CHAPTER 4

Optimal *f*

OPTIMAL FIXED FRACTION

We have seen that in order to consider betting/trading a given situation/system you must first determine if a positive mathematical expectation exists. We have seen in the previous chapter that what is seemingly a "good bet" on a mathematical expectation basis (i.e., the mathematical expectation is positive) may in fact not be such a good bet when you consider reinvestment of returns.[1] Reinvesting returns never raises the mathematical expectation (as a percentage—although it can raise the mathematical expectation in terms of dollars, which it does geometrically, which is why we want to reinvest). If there is in fact a positive mathematical expectation, however small, the next step is to exploit this positive expectation to its fullest potential. This has been shown, for an independent trials process, to be by reinvesting a fixed fraction of your total stake,[2] which leads to the following axiom: *For any given independent trials situation where you*

[1]If you are reinvesting too high a percentage of your winnings relative to the dispersion of outcomes of the system.

[2]For a dependent trials process the idea of betting a proportion of your total stake also yields the greatest exploitation of a positive mathematical expectation, just like an independent trials process. However, in a dependent trials process you optimally bet a variable fraction of your total stake, the exact fraction for each individual bet determined by the probabilities and payoffs involved for each individual bet.

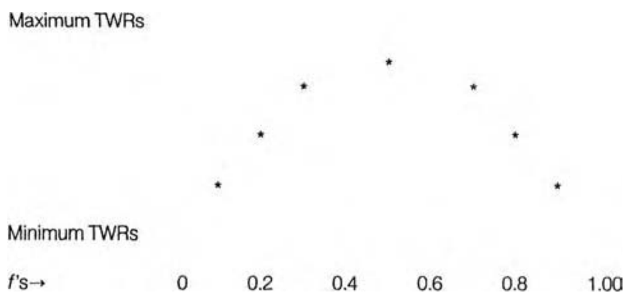

FIGURE 4.1 The curve of optimal *f*

have an edge (i.e., a positive mathematical expectation), there exists an optimal fixed fraction (f) *between 0 and 1 as a divisor of your biggest loss to bet on each and every event.*

Most people think that the optimal fixed fraction is the percentage of your total stake to bet. Optimal *f* is not in itself the percentage of our total stake to bet; it is the divisor of our biggest loss, the result of which we divide our total stake by to know how many bets to make or contracts to have on.

You will also notice that *margin has nothing to do whatsoever with what is the mathematically optimal number of contracts to have on.*

As you can see in Figure 4.1, *f* is a curve cupped downward from 0 to 1. The highest point for *f* is that fraction of your stake to bet on each and every event (bet) to maximize your winnings.

Most people incorrectly believe that *f* is a straight-line function rising up and to the right. They believe this because they think it would mean that the more you are willing to risk, the more you stand to make. People reason this way because they think that a positive mathematical expectation is just the mirror image of a negative expectancy. They mistakenly believe that if increasing your total action in a negative expectancy game results in losing faster, then increasing your total action in a positive expectancy game will result in winning faster. This is not true. At some point in a positive expectancy situation, to increase your total action further works against you. That point is a function of both the system's profitability and its consistency (i.e., its geometric mean), since you are reinvesting the returns.

ASYMMETRICAL LEVERAGE

Recall that the amount required to recoup a loss increases geometrically with the loss. We can show that the percentage gain to recoup a loss is:

$$\text{Required Gain} = (1/(1 - \text{loss in percent})) - 1 \qquad (4.01)$$

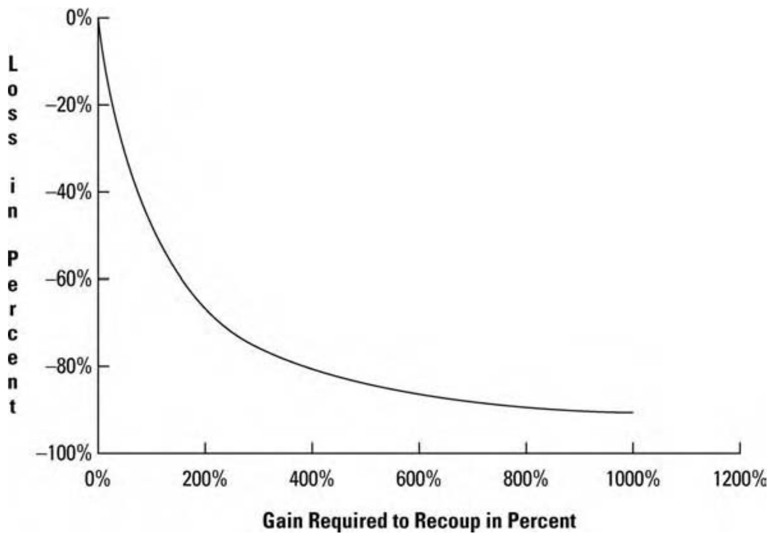

FIGURE 4.2 Asymmetrical leverage

A 20% loss requires a 25% gain afterwards to recoup. A 30% loss requires a 42% gain afterwards to recoup. This is asymmetrical leverage. In fixed fractional trading we have seen that the trader will tend to have on more contracts when she takes a loss than when she has a win. This is what amplifies the asymmetrical leverage. It is also what curves the f function, since the peak of the f function represents that point where the trader has the right amount of contracts on to go into the losses and come out of the losses (with asymmetrical leverage) and achieve the maximum growth on her money at the end of a sequence of trades (see Figure 4.2). The f value (X axis) that corresponds to the peak of this f curve will be known as the optimal f (f is always in lowercase).

So f is a curved-line function, and this is due, in part, to the fact that asymmetrical leverage is amplified when reinvesting profits.

And how do we find this optimal f? Much work has been done in recent decades on this topic in the gambling community, the most famous and accurate of which is known as the Kelly Betting System. This is actually an application of a mathematical idea developed in early 1956 by John L. Kelly, Jr., and published in the July 1956 *Bell System Technical Journal.*[3]

[3]Kelly, J. L., Jr., "A New Interpretation of Information Rate," *Bell System Technical Journal*, pp. 917–926, July 1956.

The Kelly criterion states that we should bet that fixed fraction of our stake (f) which maximizes the growth function $G(f)$:

$$G(f) = P * \ln(1 + B * f) + (1 - P) * \ln(1 - f) \tag{4.02}$$

where: f = The optimal fixed fraction.
P = The probability of a winning bet/trade.
B = The ratio of amount won on a winning bet to amount lost on a losing bet.
ln( ) = The natural logarithm function to the base $e = 2.71828\ldots$

As it turns out, for an event with two possible outcomes, this optimal f can be found quite easily with the Kelly formulas.

KELLY

Beginning around the late 1940s, Bell System engineers were working on the problem of data transmission over long distance lines. The problem facing them was that the lines were subject to seemingly random, unavoidable "noise" that would interfere with the transmission. Some rather ingenious solutions were proposed by engineers at Bell Labs. Oddly enough, there are great similarities between this data communications problem and the problem of geometric growth as it pertains to gambling money management (as both problems are the product of an environment of favorable uncertainty). The Kelly formula is one of the outgrowths of these solutions.

The first equation here is:

$$f = 2 * P - 1 \tag{4.03}$$

where: f = The optimal fixed fraction.
P = The probability of a winning bet/trade.

This formula will yield the correct answer for optimal f provided the sizes of wins and losses are the same. As an example, consider the following stream of bets:

$$-1, +1, +1, -1, -1, +1, +1, +1, +1, -1$$

There are 10 bets, 6 winners, hence:

$$\begin{aligned} f &= (.6 * 2) - 1 \\ &= 1.2 - 1 \\ &= .2 \end{aligned}$$

If the winners and losers were not all the same size, this formula would not yield the correct answer. Such a case would be our two-to-one coin-toss example, where all of the winners were for 2 units and all of the losers for 1 unit. For this situation the Kelly formula is:

$$f = ((B + 1) * P - 1)/B \tag{4.04}$$

where: f = The optimal fixed fraction.
P = The probability of a winning bet/trade.
B = The ratio of amount won on a winning bet to amount lost on a losing bet.

For the two-to-one coin toss:

$$\begin{aligned} f &= ((2 + 1) * .5 - 1)/2 \\ &= (3 * .5 - 1)/2 \\ &= (1.5 - 1)/2 \\ &= .5/2 \\ &= .25 \end{aligned}$$

This formula will yield the correct answer for optimal f, provided all wins are always for the same amount and all losses are always for the same amount. If this condition is not met, the formula will not yield the correct answer.

Consider the following sequence of bets/trades:

$$+9, +18, +7, +1, +10, -5, -3, -17, -7$$

Since all wins and all losses are of different amounts, the previous formula does not apply. However, let's try it anyway and see what we get.

Since five of the nine events are profitable, P = .555. Now let's take averages of the wins and losses to calculate B (here is where so many traders go wrong). The average win is 9 and the average loss is 8. Therefore, we will say that B = 1.125. Plugging in the values we obtain:

$$\begin{aligned} f &= ((1.125 + 1) * .555 - 1)/1.125 \\ &= (2.125 * .555 - 1)/1.125 \\ &= (1.179375 - 1)/1.125 \\ &= .179375/1.125 \\ &= .159444444 \end{aligned}$$

So we say $f = .16$. We will see later in this chapter that this is not the optimal f. The optimal f for this sequence of trades is .24. Applying Kelly when wins are not all for the same amount and/or losses are not all for the same amount is a mistake. It will not yield the optimal f.

Notice that the numerator in this formula equals the mathematical expectation for an event with two possible outcomes as defined in Chapter 1. Therefore, we can say that as long as all wins are for the same amount and all losses are for the same amount (regardless of whether the amount that can be won equals the amount that can be lost), the optimal f is:

$$f = \text{Mathematical Expectation/B} \tag{4.05}$$

where: f = The optimal fixed fraction.
B = The ratio of amount won on a winning bet to amount lost on a losing bet.

FINDING THE OPTIMAL *f* BY THE GEOMETRIC MEAN

In trading we can count on our wins being for various amounts and our losses being for various amounts. Therefore, the Kelly formula cannot give us the correct optimal f. How then can we find the optimal f to tell us how many contracts to have on and have it be mathematically correct?

As you will see later in this chapter, trading the correct quantities of contracts/shares is a far bigger problem than was previously thought. Quantity can mean the difference between winning and losing. All systems experience losing trades. All systems experience drawdown. These are givens, facts of life. Yet if you can always have the right amount of contracts on (i.e., the mathematically correct amount), then there is consolation in the losses.

Now here is the solution. To begin with, we must amend our formula for finding HPRs to incorporate f.

$$\text{HPR} = 1 + f * (-\,\text{trade/biggest loss}) \tag{4.06}$$

And again, TWR is simply the geometric product of the HPRs and working from (3.03), geometric mean is simply the Nth root of the TWR.

$$\text{TWR} = \prod_{i=1}^{N} (1 + f * (-\text{trade}_i/\text{biggest loss})) \tag{4.07}$$

$$\text{Geo. Mean} = \left(\prod_{i=1}^{N} (1 + f * (-\,\text{trade}_i/\text{biggest loss}))\right)^{1/N} \tag{4.08}$$

The geometric mean can also be calculated here by the procedure for finding the estimated geometric mean by using the HPRs as formulated above, or by taking the TWR, as formulated above, as an input to the equation:

$$\text{Geo. Mean} = \exp((1/N) * \log(\text{TWR}))$$

where: N = Total number of trades.
log(TWR) = The log base 10 of the TWR.
exp = The exponential function.

By looping through all values for f *between .01 and 1, we can find that value for* f *which results in the highest TWR.* This is the value for f that would provide us with the maximum return on our money using fixed fraction. We can also state that the optimal f is the f that yields the highest geometric mean. It matters not whether we look for highest TWR or geometric mean, as both are maximized at the same value for f.

Doing this with a computer is easy. Simply loop from $f = .01$ to $f = 1.0$ by .01. As soon as you get a TWR that is less than the previous TWR, you know that the f corresponding to the previous TWR is the optimal f. You can also calculate this by hand, but that is a lot more tedious, especially as the number of trades increases. A quicker way to do it is to use iteration to solve for the optimal f (you can use the iterative approach whether you are doing it by hand or by computer). Here, you are initially bounded on f at $f = 0$ and $f = 1.00$. Pick a start value, say $f = .10$, and find the corresponding TWR. Now step the f value up an arbitrary amount. The example that follows steps it up by .10, but you can use any amount you want to (so long as you do not have an f value greater than 1.00, the upper bound). As long as your TWRs keep increasing, you must step up the f value you are testing. Do this until $f = .30$, where your TWR is less than at $f = .20$. Now, your f bounds are .20 and .30. Keep on repeating the process until you zero in on the optimal f. The following illustration demonstrates the iterative process as well as the calculations:

At $f = .10$

TRADE	HPR	
9	1.052941	
18	1.105882	The HPRs are equal to 1 + (f * (−trade/biggest loss))
7	1.041176	
1	1.005882	
10	1.058823	
−5	0.970588	
−3	0.982352	
−17	0.9	
−7	0.958823	
	TWR = 1.062409	The TWR is all of the HPRs multiplied together

At $f = .20$

TRADE	HPR
9	1.105882
18	1.211764
7	1.082352
1	1.011764
10	1.117647
−5	0.941176
−3	0.964705
−17	0.8
−7	0.917647
	TWR = 1.093231

At $f = .30$

TRADE	HPR
9	1.158823
18	1.317647
7	1.123529
1	1.017647
10	1.176470
−5	0.911764
−3	0.947058
−17	0.7
−7	0.876470
	TWR = 1.088113

At $f = .25$

TRADE	HPR
9	1.132352
18	1.264705
7	1.102941
1	1.014705
10	1.147058
−5	0.926470
−3	0.955882
−17	0.75
−7	0.897058
	TWR = 1.095387

At $f = .23$

TRADE	HPR
9	1.121764
18	1.243529
7	1.094705
1	1.013529
10	1.135294
−5	0.932352
−3	0.959411
−17	0.77
−7	0.905294
TWR =	1.095634

At $f = .24$

TRADE	HPR
9	1.127058
18	1.254117
7	1.098823
1	1.014117
10	1.141176
−5	0.929411
−3	0.957647
−17	0.76
−7	0.901176
TWR =	1.095698

TO SUMMARIZE THUS FAR

In the previous chapter we demonstrated that a good system is the one with the highest geometric mean. Yet, to find the geometric mean you must know f. Understandably, the reader must be confused. Here now is a summary and clarification of the process:

1. Take the trade listing of a given market system.
2. Find the optimal f, either by testing various f values from 0 to 1 or through iteration. The optimal f is that which yields the highest TWR.
3. Once you have found f you can take the Nth root of that TWR corresponding to your f, where N is the total number of trades. This is your geometric mean for this market system. You can now use this geometric

mean to make apples-to-apples comparisons with other market systems, as well as use the f to know how many contracts to trade for that particular market system.

Once the highest f *is found, it can readily be turned into a dollar amount by dividing the biggest loss by the negative optimal* f. For example, if our biggest loss is \$100 and our optimal f is .25, then $-\$100/-.25 = \$400$. In other words, we should bet one unit for every \$400 we have in our stake.

In the sequence of bets that our coin-toss example would generate, we find that the optimal f value for the sequence $+2$, -1 is .25. Since our biggest loss is \$1, $1/.25 = \$4$. In other words, we should bet \$1 for every \$4 we have in our stake in order to make the most money out of this game. To bet a higher number or a lower number will not result in a greater gain! After 10 bets, for every \$4 we started out with in our stake, we will have \$9.

This approach to finding the optimal f will yield the same result as:

$$f = ((B + 1) * P - 1)/B$$

You obtain the same result, of course, when losses are all for the same amount and wins are all for the same amount. In such a case, either technique is correct. When both wins and losses are for the same amount, you can use any of the three methods—the Kelly formula just shown, the f that corresponds to the highest TWR, or:

$$f = 2 * P - 1$$

Any of the three methods will give you the same answer when all wins and losses are for the same amount.

All three methods for finding the optimal f meet the four desirable properties of a money-management strategy outlined earlier, given the constraints of the two formulas (i.e., all wins being for the same amount and all losses being for the same amount, or all wins and losses being for the same amount). Regardless of constraints, the optimal f via the highest TWR will always meet the four desirable properties of a money-management strategy.

If you're having trouble with some of these concepts, try thinking in terms of betting in units, not dollars (e.g., one \$5 chip or one futures contract or one 100-share unit of stock). The amount of dollars you allocate to each unit is calculated by figuring your largest loss divided by the negative optimal f.

The optimal f is a result of the balance between a system's profit-making ability (on a constant one-unit basis) and its risk (on a constant one-unit basis). Notice that margin doesn't matter, because the size of individual profits and losses are not the product of the amount of money put up as margin (they would be the same whatever the size of the margin). Rather, the profits and losses are the product of the exposure of one unit (one futures contract). The amount put up as margin is further made meaningless in a money-management sense, since the size of the loss is not limited to the margin.

HOW TO FIGURE THE GEOMETRIC MEAN USING SPREADSHEET LOGIC

Here is an example of how to use a spreadsheet like to calculate the geometric mean and TWR when you know the optimal f or want to test a value for f.

(Assume $f = .5$, biggest loss $= -50$)

	col A	col B	col C	col D	col E
row 1					1
row 2	15	0.3	0.15	1.15	1.15
row 3	−5	−0.1	−0.05	0.95	1.0925

cell(s) explanation

A1 through D1 are blank.
E1 Set equal to 1 to begin with.
A2 down These are the individual trade P&Ls.
B2 down = A2/abs value of (biggest loss)
C2 down = B2/f
D2 down = C2 + 1
E2 down = E1 * D2

When you get to the end of the trades (the last row), your last value in column E is your TWR. Now take the Nth root of this TWR (N is the total number of trades); that is your geometric mean. In the above example, your TWR (cell E3) raised to the power 1/2 (there are a total of two trades here) = 1.045227. That is your geometric mean.

