6

Applications of Determinants in Mathematical Physics

6.1 Introduction

This chapter is devoted to verifications of the determinantal solutions of several equations which arise in three branches of mathematical physics, namely lattice, relativity, and soliton theories. All but one are nonlinear.

Lattice theory can be defined as the study of elements in a two- or three-dimensional array under the influence of neighboring elements. For example, it may be required to determine the electromagnetic state of one loop in an electrical network under the influence of the electromagnetic field generated by neighboring loops or to study the behavior of one atom in a crystal under the influence of neighboring atoms.

Einstein's theory of general relativity has withstood the test of time and is now called classical gravity. The equations which appear in this chapter arise in that branch of the theory which deals with stationary axisymmetric gravitational fields.

A soliton is a solitary wave and soliton theory can be regarded as a branch of nonlinear wave theory.

The term *determinantal solution* needs clarification since it can be argued that any function can be expressed as a determinant and, hence, any solvable equation has a solution which can be expressed as a determinant. The term *determinantal solution* shall mean a solution containing a determinant which has not been evaluated in simple form and may possibly be the simplest form of the function it represents. A number of determinants have been evaluated in a simple form in earlier chapters and elsewhere, but they are exceptional. In general, determinants cannot be evaluated in simple form. The definition of a determinant as a sum of products of elements is not, in general, a simple form as it is not, in general, amenable to many of the processes of analysis, especially repeated differentiation.

There may exist a section of the mathematical community which believes that if an equation possesses a determinantal solution, then the determinant must emerge from a matrix like an act of birth, for it cannot materialize in any other way! This belief has not, so far, been justified. In some cases, the determinants do indeed emerge from sets of equations and hence, by implication, from matrices, but in other cases, they arise as nonlinear algebraic and differential forms with no mother matrix in sight. However, we do not exclude the possibility that new methods of solution can be devised in which every determinant emerges from a matrix.

Where the integer n appears in the equation, as in the Dale and Toda equations, n or some function of n appears in the solution as the order of the determinant. Where n does not appear in the equation, it appears in the solution as the arbitrary order of a determinant.

The equations in this chapter were originally solved by a variety of methods including the application of the Gelfand–Levitan–Marchenko (GLM) integral equation of inverse scattering theory, namely

$$K(x,y,t) + R(x+y,t) + \int_x^\infty K(x,z,t)R(y+z,t)\,dz = 0$$

in which the kernel $R(u,t)$ is given and $K(x,y,t)$ is the function to be determined. However, in this chapter, all solutions are verified by the purely determinantal techniques established in earlier chapters.

6.2 Brief Historical Notes

In order to demonstrate the extent to which determinants have entered the field of differential and other equations we now give brief historical notes on the origins and solutions of these equations. The detailed solutions follow in later sections.

6.2.1 The Dale Equation

The Dale equation is

$$(y'')^2 = y'\left(\frac{y}{x}\right)'\left(\frac{y+4n^2}{1+x}\right)',$$

where n is a positive integer. This equation arises in the theory of stationary axisymmetric gravitational fields and is the only nonlinear ordinary equation to appear in this chapter. It was solved in 1978. Two related equations, which appear in Section 4.11.4, were solved in 1980. Cosgrove has published an equation which can be transformed into the Dale equation.

6.2.2 The Kay–Moses Equation

The one-dimensional Schrödinger equation, which arises in quantum theory, is

$$\left[D^2 + \varepsilon^2 - V(x)\right] y = 0, \quad D = \frac{d}{dx},$$

and is the only linear ordinary equation to appear in this chapter.

The solution for arbitrary $V(x)$ is not known, but in a paper published in 1956 on the reflectionless transmission of plane waves through dielectrics, Kay and Moses solved it in the particular case in which

$$V(x) = -2D^2(\log A),$$

where A is a certain determinant of arbitrary order whose elements are functions of x. The equation which Kay and Moses solved is therefore

$$\left[D^2 + \varepsilon^2 + 2D^2(\log A)\right] y = 0.$$

6.2.3 The Toda Equations

The differential–difference equations

$$\begin{aligned} D(R_n) &= \exp(-R_{n-1}) - \exp(-R_{n+1}), \\ D^2(R_n) &= 2\exp(-R_n) - \exp(-R_{n-1}) - \exp(-R_{n+1}), \quad D = \frac{d}{dx}, \end{aligned}$$

arise in nonlinear lattice theory. The first appeared in 1975 in a paper by Kac and van Moerbeke and can be regarded as a discrete analog of the KdV equation (Ablowitz and Segur, 1981). The second is the simplest of a series of equations introduced by Toda in 1967 and can be regarded as a second-order development of the first. For convenience, these equations are referred to as first-order and second-order Toda equations, respectively.

The substitutions

$$\begin{aligned} R_n &= -\log y_n, \\ y_n &= D(\log u_n) \end{aligned}$$

transform the first-order equation into

$$D(\log y_n) = y_{n+1} - y_{n-1} \tag{6.2.1}$$

and then into

$$D(u_n) = \frac{u_n u_{n+1}}{u_{n-1}}. \tag{6.2.2}$$

The same substitutions transform the second-order equation first into

$$D^2(\log y_n) = y_{n+1} - 2y_n + y_{n-1}$$

and then into

$$D^2(\log u_n) = \frac{u_{n+1}u_{n-1}}{u_n^2}\,. \tag{6.2.3}$$

Other equations which are similar in nature to the transformed second-order Toda equations are

$$\begin{aligned} D_x D_y(\log u_n) &= \frac{u_{n+1}u_{n-1}}{u_n^2}, \\ (D_x^2 + D_y^2)\log u_n &= \frac{u_{n+1}u_{n-1}}{u_n^2}, \\ \frac{1}{\rho}D_\rho\bigl[\rho D_\rho(\log u_n)\bigr] &= \frac{u_{n+1}u_{n-1}}{u_n^2}\,. \end{aligned} \tag{6.2.4}$$

All these equations are solved in Section 6.5.

Note that (6.2.1) can be expressed in the form

$$D(y_n) = y_n(y_{n+1} - y_{n-1}), \tag{6.2.1a}$$

which appeared in 1974 in a paper by Zacharov, Musher, and Rubenchick on Langmuir waves in a plasma and was solved in 1987 by S. Yamazaki in terms of determinants P_{2n-1} and P_{2n} of order n. Yamazaki's analysis involves a continued fraction. The transformed equation (6.2.2) is solved below without introducing a continued fraction but with the aid of the Jacobi identity and one of its variants (Section 3.6).

The equation

$$D_x D_y(R_n) = \exp(R_{n+1} - R_n) - \exp(R_n - R_{n-1}) \tag{6.2.5}$$

appears in a 1991 paper by Kajiwara and Satsuma on the q-difference version of the second-order Toda equation.

The substitution

$$R_n = \log\left(\frac{u_{n+1}}{u_n}\right)$$

reduces it to the first line of (6.2.4).

In the chapter on reciprocal differences in his book *Calculus of Finite Differences*, Milne-Thomson defines an operator r_n by the relations

$$\begin{aligned} r_0 f(x) &= f(x), \\ r_1 f(x) &= \frac{1}{f'(x)}, \\ \bigl[r_{n+1} - r_{n-1} - (n+1)r_1 r_n\bigr] f(x) &= 0. \end{aligned}$$

Put

$$r_n f = y_n.$$

Then,

$$y_{n+1} - y_{n-1} - (n+1)r_1(y_n) = 0,$$

that is,

$$y_n'(y_{n+1} - y_{n-1}) = n + 1.$$

This equation will be referred to as the Milne-Thomson equation. Its origin is distinct from that of the Toda equations, but it is of a similar nature and clearly belongs to this section.

6.2.4 The Matsukidaira–Satsuma Equations

The following pairs of coupled differential–difference equations appeared in a paper on nonlinear lattice theory published by Matsukidaira and Satsuma in 1990.

The first pair is

$$\begin{aligned} q_r' &= q_r(u_{r+1} - u_r), \\ \frac{u_r'}{u_r - u_{r-1}} &= \frac{q_r'}{q_r - q_{r-1}}. \end{aligned}$$

These equations contain two dependent variables q and u, and two independent variables, x which is continuous and r which is discrete. The solution is expressed in terms of a Hankel–Wronskian of arbitrary order n whose elements are functions of x and r.

The second pair is

$$\begin{aligned} (q_{rs})_y &= q_{rs}(u_{r+1,s} - u_{rs}), \\ \frac{(u_{rs})_x}{u_{rs} - u_{r,s-1}} &= \frac{q_{rs}(v_{r+1,s} - v_{rs})}{q_{rs} - q_{r,s-1}}. \end{aligned}$$

These equations contain three dependent variables, q, u, and v, and four independent variables, x and y which are continuous and r and s which are discrete. The solution is expressed in terms of a two-way Wronskian of arbitrary order n whose elements are functions of x, y, r, and s.

In contrast with Toda equations, the discrete variables do not appear in the solutions as orders of determinants.

6.2.5 The Korteweg–de Vries Equation

The Korteweg–de Vries (KdV) equation, namely

$$u_t + 6uu_x + u_{xxx} = 0,$$

where the suffixes denote partial derivatives, is nonlinear and first arose in 1895 in a study of waves in shallow water. However, in the 1960s, interest in the equation was stimulated by the discovery that it also arose in studies of magnetohydrodynamic waves in a warm plasma, ion acoustic waves, and acoustic waves in an anharmonic lattice. Of all physically significant nonlinear partial differential equations with known analytic solutions, the KdV equation is one of the simplest. The KdV equation can be regarded as a particular case of the Kadomtsev–Petviashvili (KP) equation but it is of such fundamental importance that it has been given detailed individual attention in this chapter.

A method for solving the KdV equation based on the GLM integral equation was described by Gardner, Greene, Kruskal, and Miura (GGKM) in 1967. The solution is expressed in the form

$$u = 2D_x\{K(x,x,t)\}, \quad D_x = \frac{\partial}{\partial x}.$$

However, GGKM did not give an explicit solution of the integral equation and the first explicit solution of the KdV equation was given by Hirota in 1971 in terms of a determinant with well-defined elements but of arbitrary order. He used an independent method which can be described as heuristic, that is, obtained by trial and error. In another pioneering paper published the same year, Zakharov solved the KdV equation using the GGKM method. Wadati and Toda also applied the GGKM method and, in 1972, published a solution which agrees with Hirota's.

In 1979, Satsuma showed that the solution of the KdV equation can be expressed in terms of a Wronskian, again with well-defined elements but of arbitrary order. In 1982, Pöppe transformed the KdV equation into an integral equation and solved it by the Fredholm determinant method. Finally, in 1983, Freeman and Nimmo solved the KdV equation directly in Wronskian form.

6.2.6 *The Kadomtsev–Petviashvili Equation*

The Kadomtsev–Petviashvili (KP) equation, namely

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

arises in a study published in 1970 of the stability of solitary waves in weakly dispersive media. It can be regarded as a two-dimensional generalization of the KdV equation to which it reverts if u is independent of y.

The non-Wronskian solution of the KP equation was obtained from inverse scattering theory (Lamb, 1980) and verified in 1989 by Matsuno using a method based on the manipulation of bordered determinants. In 1983, Freeman and Nimmo solved the KP equation directly in Wronskian form, and in 1988, Hirota, Ohta, and Satsuma found a solution containing a two-way (right and left) Wronskian. Again, all determinants have well-defined elements but are of arbitrary order. Shortly after the Matsuno paper appeared, A. Nakamura solved the KP equation by means of four linear operators and a determinant of arbitrary order whose elements are defined as integrals.

The verifications given in Sections 6.7 and 6.8 of the non-Wronskian solutions of both the KdV and KP equations apply purely determinantal methods and are essentially those published by Vein and Dale in 1987.

6.2.7 The Benjamin–Ono Equation

The Benjamin–Ono (BO) equation is a nonlinear integro-differential equation which arises in the theory of internal waves in a stratified fluid of great depth and in the propagation of nonlinear Rossby waves in a rotating fluid. It can be expressed in the form

$$u_t + 4uu_x + H\{u_{xx}\} = 0,$$

where $H\{f(x)\}$ denotes the Hilbert transform of $f(x)$ defined as

$$H\{f(x)\} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)}{y-x}\, dy$$

and where P denotes the principal value.

In a paper published in 1988, Matsuno introduced a complex substitution into the BO equation which transformed it into a more manageable form, namely

$$2A_x A_x^* = A^*(A_{xx} + \omega A_t) + A(A_{xx} + \omega A_t)^* \quad (\omega^2 = -1),$$

where A^* is the complex conjugate of A, and found a solution in which A is a determinant of arbitrary order whose diagonal elements are linear in x and t and whose nondiagonal elements contain a sequence of distinct arbitrary constants.

6.2.8 The Einstein and Ernst Equations

In the particular case in which a relativistic gravitational field is axially symmetric, the Einstein equations can be expressed in the form

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \mathbf{P}}{\partial \rho} \mathbf{P}^{-1}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \mathbf{P}}{\partial z} \mathbf{P}^{-1}\right) = 0,$$

where the matrix $\mathbf{P}$ is defined as

$$\mathbf{P} = \frac{1}{\phi}\begin{bmatrix} 1 & \psi \\ \psi & \phi^2 + \psi^2 \end{bmatrix}. \tag{6.2.6}$$

ϕ is the gravitational potential and is real and ψ is either real, in which case it is the twist potential, or it is purely imaginary, in which case it has no physical significance. (ρ, z) are cylindrical polar coordinates, the angular coordinate being absent as the system is axially symmetric.

Since $\det \mathbf{P} = 1$,

$$\begin{aligned}
\mathbf{P}^{-1} &= \frac{1}{\phi}\begin{bmatrix} \phi^2+\psi^2 & -\psi \\ -\psi & 1 \end{bmatrix},\\
\frac{\partial \mathbf{P}}{\partial \rho} &= \frac{1}{\phi^2}\begin{bmatrix} -\phi_\rho & \phi\psi_\rho - \psi\phi_\rho \\ \phi\psi_\rho - \psi\phi_\rho & \phi^2\phi_\rho + 2\phi\psi\psi_\rho - \psi^2\phi_\rho \end{bmatrix},\\
\frac{\partial \mathbf{P}}{\partial \rho}\mathbf{P}^{-1} &= \frac{\mathbf{M}}{\phi^2},\\
\frac{\partial \mathbf{P}}{\partial z}\mathbf{P}^{-1} &= \frac{\mathbf{N}}{\phi^2},
\end{aligned}$$

where

$$\mathbf{M} = \begin{bmatrix} -(\phi\phi_\rho + \psi\psi_\rho) & \psi_\rho \\ (\phi^2-\psi^2)\psi_\rho - 2\phi\psi\phi_\rho & \phi\phi_\rho + \psi\psi_\rho \end{bmatrix}$$

and $\mathbf{N}$ is the matrix obtained from $\mathbf{M}$ by replacing ϕ_ρ by ϕ_z and ψ_ρ by ψ_z.

The equation above (6.2.6) can now be expressed in the form

$$\frac{\mathbf{M}}{\rho} - \frac{2}{\phi}(\phi_\rho\mathbf{M} + \phi_z\mathbf{N}) + (\mathbf{M}_\rho + \mathbf{N}_z) = 0 \tag{6.2.7}$$

where

$$\phi_\rho\mathbf{M} + \phi_z\mathbf{N} = \begin{bmatrix} -\left\{\begin{matrix} \phi(\phi_\rho^2+\phi_z^2) \\ +\psi(\phi_\rho\psi_\rho + \phi_z\psi_z) \end{matrix}\right\} & \{\phi_\rho\psi_\rho + \phi_z\psi_z\} \\ \left\{\begin{matrix} (\phi^2-\psi^2)(\phi_\rho\psi_\rho+\phi_z\psi_z) \\ -2\phi\psi(\phi_\rho^2+\phi_z^2) \end{matrix}\right\} & \left\{\begin{matrix} \phi(\phi_\rho^2+\phi_z^2) \\ +\psi(\phi_\rho\psi_\rho+\phi_z\psi_z) \end{matrix}\right\} \end{bmatrix},$$

$$\begin{aligned}
&\mathbf{M}_\rho + \mathbf{N}_z\\
&= \begin{bmatrix} -\left\{\begin{matrix} \phi(\phi_{\rho\rho}+\phi_{zz}) + \psi(\psi_{\rho\rho}+\psi_{zz}) \\ +\phi_\rho^2+\phi_z^2+\psi_\rho^2+\psi_z^2 \end{matrix}\right\} & \{\psi_{\rho\rho}+\psi_{zz}\} \\ \left\{\begin{matrix} (\phi^2-\psi^2)(\psi_{\rho\rho}+\psi_{zz}) - 2\phi\psi(\phi_{\rho\rho}+\phi_{zz}) \\ -2\psi(\phi_\rho^2+\phi_z^2+\psi_\rho^2+\psi_z^2) \end{matrix}\right\} & \left\{\begin{matrix} \phi(\phi_{\rho\rho}+\phi_{zz}) + \psi(\psi_{\rho\rho}+\psi_{zz}) \\ +\phi_\rho^2+\phi_z^2+\psi_\rho^2+\psi_z^2 \end{matrix}\right\} \end{bmatrix}
\end{aligned}$$

The Einstein equations can now be expressed in the form

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = 0,$$

where

$$\begin{aligned}
f_{12} &= \frac{1}{\phi}\left[\phi\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2(\phi_\rho\psi_\rho + \phi_z\psi_z)\right] = 0,\\
f_{11} &= -\psi f_{12} - \left[\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2\right] = 0,\\
f_{21} &= (\phi^2-\psi^2)f_{12} - 2\psi\left[\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2\right] = 0,\\
f_{22} &= -f_{11} = 0,
\end{aligned}$$

which yields only two independent scalar equations, namely

$$\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2 = 0, \tag{6.2.8}$$

$$\phi\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2(\phi_\rho\psi_\rho + \phi_z\psi_z) = 0. \tag{6.2.9}$$

The second equation can be rearranged into the form

$$\frac{\partial}{\partial\rho}\left(\frac{\rho\psi_\rho}{\phi^2}\right) + \frac{\partial}{\partial z}\left(\frac{\rho\psi_z}{\phi^2}\right) = 0.$$

Historically, the scalar equations (6.2.8) and (6.2.9) were formulated before the matrix equation (6.2.1), but the modern approach to relativity is to formulate the matrix equation first and to derive the scalar equations from them.

Equations (6.2.8) and (6.2.9) can be contracted into the form

$$\phi\nabla^2\phi - (\nabla\phi)^2 + (\nabla\psi)^2 = 0, \tag{6.2.10}$$

$$\phi\nabla^2\psi - 2\nabla\phi\cdot\nabla\psi = 0, \tag{6.2.11}$$

which can be contracted further into the equations

$$\tfrac{1}{2}(\zeta_+ + \zeta_-)\nabla^2\zeta_\pm = (\nabla\zeta_\pm)^2, \tag{6.2.12}$$

where

$$\begin{aligned}\zeta_+ &= \phi + \omega\psi,\\ \zeta_- &= \phi - \omega\psi \qquad (\omega^2 = -1).\end{aligned} \tag{6.2.13}$$

The notation

$$\begin{aligned}\zeta &= \phi + \omega\psi,\\ \zeta^* &= \phi - \omega\psi,\end{aligned} \tag{6.2.14}$$

where ζ^* is the complex conjugate of ζ, can be used only when ϕ and ψ are real. In that case, the two equations (6.2.12) reduce to the single equation

$$\tfrac{1}{2}(\zeta + \zeta^*)\nabla^2\zeta = (\nabla\zeta)^2. \tag{6.2.15}$$

In 1983, Y. Nakamura conjectured the existence two related infinite sets of solutions of (6.2.8) and (6.2.9). He denoted them by

$$\begin{aligned}&\phi_n',\ \psi_n', \quad n \geq 1,\\ &\phi_n,\ \psi_n, \quad n \geq 2,\end{aligned} \tag{6.2.16}$$

and deduced the first few members of each set with the aid of the pair of coupled difference–differential equations given in Appendix A.11 and the Bäcklund transformations β and γ given in Appendix A.12. The general Nakamura solutions were given by Vein in 1985 in terms of cofactors associated with a determinant of arbitrary order whose elements satisfy the difference–differential equations. These solutions are reproduced with minor modifications in Section 6.10.2. In 1986, Kyriakopoulos approached the same problem from another direction and obtained the same determinant in a different form.

The Nakamura–Vein solutions are of great interest mathematically but are not physically significant since, as can be seen from (6.10.21) and (6.10.22), ϕ_n and ψ_n can be complex functions when the elements of B_n are complex. Even when the elements are real, ψ_n and ψ'_n are purely imaginary when n is odd. The Nakamura–Vein solutions are referred to as intermediate solutions.

