CHAPTER 8

# The Geometry of Mean Variance Portfolios

*We have now covered how to find the optimal* fs *for a given market system from a number of different standpoints. Also, we have seen how to derive the efficient frontier. In this chapter we show how to combine the two notions of optimal* f *and classical portfolio theory. Furthermore, we will delve into an analytical study of the geometry of portfolio construction.*

## THE CAPITAL MARKET LINES (CMLs)

We can improve upon the performance of any given portfolio by combining a certain percentage of the portfolio with cash. Figure 8.1 shows this relationship graphically.

In Figure 8.1, point A represents the return on the risk-free asset. This would usually be the return on 91-day Treasury bills. Since the risk, the standard deviation in returns, is regarded as nonexistent, point A is at zero on the horizontal axis.

Point B represents the tangent portfolio. It is the only portfolio lying upon the efficient frontier that would be touched by a line drawn from the risk-free rate of return on the vertical axis and zero on the horizontal axis. Any point along line segment AB will be composed of the portfolio at Point B and the risk-free asset. At point B, all of the assets would be in the portfolio, and at point A all of the assets would be in the risk-free asset. Anywhere in between points A and B represents having a portion of the assets in both the portfolio and the risk-free asset. Notice that any portfolio along line segment

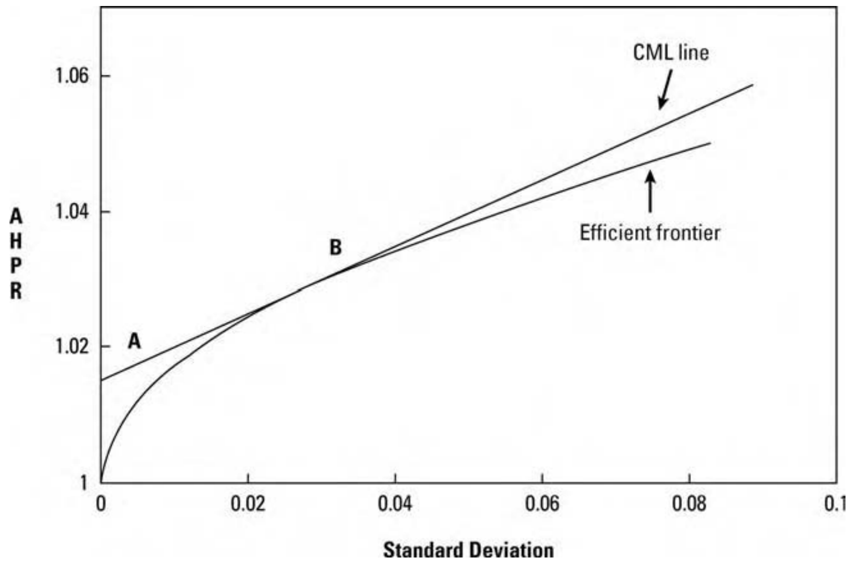


**FIGURE 8.1** Enhancing returns with the risk-free asset

AB dominates any portfolio on the efficient frontier at the same risk level, since being on the line segment AB has a higher return for the same risk. Thus, an investor who wanted a portfolio less risky than portfolio B would be better off to put a portion of his or her investable funds in portfolio B and a portion in the risk-free asset, as opposed to owning 100% of a portfolio on the efficient frontier at a point less risky than portfolio B.

The line emanating from point A, the risk-free rate on the vertical axis and zero on the horizontal axis, and emanating to the right, tangent to one point on the efficient frontier, is called the *capital market line* (CML). To the right of point B, the CML line represents portfolios where the investor has gone out and borrowed more money to invest further in portfolio B. Notice that an investor who wanted a portfolio with a greater return than portfolio B would be better off to do this, as being on the CML line right of point B dominates (has higher return than) those portfolios on the efficient frontier with the same level of risk.

Usually, point B will be a very well-diversified portfolio. Most portfolios high up and to the right and low down and to the left on the efficient frontier have very few components. Those in the middle of the efficient frontier, where the tangent point to the risk-free rate is, usually are very well diversified.

It has traditionally been assumed that all rational investors will want to get the greatest return for a given risk and take on the lowest risk for a given return. Thus, all investors would want to be somewhere on the CML line. In other words, all investors would want to own the same portfolio, only with differing degrees of leverage. This distinction between the investment decision and the financing decision is known as the *Separation Theorem.*[1]

We assume now that the vertical scale, the E in E–V theory, represents the arithmetic average HPR (AHPR) for the portfolios and the horizontal, or V, scale represents the standard deviation in the HPRs. For a given risk-free rate, we can determine where this tangent point portfolio on our efficient frontier is, as the coordinates (AHPR, V) that maximize the following function are:

$$\text{Tangent Portfolio} = \text{MAX}\{(\text{AHPR} - (1 + \text{RFR}))/\text{SD}\} \tag{8.01}$$

where: MAX{} = The maximum value.
AHPR = The arithmetic average HPR. This is the E coordinate of a given portfolio on the efficient frontier.
SD = The standard deviation in HPRs. This is the V coordinate of a given portfolio on the efficient frontier.
RFR = The risk-free rate.

In Equation (8.01), the formula inside the braces ({}) is known as the Sharpe ratio, a measurement of risk-adjusted returns. Expressed literally, the Sharpe ratio for a portfolio is a measure of the ratio of the expected excess returns to the standard deviation. The portfolio with the highest Sharpe ratio, therefore, is the portfolio where the CML line is tangent to the efficient frontier for a given RFR.

The Sharpe ratio, when multiplied by the square root of the number of periods over which it was derived, equals the $t$ statistic. From the resulting t statistic it is possible to obtain a confidence level that the AHPR exceeds the RFR by more than chance alone, assuming finite variance in the returns.

The following table shows how to use Equation (8.01) and demonstrates the entire process discussed thus far. The first two columns represent the coordinates of different portfolios on the efficient frontier. The coordinates are given in (AHPR, SD) format, which corresponds to the Y and X axes of Figure 8.1. The third column is the answer obtained for Equation (8.01) assuming a 1.5% risk-free rate (equating to an AHPR of 1.015. We assume that the HPRs here are quarterly HPRs; thus, a 1.5% risk-free rate for the

[1]See Tobin, James, "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies* 25, pp. 65–85, February 1958.

quarter equates to roughly a 6% risk-free rate for the year.). Thus, to work out (8.01a) for the third set of coordinates (.00013, 1.002):

$$\begin{aligned}(\text{AHPR} - (1 + \text{RFR}))/\text{SD} &= (1.002 - (1 + .015))/.00013\\ &= (1.002 - 1.015)/.00013\\ &= -.013/.00013\\ &= -100\end{aligned}$$

The process is completed for each point along the efficient frontier. Equation (8.01) peaks out at .502265, which is at the coordinates (02986, 1.03). These coordinates are the point where the CML line is tangent to the efficient frontier, corresponding to point B in Figure 8.1. This tangent point is a certain portfolio along the efficient frontier. The Sharpe ratio is the slope of the CML, with the steepest slope being the tangent line to the efficient frontier.

