} \psi'_m &= -m\psi_{m-1}, \\ \psi_0 &= x + c, \end{aligned} \tag{4.11.68}$$

it follows that

$$\begin{aligned} V' &= V_{11} \\ &= \left| \frac{c(1+x)^{i+j+1} + (1-c)x^{i+j+1}}{i+j+1} \right|_{n-1}. \end{aligned} \tag{4.11.69}$$

Expand $U$ and $V$ as a polynomial in $c$:

$$U(x,c) = V(x,c) = \sum_{r=0}^{n} f_r(x) c^{n-r}. \tag{4.11.70}$$

However, since

$$\psi_m = y_m c + z_m,$$

where $z_m$ is independent of $c$,

$$y_m = (-1)^m \left[ \frac{(1+x)^{m+1} - x^{m+1}}{m+1} \right], \tag{4.11.71}$$

$$\begin{aligned} y'_m &= -m y_{m-1}, \\ y_0 &= 1, \end{aligned} \tag{4.11.72}$$

it follows from the first line of (4.11.67) that $f_0$, the coefficient of $c^n$ in $V$, is given by

$$\begin{aligned} f_0 &= |y_m|_n \\ &= \text{constant}. \end{aligned} \tag{4.11.73}$$

$$c^{n-1} V(x, c^{-1}) = f_0 c^{-1} + f_1 + \sum_{r=1}^{n-1} f_{r+1} c^r,$$

where

$$\begin{aligned} f'_1 &= \left[ c^{n-1} D_x V(x, c^{-1}) \right]_{c=0}, \quad D_x = \frac{\partial}{\partial x}, \\ &= \left[ c^{n-1} V_{11}(x, c^{-1}) \right]_{c=0} \\ &= \left[ c^{n-1} \left| \frac{c^{-1}(1+x)^{i+j+1} + (1-c^{-1})x^{i+j+1}}{i+j+1} \right|_{n-1} \right]_{c=0} \\ &= E. \end{aligned} \tag{4.11.74}$$

Furthermore,

$$D_c\{c^n U(x, c^{-1})\} = D_c\{c^n V(x, c^{-1})\}, \quad D_c = \frac{\partial}{\partial c},$$

$$= D_c \sum_{r=0}^{n} f_r c^r$$
$$= f_1 + \sum_{r=2}^{n} r f_r c^{r-1}. \tag{4.11.75}$$

Hence,

$$\begin{aligned}
f_1 &= \left[D_c\{c^n U(x, c^{-1})\}\right]_{c=0} \\
&= D_c \left[ c^n \left| \frac{x^{i+j-1} - (-1)^{i+j} c^{-1}}{i+j-1} \right|_n \right]_{c=0} \\
&= \left[ D_c \left| \frac{c x^{i+j-1} - (-1)^{i+j}}{i+j-1} \right|_n \right]_{c=0} \\
&= \sum_{k=1}^{n} G_n(x, 0, k) \\
&= G_n(x, 0),
\end{aligned} \tag{4.11.76}$$

where $G_n(x, h, k)$ and $G_n(x, h)$ are defined in the first line of (4.11.55) and (4.11.56), respectively.

$$\begin{aligned}
E &= G', \\
(xE)' &= (xG')' \\
&= K_n P_n^2,
\end{aligned} \tag{4.11.77}$$

where

$$\begin{aligned}
K_n &= K_n(0), \\
P_n &= P_n(x, 0) \\
&= \frac{D^n[x^n(1+x)^{n-1}]}{(n-1)!}.
\end{aligned} \tag{4.11.78}$$

Let

$$Q_n = \frac{D^n[x^{n-1}(1+x)^n]}{(n-1)!}. \tag{4.11.79}$$

Then,

$$P_n(-1-x) = (-1)^n Q_n.$$

Since

$$E(-1-x) = E(x),$$

it follows that

$$\begin{aligned}
\{(1+x)E\}' &= K_n Q_n^2, \\
\{xE\}'\{(1+x)E\}' &= (K_n P_n Q_n)^2.
\end{aligned} \tag{4.11.80}$$

The identity

$$xD^n[x^{n-1}(1+x)^n] = nD^{n-1}[x^n(1+x)^{n-1}] \tag{4.11.81}$$

can be proved by showing that both sides are equal to the polynomial

$$n!\sum_{r=1}^{n}\binom{n}{r}\binom{n+r-1}{n}x^r.$$

It follows by differentiating (4.11.79) that

$$(xQ_n)' = nP_n. \tag{4.11.82}$$

Hence,

$$\begin{aligned}\{x(1+x)E\}' &= (1+x)E + x\{(1+x)E\}' \\ &= (1+x)E + K_n xQ_n^2, \qquad (4.11.83)\\ \{x(1+x)E\}'' &= K_n Q_n^2 + K_n(Q_n^2 + 2xQ_nQ_n') \\ &= 2K_nQ_n(xQ_n)' \\ &= 2nK_nP_nQ_n. \qquad (4.11.84)\end{aligned}$$

The theorem follows from (4.11.80). □

A polynomial solution to the differential equation in Theorem 4.47, and therefore the expansion of the determinant $E$, has been found by Chalkley using a method based on an earlier publication.

## *Exercises*

**1.** Prove that

$$\left|\frac{(1+x)^{m+1}+c-1}{m+1}\right|_n = U = V, \qquad 0 \le m \le 2n-2.$$

**2.** Prove that

$$(1+x)D^n[x^n(1+x)^{n-1}] = nD^{n-1}[x^{n-1}(1+x)^n]$$

$$[(1+x)P_n]' = nQ_n.$$

Hence, prove that

$$[X^2(X^2E)'']' = 4n^2X(XE)',$$

where

$$X = \sqrt{x(1+x)}\,.$$

## 4.12 Hankelians 5

Notes in orthogonal and other polynomials are given in Appendices A.5 and A.6. Hankelians whose elements are polynomials have been evaluated by a variety of methods by Geronimus, Beckenbach et al., Lawden, Burchnall, Seidel, Karlin and Szegö, Das, and others. Burchnall's methods apply the Appell equation but otherwise have little in common with the proof of the first theorem in which $L_m(x)$ is the simple Laguerre polynomial.

