1

Determinants, First Minors, and Cofactors

1.1 Grassmann Exterior Algebra

Let V be a finite-dimensional vector space over a field F. Then, it is known that for each non-negative integer m, it is possible to construct a vector space $\Lambda^m V$. In particular, $\Lambda^0 V = F$, $\Lambda V = V$, and for $m \geq 2$, each vector in $\Lambda^m V$ is a linear combination, with coefficients in F, of the products of m vectors from V.

If $\mathbf{x}_i \in V$, $1 \leq i \leq m$, we shall denote their vector product by $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$. Each such vector product satisfies the following identities:

i. $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{r-1}(a\mathbf{x} + b\mathbf{y})\mathbf{x}_{r+1} \cdots \mathbf{x}_n = a\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{r-1}\mathbf{x}\mathbf{x}_{r+1} \cdots \mathbf{x}_n$ $+b\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{r-1}\mathbf{y} \cdots \mathbf{x}_{r+1} \cdots \mathbf{x}_n$, where $a, b \in F$ and $\mathbf{x}, \mathbf{y} \in V$.

ii. If any two of the $\mathbf{x}$'s in the product $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_n$ are interchanged, then the product changes sign, which implies that the product is zero if two or more of the $\mathbf{x}$'s are equal.

1.2 Determinants

Let $\dim V = n$ and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a set of base vectors for V. Then, if $\mathbf{x}_i \in V$, $1 \leq i \leq n$, we can write

$$\mathbf{x}_i = \sum_{k=1}^{n} a_{ik}\mathbf{e}_k, \quad a_{ik} \in F. \tag{1.2.1}$$

It follows from (i) and (ii) that

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = \sum_{k_1=1}^{n}\cdots\sum_{k_n=1}^{n} a_{1k_1}a_{2k_2}\cdots a_{nk_n}\mathbf{e}_{k_1}\mathbf{e}_{k_2}\cdots\mathbf{e}_{k_n}. \tag{1.2.2}$$

When two or more of the k's are equal, $\mathbf{e}_{k_1}\mathbf{e}_{k_2}\cdots\mathbf{e}_{k_n} = 0$. When the k's are distinct, the product $\mathbf{e}_{k_1}\mathbf{e}_{k_2}\cdots\mathbf{e}_{k_n}$ can be transformed into $\pm\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$ by interchanging the dummy variables k_r in a suitable manner. The sign of each term is unique and is given by the formula

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = \left[\sum^{(n!\text{ terms})} \sigma_n a_{1k_1}a_{2k_2}\cdots a_{nk_n}\right]\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n, \tag{1.2.3}$$

where

$$\sigma_n = \operatorname{sgn}\left\{\begin{matrix} 1 & 2 & 3 & 4 & \cdots & (n-1) & n \\ k_1 & k_2 & k_3 & k_4 & \cdots & k_{n-1} & k_n \end{matrix}\right\} \tag{1.2.4}$$

and where the sum extends over all $n!$ permutations of the numbers k_r, $1 \le r \le n$. Notes on permutation symbols and their signs are given in Appendix A.2.

The coefficient of $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$ in (1.2.3) contains all n^2 elements a_{ij}, $1 \le i, j \le n$, which can be displayed in a square array. The coefficient is called a determinant of order n.

Definition.

$$A_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_n = \sum^{(n!\text{ terms})} \sigma_n a_{1k_1}a_{2k_2}\cdots a_{nk_n}. \tag{1.2.5}$$

The array can be abbreviated to $|a_{ij}|_n$. The corresponding matrix is denoted by $[a_{ij}]_n$. Equation (1.2.3) now becomes

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = |a_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \tag{1.2.6}$$

Exercise. If $\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ is a fixed permutation, show that

$$\begin{aligned} A_n = |a_{ij}|_n &= \sum_{k_1,\ldots,k_n}^{n!\text{ terms}} \operatorname{sgn}\begin{pmatrix} j_1 & j_2 & \cdots & j_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} a_{j_1k_1}a_{j_2k_2}\cdots a_{j_nk_n} \\ &= \sum_{k_1,\ldots,k_n}^{n!\text{ terms}} \operatorname{sgn}\begin{pmatrix} j_1 & j_2 & \cdots & j_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} a_{k_1j_1}a_{k_2j_2}\cdots a_{k_nj_n}. \end{aligned}$$