| **Efficient Frontier** | | | **CML line** | |
|---|---|---|---|---|
| **AHPR** | **SD** | **Eq. (8.1a)** | **Percentage** | **AHPR** |
| | | RFR = .015 | | |
| 1.00000 | 0.00000 | 0 | 0.00% | 1.0150 |
| 1.00100 | 0.00003 | −421.902 | 0.11% | 1.0150 |
| 1.00200 | 0.00013 | −100.000 | 0.44% | 1.0151 |
| 1.00300 | 0.00030 | −40.1812 | 1.00% | 1.0152 |
| 1.00400 | 0.00053 | −20.7184 | 1.78% | 1.0153 |
| 1.00500 | 0.00083 | −12.0543 | 2.78% | 1.0154 |
| 1.00600 | 0.00119 | −7.53397 | 4.00% | 1.0156 |
| 1.00700 | 0.00163 | −4.92014 | 5.45% | 1.0158 |
| 1.00800 | 0.00212 | −3.29611 | 7.11% | 1.0161 |
| 1.00900 | 0.00269 | −2.23228 | 9.00% | 1.0164 |
| 1.01000 | 0.00332 | −1.50679 | 11.11% | 1.0167 |
| 1.01100 | 0.00402 | −0.99622 | 13.45% | 1.0170 |
| 1.01200 | 0.00478 | −0.62783 | 16.00% | 1.0174 |
| 1.01300 | 0.00561 | −0.35663 | 18.78% | 1.0178 |
| 1.01400 | 0.00650 | −0.15375 | 21.78% | 1.0183 |
| 1.01500 | 0.00747 | 0 | 25.00% | 1.0188 |
| 1.01600 | 0.00849 | 0.117718 | 28.45% | 1.0193 |
| 1.01700 | 0.00959 | 0.208552 | 32.12% | 1.0198 |
| 1.01800 | 0.01075 | 0.279036 | 36.01% | 1.0204 |
| 1.01900 | 0.01198 | 0.333916 | 40.12% | 1.0210 |
| 1.02000 | 0.01327 | 0.376698 | 44.45% | 1.0217 |
| 1.02100 | 0.01463 | 0.410012 | 49.01% | 1.0224 |
| 1.02200 | 0.01606 | 0.435850 | 53.79% | 1.0231 |
| 1.02300 | 0.01755 | 0.455741 | 58.79% | 1.0238 |
| 1.02400 | 0.01911 | 0.470873 | 64.01% | 1.0246 |
| 1.02500 | 0.02074 | 0.482174 | 69.46% | 1.0254 |

| Efficient Frontier | | | CML line | |
|---|---|---|---|---|
| **AHPR** | **SD** | **Eq. (8.1a)** | **Percentage** | **AHPR** |
| 1.02600 | 0.02243 | 0.490377 | 75.12% | 1.0263 |
| 1.02700 | 0.02419 | 0.496064 | 81.01% | 1.0272 |
| 1.02800 | 0.02602 | 0.499702 | 87.12% | 1.0281 |
| 1.02900 | 0.02791 | 0.501667 | 93.46% | 1.0290 |
| 1.03000 | 0.02986 | 0.502265 (peak) | 100.02% | 1.0300 |
| 1.03100 | 0.03189 | 0.501742 | 106.79% | 1.0310 |
| 1.03200 | 0.03398 | 0.500303 | 113.80% | 1.0321 |
| 1.03300 | 0.03614 | 0.498114 | 121.02% | 1.0332 |
| 1.03400 | 0.03836 | 0.495313 | 128.46% | 1.0343 |
| 1.03500 | 0.04065 | 0.492014 | 136.13% | 1.0354 |
| 1.03600 | 0.04301 | 0.488313 | 144.02% | 1.0366 |
| 1.03700 | 0.04543 | 0.484287 | 152.13% | 1.0378 |
| 1.03800 | 0.04792 | 0.480004 | 160.47% | 1.0391 |
| 1.03900 | 0.05047 | 0.475517 | 169.03% | 1.0404 |
| 1.04000 | 0.05309 | 0.470873 | 177.81% | 1.0417 |
| 1.04100 | 0.05578 | 0.466111 | 186.81% | 1.0430 |
| 1.04200 | 0.05853 | 0.461264 | 196.03% | 1.0444 |
| 1.04300 | 0.06136 | 0.456357 | 205.48% | 1.0458 |
| 1.04400 | 0.06424 | 0.451416 | 215.14% | 1.0473 |
| 1.04500 | 0.06720 | 0.446458 | 225.04% | 1.0488 |
| 1.04600 | 0.07022 | 0.441499 | 235.15% | 1.0503 |
| 1.04700 | 0.07330 | 0.436554 | 245.48% | 1.0518 |
| 1.04800 | 0.07645 | 0.431634 | 256.04% | 1.0534 |
| 1.04900 | 0.07967 | 0.426747 | 266.82% | 1.0550 |
| 1.05000 | 0.08296 | 0.421902 | 277.82% | 1.0567 |

The next column over, "percentage," represents what percentage of your assets must be invested in the tangent portfolio if you are at the CML line for that standard deviation coordinate. In other words, for the last entry in the table to be on the CML line at the .08296 standard deviation level corresponds to having 277.82% of your assets in the tangent portfolio (i.e., being fully invested and borrowing another $1.7782 for every dollar already invested to invest further). This percentage value is calculated from the standard deviation of the tangent portfolio as:

$$P = SX/ST \tag{8.02}$$

where: SX = The standard deviation coordinate for a particular point on the CML line.
ST = The standard deviation coordinate of the tangent portfolio.
P = The percentage of your assets that must be invested in the tangent portfolio to be on the CML line for a given SX.

Thus, the CML line at the standard deviation coordinate .08296, the last entry in the table, is divided by the standard deviation coordinate of the tangent portfolio, .02986, yielding 2.7782, or 277.82%.

The last column in the table, the CML line AHPR, is the AHPR of the CML line at the given standard deviation coordinate. This is figured as:

$$\text{ACML} = (\text{AT} * \text{P}) + ((1 + \text{RFR}) * (1 - \text{P})) \tag{8.03}$$

where: ACML = The AHPR of the CML line at a given risk coordinate, or a corresponding percentage figured from (8.02).
AT = The AHPR at the tangent point, figured from (8.01a).
P = The percentage in the tangent portfolio, figured from (8.02).
RFR = The risk-free rate.

On occasion you may want to know the standard deviation of a certain point on the CML line for a given AHPR. This linear relationship can be obtained as:

$$\text{SD} = \text{P} * \text{ST} \tag{8.04}$$

where: SD = The standard deviation at a given point on the CML line corresponding to a certain percentage, P, corresponding to a certain AHPR.
P = The percentage in the tangent portfolio, figured from (8.02).
ST = The standard deviation coordinate of the tangent portfolio.

## THE GEOMETRIC EFFICIENT FRONTIER

The problem with Figure 8.1 is that it shows the arithmetic average HPR. When we are reinvesting profits back into the program we must look at the geometric average HPR for the vertical axis of the efficient frontier. This changes things considerably. The formula to convert a point on the efficient frontier from an arithmetic HPR to a geometric is:

$$\text{GHPR} = \sqrt{\text{AHPR}^2 - \text{V})}$$

where: GHPR = The geometric average HPR.
AHPR = The arithmetic average HPR.
V = The variance coordinate. (This is equal to the standard deviation coordinate squared.)