### *4.12.1 Orthogonal Polynomials*

**Theorem 4.54.**

$$\underset{0\le m\le 2n-2}{|L_m(x)|_n} = \frac{(-1)^{n(n-1)/2}0!\,1!\,2!\cdots(n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(2n-2)!}x^{n(n-1)}, \quad n \ge 2.$$

PROOF. Let

$$\phi_m(x) = x^m L_m\left(\frac{1}{x}\right),$$

then

$$\begin{aligned}\phi'_m(x) &= m\phi_{m-1}(x),\\ \phi_0 &= 1.\end{aligned} \tag{4.12.1}$$

Hence, $\phi_m$ is an Appell polynomial in which

$$\phi_m(0) = \frac{(-1)^m}{m!}.$$

Applying Theorem 4.33 in Section 4.9.1 on Hankelians with Appell polynomial elements and Theorem 4.47b in Section 4.11.3 on determinants with binomial and factorial elements,

$$\begin{aligned}\left|x^m L_m\left(\frac{1}{x}\right)\right|_n &= |\phi_m(x)|_n, \quad 0 \le m \le 2n-2\\ &= |\phi_m(0)|_n\\ &= \left|\frac{(-1)^m}{m!}\right|_n\\ &= \left|\frac{1}{m!}\right|_n\\ &= \frac{(-1)^{n(n-1)/2}0!\,1!\,2!\cdots(n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(2n-2)!}.\end{aligned} \tag{4.12.2}$$

But

$$\left|x^m L_m\left(\frac{1}{x}\right)\right|_n = x^{n(n-1)}\left|L_m\left(\frac{1}{x}\right)\right|_n. \tag{4.12.3}$$

The theorem follows from (4.12.2) and (4.12.3) after replacing $x$ by $x^{-1}$. $\square$

In the next theorem, $P_m(x)$ is the Legendre polynomial.

**Theorem 4.55.**

$$\underset{0\le m\le 2n-2}{|P_m(x)|_n} = 2^{-(n-1)^2}(x^2-1)^{n(n-1)/2}.$$

*First Proof.* Let

$$\phi_m(x) = (1-x^2)^{-m/2}P_m(x).$$

Then

$$\phi'_m(x) = mF\phi_{m-1}(x)$$

where

$$\begin{aligned} F &= (1-x^2)^{-3/2} \\ \phi_0 &= P_0(x) = 1. \end{aligned} \tag{4.12.4}$$

Hence, if $A = |\phi_m(x)|_n$, then $A' = 0$ and $A = |\phi_m(0)|_n$.

$$\begin{aligned} |P_m(x)|_n &= \left|(1-x^2)^{m/2}\phi_m(x)\right|_n, \quad 0 \le m \le 2n-2 \\ &= (1-x^2)^{n(n-1)/2}|\phi_m(x)|_n \\ &= (1-x^2)^{n(n-1)/2}|\phi_m(0)|_n \\ &= (1-x^2)^{n(n-1)/2}|P_m(0)|_n. \end{aligned}$$

The formula

$$|P_m(0)|_n = (-1)^{n(n-1)/2}2^{-(n-1)^2}$$

is proved in Theorem 4.50 in Section 4.11.3 on determinants with binomial and factorial elements. The theorem follows. $\square$

Other functions which contain orthogonal polynomials and which satisfy the Appell equation are given by Carlson.

The second proof, which is a modified and detailed version of a proof outlined by Burchnall with an acknowledgement to Chaundy, is preceded by two lemmas.

**Lemma 4.56.** *The Legendre polynomial $P_n(x)$ is equal to the coefficient of $t^n$ in the polynomial expansion of $[(u+t)(v+t)]^n$, where $u = \frac{1}{2}(x+1)$ and $v = \frac{1}{2}(x-1)$.*

PROOF. Applying the Rodrigues formula for $P_n(x)$ and the Cauchy integral formula for the $n$th derivative of a function,

$$P_n(x) = \frac{1}{2^n n!}D^n(x^2-1)^n$$

$$\begin{aligned}
&= \frac{1}{2^{n+1}\pi i}\int_C \frac{(\zeta^2-1)^n}{(\zeta-x)^{n+1}}\,d\zeta \qquad (\text{put } \zeta = x+2t)\\
&= \frac{1}{2^{n+1}\pi i}\int_{C'} \frac{[(x+1+2t)(x-1+2t)]^n}{(2t)^{n+1}}\,dt\\
&= \frac{1}{2\pi i}\int_{C'} \frac{g(t)}{t^{n+1}}\,dt\\
&= \frac{g^{(n)}(0)}{n!},
\end{aligned}$$

where

$$g(t) = \left[\left\{\tfrac{1}{2}(x+1)+t\right\}\left\{\tfrac{1}{2}(x-1)+t\right\}\right]^n. \tag{4.12.5}$$

The lemma follows. □

$$\begin{aligned}
[(u+t)(v+t)]^n &= \sum_{r=0}^{n}\sum_{s=0}^{n}\binom{n}{r}\binom{n}{s}u^{n-r}v^{n-s}t^{r+s}\\
&= \sum_{p=0}^{2n}\lambda_{np}t^p
\end{aligned}$$

where

$$\lambda_{np} = \sum_{s=0}^{p}\binom{n}{s}\binom{n}{p-s}u^{n-s}v^{n-p+s}, \quad 0 \le p \le 2n, \tag{4.12.6}$$

which, by symmetry, is unaltered by interchanging $u$ and $v$.

In particular,

$$\lambda_{00} = 1, \quad \lambda_{n0} = (uv)^n, \quad \lambda_{n,2n} = 1, \quad \lambda_{nn} = P_n(x). \tag{4.12.7}$$

**Lemma 4.57.**

**a.** $\lambda_{i,i-r} = (uv)^r\lambda_{i,i+r}$,
**b.** $\lambda_{i,i-r}\lambda_{j,j+r} = \lambda_{i,i+r}\lambda_{j,j-r}$.

