5

Further Determinant Theory

5.1 Determinants Which Represent Particular Polynomials

5.1.1 Appell Polynomial

Notes on Appell polynomials are given in Appendix A.4.

Let

$$\psi_n(x) = (-1)^n \sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r}(-x)^r. \tag{5.1.1}$$

Theorem.

$$\textbf{a.}\ \psi_n(x) = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ 1 & x & x^2 & x^3 & \cdots & \cdots & \binom{n}{0} x^n \\ & 1 & 2x & 3x^2 & \cdots & \cdots & \binom{n}{1} x^{n-1} \\ & & 1 & 3x & \cdots & \cdots & \binom{n}{2} x^{n-2} \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & nx \end{vmatrix}_{n+1},$$

b. $$\psi_n(x) = \frac{1}{n!}\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ n & x & & & & & \\ & n-1 & 2x & & & & \\ & & n-2 & 3x & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & nx \end{vmatrix}_{n+1}.$$

Both determinants are Hessenbergians (Section 4.6).

Proof of (a). Denote the determinant by H_{n+1}, expand it by the two elements in the last row, and repeat this operation on the determinants of lower order which appear. The result is

$$H_{n+1}(x) = \sum_{r=1}^{n} \binom{n}{r} H_{n+1-r}(-x)^r + (-1)^n \alpha_n.$$

The H_{n+1} term can be absorbed into the sum, giving

$$(-1)^n \alpha_n = \sum_{r=0}^{n} \binom{n}{r} H_{n+1-r}(-x)^r.$$

This is an Appell polynomial whose inverse relation is

$$H_{n+1}(x) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \alpha_{n-r} x^r,$$

which is equivalent to the stated result.

Proof of (b). Denote the determinant by H^*_{n+1} and note that some of its elements are functions of n, so that the minor obtained by removing its last row and column is *not* equal to H^*_n and hence there is no obvious recurrence relation linking H^*_{n+1}, H^*_n, H^*_{n-1}, etc.

The determinant H^*_{n+1} can be obtained by transforming H_{n+1} by a series of row operations which reduce some of its elements to zero. Multiply $\mathbf{R}_i$ by $(n+2-i)$, $2 \le i \le n+1$, and compensate for the unwanted factor $n!$ by dividing the determinant by that factor. Now perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \left(\frac{i-1}{n+1-i}\right) x\mathbf{R}_{i+1}$$

first with $2 \le i \le n$, which introduces $(n-1)$ zero elements into $\mathbf{C}_{n+1}$, then with $2 \le i \le n-1$, which introduces $(n-2)$ zero elements into $\mathbf{C}_n$, then with $2 \le i \le n-2$, etc., and, finally, with $i = 2$. The determinant H^*_{n+1} appears. □

5.1.2 The Generalized Geometric Series and Eulerian Polynomials

Notes on the generalized geometric series $\psi_n(x)$ and the Eulerian polynomials $A_n(x)$ are given in Appendix A.6.

$$A_n(x) = (1-x)^{n+1}\psi_n(x). \tag{5.1.2}$$

Theorem (Lawden).

$$\frac{A_n}{n!\,x} = \begin{vmatrix} 1 & 1-x \\ 1/2! & 1 & 1-x \\ 1/3! & 1/2! & 1 & 1-x \\ \cdots & \cdots & \cdots & \cdots \\ 1/(n-1)! & 1/(n-2)! & \cdots & & 1 & 1-x \\ 1/n! & 1/(n-1)! & \cdots & & 1/2! & 1 \end{vmatrix}_n .$$

The determinant is a Hessenbergian.

PROOF. It is proved in the section on differences (Appendix A.8) that

$$\Delta^m \psi_0 = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \psi_s = x\psi_m. \tag{5.1.3}$$

Put

$$\psi_s = (-1)^s s!\,\phi_s. \tag{5.1.4}$$

Then,

$$\sum_{s=0}^{m-1} \frac{\phi_s}{(m-s)!} + (1-x)\phi_m = 0, \quad m = 1,2,3,\ldots. \tag{5.1.5}$$

In some detail,

$$\begin{aligned}
&\phi_0 + (1-x)\phi_1 &&= 0,\\
&\phi_0/2! + \phi_1 + (1-x)\phi_2 &&= 0,\\
&\phi_0/3! + \phi_1/2 + \phi_2 + (1-x)\phi_3 &&= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\phi_0/n! + \phi_1/(n-1)! + \phi_2/(n-2)! + \cdots + \phi_{n-1} + (1-x)\phi_n &&= 0.
\end{aligned} \tag{5.1.6}$$

When these n equations in the $(n+1)$ variables ϕ_r, $0 \le r \le n$, are augmented by the relation

$$(1-x)\phi_0 = x, \tag{5.1.7}$$

the determinant of the coefficients is triangular so that its value is $(1-x)^{n+1}$. Solving the $(n+1)$ equations by Cramer's formula (Section 2.3.5),

$$\phi_n = \frac{1}{(1-x)^{n+1}}
\begin{vmatrix}
1-x & & & & & & x \\
1 & 1-x & & & & & 0 \\
1/2! & 1 & 1-x & & & & 0 \\
1/3! & 1/2! & 1 & 1-x & & & 0 \\
\multicolumn{7}{c}{\dotfill} \\
1/(n-1)! & 1/(n-2)! & \dots & & 1 & 1-x & 0 \\
1/n! & 1/(n-1)! & \dots & & 1/2! & 1 & 0
\end{vmatrix}_{n+1} \tag{5.1.8}$$

After expanding the determinant by the single nonzero element in the last column, the theorem follows from (5.1.2) and (5.1.4). □

Exercises

Prove that

1. $$\sum_{r=0}^{n} \alpha_r x^{n-r} y^r = \begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\
-y & x & & & & & \\
 & -y & x & & & & \\
 & & -y & x & & & \\
 & & & \dots & \dots & \dots & \dots \\
 & & & & & -y & x
\end{vmatrix}_{n+1}.$$

2. $$(x+y)^n = \begin{vmatrix}
1 & x & x^2 & x^3 & \cdots & x^{n-1} & x^n \\
-1 & y & xy & x^2y & \cdots & x^{n-2}y & x^{n-1}y \\
 & -1 & y & xy & \cdots & x^{n-3}y & x^{n-2}y \\
 & & -1 & y & \cdots & x^{n-4}y & x^{n-3}y \\
 & & \dots & \dots & \dots & \dots & \dots \\
 & & & & & -1 & y
\end{vmatrix}_{n+1}.$$

3. $$(-b)^n {}_2F_0\left(\frac{x}{a}, -n; -\frac{b}{a}\right)
= \begin{vmatrix}
-c_1 & b & & & & \\
a & -c_2 & b & & & \\
 & 2a & -c_3 & b & & \\
 & & 3a & -c_4 & & \\
 & & & \cdots & \cdots & \cdots \\
 & & & & -c_{n-1} & b \\
 & & & & (n-1)a & -c_n
\end{vmatrix}_n,$$

where

$$c_r = (r-1)a + b + x. \qquad \text{(Frost and Sackfield)}$$

and ${}_2F_0$ is the generalized hypergeometric function.

5.1.3 Orthogonal Polynomials

Determinants which represent orthogonal polynomials (Appendix A.5) have been constructed using various methods by Pandres, Rösler, Yahya, Stein et al., Schleusner, and Singhal, Frost and Sackfield and others. The following method applies the Rodrigues formulas for the polynomials.

Let

$$A_n = |a_{ij}|_n,$$

where

$$\begin{aligned} a_{ij} &= \binom{j-1}{i-1} u^{(j-i)} - \binom{j-1}{i-2} v^{(j-i+1)}, \quad u^{(r)} = D^r(u), \text{ etc.,} \\ u &= \frac{vy'}{y} = v(\log y)'. \end{aligned} \tag{5.1.9}$$

In some detail,

$$A_n = \begin{vmatrix} u & u' & u'' & u''' & \cdots & u^{(n-2)} & u^{(n-1)} \\ -v & u - v' & 2u' - v'' & 3u'' - v''' & \cdots & \cdots & \cdots \\ & -v & u - 2v' & 3u' - 3v'' & \cdots & \cdots & \cdots \\ & & -v & u - 3v' & \cdots & \cdots & \cdots \\ & & & -v & \cdots & \cdots & \cdots \\ & & & & \cdots & \cdots & \cdots \\ & & & & & -v & u-(n-1)v' \end{vmatrix}_n . \tag{5.1.10}$$

Theorem.

a. $A^{(n+1)}_{n+1,n} = -A'_n$,

b. $A_n = \dfrac{v^n D^n(y)}{y}$.

PROOF. Express A_n in column vector notation:

$$A_n = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|_n,$$

where

$$\mathbf{C}_j = \left[a_{1j}\ a_{2j}\ a_{3j} \cdots a_{j+1,j}\ O_{n-j-1}\right]^T_n \tag{5.1.11}$$

where O_r represents an unbroken sequence of r zero elements.

Let $\mathbf{C}^*_j$ denote the column vector obtained by dislocating the elements of $\mathbf{C}_j$ one position downward, leaving the uppermost position occupied by a zero element:

$$\mathbf{C}^*_j = \left[O\ a_{1j}\ a_{2j} \cdots a_{jj}\ a_{j+1,j}\ O_{n-j-2}\right]^T_n. \tag{5.1.12}$$

Then,

$$\mathbf{C}'_j + \mathbf{C}^*_j = \left[a'_{1j}\ (a'_{2j} + a_{1j})\ (a'_{3j} + a_{2j}) \cdots (a'_{j+1,j} + a_{jj})\ a_{j+1,j}\ O_{n-j-2}\right]^T_n.$$

But

$$\begin{aligned} a'_{ij} + a_{i-1,j} &= \left[\binom{j-1}{i-1} + \binom{j-1}{i-2}\right] u^{(j-i+1)} \\ &\quad - \left[\binom{j-1}{i-2} + \binom{j-1}{i-3}\right] v^{(j-i+2)} \\ &= \binom{j}{i-1} u^{(j-i+1)} - \binom{j}{i-2} v^{(j-i+2)} \\ &= a_{i,j+1}. \end{aligned} \tag{5.1.13}$$

Hence,

$$\mathbf{C}'_j + \mathbf{C}^*_j = \mathbf{C}_{j+1}, \tag{5.1.14}$$

$$\begin{aligned} A'_n &= \sum_{j=1}^{n} \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}'_j\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n, \\ A^{(n+1)}_{n+1,n} &= -\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_j\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_{n+1}\right|_n. \end{aligned} \tag{5.1.15}$$

Hence,

$$\begin{aligned} A'_n + A^{(n+1)}_{n+1,n} &= \sum_{j=1}^{n} \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots (\mathbf{C}'_j - \mathbf{C}_{j+1})\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n \\ &= -\sum_{j=1}^{n} \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}^*_j \cdots \mathbf{C}_n\right| \\ &= 0 \end{aligned}$$

by Theorem 3.1 on cyclic dislocations and generalizations in Section 3.1, which proves (a).

Expanding A_{n+1} by the two elements in its last row,

$$\begin{aligned} A_{n+1} &= (u - nv')A_n - vA^{(n+1)}_{n+1,n} \\ &= (u - nv')A_n + vA'_n \\ &= v\left[A'_n + \left(\frac{u}{v} - \frac{nv'}{v}\right)A_n\right], \\ \frac{yA_{n+1}}{v^{n+1}} &= \frac{y}{v^n}\left[A'_n + \left(\frac{y'}{y} - \frac{nv'}{v}\right)A_n\right] \\ &= \frac{yA'_n}{v^n} + \left(\frac{y}{v^n}\right)' A_n \\ &= D\left(\frac{yA_n}{v^n}\right) \\ &= D^2\left(\frac{yA_{n-1}}{v^{n-1}}\right) \\ &= D^r\left(\frac{yA_{n-r+1}}{v^{n-r+1}}\right), \quad 0 \le r \le n \end{aligned}$$

$$
\begin{aligned}
&= D^n\left(\frac{yA_1}{v}\right), \quad \left(A_1 = u = \frac{vy'}{y}\right)\\
&= D^{n+1}(y).
\end{aligned}
$$

Hence,

$$
A_{n+1} = \frac{v^{n+1}D^{n+1}(y)}{y},
$$

which is equivalent to (b).

The Rodrigues formula for the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ is

$$
L_n^{(\alpha)}(x) = \frac{x^n D^n(e^{-x}x^{n+\alpha})}{n!\, e^{-x}x^{n+\alpha}}\,. \tag{5.1.16}
$$

Hence, choosing

$$
\begin{aligned}
v &= x,\\
y &= e^{-x}x^{n+\alpha},
\end{aligned}
$$

so that

$$
u = x - n - \alpha, \tag{5.1.17}
$$

formula (b) becomes

$$
L_n^{(\alpha)}(x) = \frac{1}{n!} \times \tag{5.1.18}
$$

$$
\left|\begin{array}{cccccc}
n+\alpha-x & 1 & & & & \\
-x & n+\alpha-x-1 & 2 & & & \\
 & -x & n+\alpha-x-2 & 3 & & \\
 & & \cdots & \cdots & \cdots & \cdots \\
 & & & & 2+\alpha-x & n-1 \\
 & & & & -x & 1+\alpha-x
\end{array}\right|_n .
$$

$\square$

Exercises

Prove that

1. $$L_n^{(\alpha)}(x) = \frac{1}{n!}\left|\begin{array}{ccccc}
n+\alpha-x & n+\alpha & n+\alpha & n+\alpha & \cdots \\
1 & n+\alpha-x & n+\alpha & n+\alpha & \cdots \\
 & 2 & n+\alpha-x & n+\alpha & \cdots \\
 & & 3 & n+\alpha-x & \cdots \\
 & & & \cdots\cdots & \cdots
\end{array}\right|_n$$

(Pandres).

2. $$H_n(x) = \begin{vmatrix} 2x & 2 & & & & & & \\ 1 & 2x & 4 & & & & & \\ & 1 & 2x & 6 & & & & \\ & & 1 & 2x & 8 & & & \\ & & & \cdots & \cdots & \cdots & \cdots & \\ & & & & & & 2x & 2n-2 \\ & & & & & & 1 & 2x \end{vmatrix}_n .$$

3. $$P_n(x) = \frac{1}{n!}\begin{vmatrix} x & 1 & & & & & \\ 1 & 3x & 2 & & & & \\ & 2 & 5x & 3 & & & \\ & & 3 & 7x & 4 & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & (2n-3)x & n-1 \\ & & & & & n-1 & (2n-1)x \end{vmatrix}_n$$

(Muir and Metzler).

4. $$P_n(x) = \frac{1}{2^n n!} \begin{vmatrix} 2nx & 2n & & & & \\ 1-x^2 & (2n-2)x & 4n-2 & & & \\ & 1-x^2 & (2n-4)x & 6n-6 & & \\ & & 1-x^2 & (2n-6)x & & \\ & & & \cdots & \cdots & \cdots \\ & & & & 4x & n^2+n-2 \\ & & & & 1-x^2 & 2x \end{vmatrix}_n .$$

5. Let

$$A_n = |a_{ij}|_n = \left|\mathbf{C}_1 \ \mathbf{C}_2 \ \mathbf{C}_3 \cdots \mathbf{C}_n\right|,$$

where

$$\begin{aligned} a_{ij} &= u^{(j-1)} \\ &= \begin{cases} \dbinom{j-1}{i-2} v^{(j-i+1)}, & 2 \le i \le j+1, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

and let

$$\mathbf{C}_j^* = \left[O_2 \ a_{2j} \ a_{3j} \cdots a_{n-1,j}\right]^T.$$

Prove that

$$\mathbf{C}_j' + \mathbf{C}_j^* = \mathbf{C}_{j+1},$$
$$A_n' + A_{n+1,n}^{(n+1)} = 0$$

and hence prove that

$$A_n = (-1)^{n+1} v^n D^{n-1}\left(\frac{u}{v}\right).$$

6. Prove that the determinant A_n in (5.1.10) satisfies the relation

$$A_{n+1} = vA_n' + (u - nv')A_n.$$

Put $v = 1$ to get

$$A_{n+1} = A_n' + A_1 A_n$$

where

$$A_n = \begin{vmatrix} u & u' & u'' & u''' & \cdots \\ -1 & u & 2u' & 3u'' & \cdots \\ & -1 & u & 3u' & \cdots \\ & & -1 & u & \cdots \\ & & & \cdots & \cdots \end{vmatrix}_n .$$

These functions appear in a paper by Yebbou on the calculation of determining factors in the theory of differential equations. Yebbou uses the notation $\alpha^{[n]}$ in place of A_n.

5.2 The Generalized Cusick Identities

The principal Cusick identity in its generalized form relates a particular skew-symmetric determinant (Section 4.3) to two Hankelians (Section 4.8).

5.2.1 Three Determinants

Let ϕ_r and ψ_r, $r \geq 1$, be two sets of arbitrary functions and define three power series as follows:

$$\begin{aligned} \Phi_i &= \sum_{r=i}^{\infty} \phi_r t^{r-i}, \quad i \geq 1; \\ \Psi_i &= \sum_{r=i}^{\infty} \psi_r t^{r-i}, \quad i \geq 1; \\ G_i &= \Phi_i \Psi_i. \end{aligned} \tag{5.2.1}$$

Let

$$G_i = \sum_{j=i+1}^{\infty} a_{ij} t^{j-i-1}, \quad i \geq 1. \tag{5.2.2}$$

Then, equating coefficients of t^{j-i-1},

$$a_{ij} = \sum_{s=1}^{j-i} \phi_{s+i-1}\psi_{j-s}, \quad i < j. \tag{5.2.3}$$

In particular,

$$a_{i,2n} = \sum_{s=1}^{2n-i} \phi_{s+i-1}\psi_{2n-s}, \quad 1 \le i \le 2n-1. \tag{5.2.4}$$

Let A_{2n} denote the skew-symmetric determinant of order $2n$ defined as

$$A_{2n} = |a_{ij}|_{2n}, \tag{5.2.5}$$

where a_{ij} is defined by (5.2.3) for $1 \le i \le j \le 2n$ and $a_{ji} = -a_{ij}$, which implies $a_{ii} = 0$.