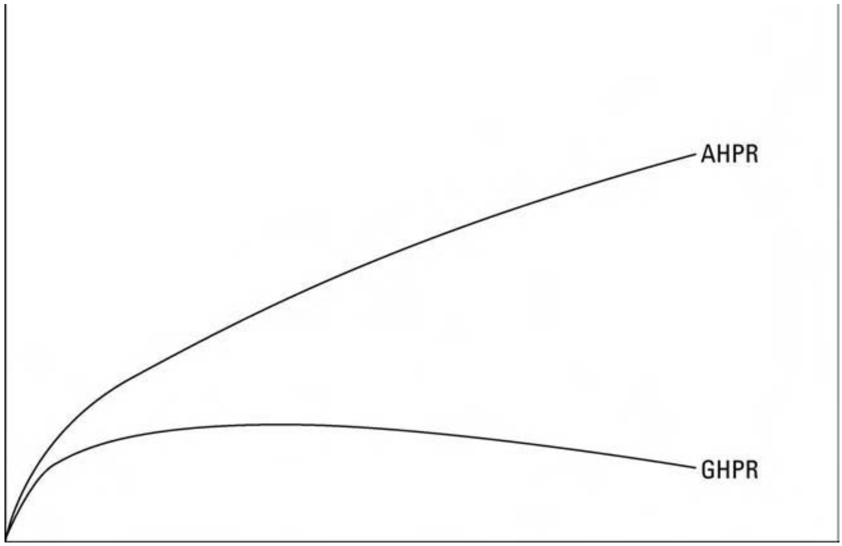


**FIGURE 8.2** The efficient frontier with/without reinvestment

In Figure 8.2 you can see the efficient frontier corresponding to the arithmetic average HPRs as well as that corresponding to the geometric average HPRs. You can see what happens to the efficient frontier when reinvestment is involved.

By graphing your GHPR line, you can see which portfolio is the geometric optimal (the highest point on the GHPR line). You could also determine this portfolio by converting the AHPRs and Vs of each portfolio along the AHPR efficient frontier into GHPRs per Equation (3.04) and see which had the highest GHPR. Again, that would be the geometric optimal. However, given the AHPRs and the Vs of the portfolios lying along the AHPR efficient frontier, we can readily discern which portfolio would be geometric optimal—the one that solves the following equality:

$$AHPR - 1 - V = 0 \tag{8.05a}$$

where: AHPR = The arithmetic average HPRs. This is the E coordinate of a given portfolio on the efficient frontier.

V = The variance in HPR. This is the V coordinate of a given portfolio on the efficient frontier. This is equal to the standard deviation squared.

Equation (8.06a) can also be written as any one of the following three forms:

$$\text{AHPR} - 1 = \text{V} \tag{8.05b}$$

$$\text{AHPR} - \text{V} = 1 \tag{8.05c}$$

$$\text{AHPR} = \text{V} + 1 \tag{8.05d}$$

A brief note on the geometric optimal portfolio is in order here. Variance in a portfolio is generally directly and positively correlated to drawdown in that higher variance is generally indicative of a portfolio with higher drawdown. Since the geometric optimal portfolio is that portfolio for which E and V are equal (with E = AHPR − 1), then we can assume that the geometric optimal portfolio will see high drawdowns. In fact, the greater the GHPR of the geometric optimal portfolio—that is, the more the portfolio makes—the greater will be its drawdown in terms of equity retracements, since the GHPR is directly positively correlated with the AHPR. Here again is a paradox. We want to be at the geometric optimal portfolio. Yet, the higher the geometric mean of a portfolio, the greater will be the drawdowns in terms of percentage equity retracements generally. Hence, when we perform the exercise of diversification, we should view it as an exercise to obtain the highest geometric mean rather than the lowest drawdown, as the two tend to pull in opposite directions! The geometrical optimal portfolio is one where a line drawn from (0,0), with slope 1, intersects the AHPR efficient frontier.

Figure 8.2 demonstrates the efficient frontiers on a one-trade basis. That is, it shows what you can expect on a one-trade basis. We can convert the geometric average HPR to a TWR by the equation:

$$\text{GTWR} = \text{GHPR}^{\text{N}}$$

where: GTWR = The vertical axis corresponding to a given GHPR after N trades.
GHPR = The geometric average HPR.
N = The number of trades we desire to observe.

Thus, after 50 trades a GHPR of 1.0154 would be a GTWR of $1.0154^{50} = 2.15$. In other words, after 50 trades we would expect our stake to have grown by a multiple of 2.15.

We can likewise project the efficient frontier of the arithmetic average HPRs into ATWRs as:

$$\text{ATWR} = 1 + \text{N} * (\text{AHPR} - 1)$$

where: ATWR = The vertical axis corresponding to a given AHPR after N trades.
AHPR = The arithmetic average HPR.
N = The number of trades we desire to observe.

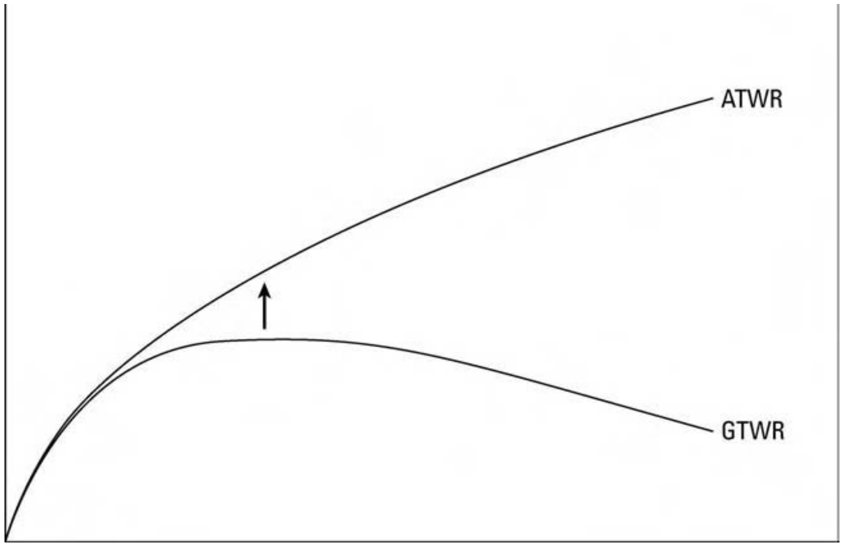


**FIGURE 8.3** The efficient frontier with/without reinvestment

Thus, after 50 trades, an arithmetic average HPR of 1.03 would have made $1 + 50 * (1.03 - 1) = 1 + 50 * .03 = 1 + 1.5 = 2.5$ times our starting stake. Note that this shows what happens when we do not reinvest our winnings back into the trading program. Equation (3.06) is the TWR you can expect when constant-contract trading.