Proof.

$$\begin{aligned}
\lambda_{n,n+r} &= \sum_{s=0}^{n+r}\binom{n}{s}\binom{n}{n+r-s}u^{n-s}v^{s-r}\\
&= \sum_{s=r}^{n+r}\binom{n}{s}\binom{n}{s-r}u^{n-s}v^{s-r}.
\end{aligned}$$

Changing the sign of $r$,

$$\lambda_{n,n-r} = \sum_{s=-r}^{n-r}\binom{n}{s}\binom{n}{s+r}u^{n-s}v^{s+r} \qquad (\text{put } s = n-\sigma)$$

$$= \sum_{\sigma=r}^{n+r} \binom{n}{n-\sigma} \binom{n}{n-\sigma+r} u^{\sigma} v^{n-\sigma+r}$$

$$= (uv)^r \sum_{\sigma=r}^{n+r} \binom{n}{\sigma} \binom{n}{\sigma-r} u^{\sigma-r} v^{n-\sigma}. \tag{4.12.8}$$

Part (a) follows after interchanging $u$ and $v$ and replacing $n$ by $i$. Part (b) then follows easily. □

It follows from Lemma 4.56 that $P_{i+j}(x)$ is equal to the coefficient of $t^{i+j}$ in the expansion of the polynomial

$$\begin{aligned}[(u+t)(v+t)]^{i+j} &= [(u+t)(v+t)]^i [(u+t)(v+t)]^j \\ &= \sum_{r=0}^{2i} \lambda_{ir} t^r \sum_{s=0}^{2j} \lambda_{js} t^s. \end{aligned} \tag{4.12.9}$$

Each sum consists of an odd number of terms, the center terms being $\lambda_{ii} t^i$ and $\lambda_{jj} t^j$ respectively. Hence, referring to Lemma 4.57,

$$\begin{aligned} P_{i+j}(x) &= \sum_{r=1}^{\min(i,j)} \lambda_{i,i-r} \lambda_{j,j+r} + \lambda_{ii}\lambda_{jj} + \sum_{r=1}^{\min(i,j)}{}^{\dagger} \lambda_{i,i+r}\lambda_{j,j-r} \\ &= 2 \sum_{r=0}^{\min(i,j)} \lambda_{i,i+r}\lambda_{j,j-r}, \end{aligned} \tag{4.12.10}$$

where the symbol † denotes that the factor 2 is omitted from the $r=0$ term. Replacing $i$ by $i-1$ and $j$ by $j-1$,

$$P_{i+j-2}(x) = 2 \sum_{r=0}^{\min(i,j)}{}^{\dagger} \lambda_{i-1,i-1+r} \lambda_{j-1,j-1-r}. \tag{4.12.11}$$

Preparations for the second proof are now complete. Adjusting the dummy variable and applying, in reverse, the formula for the product of two determinants (Section 1.4),

$$\begin{aligned} |P_{i+j-2}|_n &= \left| 2 \sum_{s=1}^{\min(i,j)}{}^{\dagger} \lambda_{i-1,i+s-2} \lambda_{j-1,j-s} \right|_n \\ &= \left|2\lambda^*_{i-1,i+j-2}\right|_n \left|\lambda_{j-1,j-i}\right|_n, \end{aligned} \tag{4.12.12}$$

where the symbol $*$ denotes that the factor 2 is omitted when $j=1$. Note that $\lambda_{np} = 0$ if $p<0$ or $p>2n$. The first determinant is lower triangular and the second is upper triangular so that the value of each determinant is given by the product of the elements in its principal diagonal:

$$|P_{i+j-2}|_n = 2^{n-1} \prod_{i=1}^{n} \lambda_{i-1,2i-2} \lambda_{j-1,0}$$

$$\begin{aligned}
&= 2^{n-1}\prod_{i=2}^{n}(uv)^{i-1}\\
&= 2^{n-1}(uv)^{1+2+3+\cdots+\overline{n-1}}\\
&= 2^{-(n-1)^2}(x^2-1)^{\frac{1}{2}n(n-1)}.
\end{aligned}$$

which completes the proof.

### *Exercises*

**1.** Prove that

$$\left|H_m(x)\right|_n = (-2)^{n(n-1)/2}1!\,2!\,3!\cdots(n-1)!,$$
$$0\le m\le 2n-2$$

where $H_m(x)$ is the Hermite polynomial.

**2.** If

$$A_n = \begin{vmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{vmatrix},$$

prove that

$$n(n+1)A_n'' = 2(P_n')^2. \qquad \text{(Beckenbach et al.)}$$

## *4.12.2 The Generalized Geometric Series and Eulerian Polynomials*

Notes on the generalized geometric series $\phi_m(x)$, the closely related function $\psi_m(x)$, the Eulerian polynomial $A_n(x)$, and Lawden's polynomial $S_n(x)$ are given in Appendix A.6.

$$\begin{aligned}
\psi_m(x) &= \sum_{r=1}^{\infty} r^m x^r,\\
x\psi_m'(x) &= \psi_{m+1}(x), &&(4.12.13)\\
S_m(x) &= (1-x)^{m+1}\psi_m, \quad m\ge 0, &&(4.12.14)\\
A_m(x) &= S_m(x), \quad m>0,\\
A_0 &= 1, \quad S_0 = x. &&(4.12.15)
\end{aligned}$$

**Theorem (Lawden).**

$$\begin{aligned}
E_n &= |\psi_{i+j-2}|_n = \frac{\lambda_n x^{n(n+1)/2}}{(1-x)^{n^2}},\\
F_n &= |\psi_{i+j-1}|_n = \frac{\lambda_n n!\,x^{n(n+1)/2}}{(1-x)^{n(n+1)}},\\
G_n &= |\psi_{i+j}|_n = \frac{\lambda_n (n!)^2 x^{n(n+1)/2}(1-x^{n+1})}{(1-x)^{(n+1)^2}},
\end{aligned}$$

$$H_n = |S_{i+j-2}|_n = \lambda_n x^{n(n+1)/2},$$
$$J_n = |A_{i+j-2}|_n = \lambda_n x^{n(n-1)/2},$$

*where*

$$\lambda_n = \big[1!\ 2!\ 3! \cdots (n-1)!\big]^2.$$

The following proofs differ from the originals in some respects.