GEOMETRIC AVERAGE TRADE

At this point you may be interested in figuring your geometric average trade. That is, what is the average garnered per contract per trade, assuming profits are always reinvested and fractional contracts can be purchased. In effect, this is the mathematical expectation when you are trading on a fixed fractional basis. *This figure shows you what effect there is from losers occurring when you have many contracts on and winners occurring when you have fewer contracts on. In effect, this approximates how a system would have fared per contract per trade doing fixed fraction.* (Actually, the geometric average trade is your mathematical expectation in dollars per contract per trade. The geometric mean minus 1 is your percent mathematical expectation per trade—e.g., a geometric mean of, say, 1.025 represents a mathematical expectation of 2.5% per trade, irrespective of size.) So many traders look simply at the average trade of a market system to see if it is high enough to justify trading the system. However, in making their decision, they should be looking at the geometric average trade (which is never greater than the average trade) as well as at the PRR.

$$\text{Geo. Avg. Trade} = G * (\text{biggest loss}/-f) \tag{4.09}$$

where: G = Geometric mean −1.
f = Optimal fixed fraction.
(And, of course, our biggest loss is always a negative number.)

For example, suppose a system has a geometric mean of 1.017238, the biggest loss is $8,000, and the optimal f is .31. Our geometric average trade would equal:

$$\begin{aligned}\text{Geo. Avg. Trade} &= (1.017238 - 1) * (-8{,}000/-.31)\\ &= .017238 * 25{,}806.45\\ &= \$444.85\end{aligned}$$

A SIMPLER METHOD FOR FINDING THE OPTIMAL *f*

There are numerous ways to arrive at the optimal value for f. The technique for finding the optimal f that has been presented thus far in this chapter is perhaps the most mathematically logical. That is to say, it is obvious upon inspection that this technique will yield the optimal f. It makes more intuitive sense when you can see the HPRs laid out than does the next and somewhat easier method. So here is another way for calculating

the optimal f, one that some readers may find simpler and more to their liking. It will give the same answer for optimal f as the technique previously described.

Under this method we still need to loop through different test values for f to see which value for f results in the highest TWR. However, we calculate our TWR without having to calculate the HPRs. Let's assume the following stream of profits and losses from a given system:

+$100
−$500
+$1500
−$600

Again, we must isolate the largest losing trade. This is −$600.

Now we want to obtain the TWR for a given test value for f. Our first step is to calculate what we'll call the *starting value*. To begin with, take the largest loss and divide it by the test value for f. Let's start out by testing a value of .01 for f. So we will divide the largest loss, –$600, by .01. This yields an answer of –$60,000. Now we make it a positive value. Therefore, our starting value for this example sequence of a .01 test value for f is $60,000.

For each trade we must now calculate a *working value*. To do this, for each trade we must take the previous working value and divide it by the starting value. (For the first trade, the answer will be 1, since the previous working value is the same as the starting value.) Next, we multiply the answer by the current trade amount. Finally, we add this answer and the previous working value to obtain the current working value.

P&L	WORKING VALUE	
	60000 ←	This is the starting value
+100	60100	
−500	59599.166667	
+1500	61089.14583	
−600	60478.25437	

Our TWR is obtained simply by taking the last entry in the working value column and dividing it by our starting value. In this instance:

$$\begin{aligned} \text{TWR} &= 60478.25437/60000 \\ &= 1.007970906 \end{aligned}$$

Now we repeat the process, only we must increment our test value for f. This time through, rather than dividing the absolute value of the largest loss of −\$600 by .01 we will divide it by .02. Therefore, we will begin this next pass through with a starting value of 600/.02 = 30000.

P&L	WORKING VALUE	
	30000	
+100	30100	((30000/30000) * 100) + 30000
−500	29598.33	((30100/30000) * −500) + 30100
+1500	31078.2465	((29598.33/30000) * 1500) + 29598.33
−600	30456.68157	((31078.2465/30000) * −600) + 31078.2465

Here the TWR = 30456.68157/30000 = 1.015222719

We keep on repeating the procedure until we obtain the value for f that results in the highest TWR. The answers we obtain for TWRs, as well as for the optimal f, will be the same with this technique as with the previous technique using the HPRs.

THE VIRTUES OF THE OPTIMAL *f*

It is a mathematical fact that when two people face the same sequence of favorable betting/trading opportunities, if one uses the optimal f and the other uses any different money-management system, then the ratio of the optimal f bettor's stake to the other person's stake will increase as time goes on, with higher and higher probability. In the long run, the optimal f bettor will have infinitely greater wealth than any other money-management-system bettor with a probability approaching one.

Furthermore, if a bettor has the goal of reaching a prespecified fortune, and is facing a series of favorable betting/trading opportunities, the expected time needed to reach the fortune will be less with optimal f than with any other betting system.

Obviously, the optimal f strategy satisfies desirable property number 1 for money management, as it makes the most amount of money that is mathematically possible using a fixed fractional betting strategy on a game where you have the edge. Since optimal f incorporates the probability of winning as well as the amounts won and lost, it also satisfies desirable property numbers 3 and 4. Not much has been discussed about desirable property number 2, the security aspect, but this will be treated in the closing chapters.

Let's go back and reconsider the following sequence of bets/trades:

$$+9, +18, +7, +1, +10, -5, -3, -17, -7$$

Recall that we determined earlier in this chapter that the Kelly formula did not apply to this sequence, since wins were not all for the same amount and losses were not all for the same amount. We also decided to average the wins and average the losses and take these averages as our values into the Kelly formula (as many traders mistakenly do). Doing this we arrived at an f value of .16. It was stated that this is an incorrect application of Kelly, that it would not yield the optimal f. The Kelly formula must be specific to a single bet. We cannot average our wins and losses from trading and obtain the true optimal f using the Kelly formula.

Our highest TWR on this sequence of bets/trades is obtained at .24, or betting \$1 for every \$71 in our stake. That is the optimal geometric growth we can squeeze out of this sequence of bets/trades trading fixed fraction. Let's look at the TWRs at different points along 100 loops through this sequence of bets.

At one loop through, nine bets/trades, the TWR for $f = .16$ is 1.085; for $f = .24$ it is 1.096. This means that for one pass through this sequence of bets an $f = .16$ made 99% of what an $f = .24$ would have made. To continue:

Passes Through	Total Bets/Trades	TWR for $f = .24$	TWR for $f = .16$	Percentage Difference
1	9	1.096	1.085	1%
10	90	2.494	2.261	9.4%
40	360	38.694	26.132	32.5%
100	900	9313.312	3490.761	62.5%

As can be seen, using an f value that we mistakenly figured from Kelly made only 37.5% as much as our optimal f of .24 after 900 bets/trades (100 cycles through the series of nine outcomes). In other words, our optimal f of .24 (which is only .08 more than .16) made almost 267% the profit that $f = .16$ did after 900 bets!

Let's go another 11 cycles through this sequence of trades, so we have a total of 999 trades. Now our TWR for $f = .16$ is 8563.302 (not even what it was for $f = .24$ at 900 trades) and our TWR for $f = .24$ is 25,451.045. At 999 trades $f = .16$ is only 33.6% of $f = .24$, or $f = .24$ is 297% of $f = .16$! Here you can see that using the Kelly formula does not yield the true optimal f for trading.

As can be seen from the above, using the optimal f *does not appear to offer much advantage over the short run, but over the long run it becomes* *more and more important to use the optimal* f. *The point is you must give the program time when trading at the optimal* f *and not expect miracles in the short run. The more time (i.e., bets/trades) that elapses, the greater the difference between using the optimal* f *and any other money-management strategy.*

WHY YOU MUST KNOW YOUR OPTIMAL *f*

Figures 4.3 through 4.6 demonstrate the importance of using optimal f in fixed fractional trading. The graphs are constructed by plotting the f values from 0 to 1.0 along the X axis and the respective TWRs along the Y axis. Values are plotted at intervals of .05.

Each graph has a corresponding spreadsheet. Each column heading in the spreadsheet has a different f value. Under each f value is the corresponding start value, figured as the biggest loss divided by the negative f value. For every unit of start value you have in your stake, you bet one unit. Along the far left is the sequence of 40 bets. This sequence is the only difference between the various spreadsheets and graphs.

As you go down through the sequence of trades you will notice that each cell equals the previous cell divided by that cell's starting value. This result is then multiplied by the outcome of the current bet, and the product added to the original value of the previous cell, to obtain the value of the current cell. When you reach the end of the column you can figure your TWR as the last value of the column divided by the start value of the column (i.e., the biggest loss divided by negative f). This is the alternative and somewhat easier way to figure your TWRs. Both methods shown thus far make the calculations non-quantum. In other words, you do not need an integer amount to multiply the current bet result by; you can use a decimal amount of the starting value as well. An example may help clarify.

In the +1.2, −1 sequence (Figure 4.3), for an f value of .05, we have a starting value of 20:

$$-1/-.05 = 20$$

In other words, we will bet one unit for every 20 units in our stake. With the first bet, a gain of 1.2, we now have 21.2 units in our stake. (Since we had 20 units in our stake prior to this bet and we bet one unit for every 20 in our stake, we bet only one unit on this bet.) Now the next bet is a loss of one unit. The question now is, "How many units were we betting on this one?"

We could argue that we were betting only one unit, since 21.20 (our stake prior to the bet) divided by 20 (the starting value) = 1.06. Since most bets must be in integer form—that is, no fractional bets (chips are not divisible and neither are futures contracts)—we could bet only one unit in real life in this situation. However, in these simulations the fractional bet is allowed. The reasoning here behind allowing the fractional bet is to keep the outcome consistent regardless of the starting stake. Notice that each simulation starts with only enough stake to make one full bet. What if each simulation started with more than that? Say each simulation started with enough to make 1.99 bets. If we were only allowing integer bets, our outcomes (TWRs) would be altogether different.

Further, the larger the amount we begin trading with is, relative to the starting value (biggest loss/ − optimal f), the closer the integer bet will be to the fractional bet. Again, clarity is provided by way of an example. What if we began trading with 400 units in the previous example? After the first bet our stake would have been:

$$\begin{aligned} \text{Stake} &= 400 + ((400/20) * 1.2) \\ &= 400 + (20 * 1.2) \\ &= 400 + 24 \\ &= 424 \end{aligned}$$

For the next bet, we would wager 21.2 units (424/20), or the integer amount of 21 units. Note that the percentage difference between the fractional and the integer bet here is only .952381% versus a 6.0% difference, had the amount we began trading with been only one starting value, 20 units. The following axiom can now be drawn: *The greater the ratio of the amount you have as a stake to begin trading relative to the starting value (biggest loss/ − optimal* f*), the more the percentage difference will tend to zero between integer and fractional betting.*

By allowing fractional bets, making the process nonquantum, we obtain a more realistic assessment of the relationship of f values to TWRs. *The fractional bets represent the average (of all possible values of the size of initial bankrolls) of the integer bets.* So the argument that we cannot make fractional bets in real life does not apply, since the fractional bet represents the average integer bet. If we made graphs of the TWRs at each f value for the +2, −1 coin toss, and used integer bets, we would have to make a different graph for each different initial bankroll. If we did this and then averaged the graphs to create a composite graph of the TWRs at each f value, we would have a graph of the fractional bet situation exactly as shown.

20 TRIALS	f VALUES ⟶								
EVENT	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
START VALUES ⟶	20.00	10.00	6.67	5.00	4.00	3.33	2.86	2.50	2.22
1.2	21.20	11.20	7.87	6.20	5.20	4.53	4.06	3.70	3.42
−1	20.14	10.08	6.69	4.96	3.90	3.17	2.64	2.22	1.88
1.2	21.35	11.29	7.89	6.15	5.07	4.32	3.74	3.29	2.90
−1	20.28	10.16	6.71	4.92	3.80	3.02	2.43	1.97	1.59
1.2	21.50	11.38	7.91	6.10	4.94	4.11	3.46	2.92	2.46
−1	20.42	10.24	6.73	4.88	3.71	2.88	2.25	1.75	1.35
1.2	21.65	11.47	7.94	6.05	4.82	3.91	3.19	2.59	2.08
−1	20.57	10.32	6.75	4.84	3.61	2.74	2.07	1.55	1.14
1.2	21.80	11.56	7.96	6.00	4.70	3.72	2.94	2.30	1.76
−1	20.71	10.41	6.77	4.80	3.52	2.61	1.91	1.38	0.97
1.2	21.95	11.66	7.99	5.96	4.58	3.54	2.72	2.04	1.49
−1	20.85	10.49	6.79	4.76	3.44	2.48	1.77	1.23	0.82
1.2	22.11	11.75	8.01	5.91	4.47	3.37	2.51	1.81	1.26
−1	21.00	10.57	6.81	4.73	3.35	2.36	1.63	1.09	0.69
1.2	22.26	11.84	8.03	5.86	4.36	3.21	2.32	1.61	1.07
−1	21.15	10.66	6.83	4.69	3.27	2.25	1.51	0.97	0.59
1.2	22.42	11.94	8.06	5.81	4.25	3.06	2.14	1.43	0.91
−1	21.30	10.74	6.85	4.65	3.18	2.14	1.39	0.86	0.50
1.2	22.57	12.03	8.08	5.77	4.14	2.91	1.97	1.27	0.77
−1	21.44	10.83	6.87	4.61	3.11	2.04	1.28	0.76	0.42
1.2	22.73	12.13	8.11	5.72	4.04	2.77	1.82	1.13	0.65
−1	21.60	10.92	6.89	4.58	3.03	1.94	1.18	0.68	0.36
1.2	22.89	12.23	8.13	5.68	3.94	2.64	1.68	1.00	0.55
−1	21.75	11.00	6.91	4.54	2.95	1.85	1.09	0.60	0.30
1.2	23.05	12.32	8.15	5.63	3.84	2.51	1.55	0.89	0.47
−1	21.90	11.09	6.93	4.50	2.88	1.76	1.01	0.53	0.26
1.2	23.21	12.42	8.18	5.59	3.74	2.39	1.43	0.79	0.40
−1	22.05	11.18	6.95	4.47	2.81	1.67	0.93	0.47	0.22
1.2	23.37	12.52	8.20	5.54	3.65	2.28	1.32	0.70	0.33
−1	22.21	11.27	6.97	4.43	2.74	1.59	0.86	0.42	0.18
1.2	23.54	12.62	8.23	5.50	3.56	2.17	1.22	0.62	0.28
−1	22.36	11.36	6.99	4.40	2.67	1.52	0.79	0.37	0.16
1.2	23.70	12.72	8.25	5.43	3.47	2.06	1.13	0.55	0.24
−1	22.52	11.45	7.01	4.36	2.60	1.44	0.73	0.33	0.13
1.2	23.87	12.82	8.28	5.41	3.38	1.96	1.04	0.49	0.20
−1	22.68	11.54	7.04	4.33	2.54	1.38	0.68	0.29	0.11
1.2	24.04	12.93	8.30	5.37	3.30	1.87	0.96	0.44	0.17
−1	22.83	11.63	7.06	4.29	2.47	1.31	0.62	0.26	0.09
1.2	24.20	13.03	8.33	5.32	3.21	1.78	0.89	0.39	0.15
−1	22.99	11.73	7.08	4.26	2.41	1.25	0.58	0.23	0.08
TWR ⟶	1.15	1.17	1.06	0.85	0.60	0.37	0.20	0.09	0.04

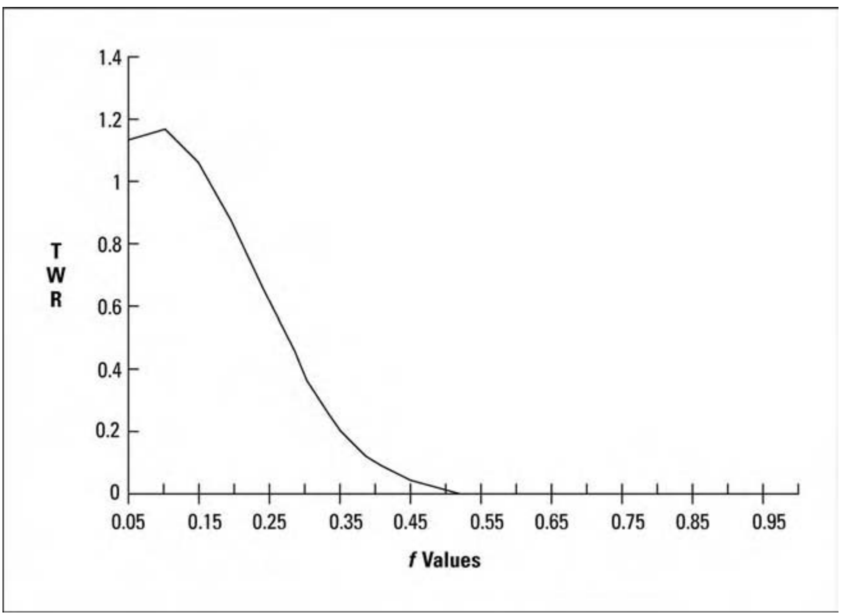

FIGURE 4.3 Values of *f* for 20 sequences at +1.2, −1

0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
2 00	1.82	1.67	1.54	1.43	1.33	1.25	1.18	1.11	1.05	1.00
3.20	3.02	2.87	2.74	2.63	2.53	2.45	2.38	2.31	2.25	2.20
1.60	1.36	1.15	0.96	0.79	0.63	0.49	0.36	0.23	0.11	0.00
2.56	2.25	1.97	1.71	1.45	1.20	0.96	0.72	0.48	0.24	0.00
1.28	1.01	0.79	0.60	0.44	0.30	0.19	0.11	0.05	0.01	0.00
2.05	1.68	1.36	1.06	0.80	0.57	0.38	0.22	0.10	0.03	0.00
1.02	0.76	0.54	0.37	0.24	0.14	0.08	0.03	0.01	.00	0.00
1.64	1.26	0.93	0.66	0.44	0.27	0.15	0.07	0.02	.00	0.00
0.82	0.57	0.37	0.23	0.13	0.07	0.03	0.01	.00	.00	0.00
1.31	0.94	0.64	0.41	0.24	0.13	0.06	0.02	.00	.00	0.00
0.66	0.42	0.26	0.14	0.07	0.03	0.01	.00	.00	.00	0.00
1.05	0.70	0.44	0.26	0.13	0.06	0.02	0.01	.00	.00	0.00
0.52	0.32	0.18	0.09	0.04	0.02	.00	.00	.00	.00	0.00
0.84	0.52	0.30	0.16	0.07	0.03	0.01	.00	.00	.00	0.00
0.42	0.24	0.12	0.06	0.02	0.01	.00	.00	.00	.00	0.00
0.67	0.39	0.21	0.10	0.04	0.01	.00	.00	.00	.00	0.00
0.34	0.18	0.08	0.03	0.01	.00	.00	.00	.00	.00	0.00
0.54	0.29	0.14	0.06	0.02	0.01	.00	.00	.00	.00	0.00
0.27	0.13	0.06	0.02	0.01	.00	.00	.00	.00	.00	0.00
0.43	0.22	0.10	0.04	0.01	.00	.00	.00	.00	.00	0.00
0.21	0.10	0.04	0.01	.00	.00	.00	.00	.00	.00	0.00
0.34	0.16	0.07	0.02	0.01	.00	.00	.00	.00	.00	0.00
0.17	0.07	0.03	0.01	.00	.00	.00	.00	.00	.00	0.00
0.27	0.12	0.05	0.02	.00	.00	.00	.00	.00	.00	0.00
0.14	0.05	0.02	0.01	.00	.00	.00	.00	.00	.00	0.00
0.22	0.09	0.03	0.01	.00	.00	.00	.00	.00	.00	0.00
0.11	0.04	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.18	0.07	0.02	0.01	.00	.00	.00	.00	.00	.00	0.00
0.09	0.03	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.14	0.05	0.02	.00	.00	.00	.00	.00	.00	.00	0.00
0.07	0.02	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.11	0.04	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.06	0.02	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.09	0.03	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.05	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.07	0.02	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.04	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.06	0.02	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.03	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.05	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.02	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00

This is not a contention that the fractional bet situation is the same as the real-life integer-bet situation. Rather, the contention is that for the purposes of studying these functions we are better off considering the fractional bet, since it represents the universe of integer bets. The fractional bet situation is what we can expect in real life in an asymptotic sense (i.e., in the long run).