Let H_n and K_n denote Hankelians of order n defined as

$$H_n = \begin{cases} |h_{ij}|_n, & h_{ij} = \phi_{i+j-1} \\ |\phi_m|_n, & 1 \le m \le 2n-1; \end{cases} \tag{5.2.6}$$

$$K_n = \begin{cases} |k_{ij}|_n, & k_{ij} = \psi_{i+j-1} \\ |\psi_m|_n, & 1 \le m \le 2n-1. \end{cases} \tag{5.2.7}$$

All the elements ϕ_r and ψ_r which appear in H_n and K_n, respectively, also appear in $a_{1,2n}$ and therefore also in A_{2n}. The principal identity is given by the following theorem.

Theorem 5.1.

$$A_{2n} = H_n^2 K_n^2.$$

However, since

$$A_{2n} = \mathrm{Pf}_n^2,$$

where Pf_n *is a Pfaffian (Section 4.3.3), the theorem can be expressed in the form*

$$\mathrm{Pf}_n = H_n K_n. \tag{5.2.8}$$

Since Pfaffians are uniquely defined, there is no ambiguity in sign in this relation.

The proof uses the method of induction. It may be verified from (4.3.25) and (5.2.3) that

$$\mathrm{Pf}_1 = a_{12} = \phi_1\psi_1 = H_1 K_1,$$

$$\begin{aligned} \mathrm{Pf}_2 &= \left| \begin{matrix} \phi_1\psi_1 & \phi_1\psi_2 + \phi_2\psi_1 & \phi_1\psi_3 + \phi_2\psi_2 + \phi_3\psi_1 \\ & \phi_2\psi_2 & \phi_2\psi_3 + \phi_3\psi_2 \\ & & \phi_3\psi_3 \end{matrix} \right| \\ &= \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{vmatrix} \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_3 \end{vmatrix} \\ &= H_2 K_2 \end{aligned} \tag{5.2.9}$$

so that the theorem is known to be true when $n = 1$ and 2.

Assume that

$$\mathrm{Pf}_m = H_m K_m, \quad m < n. \tag{5.2.10}$$

The method by which the theorem is proved for all values of n is outlined as follows.

Pf_n is expressible in terms of Pfaffians of lower order by the formula

$$\mathrm{Pf}_n = \sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} a_{i,2n}, \tag{5.2.11}$$

where, in this context, $a_{i,2n}$ is defined as a sum in (5.2.4) so that Pf_n is expressible as a double sum. The introduction of a variable x enables the inductive assumption (5.2.10) to be expressed as the equality of two polynomials in x. By equating coefficients of one particular power of x, an identity is found which expresses $\mathrm{Pf}_i^{(n)}$ as the sum of products of cofactors of H_n and K_n (Lemma 5.5). Hence, Pf_n is expressible as a triple sum containing the cofactors of H_n and K_n. Finally, with the aid of an identity in Appendix A.3, it is shown that the triple sum simplifies to the product $H_n K_n$.

The following Pfaffian identities will also be applied.

$$\mathrm{Pf}_i^{(n)} = \left(A_{ii}^{(2n-1)}\right)^{1/2}, \tag{5.2.12}$$

$$(-1)^{i+j} \mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)} = A_{ij}^{(2n-1)}, \tag{5.2.13}$$

$$\mathrm{Pf}_{2n-1}^{(n)} = \mathrm{Pf}_{n-1}. \tag{5.2.14}$$

The proof proceeds with a series of lemmas.

5.2.2 Four Lemmas

Let a_{ij}^* be the function obtained from a_{ij} by replacing each ϕ_r by $(\phi_r - x\phi_{r+1})$ and by replacing each ψ_r by $(\psi_r - x\psi_{r+1})$.

Lemma 5.2.

$$a_{ij}^* = a_{ij} - (a_{i,j+1} + a_{i+1,j})x + a_{i+1,j+1}x^2.$$

PROOF.

$$\begin{aligned} a_{ij}^* &= \sum_{s=1}^{j-i} (\phi_{s+i-1} - x\phi_{s+i})(\psi_{j-s} - x\psi_{j-s+1}) \\ &= a_{ij} - (s_1 + s_2)x + s_3 x^2, \end{aligned}$$

where

$$\begin{aligned} s_1 &= \sum_{s=1}^{j-i} \phi_{s+i-1}\psi_{j-s+1} \\ &= a_{i,j+1} - \phi_i \psi_j, \end{aligned}$$

$$s_2 = a_{i+1,j} + \phi_i \psi_j,$$
$$s_3 = a_{i+1,j+1}.$$

The lemma follows. □

Let

$$\begin{aligned} A^*_{2n} &= |a^*_{ij}|_{2n}, \\ \mathrm{Pf}^*_n &= \left(A^*_{2n}\right)^{1/2}. \end{aligned} \tag{5.2.15}$$

Lemma 5.3.

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} x^{2n-i-1} = \mathrm{Pf}^*_{n-1}.$$

Proof. Denote the sum by F_n. Then, referring to (5.2.13) and Section 3.7 on bordered determinants,

$$\begin{aligned} F_n^2 &= \sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} (-1)^{i+j} \mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)} x^{4n-i-j-2} \\ &= \sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} A_{ij}^{(2n-1)} x^{4n-i-j-2} \\ &= -\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,2n-1} & x^{2n-2} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & x^{2n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & 1 \\ x^{2n-2} & x^{2n-3} & \cdots & 1 & \bullet \end{vmatrix}_{2n}. \end{aligned} \tag{5.2.16}$$

(It is not necessary to put $a_{ii} = 0$, etc., in order to prove the lemma.)

Eliminate the x's from the last column and row by means of the row and column operations

$$\begin{aligned} \mathbf{R}'_i &= \mathbf{R}_i - x\mathbf{R}_{i+1}, \quad 1 \le i \le 2n-2, \\ \mathbf{C}'_j &= \mathbf{C}_j - x\mathbf{C}_{j+1}, \quad 1 \le j \le 2n-2. \end{aligned} \tag{5.2.17}$$

The result is

$$\begin{aligned} F_n^2 &= -\begin{vmatrix} a^*_{11} & a^*_{12} & \cdots & a^*_{1,2n-1} & \bullet \\ a^*_{21} & a^*_{22} & \cdots & a^*_{2,2n-1} & \bullet \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a^*_{2n-1,1} & a^*_{2n-1,2} & \cdots & a^*_{2n-1,2n-1} & 1 \\ \bullet & \bullet & \cdots & 1 & \bullet \end{vmatrix}_{2n} \\ &= +|a^*_{ij}|_{2n-2} \\ &= A^*_{2n-2}. \end{aligned}$$

The lemma follows by taking the square root of each side. □

Let H^*_{n-1} and K^*_{n-1} denote the determinants obtained from H_{n-1} and K_{n-1}, respectively, by again replacing each ϕ_r by $(\phi_r - x\phi_{r+1})$ and by replacing each ψ_r by $(\psi_r - x\psi_{r+1})$. In the notation of the second and fourth lines of (5.2.6),

$$\begin{aligned} H^*_{n-1} &= \left|\phi_m - x\phi_{m+1}\right|_n, \quad 1 \le m \le 2n-3, \\ K^*_{n-1} &= \left|\psi_m - x\psi_{m+1}\right|_n, \quad 1 \le m \le 2n-3. \end{aligned} \tag{5.2.18}$$

Lemma 5.4.

a. $\displaystyle\sum_{i=1}^{n} H_{in}^{(n)} x^{n-i} = H^*_{n-1}$,

b. $\displaystyle\sum_{i=1}^{n} K_{in}^{(n)} x^{n-i} = K^*_{n-1}$.

Proof of (a).

$$\sum_{i=1}^{n} H_{in}^{(n)} x^{n-i} = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_{n-1} & x^{n-1} \\ \phi_2 & \phi_3 & \cdots & \phi_n & x^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{n-1} & \phi_n & \cdots & \phi_{2n-3} & x \\ \phi_n & \phi_{n+1} & \cdots & \phi_{2n-2} & 1 \end{vmatrix}_n .$$

The result follows by eliminating the x's from the last column by means of the row operations:

$$\mathbf{R}'_i = \mathbf{R}_i - x\mathbf{R}_{i+1}, \quad 1 \le i \le n-1.$$

Part (b) is proved in a similar manner. □

Lemma 5.5.

$$(-1)^{i+1}\mathrm{Pf}_i^{(n)} = \sum_{j=1}^{n} H_{jn}^{(n)} K_{i-j+1,n}^{(n)}, \quad 1 \le i \le 2n-1.$$

Since $K_{mn}^{(n)} = 0$ when $m < 1$ and when $m > n$, the true upper limit in the sum is i, but it is convenient to retain n in order to simplify the analysis involved in its application.

Proof. It follows from the inductive assumption (5.2.10) that

$$\mathrm{Pf}^*_{n-1} = H^*_{n-1} K^*_{n-1}. \tag{5.2.19}$$

Hence, applying Lemmas 5.3 and 5.4,

$$\begin{aligned} \sum_{i=1}^{2n-1} (-1)^{i+1}\mathrm{Pf}_i^{(n)} x^{2n-i-1} &= \left[\sum_{i=1}^{n} H_{in}^{(n)} x^{n-i}\right]\left[\sum_{s=1}^{n} K_{sn}^{(n)} x^{n-s}\right] \\ &= \left[\sum_{j=1}^{n}\sum_{s=1}^{n} H_{jn}^{(n)} K_{sn}^{(n)} x^{2n-j-s}\right] \qquad \begin{bmatrix} s = i-j+1 \\ s \to i \end{bmatrix} \end{aligned}$$

$$= \sum_{j=1}^{n} \sum_{i=j}^{n+j-1} H_{jn}^{(n)} K_{i-j+1,n}^{(n)} x^{2n-i-1}$$
$$= \sum_{i=1}^{2n-1} x^{2n-i-1} \sum_{j=1}^{n} H_{jn}^{(n)} K_{i-j+1,n}^{(n)}. \tag{5.2.20}$$

Note that the changes in the limits of the i-sum have introduced only zero terms. The lemma follows by equating coefficients of x^{2n-i-1}. □

5.2.3 Proof of the Principal Theorem

A double-sum identity containing the symbols c_{ij}, f_i, and g_i is given in Appendix A.3. It follows from Lemma 5.5 that the conditions defining the validity of the double-sum identity are satisfied if

$$\begin{aligned} f_i &= (-1)^{i+1}\mathrm{Pf}_i^{(n)}, \\ c_{ij} &= H_{in}^{(n)} K_{jn}^{(n)}, \\ g_i &= a_{i,2n}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^{2n-1} (-1)^{i+1}\mathrm{Pf}_i^{(n)} a_{i,2n} &= \sum_{i=1}^{n} \sum_{j=1}^{n} H_{in}^{(n)} K_{jn}^{(n)} a_{i+j-1,2n} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} H_{in}^{(n)} K_{jn}^{(n)} \sum_{s=1}^{2n-i-j+1} \phi_{s+i+j-2}\psi_{2n-s}. \end{aligned}$$

From (5.2.11), the sum on the left is equal to Pf_n. Also, since the interval $(1, 2n-i-j+1)$ can be split into the intervals $(1, n+1-j)$ and $(n+2-j, 2n-i-j+1)$, it follows from the note in Appendix A.3 on a triple sum that

$$\mathrm{Pf}_n = \sum_{j=1}^{n} K_{jn}^{(n)} X_j + \sum_{i=1}^{n-1} H_{in}^{(n)} Y_i,$$

where

$$\begin{aligned} X_j &= \sum_{i=1}^{n} H_{in}^{(n)} \sum_{s=1}^{n+1-j} \phi_{s+i+j-2}\psi_{2n-s} \\ &= \sum_{s=1}^{n+1-j} \psi_{2n-s} \sum_{i=1}^{n} \phi_{s+i+j-2} H_{in}^{(n)} \\ &= \sum_{s=1}^{n+1-j} \psi_{2n-s} \sum_{i=1}^{n} h_{i,s+j-1} H_{in}^{(n)} \end{aligned}$$

$$
\begin{aligned}
&= H_n \sum_{s=1}^{n+1-j} \psi_{2n-s}\delta_{s,n-j+1} \\
&= H_n \psi_{n+j-1}, \quad 1 \le j \le n; && (5.2.21) \\
Y_i &= \sum_{j=1}^{n} K_{jn}^{(n)} \sum_{s=n+2-j}^{2n-i-j+1} \phi_{t+i+2}\psi_{2n-s}, && \begin{bmatrix} s = t-j \\ s \to t \end{bmatrix} \\
&= \sum_{j=1}^{n} K_{jn}^{(n)} \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2}\psi_{2n+j-t} \\
&= \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2} \sum_{j=1}^{n} \psi_{2n+j-t} K_{jn}^{(n)} \\
&= \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2} \sum_{j=1}^{n} k_{j+n+1-t,n} K_{jn}^{(n)} \\
&= K_n \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2}\delta_{t,n+1} \\
&= 0, \quad 1 \le i \le n-1, && (5.2.22)
\end{aligned}
$$

since $t > n+1$. Hence,

$$
\begin{aligned}
\mathrm{Pf}_n &= H_n \sum_{j=1}^{n} K_{jn}^{(n)} \psi_{n+j-1} \\
&= H_n \sum_{j=1}^{n} k_{jn} K_{jn}^{(n)} \\
&= H_n K_n,
\end{aligned}
$$

which completes the proof of Theorem 5.1.

5.2.4 Three Further Theorems

The principal theorem, when expressed in the form

$$
\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} a_{i,2n} = H_n K_n, \tag{5.2.23}
$$

yields two corollaries by partial differentiation. Since the only elements in Pf_n which contain ϕ_{2n-1} and ψ_{2n-1} are $a_{i,2n}$, $1 \le i \le 2n-1$, and $\mathrm{Pf}_i^{(n)}$ is independent of $a_{i,2n}$, it follows that $\mathrm{Pf}_i^{(n)}$ is independent of ϕ_{2n-1} and ψ_{2n-1}. Moreover, these two functions occur only once in H_n and K_n, respectively, both in position (n,n).

From (5.2.4),

$$\frac{\partial a_{i,2n}}{\partial \phi_{2n-1}} = \psi_i.$$

Also,

$$\frac{\partial H_n}{\partial \phi_{2n-1}} = H_{n-1}.$$

Hence,

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} \psi_i = H_{n-1} K_n. \tag{5.2.24}$$

Similarly,

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} \phi_i = H_n K_{n-1}. \tag{5.2.25}$$

The following three theorems express modified forms of $|a_{ij}|_n$ in terms of the Hankelians.

Let $B_n(\phi)$ denote the determinant which is obtained from $|a_{ij}|_n$ by replacing the last row by the row

$$\left[\phi_1 \ \phi_2 \ \phi_3 \ldots \phi_n\right].$$

Theorem 5.6.

a. $B_{2n-1}(\phi) = H_{n-1} H_n K_{n-1}^2$,
b. $B_{2n-1}(\psi) = H_{n-1}^2 K_{n-1} K_n$,
c. $B_{2n}(\phi) = -H_n^2 K_{n-1} K_n$,
d. $B_{2n}(\psi) = -H_{n-1} H_n K_n^2$.

PROOF. Expanding $B_{2n-1}(\phi)$ by elements from the last row and their cofactors and referring to (5.2.13), (5.2.14), and (5.2.25),

$$\begin{aligned} B_{2n-1}(\phi) &= \sum_{j=1}^{2n-1} \phi_j A_{2n-1,j}^{(2n-1)} \\ &= \mathrm{Pf}_{2n-1}^{(n)} \sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} \phi_i \\ &= \mathrm{Pf}_{n-1} H_n K_{n-1}. \end{aligned} \tag{5.2.26}$$

Part (a) now follows from Theorem 5.1 and (b) is proved in a similar manner.

Expanding $B_{2n}(\phi)$ with the aid of Theorem 3.9 on bordered determinants (Section 3.7) and referring to (5.2.11) and (5.2.25),

$$B_{2n}(\phi) = -\sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} a_{i,2n} \phi_j A_{ij}^{(2n-1)}$$

$$= -\left[\sum_{i=1}^{2n-1}(-1)^{i+1}\mathrm{Pf}_i^{(n)}a_{i,2n}\right]\left[\sum_{j=1}^{2n-1}(-1)^{j+1}\mathrm{Pf}_j^{(n)}\phi_j\right]$$

$$= -\mathrm{Pf}_n H_n K_{n-1}. \tag{5.2.27}$$

Part (c) now follows from Theorem 5.1 and (d) is proved in a similar manner. □

Let $\mathbf{R}(\phi)$ denote the row vector defined as

$$\mathbf{R}(\phi) = \left[\phi_1\ \phi_2\ \phi_3 \cdots \phi_{2n-1}\ \bullet\right]$$

and let $B_{2n}(\phi,\psi)$ denote the determinant of order $2n$ which is obtained from $|a_{ij}|_{2n}$ by replacing the last row by $-\mathbf{R}(\phi)$ and replacing the last column by $\mathbf{R}^T(\psi)$.

Theorem 5.7.

$$B_{2n}(\phi,\psi) = H_{n-1}H_nK_{n-1}K_n.$$

PROOF.

$$B_{2n}(\phi,\psi) = \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1}\psi_i\phi_j A_{ij}^{(2n-1)}.$$

The theorem now follows (5.2.13), (5.2.24), and (5.2.25). □

Theorem 5.8.

$$B_{2n}(\phi,\psi) = A_{2n-1,2n}^{(2n)}.$$

PROOF. Applying the Jacobi identity (Section 3.6),

$$\begin{vmatrix} A_{2n-1,2n-1}^{(2n)} & A_{2n-1,2n}^{(2n)} \\ A_{2n,2n-1}^{(2n)} & A_{2n,2n}^{(2n)} \end{vmatrix} = A_{2n}A_{2n-1,2n;2n-1,2n}^{(2n)}. \tag{5.2.28}$$

But, $A_{ii}^{(2n)}$, $i = 2n-1,\ 2n$, are skew-symmetric of odd order and are therefore zero. The other two first cofactors are equal in magnitude but opposite in sign. Hence,

$$\left(A_{2n-1,2n}^{(2n)}\right)^2 = A_{2n}A_{2n-2},$$

$$A_{2n-1,2n}^{(2n)} = \mathrm{Pf}_n\mathrm{Pf}_{n-1}. \tag{5.2.29}$$

Theorem 5.8 now follows from Theorems 5.1 and 5.7. □

If $\psi_r = \phi_r$, then $K_n = H_n$ and Theorems 5.1, 5.6a and c, and 5.7 degenerate into identities published in a different notation by Cusick, namely,

$$A_{2n} = H_n^4,$$

$$B_{2n-1}(\phi) = H_{n-1}^3 H_n,$$

$$
\begin{aligned}
B_{2n}(\phi) &= -H_{n-1}H_n^3,\\
B_{2n}(\phi,\phi) &= H_{n-1}^2 H_n^2.
\end{aligned}
\tag{5.2.30}
$$

These identities arose by a by-product in a study of Littlewood's Diophantime approximation problem.