The Neugebauer family of solutions published in 1980 contains as a particular case the Kerr–Tomimatsu–Sato class of solutions which represent the gravitational field generated by a spinning mass. The Ernst complex potential ξ in this case is given by the formula

$$\xi = F/G \tag{6.2.17}$$

where F and G are determinants of order $2n$ whose column vectors are defined as follows:

In F,

$$\mathbf{C}_j = \left[\tau_j \;\; c_j\tau_j \;\; c_j^2\tau_j \cdots c_j^{n-2}\tau_j \;\; 1 \;\; c_j \;\; c_j^2 \ldots c_j^n\right]_{2n}^T, \tag{6.2.18}$$

and in G,

$$\mathbf{C}_j = \left[\tau_j \;\; c_j\tau_j \;\; c_j^2\tau_j \cdots c_j^{n-1}\tau_j \;\; 1 \;\; c_j \;\; c_j^2 \ldots c_j^{n-1}\right]_{2n}^T, \tag{6.2.19}$$

where

$$\tau_j = e^{\omega\theta_j}\left[\rho^2 + (z + c_j)^2\right]^{\frac{1}{2}} \quad (\omega^2 = -1) \tag{6.2.20}$$

and $1 \le j \le 2n$. The c_j and θ_j are arbitrary real constants which can be specialized to give particular solutions such as the Yamazaki–Hori solutions and the Kerr–Tomimatsu–Sato solutions.

In 1993, Sasa and Satsuma used the Nakamura–Vein solutions as a starting point to obtain physically significant solutions. Their analysis included a study of Vein's quasicomplex symmetric Toeplitz determinant A_n and a related determinant E_n. They showed that A_n and E_n satisfy two equations containing Hirota operators. They then applied these equations to obtain a solution of the Einstein equations and verified with the aid of a computer that their solution is identical with the Neugebauer solution for small values of n. The equations satisfied by A_n and E_n are given as exercises at the end of Section 6.10.2 on the intermediate solutions.

A wholly analytic method of obtaining the Neugebauer solutions is given in Sections 6.10.4 and 6.10.5. It applies determinantal identities and other relations which appear in this chapter and elsewhere to modify the Nakamura–Vein solutions by means of algebraic Bäcklund transformations.

The substitution

$$\zeta = \frac{1-\xi}{1+\xi} \tag{6.2.21}$$

transforms equation (6.2.15) into the Ernst equation, namely

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*(\nabla\xi \cdot \nabla\xi) \tag{6.2.22}$$

which appeared in 1968.

In 1977, M. Yamazaki conjectured and, in 1978, Hori proved that a solution of the Ernst equation is given by

$$\xi_n = \frac{pxu_n - \omega qyv_n}{w_n} \quad (\omega^2 = -1), \tag{6.2.23}$$

where x and y are prolate spheroidal coordinates and u_n, v_n, and w_n are determinants of arbitrary order n in which the elements in the first columns of u_n and v_n are polynomials with complicated coefficients. In 1983, Vein showed that the Yamazaki–Hori solutions can be expressed in the form

$$\xi_n = \frac{pU_{n+1} - \omega qV_{n+1}}{W_{n+1}} \tag{6.2.24}$$

where U_{n+1}, V_{n+1}, and W_{n+1} are bordered determinants of order $n+1$ with comparatively simple elements. These determinants are defined in detail in Section 4.10.3.

Hori's proof of (6.2.23) is long and involved, but no neat proof has yet been found. The solution of (6.2.24) is stated in Section 6.10.6, but since it was obtained directly from (6.2.23) no neat proof is available.

6.2.9 The Relativistic Toda Equation

The relativistic Toda equation, namely

$$\begin{aligned}\ddot{R}_n &= \left(1 + \frac{\dot{R}_{n-1}}{c}\right)\left(1 + \frac{\dot{R}_n}{c}\right)\frac{\exp(R_{n-1} - R_n)}{1 + (1/c^2)\exp(R_{n-1} - R_n)} \\ &\quad - \left(1 - \frac{\dot{R}_n}{c}\right)\left(1 + \frac{\dot{R}_{n+1}}{c}\right)\frac{\exp(R_n - R_{n+1})}{1 + (1/c^2)\exp(R_n - R_{n+1})},\end{aligned} \tag{6.2.25}$$

where $\dot{R}_n = dR_n/dt$, etc., was introduced by Rujisenaars in 1990. In the limit as $c \to \infty$, (6.2.25) degenerates into the equation

$$\ddot{R}_n = \exp(R_{n-1} - R_n) - \exp(R_n - R_{n+1}). \tag{6.2.26}$$

The substitution

$$R_n = \log\left\{\frac{U_{n-1}}{U_n}\right\} \tag{6.2.27}$$

reduces (6.2.26) to (6.2.3).

Equation (6.2.25) was solved by Ohta, Kajiwara, Matsukidaira, and Satsuma in 1993. A brief note on the solutions is given in Section 6.11.

6.3 The Dale Equation

Theorem. *The Dale equation, namely*

$$(y'')^2 = y'\left(\frac{y}{x}\right)'\left(\frac{y+4n^2}{1+x}\right)',$$

where n is a positive integer, is satisfied by the function

$$y = 4(c-1)xA_n^{11},$$

where A_n^{11} is a scaled cofactor of the Hankelian $A_n = |a_{ij}|_n$ in which

$$a_{ij} = \frac{x^{i+j-1} + (-1)^{i+j}c}{i+j-1}$$

and c is an arbitrary constant. The solution is clearly defined when $n \geq 2$ but can be made valid when $n = 1$ by adopting the convention $A_{11} = 1$ so that $A^{11} = (x+c)^{-1}$.

PROOF. Using Hankelian notation (Section 4.8),

$$A = |\phi_m|_n, \quad 0 \leq m \leq 2n-2,$$

where

$$\phi_m = \frac{x^{m+1} + (-1)^m c}{m+1}. \tag{6.3.1}$$

Let

$$P = |\psi_m|_n,$$

where

$$\psi_m = (1+x)^{-m-1}\phi_m.$$

Then,

$$\psi_m' = mF\psi_{m-1}$$

(the Appell equation), where

$$F = (1+x)^{-2}. \tag{6.3.2}$$

Hence, by Theorem 4.33 in Section 4.9.1 on Hankelians with Appell elements,

$$\begin{aligned} P' &= \psi_0' P_{11} \\ &= \frac{(1-c)P_{11}}{(1+x)^2}. \end{aligned} \tag{6.3.3}$$

Note that the theorem cannot be applied to A directly since ϕ_m does not satisfy the Appell equation for any $F(x)$.

Using the identity

$$|t^{i+j-2}a_{ij}|_n = t^{n(n-1)}|a_{ij}|_n,$$

it is found that

$$\begin{aligned} P &= (1+x)^{-n^2}A, \\ P_{11} &= (1+x)^{-n^2+1}A_{11}. \end{aligned} \tag{6.3.4}$$

Hence,

$$(1+x)A' = n^2A - (c-1)A_{11}. \tag{6.3.5}$$

Let

$$\alpha_i = \sum_r x^{r-1}A^{ri}, \tag{6.3.6}$$

$$\beta_i = \sum_r (-1)^r A^{ri}, \tag{6.3.7}$$

$$\begin{aligned} \lambda &= \sum_r (-1)^r \alpha_r \\ &= \sum_r \sum_s (-1)^r x^{s-1}A^{rs} \\ &= \sum_s x^{s-1}\beta_s, \end{aligned} \tag{6.3.8}$$

where r and $s = 1, 2, 3, \ldots, n$ in all sums.

Applying double-sum identity (D) in Section 3.4 with $f_r = r$ and $g_s = s - 1$, then (B),

$$\begin{aligned} (i+j-1)A^{ij} &= \sum_r \sum_s [x^{r+s-1} + (-1)^{r+s}c]A^{ri}A^{sj} \\ &= x\alpha_i\alpha_j + c\beta_i\beta_j \end{aligned} \tag{6.3.9}$$

$$\begin{aligned} (A^{ij})' &= -\sum_r \sum_s x^{i+j-2}A^{is}A^{rj} \\ &= -\alpha_i\alpha_j. \end{aligned} \tag{6.3.10}$$

Hence,

$$\begin{aligned} x(A^{ij})' + (i+j-1)A^{ij} &= c\beta_i\beta_j, \\ (x^{i+j-1}A^{ij})' &= c(x^{i-1}\beta_i)(x^{j-1}\beta_j). \end{aligned}$$

In particular,

$$\begin{aligned} (A^{11})' &= -\alpha_1^2, \\ (xA^{11})' &= c\beta_1^2. \end{aligned} \tag{6.3.11}$$

Applying double-sum identities (C) and (A),

$$\begin{aligned}\sum_{r=1}^{n}\sum_{s=1}^{n}[x^{r+s-1}+(-1)^{r+s}c]A^{rs} &= \sum_{r=1}^{n}(2r-1)\\ &= n^2\end{aligned} \tag{6.3.12}$$

$$\begin{aligned}\frac{xA'}{A} &= \sum_{r=1}^{n}\sum_{s=1}^{n}x^{r+s-1}A^{rs}\\ &= n^2 - c\sum_{r=1}^{n}\sum_{s=1}^{n}(-1)^{r+s}A^{rs}.\end{aligned} \tag{6.3.13}$$

Differentiating and using (6.3.10),

$$\begin{aligned}\left(\frac{xA'}{A}\right)' &= c\sum_{r}^{n}\sum_{s}^{n}(-1)^{r+s}\alpha_r\alpha_s\\ &= c\lambda^2.\end{aligned} \tag{6.3.14}$$

It follows from (6.3.5) that

$$\begin{aligned}\frac{xA'}{A} &= \left[1-\frac{1}{1+x}\right][n^2-(c-1)A^{11}]\\ &= n^2-\left[\frac{(c-1)xA^{11}+n^2}{1+x}\right].\end{aligned} \tag{6.3.15}$$

Hence, eliminating xA'/A and using (6.3.14),

$$\left[\frac{(c-1)xA^{11}+n^2}{1+x}\right]' = -c\lambda^2. \tag{6.3.16}$$

Differentiating (6.3.7) and using (6.3.10) and the first equation in (6.3.8),

$$\beta_i' = \lambda\alpha_i. \tag{6.3.17}$$

Differentiating the second equation in (6.3.11) and using (6.3.17),

$$(xA^{11})'' = 2c\lambda\alpha_1\beta_1. \tag{6.3.18}$$

All preparations for proving the theorem are now complete.

Put

$$y = 4(c-1)xA^{11}.$$

Referring to the second equation in (6.3.11),

$$\begin{aligned}y' &= 4(c-1)(xA^{11})'\\ &= 4c(c-1)\beta_1^2.\end{aligned} \tag{6.3.19}$$

Referring to the first equation in (6.3.11),

$$\left(\frac{y}{x}\right)' = 4(c-1)(A^{11})'$$

$$= -4c(c-1)\alpha_1^2. \tag{6.3.20}$$

Referring to (6.3.16),

$$\begin{aligned}\left(\frac{y+4n^2}{1+x}\right)' &= 4\left[\frac{(c-1)xA^{11}+n^2}{1+x}\right]' \\ &= -4c\lambda^2. \end{aligned} \tag{6.3.21}$$

Differentiating (6.3.19) and using (6.3.17),

$$y'' = 8c(c-1)\lambda\alpha_1\beta_1. \tag{6.3.22}$$

The theorem follows from (6.3.19) and (6.3.22). □

6.4 The Kay–Moses Equation

Theorem. *The Kay–Moses equation, namely*

$$\left[D^2 + \varepsilon^2 + 2D^2(\log A)\right]y = 0 \tag{6.4.1}$$

is satisfied by the equation

$$y = e^{-\omega\varepsilon x}\left[1 - \sum_{i,j=1}^{n} \frac{e^{(c_i+c_j)\omega\varepsilon x}A^{ij}}{c_j - 1}\right], \quad \omega^2 = -1,$$

where

$$\begin{aligned} A &= |a_{rs}|_n, \\ a_{rs} &= \delta_{rs}b_r + \frac{e^{(c_r+c_s)\omega\varepsilon x}}{c_r+c_s}. \end{aligned}$$

The b_r*,* $r \geq 1$*, are arbitrary constants and the* c_r*,* $r \geq 1$*, are constants such that* $c_j \neq 1$*,* $1 \leq j \leq n$ *and* $c_r + c_s \neq 0$*,* $1 \leq r, s \leq n$*, but are otherwise arbitrary.*

The analysis which follows differs from the original both in the form of the solution and the method by which it is obtained.

Proof. Let $A = |a_{rs}(u)|_n$ denote the symmetric determinant in which

$$\begin{aligned} a_{rs} &= \delta_{rs}b_r + \frac{e^{(c_r+c_s)u}}{c_r+c_s} = a_{sr}, \\ a'_{rs} &= e^{(c_r+c_s)u}. \end{aligned} \tag{6.4.2}$$

Then the double-sum relations (A)–(D) in Section 3.4 with $f_r = g_r = c_r$ become

$$(\log A)' = \sum_{r,s} e^{(c_r+c_s)u}A^{rs}, \tag{6.4.3}$$

$$(A^{ij})' = -\sum_r e^{c_r u} A^{rj} \sum_s e^{c_s u} A^{is}, \tag{6.4.4}$$

$$2\sum_r b_r c_r A^{rr} + \sum_{r,s} e^{(c_r+c_s)u} A^{rs} = 2\sum_r c_r, \tag{6.4.5}$$

$$2\sum_r b_r c_r A^{ir} A^{rj} + \sum_r e^{c_r u} A^{rj} \sum_s e^{c_s u} A^{is} = (c_i + c_j) A^{ij}. \tag{6.4.6}$$

Put

$$\phi_i = \sum_s e^{c_s u} A^{is}. \tag{6.4.7}$$

Then (6.4.4) and (6.4.6) become

$$(A^{ij})' = -\phi_i \phi_j, \tag{6.4.8}$$

$$2\sum_r b_r c_r A^{ir} A^{rj} + \phi_i \phi_j = (c_i + c_j) A^{ij}. \tag{6.4.9}$$

Eliminating the $\phi_i \phi_j$ terms,

$$\begin{aligned}
(A^{ij})' + (c_i + c_j) A^{ij} &= 2\sum_r b_r c_r A^{ir} A^{rj}, \\
\left[e^{(c_i+c_j)u} A^{ij}\right]' &= 2e^{(c_i+c_j)u} \sum_r b_r c_r A^{ir} A^{rj}. \qquad (6.4.10)
\end{aligned}$$

Differentiating (6.4.3),

$$\begin{aligned}
(\log A)'' &= \sum_{i,j} \left[e^{(c_i+c_j)u} A^{ij}\right]' \\
&= 2\sum_r b_r c_r \sum_i e^{c_i u} A^{ir} \sum_j e^{c_j u} A^{rj} \\
&= 2\sum_r b_r c_r \phi_r^2. \qquad (6.4.11)
\end{aligned}$$

Replacing s by r in (6.4.7),

$$\begin{aligned}
e^{c_i u} \phi_i &= \sum_r e^{(c_i+c_r)u} A^{ir}, \\
\left(e^{c_j u} \phi_i\right)' &= 2\sum_r b_r c_r \left(e^{c_i u} A^{ir}\right) \sum_j e^{c_j u} A^{rj} \\
&= 2\sum_r b_r c_r \phi_r e^{c_i u} A^{ir}, \\
\phi_i' + c_i \phi_i &= 2\sum_r b_r c_r \phi_r A^{ir}.
\end{aligned}$$

Interchange i and r, multiply by $b_r c_r A^{rj}$, sum over r, and refer to (6.4.9):

$$\sum_r b_r c_r A^{rj} (\phi_r' + c_r \phi_r) = 2\sum_i b_i c_i \phi_i \sum_r b_r c_r A^{ir} A^{rj}$$

$$
\begin{aligned}
&= \sum_i b_i c_i \phi_i [(c_i + c_j) A^{ij} - \phi_i \phi_j] \\
&= \sum_r b_r c_r \phi_r [(c_r + c_j) A^{rj} - \phi_r \phi_j],
\end{aligned}
$$

$$
\sum_r b_r c_r A^{rj} \phi_r' = \sum_r b_r c_r \phi_r [c_j A^{rj} - \phi_r \phi_j], \tag{6.4.12}
$$

$$
\sum_r b_r c_r A^{rj} (\phi_r' - \phi_r) = \sum_r b_r c_r \phi_r (c_j - 1) A^{rj} - \sum_r b_r c_r \phi_r^2 \phi_j.
$$

Multiply by $e^{c_j u}/(c_j - 1)$, sum over j, and refer to (6.4.7):

$$
\begin{aligned}
\sum_{j,r} \frac{b_r c_r A^{rj} e^{c_j u} (\phi_r' - \phi_r)}{c_j - 1} &= \sum_r b_r c_r \phi_r^2 - \sum_r b_r c_r \phi_r^2 \sum_j \frac{e^{c_j u} \phi_j}{c_j - 1} \\
&= F \sum_r b_r c_r \phi_r^2 \\
&= \tfrac{1}{2} F (\log A)'',
\end{aligned} \tag{6.4.13}
$$

where

$$
\begin{aligned}
F &= 1 - \sum_j \frac{e^{c_j u} \phi_j}{c_j - 1} \\
&= 1 - \sum_{i,j} \frac{e^{(c_i + c_j) u} A^{ij}}{c_j - 1}.
\end{aligned} \tag{6.4.14}
$$

Differentiate and refer to (6.4.9):

$$
\begin{aligned}
F' &= -2 \sum_r b_r c_r \sum_j \frac{e^{c_j u} A^{rj}}{c_j - 1} \sum_i e^{c_i u} A^{ir} \\
&= -2 \sum_r b_r c_r \sum_j \frac{\phi_r e^{c_j u} A^{rj}}{c_j - 1}.
\end{aligned} \tag{6.4.15}
$$

Differentiate again and refer to (6.4.8):

$$
\begin{aligned}
F'' &= 2 \sum_r b_r c_r \sum_j \frac{e^{c_j u}}{c_j - 1} \left[\phi_r^2 \phi_j - c_j \phi_r A^{rj} - \phi_r' A^{rj} \right] \\
&= P - Q - R,
\end{aligned} \tag{6.4.16}
$$

where

$$
\begin{aligned}
P &= 2 \sum_j \frac{e^{c_j u} \phi_j}{c_j - 1} \sum_r b_r c_r \phi_r^2 \\
&= (1 - F)(\log A)''
\end{aligned} \tag{6.4.17}
$$

$$
Q = 2 \sum_{j,r} \frac{b_r c_r c_j \phi_r e^{c_j u} A^{rj}}{c_j - 1}
$$

$$\begin{aligned}
&= 2\sum_r b_r c_r \phi_r \sum_j e^{c_j u} A^{rj} + 2\sum_r b_r c_r \sum_j \frac{\phi_r e^{c_j u} A^{rj}}{c_j - 1} \\
&= 2\sum_r b_r c_r \phi_r^2 - F' \\
&= (\log A)'' - F', \qquad (6.4.18)
\end{aligned}$$

$$\begin{aligned}
R &= 2\sum_j \frac{e^{c_j u}}{c_j - 1} \sum_r b_r c_r \phi_r' A^{rj} \\
&= 2\sum_j \frac{e^{c_j u}}{c_j - 1} \sum_r b_r c_r \phi_r [c_j A^{rj} - \phi_r \phi_j] \\
&= Q - P. \qquad (6.4.19)
\end{aligned}$$

Hence, eliminating P, Q, and R from (6.4.16)–(6.4.19),

$$\frac{d^2 F}{du^2} - 2\frac{dF}{du} + 2F(\log A)'' = 0. \qquad (6.4.20)$$

Put

$$F = e^u y. \qquad (6.4.21)$$

Then, (6.4.20) is transformed into

$$\frac{d^2 y}{du^2} - y + 2y\frac{d^2}{du^2}(\log A) = 0. \qquad (6.4.22)$$

Finally, put $u = \omega\varepsilon x$, $(\omega^2 = -1)$. Then, (6.4.22) is transformed into

$$\frac{d^2 y}{dx^2} + \varepsilon^2 y + 2y\frac{d^2}{dx^2}(\log A) = 0,$$

which is identical with (6.4.1), the Kay–Moses equation. This completes the proof of the theorem. □

6.5 The Toda Equations

6.5.1 The First-Order Toda Equation

Define two Hankel determinants (Section 4.8) A_n and B_n as follows:

$$\begin{aligned}
A_n &= |\phi_m|_n, \quad 0 \le m \le 2n-2, \\
B_n &= |\phi_m|_n, \quad 1 \le m \le 2n-1, \\
A_0 &= B_0 = 1. \qquad (6.5.1)
\end{aligned}$$

The algebraic identities

$$A_n B_{n+1,n}^{(n+1)} - B_n A_{n+1,n}^{(n+1)} + A_{n+1} B_{n-1} = 0, \qquad (6.5.2)$$

$$B_{n-1} A_{n+1,n}^{(n+1)} - A_n B_{n,n-1}^{(n)} + A_{n-1} B_n = 0 \qquad (6.5.3)$$

are proved in Theorem 4.30 in Section 4.8.5 on Turanians.

