CHAPTER 1

The Random Process and Gambling Theory

We will start with the simple coin-toss case. When you toss a coin in the air there is no way to tell for certain whether it will land heads or tails. Yet over many tosses the outcome can be reasonably predicted.

This, then, is where we begin our discussion.

Certain axioms will be developed as we discuss the random process. The first of these is that *the outcome of an individual event in a random process cannot be predicted. However, we can reduce the possible outcomes to a probability statement.*

Pierre Simon Laplace (1749–1827) defined the probability of an event as the ratio of the number of ways in which the event can happen to the total possible number of events. Therefore, when a coin is tossed, the probability of getting tails is 1 (the number of tails on a coin) divided by 2 (the number of possible events), for a probability of .5. In our coin-toss example, we do not know whether the result will be heads or tails, but we do know that the probability that it will be heads is .5 and the probability it will be tails is .5. So, *a probability statement is a number between 0 (there is no chance of the event in question occurring) and 1 (the occurrence of the event is certain)*.

Often you will have to convert from a probability statement to odds and vice versa. The two are interchangeable, as the odds imply a probability, and a probability likewise implies the odds. These conversions are given now. The formula to convert to a probability statement, when you know the given odds is:

$$\text{Probability} = \text{odds for}/(\text{odds for} + \text{odds against}) \qquad (1.01)$$

If the odds on a horse, for example, are 4 to 1 (4:1), then the probability of that horse winning, as implied by the odds, is:

$$\begin{aligned} \text{Probability} &= 1/(1+4) \\ &= 1/5 \\ &= .2 \end{aligned}$$

So a horse that is 4:1 can also be said to have a probability of winning of .2. What if the odds were 5 to 2 (5:2)? In such a case the probability is:

$$\begin{aligned} \text{Probability} &= 2/(2+5) \\ &= 2/7 \\ &= .2857142857 \end{aligned}$$

The formula to convert from probability to odds is:

$$\text{Odds (against, to one)} = 1/\text{probability} - 1 \tag{1.02}$$

So, for our coin-toss example, when there is a .5 probability of the coin's coming up heads, the odds on its coming up heads are given as:

$$\begin{aligned} \text{Odds} &= 1/.5 - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

This formula always gives you the odds "to one." In this example, we would say the odds on a coin's coming up heads are 1 to 1.

How about our previous example, where we converted from odds of 5:2 to a probability of .2857142857? Let's work the probability statement back to the odds and see if it works out.

$$\begin{aligned} \text{Odds} &= 1/.2857142857 - 1 \\ &= 3.5 - 1 \\ &= 2.5 \end{aligned}$$

Here we can say that the odds in this case are 2.5 to 1, which is the same as saying that the odds are 5 to 2. So when someone speaks of odds, they are speaking of a probability statement as well.

Most people can't handle the uncertainty of a probability statement; it just doesn't sit well with them. We live in a world of exact sciences, and human beings have an innate tendency to believe they do not understand an event if it can only be reduced to a probability statement. The domain of physics seemed to be a solid one prior to the emergence of quantum physics. We had equations to account for most processes we had observed. These equations were real and provable. They repeated themselves over and over and the outcome could be exactly calculated before the event took place. With the emergence of quantum physics, suddenly a theretofore exact science could only reduce a physical phenomenon to a probability statement. Understandably, this disturbed many people.

I am not espousing the random walk concept of price action nor am I asking you to accept anything about the markets as random. Not yet, anyway. Like quantum physics, the idea that there is or is not randomness in the markets is an emotional one. At this stage, let us simply concentrate on the random process as it pertains to something we are certain is random, such as coin tossing or most casino gambling. In so doing, we can understand the process first, and later look at its applications. Whether the random process is applicable to other areas such as the markets is an issue that can be developed later.

Logically, the question must arise, "When does a random sequence begin and when does it end?" It really doesn't end. The blackjack table continues running even after you leave it. As you move from table to table in a casino, the random process can be said to follow you around. If you take a day off from the tables, the random process may be interrupted, but it continues upon your return. So, when we speak of a random process of X events in length we are arbitrarily choosing some finite length in order to study the process.

INDEPENDENT VERSUS DEPENDENT TRIALS PROCESSES

We can subdivide the random process into two categories. First are those events for which the probability statement is constant from one event to the next. These we will call independent trials processes or sampling with replacement. A coin toss is an example of just such a process. Each toss has a 50/50 probability regardless of the outcome of the prior toss. Even if the last five flips of a coin were heads, the probability of this flip being heads is unaffected, and remains .5.

Naturally, the other type of random process is one where the outcome of prior events *does* affect the probability statement and, naturally, the probability statement is not constant from one event to the next. These types of events are called dependent trials processes or sampling without replacement. Blackjack is an example of just such a process. Once a card is played, the composition of the deck for the next draw of a card is different from what it was for the previous draw. Suppose a new deck is shuffled and a card removed. Say it was the ace of diamonds. Prior to removing this card the probability of drawing an ace was 4/52 or .07692307692. Now that an ace has been drawn from the deck, and not replaced, the probability of drawing an ace on the next draw is 3/51 or .05882352941.

Some people argue that dependent trials processes such as this are really not random events. For the purposes of our discussion, though, we will assume they are—since the outcome still cannot be known beforehand. The best that can be done is to reduce the outcome to a probability statement. Try to think of the difference between independent and dependent trials processes as simply whether the probability statement is *fixed* (independent trials) or *variable* (dependent trials) from one event to the next based on prior outcomes. This is in fact the only difference.

Everything can be reduced to a probability statement. Events where the outcomes can be known prior to the fact differ from random events mathematically only in that their probability statements equal 1. For example, suppose that 51 cards have been removed from a deck of 52 cards and you know what the cards are. Therefore, you know what the one remaining card is with a probability of 1 (certainty). For the time being, we will deal with the independent trials process, particularly the simple coin toss.

MATHEMATICAL EXPECTATION

At this point it is necessary to understand the concept of mathematical expectation, sometimes known as the player's edge (if positive to the player) or the house's advantage (if negative to the player):

$$\text{Mathematical Expectation} = (1 + \text{A}) * \text{P} - 1 \tag{1.03}$$

where: P = Probability of winning.
A = Amount you can win/Amount you can lose.

So, if you are going to flip a coin and you will win \$2 if it comes up heads, but you will lose \$1 if it comes up tails, the mathematical expectation per flip is:

$$\begin{aligned}\text{Mathematical Expectation} &= (1 + 2) * .5 - 1\\ &= 3 * .5 - 1\\ &= 1.5 - 1\\ &= .5\end{aligned}$$

In other words, you would expect to make 50 cents on average each flip.

This formula just described will give us the mathematical expectation for an event that can have two possible outcomes. What about situations where there are more than two possible outcomes? The next formula will give us the mathematical expectation for an unlimited number of outcomes. It will also give us the mathematical expectation for an event with only two possible outcomes such as the 2 for 1 coin toss just described. Hence, it is the preferred formula.

$$\text{Mathematical Expectation} = \sum_{i=1}^{N} (P_i * A_i) \tag{1.03a}$$

where: P = Probability of winning or losing.
A = Amount won or lost.
N = Number of possible outcomes.

The mathematical expectation is computed by multiplying each possible gain or loss by the probability of that gain or loss, and then summing those products together.

Now look at the mathematical expectation for our 2 for 1 coin toss under the newer, more complete formula:

$$\begin{aligned}\text{Mathematical Expectation} &= .5 * 2 + .5 * (-1)\\ &= 1 + (-.5)\\ &= .5\end{aligned}$$

In such an instance, of course, your mathematical expectation is to win 50 cents per toss on average.

Suppose you are playing a game in which you must guess one of three different numbers. Each number has the same probability of occurring (.33), but if you guess one of the numbers you will lose \$1, if you guess another number you will lose \$2, and if you guess the right number you will win \$3. Given such a case, the mathematical expectation (ME) is:

$$\begin{aligned}\text{ME} &= .33 * (-1) + .33 * (-2) + .33 * 3\\ &= -.33 - .66 + .99\\ &= 0\end{aligned}$$

Consider betting on one number in roulette, where your mathematical expectation is:

$$\begin{aligned}\text{ME} &= 1/38 * 35 + 37/38 * (-1)\\ &= .02631578947 * 35 + .9736842105 * (-1)\\ &= .9210526315 + (-.9736842105)\\ &= -.05263157903\end{aligned}$$

If you bet \$1 on one number in roulette (American double-zero), you would expect to lose, on average, 5.26 cents per roll. If you bet \$5, you would expect to lose, on average, 26.3 cents per roll. Notice how *different amounts bet have different mathematical expectations in terms of amounts, but the expectation as a* percent *of the amount bet is always the same.*

The player's expectation for a series of bets is the total of the expectations for the individual bets. So if you play \$1 on a number in roulette, then \$10 on a number, then \$5 on a number, your total expectation is:

$$
\begin{aligned}
\text{ME} &= (-.0526) * 1 + (-.0526) * 10 + (-.0526) * 5 \\
&= -.0526 - .526 - .263 \\
&= -.8416
\end{aligned}
$$

You would therefore expect to lose on average 84.16 cents.

This principle explains why systems that try to change the size of their bets relative to how many wins or losses have been seen (assuming an independent trials process) are doomed to fail. The sum of negative-expectation bets is always a negative expectation!

