Session 4

Mathematical Structures

Chair: *Hiroshi Ooguri*, Caltech, USA
Rapporteur: *Robbert Dijkgraaf*, University of Amsterdam, the Netherlands
Scientific secretaries: *Philippe Spindel* (Université de Mons-Hainaut) and *Laurent Houart* (Université Libre de Bruxelles)

4.1 Rapporteur talk: Mathematical Structures, by Robbert Dijkgraaf

4.1.1 *Abstract*

The search for a quantum theory of gravity has stimulated many developments in mathematics. String theory in particular has had a profound impact, generating many new structures and concepts that extend classical geometry and give indications of what a full theory of quantum gravity should entail. I will try to put some of these ingredients in a broader mathematical context.

4.1.2 *Quantum Theory and Mathematics*

Over the years the search for a theory of quantum gravity has both depended on and enriched many fields of mathematics. String theory [1] in particular has had an enormous impact in mathematical thinking. Subjects like algebraic and differential geometry, topology, representation theory, infinite dimensional analysis and many others have been stimulated by new concepts such as mirror symmetry [2], [3], quantum cohomology [4] and conformal field theory [5]. In fact, one can argue that this influence in mathematics will be a lasting and rewarding impact of string theory, whatever its final role in fundamental physics. String theory seem to be the most complex and richest mathematical object that has so far appeared in physics and the inspiring dialogue between mathematics and physics that it has triggered is blooming and spreading in wider and wider circles.

This synergy between physics and mathematics that is driving so many developments in modern theoretical physics, in particularly in the field of quantum geometry, is definitely not a new phenomenon. Mathematics has a long history of drawing inspiration from the physical sciences, going back to astrology, architecture and land measurements in Babylonian and Egyptian times. Certainly this reached a high point in the 16th and 17th centuries with the development of what we now call classical mechanics. One of its leading architects, Galileo, has given us the famous image of the "Book of Nature" in *Il Saggiatore*, waiting to be decoded by scientists

> *Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth.*

This deep respect for mathematics didn't disappear after the 17th century. Again in the beginning of the last century we saw again a wonderful intellectual union of physics and mathematics when the great theories of general relativity and quantum mechanics were developed. In all the centers of the mathematical world this was closely watched and mathematicians actively participated. If anywhere this was so in Göttingen, where Hilbert, Minkowksi, Weyl, Von Neumann and many other mathematicians made important contributions to physics.

Theoretical physics have always been fascinated by the beauty of their equations. Here we can even quote Feynman, who was certainly not known as a fine connoisseur of higher abstract mathematics[1]

> *To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.*

But despite the warm feelings of Feynman, the paths of fundamental physics and mathematics started to diverge dramatically in the 1950s and 1960s. In the struggle with all the new subatomic particles physicists were close to giving up the hope of a beautiful underlying mathematical structure. On the other hand mathematicians were very much in an introspective mode these years. Because the fields were standing back to back, Dyson famously stated in his Gibbs Lecture in 1972:

> *I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.*

But this was a premature remark, since just at time the Standard Model was being born. This brought geometry in the form of non-abelian gauge fields, spinors

[1] But then Feynman also said "If all mathematics disappeared today, physics would be set back exactly one week." One mathematician's answer to this remark was: "This was the week God created the world."

and topology back to forefront. Indeed, it is remarkable fact, that all the ingredients of the standard model have a completely natural mathematical interpretation in terms of connections, vector bundles and Clifford algebras. Soon mathematicians and physicists started to build this dictionary and through the work of Atiyah, Singer, 't Hooft, Polyakov and many others a new period of fruitful interactions between mathematics and physics was born.

Remarkably, the recent influx of ideas from quantum theory has also led to many new developments in pure mathematics. In this regards one can paraphrase Wigner [6] and speak of the "unreasonable effectiveness of quantum physics in mathematics." There is a one immediate reason why quantum theory is so effective. Mathematics studies abstract patterns and structures. As such it has a hierarchical view of the world, where things are first put in broadly defined categories and then are more and more refined and distinguished. In topology one studies spaces in a very crude fashion, whereas in geometry the actual shape of a space matters. For example, two-dimensional (closed, connected, oriented) surfaces are topologically completely determined by their genus or number of handles $g = 0, 1, 2, \ldots$ So we have one simple topological invariant g that associates to each surface a non-negative number

$$g: \ \{Surfaces\} \ \rightarrow \ \mathbb{Z}_{\geq 0}.$$

More complicated examples are the knot invariants that distinguish embeddings of a circle in $\mathbb{R}^3$ up to isotopy. In that case there are an infinite number of such invariants

$$Z: \ \{Knots\} \ \rightarrow \ \mathbb{C}.$$

But in general such invariants are very hard to come by – the first knot invariant was discovered by J.W. Alexander in 1923, the second one sixty years later by V. Jones.

Quantum physics, in particular particle and string theory, has proven to be a remarkable fruitful source of inspiration for new topological invariants of knots and manifolds. With hindsight this should perhaps not come as a complete surprise. Roughly one can say that quantum theory takes a geometric object (a manifold, a knot, a map) and associates to it a (complex) number, that represents the probability amplitude for a certain physical process represented by the object. For example, a knot in $\mathbb{R}^3$ can stand for the world-line of a particular particle and a manifold for a particular space-time geometry. So the rules of quantum theory are perfectly set up to provide invariants.

Once we have associated concrete numbers to geometric objects one can operate on them with various algebraic operations. In knot theory one has the concept of relating knots through recursion relations (skein relations) or even differentiation (Vassiliev invariants). In this very general way quantization can be thought of as a map (functor)

$$Geometry \ \rightarrow \ Algebra.$$

that brings objects out the world of geometry into the real of algebra. This often gives powerful new perspectives, as we will see in a few examples later.

4.1.2.1 *String theory and mathematics*

First of all, it must be said that the subject of *Quantum Field Theory* (QFT) is already a powerful source for mathematical inspiration. There are important challenges in constructive and algebraic QFT: for example, the rigorous construction of four-dimensional asymptotically free non-abelian gauge theories and the establishment of a mass gap (one of the seven Millennium Prize problems of the Clay Mathematics Institute [7]).

Even in perturbative QFT there remain many beautiful mathematical structures to be discovered. Recently, a surprisingly rich algebraic structures has been discovered in the combinatorics of Feynman diagrams by Connes, Kreimer and others, relating Hopf algebras, multiple zeta-functions, and various notions from number theory [8]. Also the reinvigorated program of the twistor reformulation of (self-dual) Yang-Mills and gravity theories should be mentioned [9]. This development relates directly to the special properties of so-called MHV (maximal helicity violating) amplitudes and many other hidden mathematical structures in perturbative gauge theory [10].

In fact, a much deeper conceptual question seems to underlie the formulation of QFT. Modern developments have stressed the importance of quantum dualities, special symmetries of the quantum system that are not present in the classical system. These dualities can relate gauge theories of different gauge groups (*e.g.* Langlands dual gauge groups in the $\mathcal{N}=4$ supersymmetric Yang-Mills theory [11]) and even different matter representations (Seiberg duality [12]). All of this points to the conclusion that a QFT is more than simply the quantization of a classical (gauge) field and that even the path-integral formulation is at best one particular, duality-dependent choice of parametrization. This makes one wonder whether a formulation of QFT exist that is manifest duality invariant.

Although the mathematical aspects of quantum field theory are far from exhausted, it is fair to say, I believe, that the renewed bond between mathematics and physics has been greatly further stimulated with the advent of string theory. There is quite a history of developing and applying of new mathematical concepts in the "old days" of string theory, leading among others to representations of Kac-Moody and Virasoro algebras, vertex operators and supersymmetry. But since the seminal work of Green and Schwarz in 1984 on anomaly cancellations, these interactions have truly exploded. In particular with the discovery of Calabi-Yau manifolds as compactifications of the heterotic strings with promising phenomenological prospectives by the pioneering work of Witten and others, many techniques of algebraic geometry entered the field.

Most of these developments have been based on the perturbative formulation of string theory, either in the Lagrangian formalism in terms of maps of Riemann surfaces into manifolds or in terms of the quantization of loop spaces. This perturbative approach is however only an approximate description that appears for small values of the quantization parameter.

Recently there has been much progress in understanding a more fundamental description of string theory that is sometimes described as M-theory. It seems to unify three great ideas of twentieth century theoretical physics and their related mathematical fields:

- General relativity; the idea that gravity can be described by the Riemannian geometry of space-time. The corresponding mathematical fields are topology, differential and algebraic geometry, global analysis.
- Gauge theory; the description of forces between elementary particles using connections on vector bundles. In mathematics this involves the notions of K-theory and index theorems and more generally non-commutative algebra.
- Strings, or more generally extended objects (branes) as a natural generalization of point particles. Mathematically this means that we study spaces primarily through their (quantized) loop spaces. This relates naturally to infinite-dimensional analysis and representation theory.

At present it seems that these three independent ideas are closely related, and perhaps essentially equivalent. To some extent physics is trying to build a dictionary between geometry, gauge theory and strings. From a mathematical perspective it is extremely interesting that such diverse fields are intimately related. It makes one wonder what the overarching structure will be.

It must be said that in all developments there have been two further ingredients that are absolutely crucial. The first is quantum mechanics – the description of physical reality in terms of operator algebras acting on Hilbert spaces. In most attempts to understand string theory quantum mechanics has been the foundation, and there is little indication that this is going to change.

The second ingredient is supersymmetry – the unification of matter and forces. In mathematical terms supersymmetry is closely related to De Rham complexes and algebraic topology. In some way much of the miraculous interconnections in string theory only work if supersymmetry is present. Since we are essentially working with a complex, it should not come to a surprise to mathematicians that there are various 'topological' indices that are stable under perturbation and can be computed exactly in an appropriate limit. Indeed it is the existence of these topological quantities, that are not sensitive to the full theory, that make it possible to make precise mathematical predictions, even though the final theory is far from complete.

4.1.2.2 *What is quantum geometry?*

Physical intuition tells us that the traditional pseudo-Riemannian geometry of space-time cannot be a definite description of physical reality. Quantum corrections will change this picture at short-distances on the order of the Planck scale $\ell_P \sim 10^{-35}$ m.

Several ideas seem to be necessary ingredients of any complete quantum gravity theory.

- *Correspondence principle.* Whatever quantum geometry is, it should reduce to the classical space-times of general relativity in the limit $\ell_P \to 0$.
- *Space-time non-commutativity.* The space-time coordinates x^μ are no longer real numbers, but most likely should become eigenvalues of quantum operators. These operators should no longer commute, but instead obey relations of the form

$$[x^\mu, x^\nu] \sim \ell_P^2.$$

  In particular, space-like and time-like coordinates should no longer commute. There is a well-known simple physical argument for this: precise short-distance spatial measurements Δx require such high energy waves compressed in such a small volume, that a microscopic black hole can be formed. Due to Hawking evaporation, such a black hole is only meta-stable, and it will have a typical decay time $\Delta t \sim \ell_P^2/\Delta x$.
- *Quantum foam.* In some sense one should be able to interpret quantum geometry as a path-integral over fluctuating space-time histories. Short-distance space-time geometries should be therefore be subject to quantum corrections that have arbitrary complicated topologies. This induces some quantized, discrete structures. The sizes of the topologically non-trivial cycles (handles, loops, "holes", ...) should be quantized in units of ℓ_P. Together with the idea of non-commutativity of the space-time coordinates, this reminds one of semi-classical Bohr-Sommerfeld quantization.
- *Holography.* As is discussed in much greater details in the rapporteur talk of Seiberg [13], the ideas of holography in black hole physics [14] suggest that space-time geometry should be an *emergent* concept. It should arise in the limit $N \to \infty$, where N is some measure of the total degrees of freedom of the quantum system. In this context the analogy with the emergence of the laws of thermodynamics out of the properties of a statistical mechanical system has often been mentioned. In fact, both thermodynamics and general relativity were discover first as macroscopic theories, before the corresponding microscopic formulations were found. They are also in a precise sense universal theories: in the suitable macroscopic limit any system is subject to the laws of thermodynamics and any gravity theory will produce Einsteinian gravity.
- *Probe dependence.* Experience of string theory has taught us that the measured geometry will depend on the object that one uses to probe the system. Roughly, the metric $g_{\mu\nu}(x)$ will appear as an effective coupling constant in the worldvolume theory of a particle, string or brane that is used as probe. With Σ the worldvolume of the probe, one has

$$S_{probe}[g_{\mu\nu}] = \int_\Sigma g_{\mu\nu}(x) dx^\mu \wedge *dx^\nu + \ldots$$

As such the space-time metric $g_{\mu\nu}$ is not an invariant concept, but dependent on the "duality frame." For example, certain singularities can appear from the perspective of one kind a brane, but not from another one where the geometry seems perfectly smooth. So, even to *define* a space-time we have to split the total degrees of freedom in a large source system, that produces an emergent geometry, and a small probe system, that measures that effective geometry.

- *Alternative variables.* It is in no way obvious (and most likely simply wrong) that a suitable theory of quantum gravity can be obtained as a (non-perturbative) quantization of the metric tensor field $g_{\mu\nu}(x)$. The ultimate quantum degrees of freedom are probably not directly related to the usual quantities of classical geometry. One possible direction, as suggested for example in the case of three-dimensional geometry [15] and the Ashtekar program of loop quantum gravity [16], is that some form of gauge fields could be appropriate, possible a p-form generalization of that [17]. However, in view of the strong physical arguments for holography, it is likely that this change of dynamical variables should entail more that simply replacing the metric field with another space-time quantum field.

4.1.3 *The quantum geometry of string theory*

Let us now put the mathematical structures in some perspective. For pedagogical purposes we will consider string theory as a two parameter family of deformations of "classical" Riemannian geometry. Let us introduce these two parameters heuristically. (We will give a more precise explanation later.)

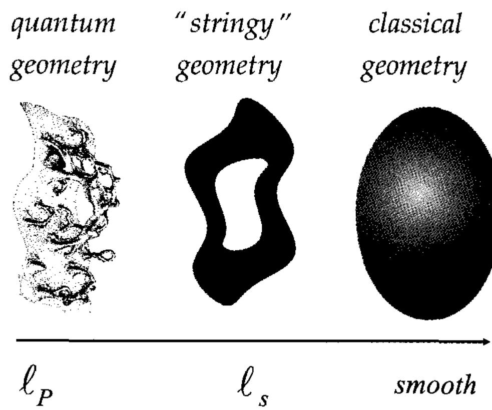
First, in perturbative string theory we study the loops in a space-time manifold. These loops can be thought to have an intrinsic length ℓ_s, the *string length.* Because of the finite extent of a string, the geometry is necessarily "fuzzy." At least at an intuitive level it is clear that in the limit $\ell_s \to 0$ the string degenerates to a point, a constant loop, and the classical geometry is recovered. The parameter ℓ_s controls the "stringyness" of the model. We will see how the quantity $\ell_s^2 = \alpha'$ plays the role of Planck's constant on the worldsheet of the string. That is, it controls the quantum correction of the two-dimensional field theory on the world-sheet of the string.

A second deformation of classical geometry has to do with the fact that strings can split and join, sweeping out a surface Σ of general topology in space-time. According to the general rules of quantum mechanics we have to include a sum over all topologies. Such a sum over topologies can be regulated if we can introduce a formal parameter g_s, the *string coupling,* such that a surface of genus h gets weighted by a factor g_s^{2h-2}. Higher genus topologies can be interpreted as virtual processes wherein strings split and join — a typical quantum phenomenon. Therefore the parameter g_s controls the quantum corrections. In fact we can equate g_s^2 with Planck's constant in space-time. Only for small values of g_s can string theory be described in terms of loop spaces and sums over surfaces.

In theories of four-dimensional gravity, the Planck scale is determined as

$$\ell_P = g_s \ell_s.$$

This is the scale at which we expect to find the effects of quantum geometry, such as non-commutativity and space-time foam. So, in a perturbative regime, where g_s is by definition small, the Planck scale will be much smaller than the string scale $\ell_P \ll \ell_s$ and we will typical have 3 regimes of geometry, depending on which length scale we will probe the space-time: a classical regime at large scales, a "stringy" regime where we study the loop space for scales around ℓ_s, finally and a truly quantum regime for scales around ℓ_P.



For large values of g_s this picture changes drastically. In the case of particles we know that for large $\hbar$ it is better to think in terms of waves, or more precisely quantum fields. So one could expect that for large g_s and ℓ_s the right framework is string field theory [18]. This is partly true, but it is in general difficult to analyze (closed) string field theory in all its generality.

Summarizing we can distinguish two kinds of deformations: *stringy* effects parameterized by ℓ_s or α', and *quantum* effects parameterized by g_s. This situation can be described with the following diagram

α' large	conformal field theory *strings*	M-theory *string fields, branes*
$\alpha' \approx 0$	quantum mechanics *particles*	quantum field theory *fields*
	$g_s \approx 0$	g_s large

4.1.3.1 *Quantum mechanics and point particles*

As a warm-up let us start by briefly reviewing the quantum mechanics of point particles in more abstract mathematical terms.

In classical mechanics we describe point particles on a Riemannian manifold X that we think of as a (Euclidean) space-time. Pedantically speaking we look at X through maps

$$x : \ pt \to X$$

of an abstract point into X. Quantum mechanics associates to the classical configuration space X the Hilbert space $\mathcal{H} = L^2(X)$ of square-integrable wavefunctions. We want to think of this Hilbert space as associated to a point

$$\mathcal{H} = \mathcal{H}_{pt}.$$

For a supersymmetric point particle, we have bosonic coordinates x^μ and fermionic variables θ^μ satisfying

$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu.$$

We can think of these fermionic variables geometrically as one-forms $\theta^\mu = dx^\mu$. So, the supersymmetric wavefunction $\Psi(x, \theta)$ can be interpreted as a linear superposition of differential forms on X

$$\Psi(x, \theta) = \sum_n \Psi_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}.$$

So, in this case the Hilbert space is given by the space of (square-integrable) de Rham differential forms $\mathcal{H} = \Omega^*(X)$.

Classically a particle can go in a time t from point x to point y along some preferred path, typically a geodesic. Quantum mechanically we instead have a linear evolution operator

$$\Phi_t : \mathcal{H} \to \mathcal{H}.$$

that describes the time evolution. Through the Feynman path-integral this operator is associated to maps of the line interval of length t into X. More precisely, the kernel $\Phi_t(x, y)$ of the operator Φ_t, that gives the probability amplitude of a particle situated at x to arrive at position y in time t, is given by the path-integral

$$\Phi_t(x, y) = \int_{x(\tau)} \mathcal{D}x \ \exp\left[-\int_0^t d\tau |\dot{x}|^2\right]$$

over all paths $x(\tau)$ with $x(0) = x$ and $x(t) = y$. Φ_t is a famous mathematical object — the integral kernel of the heat equation

$$\frac{d}{dt}\Phi_t = \Delta\Phi_t, \qquad \Phi_0(x, y) = \delta(x - y).$$

These path-integrals have a natural gluing property: if we first evolve over a time t_1 and then over a time t_2 this should be equivalent to evolving over time $t_1 + t_2$. That is, we have the composition property of the corresponding linear maps

$$\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1 + t_2}. \tag{1}$$

This allows us to write

$$\Phi_t = e^{-tH}$$

with H the Hamiltonian. In the case of a particle on X the Hamiltonian is of course simply given by (minus) the Laplacian $H = -\Delta$. The composition property (1) is a general property of quantum field theories. It leads us to Segal's functorial view of quantum field theory, as a functor between the categories of manifolds (with bordisms) to vector spaces (with linear maps) [19].

In the supersymmetric case the Hamiltonian can be written as

$$H = -\Delta = -(dd^* + d^*d)$$

Here the differentials d, d^* play the role of the supercharges

$$d = \psi^\mu \frac{\partial}{\partial x^\mu}, \qquad d^* = g^{\mu\nu} \frac{\partial^2}{\partial \psi^\mu \partial x^\nu}.$$

The ground states of the supersymmetric quantum mechanics satisfy $H\Psi = 0$ and are therefore harmonic forms

$$d\Psi = 0, \qquad d^*\Psi = 0.$$

Therefore they are in 1-to-1 correspondence with the de Rham cohomology group of the space-time manifold

$$\Psi \in \mathrm{Harm}^*(X) \cong H^*(X).$$

We want to make two additional remarks. First we can consider also a closed 1-manifold, namely a circle S^1 of length t. Since a circle is obtained by identifying two ends of an interval we can write

$$Z_{S^1} = \mathrm{Tr}_{\mathcal{H}} e^{-tH}.$$

Here the partition function Z_{S^1} is a number associated to the circle S^1 that encodes the spectrum of the operator Δ. We can also compute the supersymmetric partition function by using the fermion number F (defined as the degree of the corresponding differential form). It computes the Euler number

$$\mathrm{Tr}_{\mathcal{H}} \left((-1)^F e^{-tH} \right) = \dim H^{even}(X) - \dim H^{odd}(X) = \chi(X).$$

4.1.3.2 *Conformal field theory and strings*

We will now introduce our first deformation parameter α' and generalize from point particles and quantum mechanics to strings and conformal field theory.