1.3 First Minors and Cofactors

Referring to (1.2.1), put

$$\begin{aligned}\mathbf{y}_i &= \mathbf{x}_i - a_{ij}\mathbf{e}_j \\ &= (a_{i1}\mathbf{e}_1 + \cdots + a_{i,j-1}\mathbf{e}_{j-1}) + (a_{i,j+1}\mathbf{e}_{j+1} + \cdots + a_{in}\mathbf{e}_n) \quad (1.3.1) \\ &= \sum_{k=1}^{n-1} a'_{ik}\mathbf{e}'_k, \quad (1.3.2)\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}'_k &= \mathbf{e}_k \qquad 1 \le k \le j-1 \\ &= \mathbf{e}_{k+1}, \quad j \le k \le n-1 \quad (1.3.3) \\ a'_{ik} &= a_{ik} \qquad 1 \le k \le j-1 \\ &= a_{i,k+1}, \quad j \le k \le n-1. \quad (1.3.4)\end{aligned}$$

Note that each a'_{ik} is a function of j.

It follows from Identity (ii) that

$$\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_n = 0 \quad (1.3.5)$$

since each $\mathbf{y}_r$ is a linear combination of $(n-1)$ vectors $\mathbf{e}_k$ so that each of the $(n-1)^n$ terms in the expansion of the product on the left contains at least two identical $\mathbf{e}$'s. Referring to (1.3.1) and Identities (i) and (ii),

$$\begin{aligned}&\mathbf{x}_1\cdots\mathbf{x}_{i-1}\mathbf{e}_j\mathbf{x}_{i+1}\cdots\mathbf{x}_n \\ &\quad= (\mathbf{y}_1 + a_{1j}\mathbf{e}_j)(\mathbf{y}_2 + a_{2j}\mathbf{e}_j)\cdots(\mathbf{y}_{i-1} + a_{i-1,j}\mathbf{e}_j) \\ &\qquad \mathbf{e}_j(\mathbf{y}_{i+1} + a_{i+1,j}\mathbf{e}_j)\cdots(\mathbf{y}_n + a_{nj}\mathbf{e}_j) \\ &\quad= \mathbf{y}_1\cdots\mathbf{y}_{i-1}\mathbf{e}_j\mathbf{y}_{i+1}\cdots\mathbf{y}_n \quad (1.3.6) \\ &\quad= (-1)^{n-i}(\mathbf{y}_1\cdots\mathbf{y}_{i-1}\mathbf{y}_{i+1}\cdots\mathbf{y}_n)\mathbf{e}_j. \quad (1.3.7)\end{aligned}$$

From (1.3.2) it follows that

$$\mathbf{y}_1\cdots\mathbf{y}_{i-1}\mathbf{y}_{i+1}\cdots\mathbf{y}_n = M_{ij}(\mathbf{e}'_1\mathbf{e}'_2\cdots\mathbf{e}'_{n-1}), \quad (1.3.8)$$

where

$$M_{ij} = \sum \sigma_{n-1} a'_{1k_1} a'_{2k_2} \cdots a'_{i-1,k_{i-1}} a'_{i+1,k_{i+1}} \cdots a'_{n-1,k_{n-1}} \quad (1.3.9)$$

and where the sum extends over the $(n-1)!$ permutations of the numbers $1, 2, \ldots, (n-1)$. Comparing M_{ij} with A_n, it is seen that M_{ij} is the determinant of order $(n-1)$ which is obtained from A_n by deleting row i and column j, that is, the row and column which contain the element a_{ij}. M_{ij} is therefore associated with a_{ij} and is known as a first minor of A_n.

Hence, referring to (1.3.3),

$$\begin{aligned}&\mathbf{x}_1\cdots\mathbf{x}_{i-1}\mathbf{e}_j\mathbf{x}_{i+1}\cdots\mathbf{x}_n \\ &\quad= (-1)^{n-i}M_{ij}(\mathbf{e}'_1\mathbf{e}'_2\cdots\mathbf{e}'_{n-1})\mathbf{e}_j\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n-i} M_{ij} (\mathbf{e}'_1 \cdots \mathbf{e}'_{j-1})(\mathbf{e}'_j \cdots \mathbf{e}'_{n-1})\mathbf{e}_j \\
&= (-1)^{n-i} M_{ij} (\mathbf{e}_1 \cdots \mathbf{e}_{j-1})(\mathbf{e}_{j+1} \cdots \mathbf{e}_n)\mathbf{e}_j \\
&= (-1)^{i+j} M_{ij} (\mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n). \qquad (1.3.10)
\end{aligned}
$$

