



# 3D Point Clouds

## Lecture 2

# Nearest Neighbors

主讲人 黎嘉信

Aptiv 自动驾驶  
新加坡国立大学 博士  
清华大学 本科





**1. Binary Search Tree**



**2. Kd-tree**



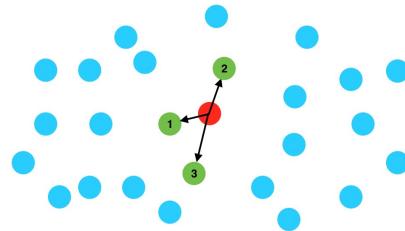
**3. Octree**



# Nearest Neighbor (NN) Problem

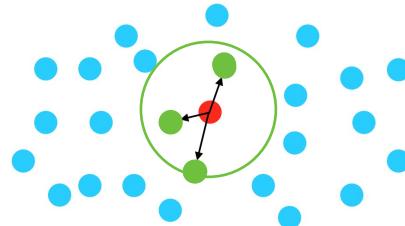
## K-NN

- Given a set of points  $S$  in a space  $M$ , a query point  $q \in M$ , find the  $k$  closest points in  $S$



## Fixed Radius-NN

- Given a set of points  $S$  in a space  $M$ , a query point  $q \in M$ , find all the points in  $S$ , s.t.,  
 $\|s - q\| < r$





# Why NN problem is important?



## It is almost everywhere

- What we have covered:
  - Surface normal estimation
  - Noise filtering
  - Sampling
- What we will cover:
  - Clustering
  - Deep learning
  - Feature detection / description
  - ... ...



## Why don't we simply call an open-source library (flann, PCL, etc.)?

- They are not efficient enough.
- They are general lib, not optimized for 2D/3D.
- Most open-source octree implementation is in-efficient, while octree is most effective for 3D.
- Few GPU based NN library is available



## Why NN is difficult for point clouds

-  For Images, a neighbor is simply  $x + \Delta x, y + \Delta y$
-  For point clouds
  - Irregular – no grid based representation
  - Curse of dimensionality
    - Non-trivial to build grids
    - Non-trivial to sort or build spatial partitions
  - Huge data throughput in real-time applications
    - Velodyne HDL-64E – 2.2 million points per second / 110,000 points at 20Hz
    - Brute-force self-NN search is  $110,000 \times 110,000 \times 0.5 = 6 \times 10^9$  comparisons @ 20Hz



# Lecture Outline



## Binary Search Tree (BST)

- Basic knowledge about trees
- 1D NN problem
- With Python codes



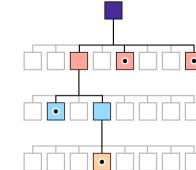
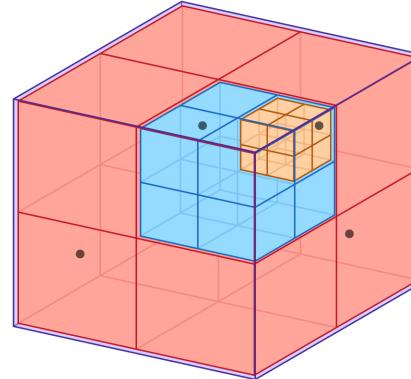
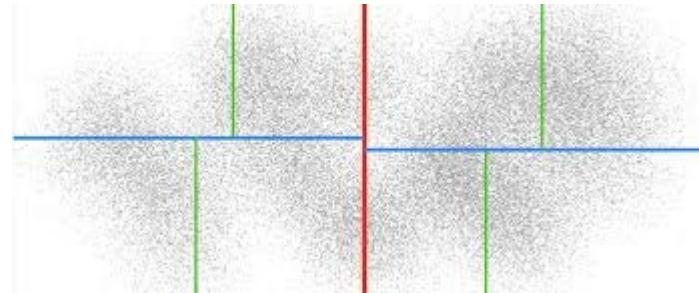
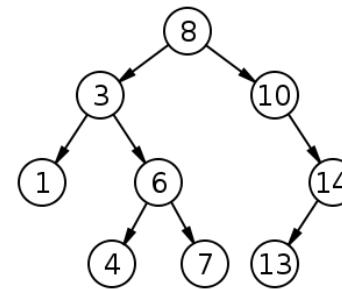
## Kd-tree

- Works for data of any dimension
- Illustrated in 2D
- With Python codes



## Octree

- Specifically designed for 3D data
- Illustrated in 2D/3D
- With Python codes





# Core Ideas Shared by BST, kd-tree, octree

## ❶ NN by space partition

- Split the space into different areas,
- Search some areas only, instead of all the data points

## ❷ Stopping criteria

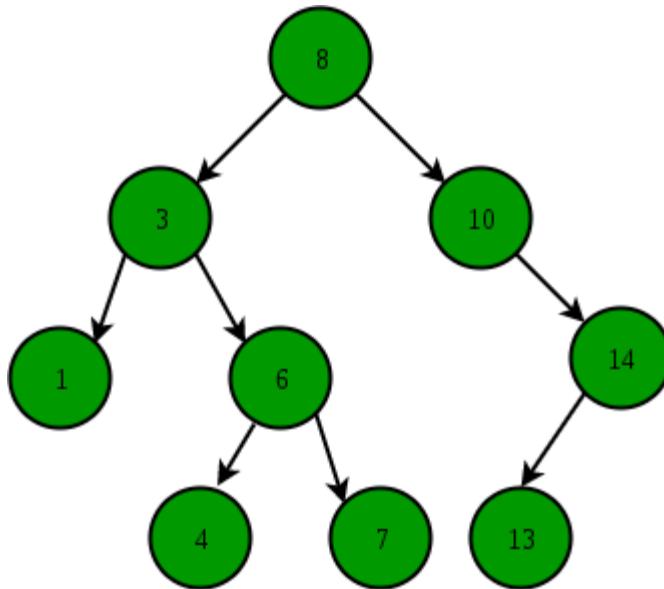
- How to skip some partitions?
  - Intersection of the “worst distance” with the partition boundaries
- How to stop the k-NN / Radius-NN search?
  - Search until the root
  - A partition completely contains the “worst distance”



# Binary Search Tree (BST)

BST is a node-based tree data structure:

1. A node' s left subtree contains nodes with keys lesser than its key
2. A node' s right subtree contains nodes with keys larger than its key
3. The left / right subtree is BST





# Binary Search Tree (BST)

From Wikipedia

## Binary search tree

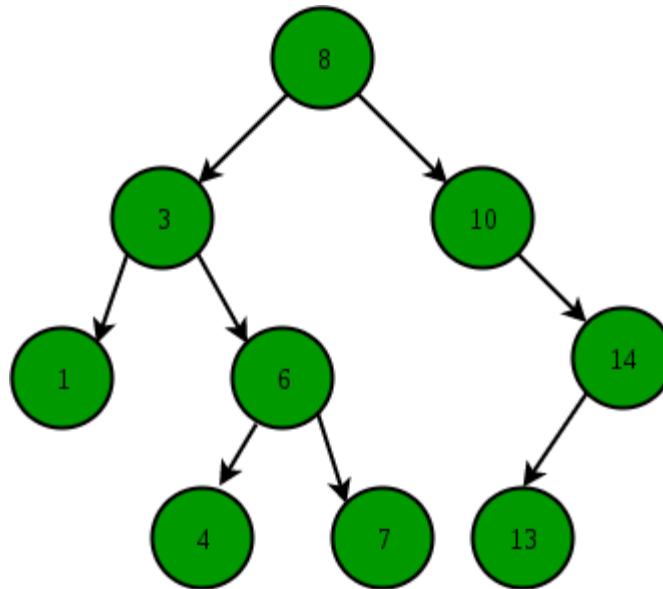
Type tree

Invented 1960

Invented P.F. Windley, A.D. Booth, A.J.T.  
by Colin, and T.N. Hibbard

### Time complexity in big O notation

Algorithm	Average	Worst case
Space	$O(n)$	$O(n)$
Search	$O(\log n)$	$O(n)$
Insert	$O(\log n)$	$O(n)$
Delete	$O(\log n)$	$O(n)$



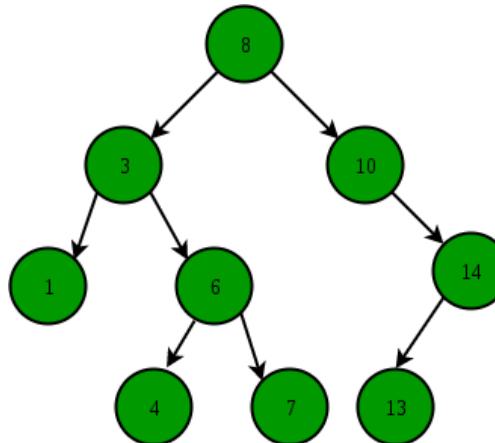


# BST – Node definition

- ➊ A node contains
  - 1. Key
  - 2. Left child
  - 3. Right child
  - 4. ... ...

- ➋ The left/right child is a Node as well

```
class Node:  
    def __init__(self, key, value=-1):  
        self.left = None  
        self.right = None  
        self.key = key  
        self.value = value
```





## BST – Construction / Insertion

- ➊ Given N 1D-points (scalar) denoted by an array
$$\{x_1, x_2, \dots, x_n\}, x_i \in \mathbb{R}$$
  
- ➋ Construct a BST that stores the points and its index in the array, e.g.  
[100, 20, 500, 10, 30, 40]

## Data generation

```
db_size = 10  
data = np.random.permutation(db_size).tolist()
```

## Recursively insert each an element

```
def insert(root, key, value=-1):  
    if root is None:  
        root = Node(key, value)  
    else:  
        if key < root.key:  
            root.left = insert(root.left, key, value)  
        elif key > root.key:  
            root.right = insert(root.right, key, value)  
        else: # don't insert if key already exist in the tree  
            pass  
    return root
```

## Insert each element

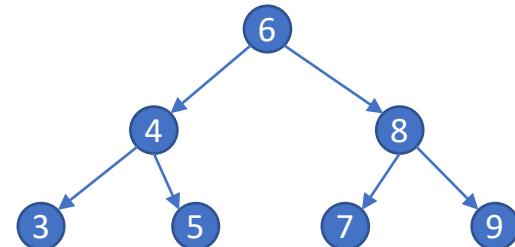
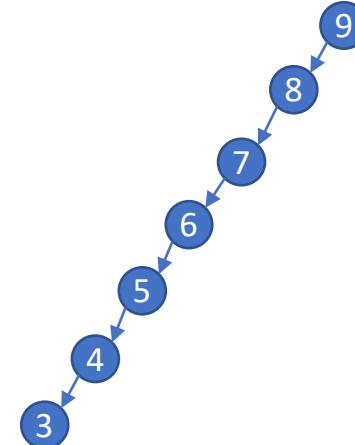
“value” in the Node is the index of a point in the array  
Useful in later NN search

```
root = None  
for i, point in enumerate(data):  
    root = insert(root, point, i)
```



# BST – Insertion Complexity

- ❖ The worst case is  $O(h)$ , where  $h$  is the height of the BST
- ❖ In the worst case,  $h$  is the number of points in BST.
  - Unbalanced tree – a chain in an extreme case
  - E.g., inserting [9, 8, 7, 6, 5, 4, 3] into an empty BST
- ❖ Tree balancing is another topic
  - Sort the array and insert as a balanced tree (select median as root)
  - AVL tree
  - Red-Black tree
  - etc.
- ❖ Best case  $h = \log_2 n$





## BST – Search

- Given a BST, and a query (key), determine *which node* equals to that query (key), if not, return *NULL*
- Can be done *recursively* or *iteratively*



# BST – Search Recursively

Search till the leaf but not found.

```
def search_recursive(root, key):
    if root is None or root.key == key:
        return root
    if key < root.key:
        return search_recursive(root.left, key)
    elif key > root.key:
        return search_recursive(root.right, key)
```

Find a match

For sure just need to look at the left

For sure just need to look at the right



## BST – Search Iteratively

- ❖ Use “current\_node” to simulate a *Stack*, so that recursion is avoided
- ❖ In any case, you can write your own *Stack* to avoid recursion, but that may be complicated sometimes.

```
def search_iterative(root, key):  
    current_node = root  
    while current_node is not None:  
        if current_node.key == key:  
            return current_node  
        elif key < current_node.key:  
            current_node = current_node.left  
        elif key > current_node.key:  
            current_node = current_node.right  
    return current_node
```

- ❖ Search recursively or iteratively complexity same as insertion worst O(h)



# BST – Search

## Recursion

### Pros:

- Easy to understand / implement
- Codes are short

### Cons:

- Hard to trace step-by-step
- $O(n)$  storage, n is number of recursion (May be optimized by compiler)

## Iteration

### Pros:

- Avoid **stack-overflow**, e.g., in embedded system / GPU
- Easier in step-by-step tracing
- $O(1)$  storage

### Cons:

- The logic is complicated



# BST – Depth First Traversal

## Hexagon icon Inorder – Left, Root, Right

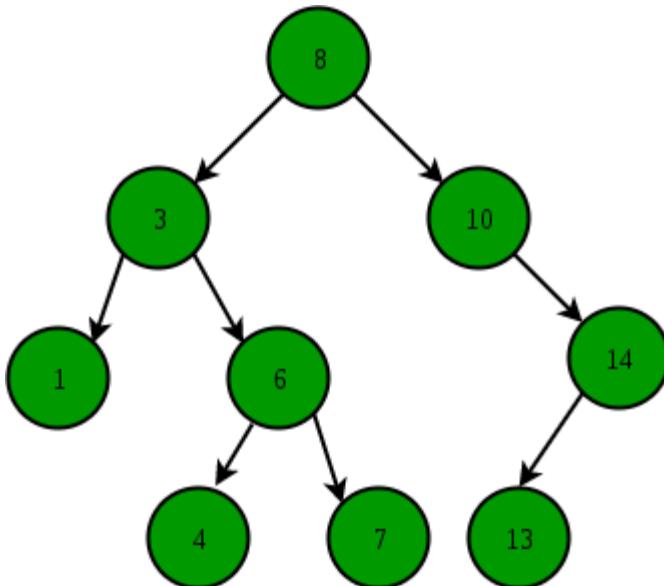
- E.g., sorting
- 1, 3, 4, 6, 7, 8, 10, 13, 14

## Hexagon icon Preorder – Root, Left, Right

- E.g., copy a tree
- 8, 3, 1, 6, 4, 7, 10, 14, 13

## Hexagon icon Postorder – Left, Right, Root

- E.g., delete a node
- 1, 4, 7, 6, 3, 13, 14, 10, 8



```

def inorder(root):
    # Inorder (Left, Root, Right)
    if root is not None:
        inorder(root.left)
        print(root)
        inorder(root.right)

    1, 3, 4, 6, 7, 8, 10, 13, 14

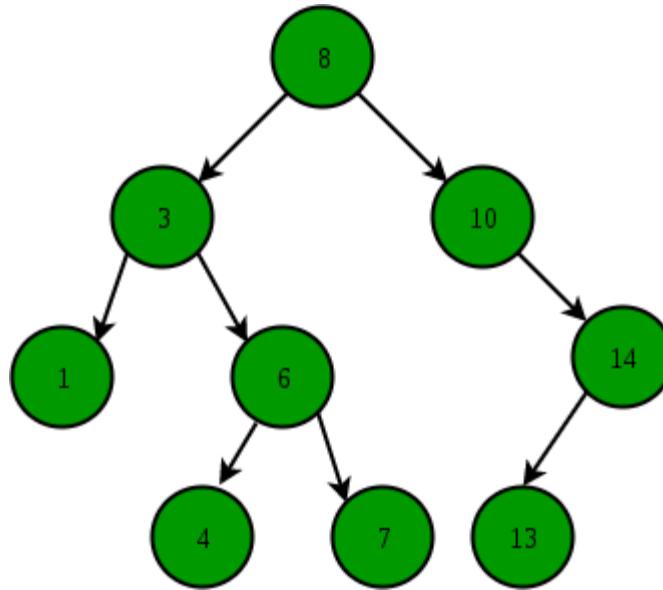
def preorder(root):
    # Preorder (Root, Left, Right)
    if root is not None:
        print(root)
        preorder(root.left)
        preorder(root.right)

    8, 3, 1, 6, 4, 7, 10, 14, 13

def postorder(root):
    # Postorder (Left, Right, Root)
    if root is not None:
        postorder(root.left)
        postorder(root.right)
        print(root)

    1, 4, 7, 6, 3, 13, 14, 10, 8

```

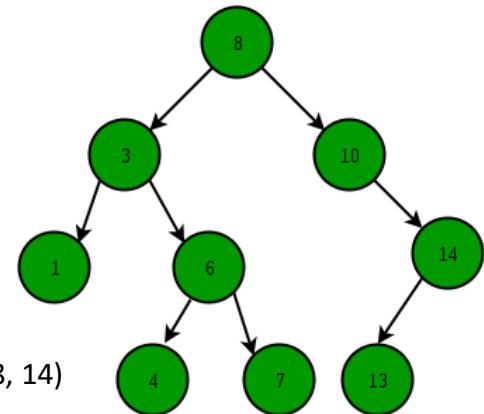




# BST – 1NN Search

- Query point – 11

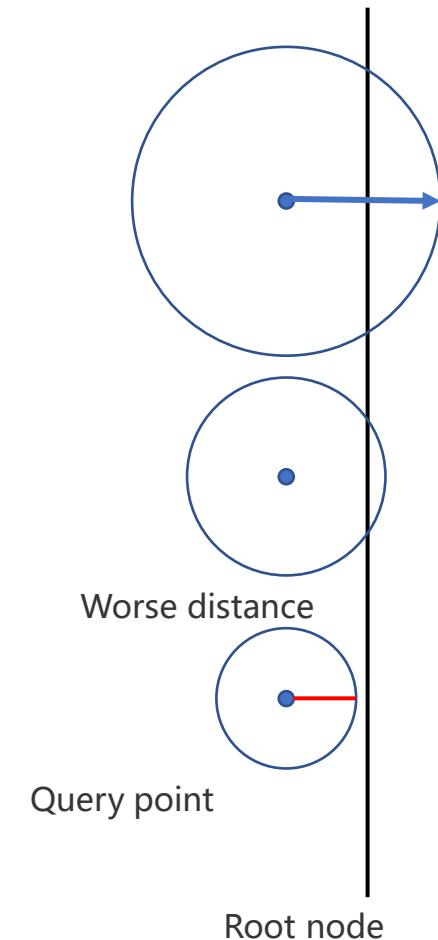
1. 8
  - a) **worst distance = 3** (11-8)
  - b) any point in 8's left tree is at least 3 away from query
  - c) do I need to go further? Yes! Right subtree is (8, +inf] but **worst distance=3** -> need to search (8, 14)
2. 10
  - a) **worst distance = 1** (11-10)
  - b) do I need to go further? Yes! Right subtree is (10, +inf] but **worst distance=1** -> need to search (10, 12)
3. 14
  - a) **worst distance = 1**
  - b) do I need to further? Yes! Left subtree is (10, 14) but **worst distance=1** -> need to search (10, 12)
4. Go back to 14
  - c) do I need to go right? No! Right subtree is (14, inf] but **worst distance=1** -> I just need to search (10, 12)
5. Go back to 10
  - c) do I need to go left? No! Left subtree is (8, 10) but **worst distance=1** -> I just need to search (10, 12)
6. Go back to 8
  - d) do I need to go left? No! Left subtree is [-inf, 8) but **worst distance=1** -> just need to search (10, 12)





## BST – kNN Search

- Almost same as 1NN search
- Difference is how to compute **worst distance**
- Worst distance** is the largest distance that you should search around the query point
- Areas outside this “worst circle” can be skipped
- In kNN search, the worst distance is dynamic





# BST – Worst Distance for kNN

- Build a container to store the kNN results
  - Example:
  - Existing container content
  - [1, 2, 3, 4, inf, inf]
- $k$  results are sorted
  - **add\_point(3.5)**
  - *Step 1.* Make space for 3.5
  - [1, 2, 3, **4, 4**, inf]
- worst\_dist is the last one
  - *Step 2.* Put 3.5 in the correct position
  - [1, 2, 3, **3.5**, 4, inf]
- Add a result if
$$dist < worse\_dist$$
  - *Step 3.* Update worst\_dist

```
class KNNResultSet:  
    def __init__(self, capacity):  
        self.capacity = capacity  
        self.count = 0  
        self.worst_dist = 1e10  
        self.dist_index_list = []  
        for i in range(capacity):  
            self.dist_index_list.append(DistIndex(self.worst_dist, 0))  
  
    self.comparison_counter = 0  
  
    def size(self):  
        return self.count  
  
    def full(self):  
        return self.count == self.capacity  
  
    def worstDist(self):  
        return self.worst_dist  
  
    def add_point(self, dist, index):  
        self.comparison_counter += 1  
        if dist > self.worst_dist:  
            return  
  
        if self.count < self.capacity:  
            self.count += 1  
  
        i = self.count - 1  
        while i > 0:  
            if self.dist_index_list[i-1].distance > dist:  
                self.dist_index_list[i] = copy.deepcopy(self.dist_index_list[i-1])  
                i -= 1  
            else:  
                break  
  
        self.dist_index_list[i].distance = dist  
        self.dist_index_list[i].index = index  
        self.worst_dist = self.dist_index_list[self.capacity-1].distance
```

Initialized to large value

Container to keep all the k neighbors

If a point is added, put it in a ordered position

```
class DistIndex:  
    def __init__(self, distance, index):  
        self.distance = distance  
        self.index = index  
  
    def __lt__(self, other):  
        return self.distance < other.distance
```

```
def knn_search(root: Node, result_set: KNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)
    if result_set.worstDist() == 0: A special case – if the worst distance is 0, no
        need to search anymore

    if root.key >= key:
        # iterate left branch first If key != query, need to go through one subtree
        if knn_search(root.left, result_set, key):
            return True      May not need to search for the other subtree, depends on worst distance
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.right, result_set, key)
        return False

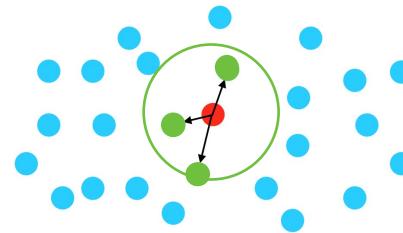
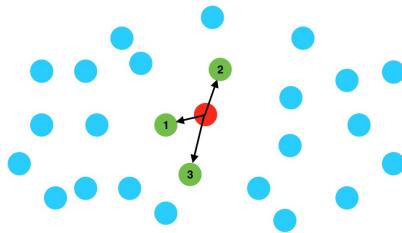
    else: # iterate right branch first
        if knn_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.left, result_set, key)
        return False
```

Similar to the “if” block above



# Radius NN Search

- Same as kNN, in the sense that,
  - Worst distance circle intersects the boundary -> search
  - If not -> skip
- In implementation, we don't need to change the BST kNN search logics, except that, the worst distance is **fixed**, instead of dynamic





## BST – Radius Result Set Manager

- ➊ Simpler than kNN result set manager.
- ➋ Worst distance is fixed.
- ➌ No need to maintain a sorted result set.

```
|     def add_point(self, dist, index):  
|         self.comparison_counter += 1  
|         if dist > self.radius:  
|             return  
  
|             self.count += 1  
|             self.dist_index_list.append(DistIndex(dist, index))
```



# BST - kNN v.s. Radius NN

```
def knn_search(root: Node, result_set: KNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)
    if result_set.worstDist() == 0:
        return True | This part is gone in
    | radius search, because
    | worst_dist = r

    if root.key >= key:
        # iterate left branch first
        if knn_search(root.left, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.right, result_set, key)
        return False
    else:
        # iterate right branch first
        if knn_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.left, result_set, key)
        return False
```

```
def radius_search(root: Node, result_set: RadiusNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)

    if root.key >= key:
        # iterate left branch first
        if radius_search(root.left, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return radius_search(root.right, result_set, key)
        return False
    else:
        # iterate right branch first
        if radius_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return radius_search(root.left, result_set, key)
        return False
```



# A complete script

```
db_size = 100
k = 5
radius = 2.0

data = np.random.permutation(db_size).tolist()

root = None
for i, point in enumerate(data):
    root = insert(root, point, i)

query_key = 6
result_set = KNNResultSet(capacity=k)
knn_search(root, result_set, query_key)
print('kNN Search:')
print('index - distance')
print(result_set)

result_set = RadiusNNResultSet(radius=radius)
radius_search(root, result_set, query_key)
print('Radius NN Search:')
print('index - distance')
print(result_set)
```

- Search in 100 points, takes 7 comparison only
- Complexity is around  $O(\log_2(n))$ , n is number of database points, if tree is balanced
- Worst  $O(N)$

kNN Search:

index - distance
73 - 0.00
5 - 1.00
12 - 1.00
1 - 2.00
98 - 2.00

In total 7 comparison operations.