A string can be considered as a parameterized loop. So, in this case we study the manifold X through maps

$$x: \ S^1 \to X,$$

that is, through the free loop space $\mathcal{L}X$.

Quantization will associate a Hilbert space to this loop space. Roughly one can think of this Hilbert space as $L^2(\mathcal{L}X)$, but it is better to think of it as a quantization of an infinitesimal thickening of the locus of constant loops $X \subset \mathcal{L}X$. These constant loops are the fixed points under the obvious S^1 action on the loop space. The normal bundle to X in $\mathcal{L}X$ decomposes into eigenspaces under this S^1 action, and this gives a description (valid for large volume of X) of the Hilbert space $\mathcal{H}_{S^1}$ associated to the circle as the normalizable sections of an infinite Fock space bundle over X.

$$\mathcal{H}_{S^1} = L^2(X, \mathcal{F}_+ \otimes \mathcal{F}_-)$$

where the Fock bundle is defined as

$$\mathcal{F} = \bigotimes_{n \geq 1} S_{q^n}(TX) = \mathbb{C} \oplus qTX \oplus \cdots$$

Here we use the formal variable q to indicate the $\mathbb{Z}$-grading of $\mathcal{F}$ and we use the standard notation

$$S_q V = \bigoplus_{N \geq 0} q^N \, S^N V$$

for the generating function of symmetric products of a vector space V.

When a string moves in time it sweeps out a surface Σ. For a free string Σ has the topology of $S^1 \times I$, but we can also consider at no extra cost interacting strings that join and split. In that case Σ will be a oriented surface of arbitrary topology. So in the Lagrangian formalism one is let to consider maps

$$x : \ \Sigma \to X.$$

There is a natural action for such a sigma model if we pick a Hogde star or conformal structure on Σ (together with of course a Riemannian metric g on X)

$$S(x) = \int_\Sigma g_{\mu\nu}(x) dx^\mu \wedge *dx^\nu$$

The critical points of $S(x)$ are the harmonic maps. In the Lagrangian quantization formalism one considers the formal path-integral over all maps $x : \Sigma \to X$

$$\Phi_\Sigma = \int_{x:\ \Sigma \to X} e^{-S/\alpha'}.$$

Here the constant α' plays the role of Planck's constant on the string worldsheet Σ. It can be absorbed in the volume of the target X by rescaling the metric as $g \to \alpha' \cdot g$. The semi-classical limit $\alpha' \to 0$ is therefore equivalent to the limit $vol(X) \to \infty$.

4.1.3.3 *Functorial description*

In the functorial description of conformal field theory the maps Φ_Σ are abstracted away from the concrete sigma model definition. Starting point is now an arbitrary (closed, oriented) Riemann surface Σ with boundary. This boundary consists of a collections of oriented circles. One declares these circles in-coming or out-going depending on whether their orientation matches that of the surface Σ or not. To a surface Σ with m in-coming and n out-going boundaries one associates a linear map

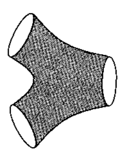
$$\Phi_\Sigma : \mathcal{H}_{S^1}^{\otimes n} \to \mathcal{H}_{S^1}^{\otimes n}.$$

These maps are not independent but satisfy gluing axioms that generalize the simple composition law (1)

$$\Phi_{\Sigma_1} \circ \Phi_{\Sigma_2} = \Phi_\Sigma, \tag{2}$$

where Σ is obtained by gluing Σ_1 and Σ_2 on their out-going and incoming boundaries respectively.

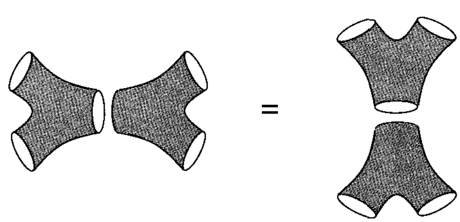
In this way we obtain what is known as a modular functor. It has a rich algebraic structure. For instance, the sphere with three holes



gives rise to a product

$$\Phi : \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1}.$$

Using the fact that a sphere with four holes can be glued together from two copies of the three-holed sphere one shows that this product is essentially commutative and associative

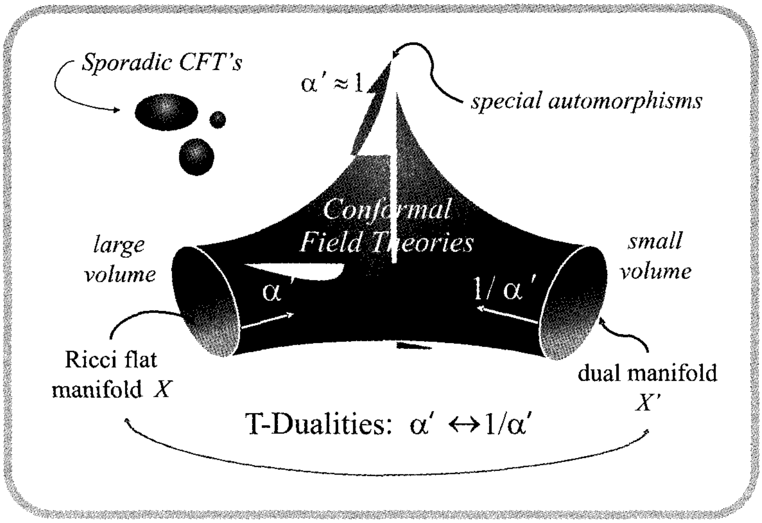


Once translated in terms of transition amplitudes, these relation lead to non-trivial differential equations and integrable hierarchies. For more details see *e.g.* [4, 20].

4.1.3.4 *Stringy geometry and T-duality*

Two-dimensional sigma models give a natural one-parameter deformation of classical geometry. The deformation parameter is Planck's constant α'. In the limit $\alpha' \to 0$ we localize on constant loops and recover quantum mechanics or point particle theory. For non-zero α' the non-constant loops contribute.

In fact we can picture the moduli space of CFT's roughly as follows.



The moduli space of conformal field theories.

It will have components that can be described in terms of a target spaces X. For these models the moduli parameterize Ricci-flat metrics plus a choice of B-field. These components have a boundary 'at infinity' which describe the large volume manifolds. We can use the parameter α' as local transverse coordinate on the collar around this boundary. If we move away from this boundary stringy corrections set in. In the middle of the moduli space exotic phenomena can take place. For example, the automorphism group of the CFT can jump, which gives rise to orbifold singularities at enhanced symmetry points.

The most striking phenomena that the moduli space can have another boundary that allows again for a semi-classical interpretation in terms of a second classical geometry $\hat{X}$. These points look like quantum or small volume in terms of the original variables on X but can also be interpreted as large volume in terms of a set of dual variables on a dual or mirror manifold $\hat{X}$. In this case we speak of a T-duality. In this way two manifold X and $\hat{X}$ are related since they give rise to the same CFT.

The most simple example of such a T-duality occurs for toroidal compactification. If $X = \mathbb{T}$ is an torus, the CFT's on $\mathbb{T}$ and its dual $\mathbb{T}^*$ are isomorphic. We will explain this now in more detail. These kind of T-dualities have led to spectacular mathematical application in mirror symmetry, as we will review after that.

Let us consider a particle or a string on a space-time that is given by a n-dimensional torus, written as the quotient

$$\mathbb{T} = \mathbb{R}^n / L$$

with L a rank n lattice. States of a quantum mechanical point particle on $\mathbb{T}$ are conveniently labeled by their momentum

$$p \in L^*.$$

The wavefunctions $\Psi(x) = e^{ipx}$ form a basis of $\mathcal{H} = L^2(\mathbb{T})$ that diagonalizes the Hamiltonian $H = -\Delta = p^2$. So we can decompose the Hilbert space as

$$\mathcal{H} = \bigoplus_{p \in L^*} \mathcal{H}_p,$$

where the graded pieces $\mathcal{H}_p$ are all one-dimensional. There is a natural action of the symmetry group

$$G = SL(n, \mathbb{Z}) = \mathrm{Aut}(L)$$

on the lattice $\Gamma = L$ and the Hilbert space $\mathcal{H}$.

In the case of a string moving on the torus $\mathbb{T}$ states are labeled by a second quantum number: their winding number

$$w \in L,$$

which is simply the class in $\pi_1 \mathbb{T}$ of the corresponding classical configuration. The winding number simply distinguishes the various connected components of the loop space $\mathcal{L}\mathbb{T}$, since $\pi_0 \mathcal{L}\mathbb{T} = \pi_1 \mathbb{T} \cong L$. We therefore see a natural occurrence of the so-called Narain lattice $\Gamma^{n,n}$, which is the set of momenta $p \in L^*$ and winding numbers $w \in L$

$$\Gamma^{n,n} = L \oplus L^*$$

This is an even self-dual lattice of signature (n, n) with inner product

$$p^2 = 2w \cdot k, \qquad p = (w, k) \in \Gamma^{n,n}.$$

It turns out that all the symmetries of the lattice $\Gamma^{n,n}$ lift to symmetries of the full conformal field theory built up by quantizing the loop space. The elements of the symmetry group of the Narain lattice

$$SO(n, n, \mathbb{Z}) = \mathrm{Aut}\,(\Gamma^{n,n})$$

are examples of $\mathbb{T}$-dualities. A particular example is the interchange of the torus with its dual

$$\mathbb{T} \leftrightarrow \mathbb{T}^*.$$

T-dualities that interchange a torus with its dual can be also applied fiberwise. If the manifold X allows for a fibration $X \to B$ whose fibers are tori, then we can produce a dual fibration where we dualize all the fibers. This gives a new manifold $\widehat{X} \to B$. Under suitable circumstances this produces an equivalent supersymmetric sigma model. The symmetry that interchanges these two manifolds

$$X \leftrightarrow \widehat{X}$$

is called mirror symmetry [2], [3].

4.1.3.5 *Topological string theory*

In the case of point particles it was instructive to consider the supersymmetric extension since we naturally produced differential form on the target space. These differential forms are able, through the De Rahm complex, to capture the topology of the manifold. In fact, reducing the theory to the ground states, we obtained exactly the harmonic forms that are unique representatives of the cohomology groups. In this way we made the step from functional analysis and operator theory to topology.

In a similar fashion there is a formulation of string theory that is able to capture the topology of string configurations. This is called topological string theory. This is quite a technical subject, that is impossible to do justice to within the confines of this survey, but I will sketch the essential features. For more details see *e.g.* [3].

Topological string theory is important for several reasons

- It is a "toy model" of string theory that allows many exact computations. In this sense, its relation to the full superstring theory is a bit like topology versus Riemannian geometry.
- It is the main connection between string theory and various fields in mathematics.
- Topological strings compute so-called BPS or supersymmetric amplitudes in the full-fledged superstring and therefore also capture exact physical information.

Roughly, the idea is the following. First, just as in the point particle case, one introduces fermion fields θ^μ. Now these are considered as spinors on the two-dimensional world-sheet and they have two components $\theta^\mu_L, \theta^\mu_R$. One furthermore assumes that the target space X is (almost) complex so that one can use holomorphic local coordinates $x^i, \overline{x}^{\overline{\imath}}$ with a similar decomposition for the fermions. When complemented with the appropriate higher order terms this gives a sigma model that has $\mathcal{N} = (2,2)$ supersymmetry.

One now changes the spins of the fermionic fields to produce the topological string. This can be done in two inequivalent ways called the A-model and the B-model. Depending on the nature of this topological twisting the path-integral of the sigma model localizes to a finite-dimensional space.

The A-model restricts to holomorphic maps

$$\frac{\partial x^i}{\partial \overline{z}} = 0$$

This reduces the full path-integral over all maps from Σ into X to a finite-dimensional integral over the moduli space $\mathcal{M}$ of *holomorphic maps.* More precisely, it is the moduli space of pairs (Σ, f) where Σ is a Riemann surface and f is a holomorphic map $f: \Sigma \to X$. The A-model only depends on the Kähler class

$$t = [\omega] \in H^2(X)$$

of the manifold X.

A-model topological strings give an important example of a typical stringy generalization of a classical geometric structure. Quantum cohomology [4] is a deformation of the De Rahm cohomology ring $H^*(X)$ of a manifold. Classically this ring captures the intersection properties of submanifolds. More precisely, if we have three cohomology classes

$$\alpha, \beta, \gamma \in H^*(X)$$

that are Poincaré dual to three subvarieties $A, B, C \subset X$, the quantity

$$I(\alpha, \beta, \gamma) = \int_X \alpha \wedge \beta \wedge \gamma$$

computes the intersection of the three classes A, B, and C. That is, it counts (with signs) the number of points in $A \cap B \cap C$.

In the case of the A-model we have to assume that X is a Kähler manifold or at least a symplectic manifold with symplectic form ω. Now the "stringy" intersection product is related to the three-string vertex. Mathematically it defined as

$$I_{qu}(\alpha, \beta, \gamma) = \sum_d q^d \int_{\overline{\mathcal{M}}_0(X,d)} \alpha \wedge \beta \wedge \gamma,$$

where we integrate our differential forms now over the moduli space of pseudo-holomorphic maps of degree d of a sphere into the manifold X. These maps are weighted by the classical instanton action

$$q^d = \exp\left[-\frac{1}{\alpha'}\int_{S^2}\omega\right] = e^{-d\cdot t/\alpha'}.$$

Clearly in the limit $\alpha' \to 0$ only the holomorphic maps of degree zero contribute. But these maps are necessarily constant and so we recover the classical definition of the intersection product by means of an integral over the space X. Geometrically, we can think of the quantum intersection product as follows: it counts the pseudo-holomorphic spheres inside X that intersect each of the three cycles A, B and C. So, in the quantum case these cycles do no longer need to actually intersect. It is enough if there is a pseudo-holomorphic sphere with points $a, b, c \in \mathbb{P}^1$ such that $a \in A$, $b \in B$ and $c \in C$, *i.e.*, if there is a string world-sheet that connect the three cycles.

For example, for the projective space $\mathbb{P}^n$ the classical cohomology ring is given by

$$H^*(\mathbb{P}^n) = \mathbb{C}[x]/(x^{n+1}).$$

The quantum ring takes the form

$$QH^*(\mathbb{P}^n) = \mathbb{C}[x]/(x^{n+1} = q),$$

with $q = e^{-t/\alpha'}$.

In the B-model one can reduce to (almost) constant maps. This model only depends on the complex structure moduli of X. It most important feature is that mirror symmetry will interchange the A-model with the B-model. A famous example of the power of mirror symmetry is the original computation of Candelas et. al. [21] of the quintic Calabi-Yau manifold given by the equation

$$X:\ x_1^5+x_2^5+x_3^5+x_4^5+x_5^5=0$$

in $\mathbb{P}^4$. In the case the A-model computation leads to an expression of the form

$$F_0(q)=\sum_d n_d\, q^d$$

where n_d computes the number of rational curves in X of degree d. These numbers are notoriously difficult to compute. The number $n_1 = 2875$ of lines is a classical result from the 19^{th} century. The next one $n_2 = 609250$ counts the different conics in the quintic and was only computed around 1980. Finally the number of twisted cubics $n_3 = 317206375$ was the result of a complicated computer program. However, now we know all these numbers and many more thanks to string theory. Here are the first ten

d	n_d
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 59999 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

How are physicists able to compute these numbers? Mirror symmetry does the job. It relates the "stringy" invariants coming from the A-model on the manifold X to the classical invariants of the B-model on the mirror manifold $\widehat{X}$. In particular this leads to a so-called Fuchsian differential equation for the function $F_0(q)$. Solving this equation one reads off the integers n_d.

4.1.4 *Non-perturbative string theory and branes*

We have seen how CFT gives rise to a rich structure in terms of the modular geometry as formulated in terms of the maps Φ_Σ. To go from CFT to string theory we have to make two more steps.

4.1.4.1 *Summing over string topologies*

First, we want to generalize to the situation where the maps Φ_Σ are not just functions on the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces but more general differential forms. In fact, we are particular interested in the case where they are volume forms since then we can define the so-called string amplitudes as

$$F_g = \int_{\mathcal{M}_g} \Phi_\Sigma$$

This is also the general definition of Gromov-Witten invariants [4] as we will come to later. Although we suppress the dependence on the CFT moduli, we should realize that the amplitudes A_g (now associated to a *topological* surface of genus g) still have (among others) α' dependence.

Secondly, it is not enough to consider a string amplitude of a given topology. Just as in field theory one sums over all possible Feynman graphs, in string theory we have to sum over all topologies of the string world-sheet. In fact, we have to ensemble these amplitudes into a generating function.

$$F(g_s) \approx \sum_{g\geq 0} {g_s}^{2g-2} F_g.$$

Here we introduce the string coupling constant g_s. Unfortunately, in general this generating function can be at best an asymptotic series expansion of an analytical function $F(g_s)$. A rough estimate of the volume of $\mathcal{M}_g$ shows that typically

$$F_g \sim 2g!$$

so the sum over string topologies will not converge. Indeed, general physics arguments tell us that the *non-perturbative* amplitudes $F(g_s)$ have corrections of the form

$$F(g_s) = \sum_{g\geq 0} {g_s}^{2g-2} F_g + \mathcal{O}(e^{-1/g_s})$$

Clearly to approach the proper definition of the string amplitudes these non-perturbative corrections have to be understood.

As will be reviewed at much greater length in other lectures, the last years have seen remarkable progress in the direction of developing such a non-perturbative formulation. Remarkable, it has brought very different kind of mathematics into the game. It involves some remarkable new ideas.

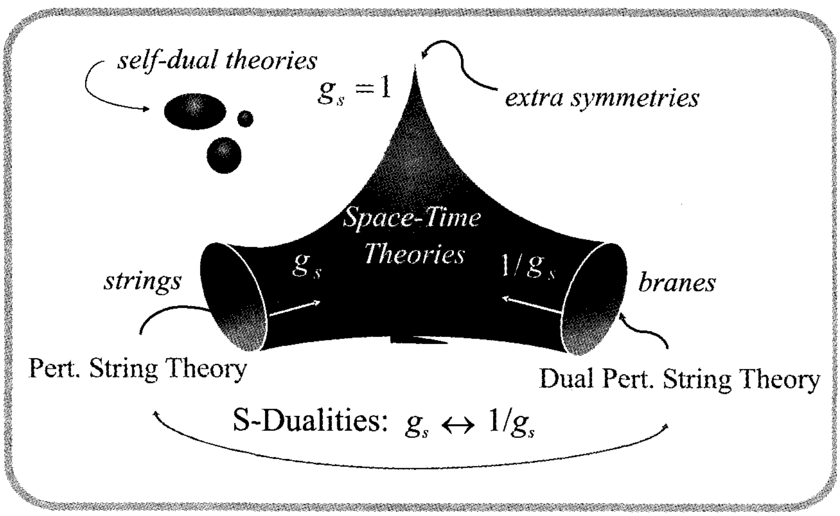
- *Branes.* String theory is not a theory of strings. It is simply not enough to consider loop spaces and their quantization. We should also include other extended objects, collectively known as branes. One can try to think of these objects as associated to more general maps $Y \to X$ where Y is a higher-dimensional space. But the problem is that there is not a consistent quantization starting from 'small' branes along the lines of string theory, that is, an expansion where we control the size of Y (through α') and the topology (through g_s). However, through the formalism of D-branes [22] these can be analyzed exactly in string perturbation theory. D-branes give contribution that are of order

$$e^{-1/g_s}$$

and therefore complement the asymptotic string perturbation series.

- *Gauge theory.* These D-branes are described by non-abelian gauge theories and therefore by definition non-commutative structures. This suggests that an alternative formulation of string theory makes use of non-commutative variables. These gauge-gravity dualities are the driving force of all recent progress in string theory [23].
- *Extra dimensions.* As we stressed, the full quantum amplitudes F depend on many parameters or moduli. Apart from the string coupling g_s all other moduli have a geometric interpretation, in terms of the metric and B-field on X. The second new ingredient is the insight that string theory on X with string coupling g_s can be given a fully geometric realization in terms of a new theory called M-theory on the manifold $X \times S^1$, where the length of the circle S^1 is g_s [24].

Summarizing, the moduli space of string theory solutions has a structure that in many aspects resembles the structures that described moduli of CFT's. In this case there are S-dualities that relate various perturbative regimes.



The moduli space of string theory vacua.

4.1.4.2 *Topological strings and quantum crystals*

The way in which quantum geometry can emerge from a non-perturbative completion of a perturbative string theory can be nicely illustrated by a topological string example.

In general the topological string partition function (of the A model) takes the form

$$Z_{top} = \exp \sum_{g \geq 0} g_s^{2g-2} F_g(t),$$

where the genus g contribution F_g can be expended as a sum over degree d maps

$$F_g(t) = \sum_d GW_{g,d}\, e^{-dt/\alpha'}.$$

Here $GW_{g,d} \in \mathbb{Q}$ denotes the Gromov-Witten invariant that "counts" the number of holomorphic maps $f: \Sigma_g \to X$ of degree d of a Riemann surface Σ_g of genus g into the Calabi-Yau manifold X.