The negative sign in the third identity, which is not required in Cusick's notation, arises from the difference between the methods by which $B_n(\phi)$ and Cusick's determinant T_n are defined. Note that $B_{2n}(\phi,\phi)$ is skew-symmetric of even order and is therefore expected to be a perfect square.

Exercises

1. Prove that

$$
\begin{aligned}
A_{1,2n}^{(2n)} &= -H_n H_{1n}^{(n)} K_n K_{1n}^{(n)},\\
A_{1,2n-1}^{(2n-1)} &= H_{n-1} H_{1n}^{(n)} K_{n-1} K_{1n}^{(n)}.
\end{aligned}
$$

2. Let $V_n(\phi)$ be the determinant obtained from $A_{1,2n}^{(2n)}$ by replacing the last row by $\mathbf{R}_{2n}(\phi)$ and let $W_n(\phi)$ be the determinant obtained from $A_{1,2n-1}^{(2n-1)}$ by replacing the last row by $\mathbf{R}_{2n-1}(\phi)$. Prove that

$$
\begin{aligned}
V_n(\phi) &= -H_n H_{1n}^{(n)} K_{n-1} K_{1n}^{(n)},\\
W_n(\phi) &= -H_{n-1} H_{1n}^{(n)} K_{n-1} K_{1,n-1}^{(n-1)}.
\end{aligned}
$$

3. Prove that

$$
A_{i,2n}^{(2n)} = (-1)^{i+1}\mathrm{Pf}_n \mathrm{Pf}_i^{(n-1)}.
$$

5.3 The Matsuno Identities

Some of the identities in this section appear in Appendix II in a book on the bilinear transformation method by Y. Matsuno, but the proofs have been modified.

5.3.1 A General Identity

Let

$$
A_n = |a_{ij}|_n,
$$

where

$$
a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \sum\limits_{\substack{r=1\\ r\neq i}}^{n} u_{ir}, & j = i, \end{cases}
\tag{5.3.1}
$$

and

$$u_{ij} = \frac{1}{x_i - x_j} = -u_{ji}, \tag{5.3.2}$$

where the x_i are distinct but otherwise arbitrary.

Illustration.

$$A_3 = \begin{vmatrix} x - u_{12} - u_{13} & u_{12} & u_{13} \\ u_{21} & x - u_{21} - u_{23} & u_{23} \\ u_{31} & u_{32} & x - u_{31} - u_{32} \end{vmatrix}.$$

Theorem.

$$A_n = x^n.$$

[This theorem appears in a section of Matsuno's book in which the x_i are the zeros of classical polynomials but, as stated above, it is valid for all x_i, provided only that they are distinct.]

PROOF. The sum of the elements in each row is x. Hence, after performing the column operations

$$\begin{aligned}\mathbf{C}'_n &= \sum_{j=1}^{n} \mathbf{C}_j \\ &= x\big[1\ 1\ 1 \cdots 1\big]^T,\end{aligned}$$

it is seen that A_n is equal to x times a determinant in which every element in the last column is 1. Now, perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_n, \quad 1 \le i \le n-1,$$

which remove every element in the last column except the element 1 in position (n, n). The result is

$$A_n = xB_{n-1},$$

where

$$\begin{aligned} B_{n-1} &= |b_{ij}|_{n-1}, \\ b_{ij} &= \begin{cases} u_{ij} - u_{nj} = \frac{u_{ij}u_{nj}}{u_{ni}}, & j \ne i \\ x - \displaystyle\sum_{\substack{r=1 \\ r \ne i}}^{n-1} u_{ir}, & j = i. \end{cases} \end{aligned}$$

It is now found that, after row i has been multiplied by the factor u_{ni}, $1 \le i \le n-1$, the same factor can be canceled from column i, $1 \le i \le n-1$, to give the result

$$B_{n-1} = A_{n-1}.$$

Hence,

$$A_n = xA_{n-1}.$$

But $A_2 = x^2$. The theorem follows. □

5.3.2 Particular Identities

It is shown in the previous section that $A_n = x^n$ provided only that the x_i are distinct. It will now be shown that the diagonal elements of A_n can be modified in such a way that $A_n = x^n$ as before, but only if the x_i are the zeros of certain orthogonal polynomials. These identities supplement those given by Matsuno.

It is well known that the zeros of the Laguerre polynomial $L_n(x)$, the Hermite polynomial $H_n(x)$, and the Legendre polynomial $P_n(x)$ are distinct. Let $p_n(x)$ represent any one of these polynomials and let its zeros be denoted by x_i, $1 \le i \le n$. Then,

$$p_n(x) = k\prod_{r=1}^{n}(x - x_r), \tag{5.3.3}$$

where k is a constant. Hence,

$$\begin{aligned}
\log p_n(x) &= \log k + \sum_{r=1}^{n} \log(x - x_r),\\
\frac{p_n'(x)}{p_n(x)} &= \sum_{r=1}^{n} \frac{1}{x - x_r}\,. \qquad (5.3.4)
\end{aligned}$$

It follows that

$$\sum_{\substack{r=1\\ r\ne i}}^{n} \frac{1}{x - x_r} = \frac{(x - x_i)p_n'(x) - p_n(x)}{(x - x_i)p_n(x)}\,. \tag{5.3.5}$$

Hence, applying the l'Hopital limit theorem twice,

$$\begin{aligned}
\sum_{\substack{r=1\\ r\ne i}}^{n} \frac{1}{x_i - x_r} &= \lim_{x\to x_i}\left[\frac{(x - x_i)p_n'(x) - p_n(x)}{(x - x_i)p_n(x)}\right]\\
&= \lim_{x\to x_i}\left[\frac{(x - x_i)p_n'''(x) + p_n''(x)}{(x - x_i)p_n''(x) + 2p_n'(x)}\right]\\
&= \frac{p_n''(x_i)}{2p_n'(x_i)}\,. \qquad (5.3.6)
\end{aligned}$$

The sum on the left appears in the diagonal elements of A_n. Now redefine A_n as follows:

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \frac{p_n''(x_i)}{2p_n'(x_i)}, & j = i. \end{cases} \tag{5.3.7}$$

This A_n clearly has the same value as the original A_n since the left-hand side of (5.3.6) has been replaced by the right-hand side, its algebraic equivalent.

The right-hand side of (5.3.6) will now be evaluated for each of the three particular polynomials mentioned above with the aid of their differential equations (Appendix A.5).

Laguerre Polynomials.

$$\begin{aligned} xL_n''(x) + (1-x)L_n'(x) + nL_n(x) &= 0, \\ L_n(x_i) = 0, \quad 1 \leq i \leq n, & \\ \frac{L_n''(x_i)}{2L_n'(x_i)} = \frac{x_i - 1}{x_i}\,. & \end{aligned} \tag{5.3.8}$$

Hence, if

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \frac{x_i - 1}{2x_i}, & j = i, \end{cases}$$

then

$$A_n = |a_{ij}|_n = x^n. \tag{5.3.9}$$

Hermite Polynomials.

$$\begin{aligned} H_n''(x) - 2xH_n'(x) + 2nH_n(x) &= 0, \\ H_n(x_i) = 0, \qquad 1 \leq i \leq n, & \\ \frac{H_n''(x_i)}{2H_n'(x_i)} = x_i. & \end{aligned} \tag{5.3.10}$$

Hence if,

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - x_i, & j = i, \end{cases}$$

then

$$A_n = |a_{ij}|_n = x^n. \tag{5.3.11}$$

Legendre Polynomials.

$$\begin{aligned} (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) &= 0, \\ P_n(x_i) = 0, \quad 1 \leq i \leq n, & \\ \frac{P_n''(x_i)}{2P_n'(x_i)} = \frac{x_i}{1 - x_i^2}\,. & \end{aligned} \tag{5.3.12}$$

Hence, if

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \frac{x_i}{1-x_i^2}, & j = i, \end{cases}$$

then

$$A_n = |a_{ij}|_n = x^n. \tag{5.3.13}$$

Exercises

1. Let A_n denote the determinant defined in (5.3.9) and let

$$B_n = |b_{ij}|_n,$$

where

$$b_{ij} = \begin{cases} \frac{2}{x_i - x_j}, & j \neq i \\ x + \frac{1}{x_i}, & j = i, \end{cases}$$

where, as for $A_n(x)$, the x_i denote the zeros of the Laguerre polynomial. Prove that

$$B_n(x-1) = 2^n A_n\left(\frac{x}{2}\right)$$

and, hence, prove that

$$B_n(x) = (x+1)^n.$$

2. Let

$$A_n^{(p)} = |a_{ij}^{(p)}|_n,$$

where

$$a_{ij}^{(p)} = \begin{cases} u_{ij}^p, & j \neq i \\ x - \sum\limits_{\substack{r=1 \\ r \neq i}}^{n} u_{ir}^p, & j = i, \end{cases}$$

$$u_{ij} = \frac{1}{x_i - x_j} = -u_{ji}$$

and the x_i are the zeros of the Hermite polynomial $H_n(x)$. Prove that

$$A_n^{(2)} = \prod_{r=1}^{n} [x - (r-1)],$$

$$A_n^{(4)} = \prod_{r=1}^{n} \left[x - \frac{1}{6}(r^2 - 1)\right].$$

5.4 The Cofactors of the Matsuno Determinant

5.4.1 *Introduction*

Let

$$E_n = |e_{ij}|_n,$$

where

$$e_{ij} = \begin{cases} \frac{1}{c_i - c_j}, & j \neq i \\ x_i, & j = i, \end{cases} \tag{5.4.1}$$

and where the c's are distinct but otherwise arbitrary and the x's are arbitrary. In some detail,

$$E_n = \begin{vmatrix} x_1 & \frac{1}{c_1-c_2} & \frac{1}{c_1-c_3} & \cdots & \frac{1}{c_1-c_n} \\ \frac{1}{c_2-c_1} & x_2 & \frac{1}{c_2-c_3} & \cdots & \cdots \\ \frac{1}{c_3-c_1} & \frac{1}{c_3-c_2} & x_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{c_n-c_1} & \cdots & \cdots & \cdots & x_n \end{vmatrix}_n . \tag{5.4.2}$$

This determinant is known here as the Matsuno determinant in recognition of Matsuno's solutions of the Kadomtsev–Petviashvili (KP) and Benjamin–Ono (BO) equations (Sections 6.8 and 6.9), where it appears in modified forms. It is shown below that the first and higher scaled cofactors of E satisfy a remarkably rich set of algebraic multiple-sum identities which can be applied to simplify the analysis in both of Matsuno's papers.

It is convenient to introduce the symbol † into a double sum to denote that those terms in which the summation variables are equal are omitted from the sum. Thus,

$$\sum_r \sum_s {}^{\dagger} u_{rs} = \sum_r \sum_s u_{rs} - \sum_r u_{rr}. \tag{5.4.3}$$

It follows from the partial derivative formulae in the first line of (3.2.4), (3.6.7), (3.2.16), and (3.2.17) that

$$\begin{aligned} \frac{\partial E_{pq}}{\partial x_i} &= E_{ip,iq}, \\ \frac{\partial E_{pr,qs}}{\partial x_i} &= E_{ipr,iqs} \\ \frac{\partial E^{pq}}{\partial x_i} &= -E^{pi}E^{iq}, \\ \left(E^{ii} + \frac{\partial}{\partial x_i}\right) E^{pq} &= E^{ip,iq}, \\ \left(E^{ii} + \frac{\partial}{\partial x_i}\right) E^{pr,qs} &= E^{ipr,iqs}, \end{aligned}$$

$$\left(E^{ii} + \frac{\partial}{\partial x_i}\right) E^{pru,qsv} = e^{ipru,iqsv}, \tag{5.4.4}$$

etc.

5.4.2 First Cofactors

When $f_r + g_r = 0$, the double-sum identities (C) and (D) in Section 3.4 become

$$\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} (f_r + g_s) a_{rs} A^{rs} = 0, \tag{C†}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} (f_r + g_s) a_{rs} A^{is} A^{rj} = (f_i + g_j) A^{ij}. \tag{D†}$$

Applying (C†) to E with $f_r = -g_r = c_r^m$,

$$\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} \left(\frac{c_r^m - c_s^m}{c_r - c_s}\right) E^{rs} = 0. \tag{5.4.5}$$

Putting $m = 1, 2, 3$ yields the following particular cases:

$m = 1$: $\quad \sum_r \sum_s{}^{\dagger} E^{rs} = 0,$

which is equivalent to

$$\sum_r \sum_s E^{rs} = \sum_r E^{rr}; \tag{5.4.6}$$

$m = 2$: $\quad \sum_r \sum_s{}^{\dagger} (c_r + c_s) E^{rs} = 0,$

which is equivalent to

$$\sum_r \sum_s (c_r + c_s) E^{rs} = 2 \sum_r c_r E^{rr}; \tag{5.4.7}$$

$m = 3$: $\quad \sum_r \sum_s{}^{\dagger} (c_r^2 + c_r c_s + c_s^2) E^{rs} = 0,$

which is equivalent to

$$\sum_r \sum_s (c_r^2 + c_r c_s + c_s^2) E^{rs} = 3 \sum_r c_r^2 E^{rr}. \tag{5.4.8}$$

Applying (D†) to E, again with $f_r = -g_r = c_r^m$,

$$\sum_r \sum_s{}^{\dagger} \left(\frac{c_r^m - c_s^m}{c_r - c_s}\right) E^{is} E^{rj} = (c_i^m - c_j^m) E^{ij}. \tag{5.4.9}$$

Putting $m = 1, 2$ yields the following particular cases:

$m = 1$: $$\sum_r \sum_s {}^{\dagger} E^{is} E^{rj} = (c_i - c_j) E^{ij},$$

which is equivalent to

$$\sum_r \sum_s E^{is} E^{rj} - \sum_r E^{ir} E^{rj} = (c_i - c_j) E^{ij}; \tag{5.4.10}$$

$m = 2$: $$\sum_r \sum_s {}^{\dagger} (c_r + c_s) E^{is} E^{rj} = (c_i^2 - c_j^2) E^{ij},$$

which is equivalent to

$$\sum_r \sum_s (c_r + c_s) E^{is} E^{rj} - 2 \sum_r c_r E^{ir} E^{rj} = (c_i^2 - c_j^2) E^{ij}, \tag{5.4.11}$$

etc. Note that the right-hand side of (5.4.9) is zero when $j = i$ for all values of m. In particular, (5.4.10) becomes

$$\sum_r \sum_s E^{is} E^{ri} = \sum_r E^{ir} E^{ri} \tag{5.4.12}$$

and the equation in item $m = 2$ becomes

$$\sum_r \sum_s (c_r + c_s) E^{is} E^{ri} = 2 \sum_r c_r E^{ir} E^{ri}. \tag{5.4.13}$$

5.4.3 First and Second Cofactors

The following five identities relate the first and second cofactors of E: They all remain valid when the parameters are lowered.

$$\sum_{r,s} {}^{\dagger} E^{ir,js} = -(c_i - c_j) E^{ij}, \tag{5.4.14}$$

$$\sum_{r,s} {}^{\dagger} (c_r + c_s) E^{ir,js} = -(c_i^2 - c_j^2) E^{ij}, \tag{5.4.15}$$

$$\sum_{r,s} (c_r - c_s) E^{rs} = \sum_{r,s} E^{rs,rs}, \tag{5.4.16}$$

$$2 \sum_{r,s} {}^{\dagger} c_r E^{rs} = -2 \sum_{r,s} {}^{\dagger} c_s E^{rs} = \sum_{r,s} E^{rs,rs}, \tag{5.4.17}$$

$$\sum_{r<s} (c_s E^{rs} + c_r E^{sr} + E^{rs,rs}) = 0. \tag{5.4.18}$$

To prove (5.4.14), apply the Jacobi identity to $E^{ir,js}$ and refer to (5.4.6) and the equation in item $m = 1$.

$$\begin{aligned}
\sum_{r,s} {}^{\dagger} E^{ir,js} &= \sum_{r,s} {}^{\dagger} \begin{vmatrix} E^{ij} & E^{is} \\ E^{rj} & E^{rs} \end{vmatrix} \\
&= E^{ij} \sum_{r,s} {}^{\dagger} E^{rs} - \sum_{r,s} {}^{\dagger} E^{is} E^{rj}
\end{aligned}$$

$$= -(c_i - c_j)E^{ij}.$$

Equation (5.4.15) can be proved in a similar manner by appling (5.4.7) and the equation in item $m = 2$. The proof of (5.4.16) is a little more difficult. Modify (5.4.12) by making the following changes in the parameters. First $i \to k$, then $(r, s) \to (i, j)$, and, finally, $k \to r$. The result is

$$\sum_{i,j}{}^{\dagger} E^{rj} E^{ir} = \sum_i E^{ri} E^{ir}. \tag{5.4.19}$$

Now sum (5.4.10) over i, j and refer to (5.4.19) and (5.4.6):

$$\begin{aligned}
\sum_{i,j}(c_i - c_j)E^{ij} &= \sum_{i,j,r,s} E^{is} E^{rj} - \sum_r \left[\sum_{i,j} E^{ir} E^{rj}\right] \\
&= \left[\sum_{i,s} E^{is}\right] \left[\sum_{r,j} E^{rj}\right] - \sum_r \sum_i E^{ri} E^{ir} \\
&= \sum_i E^{ii} \sum_r E^{rr} - \sum_{i,r} E^{ri} E^{ir} \\
&= \sum_{i,r} \begin{vmatrix} E^{ii} & E^{ir} \\ E^{ri} & E^{rr} \end{vmatrix} \\
&= \sum_{i,r} E^{ir,ir},
\end{aligned}$$

which is equivalent to (5.4.16). The symbol † can be attached to the sum on the left without affecting its value. Hence, this identity together with (5.4.7) yields (5.4.17), which can then be expressed in the symmetric form (5.4.18) in which $r < s$.