EXACT SEQUENCES, POSSIBLE OUTCOMES, AND THE NORMAL DISTRIBUTION

We have seen how flipping one coin gives us a probability statement with two possible outcomes—heads or tails. Our mathematical expectation would be the sum of these possible outcomes. Now let's flip two coins. Here the possible outcomes are:

Coin 1	Coin 2	Probability
H	H	.25
H	T	.25
T	H	.25
T	T	.25

This can also be expressed as there being a 25% chance of getting both heads, a 25% chance of getting both tails, and a 50% chance of getting a head and a tail. In tabular format:

Combination	**Probability**	
H2	.25	*
T1H1	.50	**
T2	25	*

The asterisks to the right show how many different ways the combination can be made. For example in the above two-coin flip there are two asterisks for T1H1, since there are two different ways to get this combination. Coin A could be heads and coin B tails, or the reverse, coin A tails and coin B heads. The total number of asterisks in the table (four) is the total number of different combinations you can get when flipping that many coins (two).

If we were to flip three coins, we would have:

Combination	**Probability**	
H3	.125	*
H2T1	375	***
T2H1	.375	***
T3	125	*

for four coins:

Combination	**Probability**	
H4	.0625	*
H3T1	.25	****
H2T2	.375	******
T3H1	.25	****
T4	.0625	*

and for six coins:

Combination	**Probability**	
H6	.0156	*
H5T1	.0937	******
H4T2	.2344	***************
H3T3	.3125	********************
T4H2	.2344	***************
T5H1	.0937	******
T6	.0156	*

Notice here that if we were to plot the asterisks vertically we would be developing into the familiar bell-shaped curve, also called the Normal or Gaussian Distribution (see Figure 1.1).[1]

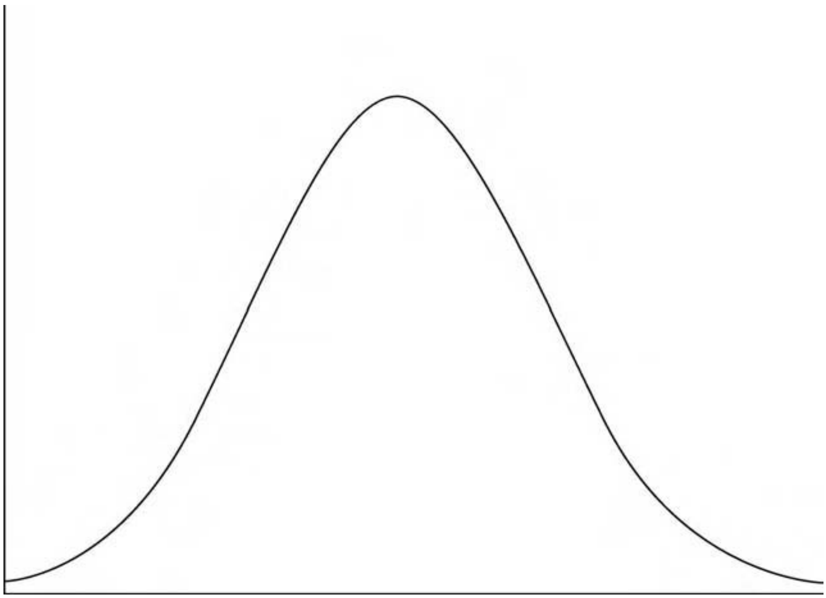

FIGURE 1.1 Normal probability function

[1]Actually, the coin toss does not conform to the Normal Probability Function in a pure statistical sense, but rather belongs to a class of distributions called the Binomial Distribution (a.k.a. Bernoulli or Coin-Toss Distributions). However, as N becomes large, the Binomial approaches the Normal Distribution as a limit (provided the probabilities involved are not close to 0 or 1). This is so because the Normal Distribution is continuous from left to right, whereas the Binomial is not, and the Normal is always symmetrical whereas the Binomial needn't be. Since we are treating a finite number of coin tosses and trying to make them representative of the universe of coin tosses, and since the probabilities are always equal to .5, we will treat the distributions of tosses as though they were Normal. As a further note, the Normal Distribution can be used as an approximation of the Binomial if both N times the probability of an event occurring and N times the complement of the probability occurring are both greater than 5. In our coin-toss example, since the probability of the event is .5 (for either heads or tails) and the complement is .5, then so long as we are dealing with N of 11 or more we can use the Normal Distribution as an approximation for the Binomial.

Finally, for 10 coins:

Combination	Probability	
H10	.001	*
H9T1	.01	**********
H8T2	.044	*****(45 different ways)
H7T3	.117	*****(120 different ways)
H6T4	205	*****(210 different ways)
H5T5	.246	*****(252 different ways)
T6H4	.205	*****(210 different ways)
T7H3	.117	*****(120 different ways)
T8H2	.044	*****(45 different ways)
T9H1	.01	**********
T10	.001	*

Notice that *as the number of coins increases, the probability of getting all heads or all tails decreases*. When we were using two coins, the probability of getting all heads or all tails was .25. For three coins it was .125, for four coins .0625; for six coins .0156, and for 10 coins it was .001.

POSSIBLE OUTCOMES AND STANDARD DEVIATIONS

So a coin flipped four times has a total of 16 possible exact sequences:

1.	H	H	H	H
2.	H	H	H	T
3.	H	H	T	H
4.	H	H	T	T
5.	H	T	H	H
6.	H	T	H	T
7.	H	T	T	H
8.	H	T	T	T
9.	T	H	H	H
10.	T	H	H	T
11.	T	H	T	H
12.	T	H	T	T
13.	T	T	H	H
14.	T	T	H	T
15.	T	T	T	H
16.	T	T	T	T

The term "exact sequence" here means the exact outcome of a random process. The set of all possible exact sequences for a given situation is called the *sample space*. Note that the four-coin flip just depicted can be four coins all flipped at once, or it can be one coin flipped four times (i.e., it can be a chronological sequence).

If we examine the exact sequence T H H T and the sequence H H T T, the outcome would be the same for a person flat-betting (i.e., betting 1 unit on each instance). However, to a person not flat-betting, the end result of these two exact sequences can be far different. To a flat bettor there are only five possible outcomes to a four-flip sequence:

4 Heads

3 Heads and 1 Tail

2 Heads and 2 Tails

1 Head and 3 Tails

4 Tails

As we have seen, there are 16 possible exact sequences for a four-coin flip. This fact would concern a person who is not flat-betting. We will refer to people who are not flat-betting as "system" players, since that is most likely what they are doing—betting variable amounts based on some scheme they think they have worked out.

If you flip a coin four times, you will of course see only one of the 16 possible exact sequences. If you flip the coin another four times, you will see another exact sequence (although you could, with a probability of $1/16 = .0625$, see the exact same sequence). If you go up to a gaming table and watch a series of four plays, you will see only one of the 16 exact sequences. You will also see one of the five possible end results. *Each exact sequence (permutation) has the same probability of occurring*, that being .0625. *But each end result (combination) does not have equal probability of occurring*:

End Result	Probability
4 Heads	.0625
3 Heads and 1 Tail	.25
2 Heads and 2 Tails	.375
1 Head and 3 Tails	.25
4 Tails	.0625

Most people do not understand the difference between exact sequences (permutation) and end results (combination) and as a result falsely conclude that exact sequences and end results are the same thing. This is a *common misconception that can lead to a great deal of trouble. It is the end results (not the exact sequences) that conform to the bell curve—the Normal Distribution*, which is a particular type of probability distribution. An interesting characteristic of all probability distributions is a statistic known as the *standard deviation*.

For the Normal Probability Distribution on a simple binomial game, such as the one being used here for the end results of coin flips, the standard deviation (SD) is:

$$D = N * \sqrt{\frac{P * (1 - P)}{N}} \tag{1.04}$$

where: P = Probability of the event (e.g., result of heads).
N = Number of trials.

For 10 coin tosses (i.e., N = 10):

$$\begin{aligned} SD &= 10 * \sqrt{.5 * (1 - .5)/10} \\ &= 10 * \sqrt{.5 * .5/10} \\ &= 10 * \sqrt{.25/10} \\ &= 10 * .158113883 \\ &= 1.58113883 \end{aligned}$$

The center line of a distribution is the peak of the distribution. In the case of the coin toss the peak is at an even number of heads and tails. So for a 10-toss sequence, the center line would be at 5 heads and 5 tails. For the Normal Probability Distribution, approximately 68.26% of the events will be + or − 1 standard deviation from the center line, 95.45% between + and − 2 standard deviations from the center line, and 99.73% between + and − 3 standard deviations from the center line (see Figure 1.2). Continuing with our 10-flip coin toss, 1 standard deviation equals approximately 1.58. We can therefore say of our 10-coin flip that 68% of the time we can expect to have our end result be composed of 3.42 (5 − 1.58) to 6.58 (5 + 1.58) being heads (or tails). So if we have 7 heads (or tails), we would be beyond 1 standard deviation of the expected outcome (the expected outcome being 5 heads and 5 tails).