To show that these invariants are very non-trivial and define some quantum geometry structure, it suffices to look at the simplest possible CY space $X = \mathbb{C}^3$. In that case only degree zero maps contribute. The corresponding Gromov-Witten invariants have been computed and can be expressed in terms of so-called Hodge integrals

$$GW_{g,0} = \int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 = \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}.$$

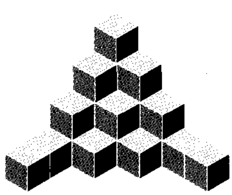
But in this case the full partition function Z_{top} simplifies considerably if it is expressed in terms of the strong coupling variable $q = e^{-g_s}$ instead of the weak coupling variable g_s:

$$Z_{top} = \exp \sum_g g_s^{2g-2} GW_{g,0} = \prod_{n>0} (1-q^n)^{-n}.$$

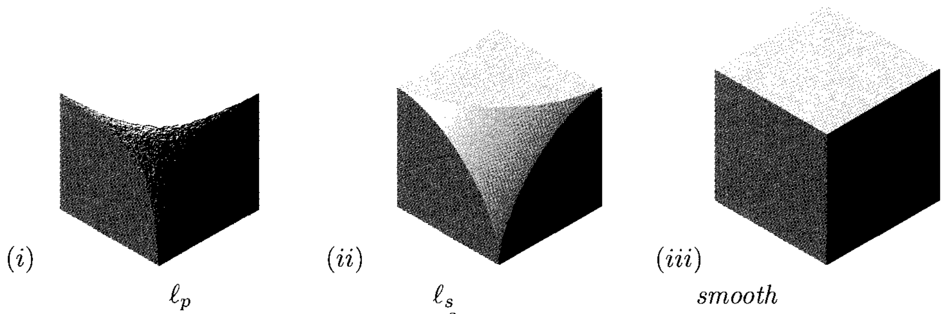
In fact, this gives a beautiful reinterpretation in terms of a statistical mechanics model. The partition function can be written as a weighted sum over all planar partitions

$$Z_{top} = \sum_\pi q^{|\pi|}.$$

Here a planar partition π is a 3d version of the usual 2d partitions [25]



Clearly, this statistical model has a granular structure that is invisible in the perturbative limit $g_s \to 0$. In fact, these quantum crystals give a very nice model in which the stringy and quantum geometry regimes can be distinguished. Here one uses a toric description of $\mathbb{C}^3$ as a $\mathbb{T}^3$ bundle over the positive octant in $\mathbb{R}^3$. In terms of pictures we have (i) the statistical model, (ii) the so-called limit space, that captures the mirror manifold, (iii) and the classical geometry



The three phases of the geometry of $\mathbb{C}^3$ *as seen through topological string theory.*

4.1.4.3 *U-dualities*

Another way to probe non-perturbative effects in string theory is to investigate the symmetries (dualities). In the case of a compactification on a torus $\mathbb{T}$ the story becomes considerably more complicated then we saw in previous section. The lattice of quantum numbers of the various objects becomes larger and so do the symmetries. For small values of the dimension n of the torus $\mathbb{T}$ $(n \leq 4)$ it turns out that the non-perturbative charge lattice M can be written as as the direct sum of the Narain lattice (the momenta and winding numbers of the strings) together with a lattice that keeps track of the homology classes of the branes

$$M = \Gamma^{n,n} \oplus H^{even/odd}(\mathbb{T})$$

Here we note that the lattice of branes (which are even or odd depending on the type of string theory that we consider)

$$H^{even/odd}(\mathbb{T}) \cong \Lambda^{even/odd} L^*$$

transform as half-spinor representations under the T-duality group $SO(n, n, \mathbb{Z})$. The full duality group turns out to be the exceptional group over the integers

$$E_{n+1}(\mathbb{Z}).$$

The lattice M will form an irreducible representation for this symmetry group. These so-called U-dualities will therefore permute strings with branes.

So we see that our hierarchy

$$\{Particles\} \subset \{Strings\} \subset \{Branes\}$$

is reflected in the corresponding sequence of symmetry (sub)groups

$$SL(n, \mathbb{Z}) \subset SO(n, n, \mathbb{Z}) \subset E_{n+1}(\mathbb{Z})$$

of rank $n-1$, n, $n+1$ respectively. Or, in terms of the Dynkin classification

$$A_{n-1} \subset D_n \subset E_{n+1}.$$

It is already a very deep (and generally unanswered) question what the 'right' mathematical structure is associated to a n-torus that gives rise to the exceptional group $E_{n+1}(\mathbb{Z})$.

4.1.5 *D-branes*

As we have mentioned, the crucial ingredient to extend string theory beyond perturbation theory are D-branes [22]. From a mathematical point of view D-branes can be considered as a relative version of Gromov-Witten theory. The starting point is now a pair of relative manifolds (X, Y) with X a d-dimensional manifold and $Y \subset X$ closed. The string worldsheets are defined to be Riemann surfaces Σ with boundary $\partial\Sigma$, and the class of maps $x : \Sigma \to X$ should satisfy

$$x(\partial\Sigma) \subset Y$$

That is, the boundary of the Riemann surfaces should be mapped to the subspace Y.

Note that in a functorial description there are now two kinds of boundaries to the surface. First there are the time-like boundaries that we just described. Here we choose a definite boundary condition, namely that the string lies on the D-brane Y. Second there are the space-like boundaries that we considered before. These are an essential ingredient in any Hamiltonian description. On these boundaries we choose initial value conditions that than propagate in time. In closed string theory these boundaries are closed and therefore a sums of circles. With D-branes there is a second kind of boundary: the open string with interval $I = [0, 1]$.

The occurrence of two kinds of space-like boundaries can be understood because there are various ways to choose a 'time' coordinate on a Riemann surface with boundary. Locally such a surface always looks like $S^1 \times \mathbb{R}$ or $I \times \mathbb{R}$. This ambiguity how to slice up the surface is a powerful new ingredient in open string theory.

To the CFT described by the pair (X, Y) we will associate an extended modular category. It has two kinds of objects or 1-manifolds: the circle S^1 (the closed string) and the interval $I = [0, 1]$ (the open string). The morphisms between two 1-manifolds are again bordisms or Riemann surfaces Σ now with a possible boundaries. We now have to kinds of Hilbert spaces: closed strings $\mathcal{H}_{S^1}$ and open strings $\mathcal{H}_I$.

Semi-classically, the open string Hilbert space is given by

$$\mathcal{H}_I = L^2(Y, \mathcal{F})$$

with Fock space bundle

$$\mathcal{F} = \bigotimes_{n \geq 1} S_{q^n}(TX)$$

Note that we have only a single copy of the Fock space $\mathcal{F}$, the boundary conditions at the end of the interval relate the left-movers and the right-movers. Also the fields are sections of the Fock space bundle over the D-brane Y, not over the full space-time manifold X. In this sense the open string states are localized on the D-brane.

4.1.5.1 *Branes and matrices*

One of the most remarkable facts is that D-branes can be given a multiplicity N which naturally leads to a non-abelian structure [26].

Given a modular category as described above there is a simple way in which this can be tensored over the $N \times N$ hermitean matrices. We simply replace the Hilbert space $\mathcal{H}_I$ associated to the interval I by

$$\mathcal{H}_I \otimes Mat_{N\times N}$$

with the hermiticity condition

$$(\psi \otimes M_{IJ})^* = \psi^* \otimes M_{JI}$$

The maps Φ_Σ are generalized as follows. Consider for simplicity first a surface Σ with a single boundary C. Let C contain n 'incoming' open string Hilbert spaces with states $\psi_1 \otimes M_1, \ldots, \psi_n \otimes M_n$. These states are now matrix valued. Then the new morphism is defined as

$$\Phi_\Sigma(\psi_1 \otimes M_1, \ldots, \psi_n \otimes M_n) = \Phi_\Sigma(\psi_1, \ldots, \psi_n)\mathrm{Tr}\,(M_1 \cdots M_n).$$

In case of more than one boundary component, we simply have an additional trace for every component.

In particular we can consider the disk diagram with three open string insertions. By considering this as a map

$$\Phi_\Sigma : \mathcal{H}_I \otimes \mathcal{H}_I \to \mathcal{H}_I$$

we see that this open string interaction vertex is now given by

$$\Phi_\Sigma(\psi_1 \otimes M_1, \psi_2 \otimes M_2) = (\psi_1 * \psi_2) \otimes (M_1 M_2).$$

So we have tensored the associate string product with matrix multiplication.

If we consider the geometric limit where the CFT is thought of as the semi-classical sigma model on X, the string fields that correspond to the states in the open string Hilbert space $\mathcal{H}_I$ will become matrix valued fields on the D-brane Y, *i.e.* they can be considered as sections of $\mathrm{End}(E)$ with E a (trivial) vector bundle over Y.

This matrix structure naturally appears if we consider N different D-branes $Y_1, \ldots, Y_N$. In that case we have a matrix of open strings that stretch from brane Y_I to Y_J. In this case there is no obvious vector bundle description. But if all the D-branes coincide $Y_1 = \ldots = Y_N$ a $U(N)$ symmetry appears.

4.1.5.2 *D-branes and K-theory*

The relation with vector bundles has proven to be extremely powerful. The next step is to consider D-branes with *non-trivial* vector bundles. It turns out that these configurations can be considered as a composite of branes of various dimensions [27]. There is a precise formula that relates the topology of the vector bundle E to the brane charge $\mu(E)$ that can be considered as a class in $H^*(X)$. (For convenience we consider first maximal branes $Y = X$.) It reads [28]

$$\mu(E) = ch(E)\widehat{A}^{1/2} \in H^*(X). \tag{3}$$

Here $ch(E)$ is the (generalized) Chern character $ch(E) = \mathrm{Tr}\exp(F/2\pi i)$ and $\widehat{A}$ is the genus that appears in the Atiyah-Singer index theorem. Note that the D-brane charge can be fractional.

Branes of lower dimension can be described by starting with two branes of top dimension, with vector bundles E_1 and E_2, of opposite charge. Physically two such branes will annihilate leaving behind a lower-dimensional collection of branes. Mathematically the resulting object should be considered as a virtual bundle $E_1 \ominus E_2$ that represents a class in the K-theory group $K^0(X)$ of X [29]. In fact the map μ in (3) is a well-known correspondence

$$\mu: \ K^0(X) \to H^{even}(X)$$

which is an isomorphism when tensored with the reals. In this sense there is a one-to-one map between D-branes and K-theory classes [29]. This relation with K-theory has proven to be very useful.

4.1.5.3 *Example: the index theorem*

A good example of the power of translating between open and closed strings is the natural emergence of the index theorem. Consider the cylinder $\Sigma = S^1 \times I$ between two D-branes described by (virtual) vector bundles E_1 and E_2. This can be seen as closed string diagram with in-state $|E_1\rangle$ and out-state $|E_2\rangle$

$$\Phi_\Sigma = \langle E_2, E_1 \rangle$$

Translating the D-brane boundary state into closed string ground states (given by cohomology classes) we have

$$|E\rangle = \mu(E) \in H^*(X)$$

so that

$$\Phi_\Sigma = \int_X ch(E_1) ch(E_2^*) \widehat{A}$$

On the other hand we can see the cylinder also as a trace over the open string states, with boundary conditions labeled by E_1 and E_2. The ground states in $\mathcal{H}_I$ are sections of the Dirac spinor bundle twisted by $E_1 \otimes E_2^*$ This gives

$$\Phi_\Sigma = \mathrm{Tr}_{\mathcal{H}_I}(-1)^F = \mathrm{index}(D_{E_1 \otimes E_2^*})$$

So the index theorem follows rather elementary.

4.1.5.4 *Non-perturbative dualities*

We indicated that in M-theory we do not want to include only strings but also D-branes (and even further objects that I will suppress in this discussion such as NS 5-branes and Kaluza-Klein monopoles). So in the limit of small string coupling g_s the full (second quantized) string Hilbert space would look something like

$$\mathcal{H} = S^*(\mathcal{H}_{string}) \otimes S^*(\mathcal{H}_{brane}).$$

Of course our discussion up to now has been very skew. In the full theory there will be U-dualities that will exchange strings and branes.

We will give a rather simple example of such a symmetry that appears when we compactify the (Type IIA) superstring on a four-torus $\mathbb{T}^4 = \mathbb{R}^4/L$. In this case the charge lattice has rank 16 and can be written as

$$\Gamma^{4,4} \oplus K^0(\mathbb{T}^4).$$

It forms an irreducible spinor representation under the U-duality group

$$SO(5,5,\mathbb{Z}).$$

Notice that the T-duality subgroup $SO(4,4,\mathbb{Z})$ has three inequivalent 8-dimensional representations (related by triality). The strings with Narain lattice $\Gamma^{4,4}$ transform in the vector representation while the even-dimensional branes labeled by the K-group $K^0(\mathbb{T}^4) \cong \Lambda^{even}L^*$ transform in the spinor representation. (The odd-dimensional D-branes that are labeled by $K^1(T)$ and that appear in the Type IIB theory transform in the conjugate spinor representation.)

To compute the spectrum of superstrings we have to introduce the corresponding Fock space. It is given by

$$\mathcal{F}_q = \bigotimes_{n=1}^{\infty} S_{q^n}(\mathbb{R}^8) \otimes \Lambda_{q^n}(\mathbb{R}^8) = \bigoplus_{N \geq 0} q^N \mathcal{F}(N).$$

The Hilbert space of BPS strings with momenta $p \in \Gamma^{4,4}$ is then given by

$$\mathcal{H}_{string}(p) = \mathcal{F}(p^2/2).$$

For the D-branes we take a completely different approach. Since we only understand the system for small string coupling we have to use semi-classical methods. Consider a D-brane that corresponds to a K-theory class E with charge vector $\mu = ch(E) \in H^*(\mathbb{T})$. To such a vector bundle we can associate a moduli space $\mathcal{M}_\mu$ of self-dual connections. (If we work in the holomorphic context we could equally well consider the moduli space of holomorphic sheaves of this topological class.) Now luckily a lot is know about these moduli spaces. They are hyper-Kähler and (for primitive μ) smooth. In fact, they are topologically Hilbert schemes which are deformations of symmetric products

$$\mathcal{M}_\mu \cong \mathrm{Hilb}^{\mu^2/2}(\mathbb{T}^4) \sim S^{\mu^2/2}\mathbb{T}^4.$$

Computing the BPS states through geometric quantization we find that

$$\mathcal{H}_{brane}(\mu) = H^*(\mathcal{M}_\mu).$$

The cohomology of these moduli spaces have been computed [30] with the result that

$$\bigoplus_{N\geq 0} q^N H^*(\mathrm{Hilb}^N(\mathbb{T}^4)) = \mathcal{F}_q.$$

This gives the final result

$$\mathcal{H}_{brane}(\mu) = \mathcal{F}(\mu^2/2) \cong \mathcal{H}_{string}(p),$$

where μ and p are related by an $SO(5,5,\mathbb{Z})$ transformation.

This is just a simple example to show that indeed the same mathematical structures (representation theory of affine lie algebras, Virasoro algebras, *etc.*) can appear both in the perturbative theory of strings and non-perturbative brane systems. Again this is a powerfull hint that a more unified mathematical structure underlies quantum gravity.

4.1.6 *The Role of Mathematics*

In this rapporteur talk we have surveyed some deep connections between physics and mathematics that have stimulated much intellectual activity. Let me finish to raise some questions about these interactions.

- First of all, it must be said that despite all these nice results, there does seem to operate a principle of complementarity (in the spirit of Niels Bohr) that makes it difficult to combine physical intuition with mathematical rigor. Quite often, deep conjectures have been proven rigorously, not be making the physical intuition more precise, but by taking completely alternative routes.
- It is not at all clear what kind of mathematical structure Nature prefers. Here there seem to be two schools of thoughts. One the one hand one can argue that it is the most universal structures that have proven to be most successful. Here one can think about the formalism of calculus, Riemannian manifolds, Hilbert spaces, *etc.* On the other hand, the philosophy behind a Grand Unified Theory or string theory, is that our world is very much described by a single unique(?) mathematical structure. This point of view seems to prefer exceptional mathematical objects, such as the Lie algebra E_8, Calabi-Yau manifolds, *etc.* Perhaps in the end a synthesis of these two points of view (that roughly correspond to the laws of Nature versus the solutions of these laws) will emerge.
- Continuing this thought it is interesting to speculate what other mathematical fields should be brought into theoretical physics. One could think of number theory and arithmetic geometry, or logic, or even subjects that have not been developed at all.

- One could also question whether we are looking for a single overarching mathematical structure or a combination of different complementary points of view. Does a fundamental theory of Nature have a global definition, or do we have to work with a series of local definitions, like the charts and maps of a manifold, that describe physics in various "duality frames." At present string theory is very much formulated in the last kind of way.
- As a whole, the study of quantum geometry takes on the form of a mathematical program, very much like the Langlands Program. There are many non-trivial examples, strange relations, dualities and automorphic forms, tying together diverse fields, with vast generalizations, all in an open ended project that seems to encompass more and more mathematics.
- Finally, there should be word of caution. To which extend should mathematics be a factor in deciding the future of theoretical physics? Is mathematical elegance a guiding light or a Siren, whose song draws the Ship of Physics onto the cliffs? Only the future will tell us.

Acknowledgement

Let me end by thanking the organizers of the XXIII Solvay Conference, in particular Marc Henneaux and Alexander Sevrin for their tremendous efforts to arrange such a unique and stimulating scientific meeting following a distinguished tradition. Let me also congratulate the Solvay family on their long-time commitment to fundamental science by which they are setting an example for the whole world.

Bibliography

[1] J. Polchinski, *String Theory* (Cambridge Monographs on Mathematical Physics), Cambridge University Press, 1998.
[2] D.A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry* (Mathematical Surveys and Monographs, No. 68.), AMS, 1999.
[3] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow, *Mirror symmetry.* Clay Mathematics Monographs, AMS 2003.
[4] Yu.I. Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces* AMS, 1999.
[5] Ph. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Graduate Texts in Contemporary Physics), Spinger, 1996.
[6] E. Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences,* Comm. Pure Appl. Math. **13** (1960), 1–14.
[7] *The Millennium Prize Problems,* Edited by: J. Carlson, A. Jaffe, and A. Wiles, AMS, 2006.
[8] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I: The Hopf algebra structure of graphs and the main theorem,* Commun. Math. Phys. **210** (2000) 249.
[9] E. Witten, *Perturbative gauge theory as a string theory in twistor space,* Commun. Math. Phys. **252**, 189 (2004).

[10] Z. Bern, L. J. Dixon and D. A. Kosower, $N = 4$ *super-Yang-Mills theory, QCD and collider physics,* Comptes Rendus Physique **5**, 955 (2004) [arXiv:hep-th/0410021].

[11] C. Vafa and E. Witten, *A Strong coupling test of S duality,* Nucl. Phys. B **431**, 3 (1994) [arXiv:hep-th/9408074].

[12] N. Seiberg, *Electric - magnetic duality in supersymmetric non Abelian gauge theories,* Nucl. Phys. B **435**, 129 (1995) [arXiv:hep-th/9411149].

[13] N. Seiberg, in this volume.

[14] G. 't Hooft, *Dimensional reduction in quantum gravity,* arXiv:gr-qc/9310026.
L. Susskind, *The world as a hologram,* J. Math. Phys. **36**, 6377 (1995) [arXiv:hep-th/9409089].

[15] E. Witten, *(2+1)-Dimensional gravity as an exactly soluble system,* Nucl. Phys. B **311**, 46 (1988).

[16] A. Ashtekar, *New variables for classical and quantum gravity,* Phys. Rev. Lett. **57** (1986) 2244.

[17] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, *Topological M-theory as unification of form theories of gravity,* Adv. Theor. Math. Phys. **9**, 593 (2005) [arXiv:hep-th/0411073].

[18] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation,* Nucl. Phys. **B390** (1993) 33–152, hep-th/9206084.

[19] G. Segal, *The definition of conformal field theory,* preprint; *Two dimensional conformal field theories and modular functors,* in *IXth International Conference on Mathematical Physics,* . B. Simon, A. Truman and I.M. Davies Eds. (Adam Hilger, Bristol, 1989).

[20] R. Dijkgraaf, *Les Houches Lectures on Fields, Strings and Duality,* in *Quantum Symmetries*, les Houches Session LXIV, Eds. A. Connes, K. Gawedzki, and J. Zinn-Justin, North-Holland, 1998, hep-th/9703136.

[21] P. Candelas, P. Green, L. Parke, and X. de la Ossa, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory,* Nucl. Phys. **B359** (1991) 21, and in [2].

[22] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges,* Phys. Rev. Lett. **75** (1995) 4724–4727, hep-th/9510017.

[23] J. Maldacena, *The large N limit of superconformal field theories and supergravity,* Adv. Theor. Math. Phys. bf 2 (1998) 231–252, hep-th/9711200.