This discussion leads to another interesting point that is true in a fixed fractional betting situation where fractional bets are allowed (think of fractional bets as the average outcome of all integer bets at different initial bankroll values, since that is what fractional betting represents here). This point is that *the TWR is the same regardless of the starting value.* In the examples just cited, if we have an initial stake of one starting value, 20 units, our TWR (ending stake divided by initial stake) is 1.15. If we have an initial stake of 400 units, 20 starting values, our TWR is still 1.15.

Figure 4.4 shows the f curve for 20 sequences of the $+1.5$, -1.

Refer now to the $+2$, -1 graph (Figure 4.5). Notice that here the optimal f is .25 where the TWR is 10.55 after 40 bets (20 sequences of $+2$, -1). Now look what happens if you bet only 15% away from the optimal .25 f. At an f

20 TRIALS	f VALVES ⟶								
EVENT	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
START VALUES ⟶	20.00	10.00	6.67	5.00	4.00	3.33	2.86	2.50	2.22
1.5	21.50	11.50	8.17	6.50	5.50	4.83	4.36	4.00	3.72
−1	20.43	10.35	6.94	5.20	4.13	3.38	2.83	2.40	2.05
1.5	21.96	11.90	8.50	6.76	5.67	4.91	4.32	3.84	3.43
−1	20.86	10.71	7.23	5.41	4.25	3.43	2.81	2.30	1.89
1.5	22.42	12.32	8.85	7.03	5.85	4.98	4.28	3.69	3.16
−1	21.30	11.09	7.53	5.62	4.39	3.49	2.78	2.21	1.74
1.5	22.90	12.75	9.22	7.31	6.03	5.05	4.24	3.54	2.91
−1	21.75	11.48	7.84	5.85	4.52	3.54	2.76	2.12	1.60
1.5	23.39	13.20	9.60	7.60	6.22	5.13	4.21	3.40	2.68
−1	22.22	11.88	8.16	6.08	4.67	3.59	2.73	2.04	1.47
1.5	23.88	13.66	10.00	7.91	6.41	5.21	4.17	3.26	2.47
−1	22.69	12.29	8.50	6.33	4.81	3.64	2.71	1.96	1.36
1.5	24.39	14.14	10.41	8.22	6.62	5.28	4.13	3.13	2.28
−1	23.17	12.72	8.85	6.58	4.96	3.70	2.69	1.88	1.25
1.5	24.91	14.63	10.84	8.55	6.82	5.36	4.10	3.01	2.10
−1	23.66	13.17	9.21	6.84	5.12	3.75	2.66	1.80	1.15
1.5	25.44	15.14	11.28	8.90	7.04	5.44	4.06	2.89	1.93
−1	24.17	13.63	9.59	7.12	5.28	3.81	2.64	1.73	1.06
1.5	25.98	15.67	11.75	9.25	7.26	5.53	4.03	2.77	1.78
−1	24.68	14.11	9.99	7.40	5.44	3.87	2.62	1.66	0.98
1.5	26.53	16.22	12.23	9.62	7.48	5.61	3.99	2.66	1.64
−1	25.20	14. 60	10.40	7.70	5.61	3.93	2.59	1.60	0.90
1.5	27.10	16.79	12.74	10.01	7.72	5.69	3.96	2.55	1.51
−1	25.74	15.11	10.83	8.01	5.79	3.99	2.57	1.53	0.83
1.5	27.67	17.38	13.26	10.41	7.96	5.78	3.92	2.45	1.39
−1	26.29	15.64	11.28	8.33	5.97	4.05	2.55	1.47	0.77
1.5	28.26	17.99	13.81	10.82	8.21	5.87	3.89	2.35	1.28
−1	26.85	16.19	11.74	8.66	6.15	4.11	2.53	1.41	0.70
1.5	28.86	18.61	14.38	11.26	8.46	5.95	3.85	2.26	1.18
−1	27.42	16.75	12.22	9.00	6.35	4.17	2.50	1.36	0.65
1.5	29.47	19.27	14.98	11.71	8.73	6.04	3.82	2.17	1.09
−1	28.00	17.34	12.73	9.36	6.54	4.23	2.48	1.30	0.60
1.5	30.10	19.94	15.59	12.17	9.00	6.13	3.79	2.08	1.00
−1	28.59	17.95	13.25	9.74	6.75	4.29	2.46	1.25	0.55
1.5	30.74	20.64	16.24	12.66	9.28	6.23	3.75	2.00	0.92
−1	29.20	18.57	13.80	10.13	6.96	4.36	2.44	1.20	0.51
1.5	31.39	21.36	16.91	13.17	9.57	6.32	3.72	1.92	0.85
−1	29.82	19.23	14.37	10.53	7.18	4.42	2.42	1.15	0.47
1.5	32.06	22.11	17.60	13.69	9.87	6.41	3.69	1.84	0.78
−1	30.46	19.90	14.96	10.96	7.40	4.49	2.40	1.11	0.43
TWR ⟶	1.52	1.99	2.24	2.19	1.85	1.35	0.84	0.44	0.19

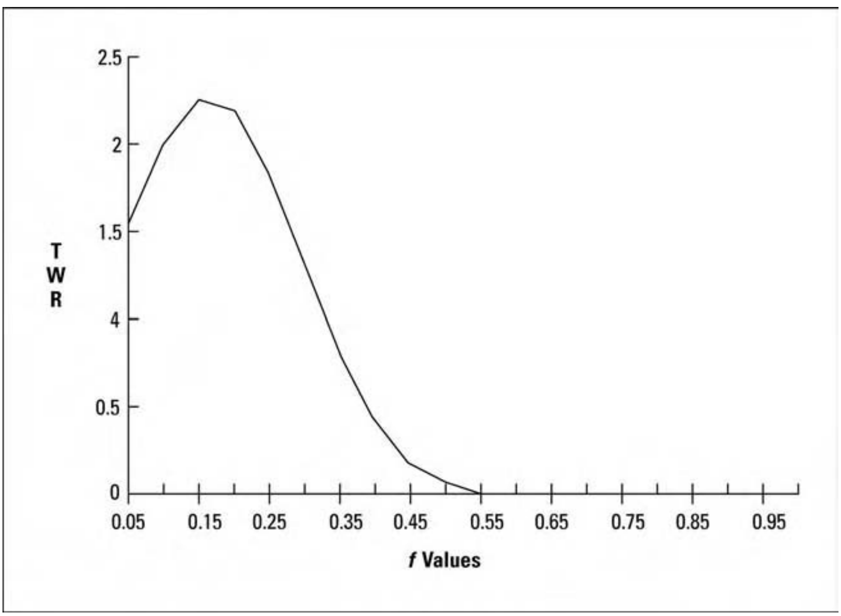

FIGURE 4.4 Values of *f* for 20 sequences at +1.5, −1

0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
2.00	1.82	1.67	1.54	1.43	1.33	1.25	1.18	1.11	1.05	1.00
3.50	3.32	3.17	3.04	2.93	2.83	2.75	2.68	2.61	2.55	2.50
1.75	1.49	1.27	1.06	0.88	0.71	0.55	0.40	0.26	0.13	0.00
3.06	2.73	2.41	2.10	l.80	1.51	1.21	0.91	0.61	0.31	0.00
1.53	1.23	0.96	0.74	0.54	0.38	0.24	0.14	0.06	0.02	0.00
2.68	2.24	1.83	1.45	1.11	0.80	0.53	0.31	0.14	0.04	0.00
1.34	1.01	0.73	0.51	0.33	0.20	0.11	0.05	0.01	.00	0.00
2.34	1.84	1.39	1.00	0.68	0.42	0.23	0.11	0.03	.00	0.00
1.17	0.83	0.56	0.35	0.20	0.11	0.05	0.02	.00	.00	0.00
2.05	1.51	1.06	0.69	0.42	0.23	0.10	0.04	0.01	.00	0.00
1.03	0.68	0.42	0.24	0.13	0.06	0.02	0.01	.00	.00	0.00
1.80	1.24	0.80	0.48	0.26	0.12	0.05	0.01	.00	.00	0.00
0.90	0.56	0.32	0.17	0.08	0 03	0.01	.00	.00	.00	0.00
1.57	1.02	0.61	0.33	0.16	0.06	0.02	.00	.00	.00	0.00
0.79	0.46	0.24	0.12	0.05	0.02	.00	.00	.00	.00	0.00
0.69	0.38	0.19	0.08	0.03	0.01	.00	.00	.00	.00	0.00
1.20	0.69	0.35	0.16	0.06	0.02	.00	.00	.00	.00	0.00
0.60	0.31	0.14	0.06	0.02	.00	.00	.00	.00	.00	0.00
1.05	0.56	0.27	0.11	0.04	0.01	.00	.00	.00	.00	0.00
0.53	0.25	0.11	0.04	0 01	.00	.00	.00	.00	.00	0.00
0.92	0.46	0.20	0.08	0.02	0.01	.00	.00	.00	.00	0.00
0.46	0.21	0.08	0.03	0.01	.00	.00	.00	.00	.00	0.00
0.81	0.38	0.15	0.05	0.01	.00	.00	.00	.00	.00	0.00
0.40	0.17	0.06	0.02	.00	.00	.00	.00	.00	.00	0.00
0.70	0.31	0.12	0.04	0.01	.00	.00	.00	.00	.00	0.00
0.35	0.14	0.05	0.01	.00	.00	.00	.00	.00	.00	0.00
0.62	0.26	0.09	0.02	0.01	.00	.00	.00	.00	.00	0.00
0.31	0.12	0.04	0.01	.00	.00	.00	.00	.00	.00	0.00
0.54	0.21	0.07	0.02	.00	.00	.00	.00	.00	.00	0.00
0.27	0.09	0.03	0.01	.00	.00	.00	.00	.00	.00	0.00
0 47	0.17	0.05	0.01	.00	.00	.00	.00	.00	.00	0.00
0.24	0.08	0.02	.00	.00	.00	.00	.00	.00	.00	0.00
0.41	0.14	0.04	0.01	.00	.00	.00	.00	.00	.00	0.00
0 21	0.06	0.02	.00	.00	.00	.00	.00	.00	.00	0.00
0.36	0.12	0.03	0.01	.00	.00	.00	.00	.00	.00	0.00
0.18	0.05	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.32	0.10	0.02	.00	.00	.00	.00	.00	.00	.00	0.00
0.16	0.04	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.28	0.08	0.02	.00	.00	.00	.00	.00	.00	.00	0.00
0.14	0.04	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.07	0.02	.00	.00	.00	.00	.00	.00	.00	.00	0.00

of .1 or .4 your TWR is 4.66. This is not even half of what it is at .25, yet you are only 15% away from the optimal and only 40 bets have elapsed! What does this mean in terms of dollars? At $f = .1$, you would be making one bet for every \$10 in your stake. At $f = .4$ you would be making one bet for every \$2.50 in your stake. Both make the same amount, with a TWR of 4.66. At $f = .25$, you are making one bet for every \$4 in your stake. Notice that if you make one bet for every \$4 in your stake, you will make more than twice as much as you would if you were making one bet for every \$2.50 in your stake! Clearly, it does not pay to overbet. At one bet for every \$10 in your stake you make the same amount as if you had bet four times that amount, one bet for every \$2.50 in your stake! Notice that in a 50/50 game where you win twice the amount that you lose, at an f of .5 you are only breaking even! That means you are only breaking even if you made one bet for every \$2 in your stake. At an f greater than .5 you are losing in this game, and it is simply a matter of time until you are completely tapped out!

Now let's increase the winning payout from two units to five units, as is demonstrated in the data in Figure 4.6. Here your optimal f is .4, or bet \$1 for every \$2.50 in your stake. After 20 sequences of $+5$, -1, 40 bets, your

20 TRIALS EVENT	f VALUES ——→ 0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
START VALUES ——→	20.00	10.00	6.67	5.00	4.00	3.33	2.86	2.50	2.22
2	22.00	12.00	8.67	7.00	6.00	5.33	4.86	4.50	4.22
−1	20.90	10.80	7.37	5.60	4.50	3.73	3.16	2.70	2.32
2	22.99	12.96	9.58	7.84	6.75	5.97	5.37	4.86	4.41
−1	21.84	11.66	8.14	6.27	5.06	4.18	3.49	2.92	2.43
2	24.02	14.00	10.59	8.78	7.59	6.69	5.93	5.25	4.61
−1	22.82	12.60	8.99	7.02	5.70	4.68	3.85	3.15	2.54
2	25.11	15.12	11.69	9.83	8.54	7.49	6.55	5.67	4.82
−1	23 85	13.60	9.94	7.87	6.41	5.25	4.26	3.40	2.65
2	26.24	16.33	12.92	11.01	9.61	8.39	7.24	6.12	5.04
−1	24.92	14.69	10.98	8.81	7.21	5.87	4.71	3.67	2.77
2	27.42	17.63	14.28	12.34	10.81	9.40	8.00	6.61	5.26
−1	26.05	15.87	12.14	9.87	8.11	6.58	5.20	3.97	2.89
2	28.65	19.04	15.78	13.82	12.16	10.53	8.84	7.14	5.50
−1	27.22	17.14	13.41	l.05	9.12	7.37	5.75	4.28	3.02
2	29.94	20.57	17.43	15.47	13.68	11.79	9.77	7.71	5.75
−1	28.44	18.51	14.82	12.38	10.26	8.25	6.35	4.63	3.16
2	31.29	22.21	19.26	17.33	15.39	13.21	10.80	8.33	6.00
−1	29.72	19.99	16.37	13.87	11.55	9.24	7.02	5.00	3.30
2	32.69	23.99	21.29	19.41	17.32	14.79	11.93	9.00	6.27
−1	31.06	21.59	18.09	15.53	12.99	10.35	7.75	5.40	3.45
2	34.17	25.91	23.52	21.74	19.48	16.56	13.18	9.72	6.56
−1	32.46	23.32	19.99	17.39	14.61	11.60	8.57	5.83	3.61
2	35.70	27.98	25.99	24.35	21.92	18.55	14.57	10.49	6.85
−1	33.92	25.18	22.09	19.48	16.44	12.99	9.47	6.30	3.77
2	37.31	30.22	28.72	27.27	24.66	20.78	16.10	11.33	7.16
−1	35.44	27.20	24.41	21.82	18.49	14.54	10.46	6.80	3.94
2	38.99	32.64	31.74	30.54	27.74	23.27	17.79	12.24	7.48
−1	37.04	29.37	26.98	24.44	20.81	16.29	11.56	7.34	4.12
2	40.74	35.25	35.07	34.21	31.21	26.06	19.65	13.22	7.82
−1	38.71	31.72	29.81	27.37	23.41	18.25	12.78	7.93	4.30
2	42.58	38.07	38.75	38.31	35.11	29.19	21.72	14.27	8.17
−1	40.45	34.26	32.94	30.65	26.33	20.43	14.12	0.56	4.49
2	44.49	41.11	42.82	42.91	39.50	32.70	24.00	15.42	8.54
−1	42.27	37.00	36.40	34.33	29.62	22.89	15.60	9.25	4.70
2	46.49	44.40	47.32	40.06	44.44	36.62	26.52	16.65	8.92
−1	44.17	39.96	40.22	38.45	33.33	25.63	17.24	9.99	4.91
2	46.59	47.95	52.28	53.83	49.99	41.01	29.30	17.98	9.32
−1	46.16	43.16	44.44	43.06	37.49	21.71	19.05	10.79	5.13
2	50.77	51.79	57.77	60.29	56.24	45.93	32.38	19.42	9.74
−1	48.23	46.61	49.11	48.23	42.18	32.15	21.05	11.65	5.36
TUR ——→	2.41	4.66	7.37	9.65	10.55	9.65	7.37	4.66	2.41