Here is another interesting phenomenon. Notice in our coin-toss examples that as the number of coins tossed increases, the probability of getting an even number of heads and tails decreases. With two coins the probability of getting H1T1 was .5. At four coins the probability of getting 50% heads and 50% tails dropped to .375. At six coins it was .3125, and at 10 coins .246. Therefore, we can state that *as the number of events increases, the probability of the end result exactly equaling the expected value decreases.*

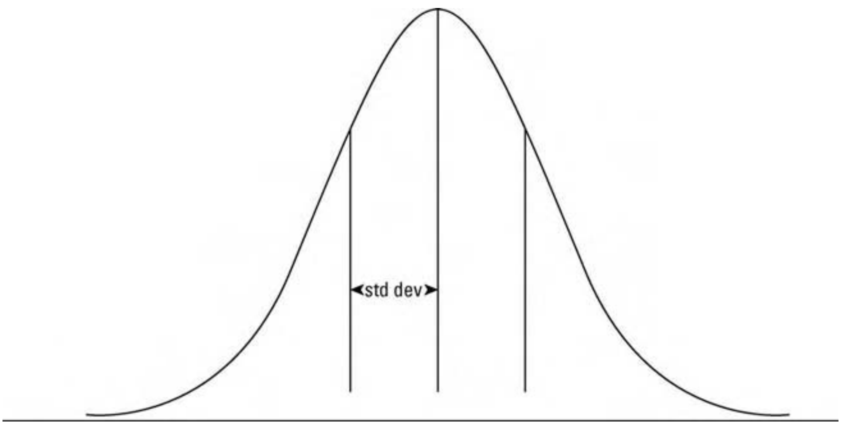

FIGURE 1.2 Normal probability function: Center line and 1 standard deviation in either direction

The mathematical expectation is what we expect to gain or lose, on average, each bet. However, it does not explain the fluctuations from bet to bet. In our coin-toss example we know that there is a 50/50 probability of a toss's coming up heads or tails. We expect that after N trials approximately $\frac{1}{2} * N$ of the tosses will be heads, and $\frac{1}{2} * N$ of the tosses will be tails. Assuming that we lose the same amount when we lose as we make when we win, we can say we have a mathematical expectation of 0, regardless of how large N is.

We also know that approximately 68% of the time we will be + or − 1 standard deviation away from our expected value. For 10 trials (N = 10) this means our standard deviation is 1.58. For 100 trials (N = 100) this means we have a standard deviation size of 5. At 1,000 (N = 1,000) trials the standard deviation is approximately 15.81. For 10,000 trials (N = 10,000) the standard deviation is 50.

N	Std Dev	Std Dev/N as%
10	1.58	15.8%
100	5	5.0%
1,000	15.81	1.581%
10,000	50	0.5%

Notice that as N increases, the standard deviation increases as well. This means that contrary to popular belief, *the longer you play, the*

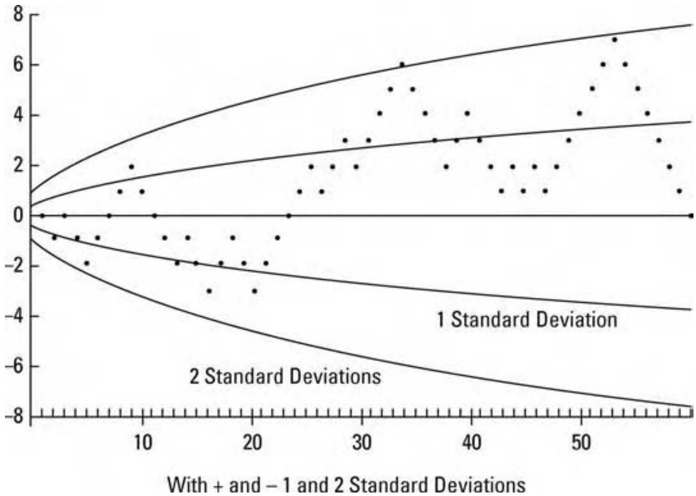

FIGURE 1.3 The random process: Results of 60 coin tosses, with 1 and 2 standard deviations in either direction

further you will be from your expected value (in terms of units won or lost). However, as N increases, the standard deviation as a percent of N decreases. This means that *the longer you play, the closer to your expected value you will be as a percent of the total action (N)*. This is the "Law of Averages" presented in its mathematically correct form. In other words, if you make a long series of bets, N, where T equals your total profit or loss and E equals your expected profit or loss, then T/N tends towards E/N as N increases. Also, the difference between E and T increases as N increases.

In Figure 1.3 we observe the random process in action with a 60-coin-toss game. Also on this chart you will see the lines for + and – 1 and 2 standard deviations. Notice how they bend in, yet continue outward forever. This conforms with what was just said about the Law of Averages.

THE HOUSE ADVANTAGE

Now let us examine what happens when there is a house advantage involved. Again, refer to our coin-toss example. We last saw 60 trials at an even or "fair" game. Let's now see what happens if the house has a 5% advantage. An example of such a game would be a coin toss where if we win, we win $1, but if we lose, we lose $1.10.

Figure 1.4 shows the same 60-coin-toss game as we previously saw, only this time there is the 5% house advantage involved. Notice how, in

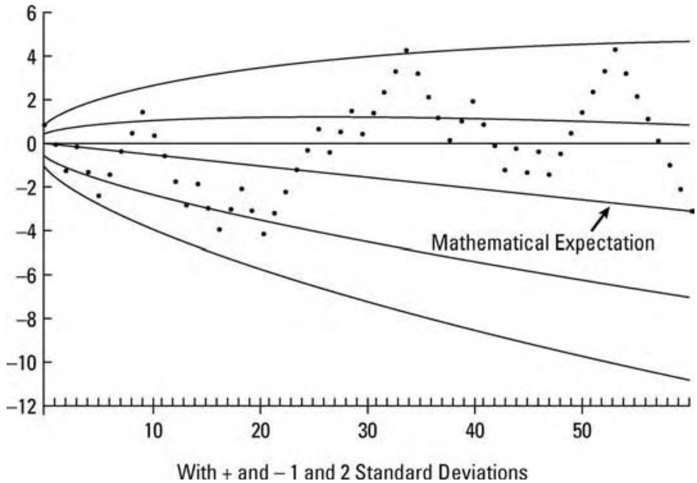

FIGURE 1.4 Results of 60 coin tosses with a 5% house advantage

this scenario, ruin is inevitable—as the upper standard deviations begin to bend down (to eventually cross below zero).

Let's examine what happens when we continue to play a game with a negative mathematical expectation.

N	Std Dev	Expectation	+ or −1 SD
10	1.58	−.5	+1.08 to −2.08
100	5.00	−5	0 to − 10
1,000	15.81	−50	−34.19 to −65.81
10,000	50.00	−500	−450 to −550
100,000	158.11	−5,000	−4,842 to −5,158
1,000,000	500.00	−50,000	−49,500 to −50,500

The principle of ergodicity is at work here. It doesn't matter if one person goes to a casino and bets $1 one million times in succession or if one million people come and bet $1 each all at once. The numbers are the same. At one million bets, it would take more than 100 standard deviations away from the expectation before the casino started to lose money! Here is the Law of Averages at work. By the same account, if you were to make one million $1 bets at a 5% house advantage, it would be equally unlikely for you to make money. Many casino games have more than a 5% house advantage, as does most sports betting. Trading the markets is a zero-sum game. However, there is a small drain involved in the way of commissions, fees, and slippage. Often these costs can run in excess of 5%.

Next, let's examine the statistics of a 100-coin-toss game with and without a 5% house advantage:

Std. Deviations from Center	Fair 50/50 Game	5% House Advantage Game
+3	+15	+10
+2	+10	+5
+1	+5	0
0	0	−5
−1	−5	−10
−2	−10	−15
−3	−15	−20

As can be seen, at 3 standard deviations, which we can expect to be the outcome 99.73% of the time, we will win or lose between +15 and −15 units in a fair game. At a house advantage of 5%, we can expect our final outcome to be between +10 and −20 units at the end of 100 trials. At 2 standard deviations, which we can expect to occur 95% of the time, we win or lose within + or −10 in a fair game. At a 5% house advantage this is +5 and −15 units. At 1 standard deviation, where we can expect the final outcome to be with 68% probability, we win or lose up to 5 units in a fair game. Yet in the game where the house has the 5% advantage we can expect the final outcome to be between winning nothing and losing 10 units! Note that at a 5% house advantage it is not impossible to win money after 100 trials, but you would have to do better than 1 whole standard deviation to do so. In the Normal Distribution, the probability of doing better than 1 whole standard deviation, you will be surprised to learn, is only .1587!

Notice in the previous example that at 0 standard deviations from the center line (that is, at the center line itself), the amount lost is equal to the house advantage. For the fair 50/50 game, this is equal to 0. You would expect neither to win nor to lose anything. In the game where the house has the 5% edge, you would expect to lose 5%, 5 units for every 100 trials, at 0 standard deviations from the center line. So you can say that *in flat-betting situations involving an independent process, you will lose at the rate of the house advantage.*

MATHEMATICAL EXPECTATION LESS THAN ZERO SPELLS DISASTER

This brings us to another axiom, which can be stated as follows: *In a negative expectancy game, there is no money management scheme that will make you a winner. If you continue to bet, regardless of how you manage your money, it is almost certain that you will be a loser, losing your entire stake regardless of how large it was to start.*

This sounds like something to think about. Negative mathematical expectations (regardless of how negative) have broken apart families and caused suicides and murders and all sorts of other things the bettors weren't bargaining for. I hope you can see what an incredibly losing proposition it is to make bets where there is a negative expectancy, for even a small negative expectancy will eventually take every cent you have. All attempts to outsmart this process are mathematically futile. Don't get this idea confused with whether or not there is a dependent or independent trials process involved; it doesn't matter. If the sum of your bets is a negative expectancy, you are in a losing proposition.