Let the elements in both A_n and B_n be defined as

$$\phi_m(x) = f^{(m)}(x), \quad f(x) \text{ arbitrary,}$$

so that

$$\phi'_m = \phi_{m+1} \tag{6.5.4}$$

and both A_n and B_n are Wronskians (Section 4.7) whose derivatives are given by

$$\begin{aligned} A'_n &= -A^{(n+1)}_{n+1,n}, \\ B'_n &= -B^{(n+1)}_{n+1,n}. \end{aligned} \tag{6.5.5}$$

Theorem 6.1. *The equation*

$$u'_n = \frac{u_n u_{n+1}}{u_{n-1}}$$

is satisfied by the function defined separately for odd and even values of n as follows:

$$\begin{aligned} u_{2n-1} &= \frac{A_n}{B_{n-1}}, \\ u_{2n} &= \frac{B_n}{A_n}\,. \end{aligned}$$

PROOF.

$$\begin{aligned} B^2_{n-1} u'_{2n-1} &= B_{n-1} A'_n - A_n B'_{n-1} \\ &= -B_{n-1} A^{(n+1)}_{n+1,n} + A_n B^{(n)}_{n,n-1} \\ B^2_{n-1}\left(\frac{u_{2n-1}u_{2n}}{u_{2n-2}}\right) &= A_{n-1}B_n. \end{aligned}$$

Hence, referring to (6.5.3),

$$\begin{aligned} B^2_{n-1}\left[\frac{u_{2n-1}u_{2n}}{u_{2n-2}} - u'_{2n-1}\right] &= A_{n-1}B_n + B_{n-1}A^{(n+1)}_{n+1,n} - A_n B^{(n)}_{n,n-1} \\ &= 0, \end{aligned}$$

which proves the theorem when n is odd.

$$\begin{aligned} A^2_n u'_{2n} &= A_n B'_n - B_n A'_n \\ &= -A_n B^{(n+1)}_{n+1,n} + B_n A^{(n+1)}_{n,n+1}, \\ A^2_n\left(\frac{u_{2n}u_{2n+1}}{u_{2n-1}}\right) &= A_{n+1}B_{n-1}. \end{aligned}$$

Hence, referring to (6.5.2),

$$A^2_n\left[\frac{u_{2n}u_{2n+1}}{u_{2n-1}} - u'_{2n}\right] = A_{n+1}B_{n-1} + A_n B^{(n+1)}_{n+1,n} - B_n A^{(n+1)}_{n,n+1}$$

$$= 0,$$

which proves the theorem when n is even. □

Theorem 6.2. *The function*

$$y_n = D(\log u_n), \quad D = \frac{d}{dx},$$

is given separately for odd and even values of n *as follows:*

$$y_{2n-1} = \frac{A_{n-1}B_n}{A_n B_{n-1}},$$
$$y_{2n} = \frac{A_{n+1}B_{n-1}}{A_n B_n}.$$

PROOF.

$$\begin{aligned} y_{2n-1} &= D \log\left(\frac{A_n}{B_{n-1}}\right) \\ &= \frac{1}{A_n B_{n-1}}\left(B_{n-1}A'_n - A_n B'_{n-1}\right) \\ &= \frac{1}{A_n B_{n-1}}\left[-B_{n-1}A^{(n+1)}_{n+1,n} + A_n B^{(n)}_{n,n-1}\right]. \end{aligned}$$

The first part of the theorem follows from (6.5.3).

$$\begin{aligned} y_{2n} &= D \log\left(\frac{B_n}{A_n}\right) \\ &= \frac{1}{A_n B_n}\left(A_n B'_n - B_n A'_n\right) \\ &= \frac{1}{A_n B_n}\left[-A_n B^{(n+1)}_{n+1,n} + B_n A^{(n+1)}_{n+1,n}\right]. \end{aligned}$$

The second part of the theorem follows from (6.5.2). □

6.5.2 The Second-Order Toda Equations

Theorem 6.3. *The equation*

$$D_x D_y(\log u_n) = \frac{u_{n+1}u_{n-1}}{u_n^2}, \quad D_x = \frac{\partial}{\partial x}, \text{ etc.}$$

is satisfied by the two-way Wronskian

$$u_n = A_n = \left|D_x^{i-1}D_y^{j-1}(f)\right|_n,$$

where the function $f = f(x,y)$ *is arbitrary.*

PROOF. The equation can be expressed in the form

$$\begin{vmatrix} D_x D_y(A_n) & D_x(A_n) \\ D_y(A_n) & A_n \end{vmatrix} = A_{n+1}A_{n-1}. \tag{6.5.6}$$

The derivative of A_n with respect to x, as obtained by differentiating the rows, consists of the sum of n determinants, only one of which is nonzero. That determinant is a cofactor of A_{n+1}:

$$D_x(A_n) = -A^{(n+1)}_{n,n+1}.$$

Differentiating the columns with respect to y and then the rows with respect to x,

$$\begin{aligned} D_y(A_n) &= -A^{(n+1)}_{n+1,n}, \\ D_x D_y(A_n) &= A^{(n+1)}_{nn}. \end{aligned} \tag{6.5.7}$$

Denote the determinant in (6.5.6) by E. Then, applying the Jacobi identity (Section 3.6) to A_{n+1},

$$\begin{aligned} E &= \begin{vmatrix} A^{(n+1)}_{nn} & -A^{(n+1)}_{n,n+1} \\ -A^{(n+1)}_{n+1,n} & A^{(n+1)}_{n+1,n+1} \end{vmatrix} \\ &= A_{n+1} A^{(n+1)}_{n,n+1;n,n+1} \end{aligned}$$

which simplifies to the right side of (6.5.6).

It follows as a corollary that the equation

$$D^2(\log u_n) = \frac{u_{n+1}u_{n-1}}{u_n^2}, \quad D = \frac{d}{dx},$$

is satisfied by the Hankel–Wronskian

$$u_n = A_n = |D^{i+j-2}(f)|_n,$$

where the function $f = f(x)$ is arbitrary. □

Theorem 6.4. *The equation*

$$\frac{1}{\rho} D_\rho\big[\rho D_\rho(\log u_n)\big] = \frac{u_{n+1}u_{n-1}}{u_n^2}, \quad D_\rho = \frac{d}{d\rho},$$

is satisfied by the function

$$u_n = A_n = e^{-n(n-1)x} B_n, \tag{6.5.8}$$

where

$$B_n = \big|(\rho D_\rho)^{i+j-2} f(\rho)\big|_n, \quad f(\rho) \text{ arbitrary.}$$

PROOF. Put $\rho = e^x$. Then, $\rho D_\rho = D_x$ and the equation becomes

$$D_x^2(\log A_n) = \frac{\rho^2 A_{n+1}A_{n-1}}{A_n^2}. \tag{6.5.9}$$

Applying (6.5.8) to transform this equation from A_n to B_n,

$$\begin{aligned} D_x^2(\log B_n) &= D_x^2(\log A_n) \\ &= \frac{\rho^2 B_{n+1}B_{n-1}}{B_n^2} e^{-[(n+1)n+(n-1)(n-2)-2n(n-1)]x} \end{aligned}$$

$$= \frac{\rho^2 B_{n+1} B_{n-1} e^{-2x}}{B_n^2}$$
$$= \frac{B_{n+1} B_{n-1}}{B_n^2}.$$

This equation is identical in form to the equation in the corollary to Theorem 6.3. Hence,

$$B_n = \left| D_x^{i+j-2} g(x) \right|_n, \quad g(x) \text{ arbitrary},$$

which is equivalent to the stated result. □

Theorem 6.5. *The equation*

$$(D_x^2 + D_y^2) \log u_n = \frac{u_{n+1} u_{n-1}}{u_n^2}$$

is satisfied by the function

$$u_n = A_n = \left| D_z^{i-1} D_{\bar{z}}^{j-1}(f) \right|_n,$$

where $z = \frac{1}{2}(x + iy)$, $\bar{z}$ *is the complex conjugate of* z *and the function* $f = f(z, \bar{z})$ *is arbitrary.*

PROOF.

$$D_x^2(\log A_n) = \tfrac{1}{4}\left(D_z^2 + 2D_z D_{\bar{z}} + D_{\bar{z}}^2\right) \log A_n,$$
$$D_y^2(\log A_n) = -\tfrac{1}{4}\left(D_z^2 - 2D_z D_{\bar{z}} + D_{\bar{z}}^2\right) \log A_n.$$

Hence, the equation is transformed into

$$D_z D_{\bar{z}}(\log A_n) = \frac{A_{n+1} A_{n-1}}{A_n^2},$$

which is identical in form to the equation in Theorem 6.3. The present theorem follows. □

6.5.3 The Milne-Thomson Equation

Theorem 6.6. *The equation*

$$y_n'(y_{n+1} - y_{n-1}) = n + 1$$

is satisfied by the function defined separately for odd and even values of n *as follows:*

$$y_{2n-1} = \frac{B_{11}^{(n)}}{B_n} = B_n^{11},$$
$$y_{2n} = \frac{A_{n+1}}{A_{11}^{(n+1)}} = \frac{1}{A_{n+1}^{11}},$$

where A_n and B_n are Hankelians defined as

$$\begin{aligned} A_n &= |\phi_m|_n, \qquad 0 \le m \le 2n-2, \\ B_n &= |\phi_m|_n, \qquad 1 \le m \le 2n-1, \\ \phi'_m &= (m+1)\phi_{m+1}. \end{aligned}$$

PROOF.

$$\begin{aligned} B_{1n}^{(n)} &= (-1)^{n+1} A_{11}^{(n)}, \\ A_{1,n+1}^{(n+1)} &= (-1)^n B_n. \end{aligned} \tag{6.5.10}$$

It follows from Theorems 4.35 and 4.36 in Section 4.9.2 on derivatives of Turanians that

$$\begin{aligned} D(A_n) &= -(2n-1)A_{n+1,n}^{(n+1)}, \\ D(B_n) &= -2nB_{n,n+1}^{(n+1)}, \\ D(A_{11}^{(n)}) &= -(2n-1)A_{1,n+1;1n}^{(n+1)}, \\ D(B_{11}^{(n)}) &= -2nB_{1,n+1;1,n}^{(n+1)}. \end{aligned} \tag{6.5.11}$$

The algebraic identity in Theorem 4.29 in Section 4.8.5 on Turanians is satisfied by both A_n and B_n.

$$\begin{aligned} B_n^2 y'_{2n-1} &= B_n D(B_{11}^{(n)}) - B_{11}^{(n)} D(B_n) \\ &= 2n\big[B_{11}^{(n)} B_{n,n+1}^{(n+1)} - B_n B_{1,n+1;1n}^{(n+1)}\big] \\ &= 2nB_{1n}^{(n)} B_{1,n+1}^{(n+1)} \\ &= -2nA_{11}^{(n)} A_{11}^{(n+1)}. \end{aligned}$$

Applying the Jacobi identity,

$$\begin{aligned} A_{11}^{(n)} A_{11}^{(n+1)} (y_{2n} - y_{2n-2}) &= A_{n+1} A_{11}^{(n)} - A_n A_{11}^{(n+1)} \\ &= A_{n+1} A_{1,n+1;1,n+1}^{(n+1)} - A_{n+1,n+1}^{n+1} A_{11}^{(n+1)} \\ &= -\big[A_{1,n+1}^{(n+1)}\big]^2 \\ &= -B_n^2. \end{aligned}$$

Hence,

$$y'_{2n-1}(y_{2n} - y_{2n-2}) = 2n,$$

which proves the theorem when n is odd.

$$\begin{aligned} \big[A_{1,n+1}^{(n+1)}\big]^2 y'_{2n} &= A_{11}^{(n+1)} D(A_{n+1}) - A_{n+1} D(A_{11}^{(n+1)}) \\ &= (2n+1)\big[A_{n+1} A_{1,n+2;1,n+1}^{(n+2)} - A_{11}^{(n+1)} A_{n+2,n+1}^{(n+2)}\big] \\ &= -(2n+1) A_{1,n+1}^{(n+1)} A_{1,n+2}^{(n+2)}. \end{aligned}$$

Hence, referring to the first equation in (4.5.10),

$$\begin{aligned}
\left[B^{(n+1)}_{1,n+1}\right]^2 y'_{2n} &= (2n+1)B_n B_{n+1},\\
B_n B_{n+1}(y_{2n-1} - y_{2n+1}) &= B_n B^{(n+1)}_{11} - B_{n+1} B^{(n)}_{11}\\
&= B^{(n+1)}_{n+1,n+1} B^{(n+1)}_{11} - B_{n+1} B^{(n+1)}_{1,n+1;1,n+1}\\
&= \left[B^{(n+1)}_{1,n+1}\right]^2.
\end{aligned}$$

Hence,

$$y'_{2n}(y_{2n-1} - y_{2n+1}) = 2n+1,$$

which proves the theorem when n is even. □

6.6 The Matsukidaira–Satsuma Equations

6.6.1 A System With One Continuous and One Discrete Variable

Let $A^{(n)}(r)$ denote the Turanian–Wronskian of order n defined as follows:

$$A^{(n)}(r) = \left|f_{r+i+j-2}\right|_n, \tag{6.6.1}$$

where $f_s = f_s(x)$ and $f'_s = f_{s+1}$. Then,

$$\begin{aligned}
A^{(n)}_{11}(r) &= A^{(n-1)}(r+2),\\
A^{(n)}_{1n}(r) &= A^{(n-1)}(r+1).
\end{aligned}$$

Let

$$\tau_r = A^{(n)}(r). \tag{6.6.2}$$

Theorem 6.7.

$$\begin{vmatrix} \tau_{r+1} & \tau_r \\ \tau_r & \tau_{r-1} \end{vmatrix} \begin{vmatrix} \tau''_r & \tau'_r \\ \tau'_r & \tau_r \end{vmatrix} = \begin{vmatrix} \tau'_{r+1} & \tau_{r+1} \\ \tau'_r & \tau_r \end{vmatrix} \begin{vmatrix} \tau'_r & \tau_r \\ \tau'_{r-1} & \tau_{r-1} \end{vmatrix}$$

for all values of n and all differentiable functions $f_s(x)$.

Proof. Each of the functions

$$\tau_{r\pm1},\ \tau_{r+2},\ \tau'_r,\ \tau''_r,\ \tau'_{r\pm1}$$

can be expressed as a cofactor of $A^{(n+1)}$ with various parameters:

$$\begin{aligned}
\tau_r &= A^{(n+1)}_{n+1,n+1}(r),\\
\tau_{r+1} &= (-1)^n A^{(n+1)}_{1,n+1}(r)\\
&= (-1)^n A^{(n+1)}_{n+1,1}(r)\\
\tau_{r+2} &= A^{(n+1)}_{11}(r).
\end{aligned}$$

Hence applying the Jacobi identity (Section 3.6),

$$\begin{aligned}\begin{vmatrix} \tau_{r+2} & \tau_{r+1} \\ \tau_{r+1} & \tau_r \end{vmatrix} &= \begin{vmatrix} A_{11}^{(n+1)}(r) & (-1)^n A_{1,n+1}^{(n+1)}(r) \\ (-1)^n A_{n+1,1}^{(n+1)}(r) & A_{n+1,n+1}^{(n+1)}(r) \end{vmatrix} \\ &= A^{(n+1)}(r) A^{(n-1)}(r+2).\end{aligned}$$

Replacing r by $r-1$,

$$\begin{vmatrix} \tau_{r+1} & \tau_r \\ \tau_r & \tau_{r-1} \end{vmatrix} = A^{(n+1)}(r-1) A^{(n-1)}(r+1) \tag{6.6.3}$$

$$\begin{aligned}\tau_r' &= -A_{n,n+1}^{(n+1)}(r) \\ &= -A_{n+1,n}^{(n+1)}(r) \\ \tau_r'' &= A_{nn}^{(n+1)}(r).\end{aligned}$$

Hence,

$$\begin{aligned}\begin{vmatrix} \tau_r'' & \tau_r' \\ \tau_r' & \tau_r \end{vmatrix} &= \begin{vmatrix} A_{nn}^{(n+1)}(r) & A_{n,n+1}^{(n+1)}(r) \\ A_{n+1,n}^{(n+1)}(r) & A_{n+1,n+1}^{(n+1)}(r) \end{vmatrix} \\ &= A^{(n+1)}(r) A_{n,n+1;n,n+1}^{(n+1)}(r) \\ &= A^{(n+1)}(r) A^{(n-1)}(r). \end{aligned} \tag{6.6.4}$$

Similarly,

$$\begin{aligned}\tau_{r+1} &= -A_{1,n+1}^{(n+1)}(r) \\ &= -A_{n+1,1}^{(n+1)}(r), \\ \tau_{r+1}' &= (-1)^{n+1} A_{1n}^{(n+1)}(r),\end{aligned}$$

$$\begin{vmatrix} \tau_{r+1}' & \tau_{r+1} \\ \tau_r' & \tau_r \end{vmatrix} = A^{(n+1)}(r) A^{(n-1)}(r+1). \tag{6.6.5}$$

Replacing r by $r-1$,

$$\begin{vmatrix} \tau_r' & \tau_r \\ \tau_{r-1}' & \tau_{r-1} \end{vmatrix} = A^{(n+1)}(r-1) A^{(n-1)}(r). \tag{6.6.6}$$

Theorem 6.7 follows from (6.6.3)–(6.6.6). □

Theorem 6.8.

$$\begin{vmatrix} \tau_r & \tau_{r+1} & \tau_{r+1}' \\ \tau_{r-1} & \tau_r & \tau_r' \\ \tau_{r-1}' & \tau_r' & \tau_r'' \end{vmatrix} = 0.$$

PROOF. Denote the determinant by F. Then, Theorem 6.7 can be expressed in the form

$$F_{33}\ F_{11} = F_{31}\ F_{13}.$$

Applying the Jacobi identity,

$$F\ F_{13,13} = \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix}$$
$$= 0.$$

But $F_{13,13} \neq 0$. The theorem follows. □

Theorem 6.9. *The Matsukidaira–Satsuma equations with one continuous independent variable, one discrete independent variable, and two dependent variables, namely*

a. $q'_r = q_r(u_{r+1} - u_r)$,

b. $\dfrac{u'_r}{u_r - u_{r-1}} = \dfrac{q'_r}{q_r - q_{r-1}}$,

where q_r and u_r are functions of x, are satisfied by the functions

$$q_r = \frac{\tau_{r+1}}{\tau_r},$$
$$u_r = \frac{\tau'_r}{\tau_r}$$

for all values of n and all differentiable functions $f_s(x)$.

PROOF.

$$q'_r = -\frac{F_{31}}{\tau_r^2},$$
$$q_r - q_{r-1} = -\frac{F_{33}}{\tau_{r-1}\tau_r},$$
$$u'_r = \frac{F_{11}}{\tau_r^2},$$
$$u_r - u_{r-1} = \frac{F_{13}}{\tau_{r-1}\tau_r},$$
$$u_{r+1} - u_r = -\frac{F_{31}}{\tau_r\tau_{r+1}}.$$

Hence,

$$\frac{q'_r}{u_{r+1} - u_r} = \frac{\tau_{r+1}}{\tau_r} = q_r,$$

which proves (a) and

$$\frac{u'_r(q_r - q_{r-1})}{q'_r(u_r - u_{r-1})} = \frac{F_{11}\ F_{33}}{F_{31}\ F_{13}}$$
$$= 1,$$

which proves (b). □

6.6.2 A System With Two Continuous and Two Discrete Variables

Let $A^{(n)}(r,s)$ denote the two-way Wronskian of order n defined as follows:

$$A^{(n)}(r,s) = \left|f_{r+i-1,s+j-1}\right|_n, \tag{6.6.7}$$

where $f_{rs} = f_{rs}(x,y)$, $(f_{rs})_x = f_{r,s+1}$, and $(f_{rs})_y = f_{r+1,s}$.

Let

$$\tau_{rs} = A^{(n)}(r,s). \tag{6.6.8}$$

Theorem 6.10.

$$\begin{aligned}\begin{vmatrix}\tau_{r+1,s} & \tau_{r+1,s-1}\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix}&\begin{vmatrix}(\tau_{rs})_{xy} & (\tau_{rs})_y\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix}\\ &= \begin{vmatrix}(\tau_{rs})_y & (\tau_{r,s-1})_y\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix}\begin{vmatrix}(\tau_{r+1,s})_x & \tau_{r+1,s}\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix}\end{aligned}$$

for all values of n and all differentiable functions $f_{rs}(x,y)$.