5.4.4 Third and Fourth Cofactors

The following identities contain third and fourth cofactors of E:

$$\sum_{r,s}(c_r - c_s)E^{rt,st} = \sum_{r,s} E^{rst,rst}, \tag{5.4.20}$$

$$\sum_{r,s}(c_r - c_s)E^{rtu,stu} = \sum_{r,s} E^{rstu,rstu}, \tag{5.4.21}$$

$$\sum_{r,s}(c_r^2 - c_s^2)E^{rs} = 2\sum_{r,s} c_r E^{rs,rs}, \tag{5.4.22}$$

$$\sum_{r,s}(c_r - c_s)^2 E^{rs} = \sum_{r,s,t} E^{rst,rst}, \tag{5.4.23}$$

$$\sum_{r,s}(c_r - c_s)^2 E^{ru,su} = \sum_{r,s,t} E^{rstu,rstu}, \tag{5.4.24}$$

$$\sum_{r,s}{}^{\dagger} c_r c_s E^{rs} = -\sum_{r,s}{}^{\dagger}(c_r^2 + c_s^2)E^{rs}$$
$$= -\tfrac{1}{3}\sum_{r,s,t} E^{rst,rst}, \tag{5.4.25}$$

$$\sum_{r,s} c_r c_s E^{rs} = \sum_r c_r^2 E^{rr} - \frac{1}{3}\sum_{r,s,t} E^{rst,rst}, \tag{5.4.26}$$

$$\sum_{r,s}(c_r^2 + c_s^2)E^{rs} = 2\sum_r c_r^2 E^{rr} + \frac{1}{3}\sum_{r,s,t} E^{rst,rst}, \tag{5.4.27}$$

$$\sum_{r,s}{}^{\dagger} c_r^2 E^{rs} = \frac{1}{6}\sum_{r,s,t} E^{rst,rst} + \sum_{r,s} c_r E^{rs,rs}, \tag{5.4.28}$$

$$\sum_{r,s}{}^{\dagger} c_s^2 E^{rs} = \frac{1}{6}\sum_{r,s,t} E^{rst,rst} - \sum_{r,s} c_r E^{rs,rs}. \tag{5.4.29}$$

To prove (5.4.20), apply the second equation of (5.4.4) and (5.4.16).

$$E_{pr,ps} = \frac{\partial E_{rs}}{\partial x_p}.$$

Multiply by $(c_r - c_s)$ and sum over r and s:

$$\sum_{r,s}(c_r - c_s)E_{pr,ps} = \frac{\partial}{\partial x_p}\sum_{r,s}(c_r - c_s)E_{rs}$$
$$= \frac{\partial}{\partial x_p}\sum_{r,s} E_{rs,rs}$$
$$= \sum_{r,s} E_{prs,prs},$$

which is equivalent to (5.4.20). The application of the fifth equation in (5.4.4) with the modification $(i, p, r, q, s) \to (u, r, t, s, t)$ to (5.4.20) yields (5.4.21).

To prove (5.4.22), sum (5.4.11) over i and j, change the dummy variables as indicated

$$\sum_{i,j}(c_i^2 - c_j^2)E^{ij} = F - G$$

where, referring to (5.4.6) and (5.4.7),

$$F = \left[\sum_{i,s} E^{is}\right]\left[\sum_{r,j} c_r E^{rj}\right] + \left[\sum_{r,j} E^{rj}\right]\left[\sum_{i,s} c_s E^{is}\right]$$
$$= \sum_i E^{ii}\left[\sum_{\substack{r,j\\(j\to s)}} c_r E^{rj} + \sum_{\substack{i,s\\(i\to r)}} c_s E^{is}\right]$$

$$
\begin{aligned}
&= \sum_i E^{ii} \sum_{r,s} (c_r + c_s) E^{rs} \\
&= 2 \sum_i E^{ii} \sum_r c_r E^{rr}, \qquad (5.4.30)
\end{aligned}
$$

$$
G = 2 \sum_{i,j,r} c_r E^{ir} E^{rj}. \qquad (5.4.31)
$$

Modify (5.4.10) with $j = i$ by making the changes $i \leftrightarrow r$ and $s \to j$. This gives

$$
G = 2 \sum_r c_r \sum_i E^{ir} E^{ri}. \qquad (5.4.32)
$$

Hence,

$$
\begin{aligned}
\sum_{i,j} (c_i^2 - c_j^2) E^{ij} &= 2 \sum_{i,r} c_r \begin{vmatrix} E^{ii} & E^{ir} \\ E^{ri} & E^{rr} \end{vmatrix} \\
&= 2 \sum_{i,r} E^{ir,ir},
\end{aligned}
$$

which is equivalent to (5.4.22).

To prove (5.4.23) multiply (5.4.10) by $(c_i - c_j)$, sum over i and j, change the dummy variables as indicated, and refer to (5.4.6):

$$
\sum_{i,j} (c_i - c_j)^2 E^{ij} = H - J, \qquad (5.4.33)
$$

where

$$
\begin{aligned}
H &= \sum_{i,j} (c_i - c_j) \sum_{r,s} E^{is} E^{rj} \\
&= \left[\sum_{r,j} E^{rj} \right] \left[\sum_{\substack{i,s \\ (s \to j)}} c_i E^{is} \right] - \left[\sum_{i,s} E^{is} \right] \left[\sum_{\substack{r,j \\ (r \to i)}} c_j E^{rj} \right] \\
&= \sum_r E^{rr} \sum_{i,j} (c_i - c_j) E^{ij}, \qquad (5.4.34)
\end{aligned}
$$

$$
J = \sum_{i,j} (c_i - c_j) \sum_r E^{ir} E^{rj}. \qquad (5.4.35)
$$

Hence, referring to (5.4.20) with suitable changes in the dummy variables,

$$
\begin{aligned}
\sum_{i,j} (c_i - c_j)^2 E^{ij} &= \sum_{i,j,r} (c_i - c_j) \begin{vmatrix} E^{ij} & E^{ir} \\ E^{rj} & E^{rr} \end{vmatrix} \\
&= \sum_{i,j,r} (c_i - c_j) E^{ir,jr}
\end{aligned}
$$

$$= \sum_{i,j,r} E^{ijr,ijr},$$

which is equivalent to (5.4.23). The application of a suitably modified the fourth line of (5.4.4) to (5.4.23) yields (5.4.24). Identities (5.4.27)–(5.4.29) follow from (5.4.8), (5.4.22), (5.4.24), and the identities

$$\begin{aligned}
3c_r c_s &= (c_r^2 + c_r c_s + c_s^2) - (c_r - c_s)^2, \\
6c_r^2 &= 2(c_r^2 + c_r c_s + c_s^2) + (c_r - c_s)^2 + 3(c_r^2 - c_s^2), \\
6c_s^2 &= 2(c_r^2 + c_r c_s + c_s^2) + (c_r - c_s)^2 - 3(c_r^2 - c_s^2).
\end{aligned}$$

5.4.5 Three Further Identities

The identities

$$\begin{aligned}
\sum_{r,s} (c_r^2 + c_s^2)(c_r - c_s) E^{rs} &= 2\sum_{r,s} c_r^2 E^{rs,rs} \\
&\quad + \frac{1}{3} \sum_{r,s,u,v} E^{rsuv,rsuv}, \qquad (5.4.36) \\
\sum_{r,s} (c_r^2 - c_s^2)(c_r + c_s) E^{rs} &= 2\sum_{r,s} c_r(c_r + c_s) E^{rs,rs} \\
&\quad - \frac{1}{6} \sum_{r,s,u,v} E^{rsuv,rsuv}, \qquad (5.4.37) \\
\sum_{r,s} c_r c_s (c_r - c_s) E^{rs} &= \sum_{r,s} c_r c_s E^{rs,rs} \\
&\quad - \frac{1}{4} \sum_{r,s,u,v} E^{rsuv,rsuv} \qquad (5.4.38)
\end{aligned}$$

are more difficult to prove than those in earlier sections. The last one has an application in Section 6.8 on the KP equation, but its proof is linked to those of the other two.

Denote the left sides of the three identities by P, Q, and R, respectively. To prove (5.4.36), multiply the second equation in (5.4.10) by $(c_i^2 + c_j^2)$, sum over i and j and refer to (5.4.4), (5.4.6), and (5.4.27):

$$\begin{aligned}
P &= \sum_{i,j,r,s} (c_i^2 + c_j^2) E^{is} E^{rj} + \sum_{i,j,r} (c_i^2 + c_j^2) E^{ir} E^{rj} \\
&= \left[\sum_{j,r} E^{rj}\right] \left[\sum_{\substack{i,s \\ (s\to j)}} c_i^2 E^{is}\right] + \left[\sum_{i,s} E^{is}\right] \left[\sum_{\substack{j,r \\ (r\to i)}} c_j^2 E^{rj}\right] \\
&\quad + \sum_{i,j,r} (c_i^2 + c_j^2) \frac{\partial E^{ij}}{\partial x_r}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_r E^{rr} \sum_{i,j} (c_i^2 + c_j^2) E^{ij} + \sum_r \frac{\partial}{\partial x_r} \sum_{i,j} (c_i^2 + c_j^2) E^{ij} \\
&= \sum_r \left(E^{rr} + \frac{\partial}{\partial x_r} \right) \sum_{i,j} (c_i^2 + c_j^2) E^{ij} \\
&= \sum_v \left(E^{vv} + \frac{\partial}{\partial x_v} \right) \left[2 \sum_r c_r^2 E^{rr} + \frac{1}{3} \sum_{r,s,t} E^{rst,rst} \right] \\
&= 2 \sum_{r,v} c_r^2 E^{rv,rv} + \frac{1}{3} \sum_{r,s,t,v} E^{rstv,rstv},
\end{aligned}
$$

which is equivalent to (5.4.36).

Since

$$
(c_r^2 - c_s^2)(c_r + c_s) - 2c_r c_s (c_r - c_s) = (c_r^2 + c_s^2)(c_r - c_s),
$$

it follows immedately that

$$
Q - 2R = P. \tag{5.4.39}
$$

A second relation between Q and R is found as follows. Let

$$
\begin{aligned}
U &= \sum_r c_r E^{rr}, \\
V &= \frac{1}{2} \sum_{r,s} E^{rs,rs}.
\end{aligned} \tag{5.4.40}
$$

It follows from (5.4.17) that

$$
\begin{aligned}
V &= \sum_{r,s} c_r E^{rs} - \sum_r c_r E^{rr} \\
&= \sum_r c_r E^{rr} - \sum_{r,s} c_s E^{rs}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{r,s} c_r E^{rs} &= U + V, \\
\sum_{r,s} c_s E^{rs} &= U - V.
\end{aligned} \tag{5.4.41}
$$

To obtain a formula for R, multiply (5.4.10) by $c_i c_j$, sum over i and j, and apply the third equation of (5.4.4):

$$
\begin{aligned}
R &= \sum_{i,j,r,s} c_i c_j E^{is} E^{rj} - \sum_{i,j,r} c_i c_j E^{ir} E^{rj} \\
&= \left[\sum_{i,s} c_i E^{is} \right] \left[\sum_{j,r} c_j E^{rj} \right] + \sum_r \frac{\partial}{\partial x_r} \sum_{i,j} c_i c_j E^{ij}
\end{aligned}
$$

$$= U^2 - V^2 + \sum_r \frac{\partial S}{\partial x_r}, \tag{5.4.42}$$

where

$$S = \sum_{i,j} c_i c_j E^{ij}. \tag{5.4.43}$$

This function is identical to the left-hand side of (5.4.26). Let

$$T = \sum_{i,j,r,s} (c_i + c_j)(c_r + c_s) E^{is} E^{rj}. \tag{5.4.44}$$

Then, applying (5.4.6),

$$\begin{aligned} T &= \sum_{i,s} c_i E^{is} \sum_{j,r} c_r E^{rj} + \sum_{j,r} E^{rj} \sum_{i,s} c_i c_s E^{is} \\ &\quad + \sum_{i,s} E^{is} \sum_{j,r} c_j c_r E^{rj} + \sum_{j,r} c_j E^{rj} \sum_{i,s} c_s E^{is} \\ &= (U+V)^2 + 2S \sum_{r,s} E^{rs} + (U-V)^2 \\ &= 2(U^2 + V^2) + 2S \sum_r E^{rr}. \end{aligned} \tag{5.4.45}$$

Eliminating V from (5.4.42),

$$T + 2R = 4U^2 + 2 \sum_r \left(E^{rr} + \frac{\partial}{\partial x_r} \right) S. \tag{5.4.46}$$

To obtain a formula for Q, multiply (5.4.11) by $(c_i + c_j)$, sum over i and j, and apply (5.4.13) with the modifications $(i, j) \leftrightarrow (r, s)$ on the left and $(i, r) \to (r, s)$ on the right:

$$\begin{aligned} Q &= \sum_{i,j,r,s} (c_i + c_j)(c_r + c_s) E^{is} E^{rj} - 2 \sum_{i,j,r} c_r (c_i + c_j) E^{ir} E^{rj} \\ &= T - 2 \sum_r c_r \sum_{i,j} (c_i + c_j) E^{ir} E^{rj} \\ &= T - 4 \sum_{r,s} c_r c_s E^{rs} E^{sr}. \end{aligned} \tag{5.4.47}$$

Eliminating T from (5.4.46) and applying (5.4.26) and the fourth and sixth lines of (5.4.4),

$$\begin{aligned} Q + 2R &= 4 \sum_{r,s} c_r c_s (E^{rr} E^{ss} - E^{sr} E^{rs}) \\ &\quad + 2 \sum_r \left(E^{rr} + \frac{\partial}{\partial x_r} \right) \left[\sum_s c_s^2 E^{ss} - \frac{1}{3} \sum_{s,t,u} E^{stu,stu} \right] \end{aligned}$$

$$= 4\sum_{r,s} c_r c_s E^{rs,rs} + 2\sum_{r,s} c_s^2 E^{rs,rs}$$
$$-\frac{2}{3}\sum_{r,s,t,u} E^{rstu,rstu}. \tag{5.4.48}$$

This is the second relation between Q and R, the first being (5.4.39). Identities (5.4.37), (5.4.38), and (5.3) follow by solving these two equations for Q and R, where P is given by (5.1).

Exercise. Prove that

$$\sum_{r,s}(c_r - c_s)\phi_n(c_r, c_s)E^{rs} = \sum_{r,s}\phi_n(c_r, c_s)E^{rs,rs}, \quad n = 1, 2,$$

where

$$\phi_1(c_r, c_s) = c_r + c_s,$$
$$\phi_2(c_r, c_s) = 3c_r^2 + 4c_r c_s + 3c_s^2.$$

Can this result be generalized?

5.5 Determinants Associated with a Continued Fraction

5.5.1 *Continuants and the Recurrence Relation*

Define a continued fraction f_n as follows:

$$f_n = \frac{1}{1+}\,\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_{n-1}}{a_{n-1}+}\,\frac{b_n}{a_n}, \quad n = 1, 2, 3, \ldots. \tag{5.5.1}$$

f_n is obtained from f_{n-1} by adding b_n/a_n to a_{n-1}.

Examples.

$$f_1 = \frac{1}{1 + \frac{b_1}{a_1}}$$
$$= \frac{a_1}{a_1 + b_1},$$
$$f_2 = \frac{1}{1 + \frac{b_1}{a_1 + \frac{b_2}{a_2}}}$$
$$= \frac{a_1 a_2 + b_2}{a_1 a_2 + b_2 + a_2 b_1},$$
$$f_3 = \frac{1}{1 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3}}}}$$

$$= \frac{a_1a_2a_3 + a_1b_3 + a_3b_2}{a_1a_2a_3 + a_1b_3 + a_3b_2 + a_2a_3b_1 + b_1b_3}.$$

Each of these fractions can be expressed in the form H_{11}/H, where H is a tridiagonal determinant:

$$f_1 = \frac{|a_1|}{\begin{vmatrix} 1 & b_1 \\ -1 & a_1 \end{vmatrix}},$$

$$f_2 = \frac{\begin{vmatrix} a_1 & b_2 \\ -1 & a_2 \end{vmatrix}}{\begin{vmatrix} 1 & b_1 & \\ -1 & a_1 & b_2 \\ & -1 & a_2 \end{vmatrix}},$$

$$f_3 = \frac{\begin{vmatrix} a_1 & b_2 & \\ -1 & a_2 & b_3 \\ & -1 & a_3 \end{vmatrix}}{\begin{vmatrix} 1 & b_1 & & \\ -1 & a_1 & b_2 & \\ & -1 & a_2 & b_3 \\ & & -1 & a_3 \end{vmatrix}}.$$

Theorem 5.9.

$$f_n = \frac{H_{11}^{(n+1)}}{H_{n+1}},$$

where

$$H_{n+1} = \begin{vmatrix} 1 & b_1 & & & & & & \\ -1 & a_1 & b_2 & & & & & \\ & -1 & a_2 & b_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -1 & a_{n-2} & b_{n-1} & \\ & & & & -1 & a_{n-1} & b_n \\ & & & & & -1 & a_n \end{vmatrix}_{n+1}. \tag{5.5.2}$$

PROOF. Use the method of induction. Assume that

$$f_{n-1} = \frac{H_{11}^{(n)}}{H_n},$$

which is known to be true for small values of n. Hence, adding b_n/a_n to a_{n-1},

$$f_n = \frac{K_{11}^{(n)}}{K_n}, \tag{5.5.3}$$

where

$$K_n = \begin{vmatrix} 1 & b_1 & & & & & \\ -1 & a_1 & b_2 & & & & \\ & -1 & a_2 & b_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & a_{n-3} & b_{n-2} & \\ & & & & -1 & a_{n-2} & b_{n-1} \\ & & & & & -1 & a_{n-1} + (b_n/a_n) \end{vmatrix}_n . \tag{5.5.4}$$

Return to H_{n+1}, remove the factor a_n from the last column, and then perform the column operation

$$\mathbf{C}_n' = \mathbf{C}_n + \mathbf{C}_{n+1}.$$

The result is a determinant of order $(n+1)$ in which the only element in the last row is 1 in the right-hand corner.