[24] E. Witten, *String theory in various dimensions,* Nucl. Phys. **B 443** (1995) 85, hep-th/9503124.

[25] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals,* arXiv:hep-th/0309208.
A. Iqbal, N. Nekrasov, A. Okounkov and C. Vafa, *Quantum foam and topological strings,* arXiv:hep-th/0312022.

[26] E. Witten, *Bound states of strings and p-branes,* Nucl. Phys. **B460** (1996) 335, hep-th/9510135.

[27] M. Douglas, *Branes within branes,* hep-th/9512077.

[28] M. Green, J. Harvey, and G. Moore, *I-brane inflow and anomalous couplings on D-branes,* Class. Quant. Grav. **14** (1997) 47–52, hep-th/9605033.

[29] E. Witten, *D-branes and K-theory,* JHEP **9812** (1998) 019, hep-th/9810188.

[30] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface,* Math. Ann. **286** (1990) 193–207; *Hilbert Schemes of Zero-dimensional Subschemes of Smooth Varieties*, Lecture Notes in Mathematics **1572**, Springer-Verlag, 1994.

4.2 Discussion

G. Horowitz At the very beginning you asked: what is quantum geometry? You said it should involve non-commutative geometry, non-commuting coordinates. It is not obvious to me that it has to involve that. Later on, you gave this example of a melting crystal and it was unclear to me where non-commutative geometry came in in that example.

R. Dijkgraaf Actually, I do not think the details have been worked out but it has a very nice interpretation: again there is a matrix model description of this melting crystal which is the reduction of supersymmetric Yang-Mills theory to zero dimension. So it is a three-matrix model where the action is $trX[Y, Z]$. If we look at the critical point of that action, it corresponds exactly to this kind of crystal configurations. So again in this example, there is a D-brane interpretation. I must say that I deliberately did not make that argument exact because usually people argue something like : "If you have to measure space you have to concentrate energy. A little black hole will form and it will have some uncertainty because it will evaporate". I never felt very comfortable with that argument.

H. Ooguri In that particular example, you are exhibiting already half of the space. In the total space, of six dimensions, you do, in fact, see the non-commutative structures?

R. Dijkgraaf Yes. I guess that is a good point.

H. Ooguri There are three torus directions and three directions he was exhibiting which are non-commuting. This quantum structure of the space, the state being represented by blocks in this three dimensions, is a reflection of this non-commutative structure. So I think the example that Dijkgraaf was describing exactly demonstrates the non-commutative feature of space time where the Planck constant is replaced by $e^{-1/g}$, with g the coupling constant.

G. Horowitz I certainly agree, any quantum description will involve non-commutativity, like some x and p do not commute. But were you suggesting that it is always some sort of X with X not commuting?

R. Dijkgraaf In this case it is indeed so. To follow what Ooguri was saying: I was talking about a six dimensional space and drawing a three dimensional picture. In some sense I was using kind of a symplectic space and I was only showing you the coordinates, not the momenta.

N. Seiberg You wrote that time does not commute with space? What did you have in mind?

R. Dijkgraaf What I just said. I am just repeating folklore here: making precise space-time measurements will create small black holes that will evaporate and give a time uncertainty. I do not have any example here. All the non-commutativity that I was discussing here was non-commutativity in the space-like directions. I did not have any examples.

N. Seiberg Do you have in mind time being an operator which does not commute with something?

R. Dijkgraaf I am completely ignorant about that. I think you know more than I.

H. Ooguri Talking about a new direction to go, I think stringy Lorentzian geometry is completely uncharted territory that we need to explore. We have gained lots of insights into quantum geometry but those are all mostly static geometries and the important question about how Lorentzian geometry can be quantized needs to be understood.

A. Ashtekar Partly going back to what Horowitz and Seiberg were saying: in loop quantum gravity we do have a quantum geometry. The coordinates are commuting. There is no problem with that, the manifold is as it is. It is the Riemannian structures which are not commuting. So for example, areas of surfaces which intersect with each other are not commuting and therefore you cannot measure the areas arbitrarily accurately, for example. So there is this other possibility also. Namely that observable quantities such as areas, etc, are not commuting, but there is commutativity for the manifold itself. The manifold itself does not go away.

R. Dijkgraaf I think that is important. One thing I did not mention, but also a good open question, is just to go to three dimensional gravity because there are many ways in which all these approaches connect. Of course, from the loop quantum gravity point of view, three dimensional gravity, written as a Chern-Simon theory, is very interesting. In fact many of these topological theories, when we reduce them down to three dimensions, you get also some Chern-Simon theories. But again, there are many open issues: "Are these Chern-Simons theories really well defined? Do they really correspond to semi-classical quantum gravity theories?" If you want to think about more precise areas, I feel that that point should be developed.

E. Rabinovici I would like to make two comments. One is that when we use strings as probes, it seems that all mathematical concepts we are used to somehow become ambiguous. When you describe T-duality, geometry becomes ambiguous or symmetric. You have two totally different representations of the same geometry. The same goes for topology and the number of dimensions. It also applies to the questions: "Is some manifold singular or not?" and "Is a manifold commutative or not commutative?" So I was wondering: "Is there anything which remains non-ambiguous when we study it with strings?" That was the first comment.

The second comment, which also relates to a discussion we had yesterday, relates to what you said about algebra and geometry. When we use the relation between affine Lie algebras and their semi-classical geometrical description, we can sometimes treat systems which have curvature singularities and large R^2 corrections. Even if we do not know what the Einstein equations are nor what the effective Lagrangian is to all orders in R^2, we still know what the answer is for g-string equals zero, that is on the sphere. For such systems, we sometimes even know the answer on the torus and for higher genus surfaces. In this way we can somehow circumvent the α'-correction problem.

R. Dijkgraaf I do not really know what to say in response. Concerning your first remark, we have indeed often asked ourselves "What is string theory?". But now, we have to ask "What is not string theory?" That is really a big question because, in some sense, finding the upper bound, finding structures which definitely are not connected in any way, is probably now more challenging than the other way round.

P. Ramond You talked, glibly, about exceptional structures. The next frontier in a sense is that, besides non-commutativity, non-associativity occurs with specific systems. Have you encountered any need for this? Does it come up geometrically?

R. Dijkgraaf That is again speculative. Roughly, look at the three levels. At the first level of particles, we have gauge fields. At the second level where we have strings, we have these kind of B-fields. That is a 2-form field that is not related to a gauge field but to something that is called gerbes. And then, in M-theory, there is the 3-form field. So the B-field is clearly related to non-commutativity. But the 3-form field, if with anything, it might be related to non-associativity. But again, there have been many isolated ideas, but I do not think there is something that ties everything together. In fact, it is also a question to the mathematicians. For instance, in string theory, the B-field is intimately connected to K-theory. It is related to some periodicity *mod* 2. The 2 is the same 2 of the B-field. In fact, in M-theory, there is almost something like a periodicity *mod* 3. So again, the question is what replaces this structure. I do not think we have any hint. There are some suggestions that E_8 plays an important rôle. So there is a direct connection between gauge theories of E_8 and 3-forms. Greg Moore has very much pushed this. But again, these are only questions.

A. Polyakov This fusion of physics and mathematics which occurred in the last few decades and which you discussed so beautifully is quite amazing of course. But it is actually bothersome to me because it seems that we start following the steps of mathematicians. I believe we are supposed to invent our own mathematics even though modern mathematics is very seductive. Mathematicians are clever enough to do their own job, I suppose. Anyway that is general philosophy. More concretely, I think there is an unexplored domain which is important for understanding gauge-string duality. It is differential geometry in loop space. In gauge theory we get some interesting differential operators in loop space, which are practically unexplored. Very few, primitive, things are known about these operators and the loop equations for gauge theories. There is obviously some deep mathematical structure. There could be some Lax pairs or integrability in the loop space. Understanding what is the right way to construct these operators is not standard mathematics, but I believe, it is important for physics.

H. Ooguri From that point of view, string field theory has been one of the directions which has been pursued in order to take these ideas seriously. Do you have any comment about that?

A. Polyakov Of course it is all very closely related to string theory. It is essentially the question how the Schrödinger equation for string theory looks like, how to effectively write it down correctly. For example, how should one write the Schrödinger equation when the boundary data are given at infinity, like in AdS space. How should one write the Wheeler-DeWitt equation? Normally, we have some boundary and the Wheeler-DeWitt equation is the equation for the partition function as a functional of this boundary. But if this boundary is at infinity then, instead of the usual Laplacian, there should be some first order operator like the loop Laplacian. How should one make this concrete? I think these are important questions, both for string theory and for gauge theory, of course.

L. Faddeev After the "insult" to mathematics from my friend, I must say that I think there are two intuitions. There is the physical intuition and the mathematical intuition. And now, in this field, they compete and they help each other. There was a try to make a kind of consensus in Princeton five years ago, not with real success. But I think we have still to work somehow together to try to have some consensus of these two different intuitions.

H. Ooguri We will hear from three panelists now and then I will open the floor for discussions. I would like to suggest that the discussion today should focus on mathematical structures of string vacua, and in particular on what we have learnt about the space of vacua and on what we need to do to better understand it. We should leave the discussion on the physical implications, such as the multi-universes and the anthropic principle to the session on cosmology tomorrow.

4.3 Prepared Comments

4.3.1 *Renata Kallosh: Stabilization of moduli in string theory*

Stabilization of moduli is necessary for string theory to describe the effective 4-dimensional particle physics and cosmology. This is a long-standing problem. Recently a significant progress towards its solution was achieved: a combination of flux compactification with non-perturbative corrections leads to stabilization of moduli, with de Sitter vacua [1]. Such vacua with positive cosmological constant can be viewed as the simplest possibility for the string theory to explain the observable dark energy. This is a prerequisite for the Landscape of String Theory [2]. Main progress was achieved in type II string theories, the heterotic case still remains unclear, which is a serious problem for particle physics models related to string theory.

In type IIB string theory flux compactification with $W_{flux} = \int G_3 \wedge \Omega_3$ leads to stabilization of the dilaton and complex structure moduli [3], whereas gaugino condensation and/or instanton corrections $W_{non-pert} = Ae^{-(\mathrm{Vol}+i\alpha)}$ stabilize the remaining Kahler moduli. The basic steps here are: i) Using the warped geometry of the compactified space and nonperturbative effects one can stabilize all moduli in anti-de-Sitter space. ii) One can uplift the AdS space to a metastable dS space by adding anti-D3 brane at the tip of the conifold (or D7 brane with fluxes [4]). More recently there was a dramatic progress in moduli stabilization in string theory and various successful possibilities were explored [5]-[6].

Examples of new tools include the recently discovered criteria, in presence of fluxes, for the instanton corrections due to branes, wrapping particular cycles [7]. This has allowed one of the simplest models with all moduli stabilized. We have found that M-theory compactified on K3xK3 is incredibly simple and elegant [6]. Without fluxes in the compactified 3d theory there are two 80-dimensional quaternionic Kähler spaces, one for each K3. With non-vanishing primitive (2,2) flux, (2,0) and (0,2), each K3 becomes an attractive K3: one-half of all moduli are fixed. 40 in each K3 still remain moduli and need to be fixed by instantons. There are 20 proper 4-cycles in each K3. They provide instanton corrections from M5-branes wrapped on these cycles: Moduli space is no more...

All cases of moduli stabilization in black holes and in flux vacua which are due to fluxes in string theory can be described by the relevant attractor equations, the so-called "new attractors" [8].

It is known for about 10 years that in extremal black holes the moduli of vector multiplets are stabilized near the horizon where they become fixed function of fluxes (p, q) independently of the values of these moduli far away from the black hole horizon. This is known as a black hole attractor mechanism [9].

$$t_{fix} = t(p,q) \ , \qquad \bar{t}_{fix} = \bar{t}(p,q) \ . \tag{1}$$

Stabilization of moduli is equivalent to minimization of the black hole potential

$$V_{BH} = |DZ|^2 + |Z|^2 \tag{2}$$

defined by the central charge Z.

In case of BPS black holes the attractor equation relating fluxes to fixed values of moduli is

$$F_3 = 2\mathrm{Im}\,(\mathrm{Z}\,\bar{\Omega}_3)_{\mathrm{DZ}=0}\ . \tag{3}$$

It has been studied extensively over the last 10 years and many interesting solutions have been found. One of the most curious solutions of the black hole attractor equation is the so-called STU black holes with three moduli [10]. It was discovered recently [11] that the entropy of such black holes is given by the Caley's hyperdeterminant of the 2x2x2 matrix describing also the 3-qubit system in quantum information theory.

The non-BPS black holes under certain conditions also exhibit the attractor phenomenon: the moduli near the horizon tend to fixed values defined by fluxes [9, 12, 13]. The corresponding attractor equation is

$$F_3 = 2\mathrm{Im}\left[\, Z\,\bar{\Omega}_3 - D^I Z\, \bar{D}_I \bar{\Omega}_3\right]_{\partial V_{BH}=0} \tag{4}$$

This equation can be also used in the form

$$H_3 = 2\mathrm{Im}\left[Z\,\overline{\Omega}_3 - \frac{(\overline{\mathcal{D}}_{\bar{a}}\overline{\mathcal{D}}_{\bar{b}}\overline{Z})g^{\bar{a}a}g^{\bar{b}b}\mathcal{D}_b Z}{2Z}\mathcal{D}_a\Omega_3\right]\ . \tag{5}$$

Stabilization of moduli is equivalent to minimization of effective N=1 supergravity potential

$$V_{flux} = |DZ|^2 - 3|Z|^2 \tag{6}$$

defined by the effective central charge Z. All supersymmetric flux vacua in type IIB string theory compactified on a Calabi-Yau manifold are subject to the attractor equations defining the values of moduli in terms of fluxes.

$$F_4 = 2\mathrm{Re}\left[\mathrm{Z}\,\bar{\Omega}_4 + \mathrm{D}^{\underline{0}\mathrm{I}}\mathrm{Z}\,\bar{\mathrm{D}}_{\underline{0}\mathrm{I}}\bar{\Omega}_4\right]_{\mathrm{DZ}=0} \tag{7}$$

We may rewrite these equations in a form in which it is easy to recognize them as generalized attractor equations. The dependence on the axion-dilaton τ is explicit, whereas the dependence on complex structure moduli is un-explicit in the section (L, M).

$$\begin{pmatrix} p_h^a \\ \\ q_{ha} \\ \\ p_f^a \\ \\ q_{fa}\end{pmatrix} = \begin{pmatrix} \bar{Z}L^a + Z\bar{L}^a \\ \\ \bar{Z}M_a + Z\bar{M}_a \\ \\ \tau\bar{Z}L^a + \bar{\tau}ZL^a \\ \\ \tau\bar{Z}M_a + \bar{\tau}Z\bar{M}_a \end{pmatrix}_{DZ=0} + \begin{pmatrix} \bar{Z}^{\underline{0}I}D_IL^a + Z^{\underline{0}I}\bar{D}_I\bar{L}^a \\ \\ \bar{Z}^{\underline{0}I}D_IM_a + Z^{\underline{0}I}\bar{D}_I\bar{M}_a \\ \\ \bar{\tau}\bar{Z}^{\underline{0}I}D_IL^a + \tau Z^{\underline{0}I}\bar{D}_IL^a \\ \\ \bar{\tau}\bar{Z}^{\underline{0}I}D_IM_a + \tau Z^{\underline{0}I}\bar{D}_I\bar{M}_a \end{pmatrix}_{DZ=0} \tag{8}$$

The second term in this equation is absent in BH case. In the black hole case $Z=0$ and $DZ=0$ conditions lead to null singularity and runaway moduli. In flux vacua, the presence of the term proportional to chiral fermion masses, $\bar{\tau}\bar{Z}^{\underline{0}I}$, permits the stabilization of moduli in Minkowski flux vacua.

Finally, we can also describe all non-supersymmetric flux vacua which minimize the effective potential (6) by the corresponding attractor equation

$$F_4 = 2\mathrm{Re}\left[\mathrm{M}_{3/2}\bar{\Omega}_4 - \mathrm{F}^{\mathrm{A}}\bar{\mathrm{D}}_{\mathrm{A}}\Omega_4 + \mathrm{M}^{\underline{0}\mathrm{I}}\,\bar{\mathrm{D}}_{\underline{0}\mathrm{I}}\bar{\Omega}_4\right]_{\partial \mathrm{V}_{\mathrm{flux}}=0} \tag{9}$$

We have presented the common features and differences in stabilization of moduli near the black hole horizon and in flux vacua.

There is an apparent similarity between non-BPS extremal black holes with stabilized moduli and the O'Raifeartaigh model of spontaneous supersymmetry breaking. In models of this type the system cannot decay to a supersymmetric ground state since such a state does not exist, so the non-SUSY vacuum is stable. The same is true of the non-BPS black hole — there is a choice of fluxes which leads to an effective superpotential such that V_{BH} does not admit a supersymmetric minimum of the potential but does admit a non-supersymmetric one, see [12], [13]. It remains a challenge to construct the analog of the stable non-BPS extremal black holes in dS flux vacua.

Bibliography

[1] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, Phys. Rev. D **68** (2003) 046005 `hep-th/0301240`.

[2] L. Susskind, *The anthropic landscape of string theory*, `hep-th/0302219`.

[3] S. B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, Phys. Rev. D **66** (2002) 106006 `hep-th/0105097`.

[4] C. P. Burgess, R. Kallosh and F. Quevedo, *de Sitter string vacua from supersymmetric D-terms*, JHEP **0310** (2003) 056 `hep-th/0309187`.

[5] F. Denef, M. R. Douglas and B. Florea, *Building a better racetrack*, JHEP **0406** (2004) 034 `hep-th/0404257`; A. Saltman and E. Silverstein, *A new handle on de Sitter compactifications*, JHEP **0601** (2006) 139 `hep-th/0411271`; V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, *Systematics of moduli stabilisation in Calabi-Yau flux compactifications*, JHEP **0503** (2005) 007 `hep-th/0502058`; B. S. Acharya, F. Denef and R. Valandro, *Statistics of M theory vacua*, JHEP **0506** (2005) 056 `hep-th/0502060`; F. Denef, M. R. Douglas, B. Florea, A. Grassi and S. Kachru, *Fixing all moduli in a simple F-theory compactification*, `hep-th/0503124`; J. P. Derendinger, C. Kounnas, P. M. Petropoulos and F. Zwirner, *Fluxes and gaugings: $N = 1$ effective superpotentials*, Fortsch. Phys. **53** (2005) 926 `hep-th/0503229`; O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, *Type IIA moduli stabilization*, JHEP **0507** (2005) 066 `hep-th/0505160`; D. Lust, S. Reffert, W. Schulgin and S. Stieberger, *Moduli stabilization in type IIB orientifolds. I: Orbifold limits*, `hep-th/0506090`.

[6] P. S. Aspinwall and R. Kallosh, *Fixing all moduli for M-theory on K3 x K3*, JHEP **0510** (2005) 001 `hep-th/0506014`.

[7] E. Bergshoeff, R. Kallosh, A. K. Kashani-Poor, D. Sorokin and A. Tomasiello, *An index for the Dirac operator on D3 branes with background fluxes*, JHEP **0510** (2005) 102 `hep-th/0507069`. R. Kallosh, A. K. Kashani-Poor and A. Tomasiello, *Counting fermionic zero modes on M5 with fluxes*, JHEP **0506** (2005) 069 `hep-th/0503138`.

[8] R. Kallosh, *New attractors*, JHEP **0512** (2005) 022 hep-th/0510024; A. Giryavets, *New attractors and area codes*, hep-th/0511215.

[9] S. Ferrara, R. Kallosh and A. Strominger, *N=2 extremal black holes*, Phys. Rev. D **52** (1995) 5412 hep-th/9508072; A. Strominger, *Macroscopic Entropy of $N = 2$ Extremal Black Holes*, Phys. Lett. B **383** (1996) 39 hep-th/9602111; S. Ferrara and R. Kallosh, *Universality of Supersymmetric Attractors*, Phys. Rev. D **54** (1996) 1525 hep-th/9603090; S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. D **54** (1996) 1514 hep-th/9602136; S. Ferrara, G. W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, Nucl. Phys. B **500** (1997) 75 hep-th/9702103; G. W. Moore, *Arithmetic and attractors*, hep-th/9807087; G. W. Moore, *Les Houches lectures on strings and arithmetic*, hep-th/0401049; F. Denef, B. R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, hep-th/0101135.

[10] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, *STU black holes and string triality*, Phys. Rev. D **54** (1996) 6293 hep-th/9608059.

[11] M. J. Duff, *String triality, black hole entropy and Cayley's hyperdeterminant*, hep-th/0601134; R. Kallosh and A. Linde, *Strings, black holes, and quantum information*, hep-th/0602061.

[12] P. K. Tripathy and S. P. Trivedi, *Non-supersymmetric attractors in string theory*, hep-th/0511117; A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, JHEP **0509** (2005) 038 hep-th/0506177.

[13] R. Kallosh, N. Sivanandam and M. Soroush, *The non-BPS black hole attractor equation*, hep-th/0602005.

