CHAPTER 6

Laws of Growth, Utility, and Finite Streams

Since this book deals with the mathematics involving growth, we must discuss the laws of growth. When dealing with growth in mathematical terms, we can discuss it in terms of growth functions or of the corresponding growth rates.

We can speak of growth functions as falling into three distinct categories, where each category is associated with a growth *rate*. Figure 6.1 portrays these three categories as lines B, C, and D, and their growth rates as A, B, and C, respectively. Each growth function has its growth rate immediately to its left.

Thus, for growth function B, the linear growth function, its growth rate is line A. Further, although B is a growth function itself, it also represents the growth rate for function C, the exponential growth rate.

Notice that there are three growth functions, *linear*, *exponential*, and *hyperbolic*. Thus, the hyperbolic growth function has an exponential growth rate, the exponential growth function has a linear growth rate, and the linear growth function has a flat-line growth rate.

The X and Y-axes are important here. If we are discussing growth functions (B, C, or D), the Y-axis represents quantity and the X-axis represents time. If we are discussing growth rates, the Y-axis represents quantity change with respect to time, and the X-axis represents quantity.

When we speak of growth rates and functions in general, we often speak of the growth of a population of something. The first of the three major growth functions is the linear growth function, line B, and its rate, line A. Members of a population characterized by linear growth tend to easily find a level of coexistence.

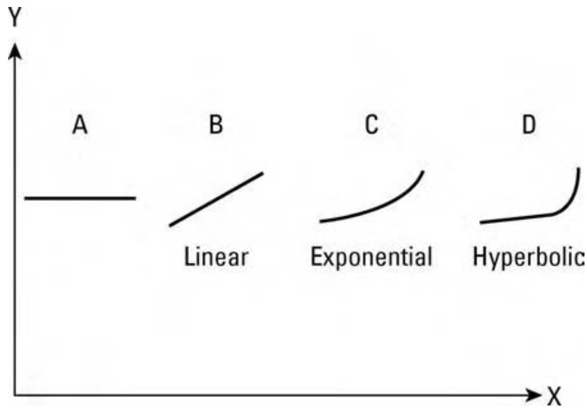

FIGURE 6.1 The three growth functions

Next, we have the exponential growth function, line C, and its growth rate, which is linear, line B. Here, we find competition among the members of the population and a survival-of-the-fittest principle setting in. In the exponential growth function, however, it is possible for a mutation to appear, which has a selective advantage, and establish itself.

Finally, in the hyperbolic growth function, line D, and its (exponential) growth rate, line C, we find a different story. Unlike the exponential growth function, which has a linear growth rate, this one's growth rate is itself exponential. That is, the greater the quantity, the faster the growth rate! Thus, the hyperbolic function, unlike the exponential function, reaches a point that we call a *singularity*. That is, it reaches a point where it becomes infinitely large, a vertical asymptote. This is not true of the exponential growth function, which simply becomes larger and larger. In the hyperbolic function, we also find competition among the members of the population, and a survival-of-the-fittest characteristic. However, at a certain point in the evolution of a hyperbolic function, it becomes nearly impossible for a mutation with a selective advantage to establish itself, since the rest of the population is growing at such a rapid rate.

In either the exponential or hyperbolic growth functions, if there are functional links between the competing species within the population, it can cause any of the following:

1. Increased competition among the partners; or
2. Mutual stabilization among the partners; or
3. Extinction of all members of the population.

The notion of populations is also a recurring theme throughout this book, and it is nearly impossible to discuss the mathematics of growth without discussing populations. The mathematics of growth is the corpus callosum between population growth and the new framework presented in this book.

Trading is exponential, *not* hyperbolic. However, if you had someone who would give you money to trade if your performance came in as promised, and that person had virtually unlimited funds, then your trading would be hyperbolic. This sounds like managed money. The problem faced by money managers is the caveat laid on the money manager by the individual of unlimited wealth: *if your performance comes in as promised*. In the later chapters in this book, we will discuss techniques to address this caveat.

MAXIMIZING EXPECTED AVERAGE COMPOUND GROWTH

Thus far, in this book, we have looked at finding a value for *f* that was asymptotically dominant. That is, we have sought a single value for *f* for a given market system, which, if there truly was independence between the trades, would maximize geometric growth with certainty as the number of trades (or holding periods) approached infinity. That is, we would end up with greater wealth in the very long run, with a probability that approached certainty, than we would using any other money management strategy.

Recall that if we have only one play, we maximize growth by maximizing the arithmetic average holding period return (i.e., $f = 1$). If we have an infinite number of plays, we maximize growth by maximizing the geometric average holding period return (i.e., $f =$ optimal *f*). However, *the f that is truly optimal is a function of the length of time—the number of finite holding period returns—that we are going to play*.

For one holding period return, the optimal *f* will always be 1.0 for a positive arithmetic mathematical expectation game. If we bet at any value for *f* other than 1.0, and quit after only one holding period, we will not have maximized our expected average geometric growth. What we regard as the optimal *f* would be optimal only if you were to play for an infinite number of holding periods. The *f* that is truly optimal starts at one for a positive arithmetic mathematical expectation game, and converges toward what we call the optimal *f* as the number of holding periods approaches infinity.