PROOF.

$$\begin{aligned}\tau_{rs} &= A^{(n+1)}_{n+1,n+1}(r,s),\\ \tau_{r+1,s} &= -A^{(n+1)}_{1,n+1}(r,s),\\ \tau_{r,s+1} &= -A^{(n+1)}_{n+1,1}(r,s),\\ \tau_{r+1,s+1} &= A^{(n+1)}_{11}(r,s).\end{aligned}$$

Hence, applying the Jacobi identity,

$$\begin{aligned}\begin{vmatrix}\tau_{r+1,s+1} & \tau_{r+1,s+1}\\ \tau_{r,s+1} & \tau_{rs}\end{vmatrix} &= A^{(n+1)}(r,s)A^{(n+1)}_{1,n+1;1,n+1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n-1)}(r+1,s+1).\end{aligned}$$

Replacing s by $s-1$,

$$\begin{vmatrix}\tau_{r+1,s} & \tau_{r+1,s-1}\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix} = A^{(n+1)}(r,s-1)A^{(n-1)}(r+1,s), \tag{6.6.9}$$

$$\begin{aligned}(\tau_{rs})_x &= -A^{(n+1)}_{n+1,n}(r,s),\\ (\tau_{rs})_y &= -A^{(n+1)}_{n,n+1}(r,s),\\ (\tau_{rs})_{xy} &= A^{(n+1)}_{nn}(r,s).\end{aligned}$$

Hence, applying the Jacobi identity,

$$\begin{aligned}\begin{vmatrix}(\tau_{rs})_{xy} & (\tau_{rs})_y\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix} &= A^{(n+1)}(r,s)A^{(n+1)}_{n,n+1;n,n+1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n-1)}(r,s) \qquad (6.6.10)\\ (\tau_{r,s+1})_y &= -A^{(n+1)}_{n1}(r,s).\end{aligned}$$

Hence,

$$\begin{aligned}\begin{vmatrix}(\tau_{r,s+1})_y & (\tau_{rs})_y\\ \tau_{r,s+1} & \tau_{rs}\end{vmatrix} &= A^{(n+1)}(r,s)A^{(n+1)}_{n,n+1;1,n+1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n)}_{n1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n-1)}(r,s+1).\end{aligned}$$

Replacing s by $s-1$,

$$\begin{vmatrix}(\tau_{rs})_y & (\tau_{r,s-1})_y\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix} = A^{(n+1)}(r,s-1)A^{(n-1)}(r,s), \tag{6.6.11}$$

$$(\tau_{r+1,s})_x = A^{(n+1)}_{1n}(r,s).$$

Hence,

$$\begin{aligned}\begin{vmatrix}(\tau_{r+1,s})_x & \tau_{r+1,s}\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix} &= A^{(n+1)}(r,s)A^{(n+1)}_{1,n+1;n,n+1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n-1)}(r+1,s). \qquad (6.6.12)\end{aligned}$$

Theorem 6.10 follows from (6.6.9)–(6.6.12). □

Theorem 6.11.

$$\begin{vmatrix}\tau_{r+1,s-1} & \tau_{r,s-1} & (\tau_{r,s-1})_y\\ \tau_{r+1,s} & \tau_{rs} & (\tau_{rs})_y\\ (\tau_{r+1,s})_x & (\tau_{rs})_x & (\tau_{rs})_{xy}\end{vmatrix} = 0.$$

Proof. Denote the determinant by G. Then, Theorem 6.10 can be expressed in the form

$$G_{33}\ G_{11} = G_{31}\ G_{13}. \tag{6.6.13}$$

Applying the Jacobi identity,

$$\begin{aligned}G\ G_{13,13} &= \begin{vmatrix}G_{11} & G_{13}\\ G_{31} & G_{33}\end{vmatrix}\\ &= 0.\end{aligned}$$

But $G_{13,13} \neq 0$. The theorem follows. □

Theorem 6.12. *The Matsukidaira–Satsuma equations with two continuous independent variables, two discrete independent variables, and three dependent variables, namely*

a. $(q_{rs})_y = q_{rs}(u_{r+1,s} - u_{rs})$,

b. $\dfrac{(u_{rs})_x}{u_{rs} - u_{r,s-1}} = \dfrac{(v_{r+1,s} - v_{rs})q_{rs}}{q_{rs} - q_{r,s-1}}$,

where q_{rs}, u_{rs}, and v_{rs} are functions of x and y, are satisfied by the functions

$$q_{rs} = \frac{\tau_{r+1,s}}{\tau_{rs}},$$

$$u_{rs} = \frac{(\tau_{rs})_y}{\tau_{rs}},$$
$$v_{rs} = \frac{(\tau_{rs})_x}{\tau_{rs}},$$

for all values of n *and all differentiable functions* $f_{rs}(x, y)$*.*

PROOF.

$$\begin{aligned}
(q_{rs})_y &= \frac{1}{\tau_{rs}^2} \begin{vmatrix} (\tau_{r+1,s})_y & (\tau_{rs})_y \\ \tau_{r+1,s} & \tau_{rs} \end{vmatrix} \\
&= \frac{\tau_{r+1,s}}{\tau_{rs}} \left[\frac{(\tau_{r+1,s})_y}{\tau_{r+1,s}} - \frac{(\tau_{rs})_y}{\tau_{rs}} \right] \\
&= q_{rs}(u_{r+1,s} - u_{rs}),
\end{aligned}$$

which proves (a).

$$\begin{aligned}
(u_{rs})_x &= \frac{G_{11}}{\tau_{rs}^2}, \\
v_{r+1,s} - v_{rs} &= -\frac{G_{13}}{\tau_{r+1,s}\tau_{rs}}, \\
u_{rs} - u_{r,s-1} &= \frac{G_{31}}{\tau_{rs}\tau_{r,s-1}}, \\
q_{rs} - q_{r,s-1} &= -\frac{G_{33}}{\tau_{rs}\tau_{r,s-1}}.
\end{aligned}$$

Hence, referring to (6.2.13),

$$\begin{aligned}
\frac{(q_{rs} - q_{r,s-1})(u_{rs})_x}{q_{rs}(u_{rs} - u_{r,s-1})(v_{r+1,s} - v_{rs})} &= \frac{G_{11}\ G_{33}}{G_{31}\ G_{13}} \\
&= 1,
\end{aligned}$$

which proves (b). □

6.7 The Korteweg–de Vries Equation

6.7.1 Introduction

The KdV equation is

$$u_t + 6uu_x + u_{xxx} = 0. \tag{6.7.1}$$

The substitution $u = 2v_x$ transforms it into

$$v_t + 6v_x^2 + v_{xxx} = 0. \tag{6.7.2}$$

Theorem 6.13. *The KdV equation in the form (6.7.2) is satisfied by the function*

$$v = D_x(\log A),$$

where

$$\begin{aligned}
A &= |a_{rs}|_n,\\
a_{rs} &= \delta_{rs}e_r + \frac{2}{b_r+b_s} = a_{sr},\\
e_r &= \exp(-b_r x + b_r^3 t + \varepsilon_r).
\end{aligned}$$

The ε_r are arbitrary constants and the b_r are constants such that the $b_r + b_s \neq 0$ but are otherwise arbitrary.

Two independent proofs of this theorem are given in Sections 6.7.2 and 6.7.3. The method of Section 6.7.2 applies nonlinear differential recurrence relations in a function of the cofactors of A. The method of Section 6.7.3 involves partial derivatives with respect to the exponential functions which appear in the elements of A.

It is shown in Section 6.7.4 that A is a simple multiple of a Wronskian and Section 6.7.5 consists of an independent proof of the Wronskian solution.

6.7.2 The First Form of Solution

First Proof of Theorem 6.1.3. The proof begins by extracting a wealth of information about the cofactors of A by applying the double-sum relations (A)–(D) in Section 3.4 in different ways. Apply (A) and (B) with f' interpreted first as f_x and then as f_t. Apply (C) and (D) first with $f_r = g_r = b_r$, then with $f_r = g_r = b_r^3$. Later, apply (D) with $f_r = -g_r = b_r^2$.

Appling (A) and (B),

$$\begin{aligned}
v = D_x(\log A) &= -\sum_r\sum_s \delta_{rs}b_r e_r A^{rs}\\
&= -\sum_r b_r e_r A^{rr}, \qquad (6.7.3)
\end{aligned}$$

$$D_x(A^{ij}) = \sum_r b_r e_r A^{ir}A^{rj}. \qquad (6.7.4)$$

Applying (C) and (D) with $f_r = g_r = b_r$,

$$\sum_r\sum_s \bigl[\delta_{rs}(b_r+b_s)e_r + 2\bigr]A^{rs} = 2\sum_r b_r,$$

which simplifies to

$$\sum_r b_r e_r A^{rr} + \sum_r\sum_s A^{rs} = \sum_r b_r, \qquad (6.7.5)$$

$$\sum_r b_r e_r A^{ir}A^{rj} + \sum_r\sum_s A^{is}A^{rj} = \tfrac{1}{2}(b_i+b_j)A^{ij}. \qquad (6.7.6)$$

Eliminating the sum common to (6.7.3) and (6.7.5) and the sum common to (6.7.4) and (6.7.6),

$$v = D_x(\log A) = \sum_r \sum_s A^{rs} - \sum_r b_r, \tag{6.7.7}$$

$$D_x(A^{ij}) = \tfrac{1}{2}(b_i + b_j)A^{ij} - \sum_r \sum_s A^{is}A^{rj}. \tag{6.7.8}$$

Returning to (A) and (B),

$$D_t(\log A) = \sum_r b_r^3 e_r A^{rr}, \tag{6.7.9}$$

$$D_t(A^{ij}) = -\sum_r b_r^3 e_r A^{ir}A^{rj}. \tag{6.7.10}$$

Now return to (C) and (D) with $f_r = g_r = b_r^3$.

$$\sum_r b_r^3 e_r A^{rr} + \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2)A^{rs} = \sum_r b_r^3, \tag{6.7.11}$$

$$\begin{aligned}\sum_r b_r^3 e_r A^{ir}A^{rj} &+ \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2)A^{is}A^{rj}\\ &= \tfrac{1}{2}(b_i^3 + b_j^3)A^{ij}.\end{aligned} \tag{6.7.12}$$

Eliminating the sum common to (6.7.9) and (6.7.11) and the sum common to (6.7.10) and (6.7.12),

$$D_t(\log A) = \sum_r b_r^3 - \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2)A^{rs}, \tag{6.7.13}$$

$$D_t(A^{ij}) = \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2)A^{is}A^{rj} - \tfrac{1}{2}(b_i^3 + b_j^3)A^{ij}. \tag{6.7.14}$$

The derivatives v_x and v_t can be evaluated in a convenient form with the aid of two functions ψ_{is} and ϕ_{ij} which are defined as follows:

$$\psi_{is} = \sum_r b_r^i A^{rs}, \tag{6.7.15}$$

$$\begin{aligned}\phi_{ij} &= \sum_s b_s^j \psi_{is},\\ &= \sum_r \sum_s b_r^i b_s^j A^{rs}\\ &= \phi_{ji}.\end{aligned} \tag{6.7.16}$$

They are definitions of ψ_{is} and ϕ_{ij}. □

Lemma. *The function ϕ_{ij} satisfies the three nonlinear recurrence relations*:

a. $\phi_{i0}\phi_{j1} - \phi_{j0}\phi_{i1} = \frac{1}{2}(\phi_{i+2,j} - \phi_{i,j+2})$,
b. $D_x(\phi_{ij}) = \frac{1}{2}(\phi_{i+1,j} + \phi_{i,j+1}) - \phi_{i0}\phi_{j0}$,

c. $D_t(\phi_{ij}) = \phi_{i0}\phi_{j2} - \phi_{i1}\phi_{j1} + \phi_{i2}\phi_{j0} - \frac{1}{2}(\phi_{i+3,j} + \phi_{i,j+3})$.

PROOF. Put $f_r = -g_r = b_r^2$ in identity (D).

$$\begin{aligned}(b_i^2 - b_j^2)A^{ij} &= \sum_r \sum_s \big[\delta_{rs}(b_r^2 - b_s^2)e_r + 2(b_r - b_s)\big]A^{is}A^{rj}\\ &= 0 + 2\sum_r \sum_s (b_r - b_s)A^{is}A^{rj}\\ &= 2\sum_s A^{is}\sum_r b_r A^{rj} - 2\sum_r A^{rj}\sum_s b_s A^{is}\\ &= 2(\psi_{0i}\psi_{1j} - \psi_{0j}\psi_{1i}).\end{aligned}$$

It follows that if

$$F_{ij} = 2\psi_{0i}\psi_{1j} - b_i^2 A^{ij},$$

then

$$F_{ji} = F_{ij}.$$

Furthermore, if G_{ij} is any function with the property

$$G_{ji} = -G_{ij},$$

then

$$\sum_i \sum_j G_{ij}F_{ij} = 0. \tag{6.7.17}$$

The proof is trivial and is obtained by interchanging the dummy suffixes.

The proof of (a) can now be obtained by expanding the quadruple series

$$S = \sum_{p,q,r,s} (b_p^i b_r^j - b_p^j b_r^i) b_s A^{pq} A^{rs}$$

in two different ways and equating the results.

$$\begin{aligned}S &= \sum_{p,q} b_p^i A^{pq} \sum_{r,s} b_r^j b_s A^{rs} - \sum_{p,q} b_p^j A^{pq} \sum_{r,s} b_r^i b_s A^{rs}\\ &= \phi_{i0}\phi_{j1} - \phi_{j0}\phi_{i1},\end{aligned}$$

which is identical to the left side of (a). Also, referring to (6.7.17) with $i, j \to p, r$,

$$\begin{aligned}S &= \sum_{p,r}(b_p^i b_r^j - b_p^j b_r^i)\sum_q A^{pq}\sum_s b_s A^{rs}\\ &= \sum_{p,r}(b_p^i b_r^j - b_p^j b_r^i)\psi_{0p}\psi_{1r}\\ &= \frac{1}{2}\sum_{p,r}(b_p^i b_r^j - b_p^j b_r^i)(F_{pr} + b_p^2 A^{pr})\end{aligned}$$

$$= 0 + \frac{1}{2}\sum_{p,r}(b_p^{i+2}b_r^j - b_p^{j+2}b_r^i)A^{pr}$$
$$= \frac{1}{2}(\phi_{i+2,j} - \phi_{i,j+2}),$$

which is identical with the right side of (a). This completes the proof of (a).

Referring to (6.7.8) with $r, s \to p, q$ and $i, j \to r, s$,

$$\begin{aligned} D_x(\phi_{ij}) &= \sum_r \sum_s b_r^i b_s^j D_x(A^{rs}) \\ &= \sum_r \sum_s b_r^i b_s^j \left[\tfrac{1}{2}(b_r + b_s)A^{rs} - \sum_p \sum_q A^{rq}A^{ps} \right] \\ &= \frac{1}{2}\sum_r \sum_s b_r^i b_s^j (b_r + b_s)A^{rs} - \sum_{q,r} b_r^i A^{rq} \sum_{p,s} b_s^j A^{ps} \\ &= \frac{1}{2}(\phi_{i+1,j} + \phi_{i,j+1}) - \phi_{i0}\phi_{j0}, \end{aligned}$$

which proves (b). Part (c) is proved in a similar manner. □

Particular cases of (a)–(c) are

$$\phi_{00}\phi_{11} - \phi_{10}^2 = \tfrac{1}{2}(\phi_{21} - \phi_{03}), \tag{6.7.18}$$
$$\begin{aligned} D_x(\phi_{00}) &= \phi_{10} - \phi_{00}^2, \\ D_t(\phi_{00}) &= 2\phi_{00}\phi_{20} - \phi_{10}^2 - \phi_{30}. \end{aligned} \tag{6.7.19}$$

The preparations for finding the derivatives of v are now complete. The formula for v given by (6.7.7) can be written

$$v = \phi_{00} - \text{constant}.$$

Differentiating with the aid of parts (b) and (c) of the lemma,

$$\begin{aligned} v_x &= \phi_{10} - \phi_{00}^2, \\ v_{xx} &= \tfrac{1}{2}(\phi_{20} + \phi_{11} - 6\phi_{00}\phi_{10} + 4\phi_{00}^3), \\ v_{xxx} &= \tfrac{1}{4}(\phi_{30} + 3\phi_{21} - 8\phi_{00}\phi_{20} - 14\phi_{10}^2, \\ &\qquad +48\phi_{00}^2\phi_{10} - 6\phi_{00}\phi_{11} - 24\phi_{00}^4) \\ v_t &= 2\phi_{00}\phi_{20} - \phi_{10}^2 - \phi_{30}. \end{aligned} \tag{6.7.20}$$

Hence, referring to (6.7.18),

$$\begin{aligned} 4(v_t + 6v_x^2 + v_{xxx}) &= 3\big[(\phi_{21} - \phi_{30}) - 2(\phi_{00}\phi_{11} - \phi_{10}^2)\big] \\ &= 0, \end{aligned}$$

which completes the verification of the first form of solution of the KdV equation by means of recurrence relations.

6.7.3 The First Form of Solution, Second Proof

Second Proof of Theorem 6.13. It can be seen from the definition of A that the variables x and t occur only in the exponential functions e_r, $1 \le r \le n$. It is therefore possible to express the derivatives A_x, v_x, A_t, and v_t in terms of partial derivatives of A and v with respect to the e_r.

The basic formulas are as follows.

If

$$y = y(e_1, e_2, \dots, e_n),$$

then

$$\begin{aligned} y_x &= \sum_r \frac{\partial y}{\partial e_r}\frac{\partial e_r}{\partial x} \\ &= -\sum_r b_r e_r \frac{\partial y}{\partial e_r}, \qquad (6.7.21) \\ y_{xx} &= -\sum_s b_s e_s \frac{\partial y_x}{\partial e_s} \\ &= \sum_s b_s e_s \sum_r b_r \frac{\partial}{\partial e_s}\left(e_r \frac{\partial y}{\partial e_r}\right) \\ &= \sum_{r,s} b_r b_s e_s \left[\delta_{rs}\frac{\partial y}{\partial e_r} + e_r \frac{\partial^2 y}{\partial e_r \partial e_s}\right] \\ &= \sum_r b_r^2 e_r \frac{\partial y}{\partial e_r} + \sum_{r,s} b_r b_s e_r e_s \frac{\partial^2 y}{\partial e_r \partial e_s}. \qquad (6.7.22) \end{aligned}$$

Further derivatives of this nature are not required. The double-sum relations (A)–(D) in Section 3.4 are applied again but this time f' is interpreted as a partial derivative with respect to an e_r.

The basic partial derivatives are as follows:

$$\begin{aligned} \frac{\partial e_r}{\partial e_s} &= \delta_{rs}, \qquad (6.7.23) \\ \frac{\partial a_{rs}}{\partial e_m} &= \delta_{rs}\frac{\partial e_r}{\partial e_m} \\ &= \delta_{rs}\delta_{rm}. \qquad (6.7.24) \end{aligned}$$

Hence, applying (A) and (B),

$$\begin{aligned} \frac{\partial}{\partial e_m}(\log A) &= \sum_{r,s} \frac{\partial a_{rs}}{\partial e_m} A^{rs} \\ &= \sum_{r,s} \delta_{rs}\delta_{rm} A^{rs} \\ &= A^{mm} \qquad (6.7.25) \end{aligned}$$

$$\frac{\partial}{\partial e_m}(A^{ij}) = -A^{im}A^{mj}. \tag{6.7.26}$$

Let

$$\psi_p = \sum_s A^{sp}. \tag{6.7.27}$$

Then, (6.7.26) can be written

$$\sum_r b_r e_r A^{ir}A^{rj} = \tfrac{1}{2}(b_i + b_j)A^{ij} - \psi_i\psi_j. \tag{6.7.28}$$

From (6.7.27) and (6.7.26),

$$\begin{aligned}\frac{\partial\psi_p}{\partial e_q} &= -A^{pq}\sum_s A^{sq} \\ &= -\psi_q A^{pq}. \end{aligned} \tag{6.7.29}$$

Let

$$\theta_p = \psi_p^2. \tag{6.7.30}$$

Then,

$$\frac{\partial\theta_p}{\partial e_q} = -2\psi_p\psi_q A^{pq} \tag{6.7.31}$$

$$= \frac{\partial\theta_q}{\partial e_p}, \tag{6.7.32}$$

$$\begin{aligned}\frac{\partial^2\theta_r}{\partial e_p\partial e_q} &= -2\frac{\partial}{\partial e_p}(\psi_q\psi_r A^{qr}) \\ &= 2(\psi_p\psi_q A^{pr}A^{qr} + \psi_q\psi_r A^{qp}A^{rp} + \psi_r\psi_p A^{rq}A^{pq}),\end{aligned}$$

which is invariant under a permutation of p, q, and r. Hence, if G_{pqr} is any function with the same property,

$$\sum_{p,q,r} G_{pqr}\frac{\partial^2\theta_r}{\partial e_p\partial e_q} = 6\sum_{p,q,r} G_{pqr}\psi_p\psi_q A^{pr}A^{qr}. \tag{6.7.33}$$

The above relations facilitate the evaluation of the derivatives of v which, from (6.7.7) and (6.7.27) can be written

$$v = \sum_m (\psi_m - b_m).$$

Referring to (6.7.29),

$$\begin{aligned}\frac{\partial v}{\partial e_r} &= -\psi_r\sum_m A^{mr} \\ &= -\psi_r^2 \\ &= -\theta_r.\end{aligned} \tag{6.7.34}$$

Hence,

$$\begin{aligned} v_x &= -\sum_r b_r e_r \frac{\partial v}{\partial e_r} \\ &= \sum_r b_r e_r \theta_r. \end{aligned} \tag{6.7.35}$$

Similarly,

$$v_t = -\sum_r b_r^3 e_r \theta_r. \tag{6.7.36}$$

From (6.7.35) and (6.7.23),

$$\begin{aligned} \frac{\partial v_x}{\partial e_q} &= \sum_r b_r \left(\delta_{qr}\theta_r + e_r \frac{\partial \theta_r}{\partial e_q} \right) \\ &= b_q\theta_q + \sum_r b_r e_r \frac{\partial \theta_r}{\partial e_q}. \end{aligned} \tag{6.7.37}$$

Referring to (6.7.32),

$$\begin{aligned} \frac{\partial^2 v_x}{\partial e_p \partial e_q} &= b_q \frac{\partial \theta_q}{\partial e_p} + \sum_r b_r \left(\delta_{pr} \frac{\partial \theta_r}{\partial e_q} + e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q} \right) \\ &= (b_p + b_q) \frac{\partial \theta_p}{\partial e_q} + \sum_r b_r e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q}. \end{aligned} \tag{6.7.38}$$

To obtain a formula for v_{xxx}, put $y = v_x$ in (6.7.22), apply (6.7.37) with $q \to p$ and $r \to q$, and then apply (6.7.38):

$$\begin{aligned} v_{xxx} &= \sum_p b_p^2 e_p \frac{\partial v_x}{\partial e_p} + \sum_{p,q} b_p b_q e_p e_q \frac{\partial^2 v_x}{\partial e_p \partial e_q} \\ &= \sum_p b_p^2 e_p \left[b_p \theta_p + \sum_q b_q e_q \frac{\partial \theta_q}{\partial e_p} \right] \\ &\quad + \sum_{p,q} b_p b_q e_p e_q \left[(b_p + b_q) \frac{\partial \theta_p}{\partial e_q} + \sum_r b_r e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q} \right] \\ &= Q + R + S + T \end{aligned} \tag{6.7.39}$$

where, from (6.7.36), (6.7.32), and (6.7.31),

$$\begin{aligned} Q &= \sum_p b_p^3 e_p \theta_p \\ &= -v_t \\ R &= \sum_{p,q} b_p^2 b_q e_p e_q \frac{\partial \theta_p}{\partial e_q} \end{aligned}$$

$$
\begin{aligned}
&= -2\sum_{p,q} b_p^2 b_q e_p e_q \psi_p \psi_q A^{pq}, \\
S &= 2R. \qquad (6.7.40)
\end{aligned}
$$

Referring to (6.7.33), (6.7.28), and (6.7.35),

$$
\begin{aligned}
T &= \sum_{p,q,r} b_p b_q b_r e_p e_q e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q} \\
&= 6\sum_{p,q} b_p b_q e_p e_q \psi_p \psi_q \sum_r b_r e_r A^{pr} A^{qr} \\
&= 6\sum_{p,q} b_p b_q e_p e_q \psi_p \psi_q \left[\tfrac{1}{2}(b_p + b_q)A^{pq} - \psi_p \psi_q\right] \\
&= 6\sum_{p,q} b_p^2 b_q e_p e_q \psi_p \psi_q A^{pq} - 6\sum_p b_p e_p \theta_p \sum_q b_q e_q \theta_q \\
&= -(3R + 6v_x^2).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
v_{xxx} &= -v_t + R + 2R - (3R + 6v_x^2) \\
&= -(v_t + 6v_x^2),
\end{aligned}
$$

which completes the verification of the first form of solution of the KdV equation by means of partial derivatives with respect to the exponential functions.

