3

Intermediate Determinant Theory

3.1 Cyclic Dislocations and Generalizations

Define column vectors $\mathbf{C}_j$ and $\mathbf{C}_j^*$ as follows:

$$\begin{aligned}\mathbf{C}_j &= \begin{bmatrix}a_{1j} \; a_{2j} \; a_{3j} \cdots a_{nj}\end{bmatrix}^T \\ \mathbf{C}_j^* &= \begin{bmatrix}a_{1j}^* \; a_{2j}^* \; a_{3j}^* \cdots a_{nj}^*\end{bmatrix}^T\end{aligned}$$

where

$$a_{ij}^* = \sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}a_{rj},$$

that is, the element a_{ij}^* in $\mathbf{C}_j^*$ is a linear combination of all the elements in $\mathbf{C}_j$ except a_{ij}, the coefficients λ_{ir} being independent of j but otherwise arbitrary.

Theorem 3.1.

$$\sum_{j=1}^{n}\bigl|\mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_j^* \cdots \mathbf{C}_n\bigr| = 0.$$

Proof.

$$\begin{aligned}\bigl|\mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_j^* \cdots \mathbf{C}_n\bigr| &= \sum_{i=1}^{n} a_{ij}^* A_{ij} \\ &= \sum_{i=1}^{n} A_{ij} \sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}a_{rj}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j=1}^{n}\left|\mathbf{C}_1\ \mathbf{C}_2\cdots\mathbf{C}_j^*\ \cdots\ \mathbf{C}_n\right| &= \sum_{i=1}^{n}\sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}\sum_{j=1}^{n}a_{rj}A_{ij}\\ &= A_n\sum_{i=1}^{n}\sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}\delta_{ir}\\ &= 0\end{aligned}$$

which completes the proof. □

If

$$\begin{aligned}\lambda_{1n} &= 1,\\ \lambda_{ir} &= \begin{cases}1, & r=i-1, \qquad i>1\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

that is,

$$[\lambda_{ir}]_n = \begin{bmatrix}0 & & & & & 1\\ 1 & 0 & & & & 0\\ & 1 & 0 & & & 0\\ & & 1 & 0 & & 0\\ & & & \cdots & \cdots & \cdots\\ & & & & 1 & 0\end{bmatrix}_n,$$

then $\mathbf{C}_j^*$ is the column vector obtained from $\mathbf{C}_j$ by dislocating or displacing the elements one place downward in a cyclic manner, the last element in $\mathbf{C}_j$ appearing as the first element in $\mathbf{C}_j^*$, that is,

$$\mathbf{C}_j^* = \left[a_{nj}\ a_{1j}\ a_{2j}\cdots a_{n-1,j}\right]^T.$$

In this particular case, Theorem 3.1 can be expressed in words as follows:

Theorem 3.1a. *Given an arbitrary determinant A_n, form n other determinants by dislocating the elements in the jth column one place downward in a cyclic manner, $1\le j\le n$. Then, the sum of the n determinants so formed is zero.*

If

$$\lambda_{ir} = \begin{cases}i-1, & r=i-1, \qquad i>1\\ 0, & \text{otherwise,}\end{cases}$$

then

$$\begin{aligned}a_{ij}^* &= (i-1)a_{i-1,j},\\ \mathbf{C}_j^* &= \left[0\ a_{1j}\ 2a_{2j}\ 3a_{3j}\cdots(n-1)a_{n-1,j}\right]^T.\end{aligned}$$

This particular case is applied in Section 4.9.2 on the derivatives of a Turanian with Appell elements and another particular case is applied in Section 5.1.3 on expressing orthogonal polynomials as determinants.

Exercises

1. Let δ^r denote an operator which, when applied to $\mathbf{C}_j$, has the effect of dislocating the elements r positions downward in a cyclic manner so that the lowest set of r elements are expelled from the bottom and reappear at the top without change of order.

$$\begin{aligned}\delta^r\mathbf{C}_j &= \left[a_{n-r+1,j}\ a_{n-r+2,j}\cdots a_{nj}\ a_{1j}\ a_{2j}\cdots a_{n-r,j}\right]^T,\\ &\qquad 1 \le r \le n-1,\\ \delta^0\mathbf{C}_j &= \delta^n\mathbf{C}_j = \mathbf{C}_j.\end{aligned}$$

Prove that

$$\sum_{j=1}^{n}\left|\mathbf{C}_1\cdots\delta^r\mathbf{C}_j\cdots\mathbf{C}_n\right| = \begin{cases}0, & 1 \le r \le n-1\\ nA, & r = 0, n.\end{cases}$$

2. Prove that

$$\sum_{r=1}^{n}\left|\mathbf{C}_1\cdots\delta^r\mathbf{C}_j\cdots\mathbf{C}_n\right| = s_j S_j,$$

where

$$\begin{aligned}s_j &= \sum_{i=1}^{n} a_{ij},\\ S_j &= \sum_{i=1}^{n} A_{ij}.\end{aligned}$$

Hence, prove that an arbitrary determinant $A_n = |a_{ij}|_n$ can be expressed in the form

$$A_n = \frac{1}{n}\sum_{j=1}^{n} s_j S_j. \qquad \text{(Trahan)}$$

3.2 Second and Higher Minors and Cofactors

3.2.1 Rejecter and Retainer Minors

It is required to generalize the concept of first minors as defined in Chapter 1.

Let $A_n = |a_{ij}|_n$, and let $\{i_s\}$ and $\{j_s\}$, $1 \le s \le r \le n$, denote two independent sets of r distinct numbers, $1 \le i_s$ and $j_s \le n$. Now let $M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ denote the subdeterminant of order $(n-r)$ which is obtained from A_n by *rejecting* rows $i_1, i_2, \ldots, i_r$ and columns $j_1, j_2, \ldots, j_r$. $M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ is known as an rth minor of A_n. It may conveniently be called a *rejecter* minor. The numbers i_s and j_s are known respectively as row and column parameters.

Now, let $N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ denote the subdeterminant of order r which is obtained from A_n by *retaining* rows $i_1, i_2, \ldots, i_r$ and columns $j_1, j_2, \ldots, j_r$ and rejecting the other rows and columns. $N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ may conveniently be called a *retainer* minor.

Examples.

$$M^{(5)}_{13,25} = \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix} = N_{245,134},$$

$$M^{(5)}_{245,134} = \begin{vmatrix} a_{12} & a_{15} \\ a_{32} & a_{35} \end{vmatrix} = N_{13,25}.$$

The minors $M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ and $N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ are said to be mutually complementary in A_n, that is, each is the complement of the other in A_n. This relationship can be expressed in the form

$$\begin{aligned} M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} &= \operatorname{comp} N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \\ N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} &= \operatorname{comp} M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}. \end{aligned} \tag{3.2.1}$$

The order and structure of rejecter minors depends on the value of n but the order and structure of retainer minors are independent of n provided only that n is sufficiently large. For this reason, the parameter n has been omitted from N.

Examples.

$$\begin{aligned} N_{ip} &= \left| a_{ip} \right|_1 = a_{ip}, \quad n \geq 1, \\ N_{ij,pq} &= \begin{vmatrix} a_{ip} & a_{iq} \\ a_{jp} & a_{jq} \end{vmatrix}, \quad n \geq 2, \\ N_{ijk,pqr} &= \begin{vmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{vmatrix}, \quad n \geq 3. \end{aligned}$$

Both rejecter and retainer minors arise in the construction of the Laplace expansion of a determinant (Section 3.3).

Exercise. Prove that

$$\begin{vmatrix} N_{ij,pq} & N_{ij,pr} \\ N_{ik,pq} & N_{ik,pr} \end{vmatrix} = N_{ip} N_{ijk,pqr}.$$

3.2.2 Second and Higher Cofactors

The first cofactor $A^{(n)}_{ij}$ is defined in Chapter 1 and appears in Chapter 2. It is now required to generalize that concept.

In the definition of rejecter and retainer minors, no restriction is made concerning the relative magnitudes of either the row parameters i_s or the column parameters j_s. Now, let each set of parameters be arranged in ascending order of magnitude, that is,

$$i_s < i_{s+1}, \; j_s < j_{s+1}, \quad 1 \leq s \leq r-1.$$

Then, the rth cofactor of A_n, denoted by $A^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ is defined as a signed rth rejecter minor:

$$A^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} = (-1)^k M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \tag{3.2.2}$$

where k is the sum of the parameters:

$$k = \sum_{s=1}^{r} (i_s + j_s).$$

However, the concept of a cofactor is more general than that of a signed minor. The definition can be extended to zero values and to all positive and negative integer values of the parameters by adopting two conventions:

i. The cofactor changes sign when any two row parameters or any two column parameters are interchanged. It follows without further assumptions that the cofactor is zero when either the row parameters or the column parameters are not distinct.

ii. The cofactor is zero when any row or column parameter is less than 1 or greater than n.

Illustration.

$$A^{(4)}_{12,23} = -A^{(4)}_{21,23} = -A^{(4)}_{12,32} = A^{(4)}_{21,32} = M^{(4)}_{12,23} = N_{34,14},$$
$$A^{(6)}_{135,235} = -A^{(6)}_{135,253} = A^{(6)}_{135,523} = A^{(6)}_{315,253} = -M^{(6)}_{135,235} = -N_{246,146},$$

$$\begin{aligned} A^{(n)}_{i_2 i_1 i_3; j_1 j_2 j_3} &= -A^{(n)}_{i_1 i_2 i_3; j_1 j_2 j_3} = A^{(n)}_{i_1 i_2 i_3; j_1 j_3 j_2}, \\ A^{(n)}_{i_1 i_2 i_3; j_1 j_2 (n-p)} &= 0 \quad \text{if } p < 0 \\ &\qquad \text{or } p \geq n \\ &\qquad \text{or } p = n - j_1 \\ &\qquad \text{or } p = n - j_2. \end{aligned}$$

3.2.3 The Expansion of Cofactors in Terms of Higher Cofactors

Since the first cofactor $A^{(n)}_{ip}$ is itself a determinant of order $(n-1)$, it can be expanded by the $(n-1)$ elements from any row or column and their first cofactors. But, first, cofactors of $A^{(n)}_{ip}$ are second cofactors of A_n. Hence, it is possible to expand $A_{ip}^{(n)}$ by elements from any row or column and second cofactors $A_{ij,pq}^{(n)}$. The formula for row expansions is

$$A_{ip}^{(n)} = \sum_{q=1}^{n} a_{jq} A_{ij,pq}^{(n)}, \quad 1 \le j \le n, \quad j \ne i. \tag{3.2.3}$$

The term in which $q = p$ is zero by the first convention for cofactors. Hence, the sum contains $(n-1)$ nonzero terms, as expected. The $(n-1)$ values of j for which the expansion is valid correspond to the $(n-1)$ possible ways of expanding a subdeterminant of order $(n-1)$ by elements from one row and their cofactors.