To see this, consider again our two-to-one coin-toss game, where we have determined the optimal *f* to be .25. That is, if the coin tosses are independent of previous tosses, by betting 25% of our stake on each and every play, we will maximize our geometric growth with certainty as the length of this game, the number of tosses (i.e., the number of holding periods) approaches infinity. That is, our expected average geometric growth—what we would expect to end up with, as an expected value, given every possible combination of outcomes—would be greatest if we bet 25% per play.

Consider the first toss. There is a 50% probability of winning \$2 and a 50% probability of losing \$2. At the second toss, there is a 25% chance of winning \$2 on the first toss and winning \$2 on the second, a 25% chance of winning \$2 on the first and losing \$1 on the second, a 25% chance of losing \$1 on the first and winning \$2 on the second, and a 25% chance of losing \$1 on the first and losing \$1 on the second (we know these probabilities to be true because we have already stated the prerequisite that these events are independent). The combinations bloom out in time in a tree-like fashion. Since we had only two scenarios (heads and tails) in this scenario spectrum, there are only two branches off of each node in the tree. If we had more scenarios in this spectrum, there would be that many more branches off of each node in this tree:

Toss#		
1	**2**	**3**
		Heads
	Heads	
		Tails
Heads		
		Heads
	Tails	
		Tails
		Heads
	Heads	
		Tails
Tails		
		Heads
	Tails	
		Tails

If we bet 25% of our stake on the first toss and quit, we will not have maximized our expected average compound growth (EACG).

What if we quit after the second toss? What, then, should we optimally bet, knowing that we maximize our expected average compound gain by betting at $f = 1$ when we are going to quit after one play, and betting at the optimal f if we are going to play for an infinite length of time?

If we go back and optimize f, allowing there to be a different f value used for the first play as well as the second play, with the intent of maximizing what our average geometric mean HPR would be at the end of the second play, we would find the following: First, the optimal f for quitting after two plays in this game approaches the asymptotic optimal, going from 1.0 if we quit after one play to .5 for both the first play and the second. That is, if we were to quit after the second play, we should optimally bet .5 on both the first and second plays to maximize growth. (Remember, we allowed for the first play to be an *f* value different from the second, yet they both came out the same: .5 in this case. It is a fact that if you are looking to maximize growth, the *f* that is optimal—for finite as well as infinite streams—is uniform.)

We can see this if we take the first two possible combinations of tosses:

Toss#	
1	**2**
	Heads
Heads	
	Tails
	Heads
Tails	
	Tails

Which can be represented by the following outcomes:

Toss#	
1	**2**
	2
2	
	−1
	2
−1	
	−1

These outcomes can be expressed as holding period returns for various *f* values. In the following, it is shown for an *f* of .5 for the first toss, as well as for an *f* of .5 for the second:

Toss#	
1	**2**
	2
2	
	.5
	2
.5	
	.5

Now, we can express all tosses subsequent to the first toss as TWRs by multiplying by the subsequent tosses on the tree. The numbers following the last toss on the tree (the numbers in parentheses) are the last TWRs taken to the root of $1/n$, where n equals the number of HPRs, or tosses—in this case two—and represents the geometric mean HPR for that terminal node on the tree:

Toss#	
1	**2**
	4 (2.0)
2	
	1 (1.0)
	1 (1.0)
.5	
	.25 (.5)

Now, if we total up the geometric mean HPRs and take their arithmetic average, we obtain the *expected average compound growth*, in this case:

$$\begin{array}{r} 2.0 \\ 1.0 \\ 1.0 \\ .5 \\ \hline \end{array}$$

$$\frac{4.5}{4} = 1.125$$

Thus, if we were to quit after two plays, and yet do this same thing over an infinite number of times (i.e., quit after two plays), we would optimally bet .5 of our stake on each and every play, thus maximizing our EACG.

Notice that we did not bet with an f of 1.0 on the first play, even though that is what would have maximized our EACG if we had quit at one play. Instead, if we are planning on quitting after two plays, we maximize our EACG growth by betting at .5 on both the first play and the second play.

Notice that the f that is optimal in order to maximize growth is uniform for all plays, yet it is a function of how long you will play. If you are to quit after only one play, the f that is optimal is the f that maximizes the arithmetic mean HPR (which is always an f of 1.0 for a positive expectation game, 0.0 for a negative expectation game). If you are playing a positive expectation game, the f that is optimal continues to decrease as the length of time after which you quit grows, and, asymptotically, if you play for an infinitely long time, the *f* that is optimal is that which maximizes the geometric mean HPR. In a negative expectation game, the *f* that is optimal simply stays at zero.

However, the *f* that you use to maximize growth is always uniform, and that uniform amount is a function of where you intend to quit the game. If you are playing the two-to-one coin-toss game, and you intend to quit after one play, you have an *f* value that provides for optimal growth of 1.0. If you intend to quit after two plays, you have an *f* that is optimal for maximizing growth of .5 on the first toss and .5 on the second. Notice that you do not bet 1.0 on the first toss if you are planning on maximizing the EACG by quitting at the end of the second play. Likewise, if you are planning on playing for an infinitely long period of time, you would optimally bet .25 on the first toss and .25 on each subsequent toss.