6.7.4 The Wronskian Solution

Theorem 6.14. *The determinant A in Theorem 6.7.1 can be expressed in the form*

$$
A = k_n (e_1\ e_2 \cdots e_n)^{1/2} W,
$$

where k_n is independent of x and t, and W is the Wronskian defined as follows:

$$
W = \left| D_x^{j-1}(\phi_i) \right|_n, \qquad (6.7.41)
$$

where

$$
\begin{aligned}
\phi_i &= \lambda_i e_i^{1/2} + \mu_i e_i^{-1/2}, && (6.7.42) \\
e_i &= \exp(-b_i x + b_i^3 t + \varepsilon_i), && (6.7.43) \\
\lambda_i &= \frac{1}{2}\prod_{p=1}^{n} (b_p + b_i), \\
\mu_i &= \prod_{\substack{p=1 \\ p \neq i}}^{n} (b_p - b_i). && (6.7.44)
\end{aligned}
$$

PROOF.

$$D_x^j(\phi_i) = \left(\tfrac{1}{2}b_i\right)^j e_i^{-1/2}[(-1)^j \lambda_i e_i + \mu_i]$$

so that every element in row i of W contains the factor $e_i^{-1/2}$. Removing all these factors from the determinant,

$$(e_1\ e_2 \ldots e_n)^{1/2} W = \begin{vmatrix} \lambda_1 e_1 + \mu_1 & \frac{1}{2}b_1(-\lambda_1 e_1 + \mu_1) & (\frac{1}{2}b_1)^2(\lambda_1 e_1 + \mu_1) & \cdots \\ \lambda_2 e_2 + \mu_2 & \frac{1}{2}b_2(-\lambda_2 e_2 + \mu_2) & (\frac{1}{2}b_2)^2(\lambda_2 e_2 + \mu_2) & \cdots \\ \lambda_3 e_3 + \mu_3 & \frac{1}{2}b_3(-\lambda_3 e_3 + \mu_3) & (\frac{1}{2}b_3)^2(\lambda_3 e_3 + \mu_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n \tag{6.7.45}$$

Now remove the fractions from the elements of the determinant by multiplying column j by 2^{j-1}, $1 \le j \le n$, and compensate for the change in the value of the determinant by multiplying the left side by

$$2^{1+2+3\cdots+(n-1)} = 2^{n(n-1)/2}.$$

The result is

$$2^{n(n-1)/2}(e_1\ e_2 \cdots e_n)^{1/2} W = |\alpha_{ij} e_i + \beta_{ij}|_n, \tag{6.7.46}$$

where

$$\begin{aligned} \alpha_{ij} &= (-b_i)^{j-1}\lambda_i, \\ \beta_{ij} &= b_i^{j-1}\mu_i. \end{aligned} \tag{6.7.47}$$

The determinants $|\alpha_{ij}|_n$, $|\beta_{ij}|_n$ are both Vandermondians. Denote them by U_n and V_n, respectively, and use the notation of Section 4.1.2:

$$\begin{aligned} U_n = |\alpha_{ij}|_n &= (\lambda_1\ \lambda_2 \cdots \lambda_n)\left|(-b_i)^{j-1}\right|_n, \\ &= (\lambda_1\ \lambda_2 \cdots \lambda_n)[X_n]_{x_i=-b_i}, \\ V_n &= |\beta_{ij}|_n. \end{aligned} \tag{6.7.48}$$

The determinant on the right-hand side of (6.7.46) is identical in form with the determinant $|a_{ij}x_i + b_{ij}|_n$ which appears in Section 3.5.3. Hence, applying the theorem given there with appropriate changes in the symbols,

$$|\alpha_{ij} e_i + \beta_{ij}|_n = U_n |E_{ij}|_n,$$

where

$$E_{ij} = \delta_{ij} e_i + \frac{K_{ij}^{(n)}}{U_n} \tag{6.7.49}$$

and where $K_{ij}^{(n)}$ is the hybrid determinant obtained by replacing row i of U_n by row j of V_n. Removing common factors from the rows of the determinant,

$$K_{ij}^{(n)} = (\lambda_1\ \lambda_2 \cdots \lambda_n)\frac{\mu_j}{\lambda_i}\left[H_{ij}^{(n)}\right]_{y_i=-x_i=b_i}.$$

Hence, from (6.7.48),

$$\begin{aligned}\frac{K_{ij}^{(n)}}{U_n} &= \frac{\mu_j}{\lambda_i}\left[\frac{H_{ij}^{(n)}}{X_n}\right]_{y_i=-x_i=b_i}\\ &= \frac{\mu_j}{\lambda_i}\,\frac{\prod_{p=1}^{n}(b_p+b_j)}{(b_i+b_j)\prod_{\substack{p=1\\p\neq i}}^{n}(b_p-b_j)}\\ &= \frac{2\lambda_j\mu_j}{(b_i+b_j)\lambda_i\mu_i}. \qquad (6.7.50)\end{aligned}$$

Hence,

$$|E_{ij}|_n = \left|\delta_{ij}e_i + \frac{2\lambda_j\mu_j}{(b_i+b_j)\lambda_i\mu_i}\right|_n. \qquad (6.7.51)$$

Multiply row i of this determinant by $\lambda_i\mu_i$, $1 \le i \le n$, and divide column j by $\lambda_j\mu_j$, $1 \le j \le n$. These operations do not affect the diagonal elements or the value of the determinant but now

$$\begin{aligned}|E_{ij}|_n &= \left|\delta_{ij}e_i + \frac{2}{b_i+b_j}\right|_n\\ &= A. \qquad (6.7.52)\end{aligned}$$

It follows from (6.7.46) and (6.7.49) that

$$2^{n(n-1)/2}(e_1\ e_2\cdots e_n)^{1/2}W = U_nA, \qquad (6.7.53)$$

which completes the proof of the theorem since U_n is independent of x and t. □

It follows that

$$\log A = \text{constant} + \frac{1}{2}\sum_i(-b_ix + b_i^3t) + \log W. \qquad (6.7.54)$$

Hence,

$$u = 2D_x^2(\log A) = 2D_x^2(\log W) \qquad (6.7.55)$$

so that solutions containing A and W have equally valid claims to be determinantal solutions of the KdV equation.

6.7.5 Direct Verification of the Wronskian Solution

The substitution

$$u = 2D_x^2(\log w)$$

into the KdV equation yields

$$u_t + 6uu_x + u_{xxx} = 2D_x\left(\frac{F}{w^2}\right), \tag{6.7.56}$$

where

$$F = ww_{xt} - w_x w_t + 3w_{xx}^2 - 4w_x w_{xxx} + ww_{xxxx}.$$

Hence, the KdV equation will be satisfied if

$$F = 0. \tag{6.7.57}$$

Theorem 6.15. *The KdV equation in the form (6.7.56) and (6.7.57) is satisfied by the Wronskian w defined as follows:*

$$w = \left|D_x^{j-1}(\psi_i)\right|_n,$$

where

$$\begin{aligned}
\psi_i &= \exp\left(\tfrac{1}{4}b_i^2 z\right)\phi_i,\\
\phi_i &= p_i e_i^{1/2} + q_i e_i^{-1/2},\\
e_i &= \exp(-b_i x + b_i^3 t).
\end{aligned}$$

z is independent of x and t but is otherwise arbitrary. b_i, p_i, and q_i are constants.

When $z = 0$, $p_i = \lambda_i$, and $q_i = \mu_i$, then $\psi_i = \phi_i$ and $w = W$ so that this theorem differs little from Theorem 6.14 but the proof of Theorem 6.15 which follows is direct and independent of the proofs of Theorems 6.13 and 6.14. It uses the column vector notation and applies the Jacobi identity.

PROOF. Since

$$(D_t + 4D_x^3)\phi_i = 0,$$

it follows that

$$(D_t + 4D_x^3)\psi_i = 0. \tag{6.7.58}$$

Also

$$(D_z - D_x^2)\psi_i = 0. \tag{6.7.59}$$

Since each row of w contains the factor $\exp\left(\frac{1}{4}b_i^2 z\right)$,

$$w = e^{Bz}W,$$

where

$$W = \left|D_x^{j-1}(\phi_i)\right|_n$$

and is independent of z and

$$B = \tfrac{1}{4}\sum_i b_i^2.$$

Hence, $ww_{zz} - w_z^2 = 0$,

$$\begin{aligned} F &= ww_{xt} - w_x w_t + 3w_{xx}^2 - 4w_x w_{xxx} + ww_{xxxx} + 3(ww_{zz} - w_z^2) \\ &= w\big[(w_t + 4w_{xxx})_x - 3(w_{xxxx} - w_{zz})\big] \\ &\quad - w_x(w_t + 4w_{xxx}) + 3(w_{xx}^2 - w_z^2). \end{aligned} \tag{6.7.60}$$

The evaluation of the derivatives of a Wronskian is facilitated by expressing it in column vector notation.

Let

$$W = \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_{n-1}\right|_n, \tag{6.7.61}$$

where

$$\mathbf{C}_j = \left[D_x^j(\psi_1)\ D_x^j(\psi_2) \cdots D_x^j(\psi_n)\right]^T.$$

The significance of the row of dots above the $(n-3)$ columns $\mathbf{C}_0$ to $\mathbf{C}_{n-4}$ will emerge shortly. It follows from (6.7.58) and (6.7.59) that

$$\begin{aligned} D_x(\mathbf{C}_j) &= \mathbf{C}_{j+1}, \\ D_z(\mathbf{C}_j) &= D_x^2(\mathbf{C}_j) = \mathbf{C}_{j+2}, \\ D_t(\mathbf{C}_j) &= -4D_x^3(\mathbf{C}_j) = -4\mathbf{C}_{j+3}. \end{aligned} \tag{6.7.62}$$

Hence, differentiating (6.7.61) and discarding determinants with two identical columns,

$$\begin{aligned} w_x &= \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_n\right|_n, \\ w_{xx} &= \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n \\ &\quad + \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_{n+1}\right|_n, \\ w_z &= \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_n\ \mathbf{C}_{n-1}\right|_n \\ &\quad + \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_{n+1}\right|_n, \end{aligned}$$

etc. The significance of the row of dots above columns $\mathbf{C}_0$ to $\mathbf{C}_{n-4}$ is beginning to emerge. These columns are common to all the determinants which arise in all the derivatives which appear in the second line of (6.7.60). They can therefore be omitted without causing confusion.

Let

$$V_{pqr} = \left|\overset{\cdots\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_p\ \mathbf{C}_q\ \mathbf{C}_r\right|_n. \tag{6.7.63}$$

Then, $V_{pqr} = 0$ if p, q, and r are not distinct and $V_{qpr} = -V_{pqr}$, etc. In this notation,

$$\begin{aligned} w &= V_{n-3,n-2,n-1}, \\ w_x &= V_{n-3,n-2,n}, \\ w_{xx} &= V_{n-3,n-1,n} + V_{n-3,n-2,n+1}, \\ w_{xxx} &= V_{n-2,n-1,n} + 2V_{n-3,n-1,n+1} + V_{n-3,n-2,n+2}, \end{aligned}$$

$$\begin{aligned}
w_{xxxx} &= 2V_{n-3,n,n+1} + 3V_{n-3,n-1,n+2} + 3V_{n-2,n-1,n+1} + V_{n-3,n-2,n+3},\\
w_z &= -V_{n-3,n-1,n} + V_{n-3,n-2,n+1},\\
w_{zz} &= 2V_{n-3,n,n+1} - V_{n-3,n-1,n+2} - V_{n-2,n-1,n+1},\\
w_t &= -4(V_{n-2,n-1,n} - V_{n-3,n-1,n+1} + V_{n-3,n-2,n+2}),\\
w_{xt} &= 4(V_{n-3,n,n+1} - V_{n-3,n-2,n+3}). \qquad (6.7.64)
\end{aligned}$$

Each of the sections in the second line of (6.7.60) simplifies as follows:

$$\begin{aligned}
w_t + 4w_{xxx} &= 12V_{n-3,n-1,n+1},\\
(w_t + 4w_{xxx})_x &= 12(V_{n-2,n-1,n+1} + V_{n-3,n,n+1} + V_{n-3,n-1,n+2}),\\
w_{xxxx} - w_{zz} &= 4(V_{n-2,n-1,n+1} + V_{n-3,n-1,n+2}),\\
(w_t + 4w_{xxx})_x - 3(w_{xxxx} - w_{zz}) &= 12V_{n-3,n,n+1}\\
w_{xx}^2 - w_z^2 &= 4V_{n-3,n-1,n}V_{n-3,n-2,n+1}. \qquad (6.7.65)
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{12}F &= V_{n-3,n-2,n-1}V_{n-3,n,n+1} + V_{n-3,n-2,n}V_{n-3,n-1,n+1}\\
&\quad + V_{n-3,n-1,n}V_{n-3,n-2,n+1}. \qquad (6.7.66)
\end{aligned}$$

Let

$$\begin{aligned}
\mathbf{C}_{n+1} &= \bigl[\alpha_1\ \alpha_2 \ldots \alpha_n\bigr]^T,\\
\mathbf{C}_{n+2} &= \bigl[\beta_1\ \beta_2 \ldots \beta_n\bigr]^T,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_r &= D_x^n(\psi_r)\\
\beta_r &= D_x^{n+1}(\psi_r).
\end{aligned}$$

Then

$$\begin{aligned}
V_{n-3,n-2,n-1} &= A_n,\\
V_{n-3,n-2,n} &= \sum_r \alpha_r A_{rn}^{(n)},\\
V_{n-3,n-1,n+1} &= -\sum_s \beta_s A_{r,n-1}^{(n)},\\
V_{n-3,n-2,n+1} &= \sum_s \beta_s A_{sn}^{(n)},\\
V_{n-3,n-1,n} &= -\sum_r \alpha_r A_{r,n-1}^{(n)},\\
V_{n-3,n,n+1} &= \sum_r \sum_s \alpha_r \beta_s A_{rs;n-1,n}^{(n)}. \qquad (6.7.67)
\end{aligned}$$

Hence, applying the Jacobi identity,

$$\frac{1}{12}F = A_n \sum_r \sum_s \alpha_r \beta_s A_{rs;n-1,n}^{(n)} + \sum_r \alpha_r A_{rn}^{(n)} \sum_s \beta_s A_{s,n-1}^{(n)}$$

$$
\begin{aligned}
&\qquad -\sum_r \alpha_r A_{r,n-1}^{(n)} \sum_s \beta_s A_{sn}^{(n)} \\
&= \sum_r \sum_s \alpha_r \beta_s \left[A_n A_{rs;n-1,n}^{(n)} - \begin{vmatrix} A_{r,n-1}^{(n)} & A_{rn}^{(n)} \\ A_{s,n-1}^{(n)} & A_{sn}^{(n)} \end{vmatrix} \right] \\
&= 0,
\end{aligned}
$$

which completes the proof of the theorem. □

Exercise. Prove that

$$\log w = k + \log W,$$

where k is independent of x and, hence, that w and W yield the same solution of the KdV equation.

6.8 The Kadomtsev–Petviashvili Equation

6.8.1 The Non-Wronskian Solution

The KP equation is

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{6.8.1}$$

The substitution $u = 2v_x$ transforms it into

$$(v_t + 6v_x^2 + v_{xxx})_x + 3v_{yy} = 0. \tag{6.8.2}$$

Theorem 6.16. *The KP equation in the form (6.8.2) is satisfied by the function*

$$v = D_x(\log A),$$

where

$$
\begin{aligned}
A &= |a_{rs}|_n, \\
a_{rs} &= \delta_{rs} e_r + \frac{1}{b_r + c_s}, \\
e_r &= \exp\bigl[-(b_r + c_r)x + (b_r^2 - c_r^2)y + 4(b_r^3 + c_r^3)t + \varepsilon_r\bigr] \\
&= \exp\bigl[-\lambda_r x + \lambda_r \mu_r y + 4\lambda_r (b_r^2 - b_r c_r + c_r^2)t + \varepsilon_r\bigr], \\
\lambda_r &= b_r + c_r, \\
\mu_r &= b_r - c_r.
\end{aligned}
$$

The ε_r are arbitrary constants and the b_r and c_s are constants such that $b_r + c_s \neq 0$, $1 \le r, s \le n$, but are otherwise arbitrary.

Proof. The proof consists of a sequence of relations similar to those which appear in Section 6.7 on the KdV equation. Those identities which arise from the double-sum relations (A)–(D) in Section 3.4 are as follows:

Applying (A),

$$v = D_x(\log A) = -\sum_r \lambda_r e_r A^{rr}, \tag{6.8.3}$$

$$D_y(\log A) = \sum_r \lambda_r \mu_r e_r A^{rr}, \tag{6.8.4}$$

$$D_t(\log A) = 4\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{rr}. \tag{6.8.5}$$

Applying (B),

$$D_x(A^{ij}) = \sum_r \lambda_r e_r A^{ir} A^{rj}, \tag{6.8.6}$$

$$D_y(A^{ij}) = -\sum_r \lambda_r \mu_r e_r A^{ir} A^{rj}, \tag{6.8.7}$$

$$D_t(A^{ij}) = -4\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{ir} A^{rj}. \tag{6.8.8}$$

Applying (C) with

i. $f_r = b_r, \qquad g_r = c_r;$
ii. $f_r = b_r^2, \qquad g_r = -c_r^2;$
iii. $f_r = b_r^3, \qquad g_r = c_r^3;$

in turn,

$$\sum_r \lambda_r e_r A^{rr} + \sum_{r,s} A^{rs} = \sum_r \lambda_r, \tag{6.8.9}$$

$$\sum_r \lambda_r \mu_r e_r A^{rr} + \sum_{r,s} (b_r - c_s) A^{rs} = \sum_r \lambda_r \mu_r, \tag{6.8.10}$$

$$\begin{aligned}\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{rr} &+ \sum_{r,s} (b_r^2 - b_r c_s + c_s^2) A^{rs} \\ &= \sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2).\end{aligned} \tag{6.8.11}$$

Applying (D) with (i)–(iii) in turn,

$$\sum_r \lambda_r e_r A^{ir} A^{rj} + \sum_{r,s} A^{is} A^{rj} = (b_i + c_j) A^{ij}, \tag{6.8.12}$$

$$\sum_r \lambda_r \mu_r e_r A^{ir} A^{rj} + \sum_{r,s} (b_r - c_s) A^{is} A^{rj} = (b_i^2 - c_j^2) A^{ij}, \tag{6.8.13}$$

$$\begin{aligned}\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{ir} A^{rj} &+ \sum_{r,s} (b_r^2 - b_r c_s + c_s^2) A^{is} A^{rj} \\ &= (b_i^3 + c_j^3) A^{ij}.\end{aligned} \tag{6.8.14}$$

Eliminating the sum common to (6.8.3) and (6.8.9), the sum common to (6.8.4) and (6.8.10) and the sum common to (6.8.5) and (6.8.11), we find new formulae for the derivatives of $\log A$:

$$v = D_x(\log A) = \sum_{r,s} A^{rs} - \sum_r \lambda_r, \tag{6.8.15}$$

$$D_y(\log A) = -\sum_{r,s}(b_r - c_s)A^{rs} + \sum_r \lambda_r\mu_r, \tag{6.8.16}$$

$$\begin{aligned} D_t(\log A) &= -4\sum_{r,s}(b_r^2 - b_r c_s + c_s^2)A^{rs} \\ &\quad +4\sum_r \lambda_r(b_r^2 - b_r c_r + c_r^2). \end{aligned} \tag{6.8.17}$$

Equations (6.8.16) and (6.8.17) are not applied below but have been included for their interest.