Just as Figure 8.2 shows the TWRs, both arithmetic and geometric, for one trade, Figure 8.3 shows them for a few trades later. Notice that the GTWR line is approaching the ATWR line. At some point for N, the geometric TWR will overtake the arithmetic TWR. Figure 8.4 shows the arithmetic and geometric TWRs after more trades have elapsed. Notice that the geometric has overtaken the arithmetic. If we were to continue with more and more trades, the geometric TWR would continue to outpace the arithmetic. Eventually, the geometric TWR becomes infinitely greater than the arithmetic.

The logical question is, "How many trades must elapse until the geometric TWR surpasses the arithmetic?" See the following equation, which tells us the number of trades required to reach a specific goal:

$$T = \ln(\text{Goal})/\ln(\text{Geometric Mean})$$

where: $T$ = The expected number of trades to reach a specific goal.
Goal = The goal in terms of a multiple on our starting stake, a TWR.
$\ln()$ = The natural logarithm function.

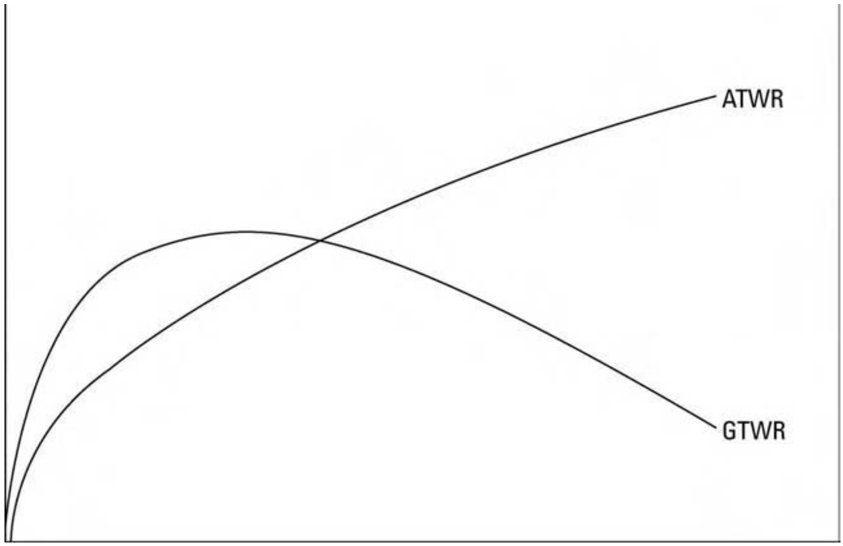


**FIGURE 8.4** The efficient frontier with/without reinvestment

We let the AHPR at the same V as our geometric optimal portfolio be our goal and use the geometric mean of our geometric optimal portfolio in the denominator of the equation just mentioned. We can now discern how many trades are required to make our geometric optimal portfolio match one trade in the corresponding arithmetic portfolio. Thus:

$$\begin{aligned} T &= \ln(1.031)/\ln(1.01542) \\ &= .035294/.0153023 \\ &= 1.995075 \end{aligned} \tag{8.06}$$

We would thus expect 1.995075, or roughly 2, trades for the optimal GHPR to be as high up as the corresponding (same V) AHPR after one trade.

The problem is that the ATWR needs to reflect the fact that two trades have elapsed. In other words, as the GTWR approaches the ATWR, the ATWR is also moving upward, albeit at a constant rate (compared to the GTWR, which is accelerating). We can relate this problem to Equations (8.07) and (8.06), the geometric and arithmetic TWRs respectively, and express it mathematically:

$$\text{GHPR}^{\text{N}} => 1 + \text{N} * (\text{AHPR} - 1) \tag{8.07}$$

Since we know that when N = 1, G will be less than A, we can rephrase the question to "At how many N will G equal A?" Mathematically this is:

$$\text{GHPR}^{\text{N}} = 1 + \text{N} * (\text{AHPR} - 1) \tag{8.08a}$$

which can be written as:

$$1 + \text{N} * (\text{AHPR} - 1) - \text{GHPR}^{\text{N}} = 0 \tag{8.08b}$$

or

$$1 + \text{N} * \text{AHPR} - \text{N} - \text{GHPR}^{\text{N}} = 0 \tag{8.08c}$$

or

$$\text{N} = (\text{GHPR} * \text{N} - 1)/(\text{AHPR} - 1) \tag{8.08d}$$

The N that solves (8.08a) through (8.08d) is the N that is required for the geometric HPR to equal the arithmetic. All three equations are equivalent. The solution must be arrived at by iteration. Taking our geometric optimal portfolio of a GHPR of 1.01542 and a corresponding AHPR of 1.031, if we were to solve for any of Equations (8.10a) through (8.10d), we would find the solution to these equations at N = 83.49894. That is, at 83.49894 elapsed trades, the geometric TWR will overtake the arithmetic TWR for those TWRs corresponding to a variance coordinate of the geometric optimal portfolio.

Just as the AHPR has a CML line, so too does the GHPR. Figure 8.5 shows both the AHPR and the GHPR with a CML line for both calculated from the same risk-free rate.

The CML for the GHPR is calculated from the CML for the AHPR by the following equation:

$$\text{CMLG} = \sqrt{\text{CMLA}^2 - \text{VT} * \text{P}} \tag{8.09}$$

where: CMLG = The E coordinate (vertical) to the CML line to the GHPR for a given V coordinate corresponding to P.
CMLA = The E coordinate (vertical) to the CML line to the AHPR for a given V coordinate corresponding to P.
P = The percentage in the tangent portfolio, figured from (8.02).
VT = The variance coordinate of the tangent portfolio.

You should know that, for any given risk-free rate, the tangent portfolio and the geometric optimal portfolio are not necessarily (and usually are

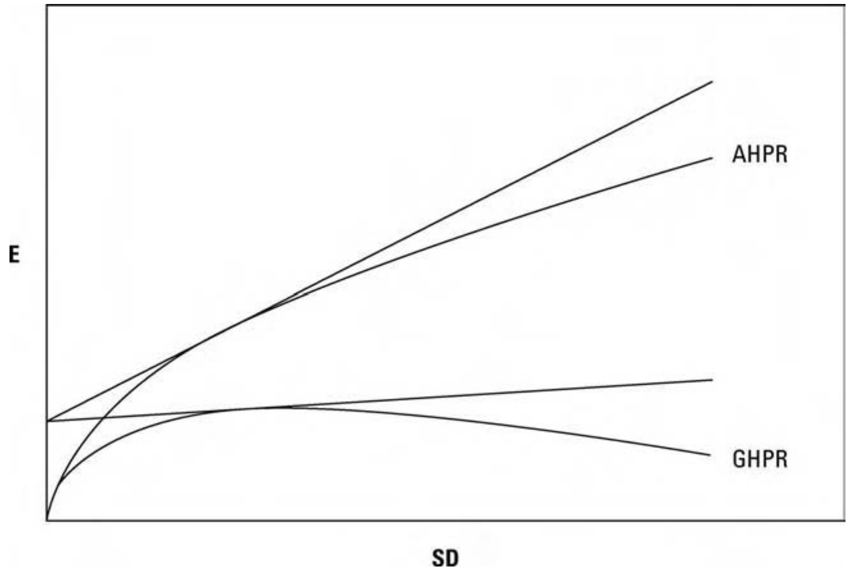


**FIGURE 8.5** AHPR, GHPR, and their CML lines

not) the same. The only time that these portfolios will be the same is when the following equation is satisfied:

$$RFR = GHPROPT - 1 \tag{8.10}$$

where: RFR = The risk-free rate.
GHPROPT = The geometric average HPR of the geometric optimal portfolio. This is the E coordinate of the portfolio on the efficient frontier.