Proof. It is proved using a slightly different notation in Theorem 4.28 in Section 4.8.5 on Turanians that

$$E_n G_n - E_{n+1} G_{n-1} = F_n^2,$$

which is equivalent to

$$E_{n-1} G_{n-1} - E_n G_{n-2} = F_{n-1}^2. \tag{4.12.16}$$

Put $x = e^t$ in (4.12.5) so that

$$\begin{aligned} D_x &= e^{-t} D_t \\ D_t &= x D_x, \quad D_x = \frac{\partial}{\partial x}, \text{ etc.} \end{aligned} \tag{4.12.17}$$

Also, put

$$\begin{aligned} \theta_m(t) &= \psi_m(e^t) \\ &= \sum_{r=1}^{\infty} r^m e^{rt} \\ \theta'_m(t) &= \theta_{m+1}(t). \end{aligned} \tag{4.12.18}$$

Define the column vector $\mathbf{C}_j(t)$ as follows:

$$\mathbf{C}_j(t) = \big[\theta_j(t)\ \theta_{j+1}(t)\ \theta_{j+2}(t) \ldots\big]^T$$

so that

$$\mathbf{C}'_j = \mathbf{C}_{j+1}(t). \tag{4.12.19}$$

The number of elements in $\mathbf{C}_j$ is equal to the order of the determinant of which it is a part, that is, $n$, $n-1$, or $n-2$ in the present context.

Let

$$Q_n(t,\tau) = \big|\mathbf{C}_0(\tau)\ \mathbf{C}_1(t)\ \mathbf{C}_2(t) \cdots \mathbf{C}_{n-1}(t)\big|_n, \tag{4.12.20}$$

where the argument in the first column is $\tau$ and the argument in each of the other columns is $t$. Then,

$$Q_n(t,t) = E_n. \tag{4.12.21}$$

Differentiate $Q_n$ repeatedly with respect to $\tau$, apply (4.12.19), and put $\tau = t$.

$$D_\tau^r\{Q_n(t,t)\} = 0, \quad 1 \le r \le n-1, \tag{4.12.22}$$

$$\begin{aligned}
D_\tau^n\{Q_n(t,t)\} &= \left|\mathbf{C}_n(t)\ \mathbf{C}_1(t)\ \mathbf{C}_2(t)\cdots\mathbf{C}_{n-1}(t)\right|_n \\
&= (-1)^{n-1}\left|\mathbf{C}_1(t)\ \mathbf{C}_2(t)\cdots\mathbf{C}_{n-1}(t)\ \mathbf{C}_n(t)\right|_n \\
&= (-1)^{n-1}F_n. \qquad (4.12.23)
\end{aligned}$$

The cofactors $Q_{i1}^{(n)}$, $1 \le i \le n$, are independent of $\tau$.

$$\begin{aligned}
Q_{11}^{(n)}(t) &= E_{11}^{(n)} = G_{n-1}, \\
Q_{n1}^{(n)}(t) &= (-1)^{n+1}\left|\mathbf{C}_1(t)\ \mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\right|_{n-1} \\
&= (-1)^{n+1}F_{n-1}, \\
Q_{1n}^{(n)}(t,\tau) &= (-1)^{n+1}\left|\mathbf{C}_1(\tau)\ \mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\right|_{n-1}. \qquad (4.12.24)
\end{aligned}$$

Hence,

$$\begin{aligned}
D_\tau^r\{Q_{1n}^{(n)}(t,t)\} &= 0, \quad 1 \le r \le n-2 \\
D_\tau^{n-1}\{Q_{1n}^{(n)}(t,t)\} &= (-1)^{n+1}\left|\mathbf{C}_n(t)\ \mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\right|_{n-1} \\
&= -\left|\mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\ \mathbf{C}_n(t)\right|_{n-1} \\
&= -G_{n-1}, \\
D_\tau^n\{Q_{1n}^{(n)}(t,t)\} &= -D_t(G_{n-1}), \\
Q_{nn}^{(n)}(t,\tau) &= Q_{n-1}(t,\tau), \\
Q_{nn}^{(n)}(t,t) &= E_{n-1}, \\
D_\tau^r\{Q_{nn}^{(n)}(t,t)\} &= \begin{cases} 0, & 1 \le r \le n-2 \\ (-1)^n F_{n-1}, & r = n-1 \\ (-1)^n D_t(F_{n-1}), & r = n. \end{cases} \qquad (4.12.25) \\
Q_{1n,1n}^{(n)}(t) &= G_{n-2}. \qquad (4.12.26)
\end{aligned}$$

Applying the Jacobi identity to the cofactors of the corner elements of $Q_n$,

$$\begin{aligned}
\begin{vmatrix} Q_{11}^{(n)}(t) & Q_{1n}^{(n)}(t,\tau) \\ Q_{n1}^{(n)}(t) & Q_{nn}^{(n)}(t,\tau) \end{vmatrix} &= Q_n(t,\tau)Q_{1n,1n}^{(n)}(t), \\
\begin{vmatrix} G_{n-1} & Q_{1n}^{(n)}(t,\tau) \\ (-1)^{n+1}F_{n-1} & Q_{nn}^{(n)}(t,\tau) \end{vmatrix} &= Q_n(t,\tau)G_{n-2}. \qquad (4.12.27)
\end{aligned}$$

The first column of the determinant is independent of $\tau$, hence, differentiating $n$ times with respect to $\tau$ and putting $\tau = t$,

$$\begin{aligned}
\begin{vmatrix} G_{n-1} & D_t(G_{n-1}) \\ (-1)^{n+1}F_{n-1} & (-1)^n D_t(F_{n-1}) \end{vmatrix} &= (-1)^{n+1}F_nG_{n-2}, \\
G_{n-1}D_t(F_{n-1}) - F_{n-1}D_t(G_{n-1}) &= -F_nG_{n-2}, \\
D_t\left[\frac{G_{n-1}}{F_{n-1}}\right] &= \frac{F_nG_{n-2}}{F_{n-1}^2}.
\end{aligned}$$

Reverting to $x$ and referring to (4.12.17),

$$xD_x\left[\frac{G_{n-1}}{F_{n-1}}\right] = \frac{F_n G_{n-2}}{F_{n-1}^2}, \tag{4.12.28}$$

where the elements in the determinants are now $\psi_m(x)$, $m = 0, 1, 2, \ldots$.