Eliminating the sum common to (6.8.6) and (6.8.12), the sum common to (6.8.7) and (6.8.13), and the sum common to (6.8.8) and (6.8.14), we find new formulas for the derivatives of A^{ij}:

$$\begin{aligned} D_x(A^{ij}) &= (b_i + c_j)A^{ij} - \sum_{r,s} A^{is}A^{rj}, \\ D_y(A^{ij}) &= -(b_i^2 - c_j^2)A^{ij} + \sum_{r,s}(b_r - c_s)A^{is}A^{rj}, \\ D_t(A^{ij}) &= -4(b_i^3 + c_j^3)A^{ij} + 4\sum_{r,s}(b_r^2 - b_r c_s + c_s^2)A^{is}A^{rj}. \end{aligned} \tag{6.8.18}$$

Define functions h_{ij}, H_{ij}, and $\overline{H}_{ij}$ as follows:

$$\begin{aligned} h_{ij} &= \sum_{r=1}^n \sum_{s=1}^n b_r^i c_s^j A^{rs}, \\ H_{ij} &= h_{ij} + h_{ji} = H_{ji}, \\ \overline{H}_{ij} &= h_{ij} - h_{ji} = -\overline{H}_{ji}. \end{aligned} \tag{6.8.19}$$

The derivatives of these functions are found by applying (6.8.18):

$$\begin{aligned} D_x(h_{ij}) &= \sum_{r,s} b_r^i c_s^j \left[(b_r + c_s)A^{rs} - \sum_{p,q} A^{rq}A^{ps}\right] \\ &= \sum_{r,s} b_r^i c_s^j (b_r + c_s)A^{rs} - \sum_{r,q} b_r^i A^{rq} \sum_{p,s} c_s^j A^{ps} \\ &= h_{i+1,j} + h_{i,j+1} - h_{i0}h_{0j}, \end{aligned}$$

which is a nonlinear differential recurrence relation. Similarly,

$$\begin{aligned} D_y(h_{ij}) &= h_{i0}h_{1j} - h_{i1}h_{0j} - h_{i+2,j} + h_{i,j+2}, \\ D_t(h_{ij}) &= 4(h_{i0}h_{2j} - h_{i1}h_{1j} + h_{i2}h_{0j} - h_{i+3,j} - h_{i,j+3}), \\ D_x(H_{ij}) &= H_{i+1,j} + H_{i,j+1} - h_{i0}h_{0j} - h_{0i}h_{j0}, \\ D_y(\overline{H}_{ij}) &= (h_{i0}h_{1j} + h_{0i}h_{j1}) - (h_{i1}h_{0j} + h_{1i}h_{j0}) \end{aligned}$$

$$-H_{i+2,j} + H_{i,j+2}. \tag{6.8.20}$$

From (6.8.15),

$$v = h_{00} - \text{constant}.$$

The derivatives of v can now be found in terms of the h_{ij} and H_{ij} with the aid of (6.8.20):

$$\begin{aligned}
v_x &= H_{10} h_{00}^2,\\
v_{xx} &= H_{20} + H_{11} - 3h_{00}H_{10} + 2h_{00}^3,\\
v_{xxx} &= 12h_{00}^2 H_{10} - 3H_{10}^2 - 4h_{00}H_{20} - 3h_{00}H_{11} + 3H_{21}\\
&\quad + H_{30} - 2h_{10}h_{01} - 6h_{00}^4,\\
v_y &= h_{00}\bar{H}_{10} - \bar{H}_{20}\\
v_{yy} &= 2(h_{10}h_{20} + h_{01}h_{02}) - (h_{10}h_{02} + h_{01}h_{20})\\
&\quad - h_{00}(h_{10}^2 - h_{10}h_{01} + h_{01}^2) + 2h_{00}^2 h_{11}\\
&\quad - 2h_{00}H_{21} + H_{22} + h_{00}H_{30} - H_{40},\\
v_t &= 4(h_{00}H_{20} - h_{10}h_{01} - H_{30}).
\end{aligned} \tag{6.8.21}$$

Hence,

$$v_t + 6v_x^2 + v_{xxx} = 3(h_{10}^2 + h_{01}^2 - h_{00}H_{11} + H_{21} - H_{30}). \tag{6.8.22}$$

The theorem appears after differentiating once again with respect to x. $\square$

6.8.2 *The Wronskian Solution*

The substitution

$$u = 2D_x^2(\log w)$$

into the KP equation yields

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 2D_x^2\left(\frac{G}{w^2}\right), \tag{6.8.23}$$

where

$$G = ww_{xt} - w_x w_t + 3w_{xx}^2 - 4w_x w_{xxx} + ww_{xxxx} + 3(ww_{yy} - w_y^2).$$

Hence, the KP equation will be satisfied if

$$G = 0. \tag{6.8.24}$$

The function G is identical in form with the function F in the first line of (6.7.60) in the section on the KdV equation, but the symbol y in this section and the symbol z in the KdV section have different origins. In this section, y is one of the three independent variables x, y, and t in the KP equation whereas x and t are the only independent variables in the KdV section and z is introduced to facilitate the analysis.

Theorem. *The KP equation in the form (6.8.2) is satisfied by the Wronskian w defined as follows:*

$$w = \left|D_x^{j-1}(\psi_i)\right|_n,$$

where

$$\begin{aligned}
\psi_i &= \exp\left(\tfrac{1}{4}b_i^2 y\right)\phi_i,\\
\phi_i &= p_i e_i^{1/2} + q_i e_i^{-1/2},\\
e_i &= \exp(-b_i x + b_i^3 t)
\end{aligned}$$

and b_i, p_i, and q_i are arbitrary functions of i.

The proof is obtained by replacing z by y in the proof of the first line of (6.7.60) with $F = 0$ in the KdV section. The reverse procedure is invalid. If the KP equation is solved first, it is not possible to solve the KdV equation by putting $y = 0$.

6.9 The Benjamin–Ono Equation

6.9.1 Introduction

The notation $\omega^2 = -1$ is used in this section, as i and j are indispensable as row and column parameters.

Theorem. *The Benjamin–Ono equation in the form*

$$A_x A_x^* - \tfrac{1}{2}\left[A^*(A_{xx} + \omega A_t) + A(A_{xx} + \omega A_t)^*\right] = 0, \tag{6.9.1}$$

where A^* is the complex conjugate of A, is satisfied for all values of n by the determinant

$$A = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} \frac{2c_i}{c_i - c_j}, & j \neq i \\ 1 + \omega\theta_i, & j = i \end{cases} \tag{6.9.2}$$

$$\theta_i = c_i x - c_i^2 t - \lambda_i, \tag{6.9.3}$$

and where the c_i are distinct but otherwise arbitrary constants and the λ_i are arbitrary constants.

The proof which follows is a modified version of the one given by Matsuno. It begins with the definitions of three determinants B, P, and Q.

6.9.2 Three Determinants

The determinant A and its cofactors are closely related to the Matsuno determinant E and its cofactor (Section 5.4)

$$\begin{aligned} A &= K_n E, \\ 2c_r A_{rs} &= K_n E_{rs}, \\ 4c_r c_s A_{rs,rs} &= K_n E_{rs,rs}, \end{aligned}$$

where

$$K_n = 2^n \prod_{r=1}^{n} c_r. \tag{6.9.4}$$

The proofs are elementary. It has been proved that

$$\begin{aligned} \sum_{r=1}^{n} E_{rr} &= \sum_{r=1}^{n} \sum_{s=1}^{n} E_{rs}, \\ \sum_{r=1}^{n} \sum_{s=1}^{n} E_{rs,rs} &= -2 \sum_{r=1}^{n} \sum_{s-1}^{n}{}^{\dagger} c_s E_{rs}. \end{aligned}$$

It follows that

$$\sum_{r=1}^{n} c_r A_{rr} = \sum_{r=1}^{n} \sum_{s=1}^{n} c_r A_{rs} \tag{6.9.5}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n} c_r c_s A_{rs,rs} = -\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} c_r c_s A_{rs}. \tag{6.9.6}$$

Define the determinant B as follows:

$$B = |b_{ij}|_n,$$

where

$$b_{ij} = \begin{cases} a_{ij} - 1 & \\ \frac{c_i + c_j}{c_i - c_j}, & j \neq i \\ \omega\theta_i, & j = i \quad (\omega^2 = -1). \end{cases} \tag{6.9.7}$$

It may be verified that, for all values of i and j,

$$\begin{aligned} b_{ji} &= -b_{ij}, \qquad j \neq i, \\ b_{ij} - 1 &= -a^*_{ji}, \\ a^*_{ij} - 1 &= -b_{ji}. \end{aligned} \tag{6.9.8}$$

When $j \neq i$, $a^*_{ij} = a_{ij}$, etc.

Notes on bordered determinants are given in Section 3.7. Let P denote the determinant of order $(n+2)$ obtained by bordering A by two rows and two columns as follows:

$$P = \begin{vmatrix} & & & & c_1 & 1 \\ & & & & c_2 & 1 \\ & & & & \cdots & \cdots \\ & [a_{ij}]_n & & & \cdots & \cdots \\ & & & & \cdots & \cdots \\ & & & & c_n & 1 \\ -c_1 & -c_2 & \cdots & -c_n & 0 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 \end{vmatrix}_{n+2} \tag{6.9.9}$$

and let Q denote the determinant of order $(n+2)$ obtained by bordering B in a similar manner. Four of the cofactors of P are

$$P_{n+1,n+1} = \begin{vmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \cdots \\ & [a_{ij}]_n & & & \cdots \\ & & & & \cdots \\ & & & & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix}_{n+1}, \tag{6.9.10}$$

$$\begin{aligned} P_{n+1,n+2} &= -\begin{vmatrix} & & & & c_1 \\ & & & & c_2 \\ & & & & \cdots \\ & [a_{ij}]_n & & & \cdots \\ & & & & \cdots \\ & & & & c_n \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix}_{n+1} \\ &= \sum_r \sum_s c_r A_{rs}, \end{aligned} \tag{6.9.11}$$

$$P_{n+2,n+1} = -\begin{vmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \cdots \\ & [a_{ij}]_n & & & \cdots \\ & & & & \cdots \\ & & & & 1 \\ -c_1 & -c_2 & \cdots & -c_n & 0 \end{vmatrix}_{n+1}, \tag{6.9.12}$$

$$P_{n+2,n+2} = \begin{vmatrix} & & & & c_1 \\ & & & & c_2 \\ & & & & \cdots \\ & [a_{ij}]_n & & & \cdots \\ & & & & \cdots \\ & & & & c_n \\ -c_1 & -c_2 & \cdots & -c_n & 0 \end{vmatrix}_{n+1}$$

$$= \sum_r \sum_s c_r c_s A_{rs}. \tag{6.9.13}$$

The determinants A, B, P, and Q, their cofactors, and their complex conjugates are related as follows:

$$\begin{aligned}
B &= Q_{n+1,n+2;n+1,n+2}, && (6.9.14)\\
A &= B + Q_{n+1,n+1}, && (6.9.15)\\
A^* &= (-1)^n (B - Q_{n+1,n+1}), && (6.9.16)\\
P_{n+1,n+2} &= Q_{n+1,n+2}, && (6.9.17)\\
P^*_{n+1,n+2} &= (-1)^{n+1} Q_{n+2,n+1}, && (6.9.18)\\
P_{n+2,n+2} &= Q_{n+2,n+2} + Q, && (6.9.19)\\
P^*_{n+2,n+2} &= (-1)^{n+1}(Q_{n+2,n+2} - Q). && (6.9.20)
\end{aligned}$$

The proof of (6.9.14) is obvious. Equation (6.9.15) can be proved as follows:

$$B + Q_{n+1,n+1} = \begin{vmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \cdots \\ & & [b_{ij}]_n & & \cdots \\ & & & & \cdots \\ & & & & 1 \\ -1 & -1 & \cdots & -1 & 1 \end{vmatrix}_{n+1}. \tag{6.9.21}$$

Note the element 1 in the bottom right-hand corner. The row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_{n+1}, \quad 1 \le i \le n, \tag{6.9.22}$$

yield

$$B + Q_{n+1,n+1} = \begin{vmatrix} & & & & 0 \\ & & & & 0 \\ & & & & \cdots \\ & & [b_{ij}+1]_n & & \cdots \\ & & & & \cdots \\ & & & & 0 \\ -1 & -1 & \cdots & -1 & 1 \end{vmatrix}_{n+1}. \tag{6.9.23}$$

Equation (6.9.15) follows by applying (6.9.7) and expanding the determinant by the single nonzero element in the last column. Equation (6.9.16) can be proved in a similar manner. Express $Q_{n+1,n+1} - B$ as a bordered determinant similar to (6.9.21) but with the element 1 in the bottom right-hand corner replaced by -1. The row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \mathbf{R}_{n+1}, \quad 1 \le i \le n, \tag{6.9.24}$$

leave a single nonzero element in the last column. The result appears after applying the second line of (6.9.8).

To prove (6.9.17), perform the row operations (6.9.24) on $P_{n+1,n+2}$ and apply (6.9.7). To prove (6.9.18), perform the same row operations on $P^*_{n+1,n+2}$, apply the third equation in (6.9.8), and transpose the result.

To prove (6.9.19), note that

$$Q + Q_{n+2,n+2} = \begin{vmatrix} & & & & c_1 & 1 \\ & & & & c_2 & 1 \\ & & & & \cdots & \cdots \\ & & [b_{ij}]_n & & \cdots & \cdots \\ & & & & \cdots & \cdots \\ & & & & c_n & 1 \\ -c_1 & -c_2 & \cdots & -c_n & 0 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 1 \end{vmatrix}_{n+2}. \tag{6.9.25}$$

The row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_{n+2}, \quad 1 \le i \le n,$$

leave a single nonzero element in the last column. The result appears after applying the second equation in (6.9.7).

To prove (6.9.20), note that $Q - Q_{n+2,n+2}$ can be expressed as a determinant similar to (6.9.25) but with the element 1 in the bottom right-hand corner replaced by -1. The row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \mathbf{R}_{n+2}, \quad 1 \le i \le n,$$

leave a single nonzero element in the last column. The result appears after applying the second equation of (6.9.8) and transposing the result.

6.9.3 Proof of the Main Theorem

Denote the left-hand side of (6.9.1) by F. Then, it is required to prove that $F = 0$. Applying (6.9.3), (6.9.5), (6.9.11), and (6.9.17),

$$\begin{aligned} A_x &= \sum_r \frac{\partial A}{\partial \theta_r} \frac{\partial \theta_r}{\partial x} \\ &= \omega \sum_r c_r A_{rr} \qquad (6.9.26) \\ &= \omega \sum_r \sum_s c_r A_{rs} \\ &= \omega P_{n+1,n+2} \\ &= \omega Q_{n+1,n+2}. \qquad (6.9.27) \end{aligned}$$

Taking the complex conjugate of (6.9.27) and referring to (6.9.18),

$$\begin{aligned} A^*_x &= -\omega P^*_{n+1,n+2} \\ &= (-1)^n \omega Q_{n+2,n+1}. \end{aligned}$$

Hence, the first term of F is given by

$$A_x A_x^* = (-1)^{n+1} Q_{n+1,n+2} Q_{n+2,n+1}. \tag{6.9.28}$$

Differentiating (6.9.26) and referring to (6.9.6),

$$\begin{aligned} A_{xx} &= \omega \sum_r c_r \frac{\partial A_{rr}}{\partial x} \\ &= \omega \sum_r c_r \sum_s \frac{\partial A_{ss}}{\partial \theta_s} \frac{\partial \theta_s}{\partial x} \\ &= -\sum_r \sum_s c_r c_s A_{rs,rs} \\ &= \sum_r \sum_s{}^{\dagger} c_r c_s A_{rs}, \end{aligned} \tag{6.9.29}$$

$$\begin{aligned} A_t &= \sum_r \frac{\partial A}{\partial \theta_r} \frac{\partial \theta_r}{\partial t} \\ &= -\omega \sum_r c_r^2 A_{rr}. \end{aligned} \tag{6.9.30}$$

Hence, applying (6.9.13) and (6.9.19),

$$\begin{aligned} A_{xx} + \omega A_t &= \sum_r \sum_s{}^{\dagger} c_r c_s A_{rs} + \sum_r c_r^2 A_{rr} \\ &= \sum_r \sum_s c_r c_s A_{rs} \\ &= P_{n+2,n+2} \\ &= Q_{n+2,n+2} + Q. \end{aligned} \tag{6.9.31}$$

Hence, the second term of F is given by

$$A^*(A_{xx} + \omega A_t) = (-1)^n (B - Q_{n+1,n+1})(Q_{n+2,n+2} + Q). \tag{6.9.32}$$

Taking the complex conjugate of (6.9.31) and applying (6.9.20) and (6.9.15),

$$\begin{aligned} (A_{xx} + \omega A_t)^* &= P^*_{n+2,n+2} \\ &= (-1)^{n+1}(Q_{n+2,n+2} - Q). \end{aligned} \tag{6.9.33}$$

Hence, the third term of F is given by

$$A(A_{xx} + \omega A_t)^* = (-1)^{n+1}(B + Q_{n+1,n+1})(Q_{n+2,n+2} - Q). \tag{6.9.34}$$

Referring to (6.9.14),

$$\begin{aligned} &\tfrac{1}{2}(-1)^n \left[A^*(A_{xx} + \omega A_t) + A(A_{xx} + \omega A_t)^* \right] \\ &\quad = BQ - Q_{n+1,n+1} Q_{n+2,n+2} \\ &\quad = Q Q_{n+1,n+2;n+1,n+2} - Q_{n+1,n+1} Q_{n+2,n+2}. \end{aligned}$$

Hence, referring to (6.9.28) and applying the Jacobi identity,

$$\begin{aligned}(-1)^n F &= \begin{vmatrix} Q_{n+1,n+1} & Q_{n+1,n+2} \\ Q_{n+2,n+1} & Q_{n+2,n+2} \end{vmatrix} - QQ_{n+1,n+2;n+1,n+2} \\ &= 0,\end{aligned}$$

which completes the proof of the theorem.

6.10 The Einstein and Ernst Equations

6.10.1 Introduction

This section is devoted to the solution of the scalar Einstein equations, namely

$$\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2 = 0, \tag{6.10.1}$$

$$\phi\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2(\phi_\rho\psi_\rho + \phi_z\psi_z) = 0, \tag{6.10.2}$$

but before the theorems can be stated and proved, it is necessary to define a function u_r, three determinants A, B, and E, and to prove some lemmas. The notation $\omega^2 = -1$ is used again as i and j are indispensable as row and column parameters, respectively.

6.10.2 Preparatory Lemmas

Let the function $u_r(\rho, z)$ be defined as any real solution of the coupled equations

$$\frac{\partial u_{r+1}}{\partial\rho} + \frac{\partial u_r}{\partial z} = -\frac{ru_{r+1}}{\rho}, \quad r = 0, 1, 2, \ldots, \tag{6.10.3}$$

$$\frac{\partial u_{r-1}}{\partial\rho} - \frac{\partial u_r}{\partial z} = \frac{ru_{r-1}}{\rho}, \quad r = 1, 2, 3 \ldots, \tag{6.10.4}$$

which are solved in Appendix A.11.

Define three determinants A_n, B_n, and E_n as follows.

$$A_n = |a_{rs}|_n$$

where

$$a_{rs} = \omega^{|r-s|}u_{|r-s|}, \quad (\omega^2 = -1). \tag{6.10.5}$$

$$B_n = |b_{rs}|_n,$$

where

$$b_{rs} = \begin{cases} u_{r-s}, & r \geq s \\ (-1)^{s-r}u_{s-r}, & r \leq s \end{cases}$$

$$b_{rs} = \omega^{s-r} a_{rs}. \tag{6.10.6}$$

$$E_n = |e_{rs}|_n = (-1)^n A_{1,n+1}^{(n+1)} = (-1)^n A_{n+1,1}^{(n+1)}. \tag{6.10.7}$$

In some detail,

$$A_n = \begin{vmatrix} u_0 & \omega u_1 & -u_2 & -\omega u_3 & \cdots \\ \omega u_1 & u_0 & \omega u_1 & -u_2 & \cdots \\ -u_2 & \omega u_1 & u_0 & \omega u_1 & \cdots \\ -\omega u_3 & -u_2 & \omega u_1 & u_0 & \cdots \\ \multicolumn{5}{c}{\dotfill} \end{vmatrix}_n \quad (\omega^2 = -1), \tag{6.10.8}$$

$$B_n = \begin{vmatrix} u_0 & -u_1 & u_2 & -u_3 & \cdots \\ u_1 & u_0 & -u_1 & u_2 & \cdots \\ u_2 & u_1 & u_0 & -u_1 & \cdots \\ u_3 & u_2 & u_1 & u_0 & \cdots \\ \multicolumn{5}{c}{\dotfill} \end{vmatrix}_n, \tag{6.10.9}$$

$$E_n = \begin{vmatrix} \omega u_1 & u_0 & \omega u_1 & -u_2 & \cdots \\ -u_2 & \omega u_1 & u_0 & \omega u_1 & \cdots \\ -\omega u_3 & -u_2 & \omega u_1 & u_0 & \cdots \\ u_4 & -\omega u_3 & -u_2 & \omega u_1 & \cdots \\ \multicolumn{5}{c}{\dotfill} \end{vmatrix}_n \quad (\omega^2 = -1), \tag{6.10.10}$$

$$A_n = (-1)^n E_{n+1,1}^{(n+1)}. \tag{6.10.11}$$

A_n is a symmetric Toeplitz determinant (Section 4.5.2) in which $t_r = \omega^r u_r$. All the elements on and below the principal diagonal of B_n are positive. Those above the principal diagonal are alternately positive and negative.