Omitting the parameter n and referring to (2.3.10), it follows that if $i < j$ and $p < q$, then

$$\begin{aligned} A_{ij,pq} &= \frac{\partial A_{ip}}{\partial a_{jq}} \\ &= \frac{\partial^2 A}{\partial a_{ip} \partial a_{jq}} \end{aligned} \tag{3.2.4}$$

which can be regarded as an alternative definition of the second cofactor $A_{ij,pq}$.

Similarly,

$$A_{ij,pq}^{(n)} = \sum_{r=1}^{n} a_{kr} A_{ijk,pqr}^{(n)}, \quad 1 \le k \le n, \quad k \ne i \text{ or } j. \tag{3.2.5}$$

Omitting the parameter n, it follows that if $i < j < k$ and $p < q < r$, then

$$\begin{aligned} A_{ijk,pqr} &= \frac{\partial A_{ij,pq}}{\partial a_{kr}} \\ &= \frac{\partial^3 A}{\partial a_{ip} \partial a_{jq} \partial a_{kr}} \end{aligned} \tag{3.2.6}$$

which can be regarded as an alternative definition of the third cofactor $A_{ijk,pqr}$.

Higher cofactors can be defined in a similar manner. Partial derivatives of this type appear in Section 3.3.2 on the Laplace expansion, in Section 3.6.2 on the Jacobi identity, and in Section 5.4.1 on the Matsuno determinant.

The expansion of an rth cofactor, a subdeterminant of order $(n-r)$, can be expressed in the form

$$\begin{aligned} A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}^{(n)} &= \sum_{q=1}^{n} a_{pq} A_{i_1 i_2 \ldots i_r p; j_1 j_2 \ldots j_r q}^{(n)}, \\ &\qquad 1 \le p \le n, \quad p \ne i_s, \quad 1 \le s \le r. \end{aligned} \tag{3.2.7}$$

The r terms in which $q = j_s$, $1 \le s \le r$, are zero by the first convention for cofactors. Hence, the sum contains $(n-r)$ nonzero terms, as expected.

The $(n-r)$ values of p for which the expansion is valid correspond to the $(n-r)$ possible ways of expanding a subdeterminant of order $(n-r)$ by elements from one row and their cofactors.

If one of the column parameters of an rth cofactor of A_{n+1} is $(n+1)$, the cofactor does not contain the element $a_{n+1,n+1}$. If none of the row parameters is $(n+1)$, then the rth cofactor can be expanded by elements from its last row and their first cofactors. But first cofactors of an rth cofactor of A_{n+1} are $(r+1)$th cofactors of A_{n+1} which, in this case, are rth cofactors of A_n. Hence, in this case, an rth cofactor of A_{n+1} can be expanded in terms of the first n elements in the last row and rth cofactors of A_n. This expansion is

$$A^{(n+1)}_{i_1i_2\ldots i_r;j_1j_2\ldots j_{r-1}(n+1)} = -\sum_{q=1}^{n} a_{n+1,q} A^{(n)}_{i_1i_2\ldots i_r;j_1j_2\ldots j_{r-1}q}. \tag{3.2.8}$$

The corresponding column expansion is

$$A^{(n+1)}_{i_1i_2\ldots i_{r-1}(n+1);j_1j_2\ldots j_r} = -\sum_{p=1}^{n} a_{p,n+1} A^{(n)}_{i_1i_2\ldots i_{r-1}p;j_1j_2\ldots j_r}. \tag{3.2.9}$$

Exercise. Prove that

$$\frac{\partial^2 A}{\partial a_{ip}\partial a_{jq}} = -\frac{\partial^2 A}{\partial a_{iq}\partial a_{jp}},$$
$$\frac{\partial^3 A}{\partial a_{ip}\partial a_{jq}\partial a_{kr}} = \frac{\partial^3 A}{\partial a_{kp}\partial a_{iq}\partial a_{jr}} = \frac{\partial^3 A}{\partial a_{jp}\partial a_{kq}\partial a_{ir}}$$

without restrictions on the relative magnitudes of the parameters.

3.2.4 Alien Second and Higher Cofactors; Sum Formulas

The $(n-2)$ elements a_{hq}, $1 \le q \le n$, $q \ne h$ or p, appear in the second cofactor $A^{(n)}_{ij,pq}$ if $h \ne i$ or j. Hence,

$$\sum_{q=1}^{n} a_{hq} A^{(n)}_{ij,pq} = 0, \quad h \ne i \text{ or } j,$$

since the sum represents a determinant of order $(n-1)$ with two identical rows. This formula is a generalization of the theorem on alien cofactors given in Chapter 2. The value of the sum of $1 \le h \le n$ is given by the sum formula for elements and cofactors, namely

$$\sum_{q=1}^{n} a_{hq} A^{(n)}_{ij,pq} = \begin{cases} A^{(n)}_{ip}, & h = j \ne i \\ -A^{(n)}_{jp}, & h = i \ne j \\ 0, & \text{otherwise} \end{cases} \tag{3.2.10}$$

which can be abbreviated with the aid of the Kronecker delta function [Appendix A]:

$$\sum_{q=1}^{n} a_{hq} A_{ij,pq}^{(n)} = A_{ip}^{(n)} \delta_{hj} - A_{jp}^{(n)} \delta_{hi}.$$

Similarly,

$$\begin{aligned}
\sum_{r=1}^{n} a_{hr} A_{ijk,pqr}^{(n)} &= A_{ij,pq}^{(n)} \delta_{hk} + A_{jk,pq}^{(n)} \delta_{hi} + A_{ki,pq}^{(n)} \delta_{hj}, \\
\sum_{s=1}^{n} a_{hs} A_{ijkm,pqrs}^{(n)} &= A_{ijk,pqr}^{(n)} \delta_{hm} - A_{jkm,pqr}^{(n)} \delta_{hi} \\
&\quad + A_{kmi,pqr}^{(n)} \delta_{hj} - A_{mij,pqr}^{(n)} \delta_{hk}
\end{aligned} \tag{3.2.11}$$

etc.

Exercise. Show that these expressions can be expressed as sums as follows:

$$\begin{aligned}
\sum_{q=1}^{n} a_{hq} A_{ij,pq}^{(n)} &= \sum_{u,v} \operatorname{sgn} \begin{Bmatrix} u & v \\ i & j \end{Bmatrix} A_{up}^{(n)} \delta_{hv}, \\
\sum_{r=1}^{n} a_{hr} A_{ijk,pqr}^{(n)} &= \sum_{u,v,w} \operatorname{sgn} \begin{Bmatrix} u & v & w \\ i & j & k \end{Bmatrix} A_{uv,pq}^{(n)} \delta_{hw}, \\
\sum_{s=1}^{n} a_{hs} A_{ijkm,pqrs}^{(n)} &= \sum_{u,v,w,x} \operatorname{sgn} \begin{Bmatrix} u & v & w & x \\ i & j & k & m \end{Bmatrix} A_{uvw,pqr}^{(n)} \delta_{hx},
\end{aligned}$$

etc., where, in each case, the sums are carried out over all possible cyclic permutations of the lower parameters in the permutation symbols. A brief note on cyclic permutations is given in Appendix A.2.

3.2.5 Scaled Cofactors

Cofactors $A_{ip}^{(n)}$, $A_{ij,pq}^{(n)}$, $A_{ijk,pqr}^{(n)}$, etc., with both row and column parameters written as subscripts have been defined in Section 3.2.2. They may conveniently be called simple cofactors. Scaled cofactors A_n^{ip}, $A_n^{ij,pq}$, $A_n^{ijk,pqr}$, etc., with row and column parameters written as superscripts are defined as follows:

$$\begin{aligned}
A_n^{ip} &= \frac{A_{ip}^{(n)}}{A_n}, \\
A_n^{ij,pq} &= \frac{A_{ij,pq}^{(n)}}{A_n}, \\
A_n^{ijk,pqr} &= \frac{A_{ijk,pqr}^{(n)}}{A_n},
\end{aligned} \tag{3.2.12}$$

etc. In simple algebraic relations such as Cramer's formula, the advantage of using scaled rather than simple cofactors is usually negligible. The Jacobi identity (Section 3.6) can be expressed in terms of unscaled or scaled cofactors, but the scaled form is simpler. In differential relations, the advantage can be considerable. For example, the sum formula

$$\sum_{j=1}^{n} a_{ij} A_{kj}^{(n)} = A_n \delta_{ki}$$

when differentiated gives rise to three terms:

$$\sum_{j=1}^{n} \left[a'_{ij} A_{kj}^{(n)} + a_{ij} (A_{kj}^{(n)})' \right] = A'_n \delta_{ki}.$$

When the cofactor is scaled, the sum formula becomes

$$\sum_{j=1}^{n} a_{ij} A_n^{kj} = \delta_{ki} \tag{3.2.13}$$

which is only slightly simpler than the original, but when it is differentiated, it gives rise to only two terms:

$$\sum_{j=1}^{n} \left[a'_{ij} A_n^{kj} + a_{ij} (A_n^{kj})' \right] = 0. \tag{3.2.14}$$

The advantage of using scaled rather than unscaled or simple cofactors will be fully appreciated in the solution of differential equations (Chapter 6).

Referring to the partial derivative formulas in (2.3.10) and Section 3.2.3,

$$\begin{aligned} \frac{\partial A^{ip}}{\partial a_{jq}} &= \frac{\partial}{\partial a_{jq}} \left(\frac{A_{ip}}{A} \right) \\ &= \frac{1}{A^2} \left[A \frac{\partial A^{ip}}{\partial a_{jq}} - A_{ip} \frac{\partial A}{\partial a_{jq}} \right] \\ &= \frac{1}{A^2} \left[A\, A_{ij,pq} - A_{ip}\, A_{jq} \right] \\ &= A^{ij,pq} - A^{ip} A^{jq}. \end{aligned} \tag{3.2.15}$$

Hence,

$$\left(A^{jq} + \frac{\partial}{\partial a_{jq}} \right) A^{ip} = A^{ij,pq}. \tag{3.2.16}$$

Similarly,

$$\left(A^{kr} + \frac{\partial}{\partial a_{kr}} \right) A^{ij,pq} = A^{ijk,pqr}. \tag{3.2.17}$$

The expressions in brackets can be regarded as operators which, when applied to a scaled cofactor, yield another scaled cofactor. Formula (3.2.15) is applied in Section 3.6.2 on the Jacobi identity. Formulas (3.2.16) and (3.2.17) are applied in Section 5.4.1 on the Matsuno determinant.

3.3 The Laplace Expansion

3.3.1 A Grassmann Proof

The following analysis applies Grassmann algebra and is similar in nature to that applied in the definition of a determinant.

Let i_s and j_s, $1 \le s \le r$, $r \le n$, denote r integers such that

$$\begin{aligned} 1 \le i_1 < i_2 < \cdots < i_r \le n, \\ 1 \le j_1 < j_2 < \cdots < j_r \le n \end{aligned}$$

and let

$$\begin{aligned} \mathbf{x}_i &= \sum_{k=1}^{n} a_{ij} \mathbf{e}_k, \quad 1 \le i \le n, \\ \mathbf{y}_i &= \sum_{t=1}^{r} a_{ij_t} \mathbf{e}_{j_t}, \quad 1 \le i \le n, \\ \mathbf{z}_i &= \mathbf{x}_i - \mathbf{y}_i. \end{aligned}$$

Then, any vector product is which the number of $\mathbf{y}$'s is greater than r or the number of $\mathbf{z}$'s is greater than $(n-r)$ is zero.