The difference formula

$$\Delta^m \psi_0 = x\psi_m, \quad m = 1, 2, 3, \ldots, \tag{4.12.29}$$

is proved in Appendix A.8. Hence, applying the theorem in Section 4.8.2 on Hankelians whose elements are differences,

$$\begin{aligned}
E_n &= |\psi_m|_n, \quad 0 \le m \le 2n-2 \\
&= |\Delta^m \psi_0|_n \\
&= \begin{vmatrix} \psi_0 & x\psi_1 & x\psi_2 & \cdots \\ x\psi_1 & x\psi_2 & x\psi_3 & \cdots \\ x\psi_2 & x\psi_3 & x\psi_4 & \cdots \\ \multicolumn{4}{c}{\dotfill} \end{vmatrix}_n .
\end{aligned} \tag{4.12.30}$$

Every element except the one in position $(1, 1)$ contains the factor $x$. Hence, removing these factors and applying the relation

$$\begin{aligned}
\psi_0/x &= \psi_0 + 1, \\
E_n &= x^n \begin{vmatrix} \psi_0 + 1 & \psi_1 & \psi_2 & \cdots \\ \psi_1 & \psi_2 & \psi_3 & \cdots \\ \psi_2 & \psi_3 & \psi_4 & \cdots \\ \multicolumn{4}{c}{\dotfill} \end{vmatrix}_n \\
&= x^n\left(E_n + E_{11}^{(n)}\right).
\end{aligned} \tag{4.12.31}$$

Hence

$$E_{11}^{(n)} = G_{n-1} = \left(\frac{1-x^n}{x^n}\right) E_n. \tag{4.12.32}$$

Put

$$\begin{aligned}
u_n &= \frac{G_n}{F_n}, \\
v_n &= \frac{E_{n-1}}{E_n}.
\end{aligned} \tag{4.12.33}$$

The theorem is proved by deducing and solving a differential–difference equation satisfied by $u_n$:

$$\frac{v_n}{v_{n+1}} = \frac{E_{n-1}E_{n+1}}{E_n^2}.$$

From (4.12.32),

$$\frac{G_{n-1}}{G_n} = \frac{x(1-x^n)v_{n+1}}{1-x^{n+1}}. \tag{4.12.34}$$

From (4.12.28) and (4.12.33),

$$xu'_{n-1} = \left(\frac{G_{n-1}}{F_{n-1}}\right)^2 \left(\frac{F_n}{G_n}\right)\left(\frac{G_{n-2}}{G_{n-1}}\right)\left(\frac{G_n}{G_{n-1}}\right),$$

$$\frac{u_n u'_{n-1}}{u_{n-1}^2} = \frac{(1-x^{n-1})(1-x^{n+1})}{x(1-x^n)^2}\left[\frac{v_n}{v_{n+1}}\right]. \tag{4.12.35}$$

From (4.12.16),

$$\begin{aligned}\left(\frac{F_{n-1}}{G_{n-1}}\right)^2 &= \frac{E_{n-1}}{G_{n-1}} - \left(\frac{E_n}{G_{n-1}}\right)\left(\frac{G_{n-2}}{G_{n-1}}\right)\\ &= \left(\frac{E_{n-1}}{E_n}\right)\left(\frac{E_n}{G_{n-1}}\right)\left[1-\left(\frac{E_n}{E_{n-1}}\right)\left(\frac{G_{n-2}}{G_{n-1}}\right)\right]\\ \frac{1}{u_{n-1}^2} &= v_n\left(\frac{x^n}{1-x^n}\right)\left[1-\frac{x(1-x^{n-1})}{1-x^n}\right]\\ &= \frac{x^n(1-x)}{(1-x^n)^2}v_n.\end{aligned}$$

Replacing $n$ by $n+1$,

$$\frac{1}{u_n^2} = \frac{x^{n+1}(1-x)}{(1-x^{n+1})^2}v_{n+1}. \tag{4.12.36}$$

Hence,

$$\left(\frac{u_n}{u_{n-1}}\right)^2 = \frac{1}{x}\left(\frac{1-x^{n+1}}{1-x^n}\right)^2\left[\frac{v_n}{v_{n+1}}\right]. \tag{4.12.37}$$

Eliminating $v_n/v_{n+1}$ from (4.12.35) yields the differential–difference equation

$$u_n = \left(\frac{1-x^{n+1}}{1-x^{n-1}}\right)u'_{n-1}. \tag{4.12.38}$$

Evaluating $u_n$ as defined by (4.12.33) for small values of $n$, it is found that

$$u_1 = \frac{1!(1-x^2)}{(1-x)^2}, \quad u_2 = \frac{2!(1-x^3)}{(1-x)^3}, \quad u_3 = \frac{3!(1-x^4)}{(1-x)^4}. \tag{4.12.39}$$

The solution which satifies (4.12.38) and (4.12.39) is

$$u_n = \frac{G_n}{F_n} = \frac{n!(1-x^{n+1})}{(1-x)^{n+1}}. \tag{4.12.40}$$

From (4.12.36),

$$v_n = \frac{E_{n-1}}{E_n} = \frac{(1-x)^{2n-1}}{(n-1)!^2 x^n},$$

which yields the difference equation

$$E_n = \frac{(n-1)!^2 x^n}{(1-x)^{2n-1}}E_{n-1}. \tag{4.12.41}$$

Evaluating $E_n$ for small values of $n$, it is found that

$$E_1 = \frac{x}{1-x}, \quad E_2 = \frac{1!^2 x^3}{(1-x)^4}, \quad E_3 = \frac{[1!\,2!]^2 x^6}{(1-x)^9}\,. \tag{4.12.42}$$

The solution which satisfies (4.12.41) and (4.12.42) is as given in the theorem. It is now a simple exercise to evaluate $F_n$ and $G_n$. $G_n$ is found in terms of $E_{n+1}$ by replacing $n$ by $n+1$ in (4.12.32) and then $F_n$ is given in terms of $G_n$ by (4.12.40). The results are as given in the theorem. The proof of the formula for $H_n$ follows from (4.12.14).

$$\begin{aligned} H_n &= |S_m|_n \\ &= |(1-x)^{m+1}\psi_m|_n \\ &= (1-x)^{n^2}|\psi_m|_n \\ &= (1-x)^{n^2} E_n. \end{aligned} \tag{4.12.43}$$