The notation is simplified by omitting the order n from a determinant or cofactor where there is no risk of confusion. Thus A_n, $A_{ij}^{(n)}$, A_n^{ij}, etc., may appear as A, A_{ij}, A^{ij}, etc. Where the order is not equal to n, the appropriate order is shown explicitly.

A and E, and their simple and scaled cofactors are related by the following identities:

$$\begin{aligned} A_{11} = A_{nn} &= A_{n-1}, \\ A_{1n} = A_{n1} &= (-1)^{n-1} E_{n-1}, \\ E_{p1} &= (-1)^{n-1} A_{pn}, \\ E_{nq} &= (-1)^{n-1} A_{1q}, \\ E_{n1} &= (-1)^{n-1} A_{n-1}, \end{aligned} \tag{6.10.12}$$

$$\left(\frac{A}{E}\right)^2 = \left(\frac{E^{n1}}{A^{11}}\right)^2, \tag{6.10.13}$$

$$E^2 E^{p1} E^{nq} = A^2 A^{pn} A^{1q}. \tag{6.10.14}$$

Lemma 6.17.

$$A = B.$$

PROOF. Multiply the rth row of A by ω^{-r}, $1 \le r \le n$ and the sth column by ω^s, $1 \le s \le n$. The effect of these operations is to multiply A by the factor 1 and to multiply the element a_{rs} by ω^{s-r}. Hence, by (6.10.6), A is transformed into B and the lemma is proved. $\square$

Unlike A, which is real, the cofactors of A are not all real. An example is given in the following lemma.

Lemma 6.18.

$$A_{1n} = \omega^{n-1} B_{1n} \quad (\omega^2 = -1).$$

PROOF.

$$A_{1n} = (-1)^{n+1} |e_{rs}|_{n-1},$$

where

$$\begin{aligned} e_{rs} &= a_{r+1,s} \\ &= \omega^{|r-s+1|} u_{|r-s+1|} \\ &= a_{r,s-1} \end{aligned}$$

and

$$B_{1n} = (-1)^{n+1} |\beta_{rs}|_{n-1},$$

where

$$\begin{aligned} \beta_{rs} &= b_{r+1,s} \\ &= b_{r,s-1}, \end{aligned}$$

that is,

$$\beta_{rs} = \omega^{s-r-1} e_{rs}.$$

Multiply the rth row of $A_{1n}^{(n)}$ by ω^{-r-1}, $1 \le r \le n-1$ and the sth column by ω^s, $1 \le s \le n-1$. The effect of these operations is to multiply $A_{1n}^{(n)}$ by the factor

$$\omega^{-(2+3+\cdots+n)+(1+2+3+\cdots+\overline{n-1})} = \omega^{1-n}$$

and to multiply the element e_{rs} by ω^{s-r-1}. The lemma follows. $\square$

Both A and B are persymmetric (Hankel) about their secondary diagonals. However, A is also symmetric about its principal diagonal, whereas B is neither symmetric nor skew-symmetric about its principal diagonal. In the analysis which follows, advantage has been taken of the fact that A with its complex elements possesses a higher degree of symmetry than B with its real elements. The expected complicated analysis has been avoided by replacing B and its cofactors by A and its cofactors.

Lemma 6.19.

a. $\dfrac{\partial e_{pq}}{\partial \rho} + \omega \dfrac{\partial a_{pq}}{\partial z} = \left(\dfrac{q-p}{\rho}\right) e_{pq},$

b. $\dfrac{\partial a_{pq}}{\partial \rho} + \omega \dfrac{\partial e_{pq}}{\partial z} = \left(\dfrac{p-q+1}{\rho}\right) a_{pq} \quad (\omega^2 = -1).$

PROOF. If $p \geq q-1$, then, applying (6.10.3) with $r \to p-q$,

$$\begin{aligned}\left(\frac{\partial}{\partial \rho} + \frac{p-q}{\rho}\right) e_{pq} &= \left(\frac{\partial}{\partial \rho} + \frac{p-q}{\rho}\right) (\omega^{p-q+1} u_{p-q+1}) \\ &= -\frac{\partial}{\partial z}(\omega^{p-q+1} u_{p-q}) \\ &= -\omega \frac{\partial a_{pq}}{\partial z}.\end{aligned}$$

If $p < q-1$, then, applying (6.10.4) with $r \to q-p$,

$$\begin{aligned}\left(\frac{\partial}{\partial \rho} + \frac{p-q}{\rho}\right) e_{pq} &= \left(\frac{\partial}{\partial \rho} - \frac{q-p}{\rho}\right) (\omega^{q-p-1} u_{q-p-1}) \\ &= \frac{\partial}{\partial z}(\omega^{q-p-1} u_{q-p}) \\ &= -\omega \frac{\partial a_{pq}}{\partial z},\end{aligned}$$

which proves (a). To prove (b) with $p \geq q-1$, apply (6.10.4) with $r \to p-q+1$. When $p < q-1$, apply (6.10.3) with $r \to q-p-1$. $\square$

Lemma 6.20.

a. $E^2 \dfrac{\partial E^{n1}}{\partial \rho} + \omega A^2 \dfrac{\partial A^{n1}}{\partial z} = \dfrac{(n-1)E^2 E^{n1}}{\rho},$

b. $A^2 \dfrac{\partial A^{n1}}{\partial \rho} + \omega E^2 \dfrac{\partial E^{n1}}{\partial z} = \dfrac{(n-2)A^2 A^{n1}}{\rho} \quad (\omega^2 = -1).$

PROOF.

$$A = |a_{pq}|_n, \qquad \sum_{p=1}^{n} a_{pq} A^{pr} = \delta_{qr},$$

$$E = |e_{pq}|_n, \qquad \sum_{p=1}^{n} e_{pq} E^{pr} = \delta_{qr}.$$

Applying the double-sum identity (B) (Section 3.4) and (6.10.12),

$$\frac{\partial E^{n1}}{\partial \rho} = -\sum_p \sum_q \frac{\partial e_{pq}}{\partial \rho} E^{p1} E^{nq},$$

$$\frac{\partial A^{n1}}{\partial z} = -\sum_p \sum_q \frac{\partial a_{pq}}{\partial z} A^{pn} A^{1q}$$

$$= -\left(\frac{E}{A}\right)^2 \sum_p \sum_q \frac{\partial a_{pq}}{\partial z} E^{p1} E^{nq}.$$

Hence, referring to Lemma 6.19,

$$\begin{aligned}
\frac{\partial E^{n1}}{\partial \rho} + \omega \left(\frac{A}{E}\right)^2 \frac{\partial A^{n1}}{\partial z} &= -\sum_p \sum_q \left(\frac{\partial e_{pq}}{\partial \rho} + \omega \frac{\partial a_{pq}}{\partial z}\right) E^{pq} E^{nq} \\
&= \frac{1}{\rho} \sum_p \sum_q (p-q) e_{pq} E^{p1} E^{nq} \\
&= \frac{1}{\rho} \left[\sum_p p E^{p1} \sum_q e_{pq} E^{nq} - \sum_q q E^{nq} \sum_p e_{pq} E^{p1}\right] \\
&= \frac{1}{\rho} \left[\sum_p p E^{p1} \delta_{pn} - \sum_q q E^{nq} \delta_{q1}\right] \\
&= \frac{1}{\rho}(n E^{n1} - E^{n1}),
\end{aligned}$$

which is equivalent to (a).

$$\begin{aligned}
\frac{\partial A^{n1}}{\partial \rho} = \frac{\partial A^{1n}}{\partial \rho} &= -\sum_p \sum_q \frac{\partial a_{pq}}{\partial \rho} A^{pn} A^{1q} \\
\frac{\partial E^{n1}}{\partial z} &= -\sum_p \sum_q \frac{\partial e_{pq}}{\partial z} E^{p1} E^{nq} \\
&= -\left(\frac{A}{E}\right)^2 \sum_p \sum_q \frac{\partial e_{pq}}{\partial z} A^{pn} A^{1q}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\partial A^{n1}}{\partial \rho} + \omega \left(\frac{E}{A}\right)^2 \frac{\partial E^{n1}}{\partial z} \\
&\quad = -\sum_p \sum_q \left(\frac{\partial a_{pq}}{\partial \rho} + \omega \frac{\partial e_{pq}}{\partial z}\right) A^{pn} A^{1q} \\
&\quad = -\frac{1}{\rho} \sum_p \sum_q (p-q+1) a_{pq} A^{pn} A^{1q} \\
&\quad = \frac{1}{\rho} \left[\sum_q q A^{1q} \sum_p a_{pq} A^{pn} - \sum_p (p+1) A^{pn} \sum_q a_{pq} A^{1q}\right] \\
&\quad = \frac{1}{\rho} \left[\sum_q q A^{1q} \delta_{qn} - \sum_p (p+1) A^{pn} \delta_{p1}\right] \\
&\quad = \frac{1}{\rho}(n A^{1n} - 2A^{1n}) \quad (A^{1n} = A^{n1}),
\end{aligned}$$

which is equivalent to (b). This completes the proof of Lemma 6.20. □

Exercise. Prove that

$$\left(\omega\frac{\partial}{\partial\rho} - \frac{p-q-1}{\rho}\right)A_n^{pq} = -\frac{A_{n+1,q}^{(n+1)}}{A_n}\frac{\partial A_n^{pn}}{\partial z} + A_n^{1q}\frac{\partial}{\partial z}\left(\frac{A_{p+1,1}^{(n+1)}}{A_n}\right) + \frac{\partial A_n^{p,q-1}}{\partial z},$$

$$\begin{aligned}\omega\frac{\partial A_n^{pq}}{\partial z} &= \frac{A_{n+1,q}^{(n+1)}}{A_n}\left(\frac{\partial}{\partial\rho} - \frac{n}{\rho}\right)A_n^{pq} - A_n^{1q}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\frac{A_{p+1,1}^{(n+1)}}{A_n}\\ &\quad - \left(\frac{\partial}{\partial\rho} - \frac{q-1}{\rho}\right)A_n^{p,q-1} - \left(\frac{p+1}{\rho}\right)A_n^{p+1,q}\\ &\qquad (\omega^2 = -1).\end{aligned}$$

Note that some cofactors are scaled but others are unscaled. Hence, prove that

$$\left(\omega\frac{\partial}{\partial\rho} - \frac{n-2}{\rho}\right)\frac{E_{n-1}}{A_n} = \frac{E_n}{A_n}\frac{\partial}{\partial z}\left(\frac{A_{n-1}}{A_n}\right) - \frac{A_{n-1}}{A_n}\frac{\partial}{\partial z}\left(\frac{E_n}{A_n}\right),$$

$$\begin{aligned}\omega\frac{\partial}{\partial z}\left(\frac{E_{n-1}}{A_n}\right) &= (-1)^n\frac{E_n}{A_n}\left(\frac{\partial}{\partial\rho} - \frac{n}{\rho}\right)\frac{E_{n-1}}{A_n}\\ &\quad + \frac{A_{n-1}}{A_n}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\frac{E_n}{A_n}.\end{aligned}$$

6.10.3 The Intermediate Solutions

The solutions given in this section are not physically significant and are called intermediate solutions. However, they are used as a starting point in Section 6.10.5 to obtain physically significant solutions.

Theorem. *Equations (6.10.1) and (6.10.2) are satisfied by the function pairs $P_n(\phi_n, \psi_n)$ and $P'_n(\phi'_n, \psi'_n)$, where*

a. $\phi_n = \dfrac{\rho^{n-2}A_{n-1}}{A_{n-2}} = \dfrac{\rho^{n-2}}{A_{n-1}^{11}}$,

b. $\psi_n = \dfrac{\omega\rho^{n-2}E_{n-1}}{A_{n-2}} = \dfrac{(-1)^n\omega\rho^{n-2}}{E_{n-1}^{n-1,1}} = \dfrac{(-1)^{n-1}\omega\rho^{n-2}A_{1n}}{A_{n-2}}$,

c. $\phi'_n = \dfrac{A^{11}}{\rho^{n-2}}$,

d. $\psi'_n = \dfrac{(-1)^n\omega A^{1n}}{\rho^{n-2}} \quad (\omega^2 = -1)$.

The first two formulas are equivalent to the pair $P_{n+1}(\phi_{n+1}, \psi_{n+1})$, where

e. $\phi_{n+1} = \dfrac{\rho^{n-1}}{A^{11}}$,

f. $\psi_{n+1} = \dfrac{(-1)^{n+1}\omega\rho^{n-1}}{E^{n1}}$.

PROOF. The proof is by induction and applies the Bäcklund transformation theorems which appear in Appendix A.12 where it is proved that if $P(\phi, \psi)$ is a solution and

$$\begin{aligned}
\phi' &= \frac{\phi}{\phi^2 + \psi^2}, \\
\psi' &= -\frac{\psi}{\phi^2 + \psi^2},
\end{aligned} \tag{6.10.15}$$

then $P'(\phi', \psi')$ is also a solution. Transformation β states that if $P(\phi, \psi)$ is a solution and

$$\begin{aligned}
\phi' &= \frac{\rho}{\phi}, \\
\frac{\partial \psi'}{\partial \rho} &= -\frac{\omega\rho}{\phi^2}\frac{\partial \psi}{\partial z}, \\
\frac{\partial \psi'}{\partial z} &= \frac{\omega\rho}{\phi^2}\frac{\partial \psi}{\partial \rho} \quad (\omega^2 = -1),
\end{aligned} \tag{6.10.16}$$

then $P'(\phi', \psi')$ is also a solution. The theorem can therefore be proved by showing that the application of transformation γ to P_n gives P'_n and that the application of Transformation β to P'_n gives P_{n+1}.

Applying the Jacobi identity (Section 3.6) to the cofactors of the corner elements of A,

$$A_{n+1}^2 - A_{1n}^2 = A_n A_{n-2}. \tag{6.10.17}$$

Hence, referring to (6.10.15),

$$\begin{aligned}
\phi_n^2 + \psi_n^2 &= \left(\frac{\rho^{n-2}}{A_{n-2}}\right)^2 \left(A_{n-1}^2 - E_{n-1}^2\right) \\
&= \left(\frac{\rho^{n-2}}{A_{n-2}}\right)^2 \left(A_{n-1}^2 - A_{1n}^2\right) \\
&= \frac{\rho^{2n-4} A_n}{A_{n-2}}, \qquad (6.10.18)\\
\frac{\phi_n}{\phi_n^2 + \psi_n^2} &= \frac{A_{n-1}}{\rho^{2n-2} A_n} \quad (A_{n-1} = A_{11}) \\
&= \frac{A^{11}}{\rho^{2n-2}} \\
&= \phi'_n, \\
\frac{\psi_n}{\phi_n^2 + \psi_n^2} &= \frac{\omega E_{n-1}}{\rho^{2n-2} A_n} \\
&= \frac{(-1)^{n-1}\omega A^{1n}}{\rho^{n-2}} \\
&= -\psi'_n.
\end{aligned}$$

Hence, the application of transformation γ to P_n gives P'_n.

In order to prove that the application of transformation β to P'_n gives P_{n+1}, it is required to prove that

$$\phi_{n+1} = \frac{\rho}{\phi'_n},$$

which is obviously satisfied, and

$$\begin{aligned}\frac{\partial \psi_{n+1}}{\partial \rho} &= -\frac{\omega\rho}{(\phi'_n)^2}\frac{\partial \psi'_n}{\partial z}\\ \frac{\partial \psi_{n+1}}{\partial z} &= \frac{\omega\rho}{(\phi'_n)^2}\frac{\partial \psi'_n}{\partial \rho},\end{aligned} \tag{6.10.19}$$

that is,

$$\begin{aligned}\frac{\partial}{\partial \rho}\left[\frac{(-1)^{n+1}\omega\rho^{n-1}}{E^{n1}}\right] &= -\omega\rho\left[\frac{\rho^{n-2}}{A^{11}}\right]^2 \frac{\partial}{\partial z}\left[\frac{(-1)^n\omega A^{1n}}{\rho^{n-2}}\right],\\ \frac{\partial}{\partial z}\left[\frac{(-1)^{n+1}\omega\rho^{n-1}}{E^{n1}}\right] &= \omega\rho\left[\frac{\rho^{n-2}}{A^{11}}\right]^2 \frac{\partial}{\partial \rho}\left[\frac{(-1)^n\omega A^{1n}}{\rho^{n-2}}\right]\\ &\qquad (\omega^2 = -1).\end{aligned} \tag{6.10.20}$$

But when the derivatives of the quotients are expanded, these two relations are found to be identical with the two identities in Lemma 6.10.4 which have already been proved. Hence, the application of transformation β to P'_n gives P_{n+1} and the theorem is proved. □

The solutions of (6.10.1) and (6.10.2) can now be expressed in terms of the determinant B and its cofactors. Referring to Lemmas 6.17 and 6.18,

$$\begin{aligned}\phi_n &= \frac{\rho^{n-2}B_{n-1}}{B_{n-2}},\\ \psi_n &= -\frac{(-\omega)^n\rho^{n-2}B_{1n}}{B_{n-2}} \quad (\omega^2=-1),\ n \geq 3,\end{aligned} \tag{6.10.21}$$

$$\begin{aligned}\phi'_n &= \frac{B_{n-1}}{\rho^{n-2}B_n},\\ \psi'_n &= \frac{(-\omega)^n B_{1n}}{\rho^{n-2}B_n}, \quad n \geq 2.\end{aligned} \tag{6.10.22}$$

The first few pairs of solutions are

$$\begin{aligned}P'_1(\phi,\psi) &= \left(\frac{\rho}{u_0}, \frac{-\omega\rho}{u_0}\right),\\ P_2(\phi,\psi) &= (u_0, -u_1),\\ P'_2(\phi,\psi) &= \left(\frac{u_0}{u_0^2+u_1^2}, \frac{u_1}{u_0^2+u_1^2}\right),\\ P_3(\phi,\psi) &= \left(\frac{\rho(u_0^2+u_1^2)}{u_0}, \frac{\omega\rho(u_0u_2-u_1^2)}{u_0}\right).\end{aligned} \tag{6.10.23}$$

Exercise. The one-variable Hirota operators H_x and H_{xx} are defined in Section 5.7 and the determinants A_n and E_n, each of which is a function of ρ and z, are defined in (6.10.8) and (6.10.10). Apply Lemma 6.20 to prove that

$$\begin{aligned} H_\rho(A_{n-1}, E_n) - \omega H_z(A_n, E_{n-1}) &= \left(\frac{n-1}{\rho}\right) A_{n-1}E_n, \\ H_\rho(A_n, E_{n-1}) - \omega H_z(A_{n-1}, E_n) &= -\left(\frac{n-2}{\rho}\right) A_n E_{n-1} \quad (\omega^2 = -1). \end{aligned}$$

Using the notation

$$K^2(f,g) = \left(H_{\rho\rho} + \frac{1}{\rho}H_\rho + H_{zz}\right)(f,g),$$

where $f = f(\rho, z)$ and $g = g(\rho, z)$, prove also that

$$\begin{aligned} K^2(E_n, A_n) &= \frac{n(n-2)}{\rho^2} E_n A_n, \\ \left\{K^2 + \frac{2n-4}{\rho}\right\}(A_n, A_{n-1}) &= -\frac{1}{\rho^2} A_n A_{n-1}, \\ K^2\left\{\rho^{n(n-2)/2}E_n, \rho^{n(n-2)/2}A_n\right\} &= 0, \\ K^2\left\{\rho^{(n^2-4n+2)/2}A_{n-1}, \rho^{n(n-2)/2}A_n\right\} &= 0, \\ K^2\left\{\rho^{(n^2-2)/2}A_{n+1}, \rho^{n(n-2)/2}A_n\right\} &= 0. \end{aligned}$$

(Sasa and Satsuma)

6.10.4 Preparatory Theorems

Define a Vandermondian (Section 4.1.2) $V_{2n}(\mathbf{x})$ as follows:

$$\begin{aligned} V_{2n}(\mathbf{x}) &= \left|x_i^{j-1}\right|_{2n} \\ &= V(x_1, x_2, \ldots, x_{2n}), \end{aligned} \tag{6.10.24}$$

and let the (unsigned) minors of $V_{2n}(\mathbf{c})$ be denoted by $M_{ij}^{(2n)}(\mathbf{c})$. Also, let

$$\begin{aligned} M_i(\mathbf{c}) &= M_{i,2n}^{(2n)}(\mathbf{c}) = V(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{2n}), \\ M_{2n}(\mathbf{c}) &= M_{2n,2n}^{(2n)}(\mathbf{c}) = V_{2n-1}(\mathbf{c}). \end{aligned} \tag{6.10.25}$$

$$\begin{aligned} x_j &= \frac{z + c_j}{\rho}, \\ \varepsilon_j &= e^{\omega\theta_j}\sqrt{1 + x_j^2} \quad (\omega^2 = -1) \\ &= \frac{\tau_j}{\rho}, \end{aligned} \tag{6.10.26}$$

where τ_j is a function which appears in the Neugebauer solution and is defined in (6.2.20).