As an example, if you are in a dependent trials process where you have an edge in 1 bet out of 10, then you must bet enough on the bet for which you have an edge so that the sum of all 10 bets is a positive expectancy situation. If you expect to lose 10 cents on average for 9 of the 10 bets, but you expect to make 10 cents on the 1 out of 10 bets where you know you have the edge, then you must bet more than 9 times as much on the bet where you know you have the edge, just to have a net expectation of coming out even. If you bet less than that, you are still in a negative expectancy situation, and complete ruin is all but certain if you continue to play.

Many people have the mistaken impression that if they play a negative expectancy game, they will lose a percentage of their capital relative to the negative expectancy. For example, when most people realize that the mathematical expectation in roulette is 5.26% they seem to think this means that if they go to a casino and play roulette they can expect to lose, on average, 5.26% of their stake. This is a dangerous misconception. The truth is that they can expect to lose 5.26% of their *total action*, not of their entire stake. Suppose they take $500 to play roulette. If they make 500 bets of $20 each, their total action is $10,000, of which they can expect to lose 5.26%, or $526, more than their entire stake.

The only smart thing to do is bet only when you have a positive expectancy. This is not so easily a winning proposition as negative expectancy betting is a losing proposition, as we shall see in a later chapter. You must bet specific quantities, which will be discussed at length. For the time being, though, resolve to bet only on positive expectancy situations.

When it comes to casino gambling, though, the only time you can find a positive expectancy situation is if you keep track of the cards in blackjack, and then only if you are a very good player, and only if you bet your money correctly. There are many good blackjack books available, so we won't delve any further into blackjack here.

BACCARAT

If you want to gamble at a casino but do not want to learn to play blackjack correctly, then baccarat has the smallest negative expectancy of any other casino game. In other words, you'll lose your money at a slower rate. Here are the probabilities in baccarat:

Banker wins 45.842% of the time.
Player wins 44.683% of the time.
A tie occurs 9.547% of the time.

Since a tie is treated as a push in baccarat (no money changes hands, the net effect is the same as if the hand were never played) the probabilities, when ties are eliminated become:

Banker wins 50.68% of the time.
Player wins 49.32% of the time.

Now let's look at the mathematical expectations. For the player side:

$$\begin{aligned} \text{ME} &= (.4932 * 1) + ((1 - .4932) * (-1)) \\ &= (.4932 * 1) + (.5068 * (-1)) \\ &= .4932 - .5068 \\ &= -.0136 \end{aligned}$$

In other words, the house advantage over the player is 1.36%.

Now for the banker side, bearing in mind that the banker side is charged a 5% commission on wins only, the mathematical expectation is:

$$\begin{aligned} \text{ME} &= (.5068 * .95) + ((1 - .5068) * (-1)) \\ &= (.5068 * .95) + (.4932 * (-1)) \\ &= .48146 - .4932 \\ &= -.01174 \end{aligned}$$

In other words, the house has an advantage, once commissions on the banker's wins are accounted for, of 1.174%.

As you can see, it makes no sense to bet on the player since the player's negative expectancy is worse than the banker's:

Player's disadvantage	−.0136
Banker's disadvantage	−.01174
Banker's edge over Player	.00186

In other words, after about 538 hands (1/.00186) the banker will be 1 unit ahead of the player. Again, the more hands that are played, the more certain this edge is.

This is not to imply that the banker has a positive mathematical expectation—he doesn't. Both banker and player have negative expectations, but the banker's is not as negative as the player's. Betting 1 unit on the banker on each hand, you can expect to lose 1 unit for approximately every 85 hands (1/.01174); whereas betting 1 unit on the player on each hand, you would expect to lose 1 unit every 74 hands (1/.0136). You will lose your money at a slower *rate*, but not necessarily a slower *pace*. Most baccarat tables have at least a $25 minimum bet. If you are betting banker, 1 unit per hand, after 85 hands you can expect to be down $25.

Let's compare this to betting red/black at roulette, where you have a mathematical expectation of −.0526, but a minimum bet size of at least $2. After 85 spins you would expect to be down about $9 ($2 * 85 * .0526). As you can see, mathematical expectation is also a function of the total amount bet, the action. If, as in baccarat, we were betting $25 per spin in red/black roulette, we would expect to be down $112 after 85 spins, compared with baccarat's expected loss of $25.

NUMBERS

Finally, let's take a look at the probabilities involved in numbers. If baccarat is the game of the rich, numbers is the game of the poor. The probabilities in the numbers game are absolutely pathetic. Here is a game where a player chooses a three-digit number between 0 and 999 and bets $1 that this number will be selected. The number that gets chosen as that day's number is usually some number that (a) cannot be rigged and (b) is well publicized. An example would be to take the first three of the last five digits of the daily stock market volume. If the player loses, then the $1 he bet is lost. If the player should happen to win, then $700 is returned, for a net profit of $699. For numbers, the mathematical expectation is:

$$\begin{aligned} \text{ME} &= (699 * (1/1000)) + ((-1) * (1 - (1/1000))) \\ &= (699 * .001) + ((-1) * (1 - .001)) \\ &= (699 * .001) + ((-1) * .999) \\ &= .699 + (-.999) \\ &= -.3 \end{aligned}$$

In other words your mathematical expectation is to lose 30 cents for every dollar of action. This is far worse than any casino game, including keno. Bad as the probabilities are in a game like roulette, the mathematical expectation in numbers is almost six times worse. The only gambling situations that are worse than this in terms of mathematical expectation are most football pools and many of the state lotteries.

PARI-MUTUEL BETTING

The games that offer seemingly the worst mathematical expectation belong to a family of what are called pari-mutuel games. Pari-mutuel means literally "to bet among ourselves." Pari-mutuel betting was originated in the 1700s by a French perfume manufacturer named Oller. Monsieur Oller, doubling as a bookie, used his perfume bottles as ticket stubs for his patrons while he booked their bets. Oller would take the bets, from this total pool he would take his cut, then he would distribute the remainder to the winners. Today we have different types of games built on this same pari-mutuel scheme, from state lotteries to football pools, from numbers to horse racing. As you have seen, the mathematical expectations on most pari-mutuel games are atrocious. Yet these very games also offer many situations that have a positive mathematical expectancy.

Let's take numbers again, for example. We can approximate how much money is bet in total by taking the average winning purse size and dividing it by 1 minus the take. In numbers, as we have said, the take is 30%, so we have 1 − .3, or .7. Dividing 1 by .7 yields 1.42857. If the average payout is, say, $1,400, then we can approximate the total purse as 1,400 times 1.42857, or roughly $2,000. So step one in finding positive mathematical expectations in pari-mutuel situations is to know or at least closely approximate the total amount in the pool.

The next step is to take this total amount and divide it by the total number of possible combinations. This gives the average amount bet per combination. In numbers there are 1,000 possible combinations, so in our example we divide the approximate total pool of $2,000 by 1,000, the total number of combinations, to obtain an average bet per combination of $2.

Now we figure the total amount bet on the number we want to play. Here we would need inside information. The purpose here is not to show how to win at numbers or any other gambling situation, but rather to show how to think correctly in approaching a given risk/reward situation. This will be made clearer as we continue with the illustration. For now, let's just assume we can get this information. Now, if we know what the average dollar bet is on any number, and we know the total amount bet on the number we want to play, we simply divide the average bet by the amount bet on our number. This gives us the ratio of what our bet size is relative to the average bet size.

Since the pool can be won by any number, and since the pool is really the average bet times all possible combinations, it stands to reason that naturally we want our bet to be relatively small compared to the average bet. Therefore, if this ratio is 1.5, it means simply that the average bet on a number is 1.5 times the amount bet on our number.

Now this can be converted into an actual mathematical expectation. We take this ratio and multiply it by the quantity (1 − takeout) where the takeout is the pari-mutuel vigorish (also known as the amount that the house skims off the top, and out of the total pool). In the case of numbers, where the takeout is 30%, then 1 minus the takeout equals .7. Multiplying our ratio in our example of 1.5 times .7 gives us 1.05. As a final step, subtracting 1 from the previous step's answer will give us the mathematical expectation, in percent. Since 1.05 − 1 is 5%, we can expect in our example situation to make 5% on our money on average if we make this play over and over.

Which brings us to an interesting proviso here. In numbers, we have probabilities of 1/1000 or .001 of winning. So, in our example, if we bet $1 for each of 1,000 plays, we would expect to be ahead by 5%, or $50, if the given parameters as we just described were always present. Since it is possible to play the number 1,000 times, the mathematical expectation is possible, too.

But let's say you try to do this on a state lottery with over 7 million possible winning combinations. Unless you have a pool together or a lot of money to cover more than one number on each drawing, it is unlikely you will see over 7 million drawings in your lifetime. Since it will take (on average) 7 million drawings until you can mathematically expect your number to have come up, your positive mathematical expectation as we described it in the numbers example is meaningless. You most likely won't be around to collect!

In order for the mathematical expectation to be meaningful (provided it is positive) you must be able to get enough trials off in your lifetime (or the pertinent time period you are considering) to have a fair mathematical chance of winning. The average number of trials needed is the total number of possible combinations divided by the number of combinations you are playing. Call this answer N. Now, if you multiply N by the length of time it takes for 1 trial to occur, you can determine the average length of time needed for you to be able to expect the mathematical expectation to manifest itself. If your chances are 1 in 7 million and the drawing is once a week, you must stick around for 7 million weeks (about 134,615 years) to expect the mathematical expectation to come into play. If you bet 10,000 of those 7 million combinations, you must stick around about 700 weeks (7 million divided by 10,000, or about $13\frac{1}{2}$ years) to expect the mathematical expectation to kick in, since that is about how long, on average, it would take until one of those 10,000 numbers won.