Note the key word there is *infinitely*, not *indefinitely*. All streams are finite—we are all going to die eventually. Therefore, when we speak of the optimal *f* as the *f* that maximizes expected average compound return, we are speaking of that value which maximizes it if played for an infinitely long period of time. Actually, it is slightly suboptimal because none of us will be able to play for an infinitely long time. And, the *f* that will maximize EACG will be slightly above—will have us take slightly heavier positions—than what we are calling the optimal *f*.

What if we were to quit after three tosses? Shouldn't the *f* which then maximizes expected average compound growth be lower still than the .5 it is when quitting after two plays, yet still be greater than the .25 optimal for an infinitely long game?

Let's examine the tree of combinations here:

Toss#		
1	**2**	**3**
		Heads
	Heads	
		Tails
Heads		
		Heads
	Tails	
		Tails
		Heads
	Heads	
		Tails
Tails		
		Heads
	Tails	
		Tails

Converting these to outcomes yields:

Toss#		
1	**2**	**3**
		2
	2	
		−1
2		
		2
	−1	
		−1
		2
	2	
		−1
−1		
		2
	−1	
		−1

If we go back with a computer and iterate to that value for f which maximizes expected average compound growth when quitting after three tosses, we find it to be .37868. Therefore, converting the outcomes to HPRs based upon a .37868 value for f at each toss yields:

Toss#		
1	**2**	**3**
		1.757369
	1.757369	
		.621316
1.757369		
		1.757369
	.621316	
		.621316
		1.757369
	1.757369	
		.621316
.621316		
		1.757369
	.621316	
		.621316

Now we can express all tosses subsequent to the first toss as TWRs by multiplying by the subsequent tosses on the tree. The numbers following the last toss on the tree (the numbers in parentheses) are the last TWRs taken to the root of $1/n$, where n equals the number of HPRs, or tosses, in this case three, and represent the geometric mean HPR for that terminal node on the tree:

Toss#

1	2	3	
		5.427324 (1.757365)	
	3.088329		
		1.918831 (1.242641)	
1.757369			
		1.918848 (1.242644)	
	1.09188		
		.678409 (.87868)	
		1.918824 (1.242639)	
	1.091875		
		.678401 (.878676)	
.621316			
		.678406 (.878678)	
	.386036		
		.239851 (.621318)	
		8.742641 / 8	= 1.09283 is the expected average compound growth (EACG)

If you are the slightest bit skeptical of this, I suggest you go back over the last few examples, either with pen and pencil or computer, and find a value for f that results in a greater EACG than the values presented. Allow yourself the liberty of a nonuniform f—that is, an f that is allowed to change at each play. You'll find that you get the same answers as we have, and that f is uniform, although a function of the length of the game.

From this, we can summarize the following conclusions:

1. To maximize the EACG, we always end up with a uniform f. That is, the value for f is uniform from one play to the next.
2. The f that is optimal in terms of maximizing the EACG is a function of the length of the game. For positive expectation games, it starts at 1.0, the value that maximizes the arithmetic mean HPR, diminishes slightly each play, and asymptotically approaches that value which maximizes the geometric mean HPR (which we have been calling—and will call throughout the sequel—the optimal f).

3. Since all streams are finite in length, regardless of how long, we will always be ever-so-slightly suboptimal by trading at what we call the optimal *f*, regardless of how long we trade. Yet, the difference diminishes with each holding period. Ultimately, we are to the left of the peak of what was truly optimal. This is not to say that everything mentioned about the $n + 1$ dimensional landscape of leverage space, to be discussed later in the text—the penalties and payoffs of where you are with respect to the optimal *f* for each market system—aren't true. It is true, however, that the landscape is a function of the number of holding periods at which you quit. The landscape we project with the techniques in this book is the asymptotic altitudes—what the landscape approaches as we continue to play.

To see this, let's continue with our two-to-one coin toss. In the graph (Figure 6.2), we can see the value for *f*, which optimally maximizes our EACG for quitting at one play through eight plays. Notice how it approaches the optimal *f* of .25, the value that maximizes growth asymptotically, as the number of holding periods approaches infinity.

Two-to-One Coin-Toss Game

Quitting after HPR #	*f* that Maximizes EACG
1	1.0
2	.5
3	.37868
4	.33626
5	.3148
6	.3019
7	.2932
8	.2871
.	.
.	.
.	.
Infinity	.25 (this is the value we refer to as the optimal *f*)

In reality, if we trade with what we are calling in this text the optimal *f*, we will always be slightly suboptimal, the degree of which diminishes as more and more holding periods elapse. If we knew exactly how many holding periods we were going to trade for, we could then use that value for *f* which maximizes EACG (which would be slightly greater than the optimal *f*) and be truly optimal. Unfortunately, we rarely know exactly how many holding periods we are going to play for, and there is consolation in

