LINMA2380 Matrix Computation

Homework 4
Group 3

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1 Exercise A: Power Method

1.1 A1

$$x^1 = Ax^0$$

Multiplying both sides by $e^\top$:

$$e^\top x^1 = e^\top Ax^0$$

Since $x^0 \in \Delta$ and A column stochastic ($e^\top A = e^\top$):

$$e^\top x^1 = e^\top x^0 = 1.$$

Now, assume this holds up to $k-1$:

$$x^k = Ax^{k-1}.$$

Multiplying both sides by $e^\top$:

$$e^\top x^k = e^\top Ax^{k-1}.$$

By the induction hypothesis (this holds for $k-1$):

$$e^\top x^k = e^\top x^{k-1} = 1.$$

Thus, the property also holds for k.

1.2 A2

Under Assumption 1, there exists one strictly positive row, thus a minimum for this row cannot be zero. Therefore, $r \neq 0$. Since $r > 0$, we have $e^r > 0$, which leads to $x_0 > 0$ from $x^0 = \frac{1}{e^\top r} r$. Now multiply the definition of x^0 by $e^\top$ in both sides.

$$e^\top x^0 = e^\top r \frac{1}{e^\top r} = 1$$

Therefore, $x^0 \in \Delta$.
We have $r \neq 0$, so $\alpha = e^\top r \neq 0$. To maximize α, we need to maximize r. This happens when all the probabilities of transition from one website to another (also to itself) are the same $a_{ij} = \frac{1}{n}, \forall i, j \in [1, n]$. Therefore, we have max α as $\alpha = e^\top r_{max} = 1$, where $r_{max} = [\frac{1}{n}, \ldots, \frac{1}{n}]$.

1.3 A3

Multiplying boths sides with $e^\top$, and using $\alpha x^0 = \alpha \frac{1}{\alpha} r = r$

$$e^\top A = e^\top (1-\alpha).\bar{A} + e^\top r e^\top.$$

Replacing $e^\top r = \alpha$, $e^\top \bar{A} = e^\top$, and $e^\top A = e^\top$

$$e^\top = e^\top - \alpha e^\top + \alpha e^\top = e^\top$$

Thus, we can write A as given.

1.4 A4

Rewriting A as in (A3):

$$||(1-\alpha)\bar{A}h + \alpha x^0 e^\top h||_1 \leq (1-\alpha)||h||_1$$

Using $e^\top h = 0$, and taking out the scalar $(1-\alpha)$:

$$||\bar{A}h||_1 \leq ||h||_1$$

When $h = 0$ we have the equality.
When $h \neq 0$, h must have negative and positive values such that $e^\top h = 0$ and $\bar{A}h$ corresponds to a weighted sum of h's entries which cannot increase its 1-norm with $\bar{A} \geq 0$ and $\bar{A}$ being column stochastic. Therefore, we get the inequality.

1.5 A5

Assume two solutions x^*, x^{**} exist such that

$$Ax^* = x^*, Ax^{**} = x^{**}$$

Setting $h = x^* - x^{**}$, we have:

$$Ah = A(x^* - x^{**}) = Ax^* - Ax^{**} = x^* - x^{**} = h$$

Which tells that h is an eigenvector corresponding to the eigenvalue 1.
However, we have:

$$e^\top h = e^\top (x^* - x^{**}) = 1 - 1 = 0$$

Since $x^*, x^{**} \in \Delta$ from (A1). By (A4) we have the inequality and replacing $Ah = h$, as shown above h being an eigenvector for eigenvalue 1, we have:

$$||h||_1 \leq (1-\alpha)||h||_1$$

Since $\alpha \in (0, 1]$ from (A2), the above inequality holds only with $\vec{h} = \vec{0}$. Therefore, $x^* = x^{**}$, the solution is unique.