The given formula follows. The formula for $J_n$ is proved as follows:

$$J_n = |A_m|_n.$$

Since

$$A_0 = 1 = (1-x)(\psi_0 + 1), \tag{4.12.44}$$

it follows by applying the second line of (4.12.31) that

$$\begin{aligned} J_n &= \begin{vmatrix} (1-x)(\psi_0+1) & (1-x)^2\psi_1 & (1-x)^3\psi_2 & \cdots \\ (1-x)^2\psi_1 & (1-x)^3\psi_2 & (1-x)^4\psi_3 & \cdots \\ (1-x)^3\psi_2 & (1-x)^4\psi_3 & (1-x)^5\psi_4 & \cdots \\ \multicolumn{4}{c}{\dotfill} \end{vmatrix}_n \\ &= (1-x)^{n^2} \begin{vmatrix} \psi_0+1 & \psi_1 & \psi_2 & \cdots \\ \psi_1 & \psi_2 & \psi_3 & \cdots \\ \psi_2 & \psi_3 & \psi_4 & \cdots \\ \multicolumn{4}{c}{\dotfill} \end{vmatrix}_n \\ &= (1-x)^{n^2} x^{-n} E_n \\ &= x^{-n} H_n, \end{aligned} \tag{4.12.45}$$

which yields the given formula and completes the proofs of all five parts of Lawden's theorem. □

### *4.12.3 A Further Generalization of the Geometric Series*

Let $A_n$ denote the Hankel–Wronskian defined as

$$A_n = |D^{i+j-2} f|_n, \quad D = \frac{d}{dt}, \quad A_0 = 1, \tag{4.12.46}$$

where $f$ is an arbitrary function of $t$. Then, it is proved that Section 6.5.2 on Toda equations that

$$D^2(\log A_n) = \frac{A_{n+1}A_{n-1}}{A_n^2}\,. \tag{4.12.47}$$

Put

$$g_n = D^2(\log A_n). \tag{4.12.48}$$

**Theorem 4.58.** *$g_n$ satisfies the differential–difference equation*

$$g_n = ng_1 + \sum_{r=1}^{n-1}(n-r)D^2(\log g_r).$$

PROOF. From (4.12.47),

$$\frac{A_{r+1}A_{r-1}}{A_r^2} = g_r,$$

$$\prod_{r=1}^{s}\frac{A_{r+1}}{A_r}\prod_{r=1}^{s}\frac{A_{r-1}}{A_r} = \prod_{r=1}^{s} g_r,$$

which simplifies to

$$\frac{A_{s+1}}{A_s} = A_1\prod_{r=1}^{s} g_r. \tag{4.12.49}$$

Hence,

$$\prod_{s=1}^{n-1}\frac{A_{s+1}}{A_s} = A_1^{n-1}\prod_{s=1}^{n-1}\prod_{r=1}^{s} g_r,$$

$$A_n = A_1^n\prod_{r=1}^{n-1} g_r^{n-r}$$

$$= A_1^n\prod_{r=1}^{n-1} g_{n-r}^r, \tag{4.12.50}$$

$$\log A_n = n\log A_1 + \sum_{r=1}^{n-1}(n-r)\log g_r. \tag{4.12.51}$$

The theorem appears after differentiating twice with respect to $t$ and referring to (4.12.48). □

In certain cases, the differential–difference equation can be solved and $A_n$ evaluated from (4.12.50). For example, let

$$f = \left(\frac{e^t}{1-e^t}\right)^p$$

$$= \sum_{r=0}^{\infty} \frac{(p)_r e^{(r+p)t}}{r!},$$

where

$$(p)_r = p(p+1)(p+2)\cdots(p+r-1) \tag{4.12.52}$$

and denote the corresponding determinant by $E_n^{(p)}$:

$$E_n^{(p)} = \left|\psi_m^{(p)}\right|_n, \quad 0 \le m \le 2n-2,$$

where

$$\begin{aligned} \psi_m^{(p)} &= D^m f \\ &= \sum_{r=0}^{\infty} \frac{(p)_r (r+p)^m e^{(r+p)t}}{r!} \,. \end{aligned} \tag{4.12.53}$$

**Theorem 4.59.**

$$E_n^{(p)} = \frac{e^{n(2p+n-1)t/2}}{(1-e^t)^{n(p+n-1)}} \prod_{r=1}^{n-1} r!(p)_r.$$

Proof. Put

$$g_r = \frac{\alpha_r e^t}{(1-e^t)^2}, \quad \alpha_r \text{ constant},$$

and note that, from (4.12.48),

$$\begin{aligned} g_1 &= D^2(\log f) \\ &= \frac{pe^t}{(1-e^t)^2}, \end{aligned}$$

so that $\alpha_1 = p$ and

$$\begin{aligned} \log g_r &= \log \alpha_r + t - 2\log(1-e^t), \\ D^2(\log g_r) &= \frac{2e^t}{(1-e^t)^2} \,. \end{aligned} \tag{4.12.54}$$

Substituting these functions into the differential–difference equation, it is found that

$$\begin{aligned} \alpha_n &= n\alpha_1 + 2\sum_{r=1}^{n-1}(n-r) \\ &= n(p+n-1). \end{aligned} \tag{4.12.55}$$

Hence,

$$\begin{aligned} g_n &= \frac{n(p+n-1)e^t}{(1-e^t)^2}, \\ g_{n-r} &= \frac{(n-r)(p+n-r-1)e^t}{(1-e^t)^2} \,. \end{aligned} \tag{4.12.56}$$

Substituting this formula into (4.12.50) with $A_n \to E_n^{(p)}$ and $E_n^{(1)} = f$,

$$E_n^{(p)} = \left(\frac{e^t}{1-e^t}\right)^p \prod_{r=1}^{n-1} \left[(n-r)(p+n-r-1)\frac{e^t}{(1-e^t)^2}\right]^r, \qquad (4.12.57)$$

which yields the stated formula. □

Note that the substitution $x = e^t$ yields

$$\begin{aligned} \psi_m^{(1)} &= \psi_m, \\ E_n^{(1)} &= E_n, \end{aligned}$$

so that $\psi_m^{(p)}$ may be regarded as a further generalization of the geometric series $\psi_m$ and $E_n^{(p)}$ is a generalization of Lawden's determinant $E_n$.