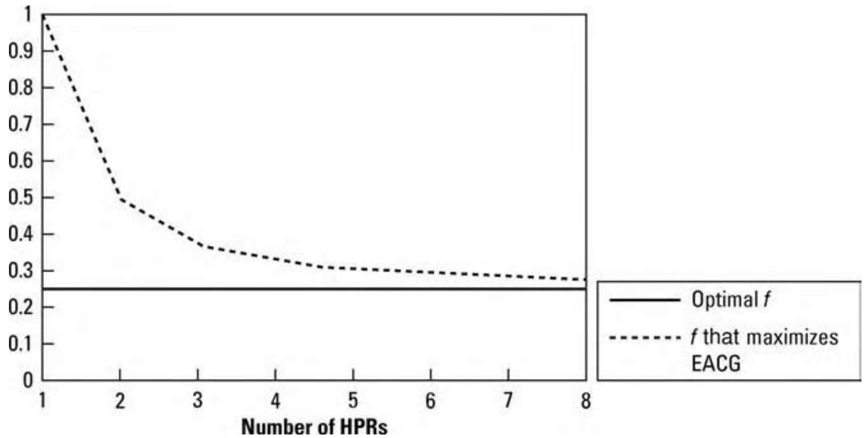

FIGURE 6.2 Optimal *f* as an asymptote

the fact that what we are calling the optimal *f* approaches what would be optimal to maximize EACG as more holding periods elapse. Later we will see the *continuous dominance* techniques, which allow us to approximate the notion of maximizing EACG when there is an active/inactive equity split (i.e., anytime someone is trading less aggressively than optimal *f*).

Note that none of these notions is addressed or even alluded to in the older mean-variance, risk-return models, which are next to be discussed in the beginning of the following chapter. The older models disregard leverage and its workings almost entirely. This is one more reason to opt for the new model of portfolio construction to be mentioned later in the text.

UTILITY THEORY

The discussion of utility theory is brought up in this book since, oftentimes, geometric mean maximizers are criticized for being able to maximize only the ln case of utility; that is, they seek to maximize only wealth, not investor satisfaction. This book attempts to show that geometric mean maximization can be applicable, regardless of one's utility preference function. Therefore, we must, at this point, discuss utility theory, in general, as a foundation.

Utility theory is often attacked as being an ivory-tower, academic construct to explain investor behavior. Unfortunately, most of these attacks come from people who have made the a priori assumption that all investor utility functions are ln; that is, they seek to maximize wealth. While this author is not a great proponent of utility theory, I accept it for lack of a better explanation for investor preferences. However, I strongly feel that if an investor's utility function is other than ln, the markets, and investing in general, are poor places to deal with this or to try to maximize one's utility—you're on the n + 1 dimensional landscape to be discussed in Chapter 9 regardless of your utility preference curve, and you will pay the consequences in real currency for being suboptimal. In short, the markets are a bad place to find out you are not a wealth maximizer. The psychiatrist's couch may be a more gentle environment in which to deal with that.

THE EXPECTED UTILITY THEOREM

A guy in an airport has $500, but needs $600 for a ticket he *must* have. He is offered a bet with a 50% probability of winning $100 and a 50% probability of losing $500. Is this a good bet? In this instance, where we assume it to be a life-and-death situation where he must have the ticket, it *is* a good bet.

The mathematical expectation of utility is vastly different in this instance than the mathematical expectation of wealth. Since, if we subscribe to utility theory, we determine *good bets* based on their mathematical expectation of *utility* rather than *wealth*, we assume that the mathematical expectation of utility in this instance is positive, even though wealth is not. Think of the words *utility* and *satisfaction* as meaning the same thing in this discussion.

Thus, we have what is called the *Expected Utility Theorem*, which states that *investors possess a utility-of-wealth function*, U(x), *where* x *is wealth, that they will seek to maximize. Thus, investors will opt for those investment decisions that maximize their utility-of-wealth function.* Only when the utility preference function $U(x) = \ln x$, that is, when the utility, or satisfaction, of wealth equals the wealth, will the expected utility theorem yield the same selection as wealth maximization.

CHARACTERISTICS OF UTILITY PREFERENCE FUNCTIONS

There are five main characteristics of utility preference functions:

1. Utility functions are unique up to a positive linear transformation. Thus, a utility preference function, such as the preceding one, $\ln x$, will lead to the same investments being selected as a utility function of $25 + \ln x$, as it would be a utility function of $71 * \ln x$ or one of the form $(\ln x)/1.453456$. That is, a utility function that is affected by a positive

constant (added, subtracted, multiplied, or divided) will result in the same investments being selected. Thus, it will lead to the same set of investments maximizing utility as before the positive constant affects the function.

2. More is preferred to less. In economic literature, this is often referred to as *nonsatiation.* In other words, a utility function must never result in a choice for less wealth over more wealth when the outcomes are certain or their probabilities equal. Since utility must, therefore, increase as wealth increases, the first derivative of utility, with respect to wealth, must be positive. That is:

$$U'(x) >= 0 \tag{6.01}$$

Given utility as the vertical axis and wealth as the horizontal axis, then the utility preference curve must never have a negative slope.

The ln x case of utility preference functions shows a first derivative of x^{-1}.