Only when the GHPR of the geometric optimal portfolio minus 1 is equal to the risk-free rate will the geometric optimal portfolio and the portfolio tangent to the CML line be the same. If RFR > GHPROPT − 1, then the geometric optimal portfolio will be to the left of (have less variance than) the tangent portfolio. If RFR < GHPROPT − 1, then the tangent portfolio will be to the left of (have less variance than) the geometric optimal portfolio. In all cases, though, the tangent portfolio will, of course, never have a higher GHPR than the geometric optimal portfolio.

Note also that the point of tangency for the CML to the GHPR and for the CML to the AHPR is at the same SD coordinate. We could use Equation

(8.01) to find the tangent portfolio of the GHPR line by substituting the AHPR in (8.11) with GHPR. The resultant equation is:

$$\text{Tangent Portfolio} = \text{MAX}\{(\text{GHPR} - (1 + \text{RFR}))/\text{SD}\} \tag{8.11}$$

where: MAX{} = The maximum value.
GHPR = The geometric average HPRs. This is the E coordinate of a given portfolio on the efficient frontier.
SD = The standard deviation in HPRs. This is the SD coordinate of a given portfolio on the efficient frontier.
RFR = The risk-free rate.

## UNCONSTRAINED PORTFOLIOS

Now we will see how to enhance returns beyond the GCML line by lifting the sum of the weights constraint. Let us return to geometric optimal portfolios. If we look for the geometric optimal portfolio among our four market systems—Toxico, Incubeast, LA Garb, and a savings account—we find it at E equal to .1688965 and V equal to .1688965, thus conforming with Equations (8.05a) through (8.05d). The geometric mean of such a portfolio would therefore be 1.094268, and the portfolio's composition would be:

| | |
|---|---|
| Toxico | 18.89891% |
| Incubeast | 19.50386% |
| LA Garb | 58.58387% |
| Savings Account | .03014% |

In using Equations (8.05a) through (8.05d), you must iterate to the solution. That is, you try a test value for E (halfway between the highest and the lowest AHPRs; −1 is a good starting point) and solve the matrix for that E. If your variance is higher than E, it means the tested for value of E was too high, and you should lower it for the next attempt. Conversely, if your variance is less than E, you should raise E for the next pass. You keep on repeating the process until whichever of Equations (8.05a) through (8.05d) you choose to use, is solved. Then you will have arrived at your geometric optimal portfolio. (Note that all of the portfolios discussed thus far, whether on the AHPR efficient frontier or the GHPR efficient frontier, are determined by constraining the sum of the percentages, the weights, to 100% or 1.00.)

See the equation used in the starting augmented matrix to find the optimal weights in a portfolio. This equation dictates that the sum of the weights equal 1:

$$\left(\sum_{i=1}^{N} X_i\right) - 1 = 0$$

where: $N$ = The number of securities comprising the portfolio.
$X_i$ = The percentage weighting of the ith security.

The equation can also be written as:

$$\left(\sum_{i=1}^{N} X_i\right) - 1$$

By allowing the left side of this equation to be greater than 1, we can find the unconstrained optimal portfolio. The easiest way to do this is to add another market system, called *non-interest-bearing cash* (NIC), into the starting augmented matrix. This market system, NIC, will have an arithmetic average daily HPR of 1.0 and a population standard deviation (as well as variance and covariances) in those daily HPRs of 0. What this means is that each day the HPR for NIC will be 1.0. The correlation coefficients for NIC to any other market system are always 0.

Now we set the sum of the weights constraint to some arbitrarily high number, greater than 1. A good initial value is three times the number of market systems (without NIC) that you are using. Since we have four market systems (when not counting NIC) we should set this sum of the weights constraint to $4 * 3 = 12$. Note that we are not really lifting the constraint that the sum of the weights be below some number, we are just setting this constraint at an arbitrarily high value. The difference between this arbitrarily high value and what the sum of the weights actually comes out to be will be the weight assigned to NIC.

We are not going to really invest in NIC, though. It's just a null entry that we are pumping through the matrix to arrive at the unconstrained weights of our market systems. Now, let's take the parameters of our four market systems from Chapter 7 and add NIC as well:

| **Investment** | **Expected Return as an HPR** | **Expected Standard Deviation of Return** |
|---|---|---|
| Toxico | 1.095 | .316227766 |
| Incubeast Corp. | 1.13 | .5 |
| LA Garb | 1.21 | .632455532 |
| Savings Account | 1.085 | 0 |
| NIC | 1.00 | 0 |

The covariances among the market systems, with NIC included, are as follows:

| | T | I | L | S | N |
|---|---|---|---|---|---|
| T | .1 | −.0237 | .01 | 0 | 0 |
| I | −.0237 | .25 | .079 | 0 | 0 |
| L | .01 | .079 | .4 | 0 | 0 |
| S | 0 | 0 | 0 | 0 | 0 |
| N | 0 | 0 | 0 | 0 | 0 |

Thus, when we include NIC, we are now dealing with five market systems; therefore, the generalized form of the starting augmented matrix is:

$$
\begin{array}{lllllll}
X_1 * U_1 & +X_2 * U_2 & +X_3 * U_3 & +X_4 * U_4 & X_5 * U_5 & = E \\
X_1 & +X_2 & +X_3 & +X_4 & X_5 & = S
\end{array}
$$

$$X_1 * COV_{1,1} + X_2 * COV_{1,2} + X_3 * COV_{1,3} + X_4 * COV_{1,4} + X_5 * COV_{1,5} + .5 * L_1 * U_1 + .5 * L_2 = 0$$

$$X_1 * COV_{2,1} + X_2 * COV_{2,2} + X_3 * COV_{2,3} + X_4 * COV_{2,4} + X_5 * COV_{2,5} + .5 * L_1 * U_2 + .5 * L_2 = 0$$

$$X_1 * COV_{3,1} + X_2 * COV_{3,2} + X_3 * COV_{3,3} + X_4 * COV_{3,4} + X_5 * COV_{3,5} + .5 * L_1 * U_3 + .5 * L_2 = 0$$

$$X_1 * COV_{4,1} + X_2 * COV_{4,2} + X_3 * COV_{4,3} + X_4 * COV_{4,4} + X_5 * COV_{4,5} + .5 * L_1 * U_4 + .5 * L_2 = 0$$

$$X_1 * COV_{5,1} + X_2 * COV_{5,2} + X_3 * COV_{5,3} + X_4 * COV_{5,4} + X_5 * COV_{5,5} + .5 * L_1 * U_5 + .5 * L_2 = 0$$

where:
$E$ = The expected return of the portfolio.
$S$ = The sum of the weights constraint.
$COV_{A,B}$ = The covariance between securities A and B.
$X_i$ = The percentage weighting of the ith security.
$U_i$ = The expected return of the ith security.
$L_1$ = The first Lagrangian multiplier.
$L_2$ = The second Lagrangian multiplier.