4.3.2 *Dieter Lüst: A short remark on flux and D-brane vacua and their statistics*

String compactifications provide a beautiful link between particle physics and the geometrical and topological structures of the corresponding background geometries. Already from the "old days" of heterotic string compactifications we know that they exist a large number of consistent string compactifications with or without (being tachyon free) space-time supersymmetry, non-Abelian gauge groups and chiral matter field representations, i.e. with more or less attractive phenomenological features. In particular, a concrete number of the order of $\mathcal{N}_{vac} \sim 10^{1500}$ of heterotic string models within the covariant lattice constructions was derived [1]. More recently, a detailed analysis of type II orientifold string compactifications with D-branes, their spectra, their effective actions and also of their statistical properties was performed, and also the study of heterotic string models and their landscape was pushed forward during the last years. In this note we will comment on type II orientifold compactifications with closed string background fluxes and with open strings ending on D-branes. Two questions will be central in our discussion: first we will briefly discuss the procedure of moduli stabilization due to background fluxes and non-perturbative superpotentials. Second, we will be interested in the question what is the fraction of all possible open string D-brane configurations within a given class of orientifold models (like the $Z_2 \times Z_2$ orientifold with background fluxes) that have realistic Standard Model like properties, such as gauge group $SU(3) \times SU(2) \times U(1)$, three generations of quarks and leptons, etc. More concretely, the following steps will be important:

- We begin with choosing a toroidal Z_N resp. $Z_N \times Z_M$ type II orbifold which preserves $\mathcal{N} = 2$ space-time supersymmtry in the closed string sector.
- A consistent orientifold projection has to be performed. This yields O-planes and in general changes the geometry. The bulk space-time supersymmetry is reduced to $\mathcal{N} = 1$ by the orientifold projection. The tadpoles due to the O-planes must be cancelled by adding D-branes and/or certain background fluxes. Then the resulting Ramond-Ramond tadpole equations as well as the NS-tadpoles, which ensure $\mathcal{N} = 1$ space-time supersymmetry on the D-branes, together with constraints from K-theory provide restrictions for the allowed D-brane configurations. For each of the allowed D-brane model one has to determine the corresponding open string spectrum, namely the gauge groups and the massless matter fields, where the chiral $\mathcal{N} = 1$ matter fields are located at the various brane intersections.
- In order to stabilize the moduli one is turning on certain background fluxes that generate a potential for the moduli. According to the KKLT proposal [2], 3-form fluxes in type IIB can fix all complex structure moduli and the dilaton. On the other hand, the Kähler moduli can be fixed by non-perturbative effects. In case the fluxes or the non-perturbative superpotential break $\mathcal{N} = 1$ supersymmetry,

one can compute the soft mass terms for the open string states on the D-branes.

- In the statistical search for D-brane models with Standard Model like properties one first has to count all possible solutions of the tadpole and K-theory constraints. Then one applies certain physical thresholds, i.e. counting those models with Standard Model gauge group or those models with a certain number of chiral matter fields. Of particular interest is the question whether certain physical observables are statistically correlated.

First let us give a few comments on the moduli stabilization process due to background fluxes and non-perturbative effects. To be specific consider type IIB Ramond and NS 3-form fluxes through 3-cycles of a Calabi-Yau space X. They give rise to the following effective flux superpotential in four dimensions [3–5]:

$$W_{\text{flux}}(\tau, U) \sim \int_X (H_3^R + \tau H_3^{NS}) \wedge \Omega\,. \tag{1}$$

It depends on the dilaton τ and also on the complex structure moduli U. However, since W_{flux} does not depend on the Kähler moduli, one needs additional non-perturbative contributions to the superpotential in order to stabilize them. These are provided by Euclidean D3-instantons [6], which are wrapped around 4-cycles (divisors) D inside X, and/or gaugino condensations in hidden gauge group sectors on the world volumes of D7-branes, which are also wrapped around certain divisors D. Both give rise to terms in the superpotential of the form

$$W_{\text{n.p.}} \sim g_i e^{-a_i V_i}\,, \tag{2}$$

where V_i is the volume of the divisor D_i, depending on the Kähler moduli T. Note that the prefactor g_i is in general not a constant, but rather depends on the complex structure moduli U. The generation of a non-perturbative superpotential crucially depends on the D-brane zero modes of the wrapping divisors, i.e. on the topology of the divisors together with their interplay with the O-planes and also with the background fluxes [7–10].

The moduli are stabilized to discrete values by solving the $\mathcal{N} = 1$ supersymmetry conditions

$$D_A W = 0 \quad (\text{vanishing} \ \ \text{F} - \text{term}). \tag{3}$$

Then typically the superpotentials of the form $W_{\text{flux}} + W_{\text{n.p.}}$ lead to stable supersymmetric AdS_4 minima. Additional restrictions on the form of the possible superpotential arise [11, 12] requiring that the mass matrix of all the fields (S, T, U) is already positive definite in the AdS vacuum (absence of tachyons), as it is necessary, if one wants to uplift the AdS vacua to a dS vacuum by a (constant) shift in the potential. These conditions cannot be satisfied in orientifold models without any complex structure moduli, i.e. for Calabi-Yau spaces with Hodge number $h^{2,1} = 0$. Alternatively one can also look for supersymmetric 4D Minkowski mimima which solve eq.(3) [13, 14]. They may exist if $W_{\text{n.p.}}$ is of the racetrack form. In this case the requirement that all flat directions are lifted in the Minkowski vacuum leads to similar constraints as the absence of tachyon condition in the AdS case.

In more concrete terms, the moduli stabilization procedure was studied in [15] for the $T^6/Z_2 \times Z_2$ orientifold, with the result that all moduli indeed can be fixed. Moreover in [12, 16, 17] all other Z_N and $Z_N \times Z_M$ orientfolds were studied in great detail, where it turns out that in order to have divisors, which contribute to the non-perturbative superpotential, one has to consider the blown-up orbifold geometries. Then the divisors originating from the blowing-ups give rise to D3-instantons and/or gaugino condensates, being rigid and hence satisfying the necessary topological conditions. As a result of this investigation of all possible orbifold models, it turns out that the $Z_2 \times Z_2$, $Z_2 \times Z_4$, Z_4, Z_{6-II} orientifolds are good candidates where all moduli can be completely stabilized.

The statistical approach to the flux vacua amounts to count all solutions of the $\mathcal{N} = 1$ supersymmetry condition eq.(3) refs. In fact, it was then shown that the number of flux vacua on a given background space is very huge [18, 19]: $\mathcal{N}_{vac} \sim 10^{500}$. In addition there is another method to assign a probability measure to flux compactifications via a black hole entropy functional $\mathcal{S}$. This method however does not apply to 3-form flux compactifications but rather to Ramond 5-form compactifications on $S^2 \times X$, hence leading to AdS_2 vacua. Specifically a connection between 4D black holes and flux compactifications is provided by type $\mathcal{N} = 2$ black hole solutions, for which the near horizon condition $DZ = 0$ can be viewed as the the extremization condition of a corresponding 5-form superpotential $W \sim \int_{S^2 \times X}(F_5 \wedge \Omega)$ [20]. In view of this connection, it was suggested in [20, 21] to interpret $\psi = e^{\mathcal{S}}$ as a probability distribution resp. wave function for flux compactifications, where ψ essentially counts the microscopic string degrees of freedom, which are associated to each flux vacuum. Maximization of the entropy $\mathcal{S}$ then shows that points in the moduli space, where a certain number of hypermultiplets become massless like the conifold point, are maxima of the entropy functional [22].

Now let us also include D-branes and discuss the statistics of D-brane models with open strings [23–25]. To be specific we discuss the toroidal type IIA orientifold $T^6/Z_2 \times Z_2$ at the orbifold point. We have to add D6-branes wrapping special Langrangian 3-cycles. They are characterized by integer-valued coefficients X^I, Y^I $(I = 0, \ldots, 3)$. The supersymmetry conditions, being equivalent to the vanishing of the D-term scalar potential, have the form:

$$\sum_{I=0}^{3} \frac{Y^I}{U_I} = 0\,, \quad \sum_{I=0}^{3} X^I U_I > 0\,. \tag{4}$$

The Ramond tadpole cancellation conditions for k stacks of N_a D6-branes are given by

$$\sum_{a=1}^{k} N_a \vec{X}_a = \vec{L}\,, \tag{5}$$

where the L^I parameterize the orientifold charge. In addition there are some more constraints from K-theory. Chiral matter in bifundamental representations originate from open strings located at the intersection of two stacks of D6-branes with a multiplicity (generation) number given by the intersection number

$$I_{ab} = \sum_{I=0}^{3}(X_a^I Y_b^I - X_b^I Y_a^I)\,. \tag{6}$$

Counting all possible solutions of the D-brane equations (4) and (5) leads to a total of $1.66 \cdot 10^8$ supersymmetric (4-stack) D-brane models on the $Z_2 \times Z_2$ orientifold. With this large sample of models we can ask the question which fraction of models satisfy several phenomenological constraints that gradually approach the spectrum of the supersymmetric MSSM. This is summarized in the following table: Therefore

Restriction	Factor
gauge factor $U(3)$	0.0816
gauge factor $U(2)/Sp(2)$	0.992
No symmetric representations	0.839
Massless $U(1)_Y$	0.423
Three generations of quarks	2.92×10^{-5}
Three generations of leptons	1.62×10^{-3}
Total	1.3×10^{-9}

only one in a billion models give rise to an MSSM like D-brane vacuum. Similar results can be obtained for models with $SU(5)$ GUT gauge group [26].

Finally we would like to pose the following question: is it possible to obtain an entropy resp. a probability wave function for D-brane vacua? To answer this question one might try to replace the D7-branes in IIB (D6-branes in IIA) by D5-branes (D4-branes). This will lead to cosmic strings in D=4. So reformulating this question would mean, can one associate an entropy to this type of cosmic string solutions? In this way one could derive, besides the statistical counting factor, a stringy probability measure for deriving the Standard Model from D-brane models.

Bibliography

[1] W. Lerche, D. Lüst and A. N. Schellekens, "Chiral Four-Dimensional Heterotic Strings From Selfdual Lattices," Nucl. Phys. B **287** (1987) 477.
[2] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, "De Sitter vacua in string theory," Phys. Rev. D **68** (2003) 046005 [arXiv:hep-th/0301240].
[3] S. Gukov, C. Vafa and E. Witten, "CFT's from Calabi-Yau four-folds," Nucl. Phys. B **584** (2000) 69 [Erratum-ibid. B **608** (2001) 477] [arXiv:hep-th/9906070].
[4] T. R. Taylor and C. Vafa, "RR flux on Calabi-Yau and partial supersymmetry breaking," Phys. Lett. B **474** (2000) 130 [arXiv:hep-th/9912152].
[5] S. B. Giddings, S. Kachru and J. Polchinski, "Hierarchies from fluxes in string compactifications," Phys. Rev. D **66** (2002) 106006 [arXiv:hep-th/0105097].

[6] E. Witten, "Non-Perturbative Superpotentials In String Theory," Nucl. Phys. B **474** (1996) 343 [arXiv:hep-th/9604030].
[7] L. Görlich, S. Kachru, P. K. Tripathy and S. P. Trivedi, "Gaugino condensation and nonperturbative superpotentials in flux compactifications," JHEP **0412** (2004) 074 [arXiv:hep-th/0407130].
[8] P. K. Tripathy and S. P. Trivedi, "D3 brane action and fermion zero modes in presence of background flux," JHEP **0506** (2005) 066 [arXiv:hep-th/0503072].
[9] E. Bergshoeff, R. Kallosh, A. K. Kashani-Poor, D. Sorokin and A. Tomasiello, "An index for the Dirac operator on D3 branes with background fluxes," JHEP **0510** (2005) 102 [arXiv:hep-th/0507069].
[10] D. Lüst, S. Reffert, W. Schulgin and P. K. Tripathy, "Fermion zero modes in the presence of fluxes and a non-perturbative arXiv:hep-th/0509082.
[11] K. Choi, A. Falkowski, H. P. Nilles, M. Olechowski and S. Pokorski, "Stability of flux compactifications and the pattern of supersymmetry JHEP **0411** (2004) 076 [arXiv:hep-th/0411066].
[12] D. Lüst, S. Reffert, W. Schulgin and S. Stieberger, "Moduli stabilization in type IIB orientifolds. I: Orbifold limits," arXiv:hep-th/0506090.
[13] J. J. Blanco-Pillado, R. Kallosh and A. Linde, "Supersymmetry and stability of flux vacua," arXiv:hep-th/0511042.
[14] D. Krefl and D. Lüst, "On supersymmetric Minkowski vacua in IIB orientifolds," arXiv:hep-th/0603166.
[15] F. Denef, M. R. Douglas, B. Florea, A. Grassi and S. Kachru, "Fixing all moduli in a simple F-theory compactification," arXiv:hep-th/0503124.
[16] S. Reffert and E. Scheidegger, "Moduli stabilization in toroidal type IIB orientifolds," arXiv:hep-th/0512287.
[17] D. Lüst, S. Reffert, E. Scheidegger, W. Schulgin and S. Stieberger, paper to appear.
[18] R. Bousso and J. Polchinski, "Quantization of four-form fluxes and dynamical neutralization of the cosmological constant," JHEP **0006** (2000) 006 [arXiv:hep-th/0004134].
[19] S. Ashok and M. R. Douglas, "Counting flux vacua," JHEP **0401** (2004) 060 [arXiv:hep-th/0307049].
[20] H. Ooguri, C. Vafa and E. P. Verlinde, "Hartle-Hawking wave-function for flux compactifications," Lett. Math. Phys. **74** (2005) 311 [arXiv:hep-th/0502211].
[21] S. Gukov, K. Saraikin and C. Vafa, "The entropic principle and asymptotic freedom," arXiv:hep-th/0509109.
[22] G. L. Cardoso, D. Lüst and J. Perz, paper to appear
[23] R. Blumenhagen, F. Gmeiner, G. Honecker, D. Lüst and T. Weigand, "The statistics of supersymmetric D-brane models," Nucl. Phys. B **713** (2005) 83 [arXiv:hep-th/0411173].
[24] F. Gmeiner, R. Blumenhagen, G. Honecker, D. Lüst and T. Weigand, "One in a billion: MSSM-like D-brane statistics," JHEP **0601** (2006) 004 [arXiv:hep-th/0510170].
[25] F. Gmeiner, "Standard model statistics of a type II orientifold," arXiv:hep-th/0512190.
[26] F. Gmeiner and M. Stein, "Statistics of SU(5) D-brane models on a type II orientifold," arXiv:hep-th/0603019.

4.3.3 *Michael Douglas: Mathematics and String Theory: Understanding the landscape*

4.3.3.1 *Historical analogies*

At a conference with such a distinguished history, one cannot help but look for analogies between the present and the past. It is tempting to compare our present struggles to understand string theory, and to find clearer evidence for or against the claim that it describes our universe, with the deep issues discussed at past Solvay Conferences, particularly in 1911 and 1927.

As was beautifully described here by Peter Galison, the 1911 meeting focused on the theory of radiation, and the quantum hypotheses invented to explain black body radiation and the photoelectric effect. These were simple descriptions of simple phenomena, which suggested a new paradigm. This was to accept the basic structure of previous models, but modify the laws of classical mechanics by inventing new, somewhat *ad hoc* rules governing quantum phenomena. This paradigm soon scored a great success in Bohr's theory of the hydrogen atom. The discovery of the electron and Rutherford's scattering experiments had suggested modeling an atom as analogous to a planetary system. But while planetary configurations are described by continuous parameters, real atoms have a unique ground state, well-defined spectral lines associated with transitions from excited states, etc. From Bohr's postulate that the action of an allowed trajectory was quantized, he was able to deduce all of these features and make precise numerical predictions.

While very successful, it was soon found that this did not work for more complicated atoms like helium. A true quantum mechanics had to be developed. Most of its essential ideas had appeared by the 1927 meeting. Although the intuitions behind the Bohr atom turned out to be correct, making them precise required existing but unfamiliar mathematics, such as the theories of infinite dimensional matrices, and wave equations in configuration space.

Are there fruitful analogies between these long-ago problems and our own? What is the key issue we should discuss in 2005? What are our hydrogen atom(s)?

If we have them, they are clearly the maximally supersymmetric theories, whose basic physics was elucidated in the second superstring revolution of 1994–98. It's too bad we can't use them to describe real world physics. But they have precise and pretty formulations, and can be used to model one system we believe exists in our universe, the near-extremal black hole. We now have microscopic models of black holes, which explain their entropy.

Perhaps we can place our position as analogous to the period between 1913 and 1927.[2] Starting from our simple and attractive maximally supersymmetric theories, we are now combining their ingredients in a somewhat *ad hoc* way, to construct $N = 1$ and nonsupersymmetric theories, loose analogs of helium, molecules, and

[2]A similar analogy was made by David Gross in talks given around 2000. However, to judge from his talk here, he now has serious reservations about it.

more complicated systems. The Standard Model, with its 19 parameters, has a complexity perhaps comparable to a large atom or small molecule. The difficulty of our present struggles to reproduce its observed intricacies and the underlying infrastructure (moduli stabilization, supersymmetry breaking), discussed here by Kallosh, Lüst and others, are probably a sign that we have not yet found the best mathematical framework.

4.3.3.2 *The chemical analogy*

What might this "best mathematical framework" be? And would knowing it help with the central problems preventing us from making definite predictions and testing the theory?

In my opinion, the most serious obstacle to testing the theory is the problem of vacuum multiplicity. This has become acute with the recent study of the string/M theory landscape. We have a good reason to think the theory has more than 10^{122} vacua, the Weinberg-Banks-Abbott-Brown-Teitelboim-Bousso-Polchinski *et al* solution to the cosmological constant problem. Present computations give estimates more like 10^{500} vacua. We do not even know the number of candidate vacua is finite. Even granting that it is, the problem of searching through all of them is daunting. Perhaps *a priori* selection principles or measure factors will help, but there is little agreement on what these might be. We should furthermore admit that the explicit constructions of vacua and other arguments supporting this picture, while improving, are not yet incontrovertible.

We will shortly survey a few mathematical frameworks which may be useful in coming to grips with the landscape, either directly or by analogy. They are generally not familiar to physicists. I think the main reason for this is that analogous problems in the past were attacked in different, non-mathematical ways. Let us expand a bit on this point.

String theory is by no means the first example of an underlying simple and unique framework describing a huge, difficult to comprehend multiplicity of distinct solutions. There is another one, very well known, which we might consider as a source of analogies.

As condensed matter physicists never tire of reminding us, all of the physical properties of matter in the everyday world, and the diversity of chemistry, follow in principle from a well established "theory of everything," the Schrödinger equations governing a collection of electrons and nuclei. Learning even the rough outlines of the classification of its solutions takes years and forms the core of entire academic disciplines: chemistry, material science, and their various interdisciplinary and applied relatives.

Of course, most of this knowledge was first discovered empirically, by finding, creating and analyzing different substances, with the theoretical framework coming much later. But suppose we were given the Schrödinger equation and Coulomb potential without this body of empirical knowledge? Discovering the basics of chemistry would be a formidable project, and there are many more layers of structure to elucidate before one would reach the phenomena usually discussed in condensed matter physics: phase transitions, strong correlations, topological structures and defects, and so on.

As in my talk at String 2003, one can develop this analogy, by imagining beings who are embedded in an effectively infinite crystal, and can only do low energy experiments. Say they can observe the low-lying phonon spectrum, measure low frequency conductivity, and so on. Suppose among their experiments they can create electron-hole bound states, and based on phenomenological models of these they hypothesize the Schrödinger equation. They would have some empirical information, but not the ability to manipulate atoms and create new molecules. How long would it take them to come up with the idea of crystal lattices of molecules, and how much longer would it take them to identify the one which matched their data?

Now, consider the impressive body of knowledge string theorists developed in the late 1990's, assembling quasi-realistic compactifications out of local constituents such as branes, singularities, and so on. Individual constituents are simple, their basic properties largely determined by the representation theory of the maximal supersymmetry algebras in various dimensions. The rules for combining pairs of objects, such as intersecting branes or branes wrapping cycles – which combinations preserve supersymmetry, and what light states appear – are not complicated either. What is complicated is the combination of the whole required to duplicate the Standard Model, stabilize moduli, break supersymmetry and the rest. Perhaps all this is more analogous to chemistry than we would like to admit.

Other parallels can be drawn. For example, as Joe Polchinski pointed out in his talk, according to standard nuclear physics, the lowest energy state of a collection of electrons, protons and neutrons is a collection of Fe_{26} atoms, and thus almost all molecules in the real world are unstable under nuclear processes. Suppose this were the case for our crystal dwellers as well. After learning about these processes, they might come to a deep paradox: how can atoms other than iron exist at all? Of course, because of Coulomb barriers, the lifetime of matter is exceedingly long, but still finite, just as is claimed for the metastable de Sitter vacua of KKLT.