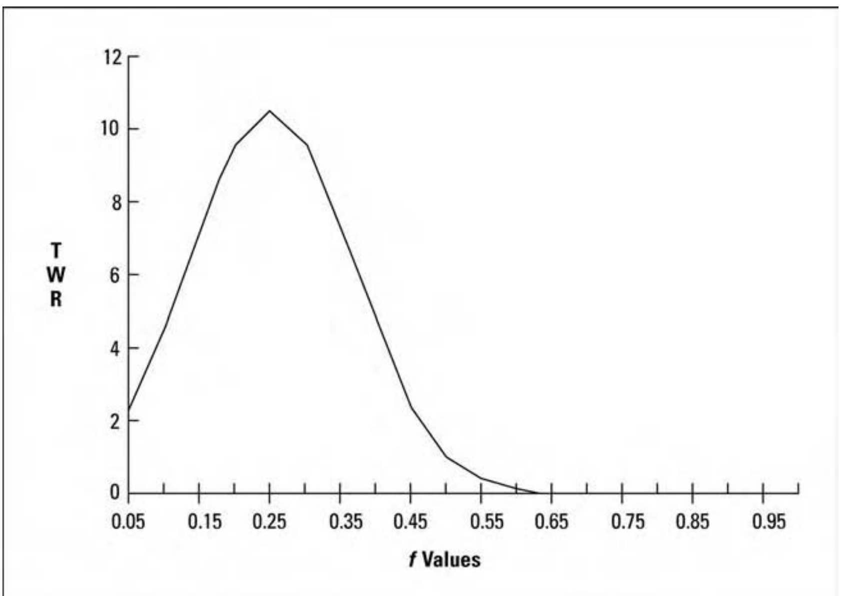

FIGURE 4.5 Values of *f* for 20 sequences at +2, −1

0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
2.00	1.82	1.67	1.54	1.43	1.33	1.25	1.18	1.11	1.05	1.00
4.00	3.82	3.67	3.54	3.43	3.33	3.25	3.18	3.11	3.05	3.00
2.00	1.72	1.47	1.24	1.03	0.83	0.65	0.48	0.31	0.15	0.00
4.00	3.61	3.23	2.85	2.47	2.08	1.69	1.29	0.87	0.44	0.00
2.00	1.62	1.29	1.00	0.74	0.52	0.34	0.19	0.09	0.02	0.00
4.00	3.41	2.84	2.29	1.78	1.30	0.88	0.52	0.24	0.06	0.00
2.00	1.53	1.14	0.80	0.53	0.33	0.18	0.08	0.02	.00	0.00
4.00	3.22	2.50	1.85	1.28	0.81	0.46	0.21	0.07	0.01	0.00
2.00	1.45	1.00	0.65	0.38	0.20	0.09	0.03	0.01	.00	0.00
4.00	3.04	.20	1.49	0.92	0.51	0.24	0.09	0.02	.00	0.00
2.00	1.37	0.88	0.52	0.28	0.13	0.05	0.01	.00	.00	0.00
4.00	2.88	1.94	1.20	0.66	0.32	0.12	0.03	0.01	.00	0.00
2.00	1.29	0.77	0.42	0.20	0.08	0.02	0.01	.00	.00	0.00
4.00	2.72	1.70	0.96	0.48	0.20	0.06	0.01	.00	.00	0.00
2.00	1.22	0.68	0.34	0.14	0.05	0.01	.00	.00	.00	0.00
4.00	2.57	1.50	0.78	0.34	0.12	0.03	0.01	.00	.00	0.00
2.00	1.16	0.60	0.25	0.08	0.03	0.01	.00	.00	.00	0.00
4.00	2.43	1.32	0.62	0.25	0.08	0.02	.00	.00	.00	0.00
2.00	1.09	0.53	0.22	0.07	0.02	0.02	.00	.00	.00	0.00
4.00	2.29	1.16	0.50	0.18	0.05	0.01	.00	.00	.00	0.00
2.00	1.03	0.46	0.18	0.05	0.01	.00	.00	.00	.00	0.00
4.00	2.17	1.02	0.40	0.13	0.03	.00	.00	.00	.00	0.00
2.00	0.98	0.41	0.14	0.04	0.01	.00	.00	.00	.00	0.00
4.00	2.05	0.90	0.33	0.09	0.02	.00	.00	.00	.00	0.00
2.00	0.92	0.36	0.11	0.03	.00	.00	.00	.00	.00	0.00
4.00	1.94	0.79	0.26	0.07	0.01	.00	.00	.00	.00	0.00
2.00	0.87	0.32	0.09	0.02	.00	.00	.00	.00	.00	0.00
4.00	1.83	0.70	0.21	0.05	0.01	.00	.00	.00	.00	0.00
2.00	0.82	0.21	0.07	0.01	.00	.00	.00	.00	.00	0.00
4.00	1.73	0.61	0.17	0.03	.00	.00	.00	.00	.00	0.00
2.00	0.78	0.24	0.06	0.01	.00	.00	.00	.00	.00	0.00
4.00	1.63	0.54	0.14	0.02	.00	.00	.00	.00	.00	0.00
2.00	0.74	0.22	0.05	0.01	.00	.00	.00	.00	.00	0.00
4.00	1.54	0.47	0.11	0.02	.00	.00	.00	.00	.00	0.00
2.00	0.69	0.19	0.04	0.01	.00	.00	.00	.00	.00	0.00
4.00	1.46	0.42	0.09	0.01	.00	.00	.00	.00	.00	0.00
2.00	0.66	0.17	0.03	.00	.00	.00	.00	.00	.00	0.00
4.00	1.38	0.37	0.07	0.01	.00	.00	.00	.00	.00	0.00
2.00	0.62	0.19	0.02	.00	.00	.00	.00	.00	.00	0.00
4.00	1.30	0.32	0.06	0.01	.00	.00	.00	.00	.00	0.00
2.00	0.59	0.13	0.02	.00	.00	.00	.00	.00	.00	0.00
1.00	0.32	0.08	0.01	.00	.00	.00	.00	.00	.00	0.00

\$2.50 stake has grown to \$127,482, thanks to optimal f. Now look what happens in this extremely favorable situation if you miss the optimal f by 20%. At f values of .6 and .2 you don't make one-tenth as much as you do at .4 in this case! This particular situation, a 50/50 bet paying 5 to 1, has a mathematical expectation of $(5 * .5) + (1 * (-.5)) = 2$. Yet if you bet using an f value greater than .8, you lose money in this situation. Clearly, the question of what is the correct quantity to bet or trade has been terribly underrated.

The graphs bear out a few more interesting points. The first is that *at no other fixed fraction will you make more money than optimal* f. In other words, it does not pay to bet \$1 for every \$2 in your stake in the above example of $+5$, -1. In such a case, you would make less money than if you bet \$1 for every \$2.50 in your stake. *It does not pay to risk more than the optimal* f*—in fact, you pay a price to do so!* Notice in Figure 4.7 that you make less at $f = .55$ than at $f = .5$. The second interesting point to notice is how important the biggest loss is in the calculations. Traders may be incorrectly inclined to use maximum drawdown rather than biggest loss.

20 TRIALS	f VALUES ⟶								
EVENT	0.06	0.1	0.15	0.2	0.29	0.3	0.35	0.4	0.45
START VALUES ⟶	20.00	10.00	6.67	5.00	4.00	3.33	2.86	2.90	2.22
5	25.00	15.00	11.67	10.00	9.00	8.33	7.86	7.50	7.22
−1	23.75	13.50	3.92	8.00	6.75	5.83	5.11	4.50	3.97
5	29.89	20.25	17.35	16.00	15.19	14.58	14.04	13.50	12.91
−1	28.20	18.23	14.75	12.80	11.39	10.21	9.13	8.10	7.10
5	35.25	27.34	25.81	25.60	25.63	25.52	25.10	24.30	23.08
−1	33.49	24.60	21.94	20.48	19.22	17.86	16.32	14.50	12.69
5	41.66	36.91	38.40	40.96	43.25	44.66	44.87	43.74	41.25
−1	39.77	33.22	32.64	32.77	32.44	31.26	29.17	26.24	22.69
5	49.71	49.82	57.12	65.54	72.98	78.16	80.21	78.73	73.73
−1	47.23	44.84	48.55	52.43	54.74	54.71	52.14	47.24	40.55
5	59.03	67.26	84.96	104.86	123.16	136.78	143.38	141.72	131.80
−1	56.08	60.53	72.22	83.89	92.37	96.74	93.20	85.03	72.49
5	70.10	90.80	126.38	167.77	207.83	239.36	256.30	255.09	230.58
−1	66.60	81.72	107.43	134.22	155.87	167.55	166.59	153.06	129.57
5	83.25	122.58	187.99	268.44	350.71	418.88	458.13	459.17	421.11
−1	79.09	110.32	199.80	214.75	263.03	293.21	297.78	275.50	231.61
5	98.86	165.49	279.64	429.50	591.82	733.03	818.90	826.50	752.73
−1	93.91	148.94	237.70	343.60	443.87	513.12	532.29	496.90	414.00
5	117.39	223.41	415.97	687.19	998.70	1282.81	1463.79	1487.69	1345.50
−1	111.52	201.07	353.57	549.76	749.03	897.96	951.46	892.62	740.03
5	139.40	301.60	618.75	1099.51	1685.31	2244.91	2616.53	2677.85	2405.09
−1	132.43	271.44	525.94	879.61	1263.98	1571.44	1700.74	1606.71	1322.60
5	165.54	407.16	920.39	1759.22	2843.96	3928.60	4677.04	4820.13	4299.10
−1	157.27	366.44	782.33	1407.37	2132.97	2750.02	3040.08	2892.08	2364.50
5	196.58	549.66	1369.09	2814.75	4799.19	6875.04	8360.21	8676.24	7684.64
−1	186.75	494.70	1163.72	2251.80	3599.39	4812.53	5434.14	5205.74	4226.55
5	233.44	742.05	2036.51	4503.60	8098.63	12031.32	14943.88	15617.22	13736.29
−1	221.77	667.84	1731.04	3602.88	6073.97	9421.93	9713.52	9370.33	7554.96
5	277.21	1001.76	3029.31	7205.76	13666.44	21054.82	26712.18	2811.00	24553.62
−1	263.35	901.58	2574.92	5764.61	10249.83	14738.37	17362.92	16866.60	13504.49
5	329.19	1352.38	4506.11	11529.22	23062.12	36845.93	47748.02	50599.80	43889.60
−1	312.73	1217.14	3830.19	9223.37	17296.59	25792.15	31036.22	30359.88	24139.28
5	390.91	1825.71	6702.83	18446.74	38917.33	64480.37	85349.59	91079.65	78452.66
−1	371.37	1643.14	5697.41	14757.40	29188.00	45136.26	55477.24	54647.79	43148.96
5	464.21	2464.71	9970.46	29514.79	65672.99	112840.65	152562.40	163943.37	140234.12
−1	441.00	2218.24	8474.89	23611.83	49254.74	78988.46	99165.56	98366.02	77128.77
5	551.25	3327.35	14831.06	47723.66	110823.17	197471.14	272705.29	295098.06	250668.50
−1	523.68	2994.62	12606.40	37778.93	83117.38	138229.80	177258.44	177058.84	137867.67
5	654.60	4491.93	22061.21	75557.86	187014.10	345574.49	487460.70	531176.51	448069.94
−1	621.87	4042.74	18752.03	60446.29	140260.58	241902.14	316849.46	318705.91	246438.47
TWR ⟶	31.09	404.27	2812.80	12089.26	35065.14	72570.64	110897.31	127482.36	110897.31

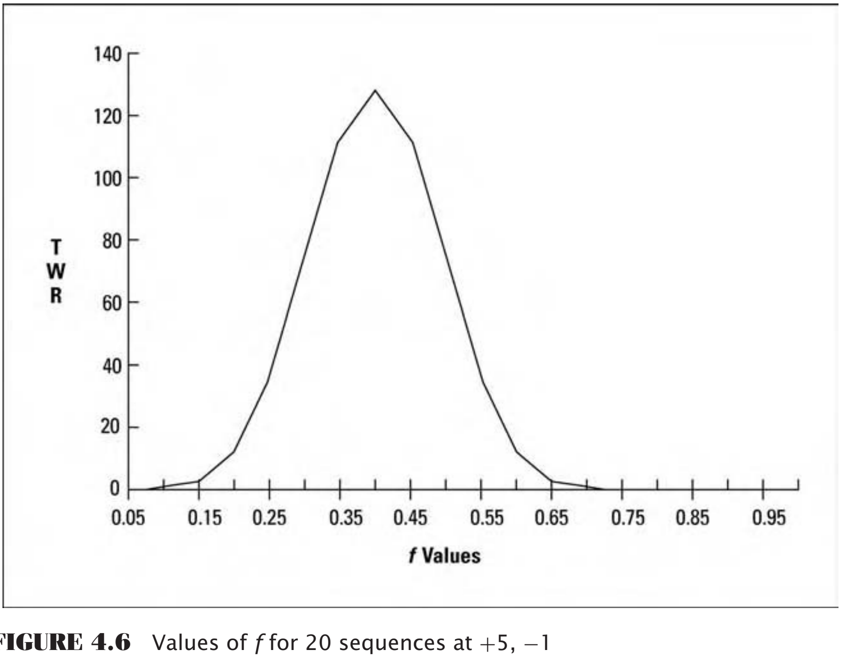

FIGURE 4.6 Values of *f* for 20 sequences at +5, −1

0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
2.00	1.82	1.67	1.54	1.43	1.33	1.25	1.18	1.11	1.05	1.00
7.00	6.82	6.67	6.54	6.43	6.33	6.25	6.18	6.11	6.05	6.00
3.50	3.07	2.67	2.29	1.93	1.59	1.25	0.93	0.61	0.30	0.00
12.25	11.51	10.67	9.73	8.68	7.52	6.25	4.86	3.36	1.74	0.00
6.13	5.18	4.27	3.40	2.60	1.88	1.25	0.73	0.34	0.09	0.00
21.44	19.42	17.07	14.47	11.72	8.93	6.25	3.83	1.85	0.50	0.00
10.72	8.74	6.83	5.06	3.51	2.23	1.25	0.57	0.18	0.03	0.00
37.52	32.76	27.31	21.52	15.82	10.61	6.25	3.02	1.02	0.14	0.00
18.76	14.74	10.92	7.53	4.75	2.65	1.25	0.45	0.10	0.01	0.00
65.65	55.29	43.69	32.01	21.35	12.59	6.25	2.38	0.56	0.04	0.00
32.83	24.88	17.48	11.20	6.41	3.15	1.25	0.36	0.06	.00	0.00
114.89	93.30	69.91	47.62	28.83	14.96	6.25	1.87	0.31	0.01	0.00
57.45	41.99	27.96	16.67	8.65	3.74	1.25	0.28	0.03	.00	0.00
201.06	157.45	111.85	70.83	38.92	17.76	6.25	1.47	0.17	.00	0.00
100.53	70.85	44.74	24.79	11.67	4.44	1.25	0.22	0.02	.00	0.00
351.86	265.69	178.96	105.36	52.54	21.09	6.25	1.16	0.09	.00	0.00
175.93	119.56	71.58	36.88	15.76	5.27	1.25	0.17	0.01	.00	0.00
615.75	448.35	286.33	156.72	70.92	25.04	6.25	0.91	0.05	.00	0.00
307.87	201.76	114.53	54.85	21.28	6.26	1.25	0.14	0.01	.00	0.00
1077.56	756.59	458.13	233.12	95.75	29.74	6.25	0.72	0.03	.00	0.00
538.78	340.47	183.25	81.59	28.72	7.43	1.25	0.11	.00	.00	0.00
1885.73	1276.75	733.01	346.77	129.26	35.32	6.25	0.57	0.02	.00	0.00
942.86	574.54	293.20	121.37	38.78	8.83	1.25	0.08	.00	.00	0.00
3300.02	2154.52	1172.81	515.82	174.50	41.94	6.25	0.45	0.01	.00	0.00
1650.01	969.53	469.12	180.54	52.35	10.48	1.25	0.07	.00	.00	0.00
5775.04	3635.75	1876.50	767.29	235.57	49.80	6.25	0.35	.00	.00	0.00
2887.52	1636.09	750.60	268.55	70.67	12.45	1.25	0.05	.00	.00	0.00
10106.31	6135.33	3002.40	1141.34	318.02	59.14	6.25	0.28	.00	.00	0.00
5053.16	2760.90	1200.96	399.47	95.41	14.78	1.25	0.04	.00	.00	0.00
17686.04	10353.36	4803.84	1697.75	429.33	70.23	6.25	0.22	.00	.00	0.00
8843.02	4659.01	1921.54	594.21	128.80	17.56	1.25	0.03	.00	.00	0.00
30950.58	17471.30	7686.14	2525.40	579.59	83.39	6.25	0.17	.00	.00	0.00
15475.29	7862.09	3074.46	883.89	173.88	20.85	1.25	0.03	.00	.00	0.00
54163.51	29482.82	12297.83	3756.53	782.45	99.03	6.25	0.14	.00	.00	0.00
27081.76	13267.27	4919.13	1314.79	234.73	24.76	1.25	0.02	.00	.00	0.00
94786.15	49752.27	19676.53	5587.84	1056.30	117.60	6.25	0.11	.00	.00	0.00
47393.07	22388.52	7870.61	1955.74	316.89	29.40	1.25	0.02	.00	.00	0.00
165875.76	83956.95	31482.44	8311.92	1426.01	139.65	6.25	0.08	.00	.00	0.00
82937.88	37780.63	12592.98	2909.17	427.80	34.91	1.25	0.01	.00	.00	0.00
290282.57	141677.35	50371.91	12363.97	1925.11	165.83	6.25	0.07	.00	.00	0.00
145141.29	63754.81	20148.76	4327.39	577.53	41.46	1.25	0.01	.00	.00	0.00
72570.64	35065.14	12089.26	2812.80	404.27	31.09	1.00	0.01	.00	.00	0.00

DRAWDOWN AND LARGEST LOSS WITH *f*

First, if you have $f = 1.00$, then as soon as the biggest loss is encountered, you would be tapped out. This is as it should be. You want f to be bounded at 0 (nothing at stake) and 1 (the lowest amount at stake where you would lose 100%).

Second, in an independent trials process the sequence of trades that results in the drawdown is, in effect, arbitrary (as a result of the independence). Suppose we toss a coin six times, and we get heads three times and tails three times. Suppose that we win \$1 every time heads comes up and lose \$1 every time tails comes up. Considering all possible sequences here our drawdown could be \$1, \$2, or \$3, the extreme case where all losses bunch together. If we went through this exercise once and came up with a \$2 drawdown, it wouldn't mean anything. Since drawdown is an *extreme* case situation, and we are speaking of exact sequences of trades that are independent, we have to assume that the extreme case can be all losses bunching together in a row (the extreme worst case in the sample space).