Hence,

$$\begin{aligned} \mathbf{x}_1 \cdots \mathbf{x}_n &= (\mathbf{y}_1 + \mathbf{z}_1)(\mathbf{y}_2 + \mathbf{z}_2) \cdots (\mathbf{y}_n + \mathbf{z}_n) \\ &= \sum_{i_1 \ldots i_r} \mathbf{z}_1 \cdots \mathbf{y}_{i_1} \cdots \mathbf{y}_{i_2} \cdots \mathbf{y}_{i_r} \cdots \mathbf{z}_n, \end{aligned} \tag{3.3.1}$$

where the vector product on the right is obtained from $(\mathbf{z}_1 \cdots \mathbf{z}_n)$ by replacing $\mathbf{z}_{i_s}$ by $\mathbf{y}_{i_s}$, $1 \le s \le r$, and the sum extends over all $\binom{n}{r}$ combinations of the numbers $1, 2, \ldots, n$ taken r at a time. The $\mathbf{y}$'s in the vector product can be separated from the $\mathbf{z}$'s by making a suitable sequence of interchanges and applying Identity (ii). The result is

$$\mathbf{z}_1 \cdots \mathbf{y}_{i_1} \cdots \mathbf{y}_{i_2} \cdots \mathbf{y}_{i_r} \cdots \mathbf{z}_n = (-1)^p \big(\mathbf{y}_{i_1} \cdots \mathbf{y}_{i_r}\big)\big(\mathbf{z}_1 \overset{*}{\cdots} \mathbf{z}_n\big), \tag{3.3.2}$$

where

$$p = \sum_{s=1}^{n} i_s - \tfrac{1}{2} r(r+1) \tag{3.3.3}$$

and the symbol $*$ denotes that those vectors with suffixes $i_1, i_2, \ldots, i_r$ are omitted.

Recalling the definitions of rejecter minors M, retainer minors N, and cofactors A, each with row and column parameters, it is found that

$$\begin{aligned}\mathbf{y}_{i_1}\cdots\mathbf{y}_{i_r} &= N_{i_1\ldots i_r;j_1\ldots j_r}\big(\mathbf{e}_{j_1}\cdots\mathbf{e}_{j_r}\big),\\ \mathbf{z}_1 \overset{*}{\cdots} \mathbf{z}_n &= M_{i_1\ldots i_r;j_1\ldots j_r}\big(\mathbf{e}_1 \overset{*}{\cdots} \mathbf{e}_n\big),\end{aligned}$$

where, in this case, the symbol $*$ denotes that those vectors with suffixes $j_1, j_2, \ldots, j_r$ are omitted. Hence,

$$\begin{aligned}&\mathbf{x}_1\cdots\mathbf{x}_n\\ &\quad= \sum_{i_1\ldots i_r}(-1)^p N_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r}\, M_{i_1 i_2\ldots, i_r;j_1 j_2\ldots j_r}\big(\mathbf{e}_{j_1}\cdots\mathbf{e}_{j_r}\big)\big(\mathbf{e}_1 \overset{*}{\cdots} \mathbf{e}_n\big).\end{aligned}$$

By applying in reverse order the sequence of interchanges used to obtain (3.3.2), it is found that

$$\big(\mathbf{e}_{j_1}\cdots\mathbf{e}_{j_r}\big)\big(\mathbf{e}_1 \overset{*}{\cdots} \mathbf{e}_n\big) = (-1)^q(\mathbf{e}_1\cdots\mathbf{e}_n),$$

where

$$q = \sum_{s=1}^{n} j_s - \tfrac{1}{2}r(r+1).$$

Hence,

$$\begin{aligned}\mathbf{x}_1\cdots\mathbf{x}_n &= \left[\sum_{i_1\ldots i_r}(-1)^{p+q} N_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r} M_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r}\right]\mathbf{e}_1\cdots\mathbf{e}_n\\ &= \left[\sum_{i_1\ldots i_r} N_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r} A_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r}\right]\mathbf{e}_1\cdots\mathbf{e}_n.\end{aligned}$$

Comparing this formula with (1.2.5) in the section on the definition of a determinant, it is seen that

$$A_n = |a_{ij}|_n = \sum_{i_1\ldots i_r} N_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r}\, A_{i_1 i_2\ldots i_r;j_1 j_2\ldots j_r}, \tag{3.3.4}$$

which is the general form of the Laplace expansion of A_n in which the sum extends over the row parameters. By a similar argument, it can be shown that A_n is also equal to the same expression in which the sum extends over the column parameters.

When $r = 1$, the Laplace expansion degenerates into a simple expansion by elements from column j or row i and their first cofactors:

$$\begin{aligned}A_n &= \sum_{i \text{ or } j} N_{ij} A_{ij},\\ &= \sum_{i \text{ or } j} a_{ij} A_{ij}.\end{aligned}$$

When $r = 2$,

$$\begin{aligned} A_n &= \sum N_{ir,js} A_{ir,js}, \quad \text{summed over } i, r \text{ or } j, s, \\ &= \sum \begin{vmatrix} a_{ij} & a_{is} \\ a_{rj} & a_{rs} \end{vmatrix} A_{ir,js}. \end{aligned}$$

3.3.2 A Classical Proof

The following proof of the Laplace expansion formula given in (3.3.4) is independent of Grassmann algebra.

Let

$$A = |a_{ij}|_n.$$

Then referring to the partial derivative formulas in Section 3.2.3,

$$A_{i_1 j_1} = \frac{\partial A}{\partial a_{i_1 j_1}} \tag{3.3.5}$$

$$\begin{aligned} A_{i_1 i_2; j_1 j_2} &= \frac{\partial A_{i_1 j_1}}{\partial a_{i_2 j_2}}, \quad i_1 < i_2 \text{ and } j_1 < j_2, \\ &= \frac{\partial^2 A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2}}. \end{aligned} \tag{3.3.6}$$

Continuing in this way,

$$A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} = \frac{\partial^r A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_r j_r}}, \tag{3.3.7}$$

provided that $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$.

Expanding A by elements from column j_1 and their cofactors and referring to (3.3.5),

$$\begin{aligned} A &= \sum_{i_1=1}^{n} a_{i_1 j_1} A_{i_1 j_1} \\ &= \sum_{i_1=1}^{n} a_{i_1 j_1} \frac{\partial A}{\partial a_{i_1 j_1}} \\ &= \sum_{i_2=1}^{n} a_{i_2 j_2} \frac{\partial A}{\partial a_{i_2 j_2}} \qquad (3.3.8) \\ \frac{\partial A}{\partial a_{i_1 j_1}} &= \sum_{i_2=1}^{n} a_{i_2 j_2} \frac{\partial^2 A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2}} \\ &= \sum_{i_2=1}^{n} a_{i_2 j_2} A_{i_1 i_2; j_1 j_2}, \quad i_1 < i_2 \text{ and } j_1 < j_2. \qquad (3.3.9) \end{aligned}$$

Substituting the first line of (3.3.9 and the second line of (3.3.8),

$$\begin{aligned} A &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} a_{i_1 j_1} a_{i_2 j_2} \frac{\partial^2 A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2}} \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} a_{i_1 j_1} a_{i_2 j_2} A_{i_1 i_2; j_1 j_2}, \quad i_1 < i_2 \text{ and } j_1 < j_2. \end{aligned} \tag{3.3.10}$$

Continuing in this way and applying (3.3.7) in reverse,

$$\begin{aligned} A &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_r=1}^{n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} \frac{\partial^r A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_r j_r}} \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_r=1}^{n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \end{aligned} \tag{3.3.11}$$

subject to the inequalities associated with (3.3.7) which require that the i_s and j_s shall be in ascending order of magnitude.

In this multiple sum, those rth cofactors in which the dummy variables are not distinct are zero so that the corresponding terms in the sum are zero. The remaining terms can be divided into a number of groups according to the relative magnitudes of the dummies. Since r distinct dummies can be arranged in a linear sequence in $r!$ ways, the number of groups is $r!$. Hence,

$$A = \sum^{(r! \text{ terms})} G_{k_1 k_2 \ldots, k_r},$$

where

$$\begin{aligned} G_{k_1 k_2 \ldots k_r} = \sum_{i \le i_{k_1} < i_{k_2} < \cdots < i_{k_r} \le n} & a_{i_{k_1} j_{k_1}} \; a_{i_{k_2} j_{k_2}} \\ & \cdots a_{i_{k_r} j_{k_r}} A_{i_{k_1} i_{k_2} \cdots i_{k_r}; j_{k_1} j_{k_2} \cdots j_{k_r}}. \end{aligned} \tag{3.3.12}$$

In one of these $r!$ terms, the dummies $i_1, i_2, \ldots, i_r$ are in ascending order of magnitude, that is, $i_s < i_{s+1}$, $1 \le s \le r-1$. However, the dummies in the other $(r! - 1)$ terms can be interchanged in such a way that the inequalities are valid for those terms too. Hence, applying those properties of rth cofactors which concern changes in sign,

$$A = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left[\sum \sigma_r a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} \right] A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r},$$

where

$$\sigma_r = \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & \cdots & r \\ i_1 & i_2 & i_3 & \cdots & i_r \end{Bmatrix}. \tag{3.3.13}$$

(Appendix A.2). But,

$$\sum \sigma_r a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} = N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}.$$

The expansion formula (3.3.4) follows.

Illustrations

1. When $r = 2$, the Laplace expansion formula can be proved as follows: Changing the notation in the second line of (3.3.10),

$$A = \sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{jq} A_{ij;pq}, \quad i < j.$$

This double sum contains n^2 terms, but the n terms in which $q = p$ are zero by the definition of a second cofactor. Hence,

$$A = \sum_{p<q} a_{ip} a_{jq} A_{ij,pq} + \sum_{q<p} a_{ip} a_{jq} A_{ij;pq}.$$

In the second double sum, interchange the dummies p and q and refer once again to the definition of a second cofactor:

$$\begin{aligned} A &= \sum_{p<q} \begin{vmatrix} a_{ip} & a_{iq} \\ a_{jp} & a_{jq} \end{vmatrix} A_{ij;pq} \\ &= \sum_{p<q} N_{ij;pq} A_{ij;pq}, \quad i < j, \end{aligned}$$

which proves the Laplace expansion formula from rows i and j. When $(n, i, j) = (4, 1, 2)$, this formula becomes

$$\begin{aligned} A = N_{12,12} A_{12,12} &+ N_{12,13} A_{12,13} + N_{12,14} A_{12,14} \\ &+ N_{12,23} A_{12,23} + N_{12,24} A_{12,24} \\ &+ N_{12,34} A_{12,34}. \end{aligned}$$

2. When $r = 3$, begin with the formula

$$A = \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} a_{ip} a_{jq} a_{kr} A_{ijk,pqr}, \quad i < j < k,$$

which is obtained from the second line of (3.3.11) with a change in notation. The triple sum contains n^3 terms, but those in which p, q, and r are not distinct are zero. Those which remain can be divided into $3! = 6$ groups according to the relative magnitudes of p, q, and r:

$$A = \left[\sum_{p<q<r} + \sum_{p<r<q} + \sum_{q<r<p} + \sum_{q<p<r} + \sum_{r<p<q} + \sum_{r<q<p} \right] a_{ip} a_{jq} a_{kr} A_{ijk,pqr}.$$

Now, interchange the dummies wherever necessary in order that $p < q < r$ in all sums. The result is

$$\begin{aligned}
A &= \sum_{p<q<r} \big[a_{ip}a_{jq}a_{kr} - a_{ip}a_{jr}a_{kq} + a_{iq}a_{jr}a_{kp} \\
&\qquad - a_{iq}a_{jp}a_{kr} + a_{ir}a_{jp}a_{kq} - a_{ir}a_{jq}a_{kp}\big] A_{ijk,pqr} \\
&= \sum_{p<q<r} \begin{vmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{vmatrix} A_{ijk,pqr} \\
&= \sum_{p<q<r} N_{ijk,pqr} A_{ijk,pqr}, \quad i < j < k,
\end{aligned}$$

which proves the Laplace expansion formula from rows i, j, and k.