$$w_r = \sum_{j=1}^{2n} \frac{(-1)^{j-1} M_j(\mathbf{c}) x_j^r}{\varepsilon_j^*}. \tag{6.10.27}$$

Then,

$$\begin{aligned} x_i - x_j &= \frac{c_i - c_j}{\rho}, \quad \text{independent of } z, \\ \varepsilon_j \varepsilon_j^* &= 1 + x_j^2. \end{aligned} \tag{6.10.28}$$

Now, let $H_{2n}^{(m)}(\varepsilon)$ denote the determinant of order $2n$ whose column vectors are defined as follows:

$$\mathbf{C}_j^{(m)}(\varepsilon) = \left[\varepsilon_j \;\; c_j\varepsilon_j \;\; c_j^2\varepsilon_j \cdots c_j^{m-1}\varepsilon_j \;\; 1 \;\; c_j \;\; c_j^2 \cdots c_j^{2n-m-1}\right]_{2n}^T, \\ 1 \le j \le 2n. \tag{6.10.29}$$

Hence,

$$\begin{aligned} \mathbf{C}_j^{(m)}\left(\frac{1}{\varepsilon}\right) &= \left[\frac{1}{\varepsilon_j} \;\; \frac{c_j}{\varepsilon_j} \;\; \frac{c_j^2}{\varepsilon_j} \cdots \frac{c_j^{m-1}}{\varepsilon_j} \;\; 1 \;\; c_j \;\; c_j^2 \cdots c_j^{2n-m-1}\right]_{2n}^T \\ &= \frac{1}{\varepsilon_j}\left[1 \;\; c_j \;\; c_j^2 \cdots c_j^{m-1} \;\; \varepsilon_j \;\; c_j\varepsilon_j \;\; c_j^2\varepsilon_j \cdots c_j^{2n-m-1}\varepsilon_j\right]_{2n}^T. \end{aligned} \tag{6.10.30}$$

But,

$$\mathbf{C}_j^{(2n-m)}(\varepsilon) = \left[\varepsilon_j \;\; c_j\varepsilon_j \;\; c_j^2\varepsilon_j \cdots c_j^{2n-m-1}\varepsilon_j \;\; 1 \; c_j \; c_j^2 \cdots c_j^{m-1}\right]_{2n}^T. \tag{6.10.31}$$

The elements in the last column vector are a cyclic permutation of the elements in the previous column vector. Hence, applying Property (c(i)) in Section 2.3.1 on the cyclic permutation of columns (or rows, as in this case),

$$\begin{aligned} H_{2n}^{(m)}\left(\frac{1}{\varepsilon}\right) &= (-1)^{m(2n-1)} \left(\prod_{j=1}^{2n} \varepsilon_j\right)^{-1} H_{2n}^{(2n-m)}(\varepsilon), \\ \frac{H_{2n}^{(n+1)}(1/\varepsilon)}{H_{2n}^{(n)}(1/\varepsilon)} &= -\frac{H_{2n}^{(n-1)}(\varepsilon)}{H_{2n}^{(n)}(\varepsilon)}. \end{aligned} \tag{6.10.32}$$

Theorem.

a. $|w_{i+j-2} + w_{i+j}|_m = (-\rho^2)^{-m(m-1)/2}\{V_{2n}(\mathbf{c})\}^{m-1} H_{2n}^{(m)}(\varepsilon)$,

b. $|w_{i+j-2}|_m = (-\rho^2)^{-m(m-1)/2}\{V_{2n}(\mathbf{c})\}^{m-1} H_{2n}^{(m)}\left(\frac{1}{\varepsilon^*}\right)$.

The determinants on the left are Hankelians.

PROOF. *Proof of (a).* Denote the determinant on the left by W_m.

$$w_{i+j-2} + w_{i+j} = \sum_{k=1}^{2n} y_k x_k^{i+j-2},$$

where

$$y_k = (-1)^{k+1}\varepsilon_k M_k(\mathbf{c}). \tag{6.10.33}$$

Hence, applying the lemma in Section 4.1.7 with $N \to 2n$ and $n \to m$,

$$\begin{aligned} W_m &= \left| \sum_{k=1}^{2n} y_k x_k^{i+j-2} \right| \\ &= \sum_{k_1,k_2,\ldots,k_m=1}^{2n} Y_m \left(\prod_{r=2}^{m} x_{k_r}^{r-1} \right) \left| x_{k_i}^{j-1} \right|_m, \end{aligned}$$

where

$$Y_m = \prod_{r=1}^{m} y_{k_r}. \tag{6.10.34}$$

Hence, applying Identity 4 in Appendix A.3,

$$W_m = \frac{1}{m!} \sum_{k_1,k_2,\ldots,k_m=1}^{2n} Y_m \sum_{j_1,j_2,\ldots,j_m}^{k_1,k_2,\ldots k_m} \left(\prod_{r=2}^{m} x_{j_r}^{r-1} \right) V(x_{j_1}, x_{j_2}, \ldots, x_{j_m}). \tag{6.10.35}$$

Applying Theorem (b) in Section 4.1.9 on Vandermondian identities,

$$W_m = \frac{1}{m!} \sum_{k_1,k_2,\ldots,k_m=1}^{2n} Y_m \bigl\{ V(x_{k_1}, x_{k_2}, \ldots, x_{k_m}) \bigr\}^2. \tag{6.10.36}$$

Due to the presence of the squared Vandermondian factor, the conditions of Identity 3 in Appendix A.3 with $N \to 2n$ are satisfied. Also, eliminating the x's using (6.10.26) and (6.10.28) and referring to Exercise 3 in Section 4.1.2,

$$\bigl\{ (V(x_{k_1}, x_{k_2}, \ldots, x_{k_m}) \bigr\}^2 = \rho^{-m(m-1)} \bigl\{ V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) \bigr\}^2. \tag{6.10.37}$$

Hence,

$$W_m = \rho^{-m(m-1)} \sum_{1 \le k_1 < k_2 < \ldots < k_m \le 2n} Y_m \bigl\{ V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) \bigr\}^2. \tag{6.10.38}$$

From (6.10.33) and (6.10.34),

$$Y_m = (-1)^K E_m \prod_{r=1}^{m} M_{k_r}(\mathbf{c}),$$

where

$$E_m = \prod_{r=1}^{m} \varepsilon_{k_r},$$
$$K = \sum_{r=1}^{m} (k_r - 1). \tag{6.10.39}$$

Applying Theorem (c) in Section 4.1.8 on Vandermondian identities,

$$Y_m = (-1)^K E_m \frac{V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}})\{V_{2n}(\mathbf{c})\}^{m-1}}{V(c_{k_1}, c_{k_2}, \ldots, c_{k_m})}. \tag{6.10.40}$$

Hence,

$$W_m = \frac{(-1)^K \{V_{2n}(\mathbf{c})\}^{m-1}}{\rho^{m(m-1)}} \sum_{1 \le k_1 < k_2 < \ldots < k_m \le 2n} \cdot E_m V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}}). \tag{6.10.41}$$

Using the Laplace formula (Section 3.3) to expand $H_{2n}^{(m)}(\varepsilon)$ by the first m rows and the remaining $(2n-m)$ rows and referring to the exercise at the end of Section 4.1.8,

$$H_{2n}^{(m)}(\varepsilon) = \sum_{1 \le k_1 < k_2 < \cdots < k_m \le 2n} N_{12\cdots m;k_1,k_2,\ldots,k_m} A_{12\cdots m;k_1,k_2,\ldots,k_m}, \tag{6.10.42}$$

where

$$\begin{aligned} N_{12\cdots m;k_1,k_2,\ldots,k_m} &= E_m V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}), \\ A_{12\cdots m;k_1,k_2,\ldots,k_m} &= (-1)^R M_{12\cdots m;k_1,k_2,\ldots,k_m} \\ &= (-1)^R V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}}), \end{aligned} \tag{6.10.43}$$

where M is the unsigned minor associated with the cofactor A and R is the sum of their parameters. Referring to (6.10.39),

$$\begin{aligned} R &= \tfrac{1}{2}m(m+1) + \sum_{r=1}^{m} k_r \\ &= K + \tfrac{1}{2}m(m-1). \end{aligned} \tag{6.10.44}$$

Hence,

$$\begin{aligned} H_{2n}^{(m)}(\varepsilon) &= (-1)^R \sum_{1 \le k_1 < k_2 < \cdots < k_m \le 2n} E_m V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) \\ &\quad \cdot V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}}) \\ &= \frac{(-\rho^2)^{m(m-1)/2}}{\{V_{2n}(\mathbf{c})\}^{m-1}} W_m, \end{aligned} \tag{6.10.45}$$

which proves part (a) of the theorem. Part (b) can be proved in a similar manner. □

6.10.5 Physically Significant Solutions

From the theorem in Section 6.10.2 on the intermediate solution,

$$\phi_{2n+1} = \frac{\rho^{2n-1} A_{2n}}{A_{2n-1}},$$
$$\psi_{2n+1} = \frac{\omega \rho^{2n-1} A^{(2n+1)}_{1,2n+1}}{A_{2n-1}} \quad (\omega^2 = -1). \tag{6.10.46}$$

Hence the functions ζ_+ and ζ_- introduced in Section 6.2.8 can be expressed as follows:

$$\begin{aligned}\zeta_+ &= \phi_{2n+1} + \omega\psi_{2n+1}\\ &= \frac{\rho^{2n-1}(A_{2n} - A^{(2n+1)}_{1,2n+1})}{A_{2n-1}},\end{aligned} \tag{6.10.47}$$
$$\begin{aligned}\zeta_- &= \phi_{2n+1} - \omega\psi_{2n+1}\\ &= \frac{\rho^{2n-1}(A_{2n} + A^{(2n+1)}_{1,2n+1})}{A_{2n-1}}.\end{aligned} \tag{6.10.48}$$

It is shown in Section 4.5.2 on symmetric Toeplitz determinants that if $A_n = |t_{|i-j|}|_n$, then

$$\begin{aligned}A_{2n-1} &= 2P_{n-1}Q_n,\\ A_{2n} &= P_nQ_n + P_{n-1}Q_{n+1},\\ A^{(2n+1)}_{1,2n+1} &= P_nQ_n - P_{n-1}Q_{n+1},\end{aligned} \tag{6.10.49}$$

where

$$\begin{aligned}P_n &= \tfrac{1}{2}\left|t_{|i-j|} - t_{i+j}\right|_n\\ Q_n &= \tfrac{1}{2}\left|t_{|i-j|} + t_{i+j-2}\right|_n.\end{aligned} \tag{6.10.50}$$

Hence,

$$\begin{aligned}\zeta_+ &= \frac{\rho^{2n-1}Q_{n+1}}{Q_n},\\ \zeta_- &= \frac{\rho^{2n-1}P_n}{P_{n-1}}.\end{aligned} \tag{6.10.51}$$

In the present problem, $t_r = \omega^r u_r$ $(\omega^2 = -1)$, where u_r is a solution of the coupled equations (6.10.3) and (6.10.4). In order to obtain the Neugebauer solutions, it is necessary first to choose the solution given by equations (A.11.8) and (A.11.9) in Appendix A.11, namely

$$u_r = (-1)^r \sum_{j=1}^{2n} \frac{e_j f_r(x_j)}{\sqrt{1+x_j^2}}, \quad x_j = \frac{z + c_j}{\rho}, \tag{6.10.52}$$

and then to choose

$$e_j = (-1)^{j-1} M_j(\mathbf{c}) e^{\omega\theta_j}. \tag{6.10.53}$$

Denote this particular solution by U_r. Then,

$$t_r = (-\omega)^r U_r,$$

where

$$U_r = \sum_{j=1}^{2n} \frac{(-1)^{j-1} M_j(\mathbf{c}) f_r(x_j)}{\varepsilon_j^*} \tag{6.10.54}$$

and the symbol $*$ denotes the complex conjugate. This function is of the form (4.13.3), where

$$a_j = \frac{(-1)^{j-1} M_j(\mathbf{c})}{\varepsilon_j^*} \tag{6.10.55}$$

and $N = 2n$. These choices of a_j and N modify the function k_r defined in (4.13.5). Denote the modified k_r by w_r, which is given explicitly in (6.10.3).

Since the results of Section 4.13.2 are unaltered by replacing ω by $(-\omega)$, it follows from (4.13.22) and (4.13.23) with $n \to m$ that

$$\begin{aligned} P_m &= (-1)^{m(m-1)/2} 2^{m^2-1} \big| w_{i+j} + w_{i+j-2} \big|_m, \\ Q_m &= (-1)^{m(m-1)/2} 2^{(m-1)^2} \big| w_{i+j-2} \big|_m. \end{aligned} \tag{6.10.56}$$

Applying the theorem in Section 6.10.4,

$$\begin{aligned} P_m &= 2^{m^2-1} \rho^{-m(m-1)} \big\{ V_{2n}(\mathbf{c}) \big\}^{m-1} H_{2n}^{(m)}(\varepsilon), \\ Q_m &= 2^{(m-1)^2} \rho^{-m(m-1)} \big\{ V_{2n}(\mathbf{c}) \big\}^{m-1} H_{2n}^{(m)} \left(\frac{1}{\varepsilon^*} \right). \end{aligned} \tag{6.10.57}$$

Hence,

$$\frac{P_n}{P_{n-1}} = 2^{2n-1} \rho^{-2(n-1)} V_{2n}(\mathbf{c}) \frac{H_{2n}^{(n)}(\varepsilon)}{H_{2n}^{(n-1)}(\varepsilon)}. \tag{6.10.58}$$

Also, applying (6.10.32),

$$\begin{aligned} \frac{Q_{n+1}}{Q_n} &= 2^{2n-1} \rho^{-2n} V_{2n}(\mathbf{c}) \frac{H_{2n}^{(n+1)}\big(1/\varepsilon^*\big)}{H_{2n}^{(n)}\big(1/\varepsilon^*\big)} \\ &= -2^{2n-1} \rho^{-2n} V_{2n}(\mathbf{c}) \frac{H_{2n}^{(n-1)}(\varepsilon^*)}{H_{2n}^{(n)}(\varepsilon^*)}. \end{aligned} \tag{6.10.59}$$

Since $\tau_j = \rho\varepsilon_j$, (the third line of (6.10.26)), the functions F and G defined in Section 6.2.8 are given by

$$\begin{aligned} F &= H_{2n}^{(n-1)}(\rho\varepsilon) = \rho^{n-1} H_{2n}^{(n-1)}(\varepsilon), \\ G &= H_{2n}^{(n)}(\rho\varepsilon) = \rho^n H_{2n}^{(n)}(\varepsilon). \end{aligned} \tag{6.10.60}$$

Hence,

$$\begin{aligned}
\frac{P_n}{P_{n-1}} &= \left\{\frac{2}{\rho}\right\}^{2n-1} V_{2n}(\mathbf{c}) \left\{\frac{G}{F}\right\}, \\
\frac{Q_{n+1}}{Q_n} &= -\left\{\frac{2}{\rho}\right\}^{2n-1} V_{2n}(\mathbf{c}) \left\{\frac{F^*}{G^*}\right\}
\end{aligned} \tag{6.10.61}$$

$$\begin{aligned}
\zeta_+ &= -2^{2n-1} V_{2n}(\mathbf{c}) \left\{\frac{F^*}{G^*}\right\}, \\
\zeta_- &= 2^{2n-1} V_{2n}(\mathbf{c}) \left\{\frac{G}{F}\right\}.
\end{aligned} \tag{6.10.62}$$

Finally, applying the Bäcklund transformation ε in Appendix A.12 with $b = 2^{2n-1} V_{2n}(\mathbf{c})$,

$$\begin{aligned}
\zeta'_+ &= \frac{\zeta_- - 2^{2n-1} V_{2n}(\mathbf{c})}{\zeta_- + 2^{2n-1} V_{2n}(\mathbf{c})} \\
&= \frac{1 - (F/G)}{1 + (F/G)}.
\end{aligned}$$

Similarly,

$$\zeta'_- = \frac{1 - (F^*/G^*)}{1 + (F^*/G^*)}. \tag{6.10.63}$$

Discarding the primes, $\zeta_- = \zeta_+^*$. Hence, referring to (6.2.13),

$$\begin{aligned}
\phi &= \tfrac{1}{2}(\zeta_+ + \zeta_-) = \tfrac{1}{2}(\zeta_+ + \zeta_+^*), \\
\psi &= \frac{1}{2\omega}(\zeta_+ - \zeta_-) = \frac{1}{2\omega}(\zeta_+ - \zeta_+^*) \quad (\omega^2 = -1),
\end{aligned} \tag{6.10.64}$$

which are both real. It follows that these solutions are physically significant.

Exercise. Prove the following identities:

$$\begin{aligned}
A_{2n} &= \alpha_n (GG^* - FF^*), \\
A_{2n+1} &= \beta_n F^* G, \\
A_{2n-1} &= \beta_{n-1} F G^*, \\
A_{1,2n+1}^{(2n+1)} &= \alpha_n (GG^* + FF^*),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_n &= \frac{(-1)^n 2^{2n(n-1)} V_{2n}^{2(n-1)}}{\rho^{2n(n-1)} \prod_{i=1}^{2n} \varepsilon_i}, \\
\beta_n &= \frac{(-1)^{n-1} 2^{2n^2} V_{2n}^{2n-1}}{\rho^{2n^2-1} \prod_{i=1}^{2n} \varepsilon_i^*}.
\end{aligned}$$

6.10.6 The Ernst Equation

The Ernst equation, namely

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*(\nabla\xi)^2,$$

is satisfied by each of the functions

$$\xi_n = \frac{pU_n(x) - \omega q U_n(y)}{U_n(1)} \quad (\omega^2 = -1), \quad n = 1, 2, 3, \ldots,$$

where $U_n(x)$ is a determinant of order $(n+1)$ obtained by bordering an nth-order Hankelian as follows:

$$U_n(x) = \left| \begin{array}{cc} & x \\ & x^3/3 \\ [a_{ij}]_n & x^5/5 \\ & \cdots \\ & x^{2n-1}/(2n-1) \\ 1\ 1\ 1 \cdots 1 & \bullet \end{array} \right|_{n+1},$$

where

$$\begin{aligned} a_{ij} &= \frac{1}{i+j-1}[p^2 x^{2(i+j-1)} + q^2 y^{2(i+j-1)} - 1], \\ p^2 + q^2 &= 1, \end{aligned}$$

and x and y are prolate spheriodal coordinates. The argument x in $U_n(x)$ refers to the elements in the last column, so that $U_n(1)$ is the determinant obtained from $U_n(x)$ by replacing the x in the last column *only* by 1. A note on this solution is given in Section 6.2 on brief historical notes. Some properties of $U_n(x)$ and a similar determinant $V_n(x)$ are proved in Section 4.10.3.

6.11 The Relativistic Toda Equation — A Brief Note

The relativistic Toda equation in a function R_n and a substitution for R_n in terms of U_{n-1} and U_n are given in Section 6.2.9. The resulting equation can be obtained by eliminating V_n and W_n from the equations

$$H_x^{(2)}(U_n, U_n) = 2(V_n W_n - U_n^2), \tag{6.11.1}$$

$$aH_x^{(2)}(U_n, U_{n-1}) = aU_n U_{n-1} + V_n W_{n-1}, \tag{6.11.2}$$

$$V_{n+1}W_{n-1} - U_n^2 = a^2(U_{n+1}U_{n-1} - U_n^2), \tag{6.11.3}$$

where $H_x^{(2)}$ is the one-variable Hirota operator (Section 5.7),

$$a = \frac{1}{\sqrt{1+c^2}},$$

$$x = t\sqrt{1-a^2} = \frac{ct}{\sqrt{1+c^2}}\,. \tag{6.11.4}$$

Equations (6.11.1)–(6.11.3) are satisfied by the functions

$$\begin{aligned} U_n &= |u_{i,n+j-1}|_m, \\ V_n &= |v_{i,n+j-1}|_m, \\ W_n &= |w_{i,n+j-1}|_m, \end{aligned} \tag{6.11.5}$$

where the determinants are Casoratians (Section 4.14) of arbitrary order m whose elements are given by

$$\begin{aligned} u_{ij} &= F_{ij} + G_{ij}, \\ v_{ij} &= a_i F_{ij} + \frac{1}{a_i} G_{ij}, \\ w_{ij} &= \frac{1}{a_i} F_{ij} + a_i G_{ij}, \end{aligned} \tag{6.11.6}$$

where

$$\begin{aligned} F_{ij} &= \left(\frac{1}{a_i - a}\right)^j \exp(\xi_i), \\ G_{ij} &= \left(\frac{a_i}{1 - aa_i}\right)^j \exp(\eta_i), \\ \xi_i &= \frac{x}{a_i} + b_i, \\ \eta_i &= a_i x + c_i, \end{aligned} \tag{6.11.7}$$

and where the a_i, b_i, and c_i are arbitrary constants.