Radius NN Search:

index - distance
73 - 0.00
5 - 1.00
12 - 1.00
1 - 2.00
98 - 2.00

In total 5 neighbors within 2.000000.  
There are 7 comparison operations.

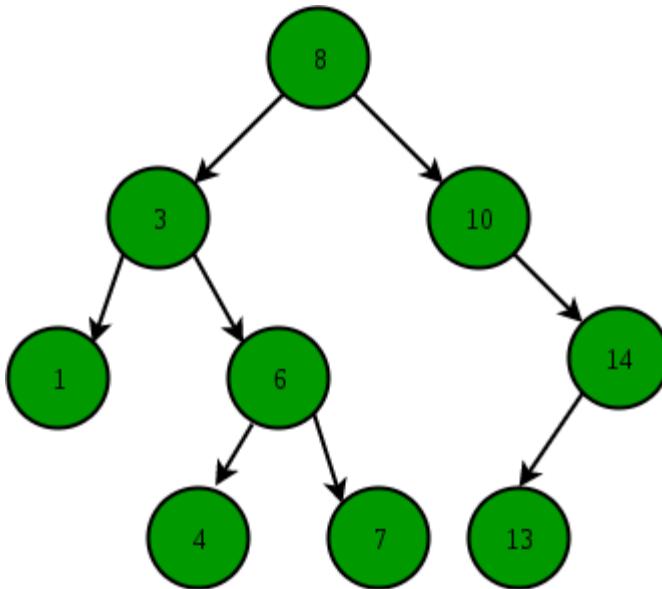


# Binary Search Tree (BST)

BST based 1D kNN/RadiusNN search

- Naïve BST is for 1D data only

Tree based kNN/RadiusNN can be viewed as a Branch-n-Bound algorithm.

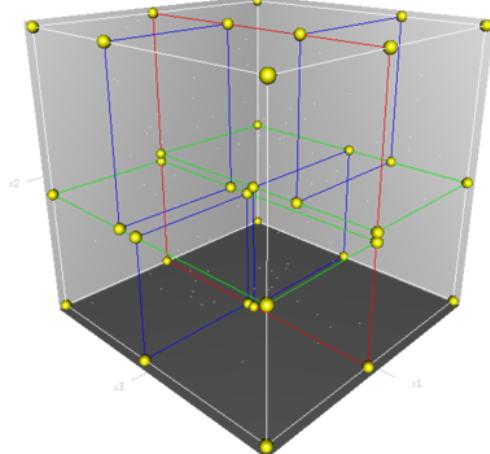




# Kd-tree (k-dimensional tree)

- ❖ It is an extension of BST into high dimension
  - BST is 1-dimensional, how to extend?
  - BST in each dimension!
- ❖ Invented by Jon Louis Bentley in 1975
- ❖ The kd-tree is a binary tree where every leaf node is a **k-dimensional point**

k-d tree		
Type	Multidimensional BST	
Invented	1975	
Invented by	Jon Louis Bentley	
Time complexity in big O notation		
Algorithm	Average	Worst case
Space	$O(n)$	$O(n)$
Search	$O(\log n)$	$O(n)$
Insert	$O(\log n)$	$O(n)$
Delete	$O(\log n)$	$O(n)$



A 3-dimensional kd tree:  
1. Red  
2. Green  
3. Blue



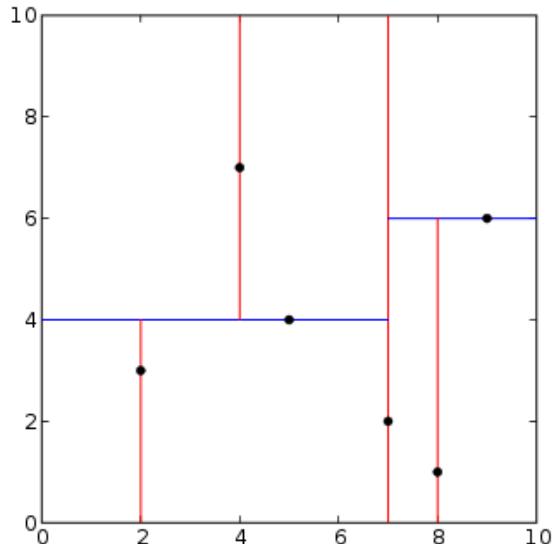
## Kd-tree Construction

- ➊ If there is only one point, or number of points < leaf\_size, build a leaf
- ➋ Otherwise, divide the points in half by a hyperplane perpendicular to the selected splitting axis
- ➌ Recursively repeat the first two steps.

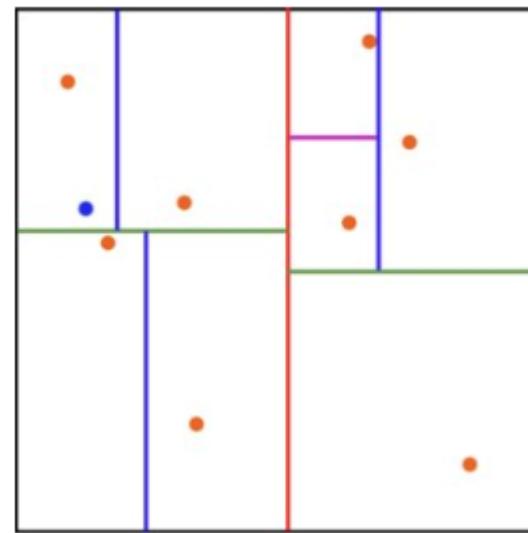


# Kd-tree Construction – Two Conventions

Splitting position is one  
of the points



Splitting position is **NOT**  
one of the points

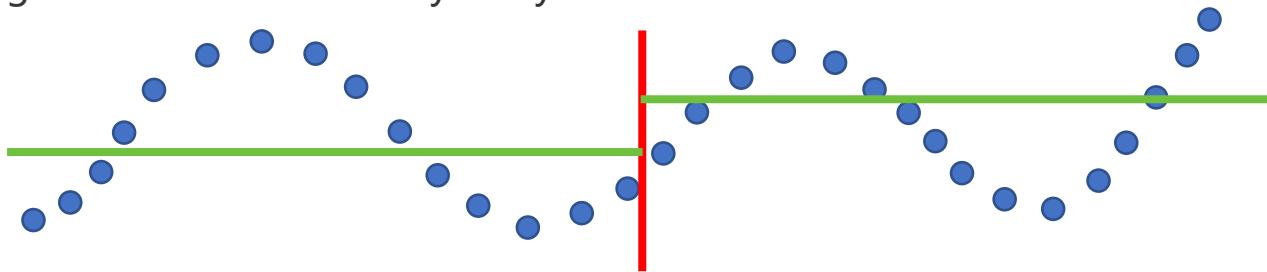




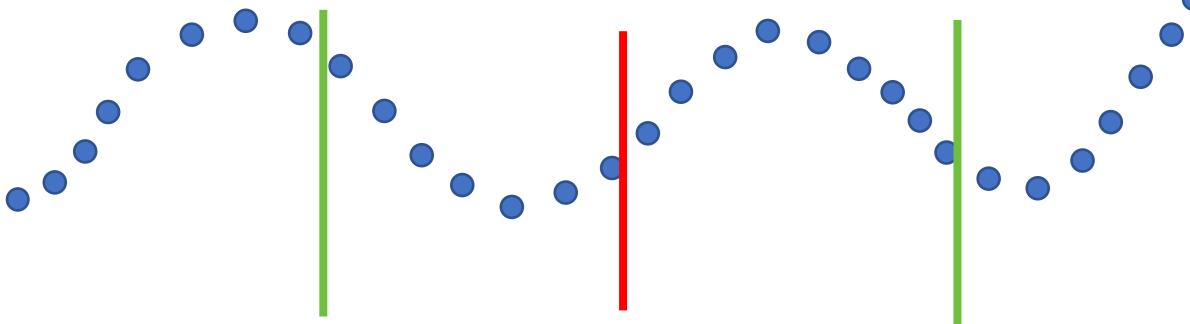
# Kd-tree Construction

## Division / Splitting Strategy

- Dividing axis is round-robin: x-y-z-x-y-z-x-.....

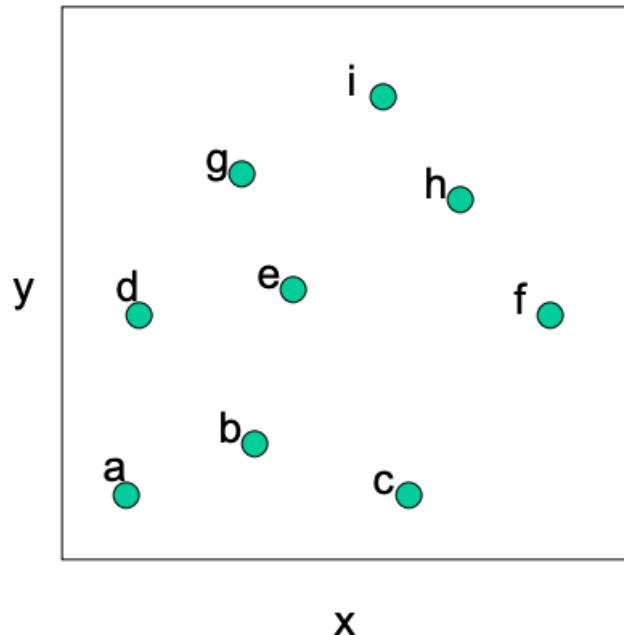


- Select the axis with the widest spread, so called "adaptive"



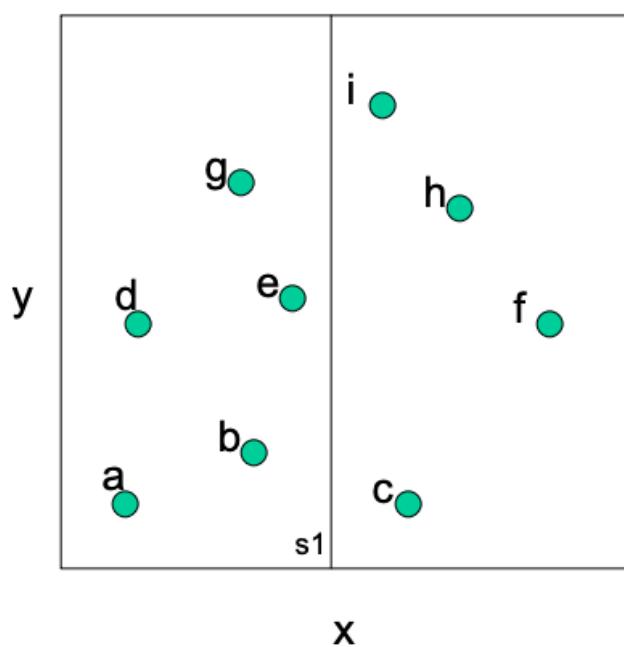


# Kd-tree Construction





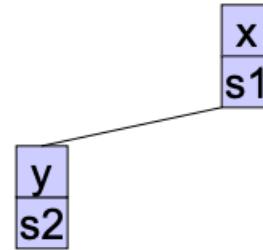
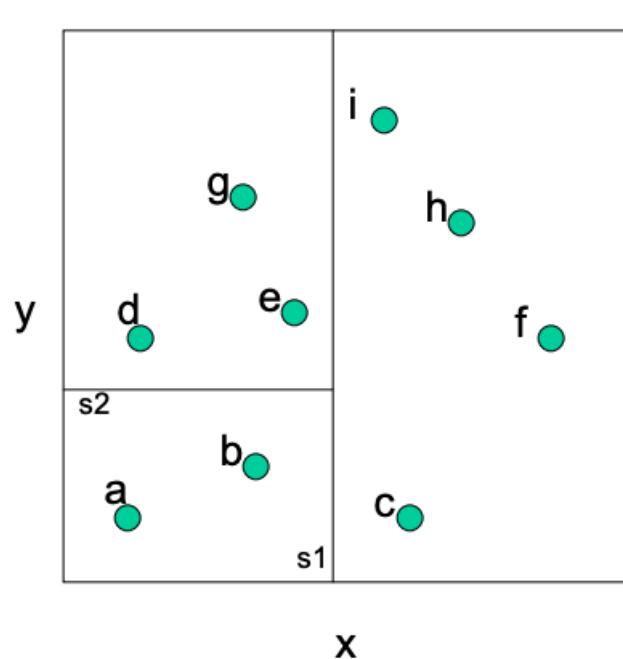
# Kd-tree Construction



x  
s1

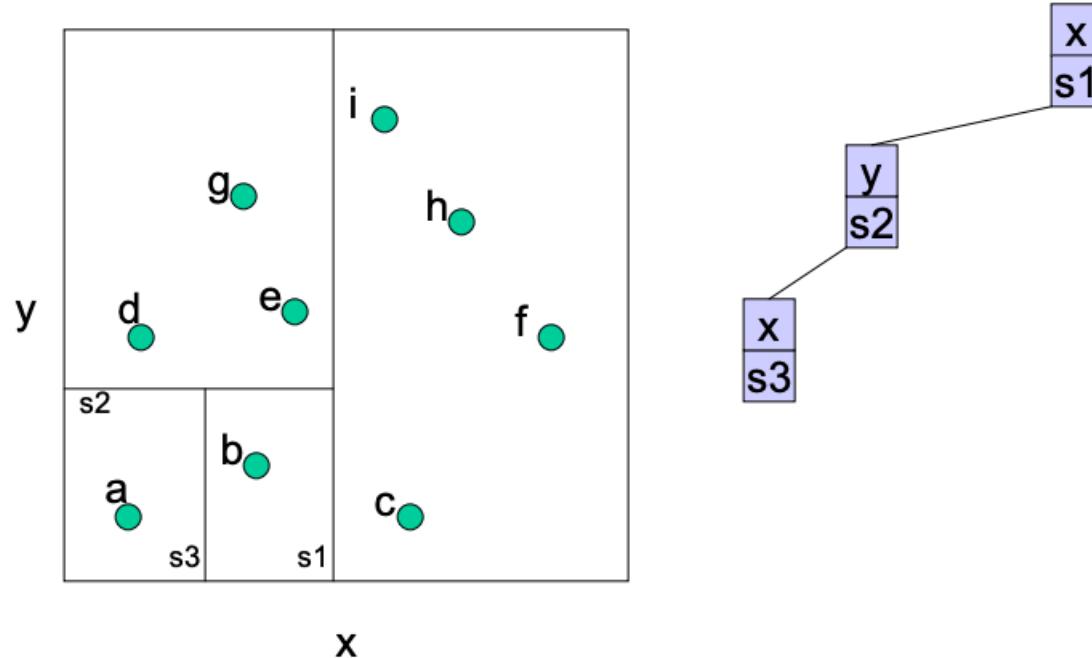


# Kd-tree Construction



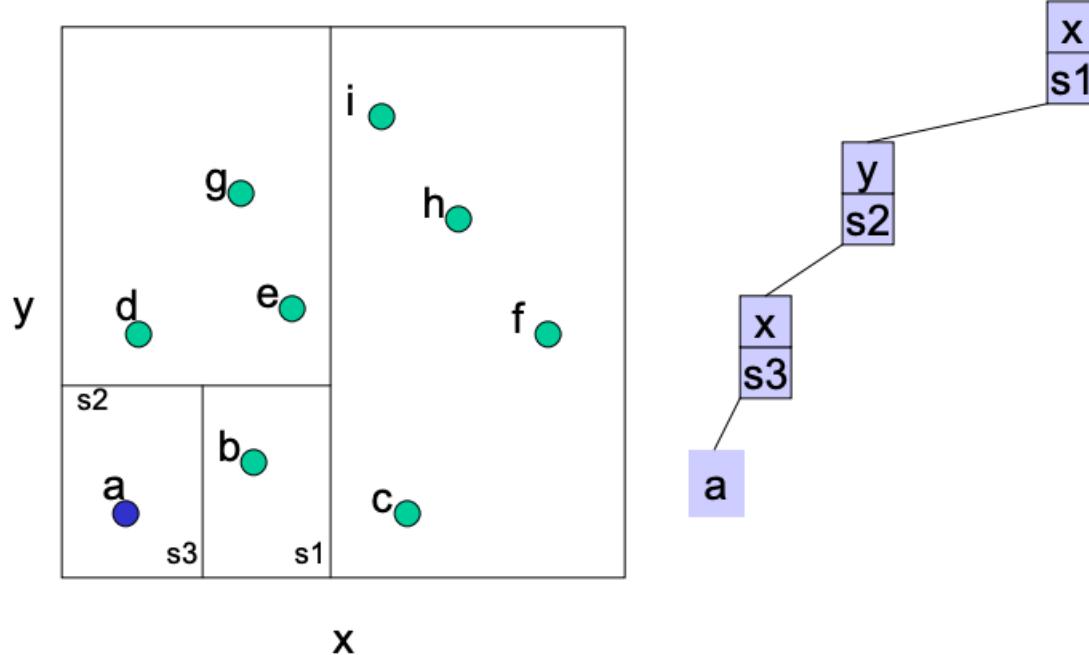


# Kd-tree Construction



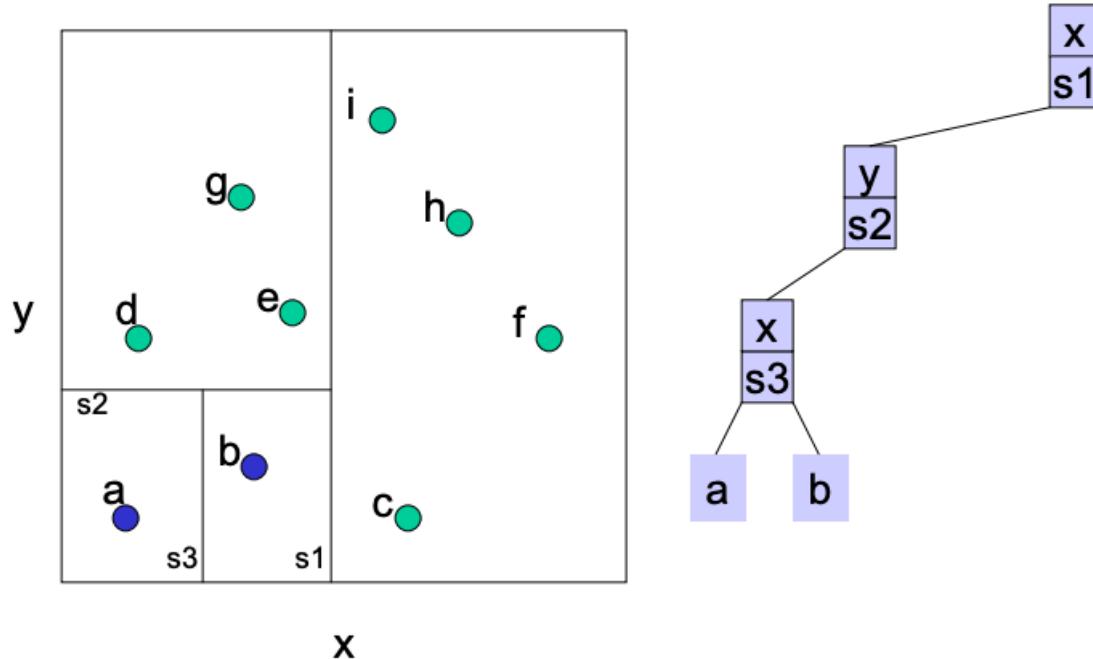


# Kd-tree Construction



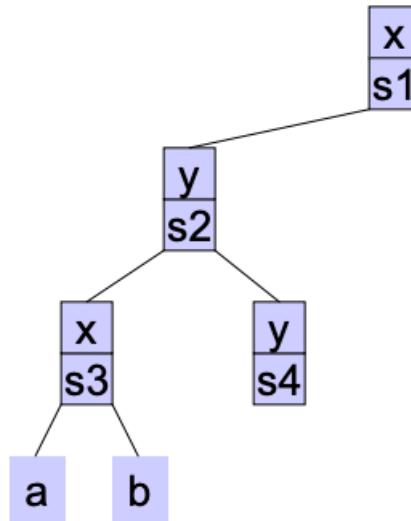
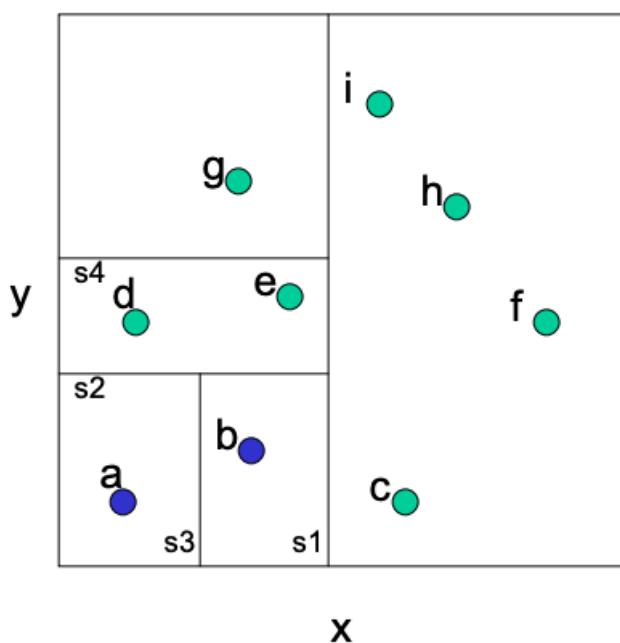