Thus, once we have included NIC, our starting augmented matrix appears as follows:

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $L_1$ | $L_2$ | Answer |
|---|---|---|---|---|---|---|---|
| .095 | .13 | .21 | .085 | 0 | | | E |
| 1 | 1 | 1 | 1 | 0 | | | 12 |
| .1 | −.0237 | .01 | 0 | 0 | .095 | 1 | 0 |
| −.0237 | .25 | .079 | 0 | 0 | .13 | 1 | 0 |
| .01 | .079 | .4 | 0 | 0 | .21 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | .085 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Note that the answer column of the second row, the sum of the weights constraint, is 12, as we determined it to be by multiplying the number of market systems (not including NIC) by 3.

When you are using NIC, it is important that you include it as the last, the Nth market system of N market systems, in the starting augmented matrix.

Now, the object is to obtain the identity matrix by using row operations to produce elementary transformations, as was detailed in Chapter 7. You can now create an unconstrained AHPR efficient frontier and an unconstrained GHPR efficient frontier. The unconstrained AHPR efficient frontier represents using leverage but not reinvesting.

The GHPR efficient frontier represents using leverage and reinvesting the profits. Ideally, we want to find the unconstrained geometric optimal portfolio. This is the portfolio that will result in the greatest geometric growth for us. We can use Equations (8.05a) through (8.05d) to solve for which of the portfolios along the efficient frontier is geometric optimal. In so doing, we find that no matter what value we try to solve E for (the value in the answer column of the first row), we get the same portfolio—comprised of only the savings account levered up to give us whatever value for E we want. This results in giving us our answer; we get the lowest V (in this case zero) for any given E.

What we must do, then, is take the savings account out of the matrix and start over. This time we will try to solve for only four market systems—Toxico, Incubeast, LA Garb, and NIC—and we set our sum of the weights constraint to nine. Whenever you have a component in the matrix with zero variance and an AHPR greater than one, you'll end up with the optimal portfolio as that component levered up to meet the required E.

Now, solving the matrix, we find Equations (8.05a) through (8.05d) satisfied at E equals .2457. Since this is the geometric optimal portfolio, V is also equal to .2457. The resultant geometric mean is 1.142833. The portfolio is:

| | |
|---|---|
| Toxico | 102.5982% |
| Incubeast | 49.00558% |
| LA Garb | 40.24979% |
| NIC | 708.14643% |

"Wait," you say. "How can you invest over 100% in certain components?" We will return to this in a moment.

If NIC is not one of the components in the geometric optimal portfolio, then you must make your sum of the weights constraint, S, higher. You must keep on making it higher until NIC becomes one of the components of the geometric optimal portfolio. Recall that if there are only two components in a portfolio, if the correlation coefficient between them is −1, and if both have positive mathematical expectation, you will be required to finance an infinite number of contracts. This is so because such a portfolio would never have a losing period. Now, the lower the correlation coefficients are between the components in the portfolio, the higher the percentage required to be invested in those components is going to be. The difference between the percentages invested and the sum of the weights constraint, S, must be filled by NIC. If NIC doesn't show up in the percentage allocations for the geometric optimal portfolio, it means that the portfolio is running into a constraint at S and is therefore not the unconstrained geometric optimal. Since you are not going to be actually investing in NIC, it doesn't matter how high a percentage it commands, as long as it is listed as part of the geometric optimal portfolio.

## HOW OPTIMAL *f* FITS IN

In Chapter 7 we saw that we must determine an expected return (as a percentage) and an expected variance in returns for each component in a portfolio. Generally, the expected returns (and the variances) are determined from the current price of the stock. An optimal percentage (weighting) is then determined for each component. The equity of the account is then multiplied by a components weighting to determine the number of dollars to allocate to that component, and this dollar allocation is then divided by the current price per share to determine how many shares to have on.

That generally is how portfolio strategies are currently practiced. But it is *not* optimal. Rather than determining the expected return and variance in expected return from the current price of the component, the expected return and variance in returns should be determined from the optimal *f*, in dollars, for the component. In other words, as input you should use the arithmetic average HPR and the variance in the HPRs. Here, the HPRs used should be not of trades, but of a fixed time length such as days, weeks, months, quarters, or years—as we did in Equation (4.14):

$$\text{Daily HPR} = (A/B) + 1$$

where: A = Dollars made or lost that day.
B = Optimal *f* in dollars.

We need not necessarily use days. We can use any time length we like so long as it is the same time length for all components in the portfolio (and the same time length is used for determining the correlation coefficients between these HPRs of the different components). Say the market system with an optimal *f* of \$2,000 made \$100 on a given day. Then the HPR for that market system for that day is 1.05.

If you are figuring your optimal *f* based on equalized data, you must use the following equation in order to obtain your daily HPRs:

$$\text{Daily HPR} = D\$/f\$ + 1$$

where: D\$ = The dollar gain or loss on 1 unit from the previous day. This is equal to
(Tonight's Close − Last Night's Close) ∗ Dollars per Point
*f*\$ = The current optimal *f* in dollars. Here, however, the current price variable is last night's close.

In other words, once you have determined the optimal *f* in dollars for one unit of a component, you then take the daily equity changes on a one-unit basis and convert them to HPRs just mentioned—or, if you are using equalized data, you can use the equation just mentioned. When you are combining market systems in a portfolio, all the market systems should be the same in terms of whether their data, and hence their optimal *f*s and by-products, has been equalized or not.

Then we take the arithmetic average of the HPRs. Subtracting 1 from the arithmetic average will give us the expected return to use for that component. Taking the variance of the daily (weekly, monthly, etc.) HPRs will give the variance input into the matrix. Lastly, we determine the correlation coefficients between the daily HPRs for each pair of market systems under consideration.

Now here is the critical point. *Portfolios whose parameters (expected returns, variance in expected returns, and correlation coefficients of the expected returns) are selected based on the current price of the component will not yield truly optimal portfolios. To discern the truly optimal portfolio you must derive the input parameters based on trading one unit at the optimal* f *for each component. You cannot be more at the peak of the* *optimal* f *curve than optimal* f *itself. To base the parameters on the current market price of the component is to base your parameters arbitrarily—and, as a consequence, not necessarily optimally.*

Now let's return to the question of how you can invest more than 100% in a certain component. One of the basic premises here is that weight and quantity are not the same thing. The weighting that you derive from solving for a geometric optimal portfolio must be reflected back into the optimal *f*s of the portfolio's components. The way to do this is to divide the optimal *f*s for each component by its corresponding weight. Assume we have the following optimal *f*s (in dollars):

| | |
|---|---|
| Toxico | $2,500 |
| Incubeast | $4,750 |
| LA Garb | $5,000 |

(Note that, if you are equalizing your data, and hence obtaining an equalized optimal *f* and by-products, then your optimal *f*s in dollars will change each day based upon the previous day's closing price and Equation [2.11].)