It then follows that

$$H_{n+1} = a_n K_n.$$

Similarly,

$$H_{11}^{(n+1)} = a_n K_{11}^{(n-1)}.$$

The theorem follows from (5.5.3). □

Tridiagonal determinants of the form H_n are called continuants. They are also simple Hessenbergians which satisfy the three-term recurrence relation. Expanding H_{n+1} by the two elements in the last row, it is found that

$$H_{n+1} = a_n H_n + b_n H_{n-1}.$$

Similarly,

$$H_{11}^{(n+1)} = a_n H_{11}^{(n)} + b_n H_{11}^{(n)}. \tag{5.5.5}$$

The theorem can therefore be reformulated as follows:

$$f_n = \frac{Q_n}{P_n}, \tag{5.5.6}$$

where P_n and Q_n each satisfy the recurrence relation

$$R_n = a_n R_{n-1} + b_n R_{n-2} \tag{5.5.7}$$

with the initial values $P_0 = 1$, $P_1 = a_1 + b_1$, $Q_0 = 1$, and $Q_1 = a_1$.

5.5.2 Polynomials and Power Series

In the continued fraction f_n defined in (5.5.1) in the previous section, replace a_r by 1 and replace b_r by $a_r x$. Then,

$$f_n = \frac{1}{1+}\,\frac{a_1 x}{1+}\,\frac{a_2 x}{1+} \cdots \frac{a_{n-1} x}{1+}\,\frac{a_n x}{1}$$

$$= \frac{Q_n}{P_n}, \tag{5.5.8}$$

where P_n and Q_n each satisfy the recurrence relation

$$R_n = R_{n-1} + a_n x R_{n-2} \tag{5.5.9}$$

with $P_0 = 1$, $P_1 = 1 + a_1 x$, $Q_0 = 1$, and $Q_1 = 1$. It follows that

$$\begin{aligned}
P_2 &= 1 + (a_1 + a_2)x, \\
Q_2 &= 1 + a_2 x, \\
P_3 &= 1 + (a_1 + a_2 + a_3)x + a_1 a_3 x^2, \\
Q_3 &= 1 + (a_2 + a_3)x, \\
P_4 &= 1 + (a_1 + a_2 + a_3 + a_4)x + (a_1 a_3 + a_1 a_4 + a_2 a_4)x^2, \\
Q_4 &= 1 + (a_2 + a_3 + a_4)x + a_2 a_4 x^2.
\end{aligned} \tag{5.5.10}$$

It also follows from the previous section that $P_n = H_{n+1}$, etc., where

$$H_{n+1} = \begin{vmatrix}
1 & a_1 x & & & & & \\
-1 & 1 & a_2 x & & & & \\
& -1 & 1 & a_3 x & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & -1 & 1 & a_{n-2} x & \\
& & & & -1 & 1 & a_{n-1} x \\
& & & & & -1 & 1
\end{vmatrix}_{n+1}. \tag{5.5.11}$$

The alternative formula

$$H_{n+1} = \begin{vmatrix}
1 & x & & & & & \\
-a_1 & 1 & x & & & & \\
& -a_2 & 1 & x & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & -a_{n-3} & 1 & x & \\
& & & & -a_{n-2} & 1 & x \\
& & & & & -a_{n-1} & 1
\end{vmatrix}_{n+1} \tag{5.5.12}$$

can be proved by showing that the second determinant satisfies the same recurrence relation as the first determinant and has the same initial values. Also,

$$Q_n = H_{11}^{(n+1)}. \tag{5.5.13}$$

Using elementary methods, it is found that

$$\begin{aligned}
f_1 &= 1 - a_1 x + a_1^2 x^2 + \cdots, \\
f_2 &= 1 - a_1 x + a_1(a_1 + a_2)x^2 - a_1(a_1^2 + 2a_1 a_2 + a_2^2)x^3 + \cdots, \\
f_3 &= 1 - a_1 x + a_1(a_1 + a_2)x^2 - a_1(a_1^2 + 2a_1 a_2 + a_2^2 + a_2 a_3)x^3 + \cdots \\
&\quad + a_1(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + 2a_2^3 + 2a_2^2 a_3 \\
&\qquad + a_2 a_3^2 + 2a_1 a_2 a_3)x^4 + \cdots,
\end{aligned} \tag{5.5.14}$$

etc. These formulas lead to the following theorem.

Theorem 5.10.

$$f_n - f_{n-1} = (-1)^n (a_1 a_2 a_3 \cdots a_n) x^n + O(x^{n+1}),$$

that is, the coefficients of x^r, $1 \le r \le n-1$, in the series expansion of f_n are identical to those in the expansion of f_{n-1}.

PROOF. Applying the recurrence relation (5.5.9),

$$\begin{aligned}
P_{n-1}Q_n - P_nQ_{n-1} &= P_{n-1}(Q_{n-1} + a_n x Q_{n-2}) - (P_{n-1} + a_n x P_{n-2})Q_{n-1} \\
&= -a_n x (P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}) \\
&= a_{n-1}a_n x^2 (P_{n-3}Q_{n-2} - P_{n-2}Q_{n-3}) \\
&\ \ \vdots \\
&= (-1)^n (a_3 a_4 \cdots a_n) x^{n-2} (P_1 Q_2 - P_2 Q_1) \\
&= (-1)^n (a_1 a_2 \cdots a_n) x^n \qquad (5.5.15)\\
f_n - f_{n-1} &= \frac{Q_n}{P_n} - \frac{Q_{n-1}}{P_{n-1}} \\
&= \frac{P_{n-1}Q_n - P_nQ_{n-1}}{P_nP_{n-1}} \\
&= \frac{(-1)^n (a_1 a_2 \cdots a_n) x^n}{P_nP_{n-1}}. \qquad (5.5.16)
\end{aligned}$$

The theorem follows since $P_n(x)$ is a polynomial with $P_n(0) = 1$. □

Let

$$f_n(x) = \sum_{r=0}^{\infty} c_r x^r. \qquad (5.5.17)$$

From the third equation in (5.5.14),

$$\begin{aligned}
c_0 &= 1, \\
c_1 &= -a_1, \\
c_2 &= a_1(a_1 + a_2), \\
c_3 &= -a_1(a_1^2 + 2a_1a_2 + a_2^2 + a_2a_3), \\
c_4 &= a_1(a_1^2a_2 + 2a_1a_2^2 + a_2^3 + 2a_2^2a_3 + a_1^2a_3 + 2a_1a_2a_3 + a_2a_3^2 \\
&\qquad + a_1^2a_4 + a_1a_2a_4 + a_2a_3a_4), \qquad (5.5.18)
\end{aligned}$$

etc. Solving these equations for the a_r,

$$\begin{aligned}
a_1 &= -|c_1|, \\
a_2 &= \frac{\begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}}{|c_1|},
\end{aligned}$$

$$a_3 = \frac{|c_0| \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix}}{|c_1| \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}}, \tag{5.5.19}$$

etc. Determinantal formulas for a_{2n-1}, a_{2n}, and two other functions will be given shortly.

Let

$$\begin{aligned} A_n &= |c_{i+j-2}|_n, \\ B_n &= |c_{i+j-1}|_n, \end{aligned} \tag{5.5.20}$$

with $A_0 = B_0 = 1$. Identities among these determinants and their cofactors appear in Hankelians 1.

It follows from the recurrence relation (5.5.9) and the initial values of P_n and Q_n that P_{2n-1}, P_{2n}, Q_{2n+1}, and Q_{2n} are polynomials of degree n. In all four polynomials, the constant term is 1. Hence, we may write

$$\begin{aligned} P_{2n-1} &= \sum_{r=0}^{n} p_{2n-1,r} x^r, \\ Q_{2n+1} &= \sum_{r=0}^{n} q_{2n+1,r} x^r, \\ P_{2n} &= \sum_{r=0}^{n} p_{2n,r} x^r, \\ Q_{2n} &= \sum_{r=0}^{n} q_{2n,r} x^r, \end{aligned} \tag{5.5.21}$$

where both p_{mr} and q_{mr} satisfy the recurrence relation

$$u_{mr} = u_{m-1,r} + a_m u_{m-2,r-1}$$

and where

$$\begin{aligned} p_{m0} = q_{m0} &= 1, \qquad \text{all } m, \\ p_{2n-1,r} = p_{2n,r} &= 0, \qquad r < 0 \text{ or } r > n. \end{aligned} \tag{5.5.22}$$

Theorem 5.11.

a. $p_{2n-1,r} = \dfrac{A^{(n+1)}_{n+1,n+1-r}}{A_n}, \quad 0 \le r \le n,$

b. $p_{2n,r} = \dfrac{B^{(n+1)}_{n+1,n+1-r}}{B_n}, \quad 0 \le r \le n,$

c. $a_{2n+1} = -\dfrac{A_n B_{n+1}}{A_{n+1} B_n},$

d. $a_{2n} = -\dfrac{A_{n+1} B_{n-1}}{A_n B_n}.$

PROOF. Let

$$f_{2n-1}P_{2n-1} - Q_{2n-1} = \sum_{r=0}^{\infty} h_{nr}x^r, \tag{5.5.23}$$

where f_n is defined by the infinite series (5.5.17). Then, from (5.5.8),

$$h_{nr} = 0, \quad \text{all } n \text{ and } r,$$

where

$$h_{nr} = \begin{cases} \sum_{t=0}^{r} c_{r-t}p_{2n-1,t} - q_{2n-1,r}, & 0 \le r \le n-1 \\ \sum_{t=0}^{r} c_{r-t}p_{2n-1,t}, & r \ge n. \end{cases} \tag{5.5.24}$$

The upper limit n in the second sum arises from (5.5.22).

The n equations

$$h_{nr} = 0, \qquad n \le r \le 2n-1,$$

yield

$$\sum_{t=1}^{n} c_{r-t}p_{2n-1,t} + c_r = 0. \tag{5.5.25}$$

Solving these equations by Cramer's formula yields part (a) of the theorem.

Part (b) is proved in a similar manner. Let

$$f_{2n}P_{2n} - Q_{2n} = \sum_{r=0}^{\infty} k_{nr}x^r. \tag{5.5.26}$$

Then,

$$k_{nr} = 0, \qquad \text{all } n \text{ and } r,$$

where

$$k_{rn} = \begin{cases} \sum_{t=0}^{r} c_{r-t}p_{2n,t} - q_{2n,r}, & 0 \le r \le n \\ \sum_{t=0}^{n} c_{r-t}p_{2n,t}, & r \ge n+1. \end{cases} \tag{5.5.27}$$

The n equations

$$k_{nr} = 0, \quad n+1 \le r \le 2n,$$

yield

$$\sum_{t=1}^{n} c_{r-t}p_{2n,t} + c_r = 0. \tag{5.5.28}$$

Solving these equations by Cramer's formula yields part (b) of the theorem.

The equation

$$h_{n,2n+1} = 0$$

yields

$$\sum_{t=0}^{n+1} c_{2n+1-t} p_{2n+1,t} = 0. \tag{5.5.29}$$

Applying the recurrence relation (5.5.22) and then parts (a) and (b) of the theorem,

$$\sum_{t=0}^{n} c_{2n+1-t} p_{2n,t} + a_{2n+1} \sum_{t=1}^{n+1} c_{2n+1-t} p_{2n-1,t-1} = 0,$$

$$\frac{1}{B_n} \sum_{t=0}^{n} c_{2n+1-t} B^{(n+1)}_{n+1,n+1-t} + \frac{a_{2n+1}}{A_n} \sum_{t=1}^{n+1} c_{2n+1-t} A^{(n+1)}_{n+1,n+2-t} = 0,$$

$$\frac{B_{n+1}}{B_n} + a_{2n+1} \frac{A_{n+1}}{A_n} = 0,$$

which proves part (c).

Part (d) is proved in a similar manner. The equation

$$k_{n,2n} = 0$$

yields

$$\sum_{t=0}^{n} c_{2n-t} p_{2n,t} = 0. \tag{5.5.30}$$

Applying the recurrence relation (5.5.22) and then parts (a) and (b) of the theorem,

$$\sum_{t=0}^{n} c_{2n-t} p_{2n-1,t} + a_{2n} \sum_{t=1}^{n} c_{2n-t} p_{2n-2,t-1} = 0,$$

$$\frac{1}{A_n} \sum_{t=0}^{n} c_{2n-t} A^{(n+1)}_{n+1,n+1-t} + \frac{a_{2n}}{B_{n-1}} \sum_{t=1}^{n} c_{2n-t} B^{(n)}_{n,n+1-t} = 0,$$

$$\frac{A_{n+1}}{A_n} + a_{2n} \frac{B_n}{B_{n-1}} = 0,$$

which proves part (d). □

Exercise. Prove that

$$P_6 = 1 + x \sum_{r=1}^{6} a_r + x^2 \sum_{r=1}^{4} a_r \sum_{s=r+2}^{6} a_s + x^3 \sum_{r=1}^{2} a_r \sum_{s=r+2}^{4} a_s \sum_{t=s+2}^{6} a_t$$

and find the corresponding formula for Q_7.

5.5.3 *Further Determinantal Formulas*

Theorem 5.12.

$$\textbf{a.}\ P_{2n-1} = \frac{1}{A_n} \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \multicolumn{5}{c}{\dotfill} \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ x^n & x^{n-1} & x^{n-2} & \cdots & 1 \end{vmatrix}_{n+1},$$

$$\textbf{b.}\ P_{2n} = \frac{1}{B_n} \begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \multicolumn{5}{c}{\dotfill} \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \\ x^n & x^{n-1} & x^{n-2} & \cdots & 1 \end{vmatrix}_{n+1}.$$

PROOF. Referring to the first line of (5.5.21) and to Theorem 5.11a,

$$\begin{aligned} P_{2n-1} &= \frac{1}{A_n} \sum_{r=0}^{n} A_{n+1,n+1-r}^{(n+1)} x^r \\ &= \frac{1}{A_n} \sum_{j=1}^{n+1} A_{n+1,j}^{(n+1)} x^{n+1-j}. \end{aligned}$$

Part (a) follows and part (b) is proved in a similar manner with the aid of the third line in (5.5.21) and Theorem 5.11b. □

Lemmas.

$$\textbf{a.}\ \sum_{r=0}^{n-1} u_r \sum_{t=0}^{r} c_{r-t} v_{n+1-t} = \sum_{j=1}^{n} v_{j+1} \sum_{r=0}^{j-1} c_r u_{n+r-j},$$

$$\textbf{b.}\ \sum_{r=0}^{n} u_r \sum_{t=0}^{r} c_{r-t} v_{n+1-t} = \sum_{j=0}^{n} v_{j+1} \sum_{r=0}^{j} c_r u_{n+r-j}.$$

These two lemmas differ only in some of their limits and could be regarded as two particular cases of one lemma whose proof is elementary and consists of showing that both double sums represent the sum of the same triangular array of terms.

Let

$$\psi_m = \sum_{r=0}^{m} c_r x^r. \tag{5.5.31}$$

Theorem 5.13.

$$\textbf{a.}\ Q_{2n-1} = \frac{1}{A_n} \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \multicolumn{5}{c}{\dotfill} \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ \psi_0 x^n & \psi_1 x^{n-1} & \psi_2 x^{n-2} & \cdots & \psi_n \end{vmatrix}_{n+1},$$

$$\textbf{b. } Q_{2n} = \frac{1}{B_n}\begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \\ \psi_0 x^n & \psi_1 x^{n-1} & \psi_2 x^{n-2} & \cdots & \psi_n \end{vmatrix}_{n+1}.$$

PROOF. From the second equation in (5.5.24) in the previous section and referring to Theorem 5.11a,

$$\begin{aligned} q_{2n-1,r} &= \sum_{t=0}^{r} c_{r-t} p_{2n-1,t}, \quad 0 \le r \le n-1 \\ &= \frac{1}{A_n} \sum_{t=0}^{r} c_{r-t} A_{n+1,n+1-t}^{(n+1)}. \end{aligned}$$

Hence, from the second equation in (5.5.21) with $n \to n-1$ and applying Lemma (a) with $u_r \to x^r$ and $v_s \to A_{n+1,s}^{(n+1)}$,

$$\begin{aligned} A_n Q_{2n-1} &= \sum_{r=0}^{n-1} x^r \sum_{t=0}^{r} c_{r-t} A_{n+1,n+1-t}^{(n+1)} \\ &= \sum_{j=1}^{n} A_{n+1,j+1}^{(n+1)} \sum_{r=0}^{j-1} c_r x^{n+r-j} \\ &= \sum_{j=1}^{n} x^{n-j} A_{n+1,j+1}^{(n+1)} \sum_{r=0}^{j-1} c_r x^r \\ &= \sum_{j=1}^{n} \psi_{j-1} x^{n-j} A_{n+1,j+1}^{(n+1)}. \end{aligned}$$

This sum represents a determinant of order $(n+1)$ whose first n rows are identical with the first n rows of the determinant in part (a) of the theorem and whose last row is

$$\begin{bmatrix} 0 & \psi_0 x^{n-1} & \psi_1 x^{n-2} & \psi_2 x^{n-3} \cdots \psi_{n-1} \end{bmatrix}_{n+1}.$$

The proof of part (a) is completed by performing the row operation

$$\mathbf{R}'_{n+1} = \mathbf{R}_{n+1} + x^n \mathbf{R}_1.$$

The proof of part (b) of the theorem applies Lemma (b) and gives the required result directly, that is, without the necessity of performing a row operation. From (5.5.27) in the previous section and referring to Theorem 5.11b,

$$q_{2n,r} = \sum_{t=0}^{r} c_{r-t} p_{2n,t}, \quad 0 \le r \le n$$

$$= \frac{1}{B_n} \sum_{t=0}^{r} c_{r-t} B_{n+1,n+1-t}^{(n+1)}.$$

Hence, from the fourth equation in (5.5.11) and applying Lemma (b) and (5.5.31),

$$\begin{aligned}
B_n Q_{2n} &= \sum_{r=0}^{n} x^r \sum_{t=0}^{r} c_{r-t} B_{n+1,n+1-t}^{(n+1)} \\
&= \sum_{j=0}^{n} B_{n+1,j+1}^{(n+1)} \sum_{r=0}^{j} c_r x^{n+r-j} \\
&= \sum_{j=0}^{n} \psi_j x^{n-j} B_{n+1,j+1}^{(n+1)}.
\end{aligned}$$

This sum is an expansion of the determinant in part (b) of the theorem. This completes the proofs of both parts of the theorem. □

Exercise. Show that the equations

$$\begin{aligned}
h_{n,2n+j} &= 0, \quad j \ge 2, \\
k_{n,2n+j} &= 0, \quad j \ge 1,
\end{aligned}$$

lead respectively to

$$S_{n+2} = 0, \quad \text{all } n, \tag{X}$$

$$T_{n+1} = 0, \quad \text{all } n, \tag{Y}$$

where S_{n+2} denotes the determinant obtained from A_{n+2} by replacing its last row by the row

$$\left[c_{n+j-1}\ c_{n+j}\ c_{n+j+1} \cdots c_{2n+j}\right]_{n+2}$$

and T_{n+1} denotes the determinant obtained from B_{n+1} by replacing its last row by the row

$$\left[c_{n+j}\ c_{n+j+1}\ c_{n+j+2} \cdots c_{2n+j}\right]_{n+1}.$$

Regarding (X) and (Y) as conditions, what is their significance?