The procedure just explained can be applied to other pari-mutuel gambling situations in a similar manner. There is really no need for inside information on certain games. Consider horse racing, another classic pari-mutuel situation. We must make one assumption here. We must assume that the money bet on a horse to win divided by the total win pool is an accurate reflection of the true probabilities of that horse winning. For instance, if the total win pool is \$25,000 and there is \$2,500 bet on our horse to win, we must assume that the probability of our horse's winning is .10. We must assume that if the same race were run 100 times with the same horses on the same track conditions with the same jockeys, and so on, our horse would win 10% of the time.

From that assumption we look now for opportunity by finding a situation where the horse's proportion of the show or place pools is much less than its proportion of the win pool. The opportunity is that if a horse has a probability of X of winning the race, then the probability of the horse's coming in second or third should not be less than X (provided, as we already stated, that X is the real probability of that horse winning). If the probability of the horse's coming in second or third is less than the probability of the horse's winning the race, an anomaly is created that we can perhaps capitalize on.

The following formula reduces what we have spoken of here to a mathematical expectation for betting a particular horse to place or show, and incorporates the track takeout. Theoretically, all we need to do is bet only on racing situations that have a positive mathematical expectation. The mathematical expectation of a show (or place) bet is given as:

$$(((W_i/\Sigma W)/(S_i/\Sigma S)) * (1 - \text{takeout}) - 1 \qquad (1.03b)$$

where: W_i = Dollars bet on the ith horse to win.
ΣW = Total dollars in the win pool—i.e., total dollars bet on all horses to win.
S_i = Dollars bet on the ith horse to show (or place).
ΣS = Total dollars in the show (or place) pool—i.e., total dollars on all horses to show (or place).
i = The horse of your choice.

If you've truly learned what is in this book you will use the Kelly formula (more on this in Chapter 4) to maximize the rate of your money's growth. How much to bet, however, becomes an iterative problem, in that the more you bet on a particular horse to show, the more you will change the mathematical expectation and payout—but not the probabilities, since they are dictated by $(W_i/\Sigma W)$. Therefore, when you bet on the horse to place, you alter the mathematical expectation of the bet and you also alter the payout on that horse to place. Since the Kelly formula is affected by the payout, you must be able to iterate to the correct amount to bet.

As in all winning gambling or trading systems, employing the winning formula just shown is far more difficult than you would think. Go to the racetrack and try to apply this method, with the pools changing every 60 seconds or so while you try to figure your formula and stand in line to make your bet and do it within seconds of the start of the race. The real-time employment of any winning system is always more difficult than you would think after seeing it on paper.

WINNING AND LOSING STREAKS IN THE RANDOM PROCESS

We have already seen that in flat-betting situations involving an independent trials process you will lose at the rate of the house advantage. To get around this rule, many gamblers then try various betting schemes that will allow them to win more during hot streaks than during losing streaks, or will allow them to bet more when they think a losing streak is likely to end and bet less when they think a winning streak is about to end. Yet another important axiom comes into play here, which is that *streaks are no more predictable than the outcome of the next event* (this is true whether we are discussing dependent or independent events). In the long run, we can predict approximately how many streaks of a given length can be expected from a given number of chances.

Imagine that we flip a coin and it lands tails. We now have a streak of one. If we flip the coin a second time, there is a 50% chance it will come up tails again, extending the streak to two events. There is also a 50% chance it will come up heads, ending the streak at one. Going into the third flip we face the same possibilities. Continuing with this logic we can construct the following table, assuming we are going to flip a coin 1,024 times:

Length of Streak	No. of Streaks Occurring	How Often Compared to Streak of One	Probability
1	512	1	.50
2	256	1/2	.25
3	128	1/4	.125
4	64	1/8	.0625
5	32	1/16	.03125
6	16	1/32	.015625
7	8	1/64	.0078125
8	4	1/128	.00390625
9	2	1/256	.001953125
10	1	1/512	.0009765625
11+	1	1/1024	.00048828125

The real pattern does not end at this point; rather it continues with smaller and smaller numbers.

Remember that this is the expected pattern. The real-life pattern, should you go out and record 1,024 coin flips, will resemble this, but most likely it won't resemble this exactly. This pattern of 1,024 coin tosses is for a fair 50/50 game. In a game where the house has the edge, you can expect the streaks to be skewed by the amount of the house advantage.

DETERMINING DEPENDENCY

As we have already explained, the coin toss is an independent trials process. This can be deduced by inspection, in that we can calculate the exact probability statement prior to each toss and it is always the same from one toss to the next. There are other events, such as blackjack, that are dependent trials processes. These, too, can be deduced by inspection, in that we can calculate the exact probability statement prior to each draw of a card, and it is not always the same from one draw to the next. For still other events, dependence on prior outcomes cannot be determined upon inspection. Such an event is the profit and loss stream of trades generated by a trading system. For these types of problems we need more tools.

Assume the following stream of coin flips where a plus (+) stands for a win and a minus (−) stands for a loss:

$$+ + - - - - - - - + - + - + - - - + + + - + + + - + ++$$

There are 28 trades, 14 wins and 14 losses. Say there is \$1 won on a win and \$1 lost on a losing flip. Hence, the net for this series is \$0.

Now assume you possess the infant's mind. You do not know if there is dependency or not in the coin-toss situation (although there isn't). Upon seeing such a stream of outcomes you deduce the following rule, which says, "Don't bet after two losers; go to the sidelines and wait for a winner to resume betting." With this new rule, the previous sequence would have been:

$$+ + - - - + - + - - + + - + + + - + ++$$

So, with this new rule the old sequence would have produced 12 winners and 8 losers for a net of \$4. You're quite confident of your new rule. You haven't learned to differentiate an exact sequence (which is all that this stream of trades is) from an end result (the end result being that this is a break-even game).

There is a major problem here, though, and that is that you do not know if there is dependency in the sequence of flips. *Unless dependency is proven, no attempt to improve performance based on the stream of profits and losses alone is of any value, and quite possibly you may do more harm than good.*[2] Let us continue with the illustration and we will see why.

[2]A distinction must be drawn between a stationary and a nonstationary distribution. A stationary distribution is one where the probability distribution does not change. An example would be a casino game such as roulette, where you are always at a .0526 disadvantage. A nonstationary distribution is one where the expectation changes over time (in fact, the entire probability distribution may change over time). Trading is just such a case. Trading is analogous in this respect to a drunk wandering through a casino, going from game to game. First he plays roulette with \$5 chips (for a −.0526 mathematical expectation), then he wanders to a blackjack table, where the deck happens to be running favorable to the player by 2%. His distribution of outcomes curve moves around as he does; the mathematical expectation and distribution of outcomes is dynamic. Contrast this to staying at one table, at one game. In such a case the distribution of outcomes is static. We say it is *stationary*. The outcomes of systems trading appear to be a nonstationary distribution, which would imply that there is perhaps some technique that may be employed to allow the trader to advantageously "trade his equity curve." Such techniques are, however, beyond the mathematical scope of this book and will not be treated here. Therefore, we will not treat nonstationary distributions any differently than stationary ones in the text, but be advised that the two are profoundly different.

Since this was a coin toss, there was in fact no dependency in the trials—that is, the outcome of each successive flip was independent of (unaffected by) the previous flips. Therefore, this exact sequence of 28 flips was totally random. (Remember, each exact sequence has an equal probability of occurring. It is the end results that follow the Normal Distribution, with the peak of the distribution occurring at the mathematical expectation. The end result in this case, the mathematical expectation, is a net profit/loss of zero.) The next exact sequence of 28 flips is going to appear randomly, and there is an equal probability of the following sequence appearing as any other:

$$- - + - - + - - + - - + - - + - - + - - + + + + + + + +$$

Once again, the net of this sequence is nothing won and nothing lost. Applying your rule here, the outcome is:

$$- - - - - - - - - - - - - - + + + + + + +$$

Fourteen losses and seven wins for a net loss of $7.

As you can see, unless dependency is proven (in a stationary process), no attempt to improve performance based on the stream of profits and losses alone is of any value, and you may do more harm than good.

THE RUNS TEST, Z SCORES, AND CONFIDENCE LIMITS

For certain events, such as the profit and loss stream of a system's trades, where dependency cannot be determined upon inspection, we have the runs test. The runs test is essentially a matter of obtaining the Z scores for the win and loss streaks of a system's trades. Here's how to do it. First, you will need a minimum of 30 closed trades. There is a very valid statistical reason for this. Z scores assume a Normal Probability Distribution (of streaks of wins and losses in this instance). Certain characteristics of the Normal Distribution are no longer valid when the number of trials is less than 30. This is because a minimum of 30 trials are necessary in order to resolve the shape of the Normal Probability Distribution clearly enough to make certain statistical measures valid.

The Z score is simply the number of standard deviations the data is from the mean of the Normal Probability Distribution. For example, a Z score of 1.00 would mean that the data you are testing is within 1 standard deviation from the mean. (Incidentally, this is perfectly normal.) The Z score is then converted into a confidence limit, sometimes also called a degree of certainty. We have seen that the area under the curve of the Normal Probability Function at 1 standard deviation on either side of the mean equals 68% of the total area under the curve. So we take our Z score and convert it to a confidence limit, the relationship being that the Z score is how many standard deviations and the confidence limit is the percentage of area under the curve occupied at so many standard deviations.