We now divide these *f*s by their respective weightings:

| | | |
|---|---|---|
| Toxico | $2,500/1.025982 = | $2,436.69 |
| Incubeast | $4,750/.4900558 = | $9,692.77 |
| LA Garb | $5,000/.4024979 = | $12,422.43 |

*Thus, by trading in these new "adjusted"* f *values, we will be at the geometric optimal portfolio in the classical portfolio sense.* In other words, suppose Toxico represents a certain market system. By trading one contract under this market system for every $2,436.69 in equity (and doing the same with the other market systems at their new adjusted *f* values) we will be at the geometric optimal unconstrained portfolio. Likewise, if Toxico is a stock, and we regard 100 shares as "one contract," we will trade 100 shares of Toxico for every $2,436.69 in account equity. For the moment, disregard margin completely. Later in the text we will address the potential problem of margin requirements.

"Wait a minute," you protest. "If you take an optimal portfolio and change it by using optimal *f*, you have to prove that it is still optimal. But if you treat the new values as a different portfolio, it must fall somewhere else on the return coordinate, not necessarily on the efficient frontier. In other words, if you keep reevaluating *f*, you cannot stay optimal, can you?"

We are not changing the *f* values. That is, our *f* values (the number of units put on for so many dollars in equity) are still the same. We are simply performing a shortcut through the calculations, which makes it appear as though we are "adjusting" our *f* values. We derive our optimal portfolios based on the expected returns and variance in returns of trading one unit of each of the components, as well as on the correlation coefficients. We thus derive optimal weights (optimal percentages of the account to trade each component with). Thus, if a market system had an optimal *f* of $2,000, and an optimal portfolio weight of .5, we would trade 50% of our account at the full optimal *f* level of $2,000 for this market system. This is exactly the same as if we said we will trade 100% of our account at the optimal *f* divided by the optimal weighting ($2,000/.5) of $4000. In other words, we are going to trade the optimal *f* of $2,000 per unit on 50% of our equity, which in turn is exactly the same as saying we are going to trade the adjusted *f* of $4,000 on 100% of our equity.

The AHPRs and SDs that you input into the matrix are determined from the optimal *f* values in dollars. If you are doing this on stocks, you can compute your values for AHPR, SD, and optimal *f* on a one-share or a 100-share basis (or any other basis you like). You dictate the size of one unit.

In a nonleveraged situation, such as a portfolio of stocks that are not on margin, weighting and quantity are synonymous. Yet in a leveraged situation, such as a portfolio of futures market systems, weighting and quantity are different indeed. You can now see the idea that optimal quantities are what we seek to know, and that this is a *function* of optimal weightings.

When we figure the correlation coefficients on the HPRs of two market systems, both with a positive arithmetic mathematical expectation, we find a slight tendency toward positive correlation. This is because the equity curves (the cumulative running sum of daily equity changes) both tend to rise up and to the right. This can be bothersome to some people. One solution is to determine a least squares regression line to each equity curve and then take the difference at each point in time on the equity curve and its regression line. Next, convert this now detrended equity curve back to simple daily equity changes (noncumulative, i.e., the daily change in the detrended equity curve). Lastly, you figure your correlations on this processed data.

This technique is valid so long as you are using the correlations of daily equity changes and not prices. If you use prices, you may do yourself more harm than good. Very often, prices and daily equity changes are linked. An example would be a long-term moving average crossover system. This detrending technique must always be used with caution. Also, the daily AHPR and standard deviation in HPRs must always be figured off of non-detrended data.

A final problem that happens when you detrend your data occurs with systems that trade infrequently. Imagine two day-trading systems that give one trade per week, both on different days. The correlation coefficient between them may be only slightly positive. Yet when we detrend their data, we get very high positive correlation. This mistakenly happens because their regression lines are rising a little each day. Yet on most days the equity change is zero. Therefore, the difference is negative. The preponderance of slightly negative days with both market systems, then, mistakenly results in high positive correlation.

## COMPLETING THE LOOP

One thing you will readily notice about unconstrained portfolios (portfolios for which the sum of the weights is greater than 1 and NIC shows up as a market system in the portfolio) is that the portfolio is exactly the same for any given level of E—the only difference being the degree of leverage. (This is *not* true for portfolios lying along the efficient frontier(s) when the sum of the weights is constrained). In other words, the ratios of the weightings of the different market systems to each other are always the same for any point along the unconstrained efficient frontiers (AHPR or GHPR).

For example, the ratios of the different weightings between the different market systems in the geometric optimal portfolio can be calculated. The ratio of Toxico to Incubeast is 102.5982% divided by 49.00558%, which equals 2.0936. We can thus determine the ratios of all the components in this portfolio to one another:

Toxico/Incubeast = 2.0936
Toxico/LA Garb = 2.5490
Incubeast/LA Garb = 1.2175

Now, we can go back to the unconstrained portfolio and solve for different values for E. What follows are the weightings for the components of the unconstrained portfolios that have the lowest variances for the given values of E. You will notice that the ratios of the weightings of the components are exactly the same:

| | E = .1 | E = .3 |
|---|---|---|
| Toxico | .4175733 | 1.252726 |
| Incubeast | .1994545 | .5983566 |
| LA Garb | .1638171 | .49145 |

Thus, we can state that *the unconstrained efficient frontiers are the same portfolio at different levels of leverage.* This portfolio, the one that gets levered up and down with E when the sum of the weights constraint is lifted, is the portfolio that has a value of zero for the second Lagrangian multiplier when the sum of the weights equals 1.

Therefore, we can readily determine what our unconstrained geometric optimal portfolio will be. First, we find the portfolio that has a value of zero for the second Lagrangian multiplier when the sum of the weights is constrained to 1.00. One way to find this is through iteration. The resultant portfolio will be that portfolio which gets levered up (or down) to satisfy any given E in the unconstrained portfolio. That value for E which satisfies any of Equations (8.05a) through (8.05d) will be the value for E that yields the unconstrained geometric optimal portfolio.

Another equation that we can use to solve for which portfolio along the unconstrained AHPR efficient frontier is geometric optimal is to use the first Lagrangian multiplier that results in determining a portfolio along any particular point on the unconstrained AHPR efficient frontier. Recall from the previous chapter that one of the by-products in determining the composition of a portfolio by the method of row-equivalent matrices is the first Lagrangian multiplier. The first Lagrangian multiplier represents the instantaneous rate of change in variance with respect to expected return, sign reversed. A first Lagrangian multiplier equal to −2 means that at that point the variance was changing at that rate (−2) opposite the expected return, sign reversed. This would result in a portfolio that was geometric optimal.

$$L1 = -2 \tag{8.12}$$

where: L1 = The first Lagrangian multiplier of a given portfolio along the unconstrained AHPR efficient frontier.[2]

Now it gets interesting as we tie these concepts together. *The portfolio that gets levered up and down the unconstrained efficient frontiers (arithmetic or geometric) is the portfolio tangent to the CML line emanating from an RFR of 0 when the sum of the weights is constrained to 1.00 and NIC is not employed.*

Therefore, we can also find the unconstrained geometric optimal portfolio by first finding the tangent portfolio to an RFR equal to 0 where the sum of the weights is constrained to 1.00, then levering this portfolio up to the point where it is the geometric optimal. But how can we determine how much to lever this constrained portfolio up to make it the equivalent of the unconstrained geometric optimal portfolio?