3.3.3 Determinants Containing Blocks of Zero Elements

Let **P**, **Q**, **R**, **S**, and **O** denote matrices of order n, where **O** is null and let

$$A_{2n} = \begin{vmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{vmatrix}_{2n}.$$

The Laplace expansion of A_{2n} taking minors from the first or last n rows or the first or last n columns consists, in general, of the sum of $\binom{2n}{n}$ nonzero products. If one of the submatrices is null, all but one of the products are zero.

Lemma.

a. $\begin{vmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{O} & \mathbf{S} \end{vmatrix}_{2n} = PS,$

b. $\begin{vmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{vmatrix}_{2n} = (-1)^n QR$

PROOF. The only nonzero term in the Laplace expansion of the first determinant is

$$N_{12\ldots n;12\ldots n} A_{12\ldots n;12\ldots n}.$$

The retainer minor is signless and equal to P. The sign of the cofactor is $(-1)^k$, where k is the sum of the row and column parameters.

$$k = 2\sum_{r=1}^{n} r = n(n+1),$$

which is even. Hence, the cofactor is equal to $+S$. Part (a) of the lemma follows.

The only nonzero term in the Laplace expansion of the second determinant is

$$N_{n+1,n+2,\ldots,2n;12\ldots n} A_{n+1,n+2,\ldots,2n;12\ldots n}.$$

The retainer minor is signless and equal to R. The sign of the cofactor is $(-1)^k$, where

$$k = \sum_{r=1}^{n}(n+2r) = 2n^2 + n.$$

Hence, the cofactor is equal to $(-1)^n Q$. Part (b) of the lemma follows. □

Similar arguments can be applied to more general determinants. Let $\mathbf{X}_{pq}$, $\mathbf{Y}_{pq}$, $\mathbf{Z}_{pq}$, and $\mathbf{O}_{pq}$ denote matrices with p rows and q columns, where $\mathbf{O}_{pq}$ is null and let

$$A_n = \begin{vmatrix} \mathbf{X}_{pq} & \mathbf{Y}_{ps} \\ \mathbf{O}_{rq} & \mathbf{Z}_{rs} \end{vmatrix}_n, \tag{3.3.14}$$

where $p + r = q + s = n$. The restriction $p \geq q$, which implies $r \leq s$, can be imposed without loss of generality. If A_n is expanded by the Laplace method taking minors from the first q columns or the last r rows, some of the minors are zero. Let U_m and V_m denote determinants of order m. Then, A_n has the following properties:

a. If $r + q > n$, then $A_n = 0$.
b. If $r + q = n$, then $p + s = n$, $q = p$, $s = r$, and $A_n = X_{pp}Z_{rr}$.
c. If $r + q < n$, then, in general,

$$\begin{aligned} A_n &= \text{sum of } \tbinom{p}{q} \text{ nonzero products each of the form } U_q V_s \\ &= \text{sum of } \tbinom{s}{r} \text{ nonzero products each of the form } U_r V_r. \end{aligned}$$

Property (a) is applied in the following examples.

Example 3.2. If $r + s = n$, then

$$U_{2n} = \begin{vmatrix} \mathbf{E}_{n,2r} & \mathbf{F}_{ns} & \mathbf{O}_{ns} \\ \mathbf{E}_{n,2r} & \mathbf{O}_{ns} & \mathbf{F}_{ns} \end{vmatrix}_{2n} = 0.$$

PROOF. It is clearly possible to perform n row operations in a single step and s column operations in a single step. Regard U_{2n} as having two "rows" and three "columns" and perform the operations

$$\begin{aligned} \mathbf{R}'_1 &= \mathbf{R}_1 - \mathbf{R}_2, \\ \mathbf{C}'_2 &= \mathbf{C}_2 + \mathbf{C}_3. \end{aligned}$$

The result is

$$\begin{aligned} U_{2n} &= \begin{vmatrix} \mathbf{O}_{n,2r} & \mathbf{F}_{ns} & -\mathbf{F}_{ns} \\ \mathbf{E}_{n,2r} & \mathbf{O}_{ns} & \mathbf{F}_{ns} \end{vmatrix}_{2n} \\ &= \begin{vmatrix} \mathbf{O}_{n,2r} & \mathbf{O}_{ns} & -\mathbf{F}_{ns} \\ \mathbf{E}_{n,2r} & \mathbf{F}_{ns} & \mathbf{F}_{ns} \end{vmatrix}_{2n} \\ &= 0 \end{aligned}$$

since the last determinant contains an $n \times (2r + s)$ block of zero elements and $n + 2r + s > 2n$. □

Example 3.3. Let

$$V_{2n} = \begin{vmatrix} \mathbf{E}_{ip} & \mathbf{F}_{iq} & \mathbf{G}_{iq} \\ \mathbf{E}_{ip} & \mathbf{G}_{iq} & \mathbf{F}_{iq} \\ \mathbf{O}_{jp} & \mathbf{H}_{jq} & \mathbf{K}_{jq} \end{vmatrix}_{2n},$$

where $2i + j = p + 2q = 2n$. Then, $V_{2n} = 0$ under each of the following independent conditions:

i. $j + p > 2n$,
ii. $p > i$,
iii. $\mathbf{H}_{jq} + \mathbf{K}_{jq} = \mathbf{O}_{jq}$.

Proof. Case (i) follows immediately from Property (a). To prove case (ii) perform row operations

$$V_{2n} = \begin{vmatrix} \mathbf{E}_{ip} & \mathbf{F}_{iq} & \mathbf{G}_{iq} \\ \mathbf{O}_{ip} & (\mathbf{G}_{iq} - \mathbf{F}_{iq}) & (\mathbf{F}_{iq} - \mathbf{G}_{iq}) \\ \mathbf{O}_{jp} & \mathbf{H}_{jq} & \mathbf{K}_{jq} \end{vmatrix}_{2n}.$$

This determinant contains an $(i + j) \times p$ block of zero elements. But, $i + j + p > 2i + j = 2n$. Case (ii) follows.

To prove case (iii), perform column operations on the last determinant:

$$V_{2n} = \begin{vmatrix} \mathbf{E}_{ip} & (\mathbf{F}_{iq} + \mathbf{G}_{iq}) & \mathbf{G}_{iq} \\ \mathbf{O}_{ip} & \mathbf{O}_{iq} & (\mathbf{F}_{iq} - \mathbf{G}_{iq}) \\ \mathbf{O}_{jp} & \mathbf{O}_{jq} & \mathbf{K}_{jq} \end{vmatrix}_{2n}.$$

This determinant contains an $(i + j) \times (p + q)$ block of zero elements. However, since $2(i+j) > 2n$ and $2(p+q) > 2n$, it follows that $i+j+p+q > 2n$. Case (iii) follows. □

3.3.4 The Laplace Sum Formula

The simple sum formula for elements and their cofactors (Section 2.3.4), which incorporates the theorem on alien cofactors, can be generalized for the case $r = 2$ as follows:

$$\sum_{p<q} N_{ij,pq} A_{rs,pq} = \delta_{ij,rs} A,$$

where $\delta_{ij,rs}$ is the generalized Kronecker delta function (Appendix A.1). The proof follows from the fact that if $r \neq i$, the sum represents a determinant in which row r = row i, and if, in addition, $s \neq j$, then, in addition, row s = row j. In either case, the determinant is zero.

Exercises

1. If $n = 4$, prove that

$$\sum_{p<q} N_{23,pq} A_{24,pq} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} = 0$$

(row 4 = row 3), by expanding the determinant from rows 2 and 3.

2. Generalize the sum formula for the case $r = 3$.

3.3.5 The Product of Two Determinants — 2

Let

$$\begin{aligned} A_n &= |a_{ij}|_n \\ B_n &= |b_{ij}|_n. \end{aligned}$$

Then

$$A_n B_n = |c_{ij}|_n,$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

A similar formula is valid for the product of two matrices. A proof has already been given by a Grassmann method in Section 1.4. The following proof applies the Laplace expansion formula and row operations but is independent of Grassmann algebra.

Applying in reverse a Laplace expansion of the type which appears in Section 3.3.3,

$$A_n B_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & & & & \\ a_{21} & a_{22} & \dots & a_{2n} & & & & \\ \dots & \dots & \dots & \dots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & & & & \\ -1 & & & & b_{11} & b_{12} & \dots & b_{1n} \\ & -1 & & & b_{21} & b_{22} & \dots & b_{2n} \\ & & \dots & & \dots & \dots & \dots & \dots \\ & & & -1 & b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}_{2n}. \tag{3.3.15}$$

Reduce all the elements in the first n rows and the first n columns, at present occupied by the a_{ij}, to zero by means of the row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \sum_{j=1}^{n} a_{ij} \mathbf{R}_{n+j}, \quad 1 \le i \le n. \tag{3.3.16}$$

The result is:

$$A_nB_n = \begin{vmatrix} & & & & c_{11} & c_{12} & \cdots & c_{1n} \\ & & & & c_{21} & c_{22} & \cdots & c_{2n} \\ & & & & \cdots & \cdots & \cdots & \cdots \\ & & & & c_{n1} & c_{n2} & \cdots & c_{nn} \\ -1 & & & & b_{11} & b_{12} & \cdots & b_{1n} \\ & -1 & & & b_{21} & b_{22} & \cdots & b_{2n} \\ & & \cdots & & \cdots & \cdots & \cdots & \cdots \\ & & & -1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}_{2n}. \tag{3.3.17}$$

The product formula follows by means of a Laplace expansion. c_{ij} is most easily remembered as a scalar product:

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \bullet \begin{bmatrix} b_{1j} \\ b_{2j} \\ \cdots \\ b_{nj} \end{bmatrix}. \tag{3.3.18}$$

Let $\mathbf{R}_i$ denote the ith row of A_n and let $\mathbf{C}_j$ denote the jth column of B_n. Then,

$$c_{ij} = \mathbf{R}_i \bullet \mathbf{C}_j.$$

Hence

$$\begin{aligned} A_nB_n &= |\mathbf{R}_i \bullet \mathbf{C}_j|_n \\ &= \begin{vmatrix} \mathbf{R}_1 \bullet \mathbf{C}_1 & \mathbf{R}_1 \bullet \mathbf{C}_2 & \cdots & \mathbf{R}_1 \bullet \mathbf{C}_n \\ \mathbf{R}_2 \bullet \mathbf{C}_1 & \mathbf{R}_2 \bullet \mathbf{C}_2 & \cdots & \mathbf{R}_2 \bullet \mathbf{C}_n \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \mathbf{R}_n \bullet \mathbf{C}_1 & \mathbf{R}_n \bullet \mathbf{C}_2 & \cdots & \mathbf{R}_n \bullet \mathbf{C}_n \end{vmatrix}_n. \end{aligned} \tag{3.3.19}$$

Exercise. If $A_n = |a_{ij}|_n$, $B_n = |b_{ij}|_n$, and $C_n = |c_{ij}|_n$, prove that

$$A_nB_nC_n = |d_{ij}|_n,$$

where

$$d_{ij} = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{ir} b_{rs} c_{sj}.$$

A similar formula is valid for the product of three matrices.