3. There are three possible assumptions regarding an investor's feelings toward risk, also called his *risk aversion.* He is either averse to, neutral to, or seeks risk. These can all be defined in terms of a fair gamble. If we assume a fair game, such as coin tossing, winning \$1 on heads and losing \$1 on tails, we can see that the arithmetic expectation of wealth is zero. A risk-averse individual would not accept this bet, whereas a risk seeker would accept it. The investor who is risk-neutral would be indifferent to accepting this bet.

Risk aversion pertains to the second derivative of the utility preference function, or $U''(x)$. A risk-averse individual will show a negative second derivative, a risk seeker a positive second derivative, and one who is risk-neutral will show a zero second derivative of the utility preference function.

Figure 6.3 depicts the three basic types of utility preference functions, based on $U''(x)$, the investor's level of risk aversion. The ln x case of utility preference functions shows neutral risk aversion. The investor is indifferent to a fair gamble.[1] The ln x case of utility preference functions shows a second derivative of $-x^{-2}$.

[1]Actually, investors should reject a fair gamble. Since the amount of money an investor has to work with is finite, there is a lower absorbing barrier. It can be shown that if an investor accepts fair gambles repeatedly, it is simply a matter of time before the lower absorbing barrier is met. That is, if you keep on accepting fair gambles, eventually you will go broke with a probability approaching certainty.

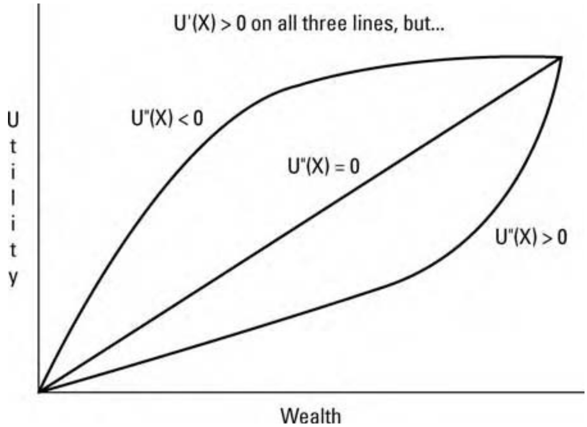

FIGURE 6.3 Three basic types of utility functions

4. The fourth characteristic of utility preference functions pertains to how an investor's levels of risk aversion change with changes in wealth. This is referred to as *absolute risk aversion.* Again, there are three possible categories. First is the individual who exhibits increasing absolute risk aversion. As wealth increases, he holds fewer dollars in risky assets. Next is the individual with constant absolute risk aversion. As his wealth increases, he holds the same dollar amount in risk assets. Last is the individual who displays decreasing absolute risk aversion. As this individual's wealth increases, he is willing to hold more dollars in risky assets.

   The mathematical formulation for defining absolute risk aversion, $A(x)$, is as follows:

$$A(x) = \frac{-U''(x)}{U'(x)} \tag{6.02}$$

   Now, if we want to see how absolute risk aversion changes with a change in wealth, we would take the first derivative of $A(x)$ with respect to x (wealth), obtaining $A'(x)$. Thus, an individual with increasing absolute risk aversion would have $A'(x) > 0$, constant absolute risk aversion would see $A'(x) = 0$, and decreasing absolute risk aversion has $A'(x) < 0$.

   The $\ln x$ case of utility preference functions shows *decreasing* absolute risk aversion. For the $\ln x$ case:

$$A(x) = \frac{-(-x^{-2})}{x^{-1}} = x^{-1} \qquad \text{and} \qquad A'(x) = -x^{-2} < 0$$

5. The fifth characteristic of utility preference functions pertains to how the percentage of wealth invested in risky assets changes with changes in wealth. This is referred to as *relative risk aversion.* That is, this pertains to how your percentages in risky assets change, rather than how your dollar amounts change, with respect to changes in wealth. Again, there are three possible categories: increasing, constant, and decreasing relative risk aversion, where the percentages invested in risky assets increase, stay the same, or decline, respectively.

   The mathematical formulation for defining relative risk aversion, $R(x)$, is as follows:

$$R(x) = \frac{(-x * U''(x))}{U'(x)} = x * A(x) \tag{6.03}$$

   Therefore, $R'(x)$, the first derivative of relative risk aversion, indicates how relative risk aversion changes with respect to changes in wealth. So, individuals who show increasing, constant, or decreasing relative risk aversion will then show positive, zero, and negative $R'(x)$, respectively.

   The $\ln x$ case of utility preference functions shows *constant* relative risk aversion. For the $\ln x$ case:

$$R(x) = \frac{(-x^*(-x^{-2}))}{x^{-1}} = 1 \qquad \text{and} \qquad R'(x) = 0$$

ALTERNATE ARGUMENTS TO CLASSICAL UTILITY THEORY

Readers should be aware that utility theory, although broadly accepted, is not universally accepted as an explanation of investor behavior. For example, R. C. Wentworth contends, with reference to the Expected Utility Theorem, that the use of the mean is an ad hoc, unjustified assumption. His theory is that players assume that the mode, rather than the mean, will prevail, and will act to maximize this.