**Exercise.** If

$$f = \begin{cases} \sec^p x \\ \operatorname{cosec}^p x \end{cases},$$

prove that

$$A_n = \begin{cases} \sec^{n(p+n-1)} x \\ \operatorname{cosec}^{n(p+n-1)} \end{cases} \prod_{r=1}^{n-1} r!\,(p)_r.$$

## 4.13 Hankelians 6

### *4.13.1 Two Matrix Identities and Their Corollaries*

Define three matrices $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{N}$ of order $n$ as follows:

$$\begin{aligned} \mathbf{M} &= [\alpha_{ij}]_n && \text{(symmetric)}, \\ \mathbf{K} &= [2^{i+j-1}k_{i+j-2}]_n && \text{(Hankel)}, \\ \mathbf{N} &= [\beta_{ij}]_n && \text{(lower triangular)}, \end{aligned} \qquad (4.13.1)$$

where

$$\alpha_{ij} = \begin{cases} (-1)^{j-1}u_{i-j} + u_{i+j-2}, & j \le i \\ (-1)^{i-1}u_{j-i} + u_{i+j+2}, & j \ge i; \end{cases} \qquad (4.13.2)$$

$$u_r = \sum_{j=1}^{N} a_j f_r(x_j), \quad a_j \text{ arbitrary}; \qquad (4.13.3)$$

$$f_r(x) = \tfrac{1}{2}\left\{(x+\sqrt{1+x^2})^r + (x-\sqrt{1+x^2})^r\right\}; \qquad (4.13.4)$$

$$k_r = \sum_{j=1}^{N} a_j x_j^r; \qquad (4.13.5)$$

$$\beta_{ij} = 0, \qquad j > i \text{ or } i+j \text{ odd},$$

$$\begin{aligned}
\beta_{2i,2i} &= \lambda_{ii}, \qquad i \geq 1,\\
\beta_{2i+1,2j+1} &= \lambda_{ij}, \qquad 0 \leq j \leq i,\\
\beta_{2i+2,2j} &= \lambda_{i+1,j} - \lambda_{ij}, \qquad 1 \leq j \leq i+1, && (4.13.6)\\
\lambda_{ij} &= \frac{i}{i+j}\binom{i+j}{2j};\\
\lambda_{ii} &= \tfrac{1}{2}, \quad i > 0; \quad \lambda_{i0} = 1,\ i \geq 0. && (4.13.7)
\end{aligned}$$

The functions $\lambda_{ij}$ and $f_r(x)$ appear in Appendix A.10.

**Theorem 4.60.**

$$\mathbf{M} = \mathbf{N}\mathbf{K}\mathbf{N}^T.$$

Proof. Let

$$\mathbf{G} = [\gamma_{ij}]_n = \mathbf{N}\mathbf{K}\mathbf{N}^T. \tag{4.13.8}$$

Then

$$\begin{aligned}
\mathbf{G}^T &= \mathbf{N}\mathbf{K}^T\mathbf{N}^T\\
&= \mathbf{N}\mathbf{K}\mathbf{N}^T\\
&= \mathbf{G}.
\end{aligned}$$

Hence, $\mathbf{G}$ is symmetric, and since $\mathbf{M}$ is also symmetric, it is sufficient to prove that $\alpha_{ij} = \gamma_{ij}$ for $j \leq i$. There are four cases to consider:

**i.** $i, j$ both odd,
**ii.** $i$ odd, $j$ even,
**iii.** $i$ even, $j$ odd,
**iv.** $i, j$ both even.

To prove case (i), put $i = 2p+1$ and $j = 2q+1$ and refer to Appendix A.10, where the definition of $g_r(x)$ is given in (A.10.7), the relationships between $f_r(x)$ and $g_r(x)$ are given in Lemmas (a) and (b) and identities among the $g_r(x)$ are given in Theorem 4.61.

$$\begin{aligned}
\alpha_{2p+1,2q+1} &= u_{2q+2p} + u_{2q-2p}\\
&= \sum_{j=1}^{N} a_j\{f_{2q+2p}(x_j) + f_{2q-2p}(x_j)\}\\
&= \sum_{j=1}^{N} a_j\{g_{q+p}(x_j) + g_{q-p}(x_j)\}\\
&= 2\sum_{j=1}^{N} a_j g_p(x_j) g_q(x_j). && (4.13.9)
\end{aligned}$$

It follows from (4.13.8) and the formula for the product of three matrices (the exercise at the end of Section 3.3.5) with appropriate adjustments to the upper limits that

$$\gamma_{ij} = \sum_{r=1}^{i} \sum_{s=1}^{j} \beta_{ir} 2^{r+s-1} k_{r+s-2} \beta_{js}.$$

Hence,

$$\gamma_{2p+1,2q+1} = 2 \sum_{r=1}^{2p+1} \sum_{s=1}^{2q+1} \beta_{2p+1,r} 2^{r+s-2} k_{r+s-2} \beta_{2q+1,s}. \tag{4.13.10}$$

From the first line of (4.13.6), the summand is zero when $r$ and $s$ are even. Hence, replace $r$ by $2r + 1$, replace $s$ by $2s + 1$ and refer to (4.13.5) and (4.13.6),

$$\begin{aligned}
\gamma_{2p+1,2q+1} &= 2 \sum_{r=0}^{p} \sum_{s=0}^{q} \beta_{2p+1,2r+1} \beta_{2q+1,2s+1} 2^{2r+2s} k_{2r+2s} \\
&= 2 \sum_{r=0}^{p} \sum_{s=0}^{q} \lambda_{pr} \lambda_{qs} \sum_{j=1}^{N} a_j (2x_j)^{2r+2s} \\
&= 2 \sum_{j=1}^{N} a_j \sum_{r=0}^{p} \lambda_{pr} (2x_j)^{2r} \sum_{s=0}^{q} \lambda_{qs} (2x_j)^{2s} \\
&= 2 \sum_{j=1}^{N} a_j g_p(x_j) g_q(x_j) \\
&= \alpha_{2p+1,2q+1},
\end{aligned} \tag{4.13.11}$$

which completes the proof of case (i). Cases (ii)–(iv) are proved in a similar manner. □

**Corollary.**

$$\begin{aligned}
|\alpha_{ij}|_n = |\mathbf{M}|_n &= |\mathbf{N}|_n^2 |\mathbf{K}|_n \\
&= |\beta_{ij}|_n^2 |2^{i+j-1} k_{i+j-2}|_n \\
&= \left( \prod_{i=1}^{n} \beta_{ii} \right)^2 2^n |2^{i+j-2} k_{i+j-2}|_n.
\end{aligned} \tag{4.13.12}$$