3.4 Double-Sum Relations for Scaled Cofactors

The following four double-sum relations are labeled (A)–(D) for easy reference in later sections, especially Chapter 6 on mathematical physics, where they are applied several times. The first two are formulas for the derivatives

A' and $(A^{ij})'$ and the other two are identities:

$$\frac{A'}{A} = (\log A)' = \sum_{r=1}^{n} \sum_{s=1}^{n} a'_{rs} A^{rs}, \tag{A}$$

$$(A^{ij})' = -\sum_{r=1}^{n} \sum_{s=1}^{n} a'_{rs} A^{is} A^{rj}, \tag{B}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r + g_s) a_{rs} A^{rs} = \sum_{r=1}^{n} (f_r + g_r), \tag{C}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r + g_s) a_{rs} A^{is} A^{rj} = (f_i + g_j) A^{ij}. \tag{D}$$

Proof. (A) follows immediately from the formula for A' in terms of unscaled cofactors in Section 2.3.7. The sum formula given in Section 2.3.4 can be expressed in the form

$$\sum_{s=1}^{n} a_{rs} A^{is} = \delta_{ri}, \tag{3.4.1}$$

which, when differentiated, gives rise to only two terms:

$$\sum_{s=1}^{n} a'_{rs} A^{is} = -\sum_{s=1}^{n} a_{rs} (A^{is})'. \tag{3.4.2}$$

Hence, beginning with the right side of (B),

$$\begin{aligned}
\sum_{r=1}^{n} \sum_{s=1}^{n} a'_{rs} A^{is} A^{rj} &= \sum_{r} A^{rj} \sum_{s} a'_{rs} A^{is} \\
&= -\sum_{r} A^{rj} \sum_{s} a_{rs} (A^{is})' \\
&= -\sum_{s} (A^{is})' \sum_{r} a_{rs} A^{rj} \\
&= -\sum_{s} (A^{is})' \delta_{sj} \\
&= -(A^{ij})'
\end{aligned}$$

which proves (B).

$$\begin{aligned}
&\sum_{r} \sum_{s} (f_r + g_s) a_{rs} A^{is} A^{rj} \\
&\qquad = \sum_{r} f_r A^{rj} \sum_{s} a_{rs} A^{is} + \sum_{s} g_s A^{is} \sum_{r} a_{rs} A^{rj}
\end{aligned}$$

$$\begin{aligned} &= \sum_r f_r A^{rj} \delta_{ri} + \sum_s g_s A^{is} \delta_{sj} \\ &= f_i A^{ij} + g_j A^{ij} \end{aligned}$$

which proves (D). The proof of (C) is similar but simpler. □

Exercises

Prove that

1. $\displaystyle\sum_{r=1}^{n} \sum_{s=1}^{n} [r^k - (r-1)^k + s^k - (s-1)^k] a_{rs} A^{rs} = 2n^k.$

2. $\displaystyle a'_{ij} = -\sum_{r=1}^{n} \sum_{s=1}^{n} a_{is} a_{rj} (A^{rs})'.$

3. $\displaystyle\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r + g_s) a_{is} a_{rj} A^{rs} = (f_i + g_j) a_{ij}.$

Note that (2) and (3) can be obtained formally from (B) and (D), respectively, by interchanging the symbols a and A and either raising or lowering all their parameters.

3.5 The Adjoint Determinant

3.5.1 Definition

The adjoint of a matrix $\mathbf{A} = [a_{ij}]_n$ is denoted by adj $\mathbf{A}$ and is defined by

$$\operatorname{adj} \mathbf{A} = [A_{ji}]_n.$$

The adjoint or adjugate or a determinant $A = |a_{ij}|_n = \det \mathbf{A}$ is denoted by adj A and is defined by

$$\begin{aligned} \operatorname{adj} A &= |A_{ji}|_n = |A_{ij}|_n \\ &= \det(\operatorname{adj} \mathbf{A}). \end{aligned} \tag{3.5.1}$$

3.5.2 The Cauchy Identity

The following theorem due to Cauchy is valid for all determinants.

Theorem.

$$\operatorname{adj} A = A^{n-1}.$$

The proof is similar to that of the matrix relation

$$\mathbf{A} \operatorname{adj} \mathbf{A} = A\mathbf{I}.$$

Proof.

$$
\begin{aligned}
A \operatorname{adj} A &= |a_{ij}|_n |A_{ji}|_n \\
&= |b_{ij}|_n,
\end{aligned}
$$

where, referring to Section 3.3.5 on the product of two determinants,

$$
\begin{aligned}
b_{ij} &= \sum_{r=1}^{n} a_{ir} A_{jr} \\
&= \delta_{ij} A.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|b_{ij}|_n &= \operatorname{diag}|A \;\; A \;\; \ldots \;\; A|_n \\
&= A^n.
\end{aligned}
$$

The theorem follows immediately if $A \neq 0$. If $A = 0$, then, applying (2.3.16) with a change in notation, $|A_{ij}|_n = 0$, that is, $\operatorname{adj} A = 0$. Hence, the Cauchy identity is valid for all A. □

3.5.3 An Identity Involving a Hybrid Determinant

Let $A_n = |a_{ij}|_n$ and $B_n = |b_{ij}|_n$, and let H_{ij} denote the hybrid determinant formed by replacing the jth row of A_n by the ith row of B_n. Then,

$$
H_{ij} = \sum_{s=1}^{n} b_{is} A_{js}. \tag{3.5.2}
$$

Theorem.

$$
|a_{ij} x_i + b_{ij}|_n = A_n \left| \delta_{ij} x_i + \frac{H_{ij}}{A_n} \right|_n, \quad A_n \neq 0.
$$

In the determinant on the right, the x_i appear only in the principal diagonal.

Proof. Applying the Cauchy identity in the form

$$
|A_{ji}|_n = A_n^{n-1}
$$

and the formula for the product of two determinants (Section 1.4),

$$
\begin{aligned}
|a_{ij} x_i + b_{ij}|_n A_n^{n-1} &= |a_{ij} x_i + b_{ij}|_n |A_{ji}|_n \\
&= |c_{ij}|_n,
\end{aligned}
$$

where

$$
\begin{aligned}
c_{ij} &= \sum_{s=1}^{n} (a_{is} x_i + b_{is}) A_{js} \\
&= x_i \sum_{s=1}^{n} a_{is} A_{js} + \sum_{s=1}^{n} b_{is} A_{js} \\
&= \delta_{ij} A_n x_i + H_{ij}.
\end{aligned}
$$

Hence, removing the factor A_n from each row,

$$|c_{ij}|_n = A_n^n \left| \delta_{ij} x_i + \frac{H_{ij}}{A_n} \right|_n$$

which yields the stated result.

This theorem is applied in Section 6.7.4 on the K dV equation. □

3.6 The Jacobi Identity and Variants

3.6.1 The Jacobi Identity — 1

Given an arbitrary determinant $A = |a_{ij}|_n$, the rejecter minor $M_{p_1 p_2 \ldots p_r; q_1 q_2 \ldots q_r}$ of order $(n-r)$ and the retainer minor $N_{p_1 p_2 \ldots p_r; q_1 q_2 \ldots q_r}$ of order r are defined in Section 3.2.1.

Define the retainer minor J of order r as follows:

$$\begin{aligned} J = J_{p_1 p_2 \ldots p_r; q_1 q_2 \ldots q_r} &= \operatorname{adj} N_{p_1 p_2 \ldots p_r; q_1 q_2 \ldots q_r} \\ &= \begin{vmatrix} A_{p_1 q_1} & A_{p_2 q_1} & \cdots & A_{p_r q_1} \\ A_{p_1 q_2} & A_{p_2 q_2} & \cdots & A_{p_r q_2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p_1 q_r} & A_{p_2 q_r} & \cdots & A_{p_r q_r} \end{vmatrix}_r . \end{aligned} \qquad (3.6.1)$$

J is a minor of adj A. For example,

$$\begin{aligned} J_{23,24} &= \operatorname{adj} N_{23,24} \\ &= \operatorname{adj} \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} \\ &= \begin{vmatrix} A_{22} & A_{32} \\ A_{24} & A_{34} \end{vmatrix}. \end{aligned}$$

The Jacobi identity on the minors of adj A is given by the following theorem:

Theorem.

$$J_{p_1 p_2 \ldots p_r; q_1 q_2 \ldots q_r} = A^{r-1} M_{p_1 p_2 \ldots p_r; q_1 q_2 \ldots q_r}, \quad 1 \le r \le n-1.$$

Referring to the section on the cofactors of a zero determinant in Section 2.3.7, it is seen that if $A = 0$, $r > 1$, then $J = 0$. The right-hand side of the above identity is also zero. Hence, in this particular case, the theorem is valid but trivial. When $r = 1$, the theorem degenerates into the definition of $A_{p_1 q_1}$ and is again trivial. It therefore remains to prove the theorem when $A \ne 0$, $r > 1$.