5.6 Distinct Matrices with Nondistinct Determinants

5.6.1 Introduction

Two matrices $[a_{ij}]_m$ and $[b_{ij}]_n$ are equal if and only if $m = n$ and $a_{ij} = b_{ij}$, $1 \le i, j \le n$. No such restriction applies to determinants. Consider determinants with constant elements. It is a trivial exercise to find two determinants $A = |a_{ij}|_n$ and $B = |b_{ij}|_n$ such that $a_{ij} \neq b_{ij}$ for any pair (i, j) and the elements a_{ij} are not merely a rearrangement of the elements b_{ij}, but $A = B$. It is an equally trivial exercise to find two determinants of different orders which have the same value. If the elements are polynomials, then the determinants are also polynomials and the exercises are more difficult.

It is the purpose of this section to show that there exist families of distinct matrices whose determinants are not distinct for the reason that they represent identical polynomials, apart from a possible change in sign. Such determinants may be described as equivalent.

5.6.2 *Determinants with Binomial Elements*

Let $\phi_m(x)$ denote an Appell polynomial (Appendix A.4):

$$\phi_m(x) = \sum_{r=0}^{m} \binom{m}{r} \alpha_{m-r} x^r. \tag{5.6.1}$$

The inverse relation is

$$\alpha_m = \sum_{r=0}^{m} \binom{m}{r} \phi_{m-r}(x)(-x)^r. \tag{5.6.2}$$

Define infinite matrices $\mathbf{P}(x)$, $\mathbf{P}^T(x)$, $\mathbf{A}$, and $\Phi(x)$ as follows:

$$\mathbf{P}(x) = \left[\overleftrightarrow{\binom{i-1}{j-1} x^{i-j}}\right], \qquad i, j \geq 1, \tag{5.6.3}$$

where the symbol $\longleftrightarrow$ denotes that the order of the columns is to be reversed. $\mathbf{P}^T$ denotes the transpose of $\mathbf{P}$. Both $\mathbf{A}$ and Φ are defined in Hankelian notation (Section 4.8):

$$\begin{aligned} \mathbf{A} &= [\alpha_m], \qquad m \geq 0, \\ \Phi(x) &= [\phi_m(x)], \quad m \geq 0. \end{aligned} \tag{5.6.4}$$

Now define block matrices $\mathbf{M}$ and $\mathbf{M}^*$ as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{O} & \mathbf{P}^T(x) \\ \mathbf{P}(x) & \Phi(x) \end{bmatrix}, \tag{5.6.5}$$

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{O} & \mathbf{P}^T(-x) \\ \mathbf{P}(-x) & \mathbf{A} \end{bmatrix}. \tag{5.6.6}$$

These matrices are shown in some detail below. They are triangular, symmetric, and infinite in all four directions. Denote the diagonals containing the unit elements in both matrices by diag(1).

It is now required to define a number of determinants of submatrices of either $\mathbf{M}$ or $\mathbf{M}^*$. Many statements are abbreviated by omitting references to submatrices and referring directly to subdeterminants.

Define a Turanian T_{nr} (Section 4.9.2) as follows:

$$T_{nr} = \begin{vmatrix} \phi_{r-2n+2} & \cdots & \phi_{r-n+2} \\ \vdots & & \vdots \\ \phi_{r-n+1} & \cdots & \phi_r \end{vmatrix}_n, \quad r \geq 2n-2, \tag{5.6.7}$$

which is a subdeterminant of $\mathbf{M}$.

$$\begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & & & & & & & & & & 1 & \cdots \\
\cdots & & & & & & & & & 1 & 4x & \cdots \\
\cdots & & & & & & & & 1 & 3x & 6x^2 & \cdots \\
\cdots & & & & & & & 1 & 2x & 3x^2 & 4x^3 & \cdots \\
\cdots & & & & & & 1 & x & x^2 & x^3 & x^4 & \cdots \\
\cdots & & & & & 1 & \phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots \\
\cdots & & & & 1 & x & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \cdots \\
\cdots & & & 1 & 2x & x^2 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \cdots \\
\cdots & & 1 & 3x & 3x^2 & x^3 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \cdots \\
\cdots & 1 & 4x & 6x^2 & 4x^3 & x^4 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}$$

The infinite matrix $\mathbf{M}$

$$\begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & & & & & & & & & & 1 & \cdots \\
\cdots & & & & & & & & & 1 & -4x & \cdots \\
\cdots & & & & & & & & 1 & -3x & 6x^2 & \cdots \\
\cdots & & & & & & & 1 & -2x & 3x^2 & -4x^3 & \cdots \\
\cdots & & & & & & 1 & -x & x^2 & -x^3 & x^4 & \cdots \\
\cdots & & & & & 1 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\
\cdots & & & & 1 & -x & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\
\cdots & & & 1 & -2x & x^2 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\
\cdots & & 1 & -3x & 3x^2 & -x^3 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \cdots \\
\cdots & 1 & -4x & 6x^2 & -4x^3 & x^4 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}$$

The infinite matrix $\mathbf{M}^*$

The element α_r occurs $(r+1)$ times in $\mathbf{M}^*$. Consider all the subdeterminants of $\mathbf{M}^*$ which contain the element α_r in the bottom right-hand corner and whose order n is sufficiently large for them to contain the element α_0 but sufficiently small for them not to have either unit or zero elements along their secondary diagonals. Denote these determinants by B_s^{nr}, $s = 1, 2, 3, \ldots$. Some of them are symmetric and unique whereas others occur in pairs, one of which is the transpose of the other. They are coaxial in the sense that all their secondary diagonals lie along the same diagonal parallel to diag(1) in $\mathbf{M}^*$.

Theorem 5.14. *The determinants B_s^{nr}, where n and r are fixed, $s = 1, 2, 3, \ldots$, represent identical polynomials of degree $(r+2-n)(2n-2-r)$.*

Denote their common polynomial by B_{nr}.

Theorem 5.15.

$$T_{r+2-n,r} = (-1)^k B_{nr}, \quad r \geq 2n-2, \quad n = 1, 2, 3, \ldots$$

where

$$k = n + r + \left[\tfrac{1}{2}(r+2)\right].$$

Both of these theorems have been proved by Fiedler using the theory of S-matrices but in order to relate the present notes to Fiedler's, it is necessary to change the sign of x.

When $r = 2n-2$, Theorem 5.15 becomes the symmetric identity

$$T_{n,2n-2} = B_{n,2n-2},$$

that is

$$\begin{vmatrix} \phi_0 & \cdots & \phi_{n-1} \\ \vdots & & \vdots \\ \phi_{n-1} & \cdots & \phi_{2n-2} \end{vmatrix}_n = \begin{vmatrix} \alpha_0 & \cdots & \alpha_{n-1} \\ \vdots & & \vdots \\ \alpha_{n-1} & \cdots & \alpha_{2n-2} \end{vmatrix}_n \quad \text{(degree 0)}$$

$$|\phi_m|_n = |\alpha_m|_n, \quad 0 \leq m \leq 2n-2,$$

which is proved by an independent method in Section 4.9 on Hankelians 2.

Theorem 5.16. *To each identity, except one, described in Theorems 5.14 and 5.15 there corresponds a dual identity obtained by reversing the role of $\mathbf{M}$ and $\mathbf{M}^*$, that is, by interchanging $\phi_m(x)$ and α_m and changing the sign of each x where it occurs explicitly. The exceptional identity is the symmetric one described above which is its own dual.*

The following particular identities illustrate all three theorems. Where $n = 1$, the determinants on the left are of unit order and contain a single element. Each identity is accompanied by its dual.

$(n, r) = (1, 1)$:

$$|\phi_1| = \begin{vmatrix} 1 & -x \\ \alpha_0 & \alpha_1 \end{vmatrix},$$

$$|\alpha_1| = \begin{vmatrix} 1 & x \\ \phi_0 & \phi_1 \end{vmatrix}; \tag{5.6.8}$$

$(n, r) = (3, 2)$:

$$|\phi_2| = -\begin{vmatrix} & 1 & -2x \\ 1 & -x & x^2 \\ \alpha_0 & \alpha_1 & \alpha_2 \end{vmatrix} = -\begin{vmatrix} & 1 & -x \\ 1 & \alpha_0 & \alpha_1 \\ -x & \alpha_1 & \alpha_2 \end{vmatrix} \text{ (symmetric)},$$

$$|\alpha_2| = -\begin{vmatrix} & 1 & 2x \\ 1 & x & x^2 \\ \phi_0 & \phi_1 & \phi_2 \end{vmatrix} = -\begin{vmatrix} & 1 & x \\ 1 & \phi_0 & \phi_1 \\ x & \phi_1 & \phi_2 \end{vmatrix} \text{ (symmetric);} \tag{5.6.9}$$

$(n, r) = (4, 3)$:

$$|\phi_3| = -\begin{vmatrix} & & 1 & -2x \\ & 1 & -x & x^2 \\ 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ -x & \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} = -\begin{vmatrix} & & 1 & -3x \\ & 1 & -2x & 3x^2 \\ 1 & -x & x^2 & -x^3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix},$$

$$|\alpha_3| = -\begin{vmatrix} & & 1 & 2x \\ & 1 & x & x^2 \\ 1 & \phi_0 & \phi_1 & \phi_2 \\ x & \phi_1 & \phi_2 & \phi_3 \end{vmatrix} = -\begin{vmatrix} & & 1 & 3x \\ & 1 & 2x & 3x^2 \\ 1 & x & x^2 & x^3 \\ \phi_0 & \phi_1 & \phi_2 & \phi_3 \end{vmatrix}; \tag{5.6.10}$$

$(n, r) = (3, 3)$:

$$\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{vmatrix} = \begin{vmatrix} 1 & -x & x^2 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix},$$

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ \phi_0 & \phi_1 & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}; \tag{5.6.11}$$

$(n, r) = (4, 4)$:

$$\begin{vmatrix} \phi_2 & \phi_3 \\ \phi_3 & \phi_4 \end{vmatrix} = -\begin{vmatrix} & 1 & -2x & 3x^2 \\ 1 & -x & x^2 & -x^3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix} = \begin{vmatrix} & 1 & -x & x^2 \\ 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ -x & \alpha_1 & \alpha_2 & \alpha_3 \\ x^2 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix},$$

$$\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = -\begin{vmatrix} & 1 & 2x & 3x^2 \\ 1 & x & x^2 & x^3 \\ \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix} = \begin{vmatrix} & 1 & x & x^2 \\ 1 & \phi_0 & \phi_1 & \phi_2 \\ x & \phi_1 & \phi_2 & \phi_3 \\ x^2 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix} \tag{5.6.12}$$

The coaxial nature of the determinants B_s^{nr} is illustrated for the case $(n, r) = (6, 6)$ as follows:

$$\begin{vmatrix} \phi_4 & \phi_5 \\ \phi_5 & \phi_6 \end{vmatrix} = \begin{cases} \text{each of the three determinants of order 6} \\ \text{enclosed within overlapping dotted frames} \\ \text{in the following display:} \end{cases}$$

$$\begin{array}{cccccccc}
 & & & & & 1 & -4x & 10x^2 \\
 & & & & 1 & -3x & 6x^2 & -10x^2 \\
 & & & 1 & -2x & 3x^2 & -4x^3 & 5x^4 \\
 & & 1 & -x & x^2 & -x^3 & x^4 & -x^5 \\
 & 1 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
1 & -x & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
-2x & x^2 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \\
3x^2 & -x^3 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & &
\end{array} \tag{5.6.13}$$

These determinants are B_s^{66}, $s = 1, 2, 3$, as indicated at the corners of the frames. B_1^{66} is symmetric and is a bordered Hankelian. The dual identities are found in the manner described in Theorem 5.16.

All the determinants described above are extracted from consecutive rows and columns of $\mathbf{M}$ or $\mathbf{M}^*$. A few illustrations are sufficient to demonstrate the existence of identities of a similar nature in which the determinants are extracted from nonconsecutive rows and columns of $\mathbf{M}$ or $\mathbf{M}^*$.

In the first two examples, either the rows or the columns are nonconsecutive:

$$\begin{vmatrix} \phi_0 & \phi_2 \\ \phi_1 & \phi_3 \end{vmatrix} = -\begin{vmatrix} & 1 & -2x \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}, \tag{5.6.14}$$

$$\begin{vmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{vmatrix} = \begin{vmatrix} & & 1 & -2x \\ 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ -x & \alpha_1 & \alpha_2 & \alpha_3 \\ x^2 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix} = \begin{vmatrix} & & 1 & -3x \\ 1 & -x & x^2 & -x^3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix}. \tag{5.6.15}$$

In the next example, both the rows and columns are nonconsecutive:

$$\begin{vmatrix} \phi_0 & \phi_2 \\ \phi_2 & \phi_4 \end{vmatrix} = -\begin{vmatrix} & & 1 & -2x \\ & \alpha_0 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ -2x & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix}. \tag{5.6.16}$$

The general form of these identities is not known and hence no theorem is known which includes them all.

In view of the wealth of interrelations between the matrices $\mathbf{M}$ and $\mathbf{M}^*$, each can be described as the dual of the other.

Exercise. Verify these identities and their duals by elementary methods. The above identities can be generalized by introducing a second variable y. A few examples are sufficient to demonstrate their form.

$$\phi_1(x+y) = \begin{vmatrix} 1 & -x \\ \phi_0(y) & \phi_1(y) \end{vmatrix} = \begin{vmatrix} 1 & -y \\ \phi_0(x) & \phi_1(x) \end{vmatrix}, \tag{5.6.17}$$

$$\phi_1(y) = \begin{vmatrix} 1 & x \\ \phi_0(x+y) & \phi_1(x+y) \end{vmatrix}, \tag{5.6.18}$$

$$\begin{vmatrix} \phi_1(x+y) & \phi_2(x+y) \\ \phi_2(x+y) & \phi_3(x+y) \end{vmatrix} = \begin{vmatrix} 1 & -x & x^2 \\ \phi_0(y) & \phi_1(y) & \phi_2(y) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) \end{vmatrix},$$

$$= \begin{vmatrix} 1 & -y & y^2 \\ \phi_0(x) & \phi_1(x) & \phi_2(x) \\ \phi_1(x) & \phi_2(x) & \phi_3(x) \end{vmatrix} \qquad (5.6.19)$$

$$\begin{vmatrix} \phi_2(x+y) & \phi_3(x+y) \\ \phi_3(x+y) & \phi_4(x+y) \end{vmatrix} = \begin{vmatrix} & 1 & -x & x^2 \\ 1 & \phi_0(y) & \phi_1(y) & \phi_2(y) \\ -x & \phi_1(y) & \phi_2(y) & \phi_3(y) \\ x^2 & \phi_2(y) & \phi_3(y) & \phi_4(y) \end{vmatrix}$$

$$= \begin{vmatrix} & 1 & -2x & 3x^2 \\ 1 & -x & x^2 & -x^3 \\ \phi_0(y) & \phi_1(y) & \phi_2(y) & \phi_3(y) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) & \phi_4(y) \end{vmatrix}. \qquad (5.6.20)$$

Do these identities possess duals?

5.6.3 Determinants with Stirling Elements

Matrices $\mathbf{s}_n(x)$ and $\mathbf{S}_n(x)$ whose elements contain Stirling numbers of the first and second kinds, s_{ij} and S_{ij}, respectively, are defined in Appendix A.1.

Let the matrix obtained by rotating $\mathbf{S}_n(x)$ through 90° in the anticlockwise direction be denotes by $\overset{\curvearrowleft}{\mathbf{S}}_n(x)$. For example,

$$\overset{\curvearrowleft}{\mathbf{S}}_5(x) = \begin{bmatrix} & & & & 1 \\ & & & 1 & 10x \\ & & 1 & 6x & 25x^2 \\ & 1 & 3x & 7x^2 & 15x^3 \\ 1 & x & x^2 & x^3 & x^4 \end{bmatrix}.$$

Define another nth-order triangular matrix $\mathbf{B}_n(x)$ as follows:

$$\mathbf{B}_n(x) = [\overleftrightarrow{b_{ij}x^{i-j}}], \quad n \geq 2, \quad 1 \leq i,j \leq n,$$

where

$$b_{ij} = \frac{1}{(j-1)!} \sum_{r=0}^{j-1} (-1)^r \binom{j-1}{r} (n-r-1)^{i-1}, \quad i \geq j. \qquad (5.6.21)$$

These numbers are integers and satisfy the recurrence relation

$$b_{ij} = b_{i-1,j-1} + (n-j)b_{i-j,j},$$

where

$$b_{11} = 1. \qquad (5.6.22)$$

Once again the symbol $\longleftrightarrow$ denotes that the columns are arranged in reverse order.