Perhaps all this is a nightmare from which we will awake, the history of Kekulé's dream being repeated as farce. If so, all our previous experience as physicists suggests that the key to the problem will be to identify some sort of **simplicity** which we have not seen in the problem so far. One might look for it in the physics of some dual or emergent formulation. But one might also look for it in mathematics. It is not crazy to suppose that the only consistent vacua are those which respect some principle or have some property which would only be apparent in an exact treatment. But what is that exact treatment going to look like? The ones we have now cannot be formulated without bringing in mathematics such as the geometry of Calabi-Yau manifolds, or the category theory underlying topological string theory. If we ever find exact descriptions of $N = 1$ or broken supersymmetry vacua, surely this will be by uncovering even more subtle mathematical structures.

But suppose the landscape in its present shape is real, and the key to the problem is to manage and abstract something useful out of its **complexity**. The tools we will need may not be those we traditionally associated with fundamental physics, but might be inspired by other parts of physics and even other disciplines. But such inspiration can not be too direct; the actual problems are too different. Again, we are probably better off looking to mathematical developments which capture the essence of the ideas and then generalize them, as more likely to be relevant.

On further developing these analogies, one realizes that we do not know even the most basic organizing principles of the stringy landscape. For the landscape of chemistry, these are the existence of atoms, the fact that each atom (independent of its type) takes up a roughly equal volume in three-dimensional space, and that binding interactions are local. This already determines the general features of matter, such as the fact that densities of solids range from 1–20 g/cm^3. Conjectures on the finite number of string vacua, on bounds on the number of massless fields or ranks of gauge groups, and so on, are suggestions for analogous general features of string vacua. But even knowing these, we would want organizing principles. The following brief overviews should be read with this question in mind.

4.3.3.3 *Two-dimensional CFT*

This is not everything, but a large swathe through the landscape. We do not understand it well enough. In particular, the often used concept of "the space of 2d CFT's," of obvious relevance for our questions, has never been given any precise meaning.

A prototype might be found in the mathematical theory of the space of all Riemannian manifolds. This exists and is useful for broad general statements. We recall Cheeger's theorem [5]:

A set of manifolds with metrics $\{X_i\}$, satisfying the following bounds,

(1) diameter$(X_i) < d_{max}$
(2) Volume $(X_i) > V_{min}$
(3) Curvature K satisfies $|K(X_i)| < K_{max}$ at every point,

contains a finite number of distinct homeomorphism types (and diffeomorphism types in $D \neq 4$).

Since (2) and (3) are conditions for validity of supergravity, while (1) with $d_{max} \sim 10\mu m$ follows from the validity of the gravitational inverse square law down to this distance, this theorem implies that there are finitely many manifolds which can be used for candidate supergravity compactifications [9, 2].

This and similar theorems are based on more general quasi-topological statements such as Cheeger-Gromov precompactness of the space of metrics – *i.e.*, infinite sequences have Cauchy subsequences, and cannot "run off to infinity." This is shown by constructions which break any manifold down into a finite number of coordinate patches, and showing that these patches and their gluing can be described by a finite amount of data.

Could we make any statement like this for the space of CFT's? (a question raised by Kontsevich). The diameter bound becomes a lower bound Δ_{min} on the operator dimensions (eigenvalues of $L_0 + \bar{L}_0$). We also need to fix c. Then, the question seems well posed, but we have no clear approach to it. Copying the approach in terms of coordinate patches does not seem right.

The key point in defining any "space" of anything is to put a topology on the set of objects. Something less abstract from which a topology can be derived is a distance between pairs of objects $d(X, Y)$ which satisfies the axioms of a metric, so that it can be used to define neighborhoods.

The usual operator approach to CFT, with a Hilbert space $\mathcal{H}$, the Virasoro algebras with $H = L_0 + \bar{L}_0$, and the operator product algebra, is very analogous to spectral geometry:

$$L_0 + \bar{L}_0 \text{ eigenvalues} \sim \text{spectrum of Laplacian } \Delta$$
$$\text{o.p.e. algebra} \sim \text{algebra of functions on a manifold}$$

Of course the o.p.e. algebra is not a standard commutative algebra and this is analogy, but a fairly close one.

A definition of a distance between a pair of manifolds with metric, based on spectral geometry, is given in Bérard, Besson, and Gallot [4]. The idea is to consider the entire list of eigenfunctions $\psi_i(x)$ of the Laplacian,

$$\Delta \psi_i = \lambda_i \psi_i,$$

as defining an embedding Ψ of the manifold into ℓ_2, the Hilbert space of semi-infinite sequences (indexed by i):

$$\Psi : x \rightarrow \{e^{-t\lambda_1}\psi_1(x), e^{-t\lambda_2}\psi_2(x), \ldots, e^{-t\lambda_n}\psi_n(x), \ldots\}.$$

We weigh by $e^{-t\lambda_i}$ for some fixed t to get convergence in ℓ_2.

Then, the distance between two manifolds M and M' is the Hausdorff distance d between their embeddings in ℓ_2. Roughly, this is the amount $\Psi(M)$ has to be "fuzzed out" to cover $\Psi(M')$.

In principle this definition might be directly adapted to CFT, where the x label boundary states $|x\rangle$ (which are the analog of points) and the $\psi_i(x)$ are their overlaps with closed string states $|\phi_i\rangle$,

$$x \rightarrow \langle \phi_i | e^{-t(L_0 + \bar{L}_0)} | x \rangle.$$

Another candidate definition would use the o.p.e. coefficients

$$\phi_i \phi_j \rightarrow \sum C_{ij}^k (z_i - z_j)^{\Delta_k - \Delta_i - \Delta_j} \ \phi_k$$

for all operators with dimensions between Δ_{min} and some Δ_{max} (one needs to show that this choice drops out), again weighted by $e^{-t(L_0 + \bar{L}_0)}$. The distance between a pair of CFT's would then be the ℓ_2 norm of the differences between these sets of numbers.

While abstract, this would make precise the idea of the "space of all 2D CFT's" and give a foundation for mapping it out.

4.3.3.4 *Topological open strings and derived categories*

This gives an example in which we actually know "the space of all X" in string theory. It is based on the discussion of boundary conditions and operators in CFT, which satisfy an operator product algebra with the usual non-commutativity of open strings. If we modify the theory to obtain a subset of dimension zero operators (by twisting to get a topological open string, taking the Seiberg-Witten limit in a B field, etc.), the o.p.e. becomes a standard associative but non-commutative algebra. This brings us into the realm of noncommutative geometry.

There are many types of noncommutative geometry. For the standard topological string obtained by twisting an $N = 2$ theory, the most appropriate is based on algebraic geometry. As described at the Van den Bergh 2004 Francqui prize colloquium, this is a highly developed subject, which forms the backdrop to quiver gauge theories, D-branes on Calabi-Yau manifolds, and so on.

One can summarize the theory of D-branes on a Calabi-Yau X in these terms as the "Pi-stable objects in the derived category $D(\text{Coh } X)$," as reviewed in [3]. Although abstract, the underlying idea is simple and physical. It is that all branes can be understood as bound states of a finite list of "generating branes," one for each generator of K theory, and their antibranes. The bound states are produced by tachyon condensation. Varying the Calabi-Yau moduli can vary masses of these condensing fields, and if one goes from tachyonic to massive, a bound state becomes unstable.

This leads to a description of all D-branes, and "geometric" pictures for all the processes of topology change which were considered "non-geometric" from the purely closed string point of view. For example, in a flop transition, an S^2 Σ is cut out and replaced with another S^2 Σ' in a topologically different embedding. In the derived category picture, what happens is that the brane wrapped on Σ, and all D0's (points) on Σ, go unstable at the flop transition, to be replaced by new branes on Σ'.

The general idea of combining classical geometric objects, using stringy rules of combination, and then extrapolating to get a more general type of geometry, should be widely useful.

4.3.3.5 *Computational complexity theory*

How hard is the problem of finding quasi-realistic string vacua? Computer scientists classify problems of varying degrees of difficulty:

- P can be solved in time polynomial in the size of the input.
- An NP problem has a solution which can be checked in polynomial time, but is far harder to find, typically requiring a search through all candidate solutions.
- An NP-complete problem is as hard as any NP problem – if any of these can be solved quickly, they all can.

It turns out that many of the problems arising in the search for string vacua are in NP or even NP-complete. [6] For example, to find the vacua in the Bousso-Polchinski model with cosmological constant $10^{-122}M_{Planck}^4$, one may need to search through 10^{122} candidates.

How did the universe do this? We usually say that the "multiverse" did it – many were tried, and we live in one that succeeded. But some problems are too difficult for the multiverse to solve in polynomial time. This is made precise by Aaronson's definition of an "anthropic computer." [1]

Using these ideas, Denef and I [7] have argued that the vacuum selected by the measure factor $\exp 1/\Lambda$ cannot be found by a quantum computer, working in polynomial time, even with anthropic postselection. Thus, if a cosmological model realizes this measure factor (and many other preselection principles which can be expressed as optimizing a function), it is doing something more powerful than such a computer.

Some cosmological models (e.g. eternal inflation) explicitly postulate exponentially long times, or other violations of our hypotheses. But for other possible theories, for example a field theory dual to eternal inflation, this might lead to a paradox.

4.3.3.6 *Conclusions*

We believe string theory has a set of solutions, some of which might describe our world. Even leaving aside the question of few vacua or many, and organizing principles, perhaps the most basic question about the landscape is whether it will turn out to be more like mathematics, or more like chemistry.

Mathematical analogy: like classification of Lie groups, finite simple groups, Calabi-Yau manifolds, etc. Characterized by simple axioms and huge symmetry groups. In this vision, the overall structure is simple, while the intricacies of our particular vacuum originate in symmetry breaking analogous to that of more familiar physical systems.

Chemical analogy: simple building blocks (atoms; here branes and extended susy gauge theory sectors) largely determined by symmetry. However, these are combined in intricate ways which defy simple characterization and require much study to master.

The current picture, as described here by Kallosh and Lüst, seems more like chemistry. Chemistry is a great science, after all the industrial chemistry of soda is what made these wonderful conferences possible. But it will surely be a long time (if ever) before we can manipulate the underlying constituents of our vacuum and produce new solutions, so this outcome would be less satisfying.

Still, our role as physicists is not to hope that one or the other picture turns out to be more correct, but to find the evidence from experiment and theory which will show us which if any of our present ideas are correct.

Acknowledgments I would like to thank the organizers and the Solvay Institute for a memorable meeting, and B. Acharya, F. Denef, M. Kontsevich, S. Zelditch and many others for collaboration and discussion of these ideas. This research was supported in part by DOE grant DE-FG02-96ER40959.

Bibliography

[1] S. Aaronson, "Quantum Computing, Postselection, and Probabilistic Polynomial-Time," arXiv:quant-ph/0412187.
[2] B. Acharya and M. R. Douglas, to appear.
[3] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," arXiv:hep-th/0403166.
[4] P. Bérard, G. Besson, and S. Gallot, "Embedding Riemannian manifolds by their heat kernel," Geom. Funct. Anal. 4 (1994), 373–398.
[5] J. Cheeger, "Finiteness theorems for Riemannian manifolds," Am. J. Math **92** (1970), 61–74.
[6] F. Denef and M. R. Douglas, "Computational Complexity of the Landscape I," hep-th/0602072.
[7] F. Denef and M. R. Douglas, "Computational Complexity of the Landscape II," to appear.
[8] M. R. Douglas, talk at Strings 2003.
[9] M. R. Douglas, talk at Strings 2005.

4.4 Discussion

A. Van Proeyen I want to make a remark or maybe a question about the structure of KKLT (Kachru, Kallosh, Linde and Trivedi). As far as I know, there is still no consistent supergravity framework for it. For the first step, there is no problem when one uses the superpotential. But then the uplifting needs D-terms which, as far as I heard, always needs the Fayet-Illiopoulos terms. But that is not consistent if one has already put a superpotential. So I do not know how this can be solved and brought into a consistent effective supergravity framework. Or is there another mechanism that someone sees? As far as I can see, the KKLT framework as a supergravity theory is just not yet consistent.

R. Kallosh In short, since 2003 when we attempted this uplifting, it was clear that the supergravity has Fayet-Illiopoulos terms, but to get them from string theory is rather difficult. So the best case we know today is when we have D7-branes at the tip of the conifold and fluxes on it. From the perspective of string theory this looks as close to a consistent D-term in supergravity as possible. At present we cannot do better, but I hope that somebody will. It needs to be done.

H. Ooguri In that spirit I would also like to note that the study of the landscape is still looking at a very limited range of the possible moduli space. There is a big territory that needs to be understood. In particular, in the context of developing mathematical tools to understand it, I would like to note that it is very important to understand the stringy corrections to this program. I hope to have some progress in that direction. I guess Kachru has a comment on that.

S. Kachru I completely agree with what you just said. This is more a comment about the status of constructions. The initial construction used some kind of configuration of branes to get a positive energy. But in fact if you look through the literature that has been generated in the last three years, there are now somehow an infinite number of proposals. The most mild one actually just uses the F-term potential coming from fluxes themselves to give the positive term. I actually do not see any possible inconsistency with embedding into supergravity. Explicit examples that give examples where the F-terms are non zero in the minima of the flux potential were actually constructed by Saltman and Silverstein. The statistics of Denef and Douglas that were quoted by Lüst in giving 10^{500} models, were actually more or less counting those vacua. So this may be relevant to Van Proeyen's question.

A. Van Proeyen As far as I can see, as long as one only has the F-terms, one still has vacua with a negative cosmological constant. You still need the uplifting.

S. Kachru No. My point was that the so-called uplifting can be done by non vanishing F-terms because of the fact that the F-terms contribute positively in the supergravity potential.

A. Strominger This sounded interesting but I did not catch the first statement: "There is no supergravity known with an F-I-term and a F-term in how many dimensions?" What was the problem you were referring to?

A. Van Proeyen $N = 1$ in four dimensions. If you want to add a Fayet-Illiopoulos term. As far as I can see, this is still necessary to uplift the potential to have it in a de Sitter vacuum. You cannot add just a Fayet-Illiopoulos term in supergravity if you have a non trivial superpotential, unless the superpotential is not invariant under the gauge symmetry that corresponds to the Fayet-Illiopoulos term. It is something which is not well known but it is a restriction in $N = 1$ supergravity.

A. Strominger And it is known to be impossible to do that or we just have not figured out how to do it yet?

A. Van Proeyen It is known to be impossible.

H. Ooguri It sounds like we need some response.

N. Seiberg Can you state very clearly what it is that is not possible?

A. Van Proeyen To add a Fayet-Illiopoulos term when you have a non trivial superpotential.

S. Kachru Can I make a comment that is relevant?

H. Ooguri That will be the last comment.

S. Kachru Of course what happens in the supergravities is that the Fayet-Illiopoulos terms become field dependent. Presumably in this model with the anti-brane, what happens is that the Kähler mode upon which the D-3-brane tension depends, which is included in the potential, has an axion partner. The coefficient of the superpotential that is used transforms by a shift under a gauge symmetry. I think this makes the structure that was used in the original model completely consistent with the field dependent F-I terms of supergravity. Do you agree with that possibility?

A. Van Proeyen Yes. I agree with the possibility. I have not seen a model, but I agree with the possibility.

J. Harvey The subject of the session is mathematical structures. I feel a certain tension between Dijkgraaf's beautiful talk about all the wonderful structures that come out and the comment that Douglas made about how our hydrogen atom is this maximally supersymmetric solutions. I have a feeling that, without actual hydrogen atoms and helium atoms and molecules, a string theorists faced with just the quantum mechanics of the hydrogen atom would discover you can use $SO(4, 2)$ as a spectrum generating algebra. He would then generalise it to $SO(p, q)$ rather than generalising to the helium atom. It is well recognised that one of the central problems facing string theory is how to narrow our research down to the investigation of the correct mathematical structures rather than this infinite sea of beautiful possibilities. The connections are wonderful and very inspiring. But how do we figure out which are the right directions to go without experiment, without data? This is not a new question, but I think it would be very welcome to have some discussion at this meeting of how we do this, how we try to connect string theory to data. Obviously these flux compactifications are an attempt, but so far they, and other things, have not really succeeded.

L. Randall I agree very much with the first part of the comments. I am surprised at the second part. Actually I thought what was surprising about this, is how narrowly defined this discussion is, in the sense that we are looking at the kind of things we know how to deal with mathematically at this point, namely things based on supersymmetric structures or even compactification. It seems to me that, before one starts restricting, one would want to make sure that we have actually covered the entire true landscape of what is possible.

H. Ooguri The second part of the program is about new geometrical structure that appears. Since string theory contains gravity, geometry has naturally played an important rôle. We expect that we learn more about geometrical structure from string theory. We first hear from Nekrasov about new applications of geometric ideas to physics.

4.5 Prepared Comments

4.5.1 *Nikita Nekrasov: On string theory applications in condensed matter physics*

Quantum field theorists have benefited from ideas originating in the condensed matter physics. In this note we present an interesting model of electrons living on a two dimensional lattice, interacting with random electric field, which can be solved using the knowledge accumulated in the studies of superstring compactifications.

4.5.1.1 *Electrons on a lattice, with noisy electric field*

Here is the model. Consider the hexagonal lattice with black and white vertices so that only the vertices of the different colors share a common edge. Let B,W denote the sets of black and white vertices, respectively. We can view the edges as the maps $e_i : B \to W$, $e_i^* : W \to B$, $i = 1, 2, 3$. The edge e_1 points northwise, e_2: southeast, and e_3 southwest. The set of edges, connecting black vertices with white ones will be denoted by E. We have two maps: $s : E \to B$ and $t : E \to W$, which send an edge to its source and target.

The free electrons on the lattice are described by the Lagrangian

$$L_0 = \sum_{b\in B} \sum_{i=1,2,3} \psi_b \psi^*_{e_i(b)} = \sum_{w\in W} \sum_{i=1,2,3} \psi_{e_i^*(w)} \psi^*_w \tag{1}$$

The variables ψ_b, ψ_w^* are fermionic variables. Our "electrons" will interact with the $U(1)$ gauge field A_e, where $e \in E$. Introduce three (complex) numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and their sum:$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. We make the free Lagrangian (1) gauge invariant, by:

$$L_{\psi A} = \sum_{b\in B} \sum_{i=1}^{3} \psi_b e^{i\varepsilon A_{e_i(b)}} \psi^*_{e_i(b)} \tag{2}$$

The gauge transformations act as follows:

$$\psi_b \mapsto e^{i\varepsilon\theta_b}, \quad \psi_w^* \mapsto e^{-i\varepsilon\theta_w} \psi_w^*, \quad A_e \mapsto A_e + \theta_{t(e)} - \theta_{s(e)} \tag{3}$$

The Lagrangian (2) is invariant under (3) but the measure $\mathcal{D}\psi\mathcal{D}\psi^*$ is not, there is an "anomaly". It can be cancelled by adding the following Chern-Simons - like term to the Lagrangian (2)

$$L_{CS} = -i \sum_{b\in B} \sum_{i=1}^{3} \varepsilon_i A_{e_i(b)} \tag{4}$$

In continuous theory in two dimensions one can write the gauge invariant Lagrangian for the gauge field using the first order formalism:

$$\mathcal{L}_{\rm 2dYM} = \int_\Sigma {\rm tr} E F_A + \sum_k t_k {\rm tr} E^k \tag{5}$$

where E is the adjoint-valued scalar, the electric field. In the conventional Yang-Mills theory only the quadratic Casimir is kept in (5), t_2 playing the role of the (square) of the gauge coupling constant. In our case, the analogue of the Lagrangian (5) would be $\mathcal{L}_{\text{latticeYM}} = \sum_f \left(h_f \sum_{e\in\partial f} \pm A_e\right) + \sum_f \mathcal{U}(h_f)$. Note that in the continuous theory one could have added more general gauge invariant expression in E, i.e. involving the derivatives. The simplest non-trivial term would be: $\mathcal{L} = \mathcal{L}_{\text{YM}} + \int \text{tr} g(E)\Delta_A E$ where g is, say, polynomial. Such terms can be generated by integrating out some charged fields. Our lattice model has the kinetic term for the electric field, as well as the linear potential (it is possible in the abelian theory):

$$L_{Ah} = i\sum_f \left(h_f \sum_{e\in\partial f} \pm A_e\right) - \sum_f \mathcal{U}(h_f)(\Delta h)_f - t\sum_f h_f \tag{6}$$

where Δ is the lattice Laplacian, and the "metric" $\mathcal{U}(x)$ is a random field, a gaussian noise with the dispersion law[3]:

$$\langle \mathcal{U}(x)\mathcal{U}(y)\rangle = D(x-y) \equiv \int_0^\infty \mathrm{d}t \frac{e^{-t(x-y)}}{t(1-e^{t\varepsilon_1})(1-e^{t\varepsilon_2})(1-e^{t\varepsilon_3})} \tag{7}$$

The partition function of our model is (we should fix some boundary conditions, see below)

$$Z(t,\varepsilon_1,\varepsilon_2,\varepsilon_3) = \int \mathcal{D}\mathcal{U} e^{-\int \mathcal{U}(x)(D^{-1}\circ\mathcal{U})(x)} \int \mathcal{D}\psi\mathcal{D}\psi^*\mathcal{D}A\mathcal{D}h\ e^{L_{\psi A}+L_{CS}+L_{Ah}} \tag{8}$$