Confidence Limit	Z Score
99.73%	3.00
99%	2.58
98%	2.33
97%	2.17
96%	2.05
95.45%	2.00
95%	1.96
90%	1.64
85%	1.44
80%	1.28
75%	1.15
70%	1.04
68.27%	1.00
65%	.94
60%	.84
50%	.67

With a minimum of 30 closed trades we can now compute our Z scores. We are trying to determine how many streaks of wins/losses we can expect from a given system. Are the win/loss streaks of the system we are testing in line with what we could expect? If not, is there a high enough confidence limit that we can assume dependency exists between trades, that is, the outcome of a trade dependent on the outcome of previous trades?

Here, then, is how to perform the runs test, how to find a system's Z score:

1. You will need to compile the following data from your run of trades:

   A. The total number of trades, hereafter called N.

   B. The total number of winning trades and the total number of losing trades. Now compute what we will call X. X = 2 ∗ Total Number of Wins ∗ Total Number of Losses.

   C. The total number of runs in a sequence. We'll call this R.

Let's construct an example to follow along with. Assume the following trades:

$$-3 \quad +2 \quad +7 \quad -4 \quad +1 \quad -1 \quad +1 \quad +6 \quad -1 \quad 0 \quad -2 \quad +1$$

The net profit is +7. The total number of trades is 12; therefore, N = 12 (we are violating the rule that there must be at least 30 trades only to keep the example simple). Now we are not concerned here with how big the wins and losses are, but rather how many wins and losses there are and how many streaks. Therefore, we can reduce our run of trades to a simple sequence of pluses and minuses. Note that a trade with a profit and loss (P&L) of 0 is regarded as a loss. We now have:

$$- + + - + - + + - - - +$$

As can be seen, there are six profits and six losses. Therefore, $X = 2 * 6 * 6 = 72$. As can also be seen, there are eight runs in this sequence, so R = 8. We will define a *run* as any time we encounter a sign change when reading the sequence as shown above from left to right (i.e., chronologically). Assume also that we start at 1. Therefore, we would count this sequence as follows:

$$\begin{array}{cccccccccccc} - & + & + & - & + & - & + & + & - & - & - & + \\ 1 & 2 & & 3 & 4 & 5 & 6 & & 7 & & & 8 \end{array}$$

2. Solve for the equation:

$$N * (R - .5) - X$$

For our example this would be:

$$\begin{aligned} &12 * (8 - .5) - 72 \\ &12 * 7.5 - 72 \\ &90 - 72 \\ &18 \end{aligned}$$

3. Solve for the equation:

$$X * (X - N)/(N - 1)$$

So for our example this would be:

$$\begin{aligned} &72 * (72 - 12)/(12 - 1) \\ &72 * 60/11 \\ &4{,}320/11 \\ &392.727272 \end{aligned}$$

4. Take the square root of the answer in number 3. For our example this would be:

$$\sqrt{392.727272} = 19.81734777$$

5. Divide the answer in number 2 by the answer in number 4. This is the Z score. For our example this would be: 18/19.81734777 = .9082951063

6. $$\begin{aligned}\text{Confidence Limit} = 1 - (2 * (X * .31938153 - Y * .356563782 \\ + (X * Y * 1.781477937 - Y^2 * 1.821255978 \\ + 1.821255978 + Y^2 * X * 1.330274429) * 1 \\ /\sqrt{\text{EXP}(Z^2) * 6.283185307}))\end{aligned}$$

where: X = 1.0/(((ABS(Z)) *.2316419) + 1.0).
Y = X ^ 2.
Z = The Z score you are converting from.
EXP( ) = The exponential function.
ABS( ) = The absolute value function.

This will give you the confidence limit for the so-called "two-tailed" test. To convert this to a confidence limit for a "one-tailed" test:

$$\text{Confidence Limit} = 1 - (1 - A)/2$$

where: A = The "two-tailed" confidence limit.

If the Z score is negative, simply convert it to positive (take the absolute value) when finding your confidence limit. A negative Z score implies positive dependency, meaning fewer streaks than the Normal Probability Function would imply, and hence that wins beget wins and losses beget losses. A positive Z score implies negative dependency, meaning more streaks than the Normal Probability Function would imply, and hence that wins beget losses and losses beget wins.

As long as the dependency is at an acceptable confidence limit, you can alter your behavior accordingly to make better trading decisions, even though you do not understand the underlying cause of the dependency. Now, if you could know the cause, you could then better estimate when the dependency was in effect and when it was not, as well as when a change in the degree of dependency could be expected.

The runs test will tell you if your sequence of wins and losses contains more or fewer streaks (of wins or losses) than would ordinarily be expected in a truly random sequence, which has no dependence between trials. Since we are at such a relatively low confidence limit, we can assume that there is no dependence between trials in this particular sequence.

What would be an acceptable confidence limit then? Dependency can never be proved nor disproved beyond a shadow of a doubt in this test; therefore, what constitutes an acceptable confidence limit is a personal choice. Statisticians generally recommend selecting a confidence limit at least in the high nineties. Some statisticians recommend a confidence limit in excess of 99% in order to assume dependency; some recommend a less stringent minimum of 95.45% (2 standard deviations).

Rarely, if ever, will you find a system that shows confidence limits in excess of 95.45%. Most frequently, the confidence limits encountered are less than 90%. Even if you find one between 90 and 95.45%, this is not exactly a nugget of gold, either. You really need to exceed 95.45% as a bare minimum to assume that there is dependency involved that can be capitalized upon to make a substantial difference.

For example, some time ago a broker friend of mine asked me to program a money management idea of his that incorporated changes in the equity curve. Before I even attempted to satisfy his request, I looked for dependency between trades, since we all know now that unless dependency is proven (in a stationary process) to a very high confidence limit, all attempts to change your trading behavior based on changes in the equity curve are futile and may even be harmful.

Well, the Z score for this system (of 423 trades) clocked in at −1.9739! This means that there is a confidence limit in excess of 95%, a very high reading compared to most trading systems, but hardly an acceptable reading for dependency in a statistical sense. The negative number meant that wins beget wins and losses beget losses in this system. Now this was a great system to start with. I immediately went to work having the system pass all trades after a loss, and continue to pass trades until it passed what would have been a winning trade, then to resume trading. Here are the results:

	Before Rule	**After Rule**
Total Profits	$71,800	$71,890
Total Trades	423	360
Winning Trades	358	310
Winning Percentage$	84.63%	86.11%
Average Trade	$169.74	$199.69
Maximum Drawdown	$4,194	$2,880
Max. Losers in Succession	4	2
4 losers in a row	2	0
3 losers in a row	1	0
2 losers in a row	7	4

All of the above is calculated with $50 commissions and slippage taken off of each trade. As you can see, this was a terrific system before this rule. So good, in fact, that it was difficult to improve upon it in any way. Yet, once the dependency was found and exploited, the system was materially improved. It was with a confidence limit of slightly over 95%. It is rare to find a confidence limit this high in futures trading systems. However, from a statistical point of view, it is hardly high enough to assume that dependency exists. Ideally, yet rarely you will find systems that have confidence limits in the high nineties.

So far we have only looked at dependency from the point of view of whether the last trade was a winner or a loser. We are trying to determine if the sequence of wins and losses exhibit dependency or not. The runs test for dependency automatically takes the percentage of wins and losses into account. However, in performing the runs test on runs of wins and losses, we have accounted for the sequence of wins and losses but not their size. For the system to be truly independent, not only must the sequence of wins and losses be independent; the sizes of the wins and losses within the sequence must also be independent. It is possible for the wins and losses to be independent, while their sizes are dependent (or vice versa).

One possible solution is to run the runs test on only the winning trades, segregating the runs in some way (e.g., those that are greater than the median win versus those that are less). Then look for dependency among the size of the winning trades; then do the same for the losing trades.

THE LINEAR CORRELATION COEFFICIENT

There is, however, a different, possibly better way to quantify this possible dependency between the size of the wins and losses. The technique to be discussed next looks at the sizes of wins and losses from an entirely different mathematical perspective than does the runs test, and when used in conjunction with the latter, measures the relationship of trades with more depth than the runs test alone could provide. This technique utilizes the linear correlation coefficient, r, sometimes called Pearson's r, to quantify the dependency/independency relationship.

Look at Figure 1.5. It depicts two sequences that are perfectly correlated with each other. We call this effect "positive" correlation.

Now look at Figure 1.6. It shows two sequences that are perfectly uncorrelated with each other. When one line is zigging, the other is zagging. We call this effect "negative" correlation.