Illustrations

$$
\begin{aligned}
\mathbf{B}_2(x) &= \begin{bmatrix} & 1 \\ 1 & x \end{bmatrix}, \\
\mathbf{B}_3(x) &= \begin{bmatrix} & & 1 \\ & 1 & 2x \\ 1 & 3x & 4x^2 \end{bmatrix}, \\
\mathbf{B}_4(x) &= \begin{bmatrix} & & & 1 \\ & & 1 & 3x \\ & 1 & 5x & 9x^2 \\ 1 & 6x & 19x^2 & 27x^3 \end{bmatrix}, \\
\mathbf{B}_5(x) &= \begin{bmatrix} & & & & 1 \\ & & & 1 & 4x \\ & & 1 & 7x & 16x^2 \\ & 1 & 9x & 37x^2 & 64x^3 \\ 1 & 10x & 55x^2 & 175x^3 & 256x^4 \end{bmatrix}.
\end{aligned}
$$

Since b_{ij} is a function of n, $\mathbf{B}_n$ is not a submatrix of $\mathbf{B}_{n+1}$. Finally, define a block matrix $\mathbf{N}_{2n}$ of order $2n$ as follows:

$$
\mathbf{N}_{2n} = \begin{bmatrix} \mathbf{O} & \overset{\curvearrowleft}{\mathbf{S}}_n(x) \\ \mathbf{B}_n(x) & \mathbf{A}_n \end{bmatrix}, \tag{5.6.23}
$$

where $\mathbf{A}_n = [\alpha_m]_n$, as before.

Illustrations

$$
\begin{aligned}
\mathbf{N}_4 &= \begin{bmatrix} & & & 1 \\ & & 1 & x \\ & 1 & \alpha_0 & \alpha_1 \\ 1 & x & \alpha_1 & \alpha_2 \end{bmatrix}, \\
\mathbf{N}_6 &= \begin{bmatrix} & & & & & 1 \\ & & & & 1 & 3x \\ & & & 1 & x & x^2 \\ & & 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ & 1 & 2x & \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 3x & 4x^2 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}
\end{aligned}
$$

$\mathbf{N}_{2n}$ is symmetric only when $n = 2$.

A subset of $\mathbf{N}_4$ is:

$$\begin{array}{ccc} & 1 & x \\ 1 & \alpha_0 & \alpha_1 \\ x & \alpha_1 & \end{array}$$

Each of the two overlapping coaxial second-order matrices indicated by frames has a determinant equal to

$$-(\alpha_0 x - \alpha_1) = \sum_{r=1}^{2} s_{2r}\alpha_{r-1}x^{2-r}. \tag{5.6.24}$$

In this case, the equality is a trivial one, as one matrix is merely the transpose of the other.

A subset of $\mathbf{N}_6$ is

$$\begin{array}{ccccc} & & & 1 & 3x \\ & & 1 & x & x^2 \\ & 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ 1 & 2x & \alpha_1 & \alpha_2 & \\ 3x & 4x^2 & \alpha_2 & & \end{array}$$

Each of the three distinct overlapping coaxial third-order matrices indicated by frames has a determinant equal to

$$-(2\alpha_0 x^2 - 3\alpha_1 x + \alpha_2) = \sum_{r=1}^{3} s_{3r}\alpha_{r-1}x^{3-r}. \tag{5.6.25}$$

A subset of $\mathbf{N}_8$ is

$$
\begin{array}{ccccccc}
 & & & & & 1 & 6x \\
 & & & & 1 & 3x & 7x^2 \\
 & & & 1 & x & x^2 & x^3 \\
 & & 1 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
 & 1 & 3x & \alpha_1 & \alpha_2 & \alpha_3 & \\
1 & 5x & 9x^2 & \alpha_2 & \alpha_3 & & \\
6x & 19x^2 & 27x^3 & \alpha_3 & & &
\end{array}
$$

Each of the four distinct overlapping coaxial fourth-order matrices indicated by frames has a determinant equal to

$$-\left(6\alpha_0 x^3 - 11\alpha_1 x^2 + 6\alpha_2 x - \alpha_3\right) = \sum_{r=1}^{4} s_{4r}\alpha_{r-1}x^{4-r}. \tag{5.6.26}$$

It does not appear to be possible to construct dual families of determinantal identities by interchanging the roles of s_{ij} and S_{ij}, but there exists the following simple identity in which the elements of the determinant contain Stirling numbers of the first kind and the sum contains Stirling numbers of the second kind:

$$\begin{vmatrix}
\alpha_0 & 1 & & & \\
\alpha_1 & s_{21}x & 1 & & \\
\alpha_2 & s_{31}x^2 & s_{32}x & 1 & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{n-2} & s_{n-1,1}x^{n-2} & \cdots & \cdots & 1 \\
\alpha_{n-1} & s_{n1}x^{n-1} & \cdots & \cdots & s_{n,n-1}x
\end{vmatrix}_n = (-1)^{n-1}\sum_{r=1}^{n} S_{nr}\alpha_{r-1}x^{n-r}. \tag{5.6.27}$$

The determinant is a Hessenbergian (Section 4.6) and is obtained from $s_n(x)$ by removing the last column, which contains a single nonzero element, and adding a column of α's on the left. The proof is left as an exercise for the reader.

5.7 The One-Variable Hirota Operator

5.7.1 Definition and Taylor Relations

Several nonlinear equations of mathematical physics, including the Korteweg–de Vries, Kadomtsev–Petviashvili, Boussinesq, and Toda equations, can be expressed neatly in terms of multivariable Hirota operators. The ability of an equation to be expressible in Hirota form is an important factor in the investigation of its integrability.

The one-variable Hirota operator, denoted here by H^n, is defined as follows: If $f = f(x)$ and $g = g(x)$, then

$$\begin{aligned} H^n(f,g) &= \left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x)g(x')\right]_{x'=x} \\ &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} D^{n-r}(f)D^r(g), \quad D = \frac{d}{dx}. \end{aligned} \tag{5.7.1}$$

The factor $(-1)^r$ distinguishes this sum from the Leibnitz formula for $D^n(fg)$. The notation H_x, H_{xx}, etc., is convenient in some applications.

Examples.

$$\begin{aligned} H_x(f,g) = H^1(f,g) &= f_x g - f g_x \\ &= -H_x(g,f), \\ H_{xx}(f,g) = H^2(f,g) &= f_{xx}g - 2f_x g_x + f g_{xx} \\ &= H_{xx}(f,g). \end{aligned}$$

Lemma.

$$e^{zH}(f,g) = f(x+z)g(x-z).$$

PROOF. Using the notation $r = i\ (\to j)$ defined in Appendix A.1,

$$\begin{aligned} e^{zH}(f,g) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} H^n(f,g) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{n(\to\infty)} (-1)^r \binom{n}{r} D^{n-r}(f)D^r(g) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r D^r(g)}{r!} \sum_{n=0(\to r)}^{\infty} \frac{z^n D^{n-r}(f)}{(n-r)!} \quad (\text{put } s = n-r) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r z^r D^r(g)}{r!} \sum_{s=0}^{\infty} \frac{z^s D^s(f)}{s!}. \end{aligned}$$

These sums are Taylor expansions of $g(x-z)$ and $f(x+z)$, respectively, which proves the lemma. □

Applying Taylor's theorem again,

$$\begin{aligned}
\tfrac{1}{2}\{\phi(x+z)-\phi(x-z)\} &= \sum_{n=0}^{\infty}\frac{z^{2n+1}D^{2n+1}(\phi)}{(2n+1)!},\\
\tfrac{1}{2}\{\psi(x+z)+\psi(x-z)\} &= \sum_{n=0}^{\infty}\frac{z^{2n}D^{2n}(\psi)}{(2n)!}. \qquad (5.7.2)
\end{aligned}$$

5.7.2 *A Determinantal Identity*

Define functions ϕ, ψ, u_n and a Hessenbergian E_n as follows:

$$\begin{aligned}
\phi &= \log(fg),\\
\psi &= \log(f/g) \qquad (5.7.3)\\
u_{2n} &= D^{2n}(\phi),\\
u_{2n+1} &= D^{2n+1}(\psi), \qquad (5.7.4)\\
E_n &= |e_{ij}|_n,
\end{aligned}$$

where

$$e_{ij} = \begin{cases} \dbinom{j-1}{i-1}u_{j-i+1}, & j \geq i,\\ -1, & j = i-1,\\ 0, & \text{otherwise.}\end{cases} \qquad (5.7.5)$$

It follows from (5.7.3) that

$$\begin{aligned}
f &= e^{(\phi+\psi)/2},\\
g &= e^{(\phi-\psi)/2},\\
fg &= e^{\phi}. \qquad (5.7.6)
\end{aligned}$$

Theorem.

$$\begin{aligned}
\frac{H^n(f,g)}{fg} &= E_n\\
&= \begin{vmatrix}
u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n\\
-1 & u_1 & 2u_2 & 3u_3 & \cdots & \cdots & \dbinom{n-1}{n-2}u_{n-1}\\
 & -1 & u_1 & 3u_2 & \cdots & \cdots & \dbinom{n-1}{n-3}u_{n-2}\\
 & & -1 & u_1 & \cdots & \cdots & \cdots\\
 & & & & \cdots & \cdots & \cdots\\
 & & & & & -1 & u_1
\end{vmatrix}_n.
\end{aligned}$$

This identity was conjectured by one of the authors and proved by Caudrey in 1984. The correspondence was private. Two proofs are given below. The first is essentially Caudrey's but with additional detail.

PROOF. *First proof* (Caudrey). The Hessenbergian satisfies the recurrence relation (Section 4.6)

$$E_{n+1} = \sum_{r=0}^{n} \binom{n}{r} u_{r+1} E_{n-r}. \tag{5.7.7}$$

Let

$$F_n = \frac{H^n(f,g)}{fg}, \quad f = f(x),\ g = g(x),\ F_0 = 1. \tag{5.7.8}$$

The theorem will be proved by showing that F_n satisfies the same recurrence relation as E_n and has the same initial values.

Let

$$K = \begin{cases} \frac{e^{zH}(f,g)}{fg} \\ \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{H^n(f,g)}{fg} \\ \sum_{n=0}^{\infty} \frac{z^n F_n}{n!}\,. \end{cases} \tag{5.7.9}$$

Then,

$$\frac{\partial K}{\partial z} = \sum_{n=1}^{\infty} \frac{z^{n-1} F_n}{(n-1)!} \tag{5.7.10}$$

$$= \sum_{n=0}^{\infty} \frac{z^n F_{n+1}}{n!}\,. \tag{5.7.11}$$

From the lemma and (5.7.6),

$$K = \frac{f(x+z)g(x-z)}{f(x)g(x)} = \exp\left[\tfrac{1}{2}\{\phi(x+z) + \phi(x-z) + \psi(x+z) - \psi(x-z) - 2\phi(x)\}\right]. \tag{5.7.12}$$

Differentiate with respect to z, refer to (5.7.11), note that

$$D_z(\phi(x-z)) = -D_x(\phi(x-z))$$

etc., and apply the Taylor relations (5.7.2) from the previous section. The result is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n F_{n+1}}{n!} &= D\left[\tfrac{1}{2}\{\phi(x+z) - \phi(x-z) + \psi(x+z) + \psi(x-z)\}\right] K \\
&= \left[\sum_{n=0}^{\infty} \frac{z^{2n+1} D^{2n+2}(\phi)}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{z^{2n} D^{2n+1}(\psi)}{(2n)!}\right] K \\
&= \left[\sum_{n=0}^{\infty} \frac{z^{2n+1} u_{2n+2}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{z^{2n} u_{2n+1}}{(2n)!}\right] K
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{z^m u_{m+1}}{m!} \sum_{r=0}^{\infty} \frac{z^r F_r}{r!}. \tag{5.7.13}$$

Equating coefficients of z^n,

$$\frac{F_{n+1}}{n!} = \sum_{r=0}^{n} \frac{u_{r+1} F_{n-r}}{r!\,(n-r)!},$$

$$F_{n+1} = \sum_{r=0}^{n} \binom{n}{r} u_{r+1} F_{n-r}. \tag{5.7.14}$$

This recurrence relation in F_n is identical in form to the recurrence relation in E_n given in (5.7.7). Furthermore,

$$E_1 = F_1 = u_1,$$
$$E_2 = F_2 = u_1^2 + u_2.$$

Hence,

$$E_n = F_n$$

which proves the theorem.

Second proof. Express the lemma in the form

$$\sum_{i=0}^{\infty} \frac{z^i}{i!} H^i(f,g) = f(x+z)g(x-z). \tag{5.7.15}$$

Hence,

$$H^i(f,g) = \left[D_z^i\{f(x+z)g(x-z)\}\right]_{z=0}. \tag{5.7.16}$$

Put

$$f(x) = e^{F(x)},$$
$$g(x) = e^{G(x)},$$
$$w = F(x+z) + G(x-z).$$

Then,

$$\begin{aligned} H^i(e^F, e^G) &= \left[D_z^i(e^w)\right]_{z=0} \\ &= \left[D_z^{i-1}(e^w w_z)\right]_{z=0} \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} \left[D_z^{i-j}(w) D_z^j(e^w)\right]_{z=0} \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} \psi_{i-j} H^j(e^F, e^G), \quad i \geq 1, \end{aligned} \tag{5.7.17}$$

where

$$\psi_r = \left[D_z^r(w)\right]_{z=0}$$

$$= D^r\{F(x) + (-1)^r G(x)\}, \quad D = \frac{d}{dx}. \tag{5.7.18}$$

Hence,

$$\begin{aligned}\psi_{2r} &= D^{2r}\log(fg)\\ &= D^{2r}(\phi)\\ &= u_{2r}.\end{aligned}$$

Similarly,

$$\psi_{2r+1} = u_{2r+1}.$$

Hence, $\psi_r = u_r$ for all values of r.

In (5.7.17), put

$$H_i = H^i(e^F, e^G),$$

so that

$$H_0 = e^{F+G}$$

and put

$$\begin{aligned}a_{ij} &= \binom{i-1}{j}\psi_{i-j}, \quad j < i,\\ a_{ii} &= -1.\end{aligned}$$

Then,

$$a_{i0} = \psi_i = u_i$$

and (5.7.17) becomes

$$\sum_{j=0}^{i} a_{ij}H_j = 0, \quad i \geq 1,$$

which can be expressed in the form

$$\begin{aligned}\sum_{j=1}^{i} a_{ij}H_j &= -a_{i0}H_0\\ &= -e^{F+G}u_i, \quad i \geq 1.\end{aligned} \tag{5.7.19}$$

This triangular system of equations in the H_j is similar in form to the triangular system in Section 2.3.5 on Cramer's formula. The solution of that system is given in terms of a Hessenbergian. Hence, the solution of (5.7.19) is also expressible in terms of a Hessenbergian,

$$H_j = e^{F+G}\begin{vmatrix} u_1 & -1 & & & \\ u_2 & u_1 & -1 & & \\ u_3 & 2u_2 & u_1 & -1 & \\ u_4 & 3u_3 & 3u_2 & u_1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n,$$

which, after transposition, is equivalent to the stated result. □

Exercises

1. Prove that

$$\sum_{k=1}^{i} b_{ik} u_k = H_i,$$

where

$$b_{ik} = \binom{i-1}{k-1} H_{i-k}$$

and hence express u_k as a Hessenbergian whose elements are the H_i.

2. Prove that

$$H(A^{is}, A^{rj}) = \sum_{p=1}^{n} \sum_{q=1}^{n} a'_{pq} \begin{vmatrix} A^{iq} & A^{ir,sq} \\ A^{pj} & A^{pr,sj} \end{vmatrix}.$$

5.8 Some Applications of Algebraic Computing

5.8.1 Introduction

In the early days of electronic digital computing, it was possible to perform, in a reasonably short time, long and complicated calculations with real numbers such as the evaluation of π to 1000 decimal places or the evaluation of a determinant of order 100 with real numerical elements, but no system was able to operate with complex numbers or to solve even the simplest of algebraic problems such as the factorization of a polynomial or the evaluation of a determinant of low order with symbolic elements.

The first software systems designed to automate symbolic or algebraic calculations began to appear in the 1950s, but for many years, the only people who were able to profit from them were those who had easy access to large, fast computers. The situation began to improve in the 1970s and by the early 1990s, small, fast personal computers loaded with sophisticated software systems had sprouted like mushrooms from thousands of desktops and it became possible for most professional mathematicians, scientists, and engineers to carry out algebraic calculations which were hitherto regarded as too complicated even to attempt.

One of the branches of mathematics which can profit from the use of computers is the investigation into the algebraic and differential properties of determinants, for the work involved in manipulating determinants of orders greater than 5 is usually too complicated to tackle unaided. Remember that the expansion of a determinant of order n whose elements are monomials consists of the sum of $n!$ terms each with n factors and that many formulas in determinant theory contain products and quotients involving several determinants of order n or some function of n.

Computers are invaluable in the initial stages of an investigation. They can be used to study the behavior of determinants as their orders increase and to assist in the search for patterns. Once a pattern has been observed, it may be possible to formulate a conjecture which, when proved analytically, becomes a theorem. In some cases, it may be necessary to evaluate determinants of order 10 or more before the nature of the conjecture becomes clear or before a previously formulated conjecture is realized to be false.

In Section 5.6 on distinct matrices with nondistinct determinants, there are two theorems which were originally published as conjectures but which have since been proved by Fiedler. However, that section also contains a set of simple isolated identities which still await unification and generalization. The nature of these identities is comparatively simple and it should not be difficult to make progress in this field with the aid of a computer.

The following pages contain several other conjectures which await proof or refutation by analytic methods and further sets of simple isolated identities which await unification and generalization. Here again the use of a computer should lead to further progress.