I personally find Wentworth's work in this area particularly interesting.[2] There are some rather interesting aspects to these papers. First, classical utility theory is directly attacked, which automatically alienates every professor in every management science department in the world. The theoretical foundation paradigm of the nonlinear *utility-of-wealth* function is sacred to these people. Wentworth draws parallels between *mode*

[2]See "Utility, Survival, and Time: Decision Strategies under Favorable Uncertainty," and "A Theory of Risk Management under Favorable Uncertainty," both by R. C. Wentworth, unpublished. 8072 Broadway Terrace, Oakland, CA 94611.

maximizers and evolution; hence, Wentworth calls his the *survival hypothesis*. A thumbnail sketch of the comparison with classical utility theory would appear as:

Utility Theory

"One-shot," risky decision making	+	Nonlinear utility-of-wealth function	=>	Observed behavior

Survival Hypothesis

"One-shot," risky decision making	+	Expansion into equivalent time series	=>	Identical observed behavior

Furthermore, there are some interesting experiments in biology that tend to support Wentworth's ideas, which ask the question why, for instance, should bumblebees search for nectar, in a controlled experiment, according to the dictates of classical utility theory?

So, why mention classical utility theory at all? It is not the purpose of this book to presuppose anything regarding utility theory. However, there is an interrelationship between utility and this new framework in asset allocation, and if one does subscribe to a utility framework notion, then they will be shown how this applies. This portion of the book is directed toward those readers unfamiliar with the notion of utility preference curves. However, it does not take a position on the validity of utility functions, and the reader should be made aware that there are other non-utility-based criteria that may explain investor behavior.

FINDING YOUR UTILITY PREFERENCE CURVE

Whether one subscribes to classical utility theory, considering that it is better to know yourself than not know yourself, we will now detail a technique for determining your own utility preference function. What follows is an adaptation from *The Commodity Futures Game, Who Wins? Who Loses? Why?* by Tewles, Harlow, and Stone.[3]

To begin with, you should determine two extreme values, one positive and the other negative, which should represent extreme trade outcomes. Typically, you should make this value be three to five times greater than the largest amounts you would typically expect to win or lose on the next trade.

[3] Richard J. Tewles, Charles V. Harlow, and Herbert L. Stone, *The Commodity Futures Game, Who Wins? Who Loses? Why?* New York: McGraw-Hill Book Company, 1977.

Let's suppose you expect, in the best case, to win $5,000 on a trade, and lose $3,000. Thus, we can make our extremes $20,000 on the upside and −$10,000 on the downside.

Next, set up a table as follows, with a leftmost column called *Probabilities of Best Outcome*, and give it 10 rows with values progressing from 1.0 to 0 by increments of .1. Your next column should be called *Probabilities of Worst Outcome*, and those probabilities are simply 1 minus the probabilities of the best outcome on that row. The third column will be labeled *Certainty Equivalent*. In the first row, you will put the value of the best outcome, and in the last row, the value of the worst outcome. Thus, your table should look like this:

P (Best Outcome)	P (Worst Outcome)	Certainty Equivalent	Computed Utility
1.0	0	$20,000	
.9	.1		
.8	.2		
.7	.3		
.6	.4		
.5	.5		
.4	.6		
.3	.7		
.2	.8		
.1	.9		
0	1.0	−$10,000	

Now, we introduce the notion of *certainty equivalents*. A certainty equivalent is an amount you would accept in lieu of a trading opportunity or an amount you might pay to sidestep a trade opportunity.

You should now fill in column three, the certainty equivalents. For the first row, the one where we entered $20,000, this simply means you would accept $20,000 in cash right now, rather than take a trade with a 100% probability of winning $20,000. Likewise, with the last row where we have filled in $10,000, this simply means you would be willing to pay $10,000 not to have to take a trade with a 100% chance of losing $10,000.

Now, on the second row, you must determine a certainty equivalent for a trade with a 90% chance of winning $20,000 and a 10% chance of losing $10,000. What would you be willing to accept in cash instead of taking this trade? Remember, this is real money with real buying power, and the rewards or consequences of this transaction will be immediate and in cash. Let's suppose it's worth $15,000 to you. That is, for $15,000 in cash, handed to you right now, you will forego this opportunity of a 90% chance of winning $20,000 and 10% chance of losing $10,000.

You should complete the table for the certainty equivalent columns. For instance, when you are on the second to last row, you are, in effect, asking yourself how much you would be willing to pay not to have to accept a 10% chance of winning $20,000 with a 90% chance of losing $10,000. Since you are willing to pay, you should enter this certainty equivalent as a negative amount.

When you have completed the third column, you must now calculate the fourth column, the *Computed Utility* column. The formula for computed utility is simply:

$$\text{Computed Utility} = U * P(\text{best outcome}) + V * P(\text{worst outcome}) \tag{6.04}$$

where: U = Given constant, equal to 1.0 in this instance.
V = Given constant, equal to -1.0 in this instance.

Thus, for the second row in the table:

$$\begin{aligned}\text{Computed utility} &= 1 * .9 - 1 * .1\\ &= .9 - .1\\ &= .8\end{aligned}$$

When you are finished calculating the computed utility columns, your table might look like this:

P (Best Outcome)	P (Worst Outcome)	Certainty Equivalent	Computed Utility
1.0	0	20,000	1.0
.9	.1	15,000	.8
.8	.2	10,000	.6
.7	.3	7,500	.4
.6	.4	5,000	.2
.5	.5	2,500	0
.4	.6	800	−.2
.3	.7	−1,500	−.4
.2	.8	−3,000	−.6
.1	.9	−4,000	−.8
0	1.0	−10,000	−1.0

You then graphically plot the certainty equivalents as the X-axis and the computed utilities as the Y-axis. Our completed utility function looks as is shown in Figure 6.4.