The proof proceeds in two stages. In the first stage, the theorem is proved in the particular case in which

$$p_s = q_s = s, \quad 1 \le s \le r.$$

It is required to prove that

$$J_{12\ldots r;12\ldots r} = A^{r-1}M_{12\ldots r;12\ldots r}$$
$$= A^{r-1}A_{12\ldots r;12\ldots r}.$$

The replacement of the minor by its corresponding cofactor is permitted since the sum of the parameters is even. In some detail, the simplified theorem states that

$$\begin{vmatrix} A_{11} & A_{21} & \ldots & A_{r1} \\ A_{12} & A_{22} & \ldots & A_{r2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1r} & A_{2r} & \ldots & A_{rr} \end{vmatrix}_r = A^{r-1} \begin{vmatrix} a_{r+1,r+1} & a_{r+1,r+2} & \cdots & a_{r+1,n} \\ a_{r+2,r+1} & a_{r+2,r+2} & \cdots & a_{r+2,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,r+1} & a_{n,r+2} & \cdots & a_{nn} \end{vmatrix}_{n-r}. \tag{3.6.2}$$

Proof. Raise the order of $J_{12\ldots r;12\ldots r}$ from r to n by applying the Laplace expansion formula in reverse as follows:

$$J_{12\ldots r;12\ldots r} = \begin{vmatrix} A_{11} & \ldots & A_{r1} & & & \\ \vdots & & \vdots & & & \\ A_{1r} & \ldots & A_{rr} & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{1,r+1} & \ldots & A_{r,r+1} & 1 & & \\ \vdots & & \vdots & & \ddots & \\ A_{1n} & \ldots & A_{rn} & & & 1 \end{vmatrix}_n \begin{array}{l} \left.\vphantom{\begin{matrix} A \\ \vdots \\ A \end{matrix}}\right\} r \text{ rows} \\ \\ \left.\vphantom{\begin{matrix} A \\ \vdots \\ A \end{matrix}}\right\} (n-r) \text{ rows} \end{array}. \tag{3.6.3}$$

Multiply the left-hand side by A, the right-hand side by $|a_{ij}|_n$, apply the formula for the product of two determinants, the sum formula for elements and cofactors, and, finally, the Laplace expansion formula again

$$AJ_{12\ldots r;12\ldots r} = \begin{vmatrix} A & & & \vdots & a_{1,r+1} & \ldots & a_{1n} \\ & \ddots & & \vdots & \vdots & & \vdots \\ & & A & \vdots & a_{r,r+1} & \ldots & a_{rn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & \vdots & a_{r+1,r+1} & \ldots & a_{r+1,n} \\ & & & \vdots & \vdots & & \vdots \\ & & & \vdots & a_{n,r+1} & \ldots & a_{nn} \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} A \\ \vdots \\ A \end{matrix}}\right\} r \text{ rows} \\ \\ \left.\vphantom{\begin{matrix} A \\ \vdots \\ A \end{matrix}}\right\} (n-r) \text{ rows} \end{array}$$
$$= A^r \begin{vmatrix} a_{r+1,r+1} & \ldots & a_{r+1,n} \\ \vdots & & \vdots \\ a_{n,r+1} & \ldots & a_{nn} \end{vmatrix}_{n-r}$$
$$= A^r A_{12\ldots r;12\ldots r}.$$

The first stage of the proof follows.

The second stage proceeds as follows. Interchange pairs of rows and then pairs of columns of adj A until the elements of J as defined in (3.6.1) appear as a block in the top left-hand corner. Denote the result by $(\operatorname{adj} A)^*$. Then,

$$(\operatorname{adj} A)^* = \sigma \operatorname{adj} A,$$

where

$$\begin{aligned}\sigma &= (-1)^{(p_1-1)+(p_2-2)+\cdots+(p_r-r)+(q_1-1)+(q_2-2)+\cdots+(q_r-r)}\\ &= (-1)^{(p_1+p_2+\cdots+p_r)+(q_1+q_2+\cdots+q_r)}.\end{aligned}$$

Now replace each A_{ij} in $(\operatorname{adj} A)^*$ by a_{ij}, transpose, and denote the result by $|a_{ij}|^*$. Then,

$$|a_{ij}|^* = \sigma|a_{ij}| = \sigma A.$$

Raise the order of J from r to n in a manner similar to that shown in (3.6.3), augmenting the first r columns until they are identical with the first r columns of $(\operatorname{adj} A)^*$, denote the result by J^*, and form the product $|a_{ij}|^* J^*$. The theorem then appears. $\square$

Illustration. Let $(n, r) = (4, 2)$ and let

$$J = J_{23,24} = \begin{vmatrix} A_{22} & A_{32} \\ A_{24} & A_{34} \end{vmatrix}.$$

Then

$$\begin{aligned}(\operatorname{adj} A)^* &= \begin{vmatrix} A_{22} & A_{32} & A_{12} & A_{42} \\ A_{24} & A_{34} & A_{14} & A_{44} \\ A_{21} & A_{31} & A_{11} & A_{41} \\ A_{23} & A_{33} & A_{13} & A_{43} \end{vmatrix}\\ &= \sigma \operatorname{adj} A,\end{aligned}$$

where

$$\sigma = (-1)^{2+3+2+4} = -1$$

and

$$\begin{aligned}|a_{ij}|^* &= \begin{vmatrix} a_{22} & a_{24} & a_{21} & a_{23} \\ a_{32} & a_{34} & a_{31} & a_{33} \\ a_{12} & a_{14} & a_{11} & a_{13} \\ a_{42} & a_{44} & a_{41} & a_{43} \end{vmatrix}\\ &= \sigma|a_{ij}| = \sigma A.\end{aligned}$$

The first two columns of J^* are identical with the first two columns of $(\operatorname{adj} A)^*$:

$$\begin{aligned}J = J^* &= \begin{vmatrix} A_{22} & A_{32} & & \\ A_{24} & A_{34} & & \\ A_{21} & A_{31} & 1 & \\ A_{23} & A_{33} & & 1 \end{vmatrix},\\ \sigma A J &= |a_{ij}|^* J^*\end{aligned}$$

$$= \begin{vmatrix} A & & a_{21} & a_{23} \\ & A & a_{31} & a_{33} \\ & & a_{11} & a_{13} \\ & & a_{41} & a_{43} \end{vmatrix}$$

$$= A^2 \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix}$$

$$= A^2 M_{23,24}$$

$$= \sigma A^2 A_{23,24}.$$

Hence, transposing J,

$$J = \begin{vmatrix} A_{22} & A_{24} \\ A_{32} & A_{34} \end{vmatrix} = A\, A_{23,24}$$

which completes the illustration.

Restoring the parameter n, the Jacobi identity with $r = 2, 3$ can be expressed as follows:

$$r = 2: \qquad \begin{vmatrix} A_{ip}^{(n)} & A_{iq}^{(n)} \\ A_{jp}^{(n)} & A_{jq}^{(n)} \end{vmatrix} = A_n A_{ij,pq}^{(n)}. \tag{3.6.4}$$

$$r = 3: \qquad \begin{vmatrix} A_{ip}^{(n)} & A_{iq}^{(n)} & A_{ir}^{(n)} \\ A_{jp}^{(n)} & A_{jq}^{(n)} & A_{jr}^{(n)} \\ A_{kp}^{(n)} & A_{kq}^{(n)} & A_{kr}^{(n)} \end{vmatrix} = A_n^2 A_{ijk,pqr}^{(n)}. \tag{3.6.5}$$

3.6.2 The Jacobi Identity — 2

The Jacobi identity for small values of r can be proved neatly by a technique involving partial derivatives with respect to the elements of A. The general result can then be proved by induction.

Theorem 3.4. *For an arbitrary determinant A_n of order n,*

$$\begin{vmatrix} A_n^{ij} & A_n^{iq} \\ A_n^{pj} & A_n^{pq} \end{vmatrix} = A_n^{ip,jq},$$

where the cofactors are scaled.

PROOF. The technique is to evaluate $\partial A^{ij}/\partial a_{pq}$ by two different methods and to equate the results. From (3.2.15),

$$\frac{\partial A^{ij}}{\partial a_{pq}} = \frac{1}{A^2}\big[A A_{ip,jq} - A_{ij} A_{pq}\big]. \tag{3.6.6}$$

Applying double-sum identity (B) in Section 3.4,

$$\frac{\partial A^{ij}}{\partial a_{pq}} = -\sum_r \sum_s \frac{\partial a_{rs}}{\partial a_{pq}} A^{is} A^{rj}$$

$$\begin{aligned}&= -\sum_r \sum_s \delta_{rp}\delta_{sq} A^{is} A^{rj}\\&= -A^{iq}A^{pj}\\&= -\frac{1}{A^2}[A_{iq}A_{pj}]. \qquad (3.6.7)\end{aligned}$$

Hence,

$$\begin{vmatrix} A_{ij} & A_{iq} \\ A_{pj} & A_{pq} \end{vmatrix} = AA_{ip,jq}, \qquad (3.6.8)$$

which, when the parameter n is restored, is equivalent to (3.6.4). The formula given in the theorem follows by scaling the cofactors. □

Theorem 3.5.

$$\begin{vmatrix} A^{ij} & A^{iq} & A^{iv} \\ A^{pj} & A^{pq} & A^{pv} \\ A^{uj} & A^{uq} & A^{uv} \end{vmatrix} = A^{ipu,jqv},$$

where the cofactors are scaled.

Proof. From (3.2.4) and Theorem 3.4,

$$\begin{aligned}\frac{\partial^2 A}{\partial a_{pq}\partial a_{uv}} &= A_{pu,qv}\\&= AA^{pu,qv}\\&= A\begin{vmatrix} A^{pq} & A^{pv} \\ A^{uq} & A^{uv} \end{vmatrix}. \qquad (3.6.9)\end{aligned}$$

Hence, referring to (3.6.7) and the formula for the derivative of a determinant (Section 2.3.7),

$$\begin{aligned}&\frac{\partial^3 A}{\partial a_{ij}\partial a_{pq}\partial a_{uv}}\\&= \frac{\partial A}{\partial a_{ij}}\begin{vmatrix} A^{pq} & A^{pv} \\ A^{uq} & A^{uv} \end{vmatrix} + A\begin{vmatrix} \frac{\partial A^{pq}}{\partial a_{ij}} & A^{pv} \\ \frac{\partial A^{uq}}{\partial a_{ij}} & A^{uv} \end{vmatrix} + A\begin{vmatrix} A^{pq} & \frac{\partial A^{pv}}{\partial a_{ij}} \\ A^{uq} & \frac{\partial A^{uv}}{\partial a_{ij}} \end{vmatrix}\\&= A_{ij}\begin{vmatrix} A^{pq} & A^{pv} \\ A^{uq} & A^{uv} \end{vmatrix} - AA^{iq}\begin{vmatrix} A^{pj} & A^{pv} \\ A^{uj} & A^{uv} \end{vmatrix} - AA^{iv}\begin{vmatrix} A^{pq} & A^{pj} \\ A^{uq} & A^{uj} \end{vmatrix}\\&= \frac{1}{A^2}\left[A_{ij}\begin{vmatrix} A_{pq} & A_{pv} \\ A_{uq} & A_{uv} \end{vmatrix} - A_{iq}\begin{vmatrix} A_{pj} & A_{pv} \\ A_{uj} & A_{uv} \end{vmatrix}\right.\\&\qquad \left. + A_{iv}\begin{vmatrix} A_{pj} & A_{pq} \\ A_{uj} & A_{uq} \end{vmatrix}\right]\\&= \frac{1}{A^2}\begin{vmatrix} A_{ij} & A_{iq} & A_{iv} \\ A_{pj} & A_{pq} & A_{pv} \\ A_{uj} & A_{uq} & A_{uv} \end{vmatrix}. \qquad (3.6.10)\end{aligned}$$

But also,

$$\frac{\partial^3 A}{\partial a_{ij}\partial a_{pq}\partial a_{uv}} = A_{ipu,jqv}. \tag{3.6.11}$$

Hence,

$$\begin{vmatrix} A_{ij} & A_{iq} & A_{iv} \\ A_{pj} & A_{pq} & A_{pv} \\ A_{uj} & A_{uq} & A_{uv} \end{vmatrix} = A^2 A_{ipujqv}, \tag{3.6.12}$$

which, when the parameter n is restored, is equivalent to (3.6.5). The formula given in the theorem follows by scaling the cofactors. Note that those Jacobi identities which contain scaled cofactors lack the factors A, A^2, etc., on the right-hand side. This simplification is significant in applications involving derivatives. □

Exercises

1. Prove that

$$\sum_{\text{ep}\{p,q,r\}} A_{pt} A_{qr,st} = 0,$$

where the symbol $\text{ep}\{p,q,r\}$ denotes that the sum is carried out over all even permutations of $\{p,q,r\}$, including the identity permutation (Appendix A.2).