1.6 A6

Starting with the power method:

$$x^k = Ax^{k-1}$$

Subtract x^* from both sides ($Ax^* = x^*$):

$$x^k - x^* = A(x^{k-1} - x^*)$$

Set $h = x^{k-1} - x^*$, and observe $e^\top h = e^\top x^{k-1} - e^\top x^* = 0$, since $x^k, x^* \in \Delta$.
From (A4) we have:

$$||x^k - x^*||_1 \leq (1-\alpha)||x^{k-1} - x^*||_1$$

For k = 1 :

$$||x^1 - x^*||_1 \leq (1-\alpha)||x^0 - x^*||_1$$

For k = 2, and using the result from $k = 1$:

$$||x^2 - x^*||_1 \leq (1-\alpha)||x^1 - x^*||_1 \leq (1-\alpha)^2||x^0 - x^*||_1$$

Assuming this holds for k up to $k-1$, and setting $k = k$:

$$\begin{aligned}
||x^k - x^*||_1 &\leq (1-\alpha)||x^{k-1} - x^*||_1 \\
&\leq (1-\alpha)(1-\alpha)^{k-1}||x^0 - x^*||_1 \text{ (using the inductive hypothesis)} \\
&\leq (1-\alpha)^k||x^0 - x^*||_1
\end{aligned}$$

Therefore, it holds for $k = k$. This result shows that the power method converges to the unique solution (A5) at a geometric rate with factor $(1-\alpha)$. Since $\alpha \in (0, 1]$ (A2), the convergence rate is strictly less than 1.

2 Exercise B: PageRank

2.1 B1

$$\begin{aligned}
(A_\beta)_{ij} &= (1-\beta)a_{ij} + \beta m_{ij} \\
&= (1-\beta)a_{ij} + \frac{\beta}{n} \\
&> 0 \text{ (A column stochastic } a_{ij} \geq 0 \text{ and } \beta \in (0,1))
\end{aligned}$$

Therefore, A_β has strictly positive rows.

2.2 B2

From (B1) we have $A_\beta > 0$.
Then, the sum of the j -th column of A_β is:

$$\sum_{i=1}^{n}(A_\beta)_{ij} = \sum_{i=1}^{n}((1-\beta)a_{ij} + \beta m_{ij})$$

Using linearity of summation:

$$\sum_{i=1}^{n}(A_\beta)_{ij} = (1-\beta)\sum_{i=1}^{n}a_{ij} + \beta\sum_{i=1}^{n}m_{ij}.$$

Since A is column stochastic, $\sum_{i=1}^{n} a_{ij} = 1$, and since M is column stochastic, $\sum_{i=1}^{n} m_{ij} = 1$, we have:

$$\sum_{i=1}^{n}(A_\beta)_{ij} = (1-\beta)(1) + \beta(1) = 1.$$

Thus, each column of A_β sums to 1. Telling us that A_β is a column-stochastic matrix.

Next, set $\alpha_\beta = e^\top r_\beta$ and $r_\beta = [\min_j(A_\beta)_{1j}, \ldots, \min_j(A_\beta)_{nj}]^\top$.
$A_\beta > 0$ from (B1) so $r_\beta > 0$. From Definition 7.2, A_β is a primitive matrix since $\forall m \in \mathbb{Z}_{>0}$ $A_\beta^m > 0$.

Then, from Theorem 7.5, where A_β is primitive (thus irreducible) and column stochastic, we have $\rho(A) = 1$ and there is a unique vector $x_\beta^* > 0$ such that $A_\beta x_\beta^* = x_\beta^*$ and $e^\top x_\beta^* = 1$. So, we have a unique solution for the Google Problem for A_β.

From (A6) the Power Method applied to A_β satisfies :

$$||x_\beta^k - x_\beta^*||_1 \leq (1-\alpha_\beta)^k ||x_\beta^0 - x_\beta^*||_1$$

Which converges to a solution at a geometric rate $(1-\alpha_\beta)$. The upper bound for the convergence rate occurs when α_β is minimized. Since $A \geq 0$, adding βM ensures that $r_\beta \geq [\frac{\beta}{n}, \ldots, \frac{\beta}{n}]^\top$.

When $A = 0$ (the zero matrix), $r_\beta = [\frac{\beta}{n}, \ldots, \frac{\beta}{n}]^\top$, leading to:

$$\alpha_\beta = e^\top r_\beta = \beta.$$

The convergence rate is then bounded above by:

$$1 - \alpha_\beta \leq 1 - \beta.$$

3 Exercise C: Collaboration network

Rank	Name	Score
1	NEWMAN, M	0.00486
2	BARABASI, A	0.00478
3	SOLE, R	0.00298
4	JEONG, H	0.00274
5	YOUNG, M	0.00255