# Kd-tree Construction



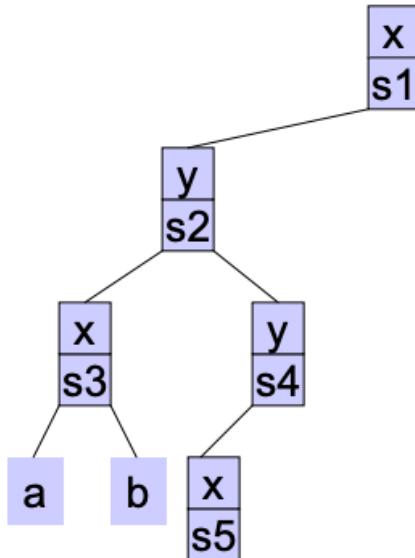
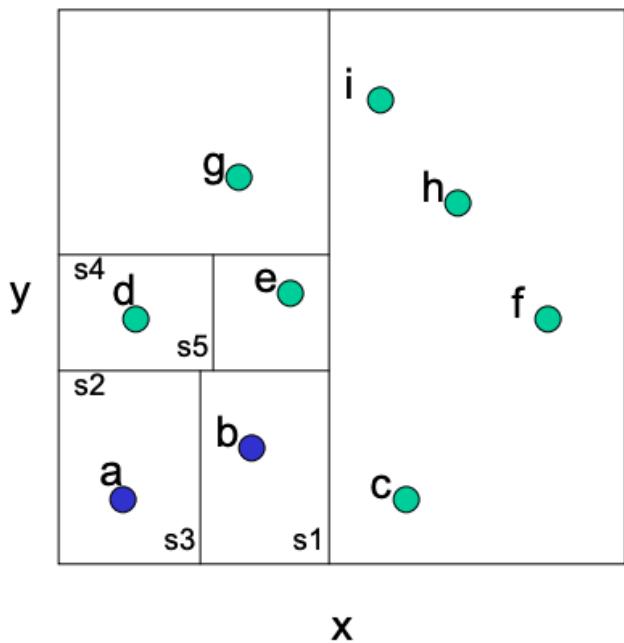


# Kd-tree Construction



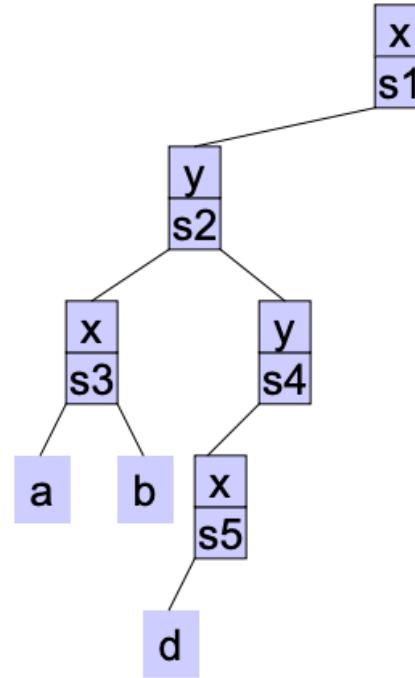
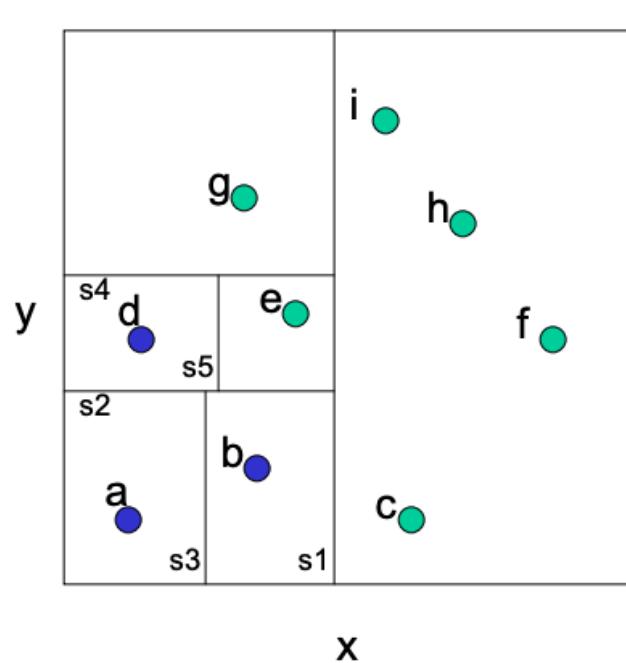


# Kd-tree Construction



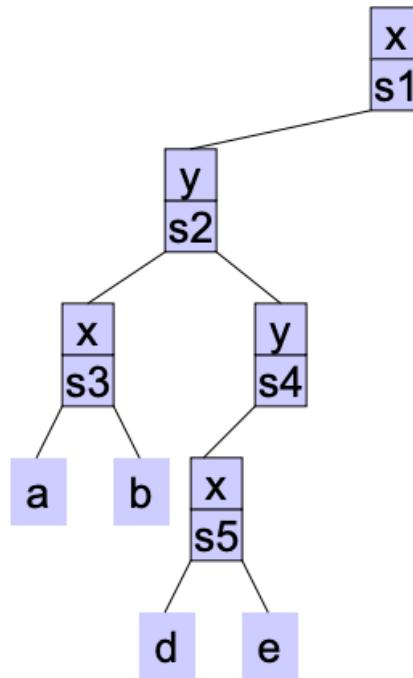
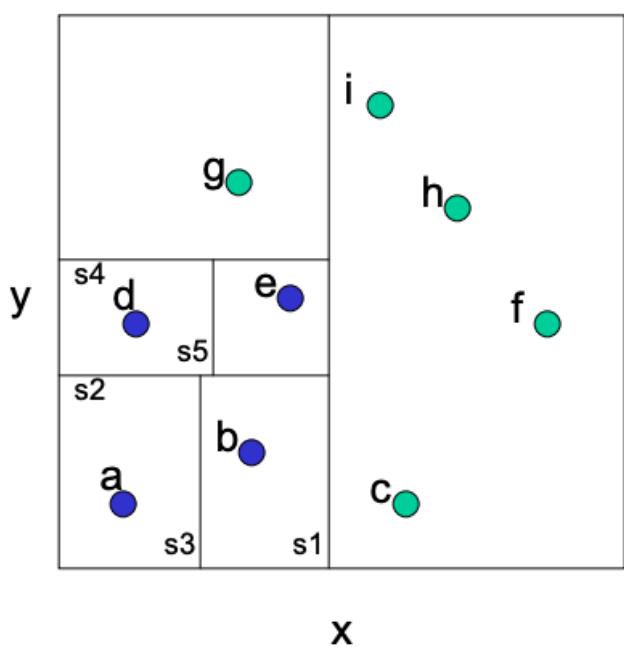


# Kd-tree Construction



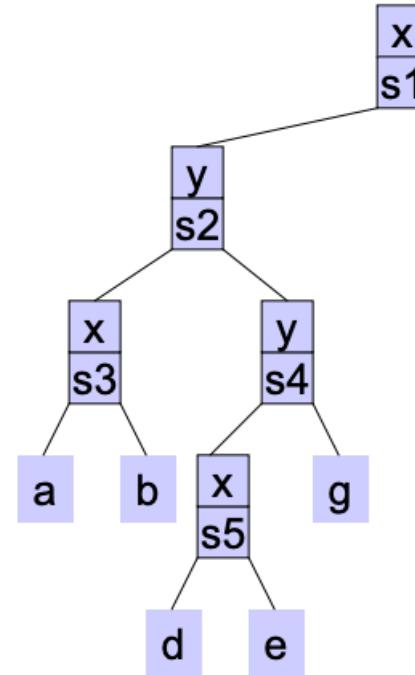
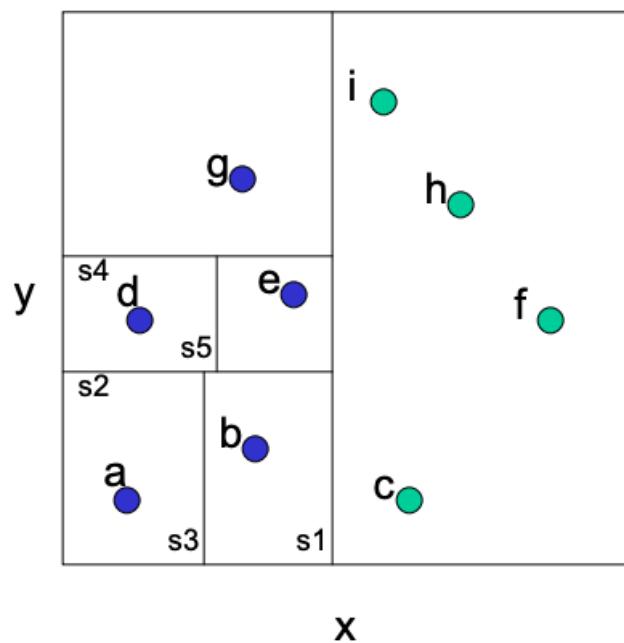


# Kd-tree Construction



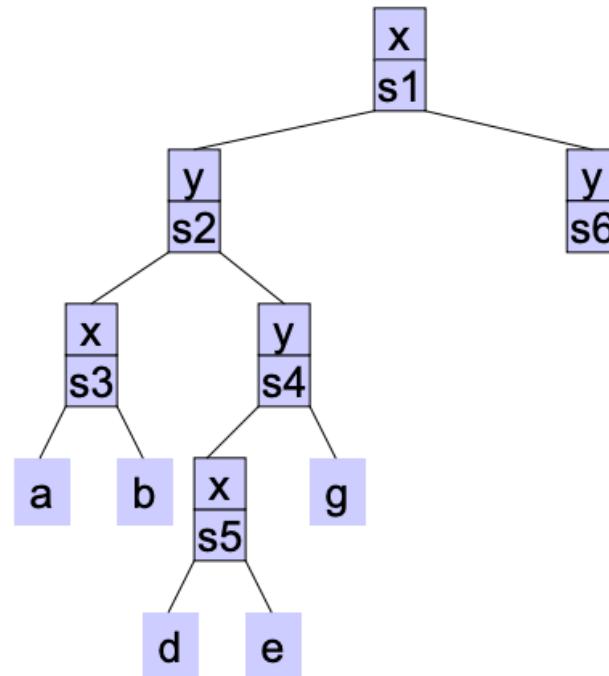
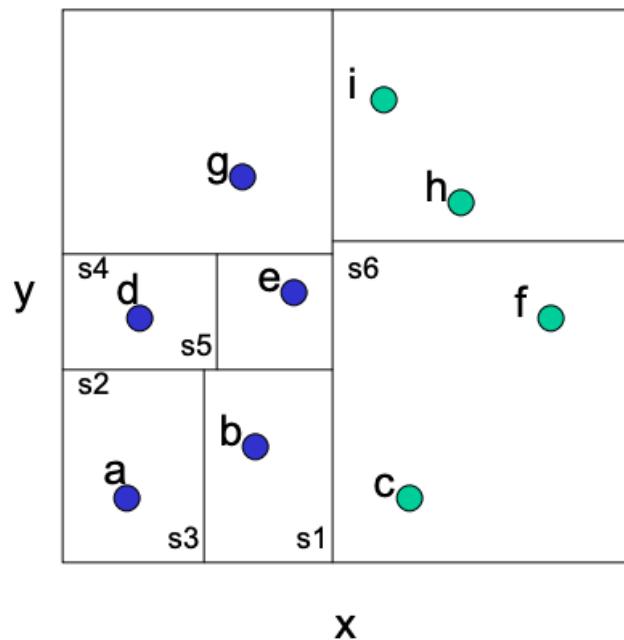


# Kd-tree Construction



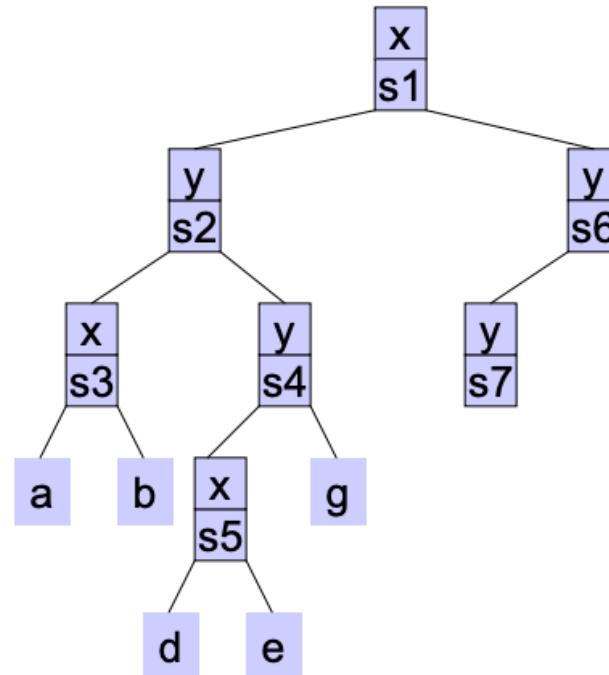
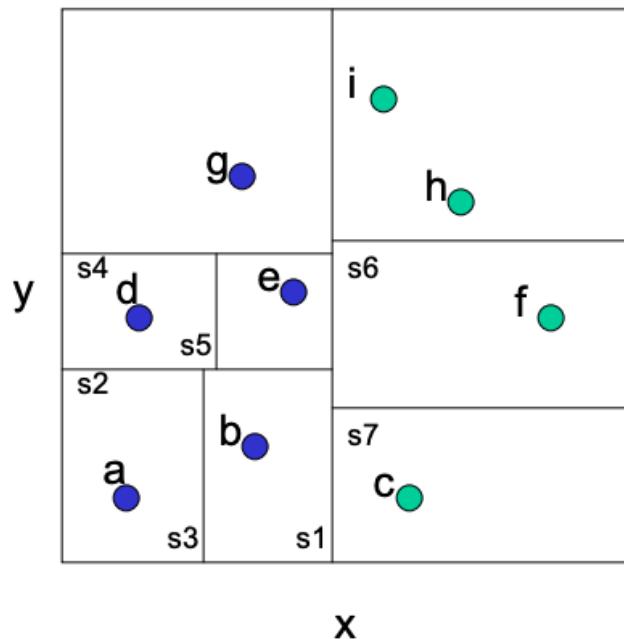


# Kd-tree Construction



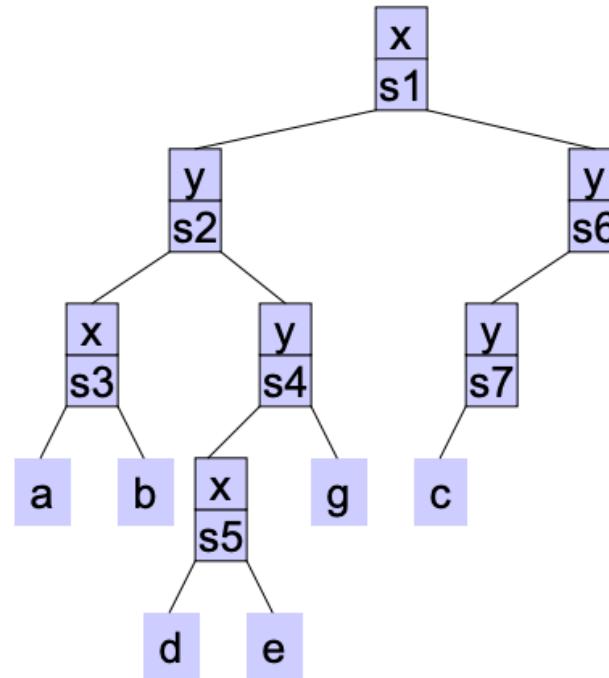
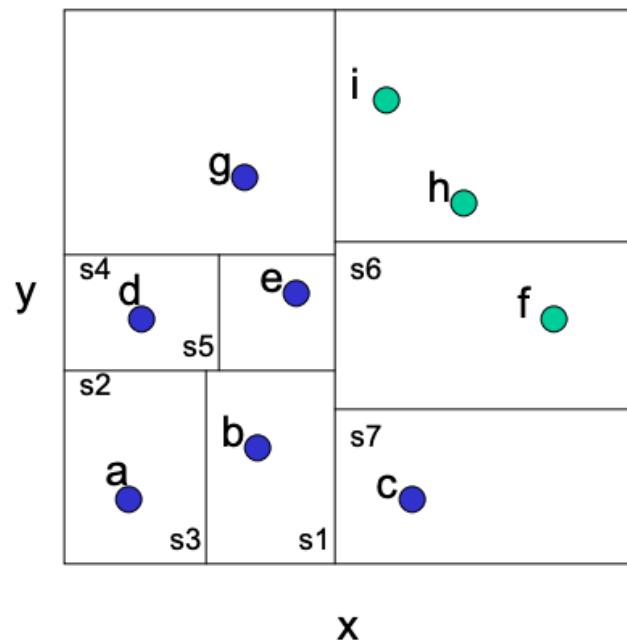


# Kd-tree Construction



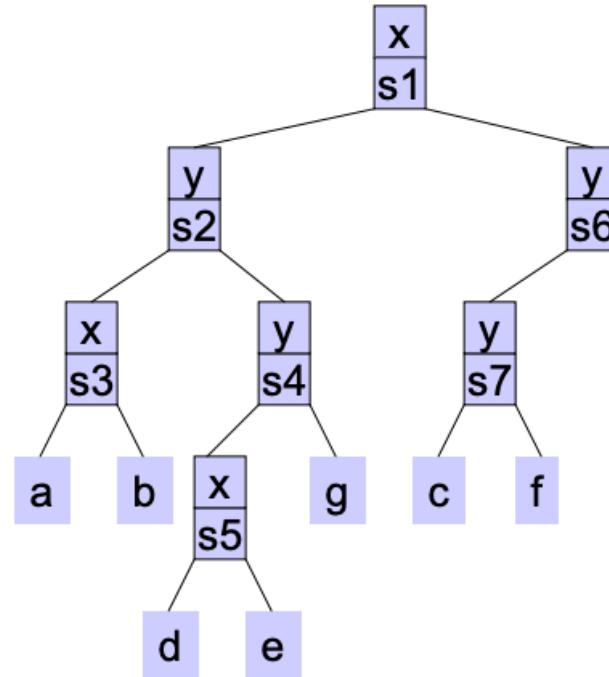
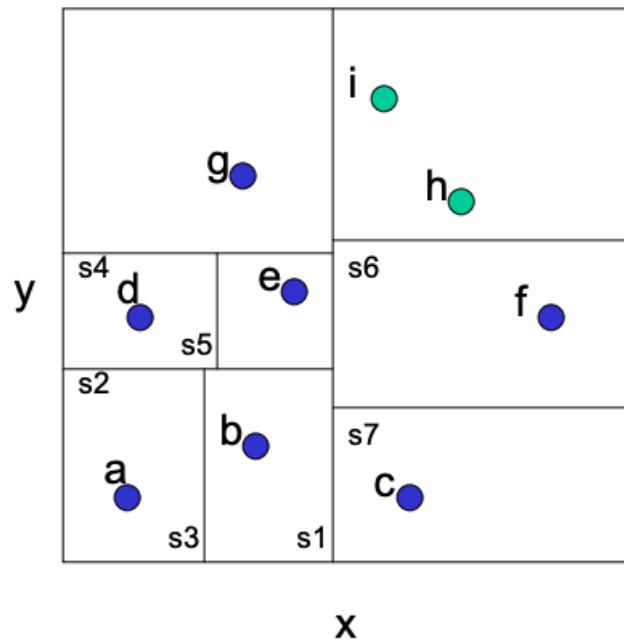


# Kd-tree Construction



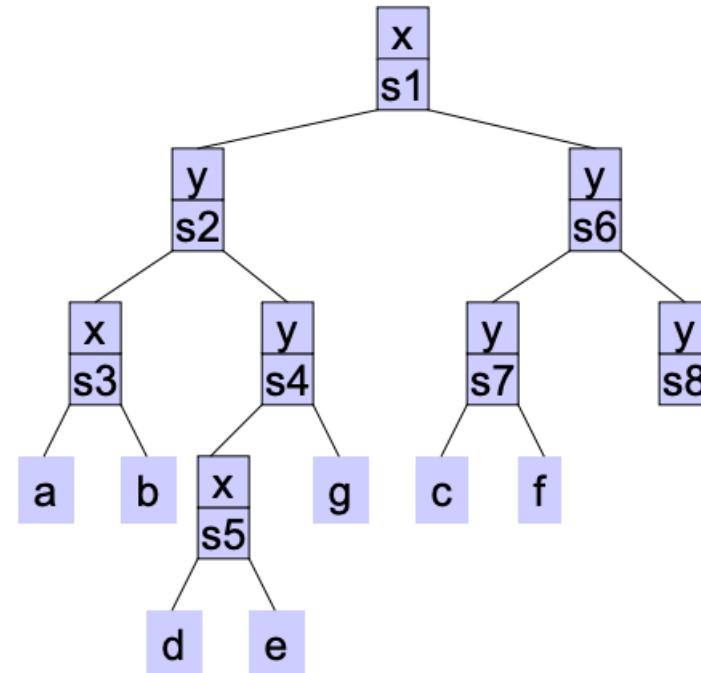
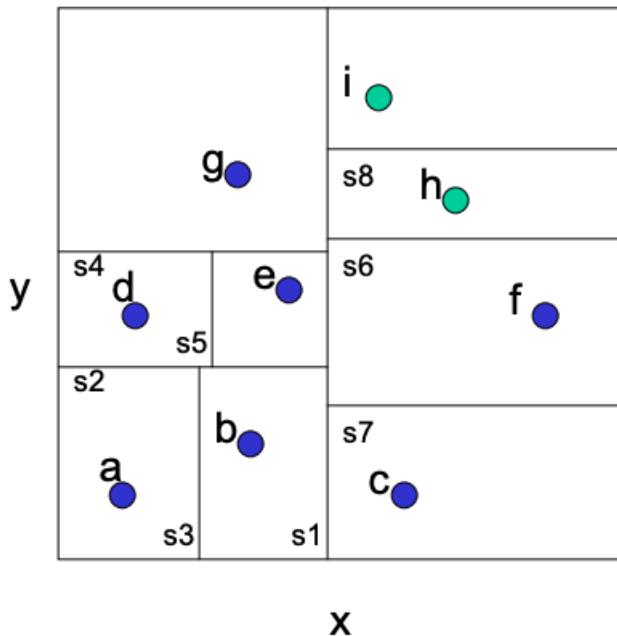