*But,* $\beta_{11} = 1$ *and* $\beta_{ii} = \frac{1}{2}$, $2 \le i \le n$. *Hence, referring to Property (e) in Section 2.3.1,*

$$|\alpha_{ij}|_n = 2^{n^2-2n+2} |k_{i+j-2}|_n. \tag{4.13.13}$$

*Thus,* $\mathbf{M}$ *can be expressed as a Hankelian.*

Define three other matrices $\mathbf{M}'$, $\mathbf{K}'$, and $\mathbf{N}'$ of order $n$ as follows:

$$\begin{aligned} \mathbf{M}' &= [\alpha'_{ij}]_n && \text{(symmetric)}, \\ \mathbf{K}' &= [2^{i+j-1}(k_{i+j} + k_{i+j-2})]_n && \text{(Hankel)}, \\ \mathbf{N}' &= [\beta'_{ij}]_n && \text{(lower triangular)}, \end{aligned} \tag{4.13.14}$$

where $k_r$ is defined in (4.13.5);

$$\alpha'_{ij} = \begin{cases} (-1)^{j-1} u_{i-j} + u_{i+j}, & j \le i \\ (-1)^{i-1} u_{j-i} + u_{i+j}, & j \ge i, \end{cases} \tag{4.13.15}$$

$$\begin{aligned} \beta'_{ij} &= 0, && j > i \text{ or } i+j \text{ odd}, \\ \beta'_{2i,2j} &= \tfrac{1}{2}\mu_{ij}, && 1 \le j \le i, \\ \beta'_{2i+1,2j+1} &= \lambda_{ij} + \tfrac{1}{2}\mu_{ij}, && 0 \le j \le i. \end{aligned} \tag{4.13.16}$$

The functions $\lambda_{ij}$ and $\frac{1}{2}\mu_{ij}$ appear in Appendix A.10. $\mu_{ij} = (2j/i)\lambda_{ij}$.

**Theorem 4.61.**

$$\mathbf{M} = \mathbf{N}'\mathbf{K}(\mathbf{N}')^T.$$

The details of the proof are similar to those of Theorem 4.60.
Let

$$\mathbf{N}'\mathbf{K}'(\mathbf{N}')^T = [\gamma'_{ij}]_n$$

and consider the four cases separately. It is found with the aid of Theorem A.8(e) in Appendix A.10 that

$$\begin{aligned} \gamma'_{2p+1,2q+1} &= \sum_{j=1}^{N} a_{ij} \{ g_{q-p}(x_j) + g_{q+p+1}(x_j) \} \\ &= \alpha'_{2p+1,2q+1} \end{aligned} \tag{4.13.17}$$

and further that $\gamma'_{ij} = \alpha'_{ij}$ for all values of $i$ and $j$.

**Corollary.**

$$\begin{aligned} |\alpha'_{ij}|_n = |\mathbf{M}'|_n &= |\mathbf{N}'|_n^2 |\mathbf{K}'|_n \\ &= |\beta'_{ij}|_n^2 2^n \left|2^{i+j-2}(k_{i+j} + k_{i+j-2})\right|_n \\ &= 2^{n^2} |k_{i+j} + k_{i+j-2}|_n \end{aligned} \tag{4.13.18}$$

since $\beta'_{ii} = 1$ for all values of $i$. Thus, $\mathbf{M}'$ can also be expressed as a Hankelian.

### *4.13.2 The Factors of a Particular Symmetric Toeplitz Determinant*

The determinants

$$P_n = \tfrac{1}{2}|p_{ij}|_n,$$

$$Q_n = \tfrac{1}{2}|q_{ij}|_n, \tag{4.13.19}$$

where

$$\begin{aligned} p_{ij} &= t_{|i-j|} - t_{i+j}, \\ q_{ij} &= t_{|i-j|} + t_{i+j-2}, \end{aligned} \tag{4.13.20}$$

appear in Section 4.5.2 as factors of a symmetric Toeplitz determinant.

Put

$$t_r = \omega^r u_r, \quad (\omega^2 = -1).$$

Then,

$$\begin{aligned} p_{ij} &= \omega^{i+j-2}\alpha'_{ij}, \\ q_{ij} &= \omega^{i+j-2}\alpha_{ij}, \end{aligned} \tag{4.13.21}$$

where $\alpha'_{ij}$ and $\alpha_{ij}$ are defined in (4.13.15) and (4.13.2), respectively. Hence, referring to the corollaries in Theorems 4.60 and 4.61,

$$\begin{aligned} P_n &= \tfrac{1}{2}\left|\omega^{i+j-2}\alpha'_{ij}\right|_n \\ &= \tfrac{1}{2}\omega^{n(n-1)}|\alpha'_{ij}|_n \\ &= (-1)^{n(n-1)/2}2^{n^2-1}|k_{i+j} + k_{i+j-2}|_n. \end{aligned} \tag{4.13.22}$$

$$\begin{aligned} Q_n &= \tfrac{1}{2}|\omega^{i+j-2}\alpha_{ij}|_n \\ &= (-1)^{n(n-1)/2}2^{(n-1)^2}|k_{i+j-2}|_n. \end{aligned} \tag{4.13.23}$$

Since $P_n$ and $Q_n$ each have a factor $\omega^{n(n-1)}$ and $n(n-1)$ is even for all values of $n$, these formulas remain valid when $\omega$ is replaced by $(-\omega)$ and are applied in Section 6.10.5 on the Einstein and Ernst equations.

## 4.14 Casoratians — A Brief Note

The Casoratian $K_n(x)$, which arises in the theory of difference equations, is defined as follows:

$$\begin{aligned} K_n(x) &= |f_i(x+j-1)|_n \\ &= \begin{vmatrix} f_1(x) & f_1(x+1) & \cdots & f_1(x+n-1) \\ f_2(x) & f_2(x+1) & \cdots & f_2(x+n-1) \\ \cdots & \cdots & \cdots & \cdots \\ f_n(x) & f_n(x+1) & \cdots & f_n(x+n-1) \end{vmatrix}_n. \end{aligned}$$

The role played by Casoratians in the theory of difference equations is similar to the role played by Wronskians in the theory of differential equations. Examples of their applications are given by Milne-Thomson, Brand, and Browne and Nillsen. Some applications of Casoratians in mathematical physics are given by Hirota, Kajiwara et al., Liu, Ohta et al., and Yuasa.