2. Prove that

$$\begin{vmatrix} A^{ps} & A^{pi,js} \\ A^{rq} & A^{ri,jq} \end{vmatrix} = \begin{vmatrix} A^{rj} & A^{rp,qj} \\ A^{is} & A^{ip,qs} \end{vmatrix} = \begin{vmatrix} A^{iq} & A^{ir,sq} \\ A^{pj} & A^{pr,sj} \end{vmatrix}.$$

3. Prove the Jacobi identity for general values of r by induction.

3.6.3 Variants

Theorem 3.6.

$$\begin{vmatrix} A_{ip}^{(n)} & A_{i,n+1}^{(n+1)} \\ A_{jp}^{(n)} & A_{j,n+1}^{(n+1)} \end{vmatrix} - A_n A_{ij;p,n+1}^{(n+1)} = 0, \tag{A}$$

$$\begin{vmatrix} A_{ip}^{(n)} & A_{iq}^{(n)} \\ A_{n+1,p}^{(n+1)} & A_{n+1,q}^{(n+1)} \end{vmatrix} - A_n A_{i,n+1;pq}^{(n+1)} = 0, \tag{B}$$

$$\begin{vmatrix} A_{rr}^{(n)} & A_{rr}^{(n+1)} \\ A_{nr}^{(n)} & A_{nr}^{(n+1)} \end{vmatrix} - A_{n+1,r}^{(n+1)} A_{rn;r,n+1}^{(n+1)} = 0. \tag{C}$$

These three identities are consequences of the Jacobi identity but are distinct from it since the elements in each of the second-order determinants are cofactors of two different orders, namely $n-1$ and n.

PROOF. Denote the left side of variant (A) by E. Then, applying the Jacobi identity,

$$\begin{aligned} A_{n+1}E &= A_{n+1}\begin{vmatrix} A_{ip}^{(n)} & A_{i,n+1}^{(n+1)} \\ A_{jp}^{(n)} & A_{j,n+1}^{(n+1)} \end{vmatrix} - A_n \begin{vmatrix} A_{ip}^{(n+1)} & A_{i,n+1}^{(n+1)} \\ A_{jp}^{(n+1)} & A_{j,n+1}^{(n+1)} \end{vmatrix} \\ &= A_{i,n+1}^{(n+1)}F_j - A_{j,n+1}^{(n+1)}F_i, \end{aligned} \tag{3.6.13}$$

where

$$\begin{aligned} F_i &= A_n A_{ip}^{(n+1)} - A_{n+1}A_{ip}^{(n)} \\ &= \left[\begin{vmatrix} A_{ip}^{(n+1)} & A_{i,n+1}^{(n+1)} \\ A_{n+1,p}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} - A_{n+1}A_{ip}^{(n)}\right] + A_{i,n+1}^{(n+1)}A_{n+1,p}^{(n+1)} \\ &= A_{i,n+1}^{(n+1)}A_{n+1,p}^{(n+1)}. \end{aligned}$$

Hence,

$$\begin{aligned} A_{n+1}E &= \left[A_{i,n+1}^{(n+1)}A_{j,n+1}^{(n+1)} - A_{j,n+1}^{(n+1)}A_{i,n+1}^{(n+1)}\right]A_{n+1,p}^{(n+1)} \\ &= 0. \end{aligned} \tag{3.6.14}$$

The result follows and variant (B) is proved in a similar manner. Variant (A) appears in Section 4.8.5 on Turanians and is applied in Section 6.5.1 on Toda equations.

The proof of (C) applies a particular case of (A) and the Jacobi identity. In (A), put $(i,j,p) = (r,n,r)$:

$$\begin{vmatrix} A_{rr}^{(n)} & A_{r,n+1}^{(n+1)} \\ A_{nr}^{(n)} & A_{n,n+1}^{(n+1)} \end{vmatrix} - A_n A_{rn;r,n+1}^{(n+1)} = 0. \tag{A$_1$}$$

Denote the left side of (C) by P

$$\begin{aligned} A_nP &= A_n\begin{vmatrix} A_{rr}^{(n)} & A_{rr}^{(n+1)} \\ A_{nr}^{(n)} & A_{nr}^{(n+1)} \end{vmatrix} - A_{n+1,r}^{(n+1)}\begin{vmatrix} A_{rr}^{(n)} & A_{r,n+1}^{(n+1)} \\ A_{nr}^{(n)} & A_{n,n+1}^{(n+1)} \end{vmatrix} \\ &= \begin{vmatrix} A_{rr}^{(n)} & A_{rr}^{(n+1)} & A_{r,n+1}^{(n+1)} \\ A_{nr}^{(n)} & A_{nr}^{(n+1)} & A_{n,n+1}^{(n+1)} \\ \bullet & A_{n+1,r}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} \\ &= A_{rr}^{(n)}G_n - A_{nr}^{(n)}G_r, \end{aligned} \tag{3.6.15}$$

where

$$\begin{aligned} G_i &= \begin{vmatrix} A_{ir}^{(n+1)} & A_{i,n+1}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} \\ &= A_{n+1}A_{i,n+1;r,n+1}^{(n+1)}. \end{aligned} \tag{3.6.16}$$

Hence,

$$A_nP = A_{n+1}\left[A_{rr}^{(n)}A_{n,n+1;r,n+1}^{(n+1)} - A_{nr}^{(n)}A_{r,n+1;r,n+1}^{(n+1)}\right].$$

But $A^{(n+1)}_{i,n+1;j,n+1} = A^{(n)}_{ij}$. Hence, $A_n P = 0$. The result follows. □

Three particular cases of (B) are required for the proof of the next theorem.

Put $(i,p,q) = (r,r,n)$, $(n-1,r,n)$, (n,r,n) in turn:

$$\begin{vmatrix} A^{(n)}_{rr} & A^{(n)}_{rn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A_n A^{(n+1)}_{r,n+1;rn} = 0, \tag{B$_1$}$$

$$\begin{vmatrix} A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A_n A^{(n+1)}_{n-1,n+1;rn} = 0, \tag{B$_2$}$$

$$\begin{vmatrix} A^{(n)}_{nr} & A^{(n)}_{nn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A_n A^{(n+1)}_{n,n+1;rn} = 0. \tag{B$_3$}$$

Theorem 3.7.

$$\begin{vmatrix} A^{(n+1)}_{r,n+1;rn} & A^{(n)}_{rr} & A^{(n)}_{rn} \\ A^{(n+1)}_{n-1,n+1;rn} & A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n+1)}_{n,n+1;rn} & A^{(n)}_{nr} & A^{(n)}_{nn} \end{vmatrix} = 0.$$

PROOF. Denote the determinant by Q. Then,

$$\begin{aligned} Q_{11} &= \begin{vmatrix} A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n)}_{nr} & A^{(n)}_{nn} \end{vmatrix} \\ &= A_n A^{(n)}_{n-1,n;rn} \\ &= A_n A^{(n-1)}_{n-1,r}, \\ Q_{21} &= -A_n A^{(n-1)}_{rr}, \\ Q_{31} &= A_n A^{(n)}_{r,n-1;rn}. \end{aligned} \tag{3.6.17}$$

Hence, expanding Q by the elements in column 1 and applying (B$_1$)–(B$_3$),

$$\begin{aligned} Q = A_n \big[&A^{(n+1)}_{r,n+1;rn} A^{(n-1)}_{n-1,r} - A^{(n+1)}_{n-1,n+1;rn} A^{(n-1)}_{rr} \\ &+ A^{(n+1)}_{n,n+1;rn} A^{(n)}_{r,n-1;rn} \big] \end{aligned} \tag{3.6.18}$$

$$\begin{aligned} &= A^{(n-1)}_{n-1,r} \begin{vmatrix} A^{(n)}_{rr} & A^{(n)}_{rn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A^{(n-1)}_{rr} \begin{vmatrix} A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} \\ &\quad + A^{(n)}_{r,n-1;rn} \begin{vmatrix} A^{(n)}_{nr} & A^{(n)}_{nn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} \\ &= A^{(n+1)}_{n+1,n} \left[A^{(n)}_{nr} A^{(n)}_{r,n-1;rn} - \begin{vmatrix} A^{(n-1)}_{rr} & A^{(n)}_{rr} \\ A^{(n-1)}_{n-1,r} & A^{(n)}_{n-1,r} \end{vmatrix} \right] \\ &\quad - A^{(n+1)}_{n+1,r} \left[A_{n-1} A^{(n)}_{r,n-1;rn} - \begin{vmatrix} A^{(n-1)}_{rr} & A^{(n)}_{rn} \\ A^{(n-1)}_{n-1,r} & A^{(n)}_{n-1,n} \end{vmatrix} \right]. \end{aligned} \tag{3.6.19}$$

The proof is completed by applying (C) and (A_1) with $n \to n-1$. Theorem 3.7 is applied in Section 6.6 on the Matsukidaira–Satsuma equations. □

Theorem 3.8.

$$A_{n+1,r}^{(n+1)} H_n = A_{n+1,n}^{(n+1)} H_r,$$

where

$$H_j = \begin{vmatrix} A_{rr}^{(n-1)} & A_{rj}^{(n+1)} \\ A_{n-1,r}^{(n-1)} & A_{n-1,j}^{(n+1)} \end{vmatrix} - A_{nj}^{(n+1)} A_{r,n-1;rn}^{(n)}.$$