Now, $\mathbf{e}_j$ can be regarded as a particular case of $\mathbf{x}_i$ as defined in (1.2.1):

$$\mathbf{e}_j = \sum_{k=1}^{n} a_{ik} \mathbf{e}_k,$$

where

$$a_{ik} = \delta_{jk}.$$

Hence, replacing $\mathbf{x}_i$ by $\mathbf{e}_j$ in (1.2.3),

$$\mathbf{x}_1 \cdots \mathbf{x}_{i-1} \mathbf{e}_j \mathbf{x}_{i+1} \cdots \mathbf{x}_n = A_{ij} (\mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n), \qquad (1.3.11)$$

where

$$A_{ij} = \sum \sigma_n a_{1k_1} a_{2k_2} \cdots a_{ik_i} \cdots a_{nk_n},$$

where

$$
\begin{aligned}
a_{ik_i} &= 0 \qquad k_i \neq j \\
&= 1 \qquad k_i = j.
\end{aligned}
$$

Referring to the definition of a determinant in (1.2.4), it is seen that A_{ij} is the determinant obtained from $|a_{ij}|_n$ by replacing row i by the row

$$[0 \ldots 0\ 1\ 0 \ldots 0],$$

where the element 1 is in column j. A_{ij} is known as the cofactor of the element a_{ij} in A_n.

Comparing (1.3.10) and (1.3.11),

$$A_{ij} = (-1)^{i+j} M_{ij}. \qquad (1.3.12)$$

Minors and cofactors should be written $M_{ij}^{(n)}$ and $A_{ij}^{(n)}$ but the parameter n can be omitted where there is no risk of confusion.

Returning to (1.2.1) and applying (1.3.11),

$$
\begin{aligned}
\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n &= \mathbf{x}_1 \cdots \mathbf{x}_{i-1} \left(\sum_{k=1}^{n} a_{ik} \mathbf{e}_k \right) \mathbf{x}_{i+1} \cdots \mathbf{x}_n \\
&= \sum_{k=1}^{n} a_{ik} (\mathbf{x}_1 \cdots \mathbf{x}_{i-1} \mathbf{e}_k \mathbf{x}_{i+1} \cdots \mathbf{x}_n) \\
&= \left[\sum_{k=1}^{n} a_{ik} A_{ik} \right] \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n. \qquad (1.3.13)
\end{aligned}
$$

Comparing this result with (1.2.5),

$$|a_{ij}|_n = \sum_{k=1}^{n} a_{ik} A_{ik} \tag{1.3.14}$$

which is the expansion of $|a_{ij}|_n$ by elements from row i and their cofactors.

From (1.3.1) and noting (1.3.5),

$$\begin{aligned}
\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n &= (\mathbf{y}_1 + a_{1j}\mathbf{e}_j)(\mathbf{y}_2 + a_{2j}\mathbf{e}_j)\cdots(\mathbf{y}_n + a_{nj}\mathbf{e}_j)\\
&= a_{1j}\mathbf{e}_j\mathbf{y}_2\mathbf{y}_3\cdots\mathbf{y}_n + a_{2j}\mathbf{y}_1\mathbf{e}_j\mathbf{y}_3\cdots\mathbf{y}_n\\
&\quad + \cdots + a_{nj}\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_{n-1}\mathbf{e}_j\\
&= (a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj})\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n\\
&= \left[\sum_{k=1}^{n} a_{kj}A_{kj}\right]\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \qquad (1.3.15)
\end{aligned}$$

Comparing this relation with (1.2.5),

$$|a_{ij}|_n = \sum_{k=1}^{n} a_{kj} A_{kj} \tag{1.3.16}$$

which is the expansion of $|a_{ij}|_n$ by elements from column j and their cofactors.

1.4 The Product of Two Determinants — 1

Put

$$\begin{aligned}
\mathbf{x}_i &= \sum_{k=1}^{n} a_{ik}\mathbf{y}_k,\\
\mathbf{y}_k &= \sum_{j=1}^{n} b_{kj}\mathbf{e}_j.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n &= |a_{ij}|_n\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_n,\\
\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_n &= |b_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n.
\end{aligned}$$

Hence,

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = |a_{ij}|_n|b_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \tag{1.4.1}$$

But,

$$\mathbf{x}_i = \sum_{k=1}^{n} a_{ik}\sum_{j=1}^{n} b_{kj}\mathbf{e}_j$$

$$= \sum_{j=1}^{n} c_{ij}\mathbf{e}_j,$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{1.4.2}$$

Hence,

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = |c_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \tag{1.4.3}$$

Comparing (1.4.1) and (1.4.3),

$$|a_{ij}|_n|b_{ij}|_n = |c_{ij}|_n. \tag{1.4.4}$$

Another proof of (1.4.4) is given in Section 3.3.5 by applying the Laplace expansion in reverse.

The Laplace expansion formula is proved by both a Grassmann and a classical method in Chapter 3 after the definitions of second and higher rejector and retainor minors and cofactors.