4.5.1.2 *Dimers and three dimensional partitions*

We now proceed with the solution of the complicated model above. The idea is to expand in the kinetic term for the $\psi\psi^*$. The non-vanishing integral comes from the terms where every vertex, both black and white, is represented by the corresponding fermions, and exactly once. Thus the integral over ψ, ψ^* is the sum over dimer configurations [5],[6], weighted with the weight

$$\sum_{\text{dimers}} \prod_{e\in\text{dimer}} e^{i\varepsilon A_e} \tag{9}$$

The gauge fields A_e enter now linearly in the exponential, integrating them out we get an equation $dh = \star\omega_{\text{dimer}}$ where ω_{dimer} is the one-form on the hexagonal lattice, whose value on the edge is equal to $\pm\varepsilon_{1,2,3}$ depending on its orientation $\pm\varepsilon$ depending on whether it belongs to the dimer configuration or not. Everything is arranged so the that at each vertex v the sum of the values of ω on the three incoming edges is equal to zero. The solution of the equation on h gives what is called *height function* in the theory of dimers. In our case it is the electric field. If we plot the graph of h_f and make it to a piecewise-linear function of two variables in an obvious way, we get a two dimensional surface – the boundary of a generalized three

[3]the integral is regularized via $\int \frac{dt}{t} \to \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int \frac{dt}{t} t^s$.

dimensional partition. In order to make it a boundary of actual three dimensional (or plane) partition, we have to impose certain boundary conditions: asymptotically the graph of h_f looks like the boundary of the positive octant $\mathbf{R}^3_+$[4]. Under these conditions, the final sum over dimers is equivalent to the sum over three dimensional partitions of the so-called *equivariant measure* [3]. The three dimensional partition is a (finite) set $\pi \subset \mathbf{Z}^3_+$ whose complement in $\bar{\pi} = \mathbf{Z}^3_+ \backslash \pi$ is invariant under the action of $\mathbf{Z}^3_+$. In other words, the space I_π of polynomials in three variables, generated by monomials $z_1^i z_2^j z_3^k$ where $(i,j,k) \in \bar{\pi}$ is an ideal, invariant under the action of the three dimensional torus $\mathbf{T}^3$. Let $ch_\pi = \sum_{(i,j,k)\in\pi} q_1^{i-1} q_2^{j-1} q_3^{k-1}$, $\quad ch_{\bar{\pi}}(q) = \frac{1}{P(q)} - ch_\pi$, $\quad |\pi| = ch_\pi(1)$, $P(q) = (1-q_1)(1-q_2)(1-q_3)$, $\qquad q_i = e^{\varepsilon_i}$. Define the "weights" x_α, y_α from $1/P(q) - P(q^{-1}) ch_{\bar{\pi}}(q) ch_{\bar{\pi}}(q^{-1}) = \sum_\alpha e^{x_\alpha} - \sum_\alpha e^{y_\alpha}$. Then,

$$\mu_\pi(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \prod_\alpha \frac{y_\alpha}{x_\alpha} \tag{10}$$

The partition function of our model reduces to:

$$Z(t, \varepsilon_1, \varepsilon_2, \varepsilon_3) = \sum_\pi \mu_\pi(\varepsilon_1, \varepsilon_2, \varepsilon_3) e^{-t|\pi|} \tag{11}$$

4.5.1.3 *Topological strings and S-duality*

The last partition function arises in the string theory context. The ideals I_π are the fixed points of the action of the torus $\mathbf{T}^3$ on the moduli space of zero dimensional D-branes in the topological string of B type on $\mathbf{C}^3$, bound to a single D5-brane, wrapping the whole space. The equivariant measure μ_π is the ratio of determinants of bosonic and fermionic fluctuations around the solution I_π in the corresponding gauge theory. The parameter t is the (complexified) theta angle, which couples to $\mathrm{tr} F^3$ instanton charge. This model is an infinite volume limit of a topological string on compact Calabi-Yau threefold. The topological string on Calabi-Yau threefold is the subsector of the physical type II superstring on Calabi-Yau $\times \mathbf{R}^4$. It inherits dualities of the physical string, like mirror symmetry and S-duality [4]. It maps the type B partition function (11) to the type A partition function. The latter counts holomorphic curves on the Calabi-Yau manifold. In the infinite volume limit it reduces to the two dimensional topological gravity contribution of the constant maps, which can be evaluated to be [3]:

$$Z(t, \varepsilon_1, \varepsilon_2, \varepsilon_3) = \exp\left(\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_3+\varepsilon_2)(\varepsilon_1+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}\right) \sum_{g=0}^\infty t^{2g-2} \frac{B_{2g-2}B_{2g}}{2g(2g-2)(2g-2)!} \tag{12}$$

$$= M(-e^{-it})^{-\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_3+\varepsilon_2)(\varepsilon_1+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}} \tag{13}$$

$$\tag{14}$$

where $M(q) = \prod_{n=1}^\infty (1-q^n)^{-n}$ is the so-called MacMahon function.

[4]i.e. as the function: $h(x,y) = \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k$, $x = i - (j+k)/2$, $y = (j-k)/2$, $i,j,k \geq 0$, $ijk = 0$

4.5.1.4 *Discussion*

We have illustrated in the simple example that the string dualities can be used to solve for partition functions of interesting statistical physics problems. The obvious hope would be that the dualities are powerful enough to provide information on the correlation functions as well. One can consider more general lattices or boundary conditions (they correspond to different toric Calabi-Yau's), more sophisticated noise functions $D(x)$ (e.g. the one coming from Z-theory [7]) . Also, it is tempting to speculate that compact CYs correspond to more interesting condensed matter problems.

I am grateful to A.Okounkov for numerous fruitful discussions.

Bibliography

[1] A. Okounkov, N. Reshetikhin, C. Vafa, *Quantum Calabi-Yau and Classical Crystals*, hep-th/0309208.
[2] A. Iqbal, N. Nekrasov, A. Okounkov, C. Vafa, *Quantum Foam and Topological Strings*, hep-th/0312022.
[3] D. Maulik, N. Nekrasov, A. Okounkov, R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I,II*, math.AG/0312059, math.AG/0406092.
[4] N. Nekrasov, H. Ooguri, C. Vafa, *S-duality and topological strings*, JHEP 0410 (2004) 009, hep-th/0403167.
[5] R. Kenyon, A. Okounkov, S. Sheffield *Dimers and amoebas*, math-ph/0311005.
[6] R. Kenyon, A. Okounkov, *Planar dimers and Harnack curves*, math.AG/0311062.
[7] N. Nekrasov, *Z-Theory*, hep-th/0412021.

4.5.2 *Shing-Tung Yau: Mathematical Structures: Geometry of Six-Dimensional String*

There has been a great deal of anxiety to find a suitable string vacuum solution or to perform statistics over the space of all such vacua. However, despite great successes in the twenty years since the first string revolution, our understanding of string vacua is far from complete. For starters, we have not achieved a satisfactory theory of computing supersymmetric cycles nor a good understanding of Hermitian-Yang-Mills fields and their instantons. Such issues pertain to deep problems in mathematics and ideas inspired from physical considerations have been essential for progress.

In the compactificaton of Candelas-Horowitz-Strominger-Witten [3], preserving supersymmetry with zero H-flux requires the compact six-manifold to be Kähler Calabi-Yau. While this class of manifold is quite large, it is believed to have a finite number of components with finite dimensions for its moduli space.

For the class of three-dimensional Kähler Calabi-Yau manifolds, there is a construction due to works of Clemens [4] and Friedman [6] where one takes a finite number of rational curves with negative normal bundle and pinch them to conifold points. Under suitable conditions for the homology class, one can deform the resulting (singular) manifold to a smooth manifold. The resulting manifold is in general non-Kähler. By repeating such procedures several times, one can obtain a smooth complex manifold with vanishing second Betti number (and hence clearly non-Kähler). If the homology of the original Calabi-Yau manifold has no torsion, a theorem of Wall [16] shows that the resulting manifold must be diffeomorphic to a connected sums of $S^3 \times S^3$. This type of manifold can be considered as a natural generalization of Riemann surfaces which are connected sums of handle bodies. These three-dimensional complex manifold also have a holomorphic three-form that is naturally inherited from the original Calabi-Yau.

There is a proposal of Reid [14] that the moduli space of all Calabi-Yau structures can be connected through such complex structures over handle bodies. Such a proposal may indeed be true. However, an immediate problem is that we are then required to analyze non-Kähler complex manifolds but we have virtually no theory for them. Non-Kähler manifolds have appeared naturally in string compactifications with fluxes. So perhaps a useful way to think about the construction of Clemens and Friedman is that the collapsing of the rational curves together with the deformation of the complex structure correspond to turning on a flux. Hence, just from mathematical considerations of the Calabi-Yau moduli space, we are led to study structures which contain fluxes and preserve supersymmetries.

A natural supersymmetric geometry with flux to consider is the one in heterotic string theory. The geometry is constrained by a system of differential equations worked out by Strominger [15] and takes the following form

$$d(\| \Omega \|_\omega \, \omega^2) = 0 \ ,$$
$$F^{2,0} = F^{0,2} = 0 \ , \quad F \wedge \omega^2 = 0 \ ,$$

$$dH = \sqrt{-1}\,\partial\bar{\partial}\omega = \alpha'(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F) \ ,$$

where ω is the hermitian metric, Ω the holomorphic three-form, and F the Hermitian-Yang-Mills field strength. In above, the H-flux is given by $H = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)\omega$.

It would be nice to understand geometrically how the flux can be turned on from a thorough analysis of the Strominger system for the non-Kähler Calabi-Yau handle bodies. As a first step, Li-Yau [12] have shown in a rather general setting, that one can always obtain a solution to the above equations by perturbing the Calabi-Yau vacuum with the gauge bundle being a sum of the tangent bundle together with copies of the trivial bundle. The deformation to non-zero H-flux will mix together the tangent and trivial bundle parts of the gauge bundle. This allows Li-Yau to construct non-zero H-flux solutions with $SU(4)$ and $SU(5)$ gauge group. In the analysis of the deformations of such gauge bundles, the deformation space of the Kähler and complex structure of the Calabi-Yau naturally arised. Therefore, studying such deformation to non-zero H-flux systems can give insights into the moduli space of Calabi-Yau.

The first equation of the Strominger system calls for the existence of a balanced metric on such manifolds. These are n-dimensional complex manifolds which admit a hermitian metric ω that satisfies $d(\omega^{n-1}) = 0$ [13]. Balanced metrics satisfy many interesting properties such as being invariant under birational transformations as was observed by Alessandrini and Bassanelli [1]. Using parallel spinors, it is possible to decompose the space of differential forms similar to that of Hodge decomposition. This has been carried out by my student C. C. Wu in her thesis.

Presently, we do not know how large is the class of balanced manifolds. Michelsohn has shown that for the twistor space of anti-self-dual four manifolds, the natural complex structure is balanced [13]. It may be useful to identify such manifolds whose anti-canonical line bundle admits a holomorphic three-form. Another well-known class of non-Kähler manifolds that is balanced consists of $K3$ surfaces fibered with a twisted torus bundle. In this special case, there is a metric ansatz [5, 8] which enabled Fu-Yau [7] to demonstrate the existence of a solution to the Strominger system that is not connected to a Calabi-Yau manifold. The existence of such a solution is consistent with duality chasing arguments from M-theory that were first discussed in detail by Becker-Dasgupta [2].

As mentioned, the theory of complex non-Kähler manifolds has not been developed much. Similar to Calabi-Yau compactification, it will be important to rephrase the four-dimensional physical quantities like the types and number of massless fields or the Yukawa coupling in terms of the properties of the non-Kähler manifold. For example, can the number massless modes or geometric moduli be expressed purely in terms of certain geometrical quantities perhaps analogous to the Hodge numbers for the Kähler case. Here, trying to answer such physical questions will compel us to seek a deeper understanding of the differential structures of non-Kähler manifold than that known currently. It is likely that fluxes and in particular the H-flux (which is the torsion in the heterotic theory) will play a central role in non-Kähler stuctures.

More importantly, the study of complex non-Kähler manifolds is another step in understanding the whole space of string solutions or vacua. The space of string vacua contains both geometrical and non-geometrical regions. But even within the geometrical region, the compactification manifold need not be Kähler nor even complex (for type IIA theory [9]) when α' corrections and branes are allowed. This seems to give many possibilities for the geometry of the internal six-manifold for different types of string theories. However, since the six different string theories are related to each other through various dualities, the geometries and structures of six-dimensional compact manifolds associated with string vacua are most likely also subtlely related. This gives hope that the space of string vacua can indeed be understood well-enough such that we can confirm or rule out that there exists at least one string vacuum that can reproduce the four-dimensional standard model of our world. Given the recent successes of compactification with fluxes - from moduli fixing [10] to addressing the cosmological constant issue [11] - we can expect that the physical real world vacuum will involve fluxes and understanding the structures of non-Kähler manifolds may prove indispensable.

Bibliography

[1] L. Alessandrini and G. Bassanelli, *Modifications of compact balanced manifolds*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995) no. 12, 1517.

[2] K. Becker and K. Dasgupta, *Heterotic strings with torsion*, JHEP **0211** (2002) 006, `hep-th/0209077`.

[3] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. B **258** (1985) 46.

[4] C. H. Clemens, *Double solids*, Adv. in Math. **47** (1983) no. 2, 107;
C. H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, IHES Publ. Math. **58** (1983) 19.

[5] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G-flux*, JHEP **9908** (1999) 023, `hep-th/9908088`.

[6] R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. **274** (1986) no. 4, 671.

[7] J.-X. Fu and S.-T. Yau, *Existence of supersymmetric Hermitian metrics with torsion on non-Kähler manifolds*, `hep-th/0509028`.

[8] E. Goldstein and S. Prokushkin, *Geometric model for complex non-Kähler manifolds with SU(3) structure*, Commun. Math. Phys. **251** (2004) 65, `hep-th/0212307`.

[9] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Generalized structures of $N = 1$ vacua*, JHEP **0511** (2005) 020, `hep-th/0505212`.

[10] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, Nucl. Phys. B **584** (2000) 69; erratum-ibid. B **608** (2001) 477, `hep-th/9906070`.

[11] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, Phys. Rev. D **68** (2003) 046005, `hep-th/0301240`.

[12] J. Li and S.-T. Yau, *The existence of supersymmetric string theorey with torsion*, J. Differential Geom. **70** (2005) 143, `hep-th/0411136`.

[13] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta

Math. **149** (1982) no. 3-4, 261.

[14] M. Reid, *The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible*, Math. Ann. **278** (1987) 329.

[15] A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (1986) 253.

[16] C. T. C. Wall, *Classification problems in differential topology. V. On certain 6-manifolds*, Invent. Math. **1** (1966) 355; corrigendum, ibid **2** (1966) 306.

4.6 Discussion

B. Greene Just in the spirit of the Mathematics-Physics interface which is the theme of the session, I might note that Dine and Seiberg some time ago have showed that the existence of certain R-symmetries allows you to prove that there can be exact flat directions. These directions can give rise to the kinds of deformations that Yau was talking about, and in particular, the one example on the three generation Calabi-Yau, the deformation $T \oplus O \oplus O$. You can in fact realise an example of that sort using the R-symmetries and prove that such a solution would exist. So you can have a physics proof, if you will, of that particular example that Yau was discussing.

A. Strominger Perhaps the most interesting new things are the ones that are not obtained by deformations like this last example and really cannot be understood by any such arguments. I have a question for Yau. Twenty years ago you made your famous estimate that there were ten thousand Calabi-Yau spaces. How many of these things do you think there are?

S.T. Yau More, I think, that is all I can say at this moment.

N. Seiberg I have two questions and I am glad that most of the relevant experts are in the audience. The first question is: What is the status of the non-perturbative existence of the topological string? The second question is: We have learned about many new Calabi-Yau spaces. How many of them look like the real world?

H. Ooguri I would like to personally respond to the first question. I can see at least two independent non-pertubative completions of the topological string in certain situations. In the case when you have an open string dual you can often use a matrix model to give a non-pertubative completion in the sense that you have a convergent matrix integral whose perturbative expansion gives rise to topological string amplitudes in the close string dual. On the other hand you can also propose to define topological string amplitudes in terms of black hole entropy, where the counting of number of states of black holes is well defined and the perturbative expansion of this counting, in particular the generating function, gives rise again to topological string partition function via the OSV conjecture. You can see in particular examples that these two give rise to different non-pertubative completions. One possible view is that the topological string is a tool to address various interesting geometric programs in physics. Depending on situations, there can be different non-perturbative completions. But there might be people with other views on that.

N. Seiberg I am not aware of an example where you have two systems which are the same to all orders in perturbation theory and their D-branes are the same, in the sense that you can probe the system with large classical field excursions, and yet they have more than one non-pertubative completion.

H. Ooguri Yes, so this might be a counterexample to that.

I. Klebanov I was also intrigued by Nekrasov's promise to repay some debts to condensed matter theorists. A question I had is: Is there any sign in the topological string of the resonating valence bond wave function, and if so, will there be a time when Phil Anderson will be learning topological string theory?

N. Nekrasov I am sure there is, but I do not know when it is.

I. Klebanov Is it a classical model or do you see the quantisation of these dimers?

N. Nekrasov Well, the model which I got is more like a statistical mechanical model.

H. Ooguri Statistical mechanical in the sense of a classical statistical mechanical model?

N. Nekrasov I am doing the functional integral over the fermions and the gauge fields and the rest. How do you call it? I call it quantum but some people may call it classical. When you reduce this problem to dimers, it is just a summation over dimer configurations. That is probably classical.

H. Ooguri At the end of the next part, I would like to also have some general discussion, and in particular I hope to identify some important physics program for which we would like to develop mathematical tools. So I would like to take some kind of informal call on this kind of questions. I have entitled the final part of this discussion session "What is M-theory?" Of course, finding a better formulation of M-theory is a very important project. We will hear two different points of view about this.

4.7 Prepared Comments

4.7.1 *Hermann Nicolai: E_{10} and $K(E_{10})$: prospects and challenges*

Definition of E_{10}: The maximal rank hyperbolic Kac-Moody algebra $\mathfrak{e}_{10} \equiv \mathrm{Lie}(E_{10})$ (in split real form) is defined via the so-called Chevalley Serre presentation in terms of generators h_i, e_i, f_i $(i = 1, \ldots, 10)$ with relations [1]

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j,$$
$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0 \ .$$

where $\{h_i\}$ span the Cartan subalgebra $\mathfrak{h}$. The entries of the Cartan matrix A_{ij} can be read off from the Dynkin diagram displayed in Figure 1 below. With all other Kac-Moody algebras, $\mathfrak{e}_{10}$ shares the following key properties:

- *Root space decomposition:* for any root $\alpha \in Q(E_{10}) = \mathrm{II}_{1,9}$ (= the unique even self-dual Lorentzian lattice in ten dimensions), we have
$$(\mathfrak{e}_{10})_\alpha = \big\{ x \in \mathfrak{e}_{10} \ : \ [h, x] = \alpha(h) x \quad \mathrm{forall}\, h \in \mathfrak{h} \big\}$$
One distinguishes *real* roots ($\alpha^2 = 2$) and *imaginary* roots ($\alpha^2 \leq 0$); the latter can be further subdivided into lightlike (null) roots ($\alpha^2 = 0$) and timelike roots ($\alpha^2 < 0$).
- *Triangular decomposition:* this is a generalization of the well known decomposition of finite dimensional matrices into (strictly) upper and lower triangular ($\mathfrak{n}^{\pm}$), and diagonal matrices ($\mathfrak{h}$), respectively.
$$\mathfrak{e}_{10} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \ , \qquad \text{with } \mathfrak{n}_\pm := \textstyle\bigoplus_{\alpha ? 0} (\mathfrak{e}_{10})_\alpha$$
This is the feature that ensures *computability* in the present context, via choice of a *triangular gauge* for the 'vielbein' $\mathcal{V}(t) \in E_{10}/K(E_{10})$.
- Existence of an *invariant bilinear form:*
$$\langle h_i | h_j \rangle = A_{ij} \quad , \quad \langle e_i | f_j \rangle = \delta_{ij} \quad , \quad \langle [x, y] | z \rangle = \langle x | [y, z] \rangle .$$
This is the feature which, in the present context, allows for the formulation of an *action principle.* Because dim $\mathfrak{e}_{10} = \infty$, this quadratic form is, in fact, the only polynomial Casimir invariant, ensuring the (essential) *uniqueness* of the σ-model action below [2, 3].

Compact subalgebra $\mathfrak{k}\mathfrak{e}_{10}$: The Chevalley involution is defined by

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h_i) = -h_i$$

and extends to all of $\mathfrak{e}_{10}$ by $\omega([x, y]) = [\omega(x), \omega(y)]$. The fixed point set $\mathfrak{k}\mathfrak{e}_{10} = \big\{ x \in \mathfrak{e}_{10} : \omega(x) = x \big\}$ is a subalgebra of $\mathfrak{e}_{10}$, which is called the *compact subalgebra.* Note that $\mathfrak{k}\mathfrak{e}_{10}$ is *not* a Kac–Moody algebra [4].