The formula for finding the linear correlation coefficient (r) between two sequences, X and Y, follows. (A bar over the variable means the mean

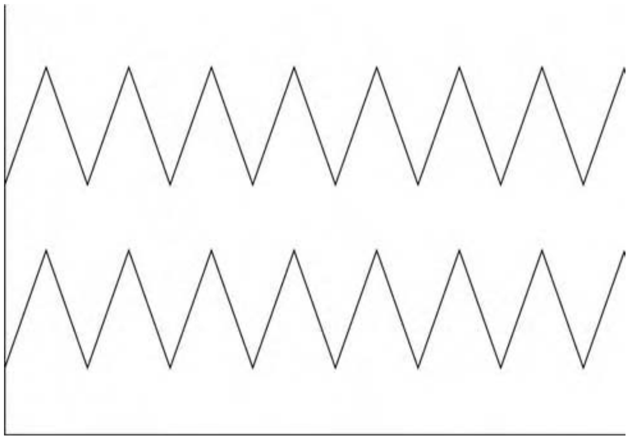

FIGURE 1.5 Perfect positive correlation (r = +1.00)

of the variables; for example, $\overline{X} = ((X_1 + X_2 + \ldots X_n)/n.)$

$$r = \frac{\sum_a (X_a - \overline{X}) * \sum_a (Y_a - \overline{Y})}{\sqrt{\sum_a (X_a - \overline{X})^2} * \sqrt{\sum_a (Y_a - \overline{Y})^2}} \tag{1.05}$$

Here is how to perform the calculation as shown in the table on page 34:

1. Average the Xs and the Ys.
2. For each period, find the difference between each X and the average X and each Y and the average Y.

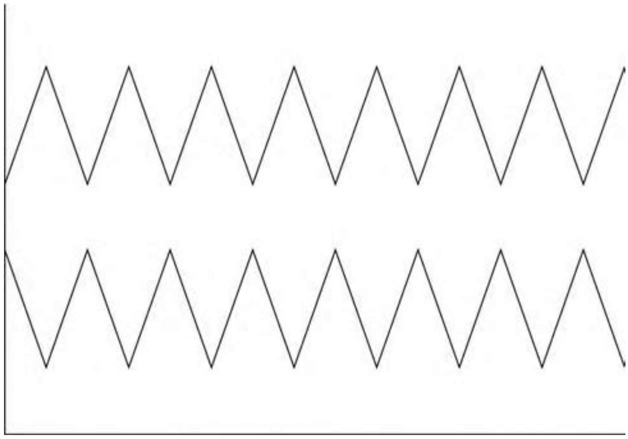

FIGURE 1.6 Perfect negative correlation (r = −1.00)

3. Now calculate the numerator. To do this, for each period, multiply the answers from step 2. In other words, for each period, multiply the difference between that period's X and the average X times the difference between that period's Y and the average Y.
4. Total up all of the answers to step 3 for all of the periods. This is the numerator.
5. Now find the denominator. To do this, take the answers to step 2 for each period, for both the X differences and the Y differences, and square them (they will now all be positive numbers).
6. Sum up the squared X differences for all periods into one final total. Do the same with the squared Y differences.
7. Take the square root of the sum of the squared X differences you just found in step 7. Now do the same with the Ys by taking the square root of the sum of the squared Y differences.
8. Multiply together the two answers you just found in step 7. That is, multiply the square root of the sum of the squared X differences by the square root of the sum of the squared Y differences. This product is your denominator.
9. Divide the numerator you found in step 4 by the denominator you found in step 8. This is your linear correlation coefficient, r.

The value for r will always be between +1.00 and −1.00. A value of 0 indicates no correlation whatsoever.

Look at Figure 1.7. It represents the following sequence of 21 trades:

$$1, 2, 1, \quad -1, 3, 2, \quad -1, -2, -3, 1, \quad -2, 3, 1, 1, 2, 3, 3, \quad -1, 2, \quad -1, 3$$

Now, here is how we use the linear correlation coefficient to see if there is any correlation between the previous trade and the current trade. The idea is to treat the trade P&Ls as the X values in the formula for r. Superimposed over that, we duplicate the same trade P&Ls, only this time we skew them by one trade, and use these as the Y values in the formula for r. In other words the Y value is the previous X value (see Figure 1.8).

The averages are different because you average only those Xs and Ys that have a corresponding X or Y value—that is, you average only those values that overlap; therefore, the last Y value (3) is not figured in the Y average, nor is the first X value (1) figured in the X average.

The numerator is the total of all entries in column E (.8). To find the denominator we take the square root of the total in column F, which is 8.555699, and we take the square root of the total in column G, which is 8.258329, and multiply them together to obtain a denominator of 70.65578. Now we divide our numerator of .8 by our denominator of 70.65578 to obtain 0.011322. This is our linear correlation coefficient, r. If you're really on top of this, you would also compute your Z score on these trades,

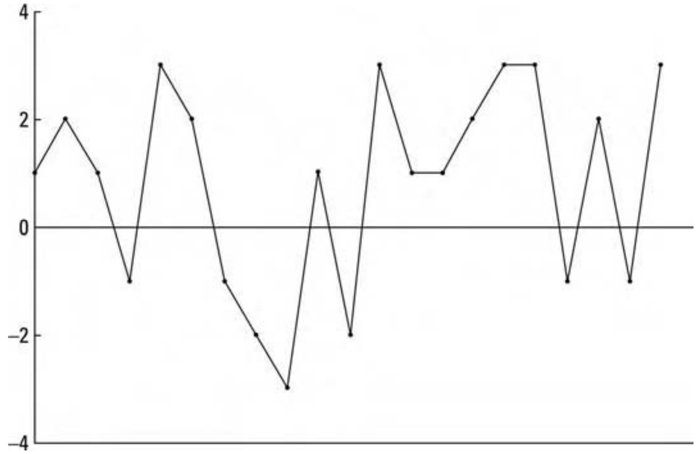

FIGURE 1.7 Individual outcomes of 21 bets/trades

which (if you want to check your work) is .5916 to four decimal places, or less than a 50% confidence limit that like begets unlike (since the Z score was positive).

The linear correlation coefficient of .011322 in this case is hardly indicative of anything, but it is pretty much in the range you can expect for most trading systems. A high correlation coefficient in futures trading systems would be one that was greater than .25 to .30 on the positive side, or less than −.25 to −.30 on the negative side. High positive correlation generally suggests that big wins are seldom followed by big losses and

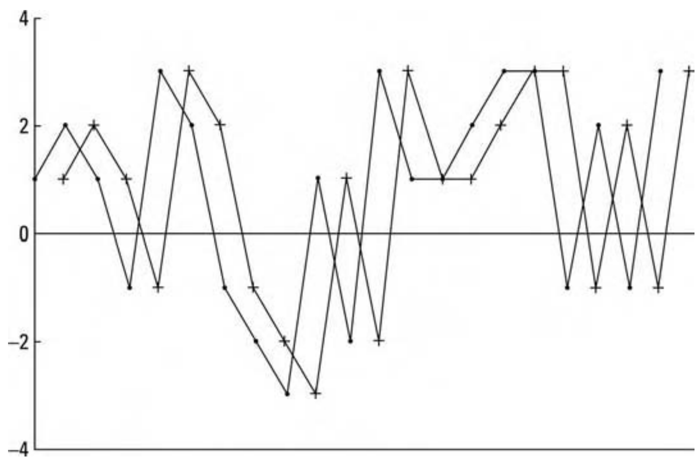

FIGURE 1.8 Individual outcomes of 21 bets/trades, skewed by 1 bet/trade

A	B	C	D	E	F	G
X	Y	X–X avg	Y–Y avg	col C times col D	col C squared	col D squared
1						
2	1	1.2	0.3	0.36	1.44	0.09
1	2	0.2	1.3	0.26	0.04	1.69
−1	1	−1.8	0.3	−0.54	3.24	0.09
3	−1	2.2	−1.7	−3.74	4.54	2.89
2	3	1.2	2.3	2.76	1.44	5.29
−1	2	−1.8	1.3	−2.34	3.24	1.69
−2	−1	−2.8	−1.7	4.76	7.84	2.89
−3	−2	−3.8	−2.7	10.26	14.44	7.29
1	−3	0.2	−3.7	−0.74	0.04	13.69
−2	1	−2.8	0.3	−0.84	7.84	0.09
3	−2	2.2	−2.7	−5.94	4.84	7.29
1	3	0.2	2.3	0.46	0.04	5.29
1	1	0.2	0.3	0.06	0.04	0.09
2	1	1.2	0.3	0.36	1.44	0.09
3	2	2.2	1.3	2.86	4.84	1.69
3	3	2.2	2.3	5.06	4.84	5.29
−1	3	−1.8	2.3	−4.14	3.24	5.29
2	−1	1.2	−1.7	−2.04	1.44	2.89
−1	2	−1.8	1.3	−2.34	3.24	1.69
3	−1	2.2	−1.7	−3.74	4.84	2.89
	3					
avg = 0.8	avg = 0.7	Totals =		0.8	73.2	68.2

vice versa. Negative correlation readings below −.25 to −.30 imply that big losses tend to be followed by big wins and vice versa.

There are a couple of reasons why it is important to use both the runs test and the linear correlation coefficient together in looking for dependency/correlation between trades. The first is that futures trading system trades (i.e., the profits and losses) do not necessarily conform to a Normal Probability Distribution. Rather, they conform pretty much to whatever the distribution is that futures prices conform to, which is as yet undetermined. Since the runs test assumes a Normal Probability Distribution, the runs test is only as accurate as the degree to which the system trade P&Ls conform to the Normal Probability Distribution.

The second reason for using the linear correlation coefficient in conjunction with the runs test is that the linear correlation coefficient is affected by the size of the trades. It not only interprets to what degree like begets like or like begets unlike, it also attempts to answer questions such as, "Are big winning trades generally followed by big losing trades?" "Are big losing trades generally followed by little losing trades?" And so on.

Negative correlation is just as helpful as positive correlation. For example, if there appears to be negative correlation, and the system has just suffered a large loss, we can expect a large win, and would therefore have more contracts on than ordinarily. Because of the negative correlation, if the trade proves to be a loss, the loss will most likely not be large.