20 TRIALS

EVENT (f VALUES ⟶)	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
START VALUES ⟶	20.00	10.00	6.67	5.00	4.00	3.33	2.86	2.50	2.22
-1	19.00	9.00	5.67	4.00	3.00	2.33	1.86	1.50	1.22
-1	18.05	8.10	4.82	3.20	2.25	1.63	1.21	0.90	1.67
-1	17.15	7.29	4.09	2.56	1.69	1.14	0.78	0.54	0.37
-1	16.29	6.56	3.48	2.05	1.27	0.80	0.51	0.32	0.20
-1	15.48	5.90	2.96	1.64	0.95	0.56	0.33	0.19	0.11
-1	14.70	5.31	2.51	1.31	0.71	0.39	0.22	0.12	0.06
-1	13.97	4.78	2.14	1.05	0.53	0.27	0.14	0.07	0.03
-1	13.27	4.30	1.82	0.84	0.40	0.19	0.09	0.04	0.02
-1	12.60	3.87	1.54	0.67	0.30	0.13	0.06	0.03	0.01
-1	11.97	3.49	1.31	0.54	0.23	0.09	0.04	0.02	0.01
-1	12.57	3.84	1.51	0.64	0.28	0.12	0.05	0.02	0.01
-1	13.20	4.22	1.74	0.77	0.35	0.16	0.07	0.03	0.01
1	13.86	4.64	2.00	0.93	0.44	0.21	0.09	0.04	0.02
1	14.56	5.11	2.30	1.11	0.55	0.27	0.13	0.06	0.02
1	15.28	5.62	2.64	1.34	0.69	0.35	0.17	0.08	0.04
1	16.05	6.18	3.04	1.60	0.86	0.45	0.23	0.11	0.05
1	16.05	6.79	3.49	1.92	1.07	0.59	0.31	0.16	0.05
1	17.69	7.47	4.01	2.31	1.34	0.77	0.42	0.22	0.11
1	18.58	8.22	4.62	2.77	1.68	1.00	0.57	0.31	0.16
1	19.51	9.04	5.31	3.32	2.10	1.30	0.77	0.44	0.23
1	20.48	9.95	6.11	3.99	2.62	1.69	1.04	0.61	0.34
1	21.50	10.94	7.02	4.79	3.28	2.19	1.41	0.86	0.49
1	22.58	12.04	8.08	5.74	4.10	2.85	1.90	1.20	0.71
1	23.71	13.24	9.29	6.89	5.12	3.71	2.57	1.68	1.02
1	24.89	14.57	10.68	8.27	6.40	4.82	3.47	2.35	1.48
1	26.14	16.02	12.28	9.93	8.00	6.27	4.68	3.29	2.15
1	27.45	17.62	14.12	11.91	10.00	8.15	6.32	4.61	3.12
1	28.82	19.39	16.24	14.29	12.50	10.59	8.53	6.45	4.52
1	30.26	21.32	18.68	17.15	15.63	13.77	11.52	9.03	6.55
1	31.77	23.46	21.48	20.58	19.54	17.89	15.55	12.65	9.50
1	33.36	25.80	24.70	24.70	24.42	23.26	20.99	17.71	13.78
1	35.03	28.38	28.41	29.64	30.53	30.24	28.34	24.79	19.98
1	36.79	31.22	32.67	35.57	38.16	39.31	38.26	34.71	28.97
1	38.62	34.34	34.57	42.68	47.70	51.11	51.65	48.59	42.00
1	40.55	37.78	43.21	51.22	59.62	66.44	69.73	68.02	60.90
1	42.58	41.56	49.69	61.46	74.53	86.37	94.13	95.23	80.30
1	44.71	45.71	57.14	73.75	93.16	112.29	127.08	133.32	128.04
1	46.94	50.28	65.71	88.50	116.45	145.97	171.56	186.65	185.66
1	49.29	55.31	75.57	106.20	145.57	189.77	231.32	261.32	269.21
1	51.75	60.84	86.90	127.44	181.96	246.69	312.66	365.84	390.35
TWR ⟶	2.59	6.08	13.04	25.49	45.49	74.01	109.43	146.34	175.66

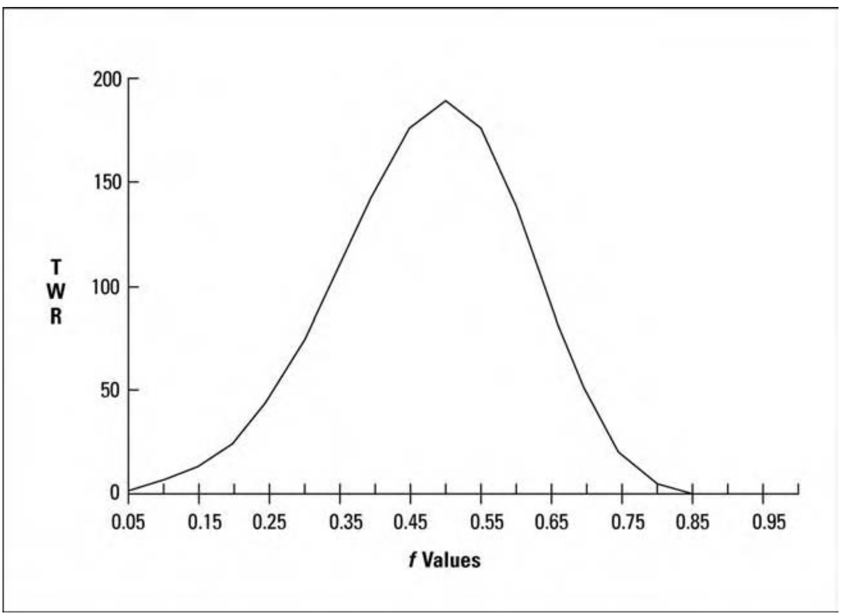

FIGURE 4.7 Values of *f* for 10 sequences at −1, 30 at +1

0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1
2.00	1.82	1.67	1.54	1.43	1.33	1.25	1.18	1.11	1.05	1.00
1.00	0.82	0.67	0.54	0.43	0.33	0.25	0.18	0.11	0.05	0.00
0.50	0.37	0.27	0.19	0.13	0.08	0.05	0.03	0.01	.00	0.00
0.25	0.17	0.11	0.07	0.04	0.02	0.01	.00	.00	.00	0.00
0.13	0.07	0.04	0.02	0.01	0.01	.00	.00	.00	.00	0.00
0.06	0.03	0.02	0.01	.00	.00	.00	.00	.00	.00	0.00
0.03	0.02	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.02	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.01	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.02	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.03	0.01	.00	.00	.00	.00	.00	.00	.00	.00	0.00
0.05	0.02	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.08	0.03	0.01	.00	.00	.00	.00	.00	.00	.00	0.00
0.11	0.05	0.02	0.01	.00	.00	.00	.00	.00	.00	0.00
0.17	0.08	0.03	0.01	.00	.00	.00	.00	.00	.00	0.00
0.25	0.12	0.05	0.02	.00	.00	.00	.00	.00	.00	0.00
0.38	0.18	0.08	0.03	0.01	.00	.00	.00	.00	.00	0.00
0.57	0.29	0.13	0.05	0.01	.00	.00	.00	.00	.00	0.00
0.86	0.44	0.20	0.08	0.02	0.01	.00	.00	.00	.00	0.00
1.28	0.69	0.32	0.13	0.04	0.01	.00	.00	.00	.00	0.00
1.92	1.07	0.52	0.21	0.07	0.02	.00	.00	.00	.00	0.00
2.89	1.65	0.83	0.35	0.12	0.03	0.01	.00	.00	.00	0.00
4.33	2.56	1.32	0.58	0.20	0.05	0.01	.00	.00	.00	0.00
6.49	3.97	2.11	0.95	0.34	0.09	0.02	.00	.00	.00	0.00
9.74	6.15	3.38	1.57	0.58	0.16	0.03	.00	.00	.00	0.00
14.61	9.53	5.41	2.58	0.99	0.28	0.05	0.01	.00	.00	0.00
21.92	14.77	8.65	4.26	1.68	0.49	0.10	0.01	.00	.00	0.00
32.88	22.89	13.85	7.04	2.86	0.87	0.17	0.02	.00	.00	0.00
49.32	35.49	22.15	11.61	4.87	1.51	8.31	0.03	.00	.00	0.00
73.98	55.00	35.45	19.16	8.28	2.65	0.56	0.06	.00	.00	0.00
110.97	85.25	56.71	31.61	14.07	4.64	1.00	0.11	.00	.00	0.00
166.45	132.14	90.74	52.16	23.92	8.12	1.80	0.21	0.01	.00	0.00
249.68	204.82	145.19	86.06	40.66	14.21	3.24	0.38	0.01	.00	0.00
374.51	317.48	232.30	142.00	69.12	24.86	5.83	0.70	0.03	.00	0.00
187.26	174.61	139.38	92.30	48.38	18.64	4.66	0.60	0.02	.00	0.00

Just because we experienced one exact sequence of six coin flips wherein the drawdown was $2 doesn't mean we can use that as any kind of a meaningful benchmark, since the next exact sequence is equally likely to be any other possible sequence as it is to be the sequence we are basing this drawdown figure on.

Return to the coin toss, whereby if we win, we win $1, and if we lose, we lose $1. Suppose 20 tosses have gone by and you have experienced a drawdown of $5 at one point. What does this mean? Does this mean that we can expect "about" a $5 drawdown on the next 20 tosses? Since coin tossing is an independent trials process (as trading is for the most part), the answer to all of these questions is no. The only estimating we can perform here is one based on the losing streaks involved. With a 20-coin toss we can figure probabilities of getting 20 tosses against us, 19 tosses, and so on. But what we are talking about with drawdown is absolute worst case—an extreme. What we are looking for is an answer to the question, "How far out on the tails of the distribution, to the adverse side, is the limit?" The answer is that there is no limit—all future coin tosses, the next 20 tosses and all sequences of 20 tosses, could go against us. It's highly unlikely, but it could happen. To assume that there is a maximum drawdown that we can expect is simply an illusion. The idea is propagated for a trader's peace of mind. Statistically, it has no significance. If we are trading on a fixed fractional basis (where the drawdown is also a function of when it happens—i.e., how big the account was when the drawdown started), then drawdown is absolutely meaningless.

Third, the drawdown under fixed fraction is not the drawdown we would encounter on a constant contract basis (i.e., nonreinvestment). This was demonstrated in the previous chapter. Fourth and finally, in this exercise we are trying to discern only how much to commit to the next trade, not the next sequence of trades. Drawdown is a sequence of trades—should the maximum drawdown occur on one trade, then that one trade would also be the biggest losing trade.

If you want to measure the downside of a system, then you should look at the biggest losing trade, since drawdown is arbitrary and, in effect, meaningless. This becomes even more so when you are considering fixed fractional (i.e., reinvestment of returns) trading. Many traders try to "limit their drawdown" either consciously (as when they are designing trading systems) or subconsciously. This is understandable, as drawdown is the trader's nemesis. Yet we see that, as a result of its arbitrary nature, drawdown is uncontrollable. What is controllable, at least to an extent, is the largest loss. As you have seen, optimal f is a function of the largest loss. It is possible to control your largest loss by many techniques, such as only day-trading, using options, and so on. The point here is that you can control your largest loss as well as your frequency of large losses (at least to some extent).

It is important to note at this point that the drawdown you can expect with fixed fractional trading, as a percentage retracement of your account equity, historically would have been at least as much as f percent. In other words, if f is .55, then your drawdown would have been at least 55% of your equity (leaving you with 45% at one point). This is so because if you are trading at the optimal f, as soon as your biggest loss is hit, you would experience the drawdown equivalent to f. Again, assuming f for a system is .55, and assuming that translates into trading one contract for every $10,000, your biggest loss would be $5,500. As should by now be obvious, when the biggest loss was encountered (again we're speaking historically, i.e., about what would have happened), you would have lost $5,500 for each contract you had on, and you would have had one contract on for every $10,000 in the account. Therefore, at that point your drawdown would have been 55% of equity. However, it is possible that the drawdown would continue, that the next trade or series of trades would draw your account down even more. Therefore, the better a system, the higher the f. The higher the f, generally the higher the drawdown, since the drawdown (as a percentage) can never be any less than the f. There is a paradox involved here, in that if a system is good enough to generate an optimal f that is a high percentage, then the drawdown for such a good system will also be quite high. While optimal f allows you to experience the greatest geometric growth, it also gives you enough rope to hang yourself.

CONSEQUENCES OF STRAYING TOO FAR FROM THE OPTIMAL *f*

The fact that the difference between being at the optimal value for f and being at any other value increases geometrically over time is particularly important to gamblers. Time in this sense is synonymous with action. For years, a simple system for blackjack has been to simply keep track of how many fives have fallen from the deck. The fewer the fives contained in the remaining deck, the greater is the player's advantage over the casino. Depending on the rules of the casino, this advantage could range to almost as high as 3.6% for a deck with no remaining fives. Roughly, then, the optimal f for this strategy would range from 0 to about .075 to .08 for each hand, depending on how many fives had fallen (i.e., you would use a different f value for each different number of remaining fives in a deck. This is a dependent trials process, and therefore your optimal betting strategy would be to trade variable fraction based on the optimal f for each scenario of the ratio of fives left in the deck). If you go into the casino and play through only one deck, you will not be penalized for deviating from the optimal f (as you would if you were to play 1,000 hands). It is incorrect to think that if you have an edge on a particular hand, you should simply increase the size of your wager. How much you increase it by is paramount.

To illustrate, if you have a stake of $500 and start playing at a table where $5 is the minimum bet, your minimum bet is therefore 1% of your stake. If you encounter, during the course of the deck, a situation where all fives are depleted from the deck, you then have an edge of anywhere from 3 to 3.6%, depending on the house rules. Say your optimal f now is .08, or one bet per every $62.50 in your stake ($5, the maximum possible loss on the next hand, divided by .08).

Suppose you had been breaking even to this point in the game and still had $500 in your stake. You would then bet $40 on the next hand ($500/$62.50 * $5). If you were to bet $45, you could expect a decrease in performance. There is no benefit to betting the extra $5 unit. This decrease in performance grows geometrically over time. If you calculate your optimal f on each hand, and slightly over- or underbet, you can expect a decrease in performance geometrically proportional to the length of the game (the action). If you were to bet, say, \$100, on the situation described above, you would be at an f factor way out to the right of the optimal f. You wouldn't stand a chance over time—no matter how good a card counter you were! If you are too far to the right of the optimal f, even if you know exactly what cards remain in the deck, you are in a losing situation!

Next are four more charts, which, if you still do not see, drive home the importance of being near the optimal f. These are equity curve charts. An equity curve is simply the total equity of an account (plotted on the Y axis) over a period of time or series of trades (the X axis). On these four charts, we assume an account starts out with 10 units. Then the following sequence of 21 trades/bets is encountered:

$$1, 2, 1, -1, 3, 2, -1, -2, -3, 1, -2, 3, 1, 1, 2, 3, 3, -1, 2, -1, 3$$

If you have done the calculations yourself, you will find that the optimal f is .6, or bet one unit for every five in your stake (since the biggest losing trade is for three units).

The first equity curve (Figure 4.8) shows this sequence on a constant one-contract basis. Nice consistency. No roller-coaster drawdowns. No geometric growth, either.

Next comes an equity curve with f at .3, or bet one unit for every 10 units in your stake (Figure 4.9). Makes a little more than constant contract.

On the third equity curve graph you see the sequence at the optimal f value of .6, or one bet for every five in your stake (Figure 4.10). Notice how much more it has made than at $f = .3$.

The final equity curve shows the sequence of bets at $f = .9$, or one bet for every $3^1/_3$ units in your stake (Figure 4.11). Notice how quickly the equity took off until it hit the drawdown periods (7 through 12). When f is too high, the market systems get beaten down so low during a drawdown that it takes far longer to come out of them, if ever, than at the optimal values.

Even at the optimal values, the drawdowns can be quite severe for any market/system. It is not unusual for a market system trading one contract under optimal f to see 80 to 95% of its equity erased in the bad drawdowns. But notice how at the optimal values the equity curve is able to recover in short order and go on to higher ground. These four charts have all traded the same sequence of trades, yet look at how using the optimal f affects performance, particularly after drawdowns.

Obviously, the greater an account's capitalization, the more accurately its traders can stick to optimal f, as the dollars per single contract required are a smaller percentage of the total equity. For example, suppose optimal f for a given market system dictates we trade one contract for every \$5,000 in an account. If an account starts out with \$10,000 equity, then it can gain (or lose) 50% before a quantity adjustment is necessary. Contrast this to a $500,000 account, where there would be a contract adjustment for every 1% change in equity. Clearly, the larger account can take advantage of the benefits provided by optimal f better than a smaller account can. Theoretically, optimal f assumes you can trade in infinitely divisible quantities, which is not the case in real life, where the smallest quantity you can trade in is a single contract. In the asymptotic sense this does not matter. In the real-life integer-bet scenario, a good case could be presented for trading a market system that requires as small a percentage of the account equity as possible, especially for smaller accounts. But there is a trade-off here as well. Since we are striving to trade in markets that would require us to trade in greater multiples, we will be paying greater commissions, execution costs, and slippage. Bear in mind that the amount required per contract in real life is the greater of the initial margin requirement or the dollar amount per contract dictated by the optimal f.

As the charts bear out, you pay a substantial penalty for deviating from the optimal fixed dollar fraction. *Being at the right value for* f *is more important than how good your trading system is* (provided of course that the system is profitable on a single-contract basis)! Therefore, the finer you can cut it (i.e., the more frequently you adjust the size of the positions you are trading, so as to align yourself with what the optimal f dictates), the better off you are. Most accounts, therefore, would be better off trading the smaller markets. Corn may not seem like a very exciting market to you compared to the S&Ps. Yet, for most people, the corn market can get awfully exciting if they have a few hundred contracts on.

Throughout the text, we refigure the amount of contracts you should have on for the next trade based on the dictates of the optimal f for a given market system. However, the finer you can cut it, the better. If you refigure how many contracts you should have on every day as opposed to simply every week, you will be that much better off. If you refigure how many contracts you should have on every hour as opposed to every day, you will be even better off. However, there is the old trade-off of commissions, slippage, and fees, not to mention the cost of mistakes, which will be more likely the more frequently you realign yourself with the dictates of the optimal f. Bear in mind that realigning yourself on every trade is not the only way to do it, and the finer (more frequently) you can cut it—the more often you realign yourself with the dictates of the optimal f—the more the benefits of the optimal f will work for you. Ideally, you will realign yourself with optimal f on as close to a continuous basis as possible with respect to the trade-offs of commissions, fees, slippage, and the costs of human error.