Level decomposition: No closed formulas exist for the dimensions of the root spaces, although the root multiplicities are in principle computable recursively [1].

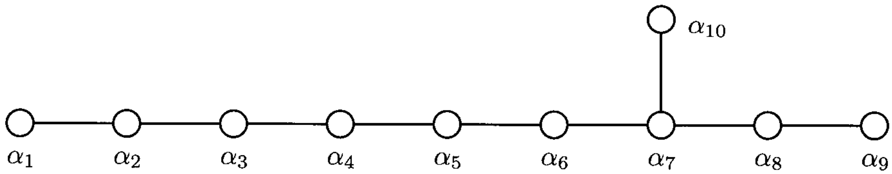


Fig. 4.1 *Dynkin diagram of* $\mathfrak{e}_{10}$.

However, it is known that, generically, they *grow exponentially* as $\alpha^2 \to -\infty$, like the number of massive string states. In order to get a handle at least on low-lying generators, one analyzes $\mathfrak{e}_{10}$ w.r.t. certain finite-dimensional regular subalgebras by means of a *level decomposition*: pick one special node α_0, and write a given root as $\alpha = \sum_j m^j \alpha_j + \ell \alpha_0$ with $\alpha_j \in$ subalgebra, and where ℓ is the 'level'. For instance, decomposing w.r.t. the subalgebra $A_9 \equiv \mathfrak{sl}_{10}$ (*i.e.* $\alpha_0 = \alpha_{10}$) we obtain the following table for $\ell \leq 3$:

ℓ	A_9 module	Tensor
0	$[100000001] \oplus [000000000]$	$K^a{}_b$
1	$[000000100]$	E^{abc}
2	$[000100000]$	$E^{a_1 \dots a_6}$
3	$[010000001]$	$E^{a_1 \dots a_8 \vert a_9}$

These tensors correspond to the bosonic fields of $D = 11$ supergravity and their 'magnetic' duals. Similar low level decompositions of $\mathfrak{e}_{10}$ w.r.t. its other distinguished rank-9 subalgebras $D_9 \equiv \mathfrak{so}(9,9)$ and $A_8 \oplus A_1 \equiv \mathfrak{sl}_9 \oplus \mathfrak{sl}_2$ yield the correct bosonic multiplets, again with 'magnetic' duals, of (massive) type IIA and type IIB supergravity, respectively. Furthermore, for the D_9 decomposition, one finds that the (Neveu-Schwarz)^2 states (at even levels) and the (Ramond)^2 states (at odd levels), respectively, belong to tensorial and spinorial representations of the T-duality group $SO(9,9)$, and that the truncation to even levels contains the rank-10 hyperbolic Kac–Moody algebra DE_{10}, corresponding to type-I supergravity, as a subalgebra.

Dynamics (cf. [2, 3, 5]): The equations of motion are derived from the following (essentially unique) 'geodesic' σ-model over $E_{10}/K(E_{10})$:

$$\int dt\, \mathcal{L}(t) = \int \frac{dt}{n(t)} \left\langle \mathcal{P}(t) \vert \mathcal{P}(t) \right\rangle ,$$

where $\mathcal{P} := \dot{\mathcal{V}}\mathcal{V}^{-1} - \omega(\dot{\mathcal{V}}\mathcal{V}^{-1})$ is the 'velocity', $\langle . \vert . \rangle$ is the standard invariant bilinear form, and $n(t)$ a one-dimensional 'lapse' needed to ensure (time) reparametrisation invariance. When truncated to levels ≤ 3, the corresponding equations of motion coincide with the appropriately truncated bosonic supergravity equations of motion, where only first order spatial gradients are retained [2]. Analogous results hold for the D_9 and $A_8 \times A_1$ decompositions. Remarkably, for the bosonic sector of these theories, E_{10} yields exactly the same information as local supersymmetry, namely

- unique (bosonic) actions with the correct (Chern–Simons) couplings;
- incompatibility of $D = 11$ supergravity with a cosmological constant;
- self-duality of the 5-form field strength in IIB supergravity;
- information about the higher order $R^4, R^7, \ldots$ corrections to M theory.

This may indicate, perhaps surprisingly, that (maximal local) supersymmetry may not play such a prominent role at the most fundamental level. Indeed, in a theory, where space and time are 'emergent', the distinction between bosons and fermions must also be regarded an 'emergent' phenomenon, and not as a feature of the theory at its most basic level [5].

Towards higher levels: Understanding the (exponentially growing) spectrum of higher level representations and their correct physical interpretation is the key problem; on the mathematical side, this is the (still unsolved) problem of finding a manageable realization for indefinite Kac–Moody algebras. An analysis of the higher level representations w.r.t. A_9 has revealed the existence of a distinguished series of representations, the so-called *gradient representations*, which we tentatively associate with (non-local functionals of) the higher order spatial gradients, corresponding to the differential operators $\partial_{a_1} \cdots \partial_{a_k}$ acting on the $D = 11$ supergravity fields.

ℓ	A_9 module	Tensor
$3k+1$	$[k00000100]$	$E_{a_1 \ldots a_k}{}^{b_1 b_2 b_3}$
$3k+2$	$[k00100000]$	$E_{a_1 \ldots a_k}{}^{b_1 \ldots b_6}$
$3k+3$	$[k10000001]$	$E_{a_1 \ldots a_k}{}^{b_1 \ldots b_8 \mid b_9}$

(All tensors in the above table are *symmetric* in the lower indices $a_1, \ldots, a_k$.) We thus face the following questions:

- Can the complete evolution of $D = 11$ supergravity (or some M theoretic extension thereof) be mapped to a *null geodesic motion* on the infinite-dimensional coset space $E_{10}/K(E_{10})$?
- Can the expansion in spatial gradients be understood in terms of a 'zero tension limit', where space-time is regarded as some kind of 'elastic medium'? And is the level expansion (or an expansion in the height of the roots) the correct mathematical framework for studying this limit?
- Does E_{10} thereby provide a new *Lie algebraic mechanism* for the emergence of space [6], and can the initial singularity be described in these terms as the place

[5]This point of view might receive further support if, contrary to widespread expectations, no evidence for supersymmetry is found at the Large Hadron Collider.

[6]Whereas time could emerge 'operationally', as it is expected to in canonical approaches based on a Wheeler-DeWitt equation.

where space 'de-emerges'?

Even if some variant of the gradient hypothesis turns out to be correct, there remains the question how to interpret the remaining (M theoretic?) degrees of freedom. E.g., up to level $\ell = 28$, there are already 4 400 752 653 representations of A_9, out of which only 28 qualify as gradient representations!

Fermions: The above considerations can be extended to the fermionic sector, and it can be shown that $K(E_{10})$ indeed plays the role of a generalised 'R symmetry' [6]. Because $K(E_{10})$ is not a Kac–Moody group, many of the standard tools are not available. This applies in particular to *fermionic* (*i.e.* double-valued) representations which cannot be obtained from (or lifted to) representations of E_{10}. Remarkably, it has been shown very recently [6, 7] that the gravitino field of $D = 11$ supergravity (at a fixed spatial point) can be promoted to a *bona fide*, albeit unfaithful, spinorial representation of $K(E_{10})$. This result strengthens the evidence for the correspondence proposed in [2], and for the existence of a map between the time evolution of the bosonic and fermionic fields of $D = 11$ supergravity and the dynamics of a *massless spinning particle* on $E_{10}/K(E_{10})$. However, the existence (and explicit construction) of a *faithful* spinorial representation, which might also accommodate spatially dependent fermionic degrees of freedom, remains an open problem.

For further references and details on the results reported in this comment see [2, 3, 6]. The potential relevance of E_{10} was first recognized in [8]; an alternative proposal based on E_{11} has been developed in [9].

Acknowledgments: It is a great pleasure to thank T. Damour, T. Fischbacher, M. Henneaux and A. Kleinschmidt for enjoyable collaborations and innumerable discussions, which have shaped my understanding of the results reported here.

Bibliography

[1] V. Kac, *Infinite Dimensional Lie Algebras*, 3rd edition, C.U.P. (1990).
[2] T. Damour, M. Henneaux and H. Nicolai, Phys. Rev. Lett. **89** (2002) 221601, `hep-th/0207267`.
[3] T. Damour and H. Nicolai, `hep-th/0410245`.
[4] A. Kleinschmidt and H. Nicolai, `hep-th/0407101`.
[5] T. Damour, prepared comment in this volume.
[6] T. Damour, A. Kleinschmidt and H. Nicolai, `hep-th/0512163`.
[7] S. de Buyl, M. Henneaux and L. Paulot, `hep-th/0512292`.
[8] B. Julia, in: Lectures in Applied Mathematics, Vol. 21 (1985), AMS-SIAM, p. 335; preprint LPTENS 80/16.
[9] P. C. West, Class. Quant. Grav. **18** (2001) 4443, `hep-th/0104081`; `hep-th/0407088`.

4.7.2 *Michael Atiyah: Beyond string theory?*

String theory (and M-theory) is a remarkably sophisticated mathematical structure, as yet incomplete. While it has led to exciting new results in mathematics it is not yet clear what shape the final physical theory will take. Perhaps it will only be a modest extension of what we now have, but perhaps it will have to undergo some radical reformulation.

As we know the great aim is to combine quantum theory with gravity. String theory maintains the formal apparatus of quantum theory unchanged but modifies Einstein's theory of gravitation. Perhaps both sides need modification?

I confess to being a believer in Occam's razor, in which the simple solution is always preferred. Although string theory is an impressive structure it still lacks the overall simplicity that we should aim at.

We may need radically new ideas and I think it is worth investigating whether **retarded differential equations** should be seriously considered at a fundamental level. This idea has been put forward by Raju [1], although my ideas are somewhat different.

Consider as a simple example the linear equation for the function $x(t)$.

$$\dot{x}(t) + kx(t-r) = 0 \qquad\qquad r > 0 \tag{1}$$

where r, k are fixed constants. Such an equation can be solved for $t > 0$ with **initial data** being a function on the interval $[-r, 0]$. This is very different from the usual differential equations of mathematical physics (for which $r = 0$), which have been our paradigm since Isaac Newton.

While this equation makes sense for $x(t)$ in Minkowski space, with t being proper time along the trajectory, it is not clear how to extend it to wave-propagation in a relativistic manner. For this purpose note that $t \longrightarrow t - r$ has infinitesimal generator $-r\dfrac{\partial}{\partial t}$, so that translation by $-r$ is just $\exp\left(-r\dfrac{\partial}{\partial t}\right)$. Guided by this we can consider formally the relativistically invariant retarded Dirac operator (as suggested to me by G. Moore).

$$i \;\; D - mc + k\exp(-rD) \tag{2}$$

where D is the Dirac operator in Minkowski space. In the non-relativistic limit D reduces to $\dfrac{\gamma^\circ}{c}\dfrac{\partial}{\partial t}$ where γ° is the 4×4 matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, showing that we have a retarded equation for positive energy and an advanced equation for negative energy.

There are problems in interpreting $\exp(-rD)$ which I will pass over, but applied to the plane wave solution

$$\exp\left(\frac{-iEt}{}\right)\begin{pmatrix} \phi \\ 0 \end{pmatrix}$$

(2) leads to the dispersion relation

$$\frac{E}{c} - mc + k\exp\left(\frac{irE}{c}\right) = 0 \tag{3}$$

giving the quantization condition

$$r = \frac{nc}{E}\pi \qquad (n \text{ an integer}) \tag{4}$$

which shows that r is an integer multiple of the Compton wavelength of the electron, so is of order 10^{-12} m.

This elementary argument giving the scale of the retardation parameter r/c is striking. Note that the real part of the equation gives

$$E = mc^2 + (-1)^{n-1}kc \tag{5}$$

so that the constant k will have to be extremely small.

Analysis of $\exp(-rD)$ raises many challenging problems which are related to the mixing of positive and negative energy states. On the other hand the equation makes sense on a curved background. Coupling the Dirac operator to other gauge fields can be treated in a similar way, though a gravitational counterpart presents further problems.

I finish by simply asking whether these ideas have any relevance to string theory or M-theory. In particular is the initial data problem for retarded equations related in some way to quantum theory?

My purpose is not to make any claims but to stimulate thought†.

Bibliography

[1] C.K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, (1994) Fundamental Theories of Physics, Vol. 65.

† I am grateful to the participants of the Solvay Conference for a number of cogent and constructive criticisms.

4.8 Discussion

N. Seiberg I have a question to the current speaker. Putting derivatives in exponentials is very common when you consider fields in non-commutative spaces. Is there any connection to your work?

M. Atiyah I do not know. Non-commutative geometry/analysis is a very interesting part of mathematics and physics has a link to all these things. It could well be. I approach this from a very different point of view. I just naively ask certain philosophical questions and I am led to this by the nature of the formalism. I have not had time to search the literature, but I would be delighted if it links up with anything else you or anybody else knows in physics, or in non-commutative geometry. The hope is, of course, that all the ideas we have been talking about, string theory, non-commutative geometry, and so on, are obviously related in some way. We want to clear the ground and find out what the real relations are. If this plays any role, I would be delighted.

H. Ooguri I would like to reserve some time for general discussions. I recently read the history of this conference and there is a preface that was written by Werner Heisenberg who commented that this conference has been held for the purpose of attacking problems of unusual difficulty rather than exchanging the results of recent scientific work. In that spirit I would like to raise the question: What would be the important physics programs that are still waiting for some new mathematical tool? Or maybe are there some hidden tools that we are not aware of, that we should try to make use of?

M. Douglas I am coming back actually to answer the question of Harvey and also Seiberg's second question, which I hope are the kind of general questions you were asking. There is all this wealth of mathematics and structures, but we are physicists and we have to address some physical question to make progress. The basic physical questions are the combination of the ones that we started to work on twenty years ago: trying to get the standard model out of string compactifications. This has made twenty years of progress and inspired a lot of the mathematics that Dijkgraaf talked about. Also, there is the recent discovery of the dark energy, which in the simplest models is a positive cosmological constant. Those are the experimental facts that seem the most salient to the type of work that we were discussing.

Let me turn then to Seiberg's second question. We have this big number of Calabi-Yaus, and this potentially vaster number of non Calabi-Yaus. We have an even vaster number of flux vacua. What number of them looks like the real world? That is obviously a very hard question. I think this mathematics is relevant because it gives us tools for addressing problems that we, as physicists, have had very little experience with. Namely, exploring this vast mathematical space of possibilities in string theory. Experimental input is essential, such as the standard model and what will be discovered at LHC, but also this mathematical space to some extent has to be explored. Then, just to give a glib answer to Seiberg's question: if you make the zeroth order sorts of estimates that I made in my work of two years ago and if you would incorporate both the issues that Lüst talked about, then the difficulty of getting the standard model is one in a billion. It is one in 10^{200} taking into account the cosmological constant and other factors. You then decide that out of this number of Calabi-Yaus and fluxes and the rest, something like one in 10^{200} to 10^{300} should work. Although you may say: "How can you say such a thing?", the arguments have the virtue that they are very simple. They are using very little input and so one can take them as a zeroth order starting point. I certainly hope that is not the right answer and that there are far more features that have not been exploited yet in terms of the structure of the string vacua. To the extent that there is structural peaking, then there might be far fewer that match the standard model, or far more. Both of those possibilities would be interesting. It could be that there is information from early cosmology. Just the difficulty of getting a viable cosmology that will fit the data, or these more speculative considerations about the wave function measure factors and so forth, all this could drastically affect this calculation. All I am saying by throwing out a number like this is the following: here is a framework in which to think about the problem of combining these many disparate ingredients and talk about them together.

D. Gross It seems to be clear that the one thing that was missing from this session, in which we focused partly on mathematical structures that might reveal the nature of space and time, was time, except in the last talk which could have been delivered back in 1911. It is clear that elliptic equations are easier than hyperbolic, and Euclidean metrics easier than Lorentzian, and ignoring time easier than taking it into account. But for example the discussion of what some people mistakenly call vacua, which are really metastable states, should illustrate that they are discussing things which are of course time dependent. Yet, nothing is known about the time evolution. This indicates that, surely, the big open issue in string theory is time. What does it mean for time to be emergent? Non-locality in time, how do we deal with that? How do we make or have a causal structure? How do we discuss metastable states whose beginning we know nothing about, and so on. It seems that a lot of tools we are focusing on are avoiding the tough questions because the mathematics is simpler. As I said, we prefer to study elliptic equations rather than the hard case of hyperbolic equations.

N. Arkani-Hamed I just wanted to ask something about the status of large numbers of vacua in the heterotic context. It seems that a lot of the studies of the statistics for type IIB vacua for example, which are definitely interesting, are less likely to apply to the real world if we take the hint from gauge coupling unification seriously. Everything seemed to be going swimmingly in the perturbative heterotic string, except for not being able to find a mechanism to find a small cosmological constant. So, I would like to know what the status of statistics is in the heterotic context.

S. Kachru I can say one thing. One of the huge advances in the mid-nineties was this duality revolution. After the duality revolution, Freedman, Morgan, Witten and many others developed very powerful techniques to take heterotic strings on certain Calabi-Yau three-folds, elliptic Calabi-Yaus, and Fourier transform them over to type II strings, F-theory or type IIB string theory. Actually the groups that have been making the most progress in constructing realistic GUT models in the heterotic string, the Penn group for instance, worked precisely in this elliptic Calabi-Yaus. Now you can then ask: "What happens if you dualise these over to the type II context where people have started counting vacua with fluxes?" It seems quite clear that the same kind of structure will emerge in the heterotic theory. The reason that it is much easier to study in the type II context is that what these fluxes in the type II theory map back to in the heterotic theory correspond to deformations of the Calabi-Yau into a non-Kähler geometry. As should have been clear from Yau's talk, although that is a very interesting subject, we know almost nothing about it. That makes it clear that you can import the best features of GUTs into the type II context and the best features of the type II context back into the heterotic theory. But different sides are definitely better suited to describing different phenomena.

A. Strominger I think that is very misleading. There is no reason to believe that there are not also non-Kähler types of geometries on the type II side. We just are sticking with those because we are looking under the lamp post.

S. Kachru I completely agree with what you have said.

A. Polyakov Two brief comments. First of all it seems to me that one should not be overfixated on Calabi-Yau compactifications, or on compactifications at all. We do not really know how string theory applies to the real world. It is quite possible that we have some non-critical string theory working directly in four dimensions. I think that the important thing to do is to realise what mechanisms, what possibilities, we have in string theory, and it might be premature to try to get too much, to directly derive the standard model, etc. That is the first comment. The second comment is about a physical problem. It is just to inform you about things that are not widely known. There is an interesting application of the methods which we developed in string theory and conformal field theory in the theory of turbulence. There are recent numerical results which indicate that, in two dimensional turbulence, in some cases there are very conclusive signs of conformal theories. So that might be repaying debts which Nekrasov mentioned.

B. Julia I would like just to make a general remark in the same spirit of being useful. I think the most interesting thing I have learned, and everybody has learned, here is that the classical world does not exist. There is no unique classical limit. I think we should really develop more powerful tools to decide how good any classical limit is and which classical limit applies in any given computation. That would be useful for many people.

P. Ramond I do not know about being useful. First a remark about landscape. I just saw in a museum of ancient art Hieronymous Bosch's temptation of Saint Anthony, let me not say more.

The second thing is about some no-go theorems that we should be aware of. There are no-go theorems about massless particles of spin higher than two. If they are taken one at a time, there is no doubt that the no go-theorems apply. If there are an infinite amount, it is an open question. Moreover, there are mathematical structures based on the coset $F_4/SO(9)$, which is of great mathematical interest actually, that basically contain the fields of $N = 1$ supergravity in eleven dimension as its lowest level. One should not think this is completely about things that are known. This presents great difficulties, but one should keep an open mind and try to look at these things without giving up.

J. Harvey I think of the areas of mathematics that have been discussed here, the one that to me seems to have some of the strongest hints, but the least understood, is the general question of E_{10}, Borcherds algebras, generalized Kac-Moody algebras. In Nicolai's talk, to the degree I understood it, there definitely seems to be some structure going on, where E_{10} is reflected in eleven dimensional supergravity. But when he was talking about higher level in the landscape of these representations, I think it was about one in a billion that actually fit into the structure that we know so far, and we don't know what the rest of these are doing. In calculations I did with Greg Moore number of years ago, we found denominator formulas for Borcherds algebras coming out of definite one-loop string integrals. We tried to give an algebraic explanation of this but we failed. I do not think that these things can be coincidences. I think there is a very general algebraic structure which will allow us to have much greater control over some supersymmetric theories. I do not think it will solve the real world problem of what we do without supersymmetry or time evolution. But it seems to me to be an example of not drilling where the board is thinnest, but where it is thin enough that we might actually get through it.