Finally, in determining dependency you should also consider out-of-sample tests. That is, break your data segment into two or more parts. If you see dependency in the first part, then see if that dependency also exists in the second part, and so on. This will help eliminate cases where there appears to be dependency when in fact no dependency exists.

Using these two tools (the runs test and the linear correlation coefficient) can help answer many of these questions. However, they can answer them only if you have a high enough confidence limit and/or a high enough correlation coefficient (incidentally, the system we used earlier in this chapter, which had a confidence limit greater than 95%, had a correlation coefficient of only .0482). Most of the time, these tools are of little help, since all too often the universe of futures system trades is dominated by independence.

Recall the system mentioned in the discussion of Z scores that showed dependency to the 95% confidence limit. Based upon this statistic, we were able to improve this system by developing rules for passing trades. Now here is an interesting but disturbing fact. That system had one optimizeable parameter. When the system was run with a different value for that parameter, the dependency vanished! Was this saying that the appearance of dependency in our cited example was an illusion? Was it saying that only if you keep the value of this parameter within certain bounds can you have any dependency? If so, then isn't it possible that the appearance of dependency can be deceiving? To an extent this seems to be true.

Unfortunately, as traders, we most often must assume that dependency does not exist in the marketplace for the majority of market systems. That is, when trading a given market system, we will usually be operating in an environment where the outcome of the next trade is not predicated upon the outcome(s) of the preceding trade(s). This is not to say that there is never dependency between trades for some market systems (because for some market systems dependency does exist), only that we should act as though dependency does not exist unless there is very strong evidence to the contrary. Such would be the case if the Z score and the linear correlation coefficient indicated dependency, and the dependency held up across markets and across optimizeable parameter values. If we act as though there is dependency when the evidence is not overwhelming, we may well just be fooling ourselves and cause more self-inflicted harm than good.

Even if a system showed dependency to a 95% confidence limit for all values of a parameter, that confidence limit is hardly high enough for us to assume that dependency does in fact exist between the trades of a given market/system.

Yet the confidence limits and linear correlation coefficients are tools that should be used, because on rare occasions they may turn up a diamond in the rough, which can then possibly be exploited. Furthermore, and perhaps more importantly, they increase our understanding of the environment in which we are trying to operate.

On occasion, particularly in longer-term trading systems, you will encounter cases where the Z score and the linear correlation coefficient indicate dependency, and the dependency holds up across markets and across optimizeable parameter values. In such rare cases, you can take advantage of this dependency by either passing certain trades or altering your commitment on certain trades.

By studying these examples, you will better understand the subject matter.

−10, 10, −1, 1
Linear Correlation = −.9172
Z score = 1.8371 or 90 to 95% confidence limit that like begets unlike.

10, − 1, 1, −10
Linear Correlation = .1796
Z score = 1.8371 or 90 to 95% confidence limit that like begets unlike.

10, −10, 10, −10
Linear Correlation = −1.0000
Z score = 1.8371 or 90 to 95% confidence limit that like begets unlike.

−1, 1, −1, 1
Linear Correlation = −1.0000
Z score = 1.8371 or 90 to 95% confidence limit that like begets unlike.

1, 1, −1, −1
Linear Correlation = .5000
Z score = −.6124 or less than 50% confidence limit that like begets like.

100, −1, 50, −100, 1, −50
Linear Correlation = −.2542
Z score = 2.2822 or more than 97% confidence limit that like begets unlike.

The *turning points test* is an altogether different test for dependency. Going through the stream of trades, a turning point is counted if a trade is for a greater P&L value than both the trade before it and the trade after it. A trade can also be counted as a turning point if it is for a lesser P&L value than both the trade before it and the trade after it. Notice that we are using the individual trades, not the equity curve (the cumulative values of the trades). The number of turning points is totaled up for the entire stream of trades. Note that we must start with the second trade and end with the next to last trade, as we need a trade on either side of the trade we are considering as a turning point.

Consider now three values (1, 2, 3) in a random series, whereby each of the six possible orderings are equally likely:

$$1, 2, 3 \qquad 2, 3, 1 \qquad 1, 3, 2 \qquad 3, 1, 2 \qquad 2, 1, 3 \qquad 3, 2, 1$$

Of these six, four will result in a turning point. Thus, for a random stream of trades, the expected number of turning points is given as:

$$\text{Expected number of turning points} = 2/3 * (N - 2) \tag{1.06}$$

where: N = The total number of trades

We can derive the variance in the number of turning points of a random series as:

$$\text{Variance} = (16 * \text{N} - 29)/90 \tag{1.07}$$

The standard deviation is the square root of the variance. Taking the difference between the actual number of turning points counted in the stream of trades and the expected number and then dividing the difference by the standard deviation will give us a Z score, which is then expressed as a confidence limit. The confidence limit is discerned from Equation (2.22) for two-tailed Normal probabilities. Thus, if our stream of trades is very far away (very many standard deviations from the expected number), it is unlikely that our stream of trades is random; rather, dependency is present. If dependency appears to a high confidence limit (at least 95%) with the turning points test, you can determine from inspection whether like begets like (if there are fewer actual turning points than expected) or whether like begets unlike (if there are more actual turning points than expected).

Another test for dependence is the *phase length test*. This is a statistical test similar to the turning points test. Rather than counting up the number of turning points between (but not including) trade 1 and the last trade, the phase length test looks at how many trades have elapsed between turning points. A “phase” is the number of trades that elapse between a turning point high and a turning point low, or a turning point low and a turning point high. It doesn’t matter which occurs first, the high turning point or the low turning point. Thus, if trade number 4 is a turning point (high or low) and trade number 5 is a turning point (high or low, so long as it's the opposite of what the last turning point was), then the phase length is 1, since the difference between 5 and 4 is 1.

With the phase length test you add up the number of phases of length 1, 2, and 3 or more. Therefore, you will have three categories: 1, 2, and 3+. Thus, phase lengths of 4 or 5, and so on, are all totaled under the group of 3+. It doesn't matter if a phase goes from a high turning point to a low turning point or from a low turning point to a high turning point; the only thing that matters is how many trades the phase is comprised of. To figure the phase length, simply take the trade number of the latter phase (what number it is in sequence from 1 to N, where N is the total number of trades) and subtract the trade number of the prior phase. For each of the three categories you will have the total number of complete phases that occurred between (but not including) the first and the last trades.

Each of these three categories also has an expected number of trades for that category. The expected number of trades of phase length D is:

$$E(D) = 2 * (N - D - 2) * (D \wedge 2 * 3 * D + 1)/(D + 3)! \qquad (1.08)$$

where: D = The length of the phase.
$E(D)$ = The expected number of counts.
N = The total number of trades.

Once you have calculated the expected number of counts for the three categories of phase length (1, 2, and 3+), you can perform the chi-square test. According to Kendall and colleagues,[3] you should use 2.5 degrees of freedom here in determining the significance levels, as the lengths of the phases are not independent. Remember that the phase length test doesn't tell you about the dependence (like begetting like, etc.), but rather whether or not there is dependence or randomness.

Lastly, this discussion of dependence addresses converting a correlation coefficient to a confidence limit. The technique employs what is known as *Fisher's Z transformation*, which converts a correlation coefficient, r, to a Normally distributed variable:

$$F = .5 * \ln(1 + r)/(1 - r)) \qquad (1.09)$$

where: F = The transformed variable, now Normally distributed.
r = The correlation coefficient of the sample.
$\ln(\,)$ = The natural logarithm function.

[3] Kendall, M. G., A. Stuart, and J. K. Ord. *The Advanced Theory of Statistics*, Vol. III. New York: Hafner Publishing, 1983.

The distribution of these transformed variables will have a variance of:

$$V = 1/(N - 3) \tag{1.10}$$

where: V = The variance of the transformed variables.
N = The number of elements in the sample.

The mean of the distribution of these transformed variables is discerned by Equation (1.09), only instead of being the correlation coefficient of the sample, r is the correlation coefficient of the population. Thus, since our population has a correlation coefficient of 0 (which we assume, since we are testing deviation from randomness), then Equation (1.09) gives us a value of 0 for the mean of the population.

Now we can determine how many standard deviations the adjusted variable is from the mean by dividing the adjusted variable by the square root of the variance, Equation (1.10). The result is the Z score associated with a given correlation coefficient and sample size. For example, suppose we had a correlation coefficient of .25, and this was discerned over 100 trades. Thus, we can find our Z score as Equation (1.9) divided by the square root of Equation (1.10), or:

$$Z = .5 * \ln((1 + r)/(1 - r))/\sqrt{1/(N - 3)} \tag{1.11}$$

Which, for our example, is:

$$\begin{aligned}
Z &= (.5 * \ln((1 + .25)/(1 - .25)))/(1/(100 - 3)) \wedge .5 \\
&= (.5 * \ln(1.25/.75))/(1/97) \wedge .5 \\
&= (.5 * \ln(1.6667))/.010309 \wedge .5 \\
&= (.5 * .51085)/.1015346165 \\
&= .25541275/.1015346165 \\
&= 2.515523856
\end{aligned}$$

Now we can translate this into a confidence limit by using Equation (2.22) for a Normal Distribution two-tailed confidence limit. For our example this works out to a confidence limit in excess of 98.8%. If we had had 30 trades or less, we would have had to discern our confidence limit by using the Student's Distribution with N −1 degrees of freedom.