5.8.2 Hankel Determinants with Hessenberg Elements

Define a Hessenberg determinant H_n (Section 4.6) as follows:

$$H_n = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 & \cdots & h_{n-1} & h_n \\ 1 & h_1 & h_2 & h_3 & \cdots & \cdots & \cdots \\ & 1 & h_1 & h_2 & \cdots & \cdots & \cdots \\ & & 1 & h_1 & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & h_1 \end{vmatrix}_n,$$

$$H_0 = 1. \tag{5.8.1}$$

Conjecture 1.

$$\begin{vmatrix} H_{n+r} & H_{n+r+1} & \cdots & H_{2n+r-1} \\ H_{n+r-1} & H_{n+r} & \cdots & H_{2n+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ H_{r+1} & H_{r+2} & \cdots & H_{n+r} \end{vmatrix}_n = \begin{vmatrix} h_n & h_{n+1} & \cdots & h_{2n+r-1} \\ h_{n-1} & h_n & \cdots & h_{2n+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ h_{1-r} & h_{2-r} & \cdots & h_n \end{vmatrix}_{n+r}.$$

$h_0 = 1$, $h_m = 0$, $m < 0$.

Both determinants are of Hankel form (Section 4.8) but have been rotated through $90°$ from their normal orientations. Restoration of normal orientations introduces negative signs to determinants of orders $4m$ and $4m+1$, $m \geq 1$. When $r = 0$, the identity is unaltered by interchanging H_s and h_s, $s = 1, 2, 3 \ldots$. The two determinants merely change sides. The identities in which $r = \pm 1$ form a dual pair in the sense that one can be transformed into the other by interchanging H_s and h_s, $s = 0, 1, 2, \ldots$.

Examples.

$(n, r) = (2, 0)$:

$$\begin{vmatrix} H_2 & H_3 \\ H_1 & H_2 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 \\ h_1 & h_2 \end{vmatrix};$$

$(n, r) = (3, 0)$:

$$\begin{vmatrix} H_3 & H_4 & H_5 \\ H_2 & H_3 & H_4 \\ H_1 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \end{vmatrix};$$

$(n, r) = (2, 1)$:

$$\begin{vmatrix} H_3 & H_4 \\ H_2 & H_3 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \end{vmatrix};$$

$(n, r) = (3, -1)$:

$$\begin{vmatrix} H_2 & H_3 & H_4 \\ H_1 & H_2 & H_3 \\ 1 & H_1 & H_2 \end{vmatrix} = \begin{vmatrix} h_3 & h_4 \\ h_2 & h_3 \end{vmatrix}.$$

Conjecture 2.

$$\begin{vmatrix} H_n & H_{n+1} \\ 1 & H_1 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 & h_4 & h_5 & \cdots & h_n & h_{n+1} \\ 1 & h_1 & h_2 & h_3 & \cdots & h_{n-2} & h_{n-1} \\ & 1 & h_1 & h_2 & \cdots & h_{n-3} & h_{n-2} \\ & & 1 & h_1 & \cdots & h_{n-4} & h_{n-3} \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & h_1 \end{vmatrix}_n.$$

Note that, in the determinant on the right, there is a break in the sequence of suffixes from the first row to the second.

The following set of identities suggest the existence of a more general relation involving determinants in which the sequence of suffixes from one row to the next or from one column to the next is broken.

$$\begin{vmatrix} H_1 & H_3 \\ 1 & H_2 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 \\ 1 & h_2 \end{vmatrix},$$

$$\begin{vmatrix} H_2 & H_4 \\ H_1 & H_3 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_4 \\ 1 & h_2 & h_3 \\ & h_1 & h_2 \end{vmatrix},$$

$$\begin{vmatrix} H_3 & H_5 \\ H_2 & H_4 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_4 & h_5 \\ 1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3 \\ & 1 & h_1 & h_2 \end{vmatrix},$$

$$\begin{vmatrix} H_2 & H_4 & H_5 \\ H_1 & H_3 & H_4 \\ 1 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} h_2 & h_4 & h_5 \\ h_1 & h_3 & h_4 \\ 1 & h_2 & h_3 \end{vmatrix},$$

$$\begin{vmatrix} H_3 & H_5 \\ H_1 & H_3 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_4 & h_5 \\ 1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3 \\ & & 1 & h_1 \end{vmatrix}. \tag{5.8.2}$$

5.8.3 Hankel Determinants with Hankel Elements

Let

$$A_n = |\phi_{r+m}|_n, \quad 0 \le m \le 2n-2, \tag{5.8.3}$$

which is an Hankelian (or a Turanian).

Let

$$\begin{aligned} B_r &= A_2 \\ &= \begin{vmatrix} \phi_r & \phi_{r+1} \\ \phi_{r+1} & \phi_{r+2} \end{vmatrix}. \end{aligned} \tag{5.8.4}$$

Then B_r, B_{r+1}, and B_{r+2} are each Hankelians of order 2 and are each minors of A_3:

$$\begin{aligned} B_r &= A_{33}^{(3)}, \\ B_{r+1} &= A_{31}^{(3)} = A_{13}^{(3)}, \\ B_{r+2} &= A_{11}^{(3)}. \end{aligned} \tag{5.8.5}$$

Hence, applying the Jacobi identity (Section 3.6),

$$\begin{aligned} \begin{vmatrix} B_{r+2} & B_{r+1} \\ B_{r+1} & B_r \end{vmatrix} &= \begin{vmatrix} A_{11}^{(3)} & A_{13}^{(3)} \\ A_{31}^{(3)} & A_{33}^{(3)} \end{vmatrix} \\ &= A_3 A_{13,13}^{(3)} \\ &= \phi_2 A_3. \end{aligned} \tag{5.8.6}$$

Now redefine B_r. Let $B_r = A_3$. Then, $B_r, B_{r+1}, \ldots, B_{r+4}$ are each second minors of A_5:

$$\begin{aligned} B_r &= A_{45,45}^{(5)}, \\ B_{r+1} &= -A_{15,45}^{(5)} = -A_{45,15}^{(5)}, \\ B_{r+2} &= A_{12,45}^{(5)} = A_{15,15}^{(5)} = A_{45,12}^{(5)}, \\ B_{r+3} &= -A_{12,15}^{(5)} = -A_{15,12}^{(5)}, \\ B_{r+4} &= A_{12,12}^{(5)}. \end{aligned} \tag{5.8.7}$$

Hence,

$$\begin{vmatrix} B_{r+4} & B_{r+3} & B_{r+2} \\ B_{r+3} & B_{r+2} & B_{r+1} \\ B_{r+2} & B_{r+1} & B_r \end{vmatrix} = \begin{vmatrix} A^{(5)}_{12,12} & A^{(5)}_{12,15} & A^{(5)}_{12,45} \\ A^{(5)}_{15,12} & A^{(5)}_{15,15} & A^{(5)}_{15,45} \\ A^{(5)}_{45,12} & A^{(5)}_{45,15} & A^{(5)}_{45,45} \end{vmatrix}. \tag{5.8.8}$$

Denote the determinant on the right by V_3. Then, V_3 is not a standard third-order Jacobi determinant which is of the form

$$|A^{(n)}_{pq}|_3 \quad \text{or} \quad |A^{(n)}_{gp,hq}|_3, \quad p = i, j, k, \quad q = r, s, t.$$

However, V_3 can be regarded as a generalized Jacobi determinant in which the elements have vector parameters:

$$V_3 = |A^{(5)}_{\mathbf{uv}}|_3, \tag{5.8.9}$$

where $\mathbf{u}$ and $\mathbf{v} = [1, 2]$, $[1, 5]$, and $[4, 5]$, and $A^{(5)}_{\mathbf{uv}}$ is interpreted as a second cofactor of A_5. It may be verified that

$$V_3 = A^{(5)}_{125;125} A^{(5)}_{145;145} A_5 + \phi_4 (A^{(5)}_{15})^2 \tag{5.8.10}$$

and that if

$$V_3 = |A^{(4)}_{\mathbf{uv}}|_3, \tag{5.8.11}$$

where $\mathbf{u}$ and $\mathbf{v} = [1, 2]$, $[1, 4]$, and $[3, 4]$, then

$$V_3 = A^{(4)}_{124;124} A^{(4)}_{134;134} A_4 + (A^{(4)}_{14})^2. \tag{5.8.12}$$

These results suggest the following conjecture:

Conjecture. If

$$V_3 = |A^{(n)}_{\mathbf{uv}}|_3,$$

where $\mathbf{u}$ and $\mathbf{v} = [1, 2]$, $[1, n]$, and $[n-1, n]$, then

$$V_3 = A^{(n)}_{12n;12n} A^{(n)}_{1,n-1,n;1,n-1,n} A_n + A^{(n)}_{12,n-1,n;12,n-1,n} (A^{(n)}_{1n})^2.$$

Exercise. If

$$V_3 = |A^{(4)}_{\mathbf{uv}}|,$$

where

$$\begin{aligned} \mathbf{u} &= [1, 2], [1, 3], \text{ and } [2, 4], \\ \mathbf{v} &= [1, 2], [1, 3], \text{ and } [2, 3]. \end{aligned}$$

prove that

$$V_3 = -\phi_5 \phi_6 A_4.$$

5.8.4 Hankel Determinants with Symmetric Toeplitz Elements

The symmetric Toeplitz determinant T_n (Section 4.5.2) is defined as follows:

$$T_n = |t_{|i-j|}|_n,$$

with

$$T_0 = 1. \tag{5.8.13}$$

For example,

$$\begin{aligned}
T_1 &= t_0, \\
T_2 &= t_0^2 - t_1^2, \\
T_3 &= t_0^3 - 2t_0t_1^2 - t_0t_2^2 + 2t_1^2t_2,
\end{aligned} \tag{5.8.14}$$

etc. In each of the following three identities, the determinant on the left is a Hankelian with symmetric Toeplitz elements, but when the rows or columns are interchanged they can also be regarded as second-order subdeterminants of $|T_{|i-j|}|_n$, which is a symmetric Toeplitz determinant with symmetric Toeplitz elements. The determinants on the right are subdeterminants of T_n with a common principal diagonal.

$$\begin{aligned}
\begin{vmatrix} T_0 & T_1 \\ T_1 & T_2 \end{vmatrix} &= -|t_1|^2, \\
\begin{vmatrix} T_1 & T_2 \\ T_2 & T_3 \end{vmatrix} &= -\begin{vmatrix} t_1 & t_0 \\ t_2 & t_1 \end{vmatrix}^2, \\
\begin{vmatrix} T_2 & T_3 \\ T_3 & T_4 \end{vmatrix} &= -\begin{vmatrix} t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 \\ t_3 & t_2 & t_1 \end{vmatrix}^2.
\end{aligned} \tag{5.8.15}$$

Conjecture.

$$\begin{vmatrix} T_{n-1} & T_n \\ T_n & T_{n+1} \end{vmatrix} = -\begin{vmatrix} t_1 & t_0 & t_1 & t_2 & \cdots & t_{n-2} \\ t_2 & t_1 & t_0 & t_1 & \cdots & t_{n-3} \\ t_3 & t_2 & t_1 & t_0 & \cdots & t_{n-4} \\ t_4 & t_3 & t_2 & t_1 & \cdots & t_{n-5} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}_n^2.$$

Other relations of a similar nature include the following:

$$\begin{vmatrix} T_0 & T_1 \\ T_2 & T_3 \end{vmatrix} = \begin{vmatrix} t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 \\ t_2 & t_1 & \end{vmatrix},$$

$$\begin{vmatrix} T_1 & T_2 & T_3 \\ T_2 & T_3 & T_4 \\ T_3 & T_4 & T_5 \end{vmatrix} \quad \text{has a factor} \quad \begin{vmatrix} t_0 & t_1 & t_2 \\ t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \end{vmatrix}. \tag{5.8.16}$$

5.8.5 Hessenberg Determinants with Prime Elements

Let the sequence of prime numbers be denoted by $\{p_n\}$ and define a Hessenberg determinant H_n (Section 4.6) as follows:

$$H_n = \begin{vmatrix} p_1 & p_2 & p_3 & p_4 & \cdots \\ 1 & p_1 & p_2 & p_3 & \cdots \\ & 1 & p_1 & p_2 & \cdots \\ & & 1 & p_1 & \cdots \\ & & & \cdots & \cdots \end{vmatrix}_n .$$

This determinant satisfies the recurrence relation

$$H_n = \sum_{r=0}^{n-1} (-1)^r p_{r+1} H_{n-1-r}, \quad H_0 = 1.$$

A short list of primes and their associated Hessenberg numbers is given in the following table:

n	1	2	3	4	5	6	7	8	9	10
p_n	2	3	5	7	11	13	17	19	23	29
H_n	2	1	1	2	3	7	10	13	21	26

n	11	12	13	14	15	16	17	18	19	20
p_n	31	37	41	43	47	53	59	61	67	71
H_n	33	53	80	127	193	254	355	527	764	1149

Conjecture. The sequence $\{H_n\}$ is monotonic from H_3 onward.

This conjecture was contributed by one of the authors to an article entitled "Numbers Count" in the journal *Personal Computer World* and was published in June 1991. Several readers checked its validity on computers, but none of them found it to be false. The article is a regular one for computer buffs and is conducted by Mike Mudge, a former colleague of the author.

Exercise. Prove or refute the conjecture analytically.

5.8.6 Bordered Yamazaki–Hori Determinants — 2

A bordered determinant W of order $(n+1)$ is defined in Section 4.10.3 and is evaluated in Theorem 4.42 in the same section. Let that determinant be denoted here by W_{n+1} and verify the formula

$$W_{n+1} = -\frac{K_n}{4}(x^2-1)^{n(n-1)}\{(x+1)^n - (x-1)^n\}^2$$

for several values of n. K_n is the simple Hilbert determinant.

Replace the last column of W_{n+1} by the column

$$\left[1\ 3\ 5 \cdots (2n-1)\ \bullet\right]^T$$

and denote the result by Z_{n+1}. Verify the formula

$$Z_{n+1} = -n^2 K_n (x^2-1)^{n^2-2}(x^2-n^2)$$

for several values of n.

Both formulas have been proved analytically, but the details are complicated and it has been decided to omit them.

Exercise. Show that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & x \\ a_{21} & a_{22} & \cdots & a_{2n} & x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x^n \\ 1 & \frac{x}{3} & \cdots & \frac{x^{n-1}}{2n-1} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{2} K_n F\left(n, -n; \tfrac{1}{2}; -x\right),$$

where

$$a_{ij} = \frac{(1+x)^{i+j-1} - x^{i+j-1}}{i+j-1}$$

and where $F(a,b;c;x)$ is the hypergeometric function.

5.8.7 Determinantal Identities Related to Matrix Identities

If $\mathbf{M}_r$, $1 \le r \le s$, denote matrices of order n and

$$\sum_{r=1}^{s} \mathbf{M}_r = \mathbf{0}, \quad s > 2,$$

then, in general,

$$\sum_{r=1}^{s} |\mathbf{M}_r| \ne 0, \quad s > 2,$$

that is, the corresponding determinantal identity is *not* valid. However, there are nontrivial exceptions to this rule.

Let $\mathbf{P}$ and $\mathbf{Q}$ denote arbitrary matrices of order n. Then

1. a. $(\mathbf{PQ}+\mathbf{QP}) + (\mathbf{PQ}-\mathbf{QP}) - 2\mathbf{PQ} = \mathbf{0}$, all n,
b. $|\mathbf{PQ}+\mathbf{QP}| + |\mathbf{PQ}-\mathbf{QP}| - |2\mathbf{PQ}| = 0$, $n=2$.
2. a. $(\mathbf{P}-\mathbf{Q})(\mathbf{P}+\mathbf{Q}) - (\mathbf{P}^2-\mathbf{Q}^2) - (\mathbf{PQ}-\mathbf{QP}) = \mathbf{0}$, all n,
b. $|(\mathbf{P}-\mathbf{Q})(\mathbf{P}+\mathbf{Q})| - |\mathbf{P}^2-\mathbf{Q}^2| - |\mathbf{PQ}-\mathbf{QP}| = 0$, $n=2$.
3. a. $(\mathbf{P}-\mathbf{Q})(\mathbf{P}+\mathbf{Q}) - (\mathbf{P}^2-\mathbf{Q}^2) + (\mathbf{PQ}+\mathbf{QP}) - 2\mathbf{PQ} = \mathbf{0}$, all n,
b. $|(\mathbf{P}-\mathbf{Q})(\mathbf{P}+\mathbf{Q})| - |\mathbf{P}^2-\mathbf{Q}^2| + |\mathbf{PQ}+\mathbf{QP}| - |2\mathbf{PQ}| = 0$, $n=2$.

The matrix identities 1(a), 2(a), and 3(a) are obvious. The corresponding determinantal identities 1(b), 2(b), and 3(b) are not obvious and no neat proofs have been found, but they can be verified manually or on a computer. Identity 3(b) can be obtained from 1(b) and 2(b) by eliminating $|\mathbf{PQ}-\mathbf{QP}|$.

It follows that there exist at least two solutions of the equation

$$|\mathbf{X}+\mathbf{Y}| = |\mathbf{X}| + |\mathbf{Y}|, \quad n = 2,$$

namely

$$\begin{aligned}\mathbf{X} &= \mathbf{PQ}+\mathbf{QP} \quad \textit{or} \quad \mathbf{P}^2-\mathbf{Q}^2,\\ \mathbf{Y} &= \mathbf{PQ}-\mathbf{QP}.\end{aligned}$$

Furthermore, the equation

$$|\mathbf{X}-\mathbf{Y}+\mathbf{Z}| = |\mathbf{X}| - |\mathbf{Y}| + |\mathbf{Z}|, \quad n = 2,$$

is satisfied by

$$\begin{aligned}\mathbf{X} &= \mathbf{P}^2-\mathbf{Q}^2,\\ \mathbf{Y} &= \mathbf{PQ}+\mathbf{QP},\\ \mathbf{Z} &= 2\mathbf{PQ}.\end{aligned}$$

Are there any other determinantal identities of a similar nature?