# Kd-tree Construction



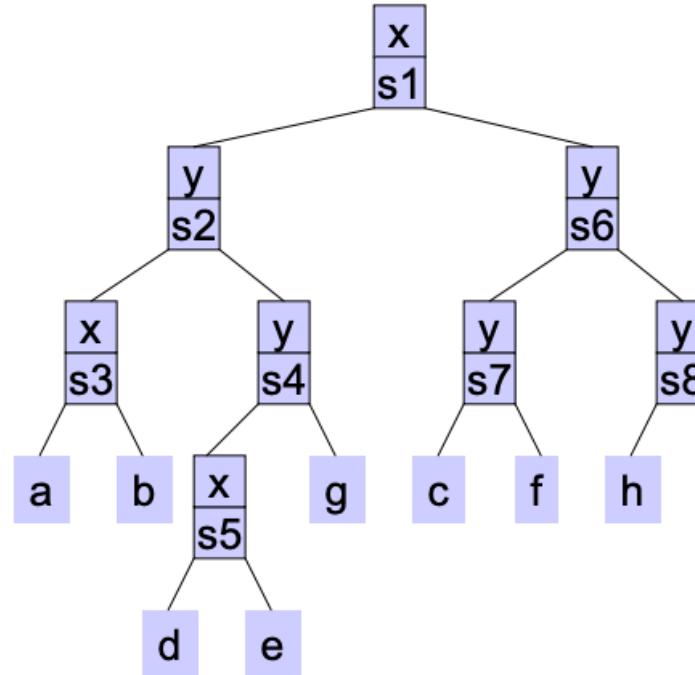
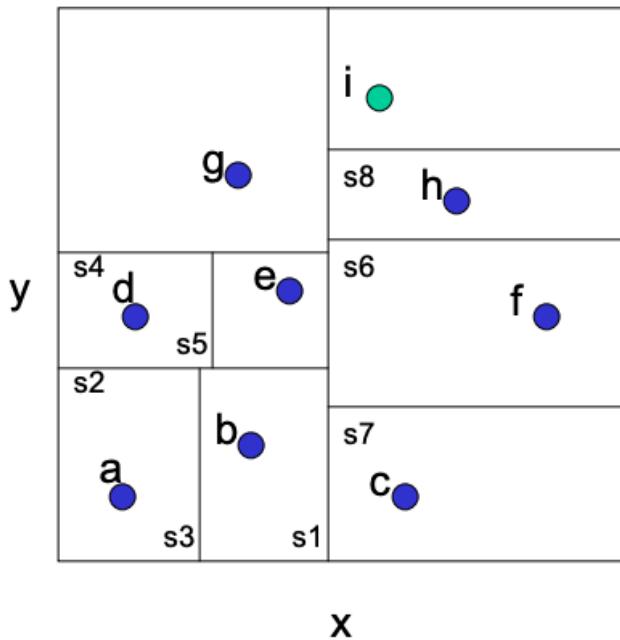


# Kd-tree Construction



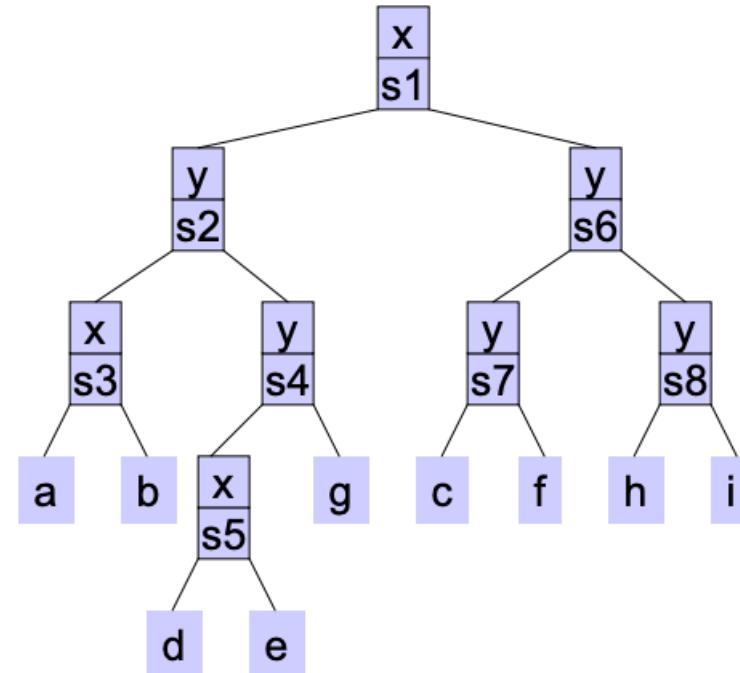
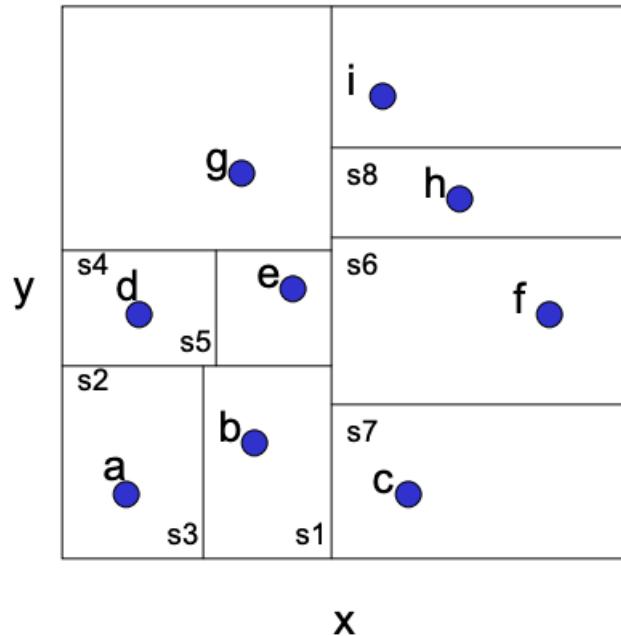


# Kd-tree Construction





# Kd-tree Construction





# Kd-tree Construction



Talk is cheap, show me the code.  
**Linus Torvalds**

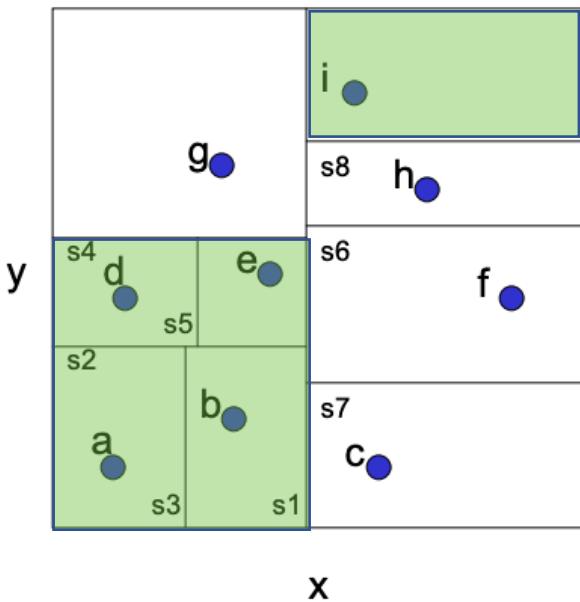


# Kd-tree Node Representation

```
class Node:  
    def __init__(self, axis, value, left, right, point_indices):  
        self.axis = axis          Splitting position  
        self.value = value  
        self.left = left  
        self.right = right  
        self.point_indices = point_indices  
    def is_leaf(self):           Stores points that belongs to  
        if self.value is None:  
            return True  
        else:  
            return False
```

A node with  
axis = x  
value = \*\*\*  
points = [a, b, d, e]

A leaf node with  
axis = y  
value = None  
points = i



```

def kdrtree_recursive_build(root, db, point_indices, axis, leaf_size):
    """
    :param root:
    :param db: NxD
    :param db_sorted_idx_inv: NxD
    :param point_idx: M
    :param axis: scalar
    :param leaf_size: scalar
    :return:
    """

    if root is None:
        root = Node(axis, None, None, None, point_indices)

    # determine whether to split into left and right
    if len(point_indices) > leaf_size:
        # --- get the split position ---
        point_indices_sorted, _ = sort_key_by_val(db[point_indices, axis]) # M
        middle_left_idx = math.ceil(point_indices_sorted.shape[0] / 2) - 1
        middle_left_point_idx = point_indices_sorted[middle_left_idx]
        middle_left_point_value = db[middle_left_point_idx, axis]

        middle_right_idx = middle_left_idx + 1
        middle_right_point_idx = point_indices_sorted[middle_right_idx]
        middle_right_point_value = db[middle_right_point_idx, axis]

        root.value = (middle_left_point_value + middle_right_point_value) * 0.5
        # === get the split position ===
        root.left = kdrtree_recursive_build(root.left,
                                             db,
                                             point_indices_sorted[0:middle_right_idx],
                                             axis_round_robin(axis, dim=db.shape[1]),
                                             leaf_size)
        root.right = kdrtree_recursive_build(root.right,
                                              db,
                                              point_indices_sorted[middle_right_idx:],
                                              axis_round_robin(axis, dim=db.shape[1]),
                                              leaf_size)

    return root

```

A leaf node can contain  
more than 1 point

Sort the points in this node, get the median position

```

def axis_round_robin(axis, dim):
    if axis == dim-1:
        return 0
    else:
        return axis + 1

```



# Kd-tree Construction Complexity

❖ The example shown here is not optimal because of sorting at each level of the tree

- Time complexity of around  $O(n \log n \log n)$
- Space complexity of  $O(kn + n \log n)$  → can be easily reduced to  $O(kn + n)$ 
  - Only store points at leaf

❖ Can we select median instead of sorting?

- If median finding is  $O(n)$
- Kd-tree is  $O(n \log n)$
- Median finding in  $O(n)$  is complicated, but **possible!**

❖  $O(kn \log n)$  method

- Building a Balanced k-d Tree in  $O(kn \log n)$  Time
  - Russel A. Brown, Journal of Computer Graphics Techniques, 2015



# Kd-tree Construction Complexity

## Simple methods that work well in practice

- Sample a **subset** of point in each node for sorting, instead of sorting all points
  - $O(n' \log n)$  or  $O(n' \log n' \log n)$
- Use **mean** instead of median
  - An easy way to achieve  $O(n \log n)$

## They don't guarantee balanced kd-tree

- Balanced tree - each leaf node is approximately the same distance from the root
- Similar to the "chain" example in BST.

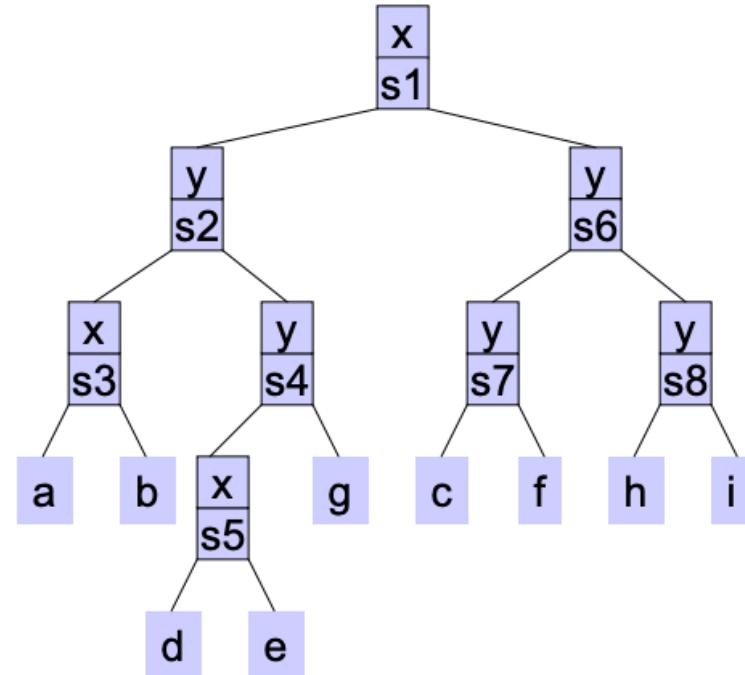
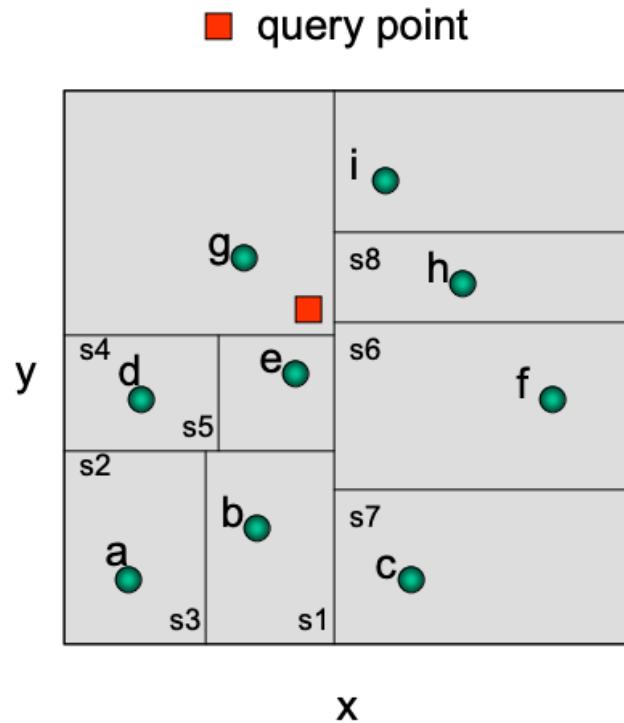


## Kd-tree – kNN Search

-  Start from root
-  Reach the leaf node than covers the query point
  - Compare all points in the leaf node
-  Go up and traverse the tree

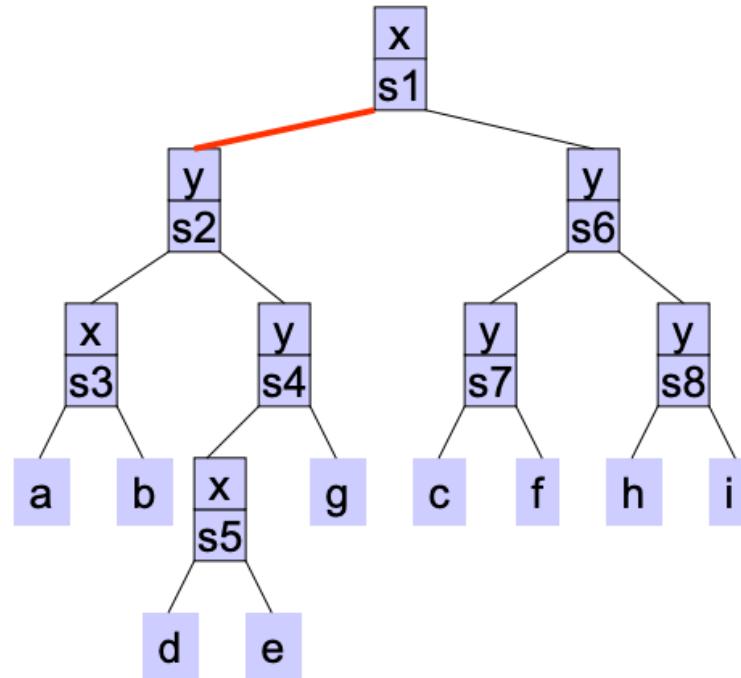
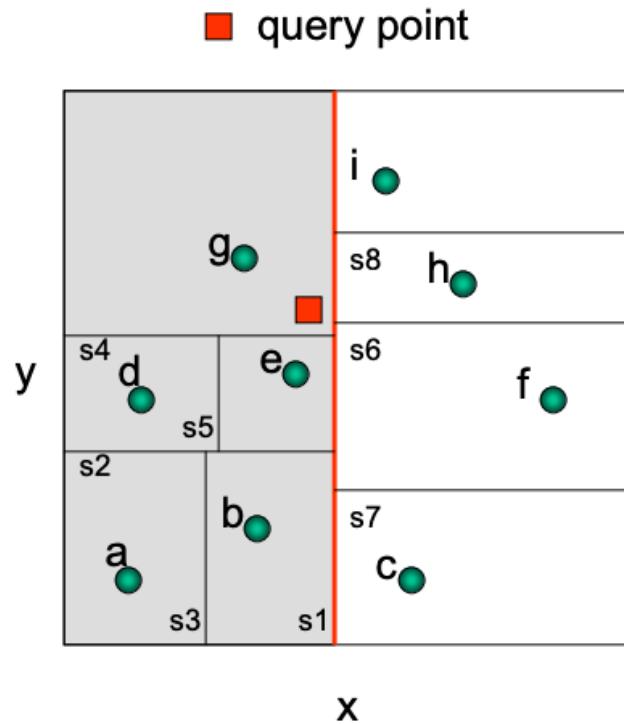


# Kd-tree – kNN Search



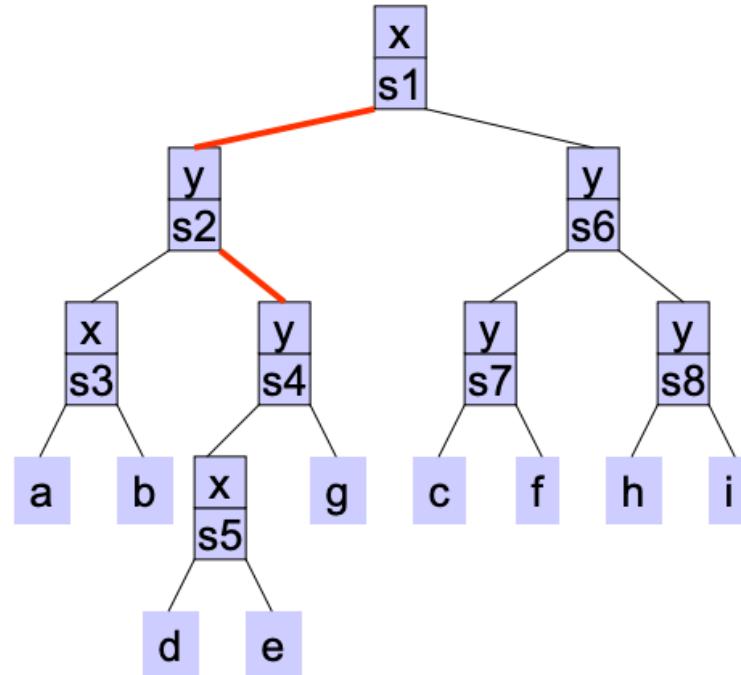
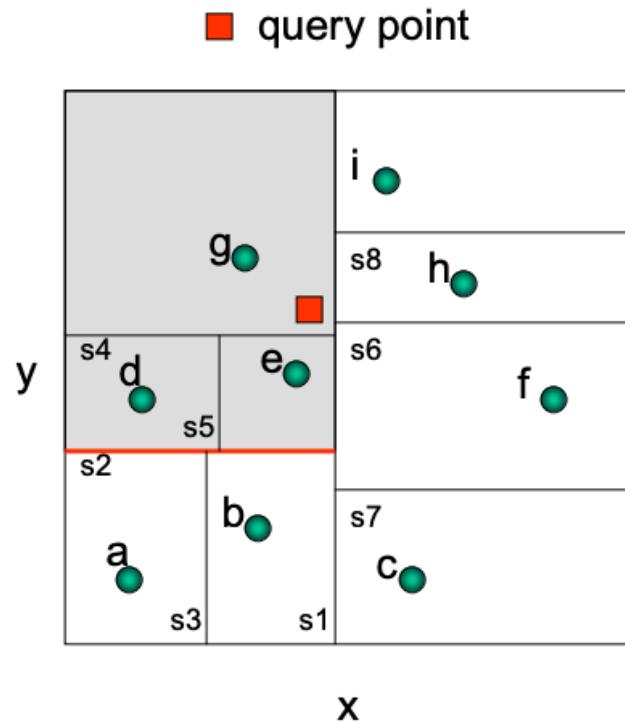


# Kd-tree – kNN Search



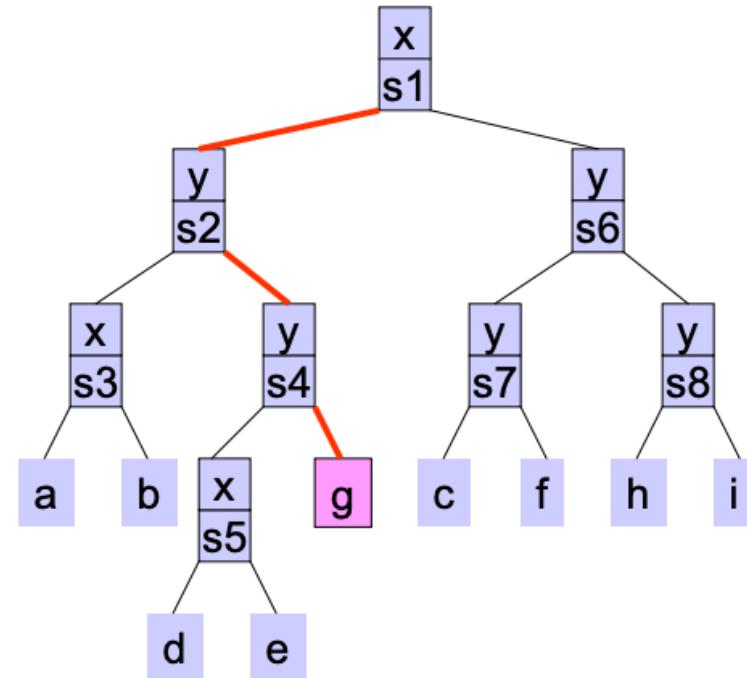
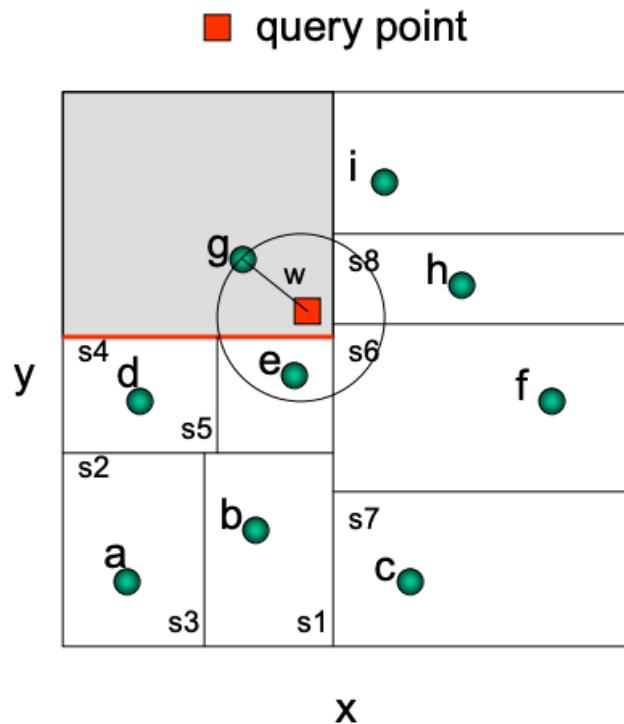