It is doubtful whether anyone in the history of the markets has been able to religiously stick to trading on a constant contract basis. If someone quadrupled their money, would they still stick to trading in the same exact

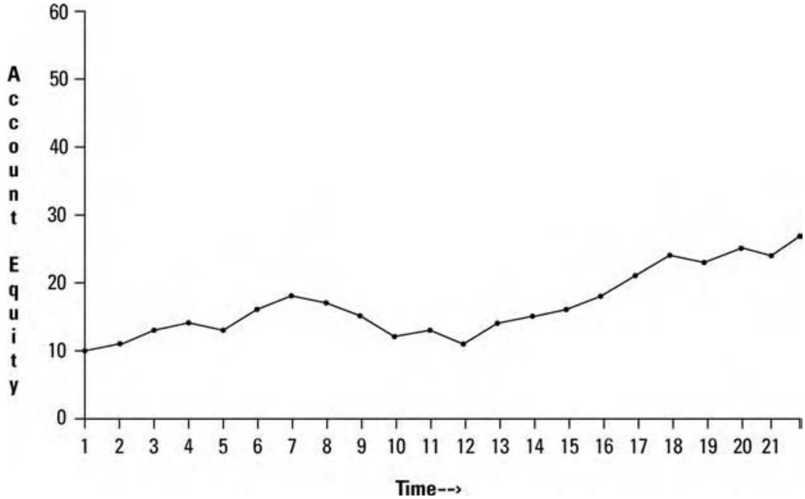

FIGURE 4.8 Equity curve for 21 trades on a constant contract basis

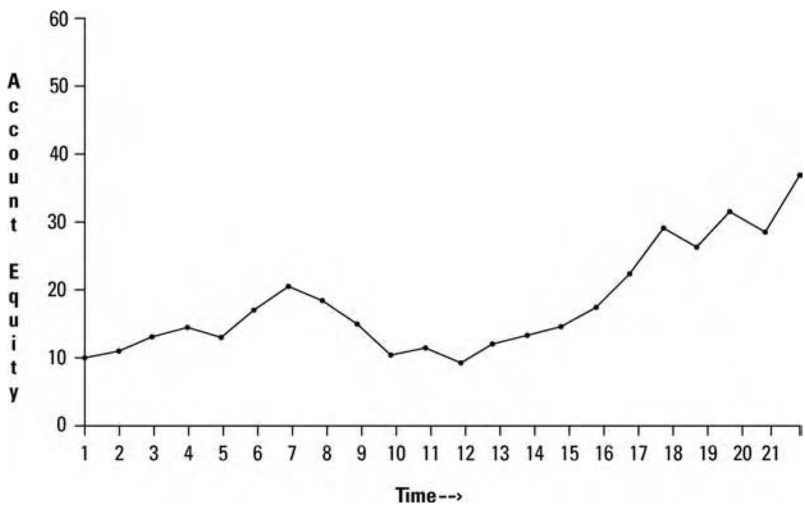

FIGURE 4.9 Equity curve for 21 trades with $f = .30$, or 1 contract for every 10 units in the stake

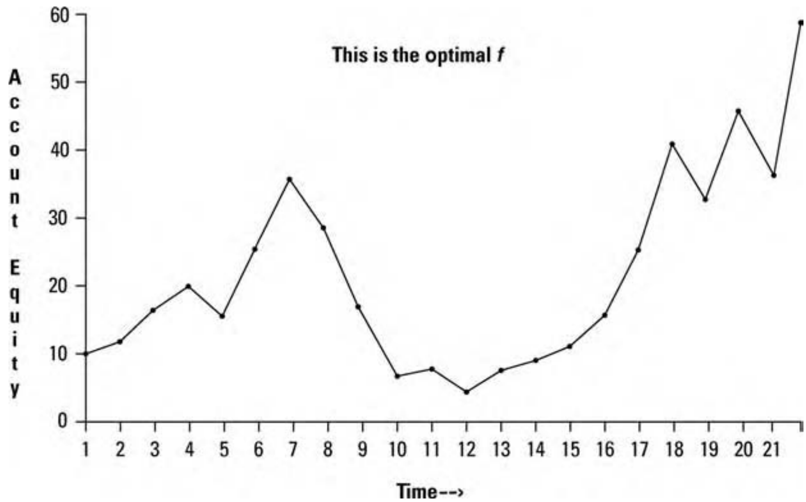

FIGURE 4.10 Equity curve for 21 trades with $f = .60$, or 1 contract for every 5 units in the stake

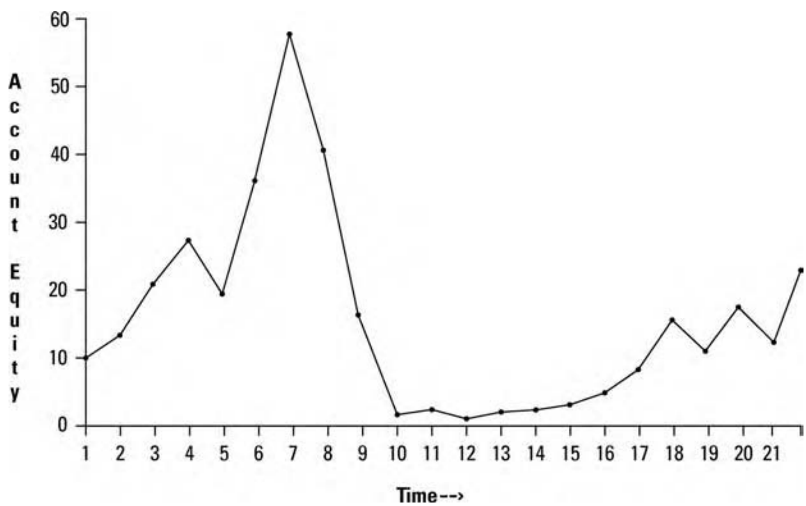

FIGURE 4.11 Equity curve for 21 trades with $f = .90$, or 1 contract for every 3.33 units in the stake

size? Conversely, would someone trading 10 contracts on every trade who was suddenly cut down to trading less than 10 contracts inject enough capital into the account to margin 10 contracts again on the next trade? It's quite unlikely. Any time a trader trading on a constant contract basis deviates from always trading the same constant contract size, the problem of what quantities to trade in arises. This is so whether the trader recognizes this problem or not. As you have seen demonstrated in this chapter, this is a problem for the trader. Constant contract trading is not the solution, because you can never experience geometric growth trading constant contract. So, like it or not, the question of what quantity to take on the next trade is inevitable for everyone. To simply select an arbitrary quantity is a costly mistake. Optimal f is factual; it is mathematically correct.

Are there traders out there who aren't planning on reinvesting their profits? Unless we're looking at optimal f via the highest TWR, we wouldn't know a good market system.

If a system is good enough, it is often possible to have a value for f that implies applying a dollar amount per contract that is less than the initial margin. Remember that f gives us the peak of the curve; to go off to the right of the peak (take on more contracts) provides no benefit. But the trader need not use that value for f that puts him at the peak; he may want to go to the left of the peak (i.e., apply more dollars in equity to each contract he puts on). You could, for instance, divide your account into two equal parts and resolve to keep one part cash and one part as dollars to apply to trading positions and use f on that half. This in effect would amount to a half f or fractional f strategy.

By now it should be obvious that we have a working range for usable values of f, that range being from zero to the optimal value. The higher you go within this range, the greater the return (up to but not beyond the optimal f) and the greater the risk (the greater the expected drawdowns in size—not, however, in frequency). The lower you go in this range, the less the risk (again in terms of extent but not frequency of drawdowns), and the less the potential returns. However, as you move down this range toward zero, the greater the probability is that an account will be profitable (remember that a constant-contract-based account has a greater probability of being profitable than a fractional f one). Ziemba's *Gambling Times* articles on Kelly demonstrated that *at smaller profit targets the half Kelly was more apt to reach these levels before halving than was a full Kelly bet. In other words, the fractional Kelly (fractional* f*) bet is safer—it has less variance in the final outcome after X bets. This ability to choose a fraction of the optimal* f *(choosing a value between 0 and the optimal* f*) allows you to have any desired risk/return trade-off that you like.*

Referring back to our four equity curve charts where the optimal $f = .60$, notice how nice and smooth the half f chart of $f = .30$ is. Half f makes for a much smoother equity curve than does full *f*. Of course, the trade-off is less return—again, a difference that grows as time passes.

Here, a word of caution is in order. Just as there is a price to be paid (in reduced return and greater drawdowns) for being too far to the right of the peak of the *f* curve (i.e., too many contracts on), there is also a price to be paid for being to the left of the peak of the *f* curve (i.e., too few contracts on). This price is not as steep as being too far to the right, so if you must err, err to the left.

As you move to the left of the peak of the curve (i.e., allocate more dollars per contract) you reduce your drawdowns arithmetically. However, you also reduce your returns geometrically. Reducing your returns geometrically is the price you pay for being to the left of the optimal *f* on the *f* curve. However, using the fractional *f* still makes good sense in many cases. When viewed from the perspective of time required to reach a specific goal (as opposed to absolute gain), the fractional *f* strategy makes a great deal of sense. Very often, a fractional *f* strategy will not take much longer to reach a specific goal than will the full *f* (the height of the goal and what specific fraction of *f* you choose will determine how much longer). If minimizing the time required to reach a specific goal times the potential drawdown as a percentage of equity retracement is your priority, then the fractional *f* strategy is most likely for you.

Aside from, or in addition to, diluting the optimal *f* by using a percentage or fraction of the optimal *f*, you could diversify into other markets and systems (as was just done by putting 50% of the account into cash, as if cash were another market or system).

EQUALIZING OPTIMAL *f*

Optimal *f* will yield the greatest geometric growth on a stream of outcomes. This is a mathematical fact. Consider the hypothetical stream of outcomes:

$$+2, -3, +10, -5$$

This is a stream from which we can determine our optimal *f* as .17, or to bet one unit for every $29.41 in equity. Doing so on such a stream will yield the greatest growth on our equity.

Consider for a moment that this stream represents the trade profits and losses (P&Ls) on one share of stock. Optimally, we should buy one share of stock for every $29.41 that we have in account equity, regardless of what the current stock price is. But suppose the current stock price is $100 per share. Further, suppose the stock was $20 per share when the first two trades occurred and was $50 per share when the last two trades occurred.

Recall that with optimal f we are using the stream of past trade P&Ls as a proxy for the distribution of expected trade P&Ls currently. Therefore, we can preprocess the trade P&L data to reflect this by converting the past trade P&L data to reflect a commensurate percentage gain or loss based upon the current price.

For our first two trades, which occurred at a stock price of \$20 per share, the \$2 gain corresponds to a 10% gain and the \$3 loss corresponds to a 15% loss. For the last two trades, taken at a stock price of \$50 per share, the \$10 gain corresponds to a 20% gain and the \$5 loss corresponds to a 10% loss.

The formulas to convert raw trade P&Ls to percentage gains and losses for longs and shorts are as follows:

$$\text{P\&L\%} = \text{Exit Price/Entry Price} - 1 \quad \text{(for longs)} \tag{4.10a}$$

$$\text{P\&L\%} = \text{Entry Price/Exit Price} - 1 \quad \text{(for shorts)} \tag{4.10b}$$

or we can use the following formula to convert both longs and shorts:

$$\text{P\&L\%} = \text{P\&L in Points/Entry Price} \tag{4.11}$$

Thus, for our four hypothetical trades, we now have the following stream of *percentage* gains and losses (assuming all trades are long trades):

$$+.1, -.15, +.2, -.1$$

We call this new stream of translated P&Ls the *equalized data*, because it is equalized to the price of the underlying instrument when the trade occurred.

To account for commissions and slippage, you must adjust the exit price downward in Equation (4.10a) for an amount commensurate with the amount of the commissions and slippage. Likewise, you should adjust the exit price upward in (4.10b). If you are using (4.11), you must deduct the amount of the commissions and slippage (in points again) from the numerator P&L in Points.

Next, we determine our optimal f on these percentage gains and losses. The f that is optimal is .09. We must now convert this optimal f of .09 into a dollar amount based upon the current stock price. This is accomplished by the following formula:

$$f\$ = \text{Biggest \% Loss} * \text{Current Price} * \text{\$per Point}/-f \tag{4.12}$$

Thus, since our biggest percentage loss was $-.15$, the current price is \$100 per share, and the number of dollars per full point is 1 (since we are dealing with buying only one share), we can determine our $f\$$ as:

$$\begin{aligned} f\$ &= -.15 * 100 * 1/-.09 \\ &= -15/-.09 \\ &= 166.67 \end{aligned}$$

Thus, we would optimally buy one share for every \$166.67 in account equity. If we used 100 shares as our unit size, the only variable affected would have been the number of dollars per full point, which would have been 100. The resulting $f\$$ would have been \$16,666.67 in equity for every 100 shares.

Suppose now that the stock went down to \$3 per share. Our $f\$$ equation would be exactly the same except for the current price variable, which would now be 3. Thus, the amount to finance one share by becomes:

$$\begin{aligned} f\$ &= -.15 * 3 * 1/-.09 \\ &= -.45/-.09 \\ &= 5 \end{aligned}$$

We optimally would buy one share for every \$5 we had in account equity.

Notice that the optimal f does not change with the current price of the stock. It remains at .09. However, the $f\$$ changes continuously as the price of the stock changes. This doesn't mean that you must alter a position you are already in on a daily basis, but it does make it more likely to be beneficial that you do so. As an example, if you are long a given stock and it declines, the dollars that you should allocate to one unit (100 shares in this case) of this stock will decline as well, with the optimal f determined off of equalized data. If your optimal f is determined off of the raw trade P&L data, it will not decline. In both cases, your daily equity is declining. Using the equalized optimal f makes it more likely that adjusting your position size daily will be beneficial.

Equalizing the data for your optimal f necessitates changes in the by-products. We have already seen that both the optimal f and the geometric mean (and hence the TWR) change. The arithmetic average trade changes because now it, too, must be based on the idea that all trades in the past must be adjusted as if they had occurred from the current price. Thus, in our hypothetical example of outcomes on one share of +2, −3, +10, and −5, we have an average trade of \$1. When we take our percentage gains and losses of +.1, −15, +.2, and −.1, we have an average trade (in percent) of +.5. At \$100 per share, this translates into an average trade of 100 ∗ .05 or \$5 per trade. At \$3 per share, the average trade becomes \$.15(3 ∗ .05).

The geometric average trade changes as well.

$$\text{GAT} = \text{G} * (\text{Biggest Loss}/-f)$$

where: G = Geometric mean − 1.
f = Optimal fixed fraction.

(and, of course, our biggest loss is always a negative number). This equation is the equivalent of:

$$\text{GAT} = (\text{geometric mean} - 1) * f\$$$

We have already obtained a new geometric mean by equalizing the past data. The $f\$$ variable, which is constant when we do not equalize the past data, now changes continuously, as it is a function of the current underlying price. Hence, our geometric average trade changes continuously as the price of the underlying instrument changes.

Our threshold to the geometric also must be changed to reflect the equalized data.

$$T = AAT/GAT * \text{Biggest Loss}/-f \tag{4.13}$$

where: T = The threshold to the geometric.
AAT = The arithmetic average trade.
GAT = The geometric average trade.
f = The optimal f (0 to 1).

This equation can also be rewritten as:

$$T = AAT/GAT * f\$ \tag{4.13a}$$

Now, not only do the AAT and GAT variables change continuously as the price of the underlying changes, so too does the $f\$$ variable.

Finally, when putting together a portfolio of market systems we must figure daily HPRs. These too are a function of $f\$$:

$$\text{Daily HPR} = D\$/f\$ + 1 \tag{4.14}$$

where: D\$ = The dollar gain or loss on 1 unit from the previous day. This is equal to (Tonight's Close – Last Night's Close) * Dollars per Point.
$f\$$ = The current optimal f in dollars, calculated from Equation (4.12). Here, however, the current price variable is last night's close.

For example, suppose a stock tonight closed at \$99 per share. Last night it was \$102 per share. Our biggest percentage loss is −15. If our f is .09, then our $f\$$ is:

$$\begin{aligned} f\$ &= -.15 * 102 * 1/-.09 \\ &= -15.3/-.09 \\ &= 170 \end{aligned}$$

Since we are dealing with only 1 share, our dollars per point value is \$1. We can now determine our daily HPR for today as:

$$\begin{aligned} \text{Daily HPR} &= (99 - 102) * 1/170 + 1 \\ &= -3/170 + 1 \\ &= -.01764705882 + 1 \\ &= .9823529412 \end{aligned}$$

Return now to what was said at the outset of this discussion. Given a stream of trade P&Ls, the optimal f will make the greatest geometric growth on that stream (provided it has a positive arithmetic mathematical expectation). We use the stream of trade P&Ls as a proxy for the distribution of possible outcomes on the next trade. Along this line of reasoning, it may be advantageous for us to equalize the stream of past trade profits and losses to be what they would be if they were performed at the current market price. In so doing, we may obtain a more realistic proxy of the distribution of potential trade profits and losses on the next trade. Therefore, we should figure our optimal f from this adjusted distribution of trade profits and losses.

This does not mean that we would have made more by using the optimal f off of the equalized data. We would *not* have, as the following demonstration shows:

P&L	Percentage	Underlying Price	*f*$	Number of Shares	Cumulative
At $f = .09$, trading the equalized method:					$10,000
+2	.1	20	$33.33	300	$10,600
−3	−.15	20	$33.33	318	$9,646
+10	.2	50	$83.33	115.752	$10,803.52
−5	−.1	50	$83.33	129.642	$10,155.31

P&L	Percentage	Underlying Price	*f*$	Number of Shares	Cumulative
At $f = .17$, trading the nonequalized method:					$10,000
+2	.1	20	$29.41	340.02	$10,680.04
−3	−.15	20	$29.41	363.14	$9,590.61
+10	.2	50	$29.41	326.1	$12,851.61
−5	−.1	50	$29.41	436.98	$10,666.71

However, if all of the trades were figured off of the current price (say $100 per share), the equalized optimal f would have made more than the raw optimal f.

Which, then, is the better to use? Should we equalize our data and determine our optimal f (and its by-products), or should we just run everything as it is? This is more a matter of your beliefs than it is mathematical fact. It is a matter of what is more pertinent in the item you are trading, percentage changes or absolute changes. Is a $2 move in a $20 stock the same as a $10 move in a $100 stock? What if we are discussing dollars and euros? Is a .30-point move at .4500 the same as a .40-point move at .6000?

My personal opinion is that you are probably better off with the equalized data. Often, the matter is moot, in that if a stock has moved from \$20 per share to \$100 per share and we want to determine the optimal f, we want to use current data. The trades that occurred at \$20 per share may not be representative of the way the stock is presently trading, regardless of whether they are equalized or not.

Generally, then, you are better off not using data where the underlying was at a dramatically different price than it presently is, as the characteristics of the way the item trades may have changed as well. In that sense, the optimal f off of the raw data and the optimal f off of the equalized data will be identical if all trades occurred at the same underlying price.