Now you should repeat the test, only with different best and worst outcomes. Select a certainty equivalent from the preceding table to act as *best*

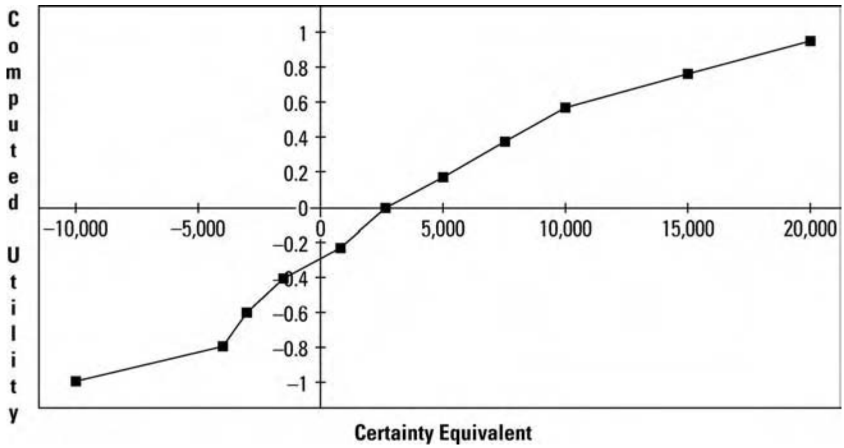

FIGURE 6.4 Example utility function

outcome, and one for *worst outcome* as well. Suppose we choose \$10,000 and −\$4,000. Notice that the computed utilities associated with certainty equivalents are .6 with \$10,000 and −.8 with −\$4,000. Thus, *U* and *V*, in determining computed utilities in this next table, will be .6 and −.8, respectively. Again, assign certainty equivalents and calculate the corresponding computed utilities:

P (Best Outcome)	P (Worst Outcome)	Certainty Equivalent	Computed Utility
1.0	0	10,000	.6
.9	.1	8,000	.46
.8	.2	6,000	.32
.7	.3	5,000	.18
.6	.4	4,000	.04
.5	.5	2,500	−.10
.4	.6	500	−.24
.3	.7	−1,000	−.38
.2	.8	−2,000	−.52
.1	.9	−3,000	−.66
0	1.0	−4,000	−.80

And, again, you should plot these values. You should repeat the process a few times, and keep plotting all the values on the same chart. What you will probably begin to see is some scattering to the values; that is, they will not all neatly fit on the same line. The scattering of these values reveals information about yourself, in that the scattering represents inconsistencies in your decisions. Usually, scattering is more pronounced near the extremes (left and right) of the chart. This is normal and simply indicates areas where you have probably not had a lot of experience winning and losing money.

The shape of the curve is also important, and should be looked at with respect to the earlier section entitled "Characteristics of Utility Preference Functions." It is not at all uncommon for the curve to be imperfect, not simply the textbook concave-up, concave-down, or straight-line shape. Again, this reveals information about yourself, and warrants careful analysis.

Ultimately, the most conducive form of utility preference function for maximizing wealth is a straight line pointing upwards, decreasing absolute risk aversion, constant relative risk aversion, and near indifference to a fair gamble; i.e., we are indifferent to a gamble with anything less than the very slightest positive arithmetic mathematical expectation. If your line is anything less than this, then this may be the time for you to reflect upon what you want as well as *why*, and perhaps make necessary personal changes.

UTILITY AND THE NEW FRAMEWORK

This book does not take a stand regarding utility theory, other than this: Regardless *of your utility preference curve, you are somewhere in the leverage space, described later in the text, of Figure 9.2 for individual games, and somewhere in the* $n + 1$ *dimensional leverage space for multiple simultaneous games, and you reap the benefits of this as well as pay the consequences* no matter what your utility preference.

Oftentimes, the geometric mean criterion is criticized as it only strives to maximize wealth, and it maximizes utility only for the ln function.

Actually, if someone does not subscribe to an ln utility preference function, they can still maximize utility much as we are maximizing wealth with optimal f, except they will have a different value for optimal f at each holding period. That is, if someone's utility preference function is other than ln (wealth maximization), then their optimal f to (asymptotically) maximize utility is uniform, while at the same time, their optimal f to maximize wealth is nonuniform. In other words, if, as you make more money, your utility is such that you are willing to risk less, then your optimal f will decrease as each holding period elapses.

Do not get this confused with the notion, presented earlier, that the f that is optimal for maximizing expected average compound growth is a function of the number of holding periods at which you quit. It still is, but the idea presented here is that the f that is optimal to maximize utility is not uniform throughout the time period. For example, we have seen in our two-to-one coin toss game that if we were planning on quitting after three plays, three holding periods, we would maximize growth by betting .37868 on each and every play. That is, we uniformly bet .37868 for all three plays.