# Kd-tree – kNN Search



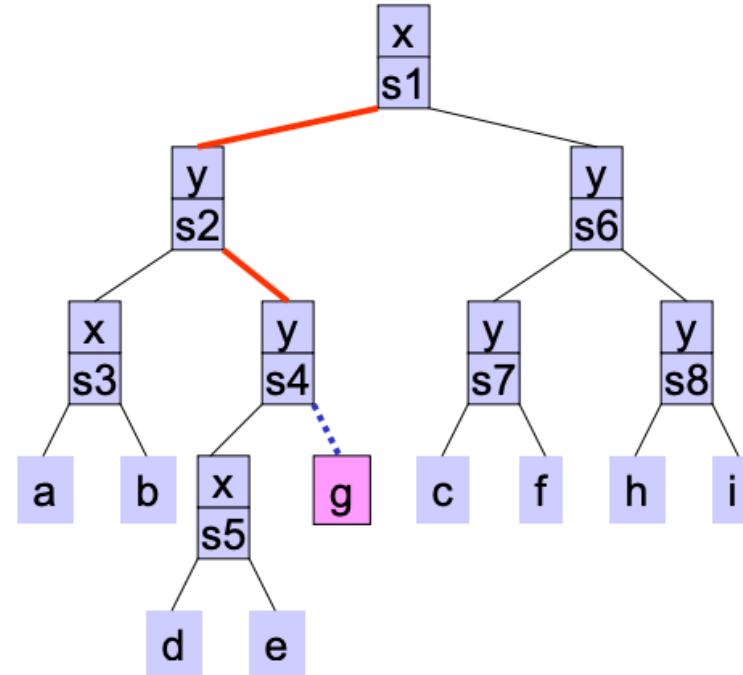
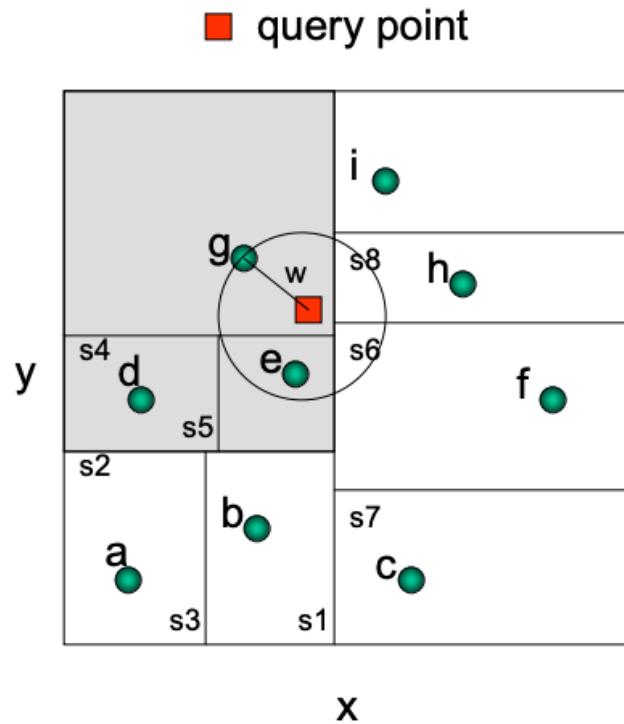


# Kd-tree – kNN Search



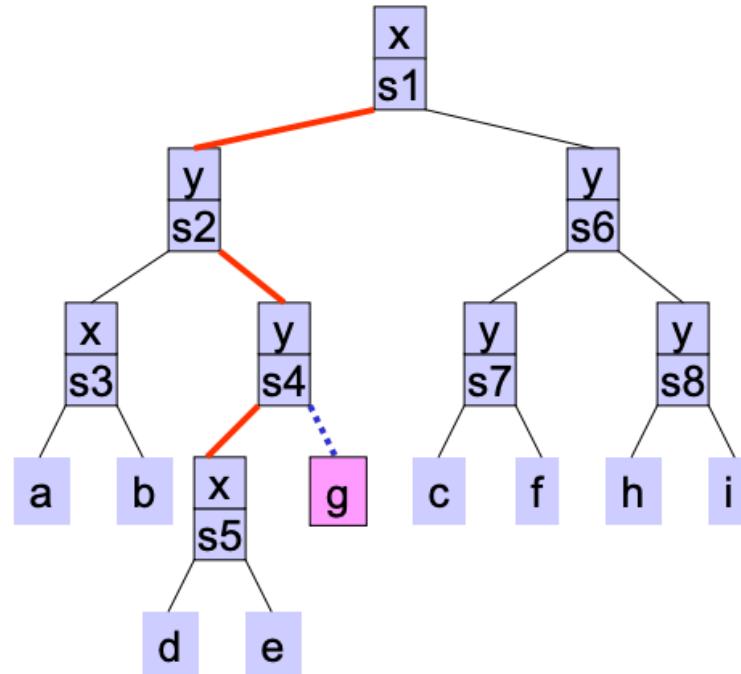
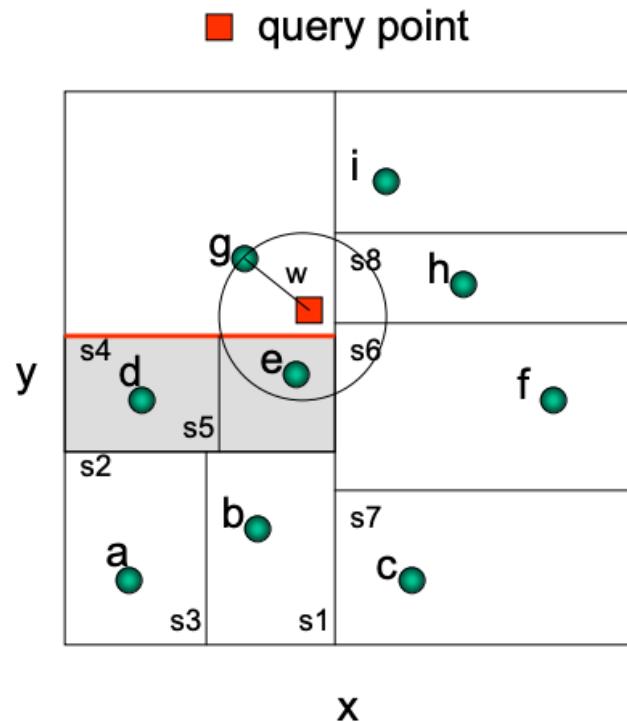


# Kd-tree – kNN Search



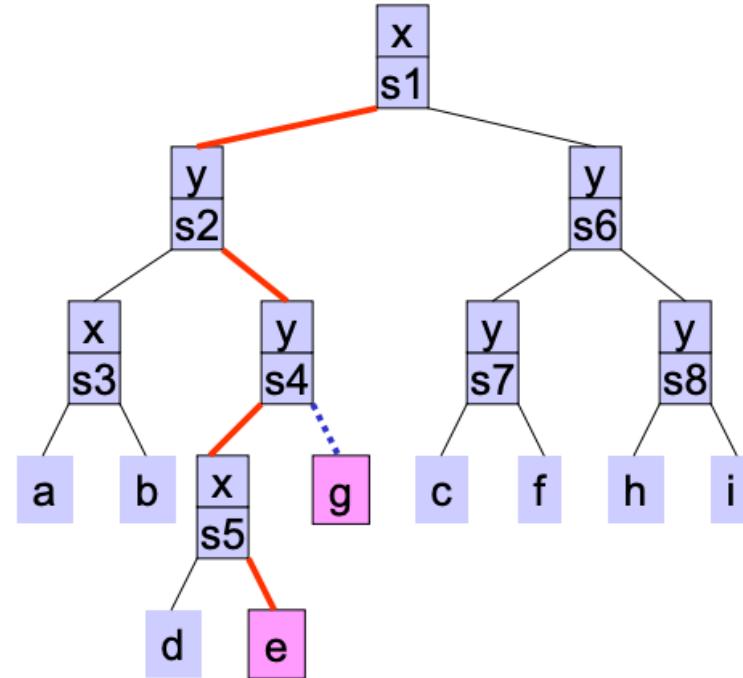
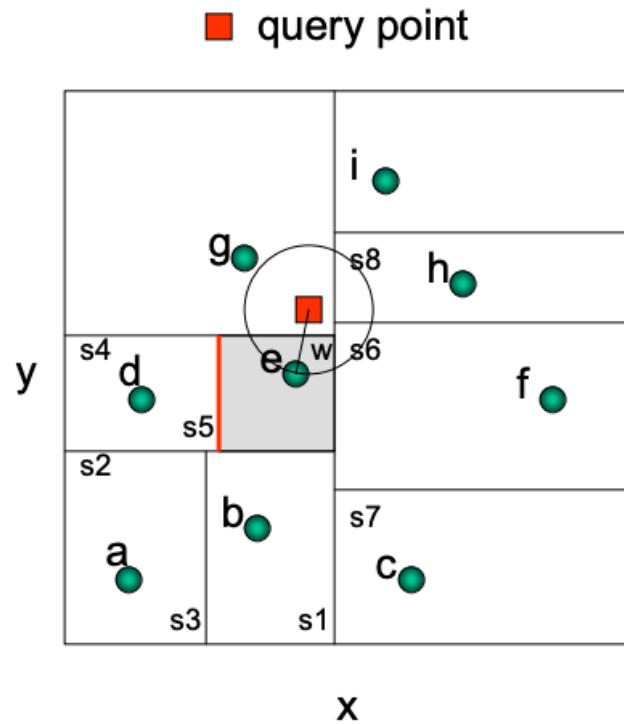


# Kd-tree – kNN Search



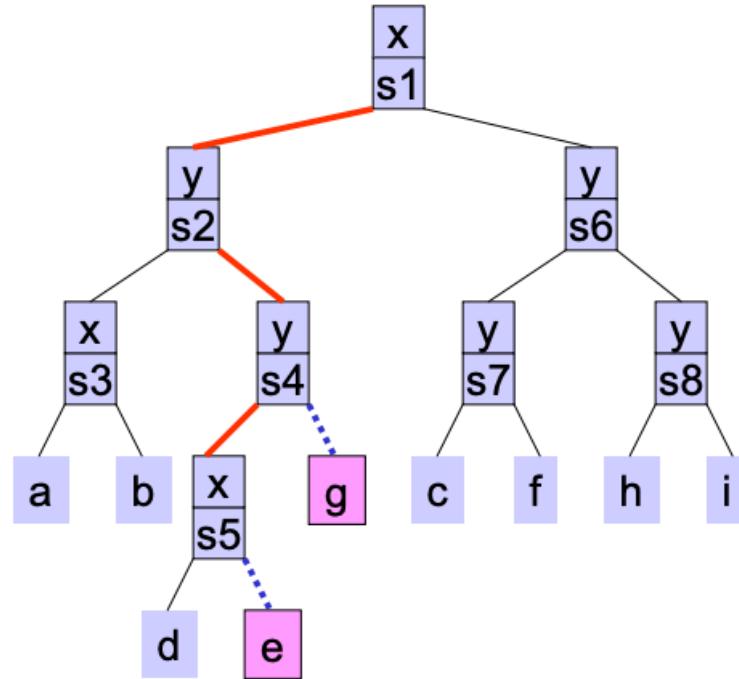
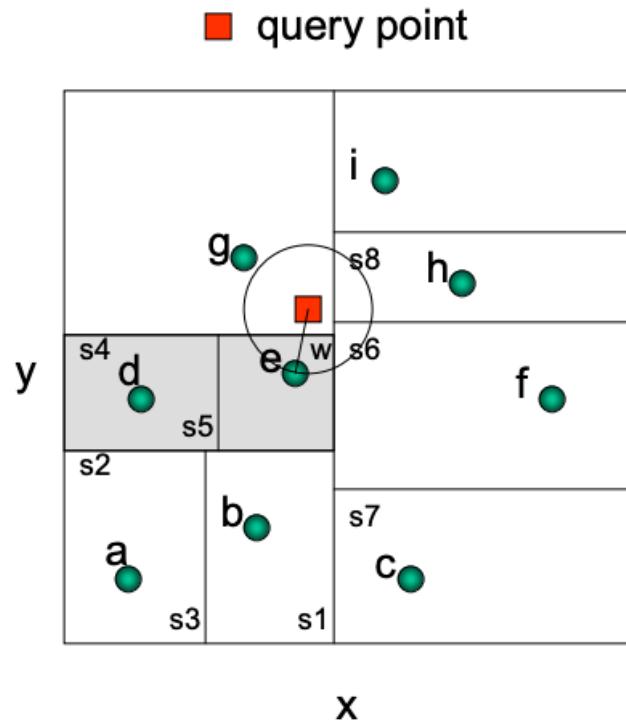


# Kd-tree – kNN Search



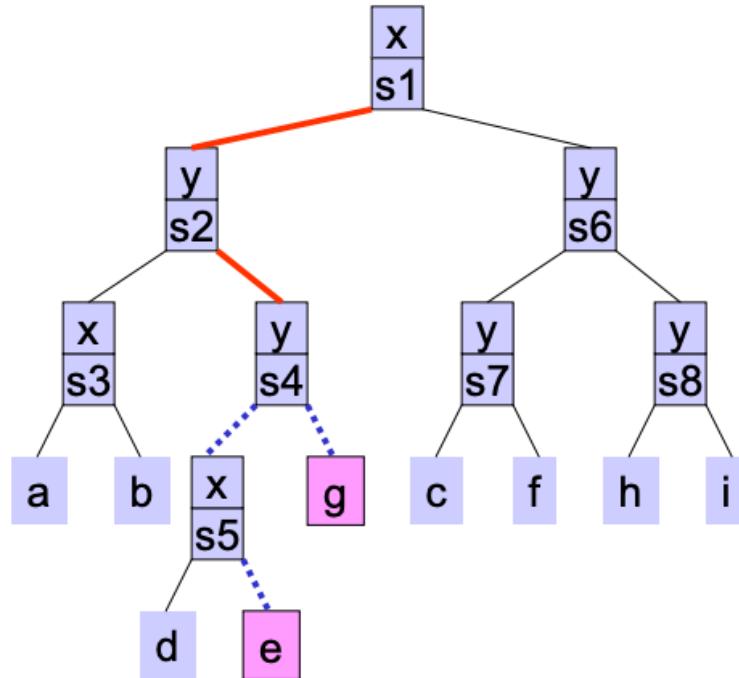
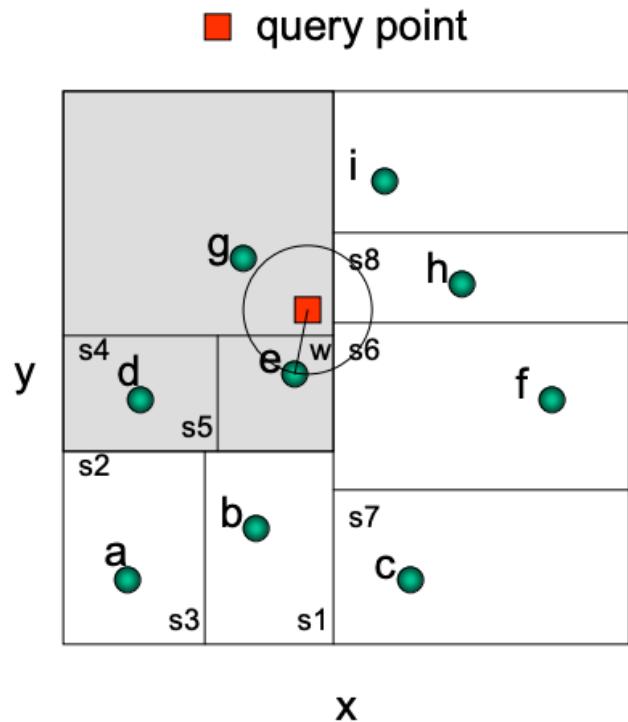


# Kd-tree – kNN Search



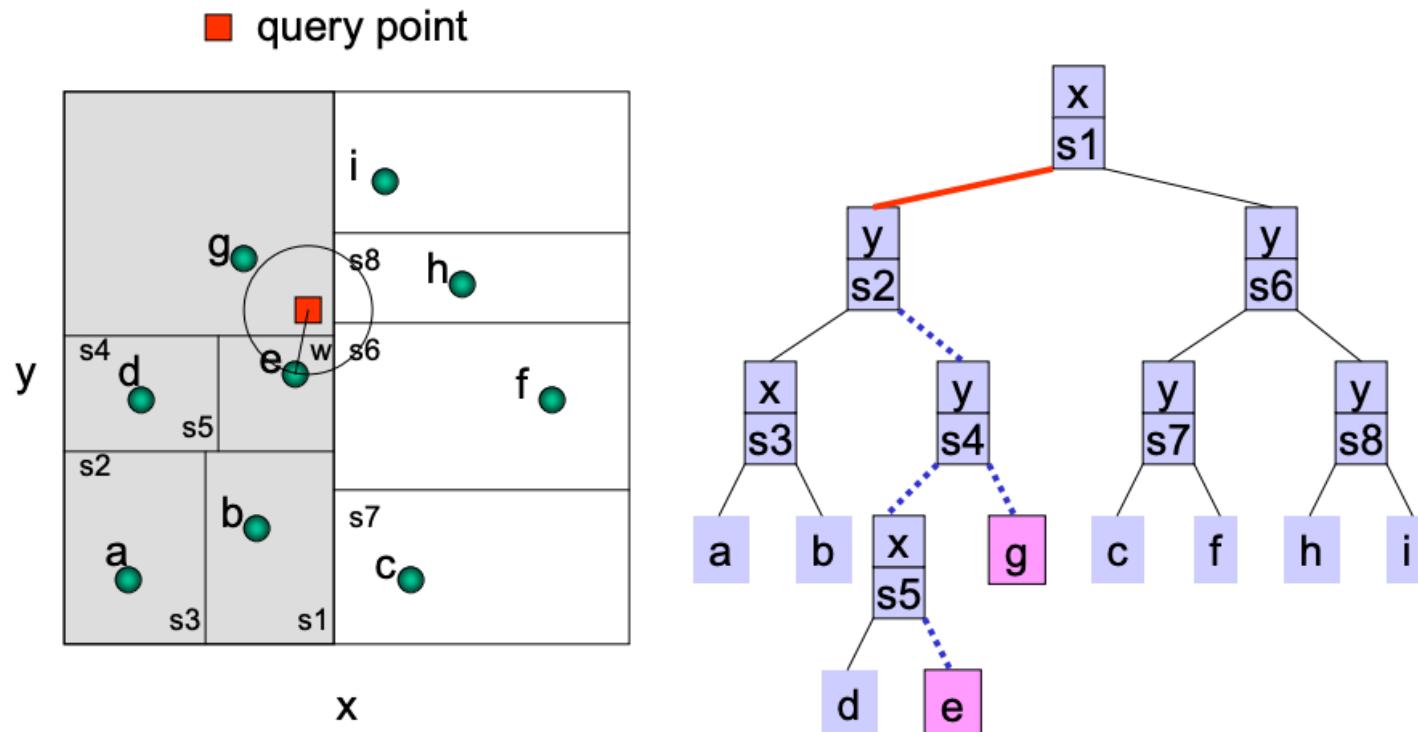


# Kd-tree – kNN Search



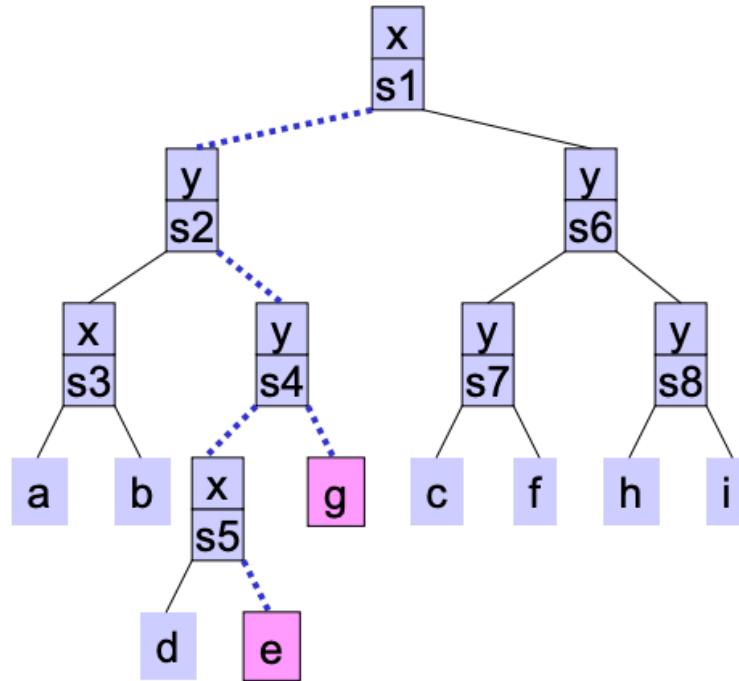
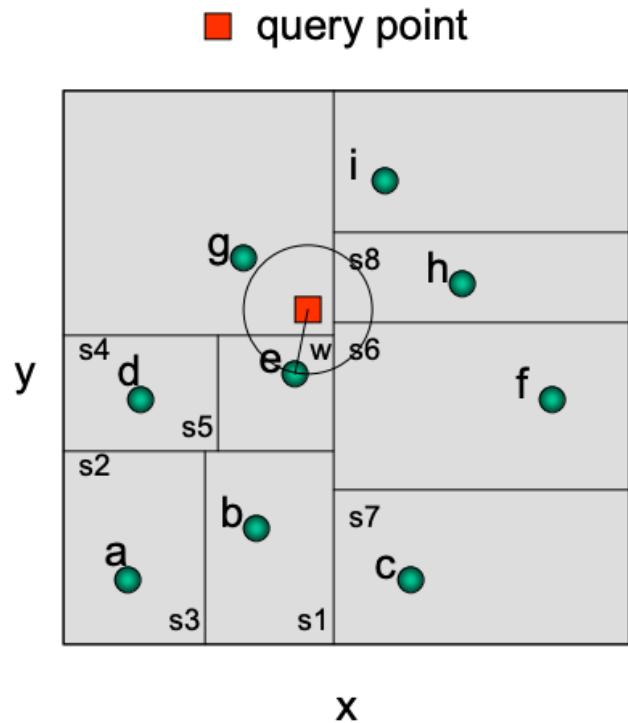


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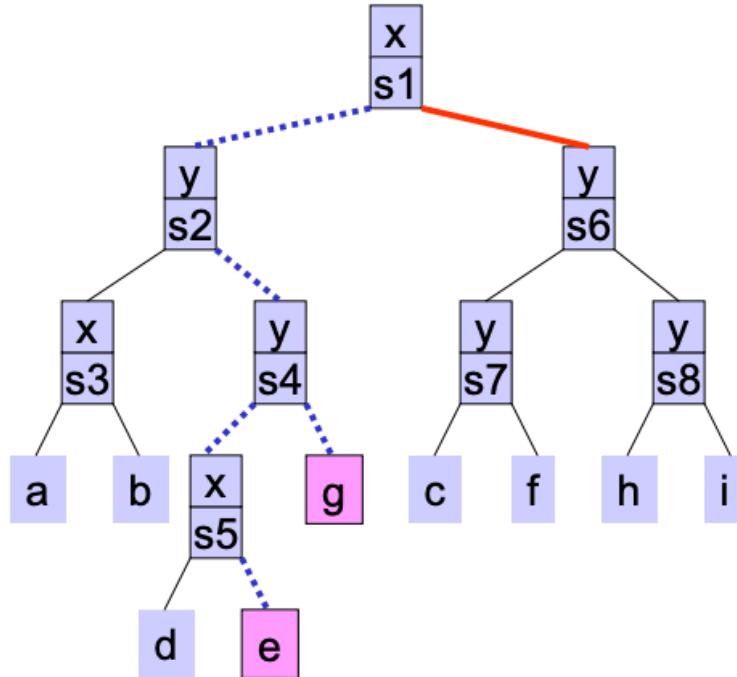
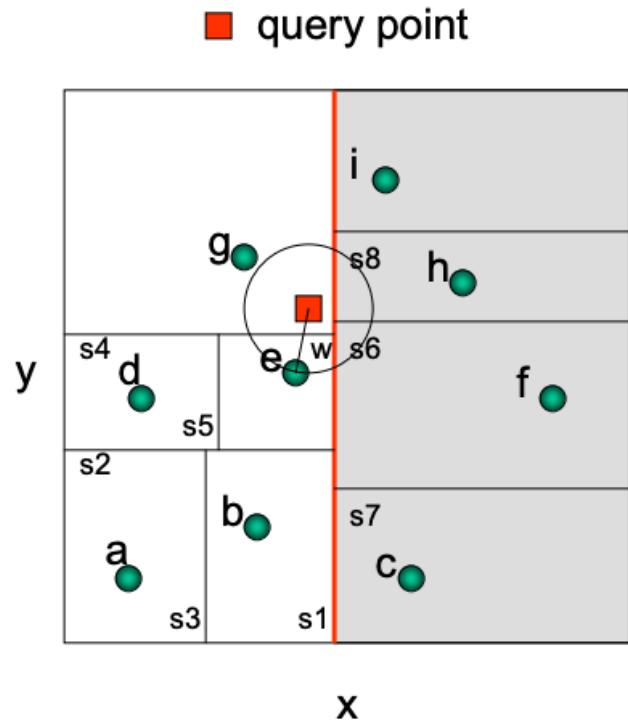