So we can state that if it does matter a great deal whether you equalize your data or not, then you're probably using too much data anyway. You've gone so far into the past that the trades generated back then probably are not very representative of the next trade. In short, we can say that it doesn't much matter whether you use equalized data or not, and if it does, there's probably a problem. If there isn't a problem, and there is a difference between using the equalized data and the raw data, you should opt for the equalized data. This does not mean that the optimal f figured off of the equalized data would have been optimal in the past. It would not have been. The optimal f figured off of the raw data would have been the optimal in the past. However, in terms of determining the as-yet-unknown answer to the question of what will be the optimal f (or closer to it tomorrow), the optimal f figured off of the equalized data makes better sense, as the equalized data is a fairer representation of the distribution of possible outcomes on the next trade.

Equations (4.10a) through (4.11) will give different answers depending upon whether the trade was initiated as a long or a short. For example, if a stock is bought at 80 and sold at 100, the percentage gain is 25. However, if a stock is sold short at 100 and covered at 80, the gain is only 20%. In both cases, the stock was bought at 80 and sold at 100, but the sequence—the chronology of these transactions—must be accounted for. As the chronology of transactions affects the distribution of percentage gains and losses, we assume that the chronology of transactions in the future will be more like the chronology in the past than not. Thus, Equations (4.10a) through (4.11) will give different answers for longs and shorts.

Of course, we could ignore the chronology of the trades (using 4.01c for longs and using the exit price in the denominator of 4.01c for shorts), but to do so would be to reduce the information content of the trade's history.

Further, the risk involved with a trade is a function of the chronology of the trade, a fact we would be forced to ignore.

FINDING OPTIMAL *f* VIA PARABOLIC INTERPOLATION

Originally, I had hoped to find a method of finding the optimal f by way of a single equation like the Kelly formula. In finding the optimal f we are looking for that value for f which generates the highest TWR in the domain 0 to 1.0 for f. Since f is the only variable we have to maximize the TWR for, we say that we are *maximizing in one dimension*.

We can use another technique to iterate to the optimal f with a little more style than the brute methods already described. Recall that in the iterative technique we bracket an intermediate point (A, B); test a point within the bracket (X); and obtain a new, smaller bracketing interval (either A, X or X, B). This process continues until the answer is converged upon. This is still brutish, but not so brutish as the simple 0 to 1 by .01 loop method.

The best (i.e., fastest and most elegant) way to find a maximum in one dimension, when you are certain that only one maximum exists, that each successive point to the left of the maximum lessens, and that each successive point to the right of the maximum lessens (as is the case with the shape of the f curve), is to use *parabolic interpolation.* When there is only one local extreme (be it a maximum or a minimum) in the range you are searching, parabolic interpolation will work. If there is more than one local extreme, parabolic interpolation will not work (see Figure 4.12).

With this technique we simply input three coordinate points. The axes of these points are the TWRs (Y axis) and the f values (X axis). We can find the abscissa (the X axis, or f value corresponding to the peak of a parabola) by the following formula, given the three coordinates:

$$\text{ABSCISSA} = X2 - .5 * \frac{(X2 - X1)^2 * (Y2 - Y3) - (X2 - X3)^2 * (Y2 - Y1)}{(X2 - X1) * (Y2 - Y3) - (X2 - X3) * (Y2 - Y1)} \quad (4.15)$$

The result returned by this equation is the value for f (or X if you will) that corresponds to the abscissa of a parabola where the three coordinates (X1, Y1), (X2, Y2), (X3, Y3) lie on the parabola.

The object now is to superimpose a parabola over the f curve, change one of the input coordinates to draw an amended parabola, and keep on doing this until the abscissa of the most recent parabola converges with the

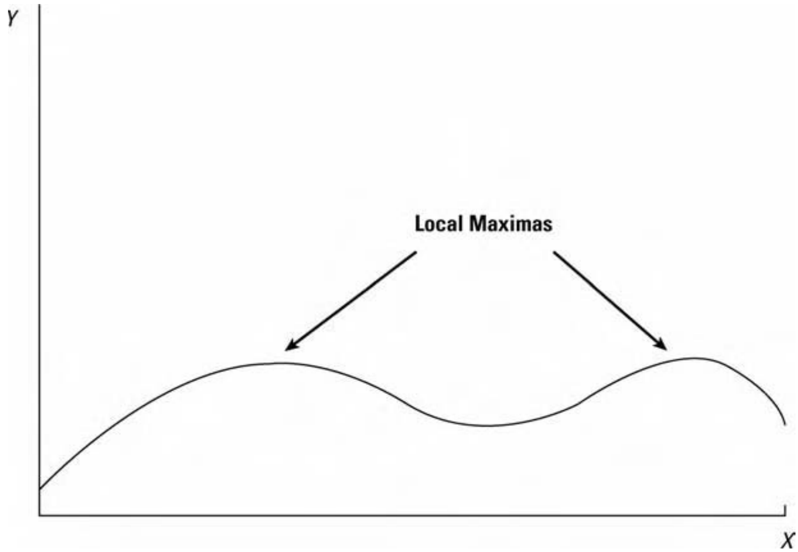

FIGURE 4.12 A function with two local extremes

previous parabola's abscissa. Convergence is determined when the absolute value of the difference between two abscissas is less than a prescribed amount called the tolerance, or TOL for short. This amount should be chosen with respect to how accurate you want your f to be. Generally, I use a value of .005 for TOL. This gives the same accuracy in searching for the optimal f as the brute force techniques described earlier.

We can start with two of the three coordinate points as (0, 0), (1.0, 0). The third coordinate point must be a point that lies on the actual f curve itself. Let us choose the X value here to be 1 – TOL, or .995. To make sure that the coordinate lies on the f curve, we determine our Y value by finding what the TWR is at $f = .995$. Assume we are looking for the optimal f for the four-trade example −1, −3, 3, 5. For these four trades the TWR at $f = .995$ is .017722. Now we have the three coordinates: (0, 0), (.995, .017722), (1.0, 0). We plug them into the above described equation to find the abscissa of a parabola that contains these three points, and our result is .5.

Now we compute the TWR corresponding to this abscissa; this equals 1.145833. Since the X value here now (.5) is to the left of the value for X2 previously (.995), we move our three points over to the left, and compute a new abscissa to the parabola that contains the three points (0, 0), (.5, 1.145833), (.995, .017722).

This abscissa is at .499439. The TWR corresponding to this f value is 1.146363. When we encounter a difference in abscissas that is less than or equal to TOL, we will have converged to the optimal f.

Shown here are the full seven passes and the values used in each pass so that you may better understand this technique.

PARABOLIC INTERPOLATION

Pass#	x1	y1	x2	y2	x3	y3	abscissa
1	0	0	0.995	0.017722	1	0	0.5
2	0	0	0.5	1.145833	0.995	0.017722	0.499439
3	0	0	0.499439	1.146363	0.5	1.145833	0.426923
4	0	0	0.426923	1.200415	0.499439	1.146363	0.410853
5	0	0	0.410853	1.208586	0.426923	1.200415	0.387431
6	0	0	0.387431	1.218059	0.410853	1.208586	0.375727
7	0	0	0.375727	1.22172	0.387431	1.218059	0.364581
8	0	0	0.364581	1.224547	0.375727	1.22172	0.356964
9	0	0	0.356964	1.226111	0.364581	1.224547	0.350489

Convergence is extremely rapid. Typically, the more peaked the curve for the TWR (i.e., the more plays which comprise the TWR) the faster convergence is attained.

Refer now to Figure 4.13. This graphically shows the parabolic interpolation process for the coin-toss example with a 2:1 payoff, where the optimal f is .25. On the graph, notice the familiar f curve, which peaks out at .25. The first step here is to draw a parabola through three points: A, B, and C. The coordinates for A are (0, 0). For C the coordinates are (1, 0). For point B we now pick a point whose coordinates lie on the f curve itself. Once parabola ABC is drawn, we obtain its abscissa (the f value corresponding to the peak of the parabola ABC). We find what the TWR is for this f value. This gives us coordinates for point D. We repeat the process, this time drawing a parabola through points A, B, and D. Once the abscissa to parabola ABD is found, we can find the TWR that corresponds to an f value of the abscissa of parabola ABD. These coordinates (f value, TWR) give us point E.

Notice how quickly we are converging to the peak of the f curve at $f = .25$. If we were to continue with the exercise in Figure 4.12, we would next draw a parabola through points E, B, and D, and continue until we converged upon the peak of the f curve.

One potential problem with this technique from a computer standpoint is that the denominator in the equation that solves for the abscissa might

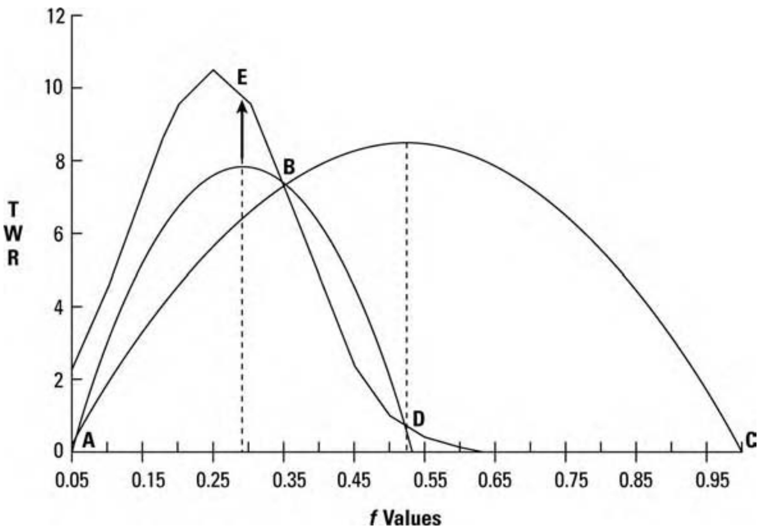

FIGURE 4.13 Parabolic interpolation performed on TWRs of 20 sequences of +2, −1

equal zero while running. One possible solution is the following fast and dirty patch in Java:

$$\text{dm} = (\text{x2} - \text{x1}) * (\text{y2} - \text{y3}) - (\text{x2} - \text{x3}) * (\text{y2} - \text{y1});$$
$$\text{If (dm} == 0.0)$$
$$\text{dm} = .00001;$$
$$\text{abscissa} = \text{x2} - 0.5 * (((\text{x2} - \text{x1}) * (\text{x2} - \text{x1}) * (\text{y2} - \text{y3}) - (\text{x2} - \text{x3})$$
$$(\text{x2} - \text{x3}) * (\text{y2} - \text{y1})/\text{dm};$$

This patch will not detract from the integrity of the results.

Note that this method can be used to find a local maximum for a given function, provided only one maximum exists within the range. The same technique could be used to find a local minimum for a function that opened upward (for example, the function Y equals X squared is such a function). Again, the technique will work provided there is only one local minimum (as is the case with our example). The only change from looking for a local maximum is in the equation for finding the abscissa:

$$\text{ABSCISSA} = \text{X2} + .5 * \frac{(\text{X2} - \text{X1})^2 * (\text{Y2} - \text{Y3}) - (\text{X2} - \text{X3})^2 * (\text{Y2} - \text{Y1})}{(\text{X2} - \text{X1}) * (\text{Y2} - \text{Y3}) - (\text{X2} - \text{X3}) * (\text{Y2} - \text{Y1})}$$

Note that here, for a local minimum, the first operator is a plus (+) sign, not a minus (−) sign as when we were looking for a local maximum.

THE NEXT STEP

The real problem with the formula to this point is that it makes the assumption that all HPRs have an equal probability of occurrence. What is needed is a new formula that allows for different probabilities associated with different HPRs. Such a formula would allow you to find an optimal f given a description of a probability distribution of HPRs. To accommodate this, we need to rework (4.06) to:

$$\text{HPR} = \left(1 + \left(\frac{A}{\left(\frac{W}{f}\right)}\right)\right)^{P} \tag{4.16}$$

where A = outcome of the scenario
P = probability of the scenario
W = worst outcome of all n scenarios
f = value for f which we are testing

Now, we obtain the terminal wealth relative, or TWR[4], originally given by (3.03) and (4.07) to:

$$\text{TWR} = \prod_{i=1}^{T} \text{HPR}_i$$

or

$$\text{TWR} = \prod_{i=1}^{T} \left(1 + \left(\frac{A_i}{\left(\frac{W}{f}\right)}\right)\right)^{P_i} \tag{4.17}$$

Finally, if we take Equation (4.18) to the Σ_{p_i} root, we can find our average compound growth per play, also called the geometric mean HPR, and replace that given in (4.08) which will become more important later on:

$$G = \text{TWR}^{1/\Sigma_{p_i}} \tag{4.18}$$

[4]In this formulation, unlike the 1990 formulations, the TWR has no special meaning. In this instance, it is simply an interim value used to find G, and it does *not* represent the multiple made on our starting stake. The variable named "TWR" is maintained solely for consistency's sake.

or

$$G = \left(\prod_{i=1}^{T}\left(\left(1+\left(\frac{A_i}{\frac{W}{f}}\right)\right)^{P_i}\right)\right)^{1/\Sigma P_i} \tag{4.18a}$$

where: T = Number of different scenarios.
TWR = Terminal wealth relative.
HPR_i = Holding period return of the ith scenario.
A_i = Outcome of the ith scenario.
P_i = Probability of the ith scenario.
W = Worst outcome of all n scenarios.
f = Value for f which we are testing.

Just as you could use Equation (4.04) to solve Equation (4.03), likewise you can use Equation (4.18a) to solve *any* optimal f problem. It will yield the same answers as the Kelly formulas when the data correctly has a Bernoulli distribution. It will yield the same answers as previously mentioned formulas if you pump a distribution of trades through it (where the probability of each trade is $1/T$). This formula can be used to maximize the expected value of the logarithm of any starting quantity of anything when there is exponential growth involved. We will now see how to employ this formula in the context of *scenario planning.*

SCENARIO PLANNING

People who forecast for a living, be they economists, stock market forecasters, meteorologists, government agencies, or the like, have a notorious history for incorrect forecasts. Most decisions anyone must make in life usually require that the individual make a forecast about the future.

There are a couple of pitfalls that immediately crop up. To begin with, people generally make more optimistic assumptions about the future than the actual probabilities. Most people feel that they are far more likely to win the lottery this month than they are to die in an auto accident, even though the probabilities of the latter are greater. This is true not only on the level of the individual; it is even more pronounced at the group level. When people work together, they tend to see a favorable outcome as the most likely result.

The second pitfall—and the more harmful—is that people make straight-line forecasts into the future. People predict what the price of a gallon of gas will be two years from now; they predict what will happen with their jobs, who the next president will be, what the next styles will be, and on and on. Whenever we think of the future, we tend to think in terms of a single most likely outcome. As a result, whenever we must make decisions, whether as an individual or a group, we tend to make these decisions based on what we think will be the single most likely outcome in the future. As a consequence, we are extremely vulnerable to unpleasant surprises.

Scenario planning is a partial solution to this problem. A scenario is simply a possible forecast, a story about one way that the future might unfold. Scenario planning is a collection of scenarios, to cover the spectrum of possibilities. Of course, the complete spectrum can never be covered, but the scenario planner wants to cover as many possibilities as he or she can. By acting in this manner, as opposed to using a straight-line forecast of the most likely outcome, the scenario planner can prepare for the future as it unfolds. Furthermore, scenario planning allows the planner to be prepared for what might otherwise be an unexpected event. Scenario planning is tuned to reality in that it recognizes that *certainty is an illusion.*

Suppose you are in a position where you are involved in the long-run planning for your company. Let's say you make a particular product. Rather than making a single most likely straight-line forecast, you decide to exercise scenario planning. You will need to sit down with the other planners and brainstorm for possible scenarios. What if you cannot get enough of the raw materials to make your product? What if one of your competitors fails? What if a new competitor emerges? What if you have severely underestimated demand for this product? What if a war breaks out on such and such a continent? What if it is a nuclear war?

Because each scenario is only one of several possible, each scenario can be considered seriously. But what do you do once you have defined these scenarios?

To begin with, you must determine what goal you would like to achieve for each given scenario. Depending upon the scenario, the goal need not be a positive one. For instance, under a bleak scenario, your goal may simply be damage control. Once you have defined a goal for a given scenario, you then need to draw up the contingency plans pertaining to that scenario to achieve the desired goal. For instance, in the rather unlikely bleak scenario where your goal is damage control, you need to have plans to go to should this scenario manifest itself so that you can minimize the damage. Scenario planning, above all else, provides the planner with a course of action to take should a certain scenario develop. It forces you to make plans before the fact; it forces you to be prepared for the unexpected.

Scenario planning, however, can do a lot more. There is a hand-in-glove fit between scenario planning and optimal f. Optimal f allows us to determine the optimal quantity to allocate to a given set of possible scenarios. Our existence limits us to existing in only one scenario at a time, even though we are planning for multiple futures, multiple scenarios.

Therefore, oftentimes, scenario planning puts us in a position where we must make a decision regarding how much of a resource to allocate today, given the possible scenarios of tomorrow. This is the true heart of scenario planning: quantifying it.

First, we must define each unique scenario. Second, we must assign a probability of that scenario's occurrence. Being a probability means that this number is between 0 and 1. We need not consider any further scenarios with a probability of 0. Note that these probabilities are not cumulative. In other words, the probability assigned to a given scenario is unique to that scenario. Suppose we are decision makers for XYZ Manufacturing Corporation. Two of the many scenarios we have are as follows. In one scenario, we have the probability of XYZ Manufacturing filing for bankruptcy with a probability of .15, and, in another scenario, we have XYZ being put out of business by intense foreign competition with a probability of .07. Now, we must ask if the first scenario, filing for bankruptcy, includes filing for bankruptcy due to the second scenario, intense foreign competition. If it does, then the probabilities in the first scenario must not take the probabilities of the second scenario into account, and we must amend the probabilities of the first scenario to be.08 (.15 − .07).

Just as important as the uniqueness of each probability to each scenario is that the sum of the probabilities of all of the scenarios we are considering must equal 1 exactly. They must equal not 1.01 nor .99, but 1.

For each scenario, we now have a probability of just that scenario assigned. We must now also assign an outcome result. This is a numerical value. It can be dollars made or lost as a result of a scenario's manifesting itself; it can be units of utility or medication or anything. However, our output is going to be in the same units that we put in.

You must have at least one scenario with a negative outcome in order to use this technique. This is mandatory.