Now, if we're looking to maximize utility, and our utility function were other than that of maximizing wealth, we would not have a uniform *f* value to bet on each and every play. Rather, we would have a different *f* value to bet on each and every play.

Thus, it is possible to maximize utility with the given approach (for utility preference functions other than ln), provided you use a *nonuniform* value for *f* from one holding period to the next. When utility preference is ln—that is, when one prefers wealth maximization—the *f* that is optimal is always uniform. Thus, the optimal *f* is the same from one play to the next. When utility preference is other than ln, wealth maximization, a nonuniform optimal *f* value from one holding period to the next is called for.

Like maximizing wealth, utility can also be maximized in the very same fashion that we are maximizing wealth. This can be accomplished by assigning *utils*, rather than a dollar value for the outcomes to each scenario. A util is simply a unit of satisfaction. The scenario set must also contain negative util scenarios, just as in wealth maximization, you must have a scenario that encompasses losing money. Also, the (arithmetic) mathematical expectation of the scenario set must be positive in terms of utils, or negative if it improves the overall mix of components.

But, how do you determine the nonuniform value for *f* as you go through holding periods when your utility preference curve is other than ln? As each new holding period is encountered, and you update the outcome values (specified in utils) as your account equity itself changes, you will get a new optimal *f* value, which, divided by the largest losing scenario (specified in utils), yields an optimal *f* $ value (also specified in utils), and you will know how many contracts to trade. The process is simple; you simply substitute utils in lieu of dollars. The only other thing you need to do is keep track of your account's cumulative utils (i.e., the surrogate for equity). Notice that, if you do this and your utility preference function is other than ln, you will actually end up with a nonuniform optimal *f*, in terms of *dollars*, from one holding period to the next.

For example, if we are again faced with a coin-toss game that offers us $2 on heads being tossed, and a loss of $1 if tails is tossed, how much should we bet? We know that if we want to maximize wealth, and we are going to play this game repeatedly, and we have to play each subsequent play with money that we started the game with, we should optimally bet 25% of our stake on each and every play. Not only would this maximize wealth; it would also maximize utility if we determined that a win of $2 were twice as valuable to us as a loss of $1.

But what if a win of $2 were only one-and-a-half times as valuable to us as a loss of $1? To determine how to maximize utility then, we assign a util value of -1 to the losing scenario, the tails scenario, and a utils value of 1.5 to the winning scenario, the heads scenario. Now, we determine the optimal *f* based upon these utils rather than dollars, and we find it to be .166666, or to bet $16\frac{2}{3}$% on each and every play to maximize our geometric average utility. That means we divide our total cumulative utils to this point by .166666 to determine the number of contracts.

We can then translate this into how many contracts we have per dollars in our account, and, from there, figure what the *f* value (between zero and one) is that we are really using (based on dollars, not utils).

If we do this, then the original two-to-one coin-toss curve of wealth maximization, which peaks at .25 (Figure 9.2), still applies, and we are at the .166666 *f* abscissa. Thus, we pay the consequences of being suboptimal in terms of *f* on our wealth. However, there is a second *f* curve—one based on our utility—which peaks at .166666, and we are at the optimal *f* on this curve. Notice that, if we were to accept the .25 optimal *f* on this curve, we would be way right of the peak and would pay the concomitant consequences of being right of the peak with respect to our utility.

Now, suppose we were profitable in this holding period, and we go in and update the outcomes of the scenarios based on utils, only this time, since we have more wealth, the utility of a winning scenario in the next holding period is only 1.4 utils. Again, we would find our optimal *f* based on utils. Again, once we determined how many units to trade in the next holding period based on our cumulative utils, we could translate it into what the *f* (between zero and one) is for dollars, and we would find it to be nonuniform from the previous holding period.

The example shown is one in which we assume a sequence of more than one play, where we are reusing the same money we started with. If there was only one play, one holding period, or we received new money to play at each holding period, maximizing the arithmetic expected utility would be the optimal strategy. However, in most cases we must reuse the money on the next play, the next holding period, that we have used on this last play, and, therefore, we must strive to maximize geometric expected growth. To some, this might mean maximizing the geometric expected growth of wealth; to others, the geometric expected growth of utility. The mathematics is the same for both. Both have two curves in $n + 1$ space: a wealth maximization curve and a utility maximization curve. For those maximizing the expected growth of wealth, the two are the same.

If the reader has a different utility preference curve other than ln (wealth maximization), he may apply the techniques herein, provided he substitutes a *utils* quantity for the outcome of each scenario rather than a monetary value, which will then yield a nonuniform optimal *f* value (one whose value changes from one holding period to the next).

Such readers are forewarned, however, that they will still pay the consequences, in terms of their wealth, for being suboptimal in the $n+1$ dimensional leverage space of wealth maximization. Again, this is so because, regardless of your utility preference curve, you are somewhere in the leverage space of Figure 9.2 for individual games, and somewhere in the $n+1$ dimensional leverage space for multiple simultaneous games. You reap the benefits of this, as well as pay the consequences, no matter what your utility preference function. Ideally, you will have a utility preference function and it will be ln.