Proof. Return to (3.6.18), multiply by A_{n+1}/A_n and apply the Jacobi identity:

$$\begin{aligned}
&A_{n-1,r}^{(n-1)} \begin{vmatrix} A_{rr}^{(n+1)} & A_{rn}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n}^{(n+1)} \end{vmatrix} - A_{rr}^{(n-1)} \begin{vmatrix} A_{n-1,r}^{(n+1)} & A_{n-1,n}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n}^{(n+1)} \end{vmatrix} \\
&\qquad + A_{r,n-1;rn}^{(n)} \begin{vmatrix} A_{nr}^{(n+1)} & A_{nn}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n}^{(n+1)} \end{vmatrix} = 0, \\
&A_{n+1,r}^{(n+1)} \left[A_{rr}^{(n-1)} A_{n-1,n}^{(n+1)} - A_{n-1,r}^{(n-1)} A_{rn}^{(n+1)} - A_{nn}^{(n+1)} A_{r,n-1;rn}^{(n)} \right] \\
&\quad = A_{n+1,n}^{(n+1)} \left[A_{rr}^{(n-1)} A_{n-1,r}^{(n+1)} - A_{rr}^{(n+1)} A_{n-1,r}^{(n-1)} - A_{r,n-1;rn}^{(n)} A_{nr}^{(n+1)} \right], \\
&A_{n+1,r}^{(n+1)} \left[\begin{vmatrix} A_{rr}^{(n-1)} & A_{rn}^{(n+1)} \\ A_{n-1,r}^{(n-1)} & A_{n-1,n}^{(n+1)} \end{vmatrix} - A_{nn}^{(n+1)} A_{r,n-1;rn}^{(n)} \right] \\
&\quad = A_{n+1,n}^{(n+1)} \left[\begin{vmatrix} A_{rr}^{(n-1)} & A_{rr}^{(n+1)} \\ A_{n-1,r}^{(n-1)} & A_{n-1,r}^{(n+1)} \end{vmatrix} - A_{nr}^{(n+1)} A_{r,n-1;rn}^{(n)} \right].
\end{aligned}$$

The theorem follows. □

Exercise. Prove that

$$\begin{vmatrix} A_{i_1 j_1}^{(n)} & A_{j_1 j_2}^{(n)} & \dots & A_{i_1 j_{r-1}}^{(n)} & A_{i_1,n+1}^{(n+1)} \\ A_{i_2 j_1}^{(n)} & A_{i_2 j_2}^{(n)} & \dots & A_{i_2 j_{r-1}}^{(n)} & A_{i_2,n+1}^{(n+1)} \\ \dots & \dots & \dots & \dots & \dots \\ A_{i_r j_1}^{(n)} & A_{i_r j_2}^{(n)} & \dots & A_{i_r j_{r-1}}^{(n)} & A_{i_r,n+1}^{(n+1)} \end{vmatrix}_r = A_n^{r-1} A_{i_1 i_2 \dots i_r; j_1 j_2 \dots j_{r-1},n+1}^{(n+1)}.$$

When $r = 2$, this identity degenerates into Variant (A). Generalize Variant (B) in a similar manner.

3.7 Bordered Determinants

3.7.1 Basic Formulas; The Cauchy Expansion

Let

$$\begin{aligned} A_n &= |a_{ij}|_n \\ &= \left| \mathbf{C}_1 \ \mathbf{C}_2 \ \mathbf{C}_3 \cdots \mathbf{C}_n \right|_n \end{aligned}$$

and let B_n denote the determinant of order $(n+1)$ obtained by bordering A_n by the column

$$\mathbf{X} = \begin{bmatrix} x_1 \ x_2 \ x_3 \cdots x_n \end{bmatrix}^T$$

on the right, the row

$$\mathbf{Y} = \begin{bmatrix} y_1 \ y_2 \ y_3 \cdots y_n \end{bmatrix}$$

at the bottom and the element z in position $(n+1, n+1)$. In some detail,

$$B_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & x_2 \\ \multicolumn{5}{c}{\dotfill} \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_n & z \end{vmatrix}_{n+1}. \tag{3.7.1}$$

Some authors border on the left and at the top but this method displaces the element a_{ij} to the position $(i+1, j+1)$, which is undesirable for both practical and aesthetic reasons except in a few special cases.

In the theorems which follow, the notation is simplified by discarding the suffix n.

Theorem 3.9.

$$B = zA - \sum_{r=1}^{n} \sum_{s=1}^{n} A_{rs} x_r y_s.$$

PROOF. The coefficient of y_s in B is $(-1)^{n+s+1} F$, where

$$\begin{aligned} F &= \left| \mathbf{C}_1 \ \ldots \ \mathbf{C}_{s-1} \ \mathbf{C}_{s+1} \ldots \mathbf{C}_n \ \mathbf{X} \right|_n \\ &= (-1)^{n+s} G, \end{aligned}$$

where

$$G = \left| \mathbf{C}_1 \ldots \mathbf{C}_{s-1} \ \mathbf{X} \ \mathbf{C}_{s+1} \ldots \mathbf{C}_n \right|_n.$$

The coefficient of x_r in G is A_{rs}. Hence, the coefficient of $x_r y_s$ in B is

$$(-1)^{n+s+1+n+s} A_{rs} = -A_{rs}.$$

The only term independent of the x's and y's is zA. The theorem follows. □

Let E_{ij} denote the determinant obtained from A by

a. replacing a_{ij} by z, i, j fixed,
b. replacing a_{rj} by x_r, $1 \le r \le n$, $r \ne i$,
c. replacing a_{is} by y_s, $1 \le s \le n$, $s \ne j$.

Theorem 3.10.

$$B_{ij} = zA_{ij} - \sum_{r=1}^{n} \sum_{s=1}^{n} A_{ir,js} x_r y_s = E_{ij}.$$

PROOF.

$$B_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} & x_{i-1} \\ a_{i+1,i} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} & x_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_{j-1} & y_{j+1} & \cdots & y_n & z \end{vmatrix}_n .$$

The expansion is obtained by applying arguments to B_{ij} similar to those applied to B in Theorem 3.9. Since the second cofactor is zero when $r = i$ or $s = j$ the double sum contains $(n-1)^2$ nonzero terms, as expected. It remains to prove that $B_{ij} = E_{ij}$.

Transfer the last row of B_{ij} to the ith position, which introduces the sign $(-1)^{n-i}$ and transfer the last column to the jth position, which introduces the sign $(-1)^{n-j}$. The result is E_{ij}, which completes the proof. □

The Cauchy expansion of an arbitrary determinant focuses attention on one arbitrarily chosen element a_{ij} and its cofactor.

Theorem 3.11. *The Cauchy expansion*

$$A = a_{ij}A_{ij} + \sum_{r=1}^{n} \sum_{s=1}^{n} a_{is}a_{rj}A_{ir,sj}.$$

First Proof. The expansion is essentially the same as that given in Theorem 3.10. Transform E_{ij} back to A by replacing z by a_{ij}, x_r by a_{rj} and y_s by a_{is}. The theorem appears after applying the relation

$$A_{ir,js} = -A_{ir,sj}. \tag{3.7.2}$$

Second Proof. It follows from (3.2.3) that

$$\sum_{r=1}^{n} a_{rj}A_{ir,sj} = (1 - \delta_{js})A_{is}.$$

Multiply by a_{is} and sum over s:

$$\begin{aligned} \sum_{r=1}^{n} \sum_{s=1}^{n} a_{is}a_{rj}A_{ir,sj} &= \sum_{s=1}^{n} a_{is}A_{is} - \sum_{s=1}^{n} \delta_{js}a_{is}A_{is} \\ &= A - a_{ij}A_{ij}, \end{aligned}$$

which is equivalent to the stated result. □

Theorem 3.12. *If $y_s = 1$, $1 \le s \le n$, and $z = 0$, then*

$$\sum_{j=1}^{n} B_{ij} = 0, \quad 1 \le i \le n.$$

PROOF. It follows from (3.7.2) that

$$\sum_{j=1}^{n}\sum_{s=1}^{n} A_{ir,js} = 0, \quad 1 \le i, r \le n.$$

Expanding B_{ij} by elements from the last column,

$$B_{ij} = -\sum_{r=1}^{n} x_r \sum_{s=1}^{n} A_{ir,js}.$$

Hence

$$\begin{aligned}\sum_{j=1}^{n} B_{ij} &= -\sum_{r=1}^{n} x_r \sum_{j=1}^{n}\sum_{s=1}^{n} A_{ir,js}\\ &= 0.\end{aligned}$$

Bordered determinants appear in other sections including Section 4.10.3 on the Yamazaki–Hori determinant and Section 6.9 on the Benjamin–Ono equation. □

3.7.2 A Determinant with Double Borders

Theorem 3.13.

$$\left|\begin{array}{cccc|cc} & & & & u_1 & v_1\\ & & & & u_2 & v_2\\ & [a_{ij}]_n & & & \cdots & \cdots\\ & & & & u_n & v_n\\ \hline x_1 & x_2 & \cdots & x_n & \bullet & \bullet\\ y_1 & y_2 & \cdots & y_n & \bullet & \bullet \end{array}\right|_{n+2} = \sum_{p,q,r,s=1}^{n} u_p v_q x_r y_s A_{pq,rs},$$

where

$$A = |a_{ij}|_n.$$

PROOF. Denote the determinant by B and apply the Jacobi identity to cofactors obtained by deleting one of the last two rows and one of the last two columns

$$\begin{vmatrix} B_{n+1,n+1} & B_{n+1,n+2}\\ B_{n+2,n+1} & B_{n+2,n+2} \end{vmatrix} = \frac{BB_{n+1,n+2;n+1,n+2}}{BA.} \tag{3.7.3}$$

Each of the first cofactors is a determinant with single borders

$$B_{n+1,n+1} = \left|\begin{array}{cccc|c} & & & & v_1\\ & & & & v_2\\ & [a_{ij}]_n & & & \cdots\\ & & & & v_n\\ \hline y_1 & y_2 & \cdots & y_n & \bullet \end{array}\right|_{n+1}$$

$$= -\sum_{q=1}^{n}\sum_{s=1}^{n} v_q y_s A_{qs}.$$

Similarly,

$$\begin{aligned}
B_{n+1,n+2} &= +\sum_{p=1}^{n}\sum_{s=1}^{n} u_p y_s A_{ps},\\
B_{n+2,n+1} &= +\sum_{q=1}^{n}\sum_{r=1}^{n} v_q x_r A_{qr},\\
B_{n+2,n+2} &= -\sum_{p=1}^{n}\sum_{r=1}^{n} u_p x_r A_{pr}.
\end{aligned}$$

Note the variations in the choice of dummy variables. Hence, (3.7.3) becomes

$$BA = \sum_{p,q,r,s=1}^{n} u_p v_q x_r y_s \begin{vmatrix} A_{pr} & A_{ps} \\ A_{qr} & A_{qs} \end{vmatrix}.$$

The theorem appears after applying the Jacobi identity and dividing by A. □

Exercises

1. Prove the Cauchy expansion formula for A_{ij}, namely

$$A_{ij} = a_{pq}A_{ip,jq} - \sum_{r=1}^{n}\sum_{s=1}^{n} a_{ps}a_{rq}A_{ipr,jqs},$$

where $(p,q) \neq (i,j)$ but are otherwise arbitrary. Those terms in which $r = i$ or p or those in which $s = j$ or q are zero by the definition of higher cofactors.

2. Prove the generalized Cauchy expansion formula, namely

$$A = N_{ij,hk}A_{ij,hk} + \sum_{1\le p\le q\le n}\;\sum_{1\le r\le s\le n} N_{ij,rs}N_{pq,hk}A_{ijpq,rshk},$$

where $N_{ij,hk}$ is a retainer minor and $A_{ij,hk}$ is its complementary cofactor.