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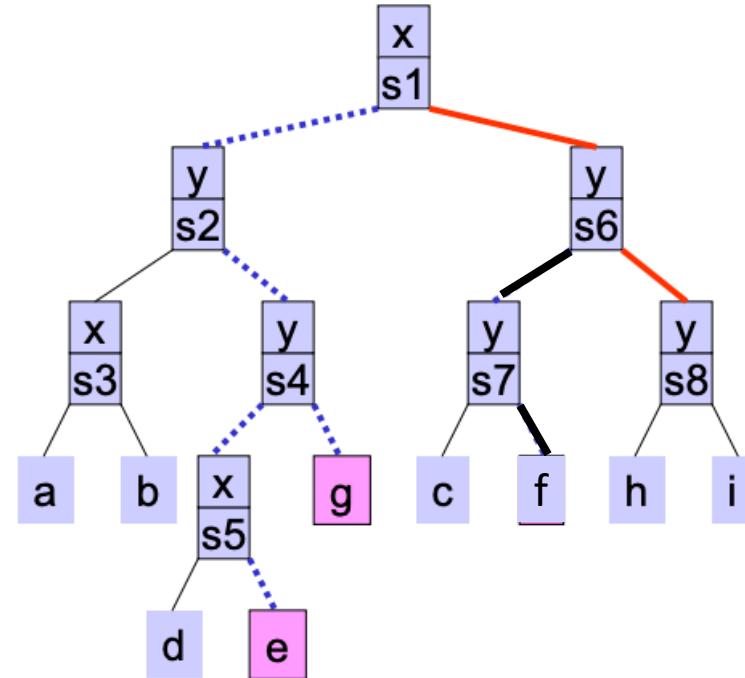
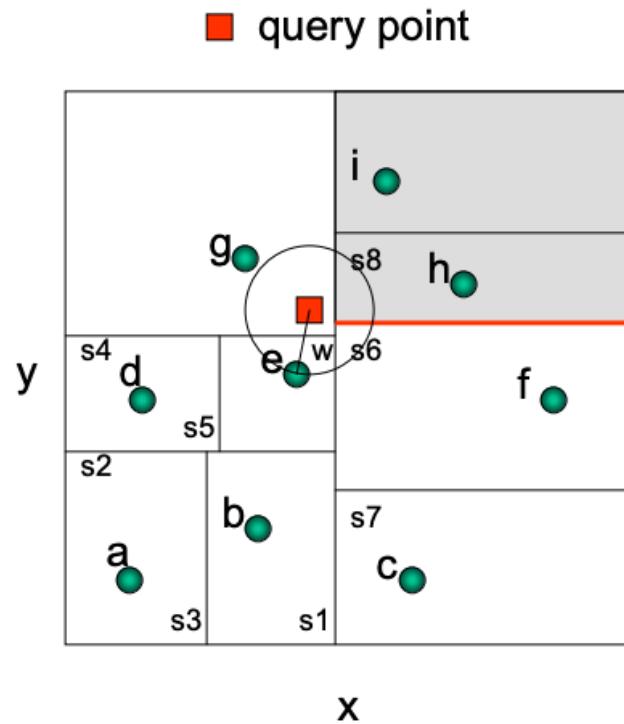


# Kd-tree – kNN Search



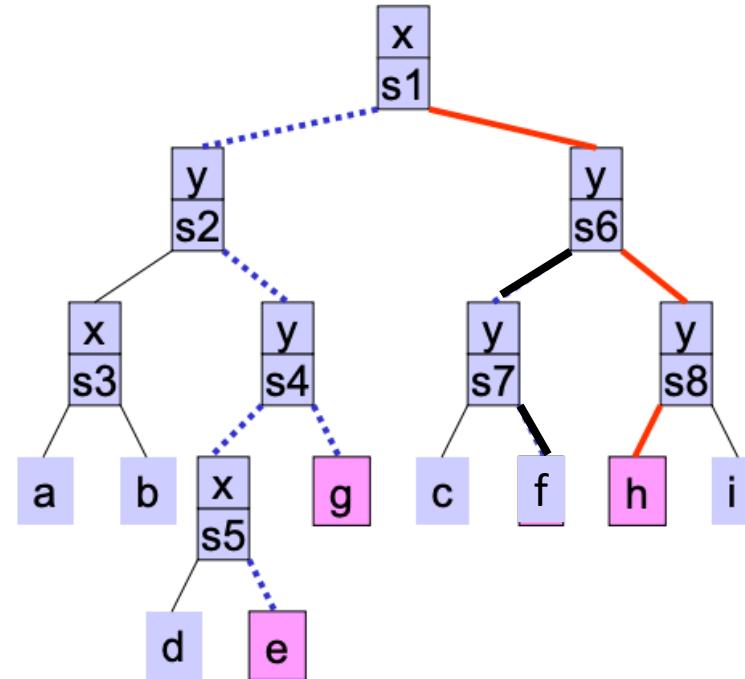
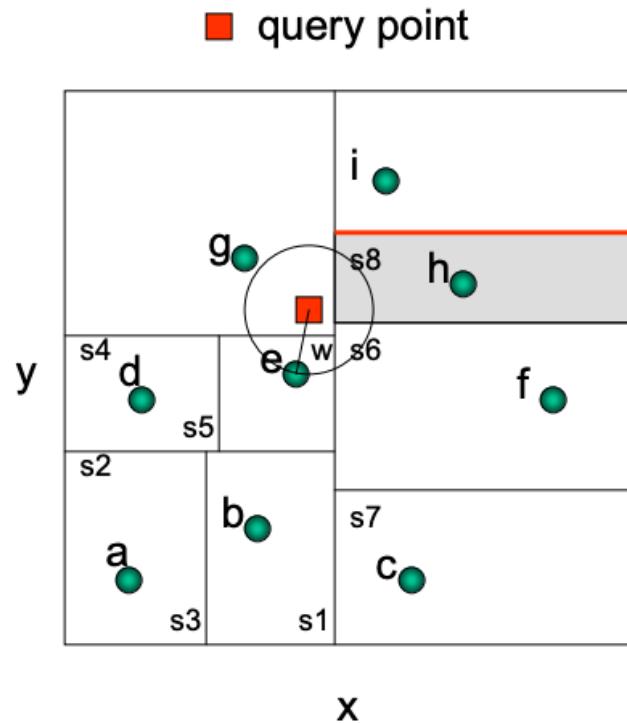


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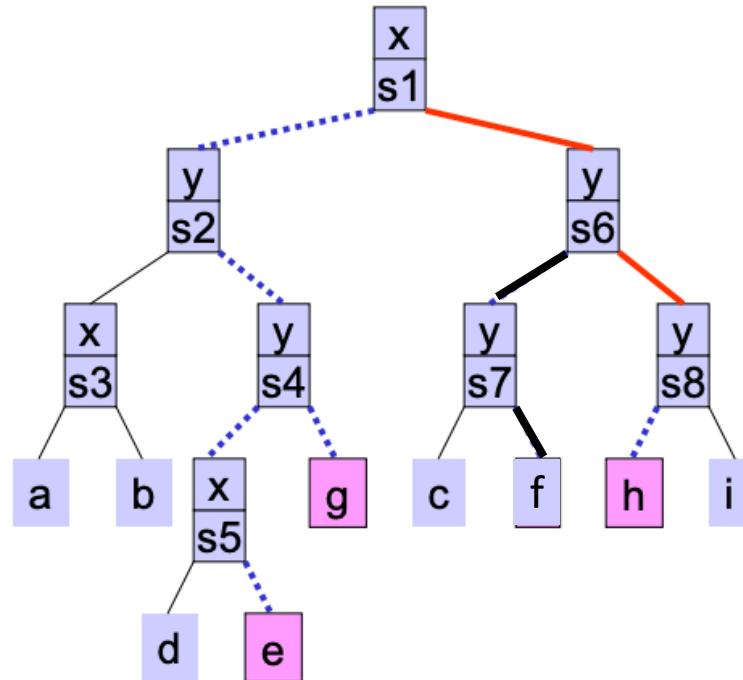
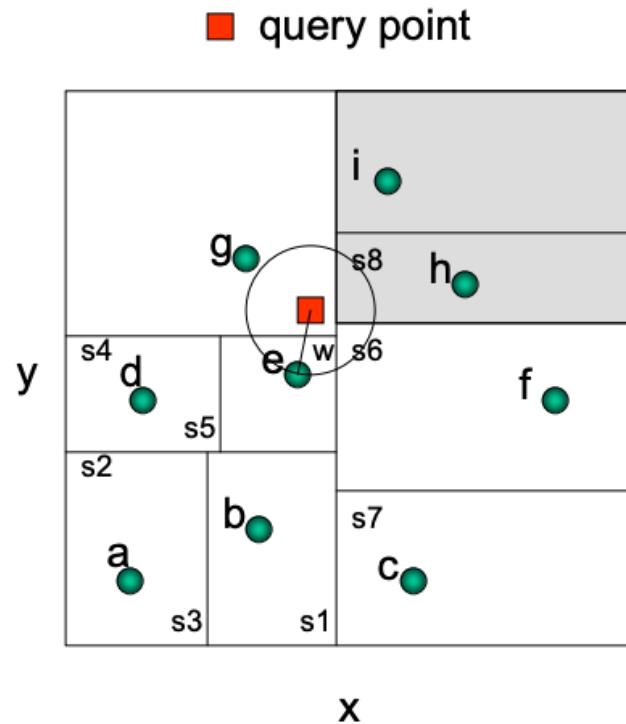


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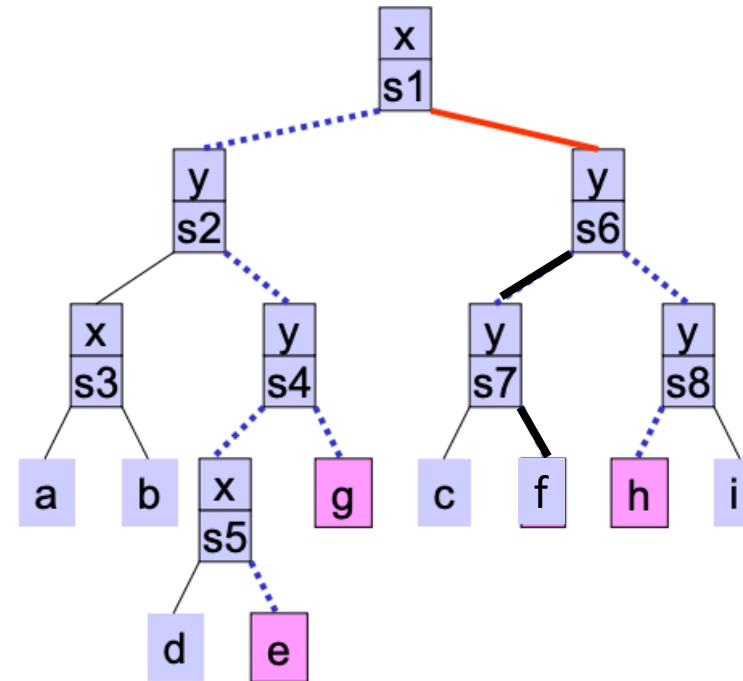
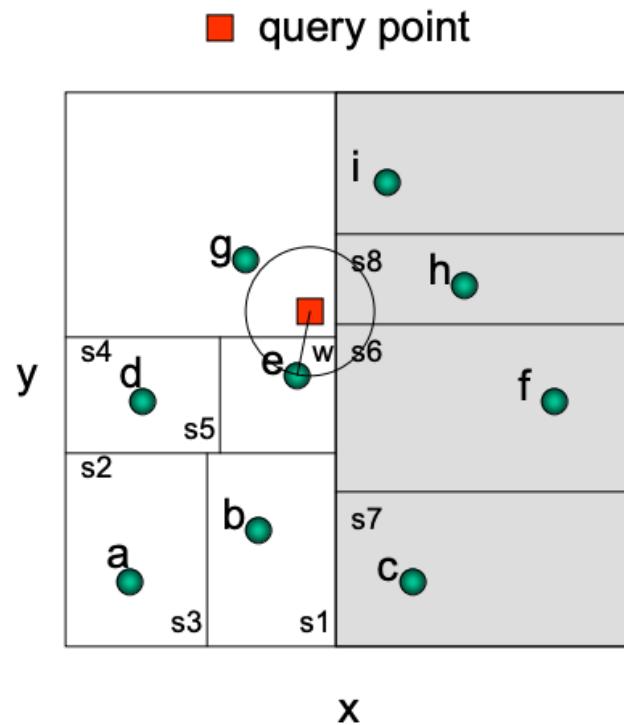


# Kd-tree – kNN Search



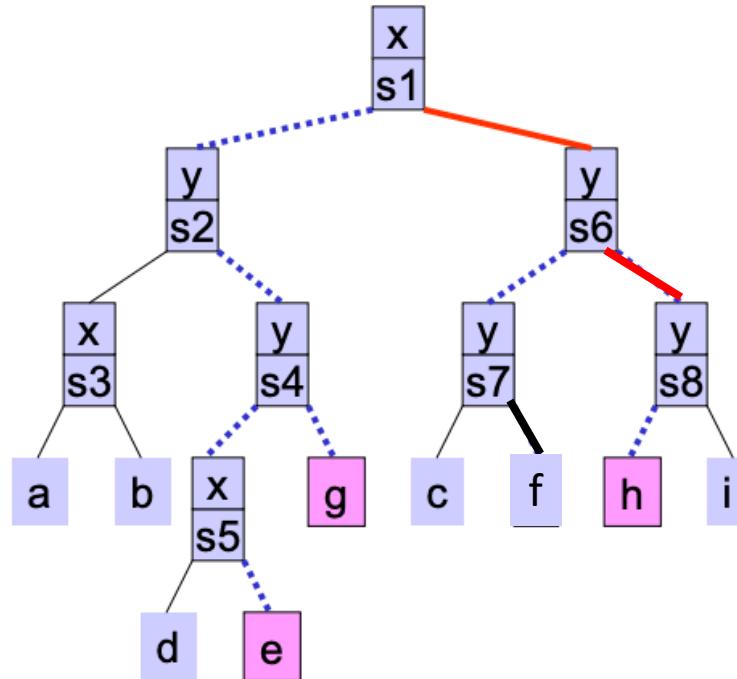
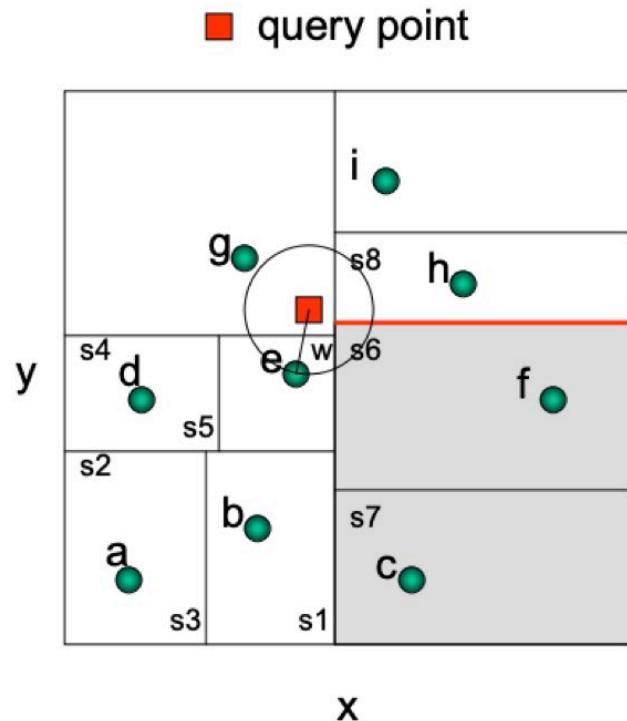


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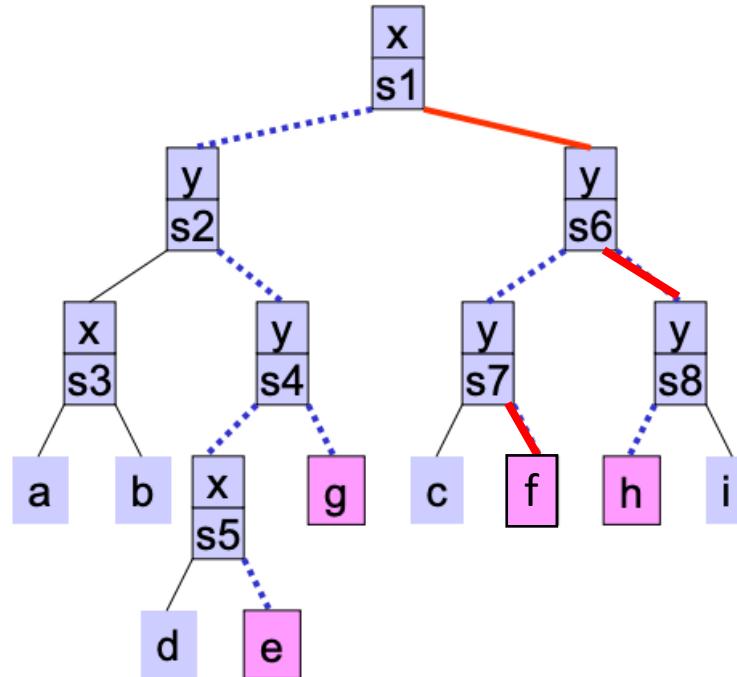
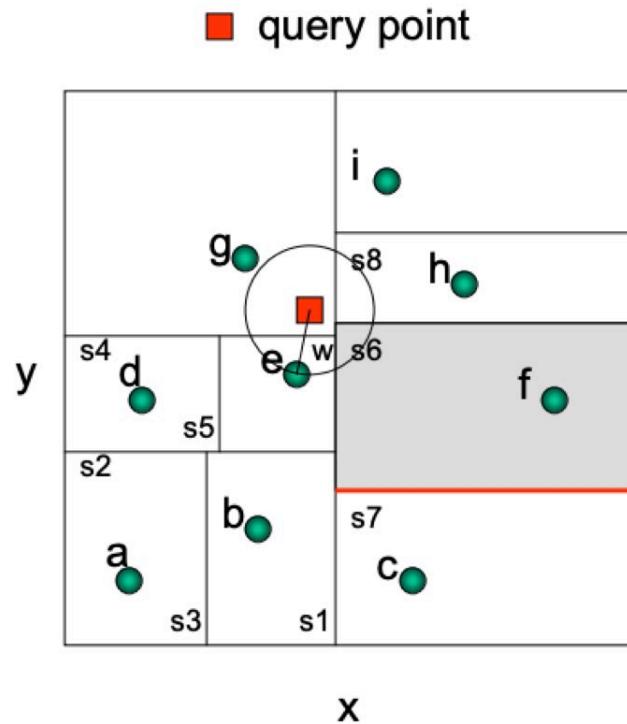


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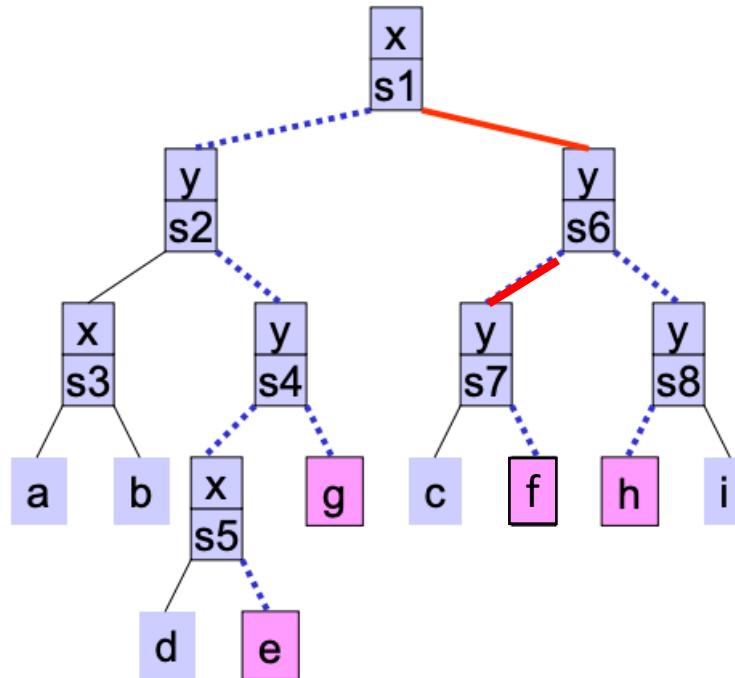
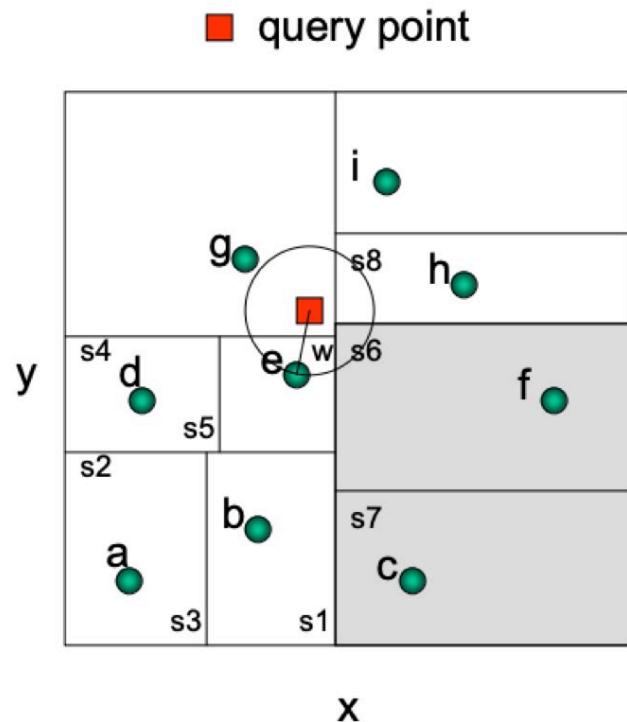