A last prerequisite to using this technique is that the arithmetic mathematical expectation, the sum of all of the outcome results times their respective probabilities [Equation (1.01a)], must be greater than zero. If the arithmetic mathematical expectation equals zero or is negative, the following technique cannot be used.[5] That is not to say that scenario planning itself cannot be used. It can and should. However, optimal f can be incorporated with scenario planning only when there is a positive, mathematical expectation.

Lastly, you must try to cover as much of the spectrum of outcomes as possible. In other words, you really want to account for 99% of the possible outcomes. This may sound nearly impossible, but many scenarios can be

[5]However, later in the text we will be using scenario planning for portfolios, and, therein, a negative arithmetic mathematical expectation will be allowed and can possibly benefit the portfolio as a whole.

made broader so that you don't need 10,000 scenarios to cover 99% of the spectrum.

In making your scenarios broader, you must avoid the common pitfall of three scenarios: an optimistic one, a pessimistic one, and a third in which things remain the same. This is too simple, and the answers derived therefrom are often too crude to be of any value. Would you want to find your optimal f for a trading system based on only three trades?

So, even though there may be an unknowably large number of scenarios to cover the entire spectrum, we can cover what we believe to be about 99% of the spectrum of outcomes. If this makes for an unmanageably large number of scenarios, we can make the scenarios broader to trim down their number. However, by trimming down their number, we lose a certain amount of information. When we trim down the number of scenarios (by broadening them) to only three (a common pitfall), we have effectively eliminated so much information that the effectiveness of this technique is severely hampered.

What, then, is a good number of scenarios to have? As many as you can and still manage them.

Think of the two-to-one coin toss as a spectrum of two scenarios. Each has a probability, and that probability is .5 for each scenario, labeled heads and tails. Each has an outcome, +2 and −1, respectively:

Scenario	**Probability**	**Outcome**
Heads	.5	2
Tails	.5	−1

Assume again that we are decision making for XYZ. We are looking at marketing a new product of ours in a primitive, remote little country. Assume we have five possible scenarios we are looking at (in reality, you would have many more than this, but we'll use five for the sake of simplicity). These five scenarios portray what we perceive as possible futures for this primitive remote country, their probabilities of occurrence, and the gain or loss of investing there.

Scenario	**Probability**	**Result**
War	.1	−$500,000
Trouble	.2	−$200,000
Stagnation	.2	0
Peace	.45	$500,000
Prosperity	.05	$1,000,000
Sum	1.00	

The sum of our probabilities equals 1. We have at least one scenario with a negative result, and our mathematical expectation is positive:

$$(.1 * -500{,}000) + (.2 * -200{,}000) + \ldots \text{etc.} = 185{,}000$$

We can, therefore, use the technique on this set of scenarios.

Notice first, however, that if we used the single most likely outcome method, we would conclude that peace will be the future of this country, and we would then act as though peace were to occur, as though it were a certainty, only vaguely remaining aware of the other possibilities.

Returning to the technique, we must determine the optimal f. The optimal f is that value for f (between zero and one) which maximizes the geometric mean, using Equations (4.16 to 4.18). Now, we obtain the terminal wealth relative, or TWR using Equation (4.17). Finally, if we take Equation (4.17) to the Σ_{p_i} root, we can find our average compound growth per play, also called the geometric mean HPR, which will become more important later on. We use Equation (4.18) for this.

Here is how to perform these equations. To begin with, we must decide on an optimization scheme, a way of searching through the f values to find that f which maximizes our equation. Again, we can do this with a straight loop with f from .01 to 1, through iteration, or through parabolic interpolation.

Next, we must determine the worst possible result for a scenario among all of the scenarios we are looking at, regardless of how small the probability of that scenario's occurrence are. In the example of XYZ Corporation, this is −\$500,000.

Now, for each possible scenario, we must first divide the worst possible outcome by negative f. In our XYZ Corporation example, we will assume that we are going to loop through f values from .01 to 1. Therefore, we start out with an f value of .01. Now, if we divide the worst possible outcome of the scenarios under consideration by the negative value for f, we get the following:

$$\frac{-\$500{,}000}{-.01} = 50{,}000{,}000$$

Notice how negative values divided by negative values yield positive results, and vice versa. Therefore, our result in this case is positive. Now, as we go through each scenario, we will divide the outcome of the scenario by the result just obtained. Since the outcome to the first scenario is also the worst scenario—a loss of \$500,000—we now have:

$$\frac{-\$500{,}000}{50{,}000{,}000} = -.01$$

The next step is to add this value to 1. This gives us:

$$1 + (-.01) = .99$$

Last, we take this answer to the power of the probability of its occurrence, which in our example is .1:

$$.99^{.1} = .9989954713$$

Next, we go to the next scenario labeled *Trouble*, where there is a .2 loss of $200,000. Our worst-case result is still −$500,000. The f value we are working on is still .01, so the value we want to divide this scenario's result by is still 50 million:

$$\frac{-200,000}{50,000,000} = -.004$$

Working through the rest of the steps to obtain our HPR:

$$1 + (-.004) = .996$$
$$.996^{.2} = .9991987169$$

If we continue through the scenarios for this test value of .01 for f, we will find the three HPRs corresponding to the last three scenarios:

Stagnation	1.0
Peace	1.004487689
Prosperity	1.000990622

Once we have turned each scenario into an HPR for the given f value, we must multiply these HPRs together:

$$\begin{array}{l} \ .9989954713 \\ *\ .9991987169 \\ *\ 1.0 \\ *\ 1.004487689 \\ *\ 1.000990622 \\ \hline \ 1.003667853 \end{array}$$

This gives us the interim TWR, which in this case is 1.003667853. Our next step is to take this to the power of 1 divided by the sum of the probabilities. Since the sum of the probabilities will always equal 1 the way we are calculating this, we can state that we must raise the TWR to the power of 1 to give us the geometric mean. Since anything raised to the power of 1 equals itself, we can say that, in this case, our geometric mean equals the TWR. We therefore have a geometric mean of 1.003667853.

The answer we have just obtained in our example is our geometric mean corresponding to an f value of .01. Now we move on to an f value of .02, and repeat the whole process until we have found the geometric mean corresponding to an f value of .02. We will proceed as such until we arrive at that value for f which yields the highest geometric mean.

In the case of our example, we find that the highest geometric mean is obtained at an f value of .57, which yields a geometric mean of 1.1106. Dividing our worst possible outcome to a scenario (−500,000) by the negative optimal f yields a result of \$877,192.35. In other words, if XYZ Corporation wants to commit to marketing this new product in this remote country, they will optimally commit this amount to this venture at this time. As time goes by and things develop, the scenarios, their resultant outcomes, and probabilities will likewise change. This f amount will then change as well. The more XYZ Corporation keeps abreast of these changing scenarios, as well as the more accurate the scenarios they develop as input are, the more accurate their decisions will be. Note that if XYZ Corporation cannot commit this \$877,192.35 to this undertaking at this time, then they are too far beyond the peak of the f curve. It is the equivalent of the guy who has too many commodity contracts with respect to what the optimal f says he should have. If XYZ Corporation commits more than this amount to this project at this time, the situation would be analogous to a commodity trader with too few contracts.

There is an important point to note about scenarios and trading. What you use for a scenario can be any of a number of things:

1. It can be, as in the previous example, the outcomes that a given trade may take. This is useful if you are trading only one item. However, when you trade a portfolio of items, you violate the rule that all holding period lengths must be uniform.
2. If you know what the distribution of price outcomes will be, you can use that for scenarios. For example, suppose you have reason to believe that prices for the next day for a given item are normally distributed. Therefore, you can discern your scenarios based on the normal distribution. For example, in the normal distribution, 97.72% of the time, prices will not exceed 2 standard deviations to the upside, and 99.86% of the time they will not exceed 3 standard deviations to the upside. Therefore, as one scenario, you can have as the result something

between 2 and 3 standard deviations in price to the upside (whatever dollar amount that would be to you trading one unit over the next day, holding period), whose probability would be .9986 − .9772 = .0214, or 2.14% probability.

3. You can use the distributions of possible monetary outcomes for trading one unit with the given market approach over the next holding period. This is my preferred method, and it lends itself well to portfolio construction under the new framework.

Although I strongly recommend using the third item from the preceding list, whichever method you use, remember that *you want to be constantly updating your scenarios, their outcomes, and the probability of occurrences as conditions change. Then, you always want to go into the next holding period with what the formulas presently tell you is optimal.* The situation is analogous to that of a blackjack player. As the composition of the deck changes with each card drawn, so, too, do the player's probabilities. However, he must always adjust to what the probabilities currently dictate.

Although the quantity discussed here is a quantity of money, it can be a quantity of anything and the technique is just as valid.

If you create different scenarios for the stock market, the optimal f derived from this methodology will give you the correct percentage to be invested in the stock market at any given time. For instance, if the f returned is .65, then that means that 65% of your equity should be in the stock market, with the remaining 35% in, say, cash. This approach will provide you with the greatest geometric growth of your capital in a long-run sense. Of course, again, the output is only as accurate as the input you have provided the system with in terms of scenarios, their probabilities of occurrence, and resultant payoffs and costs.

This same process can be used as an alternative parametric technique for determining the optimal f for a given trade. Suppose you are making your trading decisions based on fundamentals. You could, if you wanted, outline the different scenarios that the trade may take. The more scenarios, and the more accurate the scenarios, the more accurate your results would be. Let's say you are looking to buy a municipal bond for income, but you're not planning on holding the bond to maturity. You could outline numerous different scenarios of how the future might unfold. Now, you can use these scenarios to determine how much to invest in this particular bond issue.

Suppose a trader is presented with a decision to buy soybeans. He may be using Elliot Wave, he may be using weather forecasts, but whatever he is using, let's say he can discern the following scenarios for this potential trade:

Scenario	Probability	Result
Best-case outcome	.05	150/cent bushel (profit)
Quite likely	.4	10/cent bushel (profit)
Typical	.45	−5/cent bushel (loss)
Not good	.05	−30/cent bushel (loss)
Disastrous	.05	−150/cent bushel (loss)

Now, when our Elliot Wave soybean trader (or weather forecaster soybean trader) paints this set of scenarios, this set of possible outcomes to this trade, and, in order to maximize his long-run growth (and survival), assumes that he must make this same trading decision an infinite number of times into the future, he will find, using this scenario planning approach, that optimally he should bet .02 (2%) of his stake on this trade. This translates into putting on one soybean contract for every $375,000 in equity, since the scenario with the largest loss, −150/cent bushel, divided by the optimal f for this scenario set, .02, results in $\$7{,}500/.02 = \$375{,}000$. Thus, at one contract for every $375,000 in equity, the trader can be said to be risking 2% of his stake on the next trade.

For each trade, regardless of the basis the trader uses for making the trade (i.e., Elliot Wave, weather, etc.), the scenario parameters may change. Yet the trader must maximize the long-run geometric growth of his account by assuming that the same scenario parameters will be infinitely repeated. Otherwise, the trader pays a severe price. Notice in our soybean trader example, if the trader were to go *to the right of the peak of the f curve* (that is, have slightly too many contracts), he gains no benefit. In other words, if our soybean trader were to put on one contract for every $300,000 in account equity, he would actually make less money in the long run than putting on one contract for every $375,000.

When we are presented with a decision in which there is a different set of scenarios for each facet of the decision, selecting the scenario whose geometric mean corresponding to its optimal f is greatest will maximize our decision in an asymptotic sense.

For example, suppose we are presented with a decision that involves two possible choices. It could have many possible choices, but for the sake of simplicity we will say it has two possible choices, which we will call "white" and "black." If we choose the decision labeled white, we determine that it will present the possible future scenarios to us:

Scenario	Probability	Result
A	.3	−20
B	.4	0
C	.3	30

Mathematical expectation = \$3.00
Optimal f= .17
Geometric mean = 1.0123

It doesn't matter what these scenarios are, they can be anything. To further illustrate this, they will simply be assigned letters, A, B, C in this discussion. Further, it doesn't matter what the result is; it can be just about anything.

Our analysis determines that the black decision will present the following scenarios:

Scenario	Probability	Result
A	.3	−10
B	.4	5
C	.15	6
D	.15	20

Mathematical expectation = \$2.90
Optimal f = .31
Geometric mean = 1.0453

Many people would opt for the white decision, since it is the decision with the higher mathematical expectation. With the white decision, you can expect, *on average*, a \$3.00 gain versus black's \$2.90 gain. Yet the black decision is actually the correct decision because it results in a greater geometric mean. With the black decision, you would expect to make 4.53% (1.0453 – 1) *on average* as opposed to white's 1.23% gain. When you consider the effects of reinvestment, the black decision makes more than three times as much, on average, as does the white decision!

The reader may protest at this point that, "We're not doing this thing over again; we're only doing it once. We're not reinvesting back into the same future scenarios here. Won't we come out ahead if we always select the highest arithmetic mathematical expectation for each set of decisions that present themselves to us?"

The only time we want to be making decisions based on greatest arithmetic mathematical expectation is if we are planning on not reinvesting the money risked on the decision at hand. Since, in almost every case, the money risked on an event today will be risked again on a different event in the future, and money made or lost in the past affects what we have available to risk today, we should decide, based on geometric mean, to maximize the long-run growth of our money. Even though the scenarios that present themselves tomorrow won't be the same as those today, by always deciding based on greatest geometric mean, we are maximizing our decisions. It is analogous to a dependent trials process, like a game of blackjack. In each hand, the probabilities change and, therefore, the optimal fraction to bet changes as well. By always betting what is optimal for that hand, however, we maximize our long-run growth. Remember that, to maximize long-run growth, we must look at the current contest as one that expands infinitely into the future. In other words, we must look at each individual event as though we were to play it an infinite number of times if we wanted to maximize growth over many plays of different contests.

As a generalization, whenever the outcome of an event has an effect on the outcome(s) of subsequent event(s), we are better off to maximize for greatest geometric expectation. In the rare cases where the outcome of an event has no effect on subsequent events, we are then better off to maximize for greatest arithmetic expectation.

Mathematical expectation (arithmetic) does not take the dispersion between the outcomes of the different scenarios into account and, therefore, can lead to incorrect decisions when reinvestment is considered.

Using this method of scenario planning gets you quantitatively positioned with respect to the possible scenarios, their outcomes, and the likelihood of their occurrence. The method is inherently more conservative than positioning yourself per the greatest arithmetic mathematical expectation. The geometric mean of a data set is never greater than the arithmetic mean. Likewise, this method can never have you position yourself (have a greater commitment) otherwise than selecting by the greatest arithmetic mathematical expectation would. In the asymptotic sense (the long-run sense), this is not only the superior method of positioning yourself as it achieves greatest geometric growth; it is also a more conservative one than positioning yourself per the greatest arithmetic mathematical expectation.

Since reinvestment is almost always a fact of life (except on the day before you retire)—that is, you reuse the money that you are using today—we must make today's decision under the assumption that the same decision will present itself a thousand times over, in order to maximize the results of our decision. We must make our decisions and position ourselves in order to maximize geometric expectation. Further, since the outcomes of most events do, in fact, have an effect on the outcomes of subsequent events, we should make our decisions and position ourselves based on maximum geometric expectation. This tends to lead to decisions and positions that are not always obvious.

Note that we have created our own binned distribution in creating our scenarios here. Similarly, if we know the distributional form of the data, we can use that and the probabilities associated, with that distribution with this technique for finding the optimal f. Such techniques we call "parametric techniques," as opposed to the "Empirical Techniques" described prior to this section in the text. The Scenario Planning Approach, as described here, where we create the data bins from empirical data, being therefore a hybrid approach between an empirical means of determining optimal f and a parametric one.

SCENARIO SPECTRUMS

We now must become familiar with the notion of a *scenario spectrum*. A scenario spectrum is a set of scenarios, aligned in succession, left to right, from worst outcome to best, which range in probability from 0% to 100%. For example, consider the scenario spectrum for a simple coin toss whereby we lose on heads and win on tails, and both have a .5 probability of occurrence (Figure 4.14).

A scenario spectrum can have more than two scenarios—you can have as many scenarios as you like (see Figure 4.15).

This scenario spectrum corresponds to the following scenarios, taken from the previous section pertaining to XYZ Manufacturing Corporation's assessment of marketing a new product in a remote little country:

Scenario	Probability	Result	Prob × Result
War	.1	−$500,000	−$50,000
Trouble	.2	−$200,000	−$40,000
Stagnation	.2	$0	$0
Peace	.45	$500,000	$225,000
Prosperity	.05	$1,000,000	$50,000
Sum	1.00	Expectation	$185,000

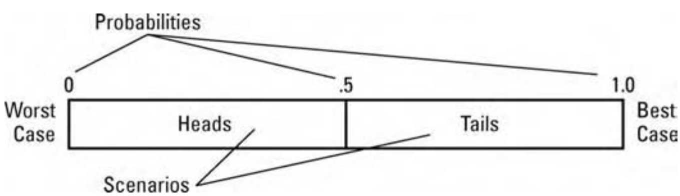

FIGURE 4.14 Scenario spectrum for a simple coin toss in which tails wins

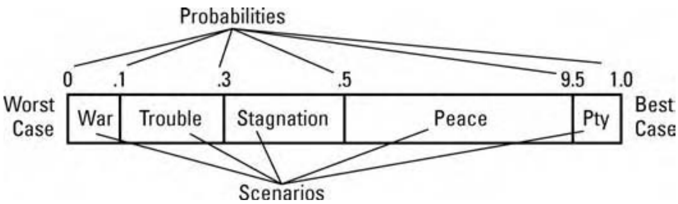

FIGURE 4.15 Scenario spectrum with multiple scenarios

Notice that this is a *valid scenario spectrum* since:

A. There is at least one scenario with a negative result.
B. The sum of the probabilities equals 1.00.
C. The scenarios within the spectrum do not overlap.

For example the stagnation scenario implies peace. However, the stagnation scenario implies peace with zero economic growth. The peace scenario is separate and apart from this, and implies peace with at least some economic growth. In other words, the stagnation scenario is *not* encapsulated in the peace scenario, nor is any scenario encapsulated in another.

One last point about scenario spectrums, and this is very important: All scenarios within a given spectrum must pertain to outcomes of a given holding period. Again, the length of the holding period can be any length you choose—it can be one day, one week, quarter, month, year, whatever, but the holding period must be decided upon. Once decided upon, all scenarios in a given spectrum must pertain to possible outcomes over the *next* holding period, and all scenario spectrums must be for the same length holding period. This is critical. Thus, if you decide upon one day for the holding period length, then all of your scenarios in all of your scenario spectrums must pertain to possible outcomes for the next day.