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[2]Thus, we can state that the geometric optimal portfolio is that portfolio which, when the sum of the weights is constrained to 1, has a second Lagrangian multiplier equal to 0, and when unconstrained has a first Lagrangian multiplier of −2. Such a portfolio will also have a second Lagrangian multiplier equal to 0 when unconstrained.

Recall that the tangent portfolio is found by taking the portfolio along the constrained efficient frontier (arithmetic or geometric) that has the highest Sharpe ratio, which is Equation (8.01). Now we lever this portfolio up, and we multiply the weights of each of its components by a variable named $q$, which can be approximated by:

$$q = (\mathrm{E} - \mathrm{RFR})/\mathrm{V} \tag{8.13}$$

where: E = The expected return (arithmetic) of the tangent portfolio.
RFR = The risk-free rate at which we assume you can borrow or loan.
V = The variance in the tangent portfolio.

Equation (8.13) actually is a very close approximation for the actual optimal $q$.

An example may help illustrate the role of optimal $q$. Recall that our unconstrained geometric optimal portfolio is as follows:

| Component | Weight |
|---|---|
| Toxico | 1.025955 |
| Incubeast | .4900436 |
| LA Garb | .4024874 |

This portfolio, we found, has an AHPR of 1.245694 and variance of .2456941. Throughout the remainder of this discussion we will assume for simplicity's sake an RFR of zero. (Incidentally, the Sharpe ratio of this portfolio, (AHPR − (1 + RFR))/SD, is .49568.)

Now, if we were to input the same returns, variances, and correlation coefficients of these components into the matrix and solve for which portfolio was tangent to an RFR of zero when the sum of the weights is constrained to 1.00 and we do not include NIC, we would obtain the following portfolio:

| Component | Weight |
|---|---|
| Toxico | .5344908 |
| Incubeast | .2552975 |
| LA Garb | .2102117 |

This particular portfolio has an AHPR of 1.128, a variance of .066683, and a Sharpe ratio of .49568. It is interesting to note that *the Sharpe ratio* *of the tangent portfolio, a portfolio for which the sum of the weights is constrained to 1.00 and we do not include NIC, is exactly the same as the Sharpe ratio for our unconstrained geometric optimal portfolio.*

Subtracting 1 from our AHPRs gives us the arithmetic average return of the portfolio. Doing so we notice that in order to obtain the same return for the constrained tangent portfolio as for the unconstrained geometric optimal portfolio, we must multiply the former by 1.9195.

$$.245694/.128 = 1.9195$$

Now if we multiply each of the weights of the constrained tangent portfolio, the portfolio we obtain is virtually identical to the unconstrained geometric optimal portfolio:

| Component | Weight | * 1.9195 = Weight |
|---|---|---|
| Toxico | .5344908 | 1.025955 |
| Incubeast | .2552975 | .4900436 |
| LA Garb | .2102117 | .4035013 |

The factor 1.9195 was arrived at by dividing the return on the unconstrained geometric optimal portfolio by the return on the constrained tangent portfolio. Usually, though, we will want to find the unconstrained geometric optimal portfolio knowing only the constrained tangent portfolio. This is where optimal $q$ comes in.[3] If we assume an RFR of zero, we can determine the optimal $q$ on our constrained tangent portfolio as:

$$\begin{aligned} q &= (\mathrm{E} - \mathrm{RFR})/\mathrm{V} \\ &= (.128 - 0)/.066683 \\ &= 1.919529715 \end{aligned}$$

A few notes on the RFR. To begin with, we should always assume an RFR of zero when we are dealing with futures contracts. Since we are not actually borrowing or lending funds to lever our portfolio up or down, there is effectively an RFR of zero. With stocks, however, it is a different story. The RFR you use should be determined with this fact in mind. Quite possibly, the leverage you employ does not require you to use an RFR other than zero.

You will often be using AHPRs and variances for portfolios that were determined by using daily HPRs of the components. In such cases, you

[3]Latane, Henry, and Donald Tuttle, "Criteria for Portfolio Building," *Journal of Finance* 22, September 1967, pp. 362–363.

must adjust the RFR from an annual rate to a daily one. This is quite easy to accomplish. First, you must be certain that this annual rate is what is called the *effective annual interest rate.* Interest rates are typically stated as annual percentages, but frequently these annual percentages are what is referred to as the *nominal annual interest rate.* When interest is compounded semiannually, quarterly, monthly, and so on, the interest earned during a year is greater than if compounded annually (the nominal rate is based on compounding annually). When interest is compounded more frequently than annually, an effective annual interest rate can be determined from the nominal interest rate. It is the effective annual interest rate that concerns us and that we will use in our calculations. To convert the nominal rate to an effective rate we can use:

$$\mathrm{E} = (1 + \mathrm{R}/\mathrm{M})^{\mathrm{M}} - 1 \tag{8.14}$$

where: E = The effective annual interest rate.
R = The nominal annual interest rate.
M = The number of compounding periods per year.

Assume that the nominal annual interest rate is 9%, and suppose that it is compounded monthly. Therefore, the corresponding effective annual interest rate is:

$$\begin{aligned}
\mathrm{E} &= (1 + .09/12)^{12} - 1 \\
&= (1 + .0075)^{12} - 1 \\
&= 1.0075^{12} - 1 \\
&= 1.093806898 - 1 \\
&= .093806898
\end{aligned}$$

Therefore, our effective annual interest rate is a little over 9.38%. Now if we figure our HPRs on the basis of weekdays, we can state that there are $365.2425/7 * 5 = 260.8875$ weekdays, on average, in a year. Dividing .093806898 by 260.8875 gives us a daily RFR of .0003595683887.

If we determine that we are actually paying interest to lever our portfolio up, and we want to determine from the constrained tangent portfolio what the unconstrained geometric optimal portfolio is, we simply input the value for the RFR into the Sharpe ratio, Equation (8.01), and the optimal $q$, Equation (8.13).

Now to close the loop. Suppose you determine that the RFR for your portfolio is not zero, and you want to find the geometric optimal portfolio without first having to find the constrained portfolio tangent to your applicable RFR. Can you just go straight to the matrix, set the sum of the weights to some arbitrarily high number, include NIC, and find the unconstrained geometric optimal portfolio when the RFR is greater than zero? Yes, this is easily accomplished by subtracting the RFR from the expected returns of each of the components, but not from NIC (i.e., the expected return for NIC remains at zero, or an arithmetic average HPR of 1.00). Now, solving the matrix will yield the unconstrained geometric optimal portfolio when the RFR is greater than zero.

Since the unconstrained efficient frontier is the same portfolio at different levels of leverage, you cannot put a CML line on the unconstrained efficient frontier. You can only put CML lines on the AHPR or GHPR efficient frontiers if they are constrained (i.e., if the sum of the weights equals 1). It is not logical to put CML lines on the AHPR or GHPR unconstrained efficient frontiers.