# Kd-tree – kNN Search



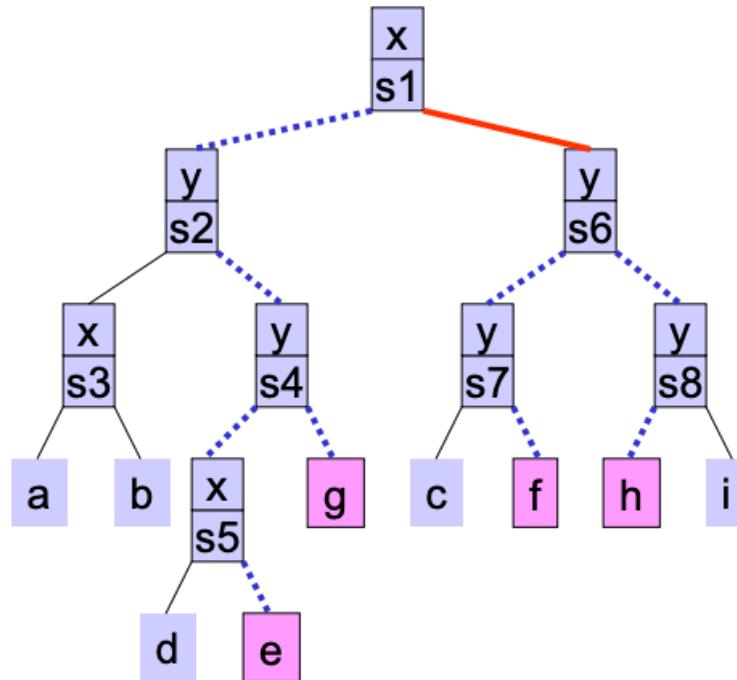
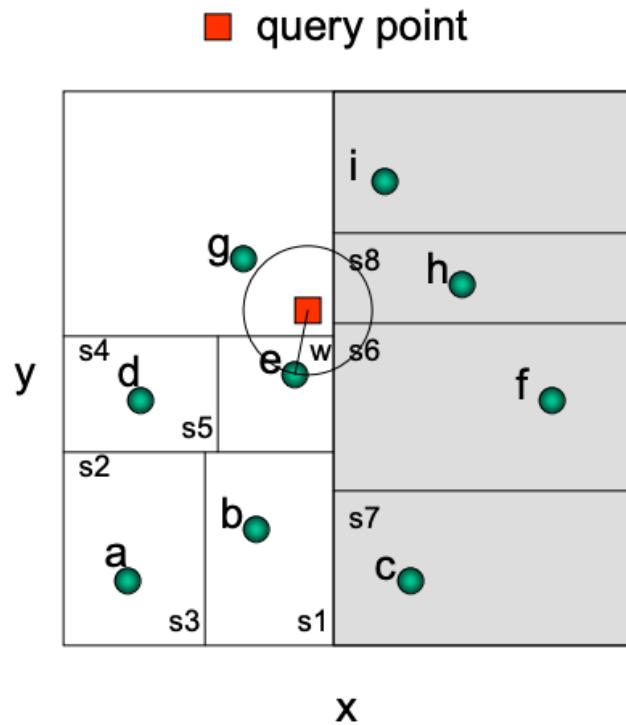


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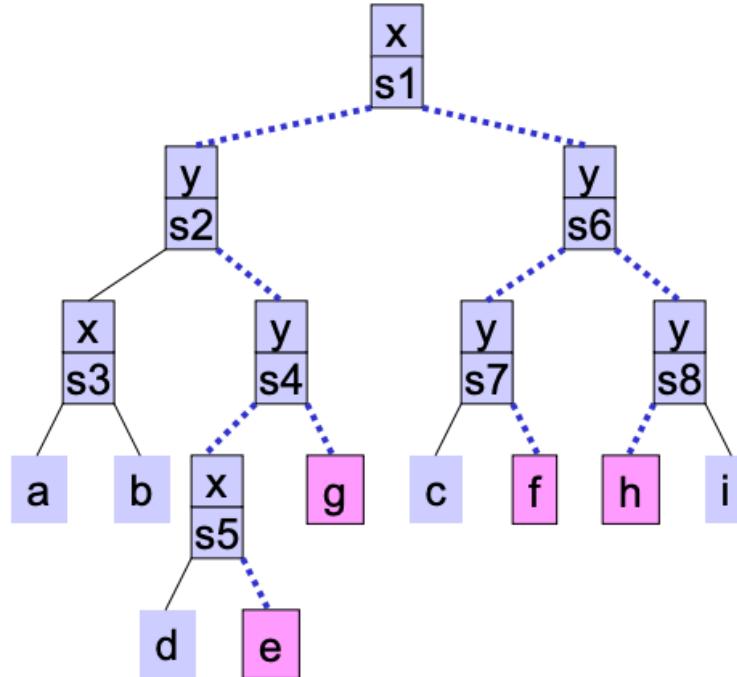
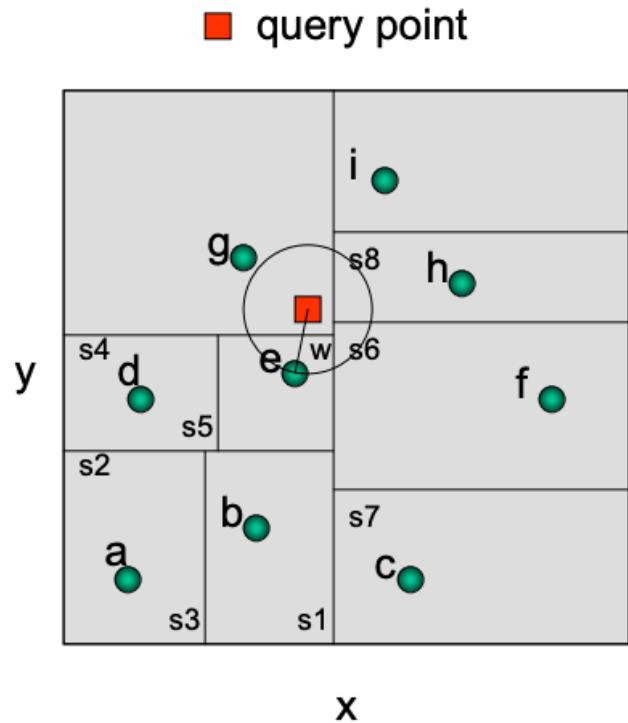


# Kd-tree – kNN Search



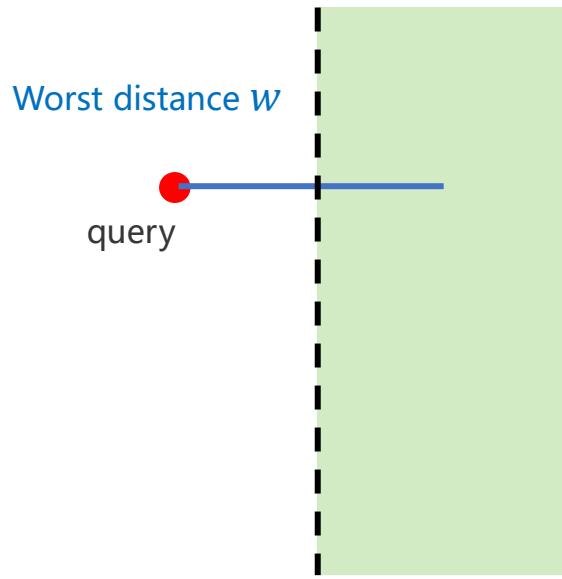


# Kd-tree – kNN Search





# Kd-tree – kNN Search



*splitting\_value*

Criteria of a partition intersects with the worst-distance range:

$q[\text{axis}]$  inside the **partition**

OR

$|q[\text{axis}] - \text{splitting\_value}| < w$

```
def knn_search(root: Node, db: np.ndarray, result_set: KNNResultSet, query: np.ndarray):
    if root is None:
        return False

    if root.is_leaf():          Compare query to every point inside the leaf, put into the result set
        # compare the contents of a leaf
        leaf_points = db[root.point_indices, :]
        diff = np.linalg.norm(np.expand_dims(query, 0) - leaf_points, axis=1)
        for i in range(diff.shape[0]):
            result_set.add_point(diff[i], root.point_indices[i])
        return False

    if query[root.axis] <= root.value:      q[axis] inside the partition
        knn_search(root.left, db, result_set, query)
        if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
            knn_search(root.right, db, result_set, query)

    else:                                |q[axis] – splitting_value| < w
        knn_search(root.right, db, result_set, query)
        if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
            knn_search(root.left, db, result_set, query)

    return False
```



# Kd-tree Radius-NN Search

- Exactly the same as kNN search except:

- Use *RadiusNNResultSet*, similar to BST search
- Fixed worst distance, instead of dynamic

```
if query[root.axis] <= root.value:  
    radius_search(root.left, db, result_set, query)  
    if math.fabs(query[root.axis] - root.value) < result_set.worstDist():  
        [radius_search(root.right, db, result_set, query)]  
else:  
    radius_search(root.right, db, result_set, query)  
    if math.fabs(query[root.axis] - root.value) < result_set.worstDist():  
        [radius_search(root.left, db, result_set, query)]  
  
return False
```



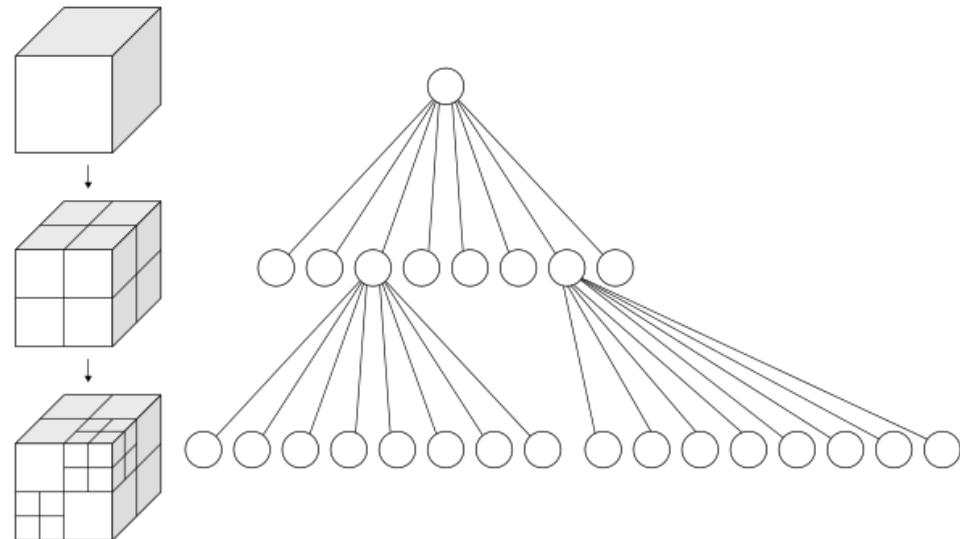
# Kd-tree Search Complexity

- ➊ 1NN search is  $O(\log n)$  for a balanced kd-tree
- ➋ kNN/radiusNN complexity is hard to analyze
  - Depends on the distribution of points
  - Depends on  $k$  or  $r$
  - Varies from  $O(\log n)$  to  $O(n)$



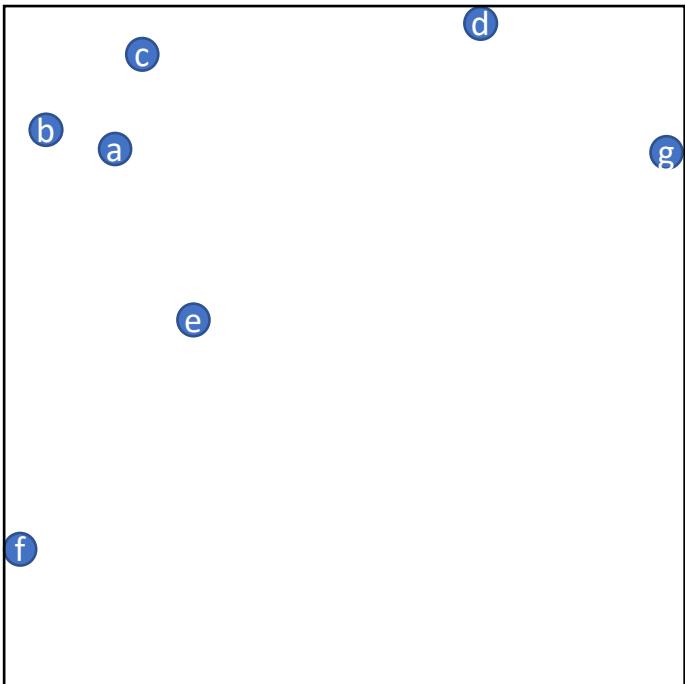
# Octree

- ❖ Each node has 8 children
- ❖ oct – tree
- ❖ Specifically for 3D,  $2^3=8$
- ❖ In kd-tree, it is non-trivial to determine whether the NN search is done, so we have to go back to root every time
- ❖ Octree is more efficient because we can stop without going back to root





# Octree Construction

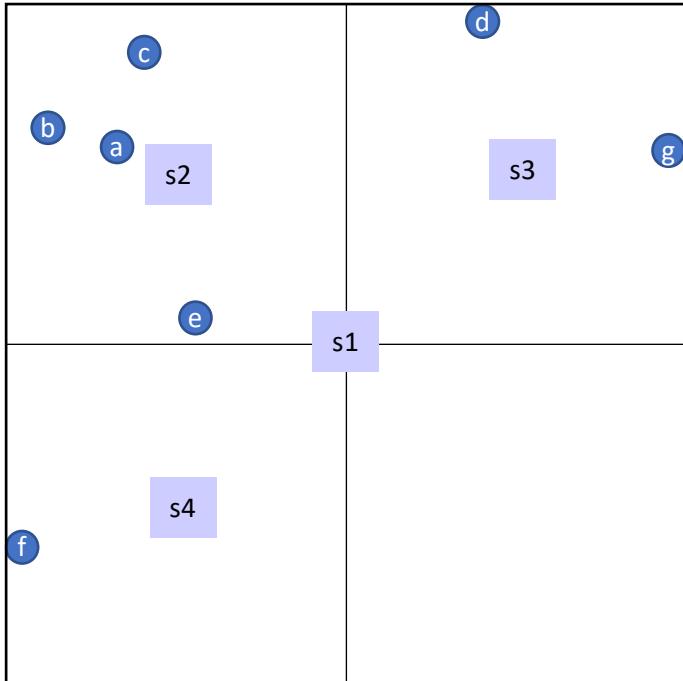


- Determine the extent of the first octant
- Octant is an element in the octree
- Octant is a cube.

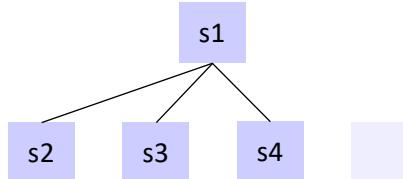
s1



# Octree Construction

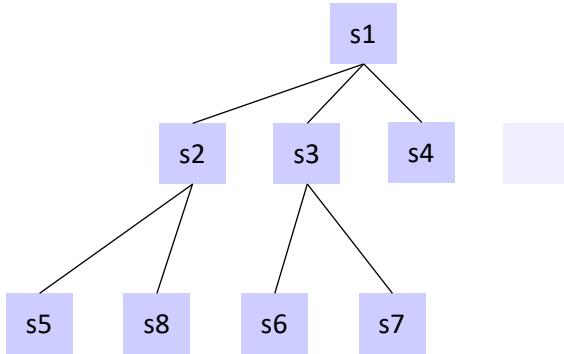
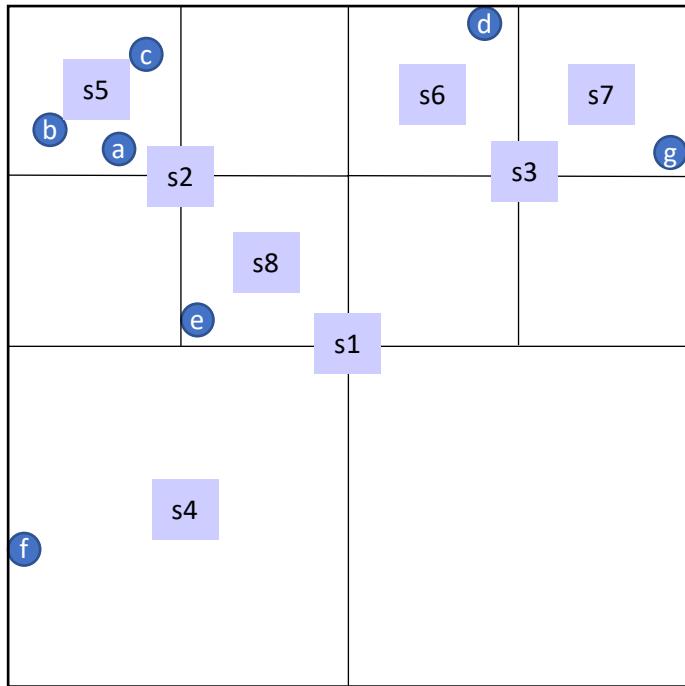


- Determine whether to further split the octant
- `leaf_size` = 1 here
- `min_extent` – avoid infinite splitting when there are repeated points



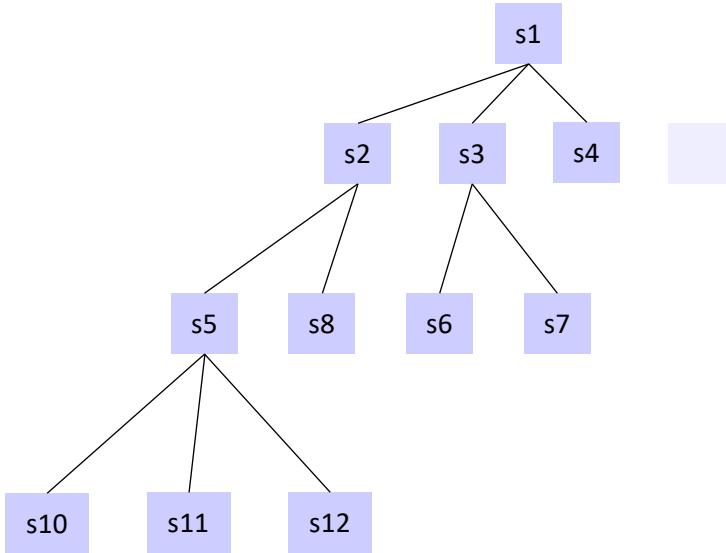
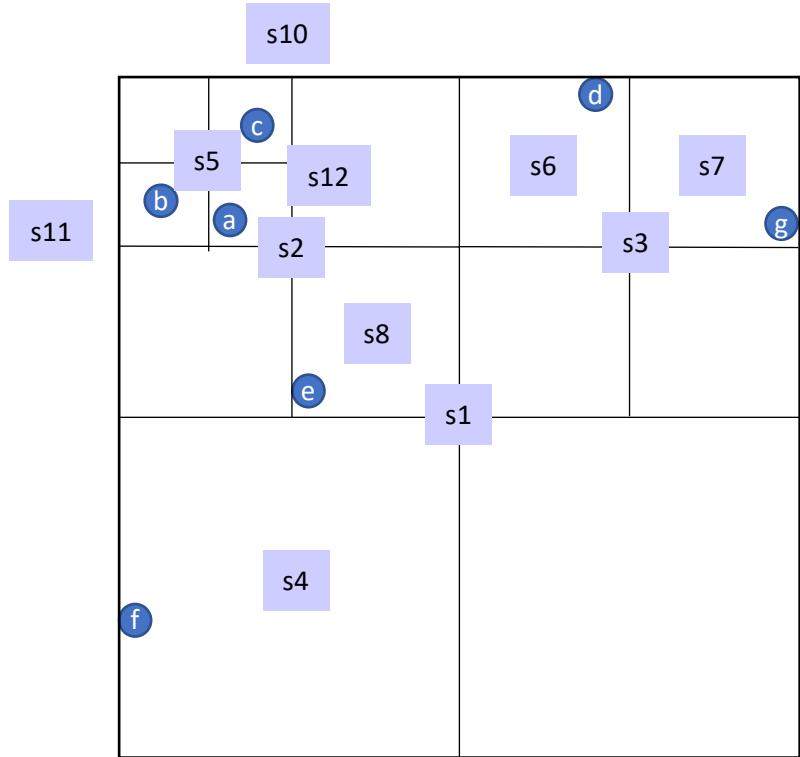


# Octree Construction





# Octree Construction





# Octree Construction

```
class Octant:  
    def __init__(self, children, center, extent, point_indices, is_leaf):  
        self.children = children  
        self.center = center  
        self.extent = extent  
        self.point_indices = point_indices  
        self.is_leaf = is_leaf
```

Annotations pointing to specific variables:

- Array of length 8: points to `point_indices`
- Center of the cube: points to `center`
- Point inside octant: points to `is_leaf`
- 0.5 \* length: points to `extent`

```

def octree_recursive_build(root, db, center, extent, point_indices, leaf_size, min_extent):
    if len(point_indices) == 0:
        return None

    if root is None:
        root = Octant([None for i in range(8)], center, extent, point_indices, is_leaf=True)

    # determine whether to split this octant
    if len(point_indices) <= leaf_size or extent <= min_extent:
        root.is_leaf = True
    else:
        root.is_leaf = False
        children_point_indices = [[] for i in range(8)]
        for point_idx in point_indices:
            point_db = db[point_idx]
            morton_code = 0
            if point_db[0] > center[0]:
                morton_code = morton_code | 1
            if point_db[1] > center[1]:
                morton_code = morton_code | 2
            if point_db[2] > center[2]:
                morton_code = morton_code | 4
            children_point_indices[morton_code].append(point_idx)

    # create children
    factor = [-0.5, 0.5]
    for i in range(8):
        child_center_x = center[0] + factor[(i & 1) > 0] * extent
        child_center_y = center[1] + factor[(i & 2) > 0] * extent
        child_center_z = center[2] + factor[(i & 4) > 0] * extent
        child_extent = 0.5 * extent
        child_center = np.asarray([child_center_x, child_center_y, child_center_z])
        root.children[i] = octree_recursive_build(root.children[i],
                                                db,
                                                child_center,
                                                child_extent,
                                                children_point_indices[i],
                                                leaf_size,
                                                min_extent)

    return root

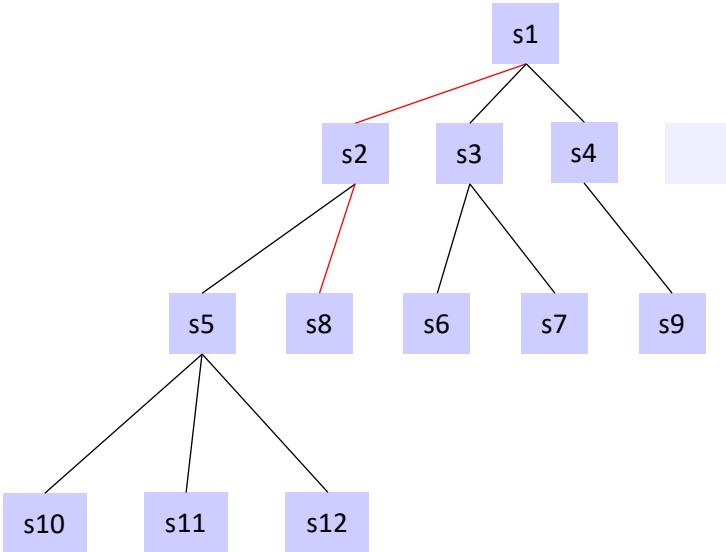
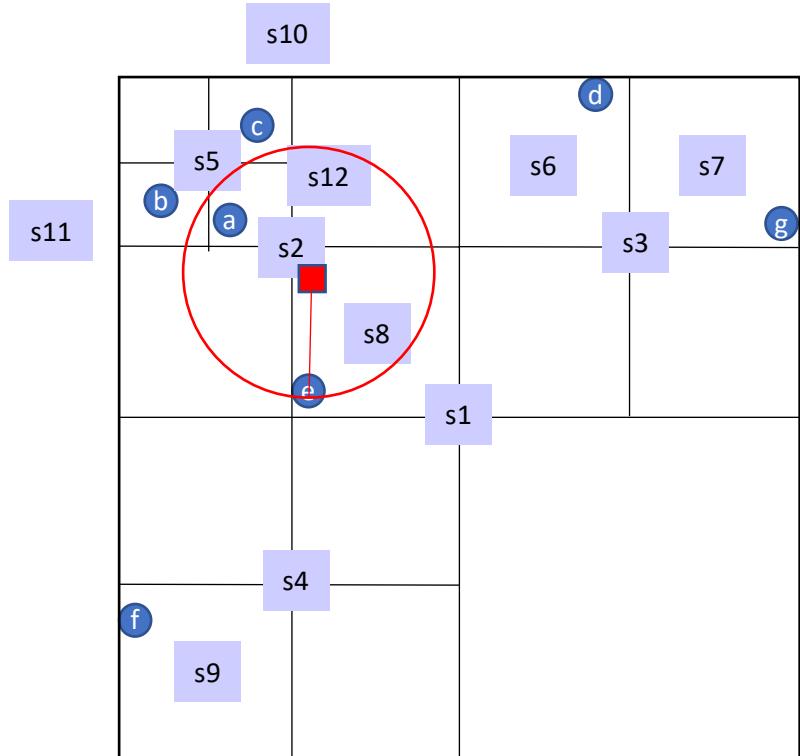
```

Determine which child a point belongs to

Determine child center & extent

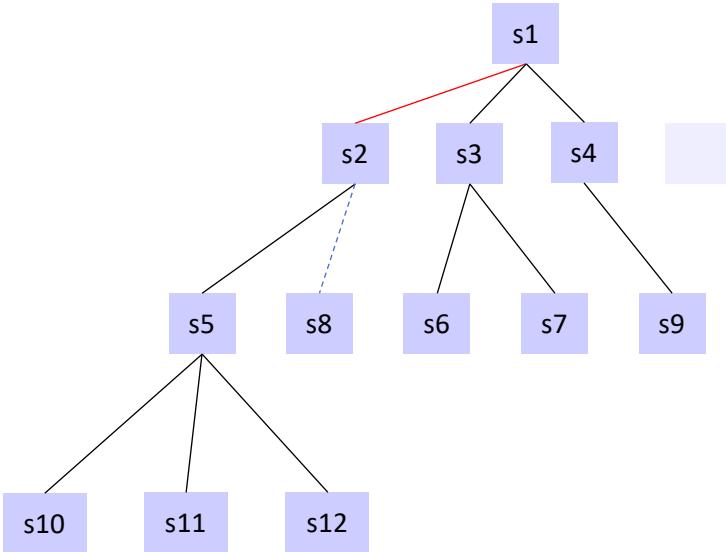
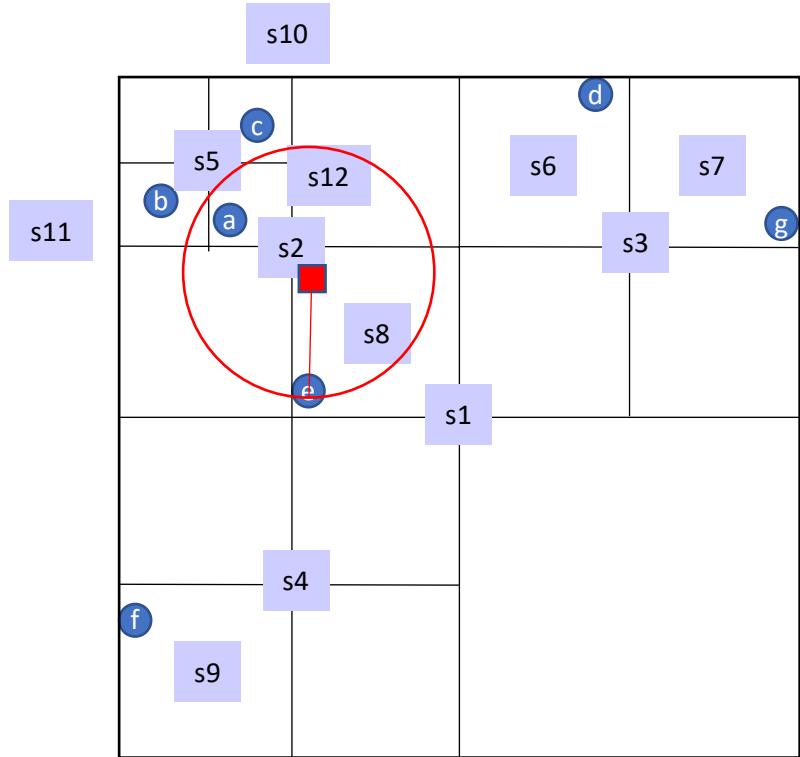


# Octree kNN Search



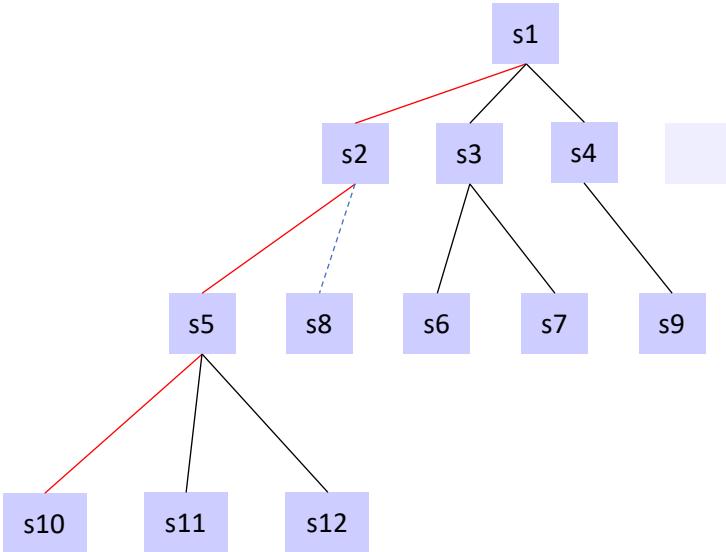
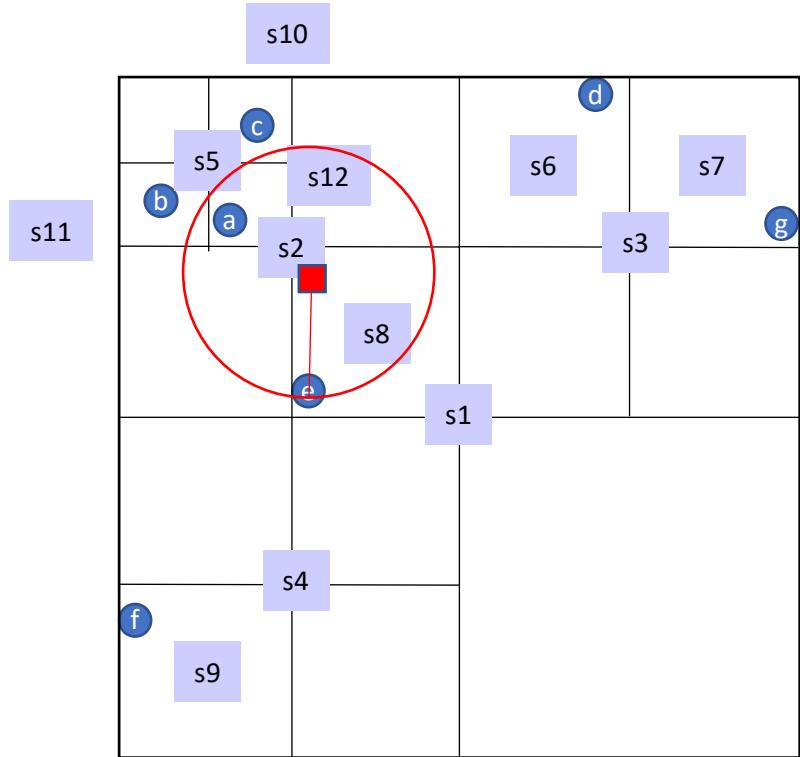


# Octree kNN Search



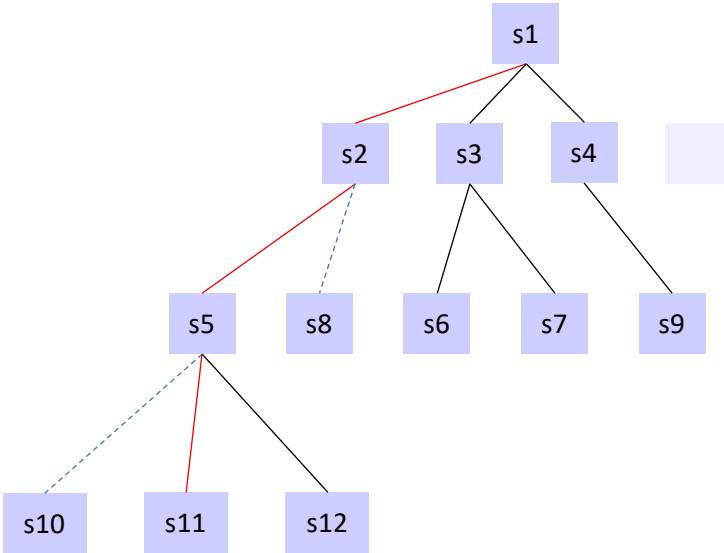
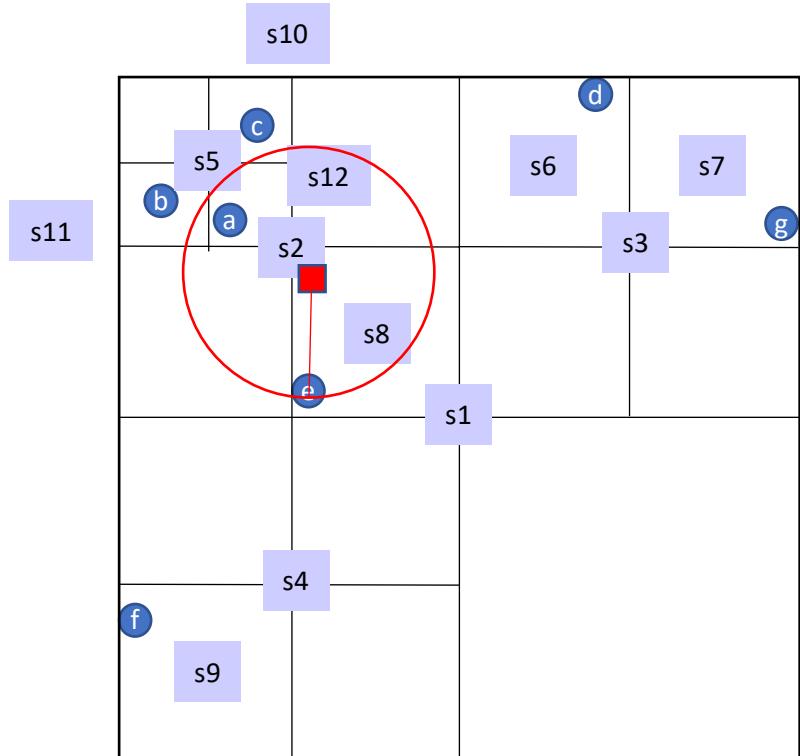


# Octree kNN Search



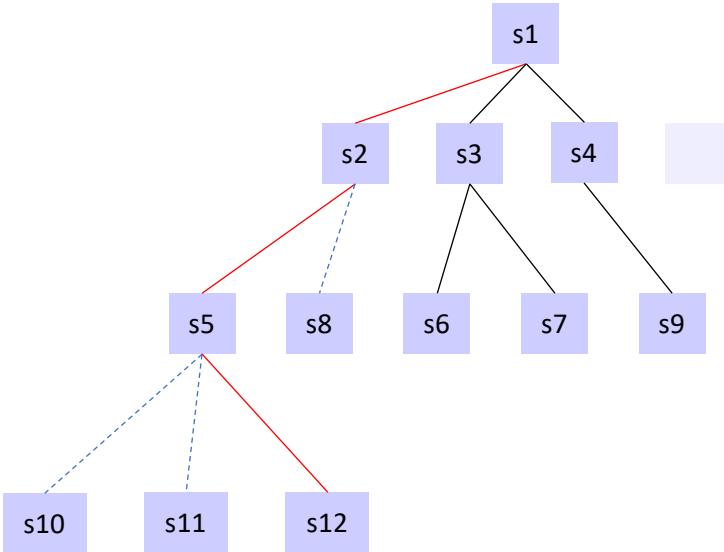
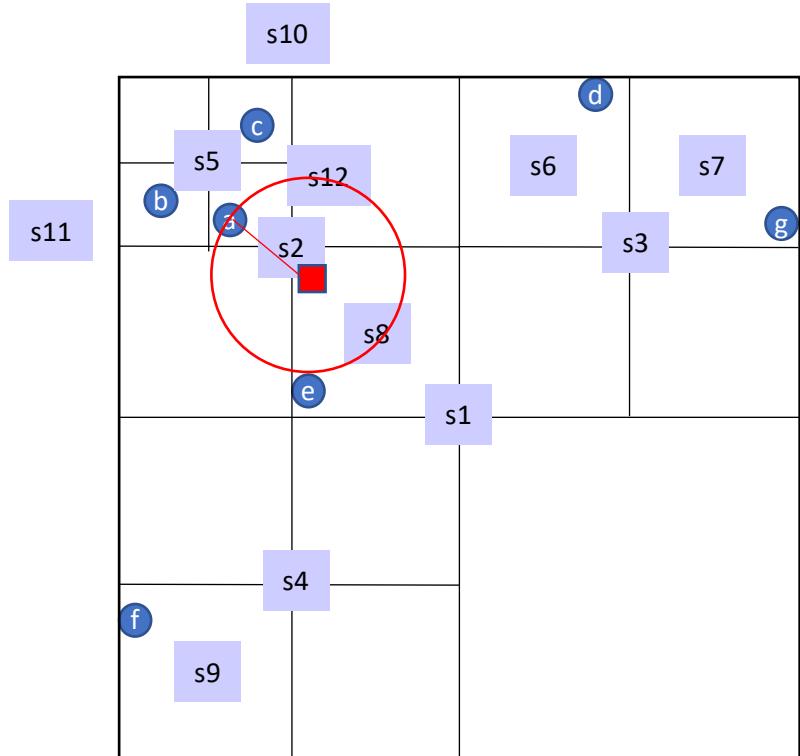


# Octree kNN Search



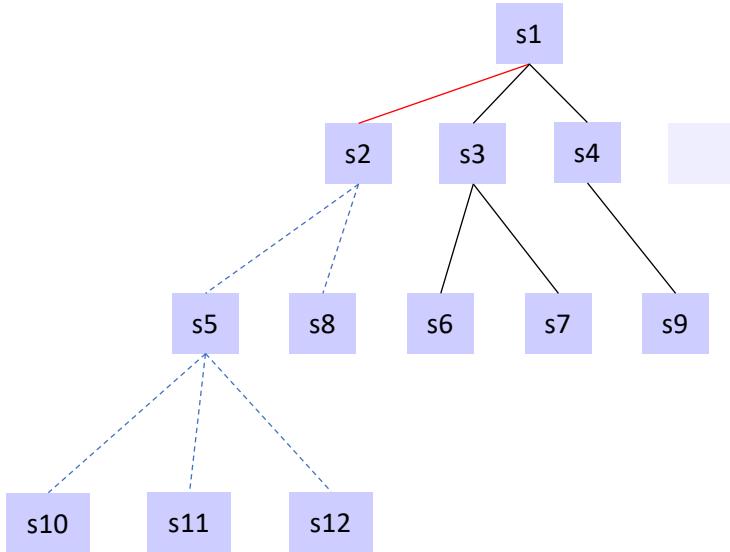
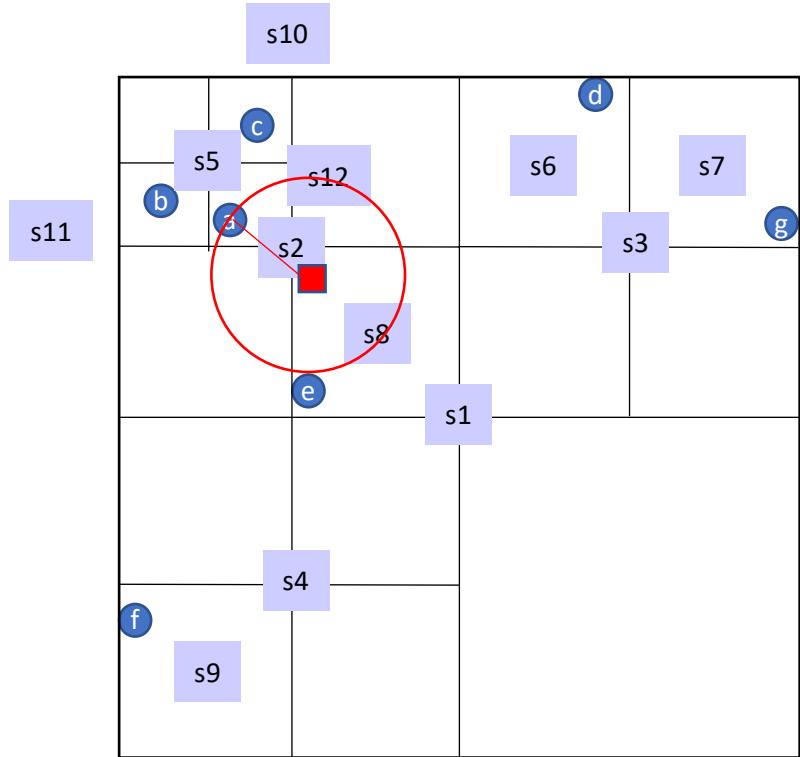


# Octree kNN Search





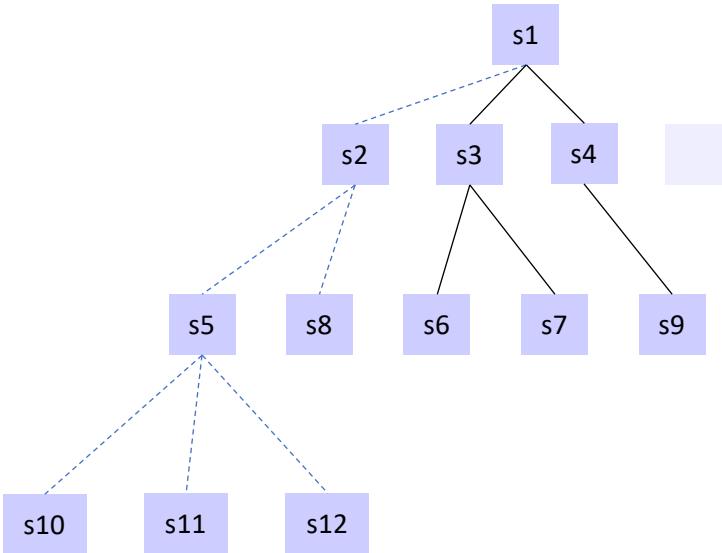
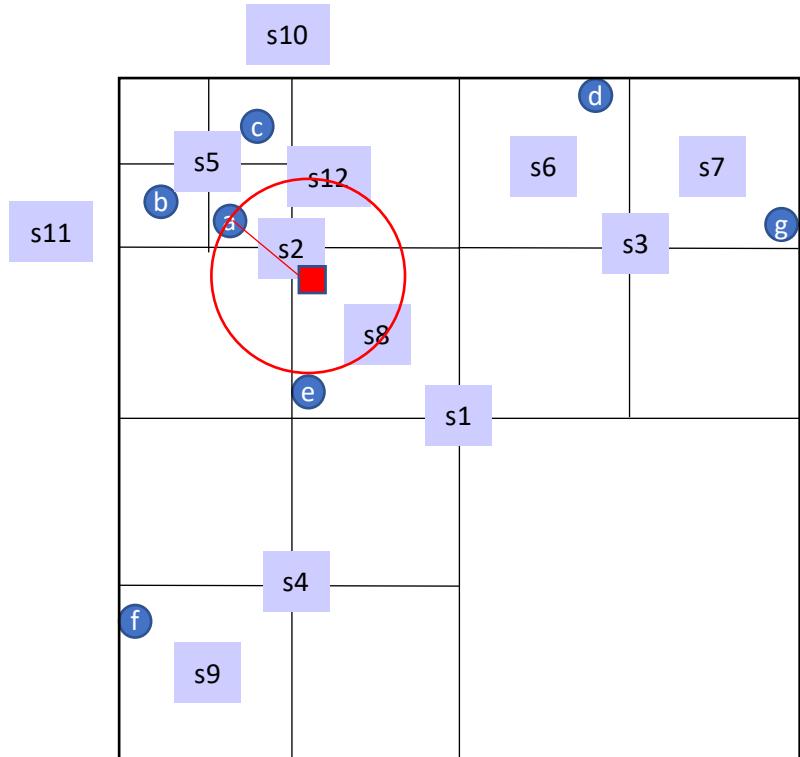
# Octree kNN Search





# Octree kNN Search

Query ball inside s2, search end!



```
def octree_knn_search(root: Octant, db: np.ndarray, result_set: KNNResultSet, query: np.ndarray):
    if root is None:
        return False

    if root.is_leaf and len(root.point_indices) > 0:
        # compare the contents of a leaf
        leaf_points = db[root.point_indices, :]
        diff = np.linalg.norm(np.expand_dims(query, 0) - leaf_points, axis=1)
        for i in range(diff.shape[0]):
            result_set.add_point(diff[i], root.point_indices[i])
        # check whether we can stop search now
        return inside(query, result_set.worstDist(), root)
```

Compare all points in a leaf

```
# go to the relevant child first
morton_code = 0
if query[0] > root.center[0]:
    morton_code = morton_code | 1
if query[1] > root.center[1]:
    morton_code = morton_code | 2
if query[2] > root.center[2]:
    morton_code = morton_code | 4

if octree_knn_search(root.children[morton_code], db, result_set, query):
    return True
```

Determine & search the most relevant child

```
# check other children
for c, child in enumerate(root.children):
    if c == morton_code or child is None:
        continue
    if False == overlaps(query, result_set.worstDist(), child):
        continue
    if octree_knn_search(child, db, result_set, query):
        return True
# final check of if we can stop search
return inside(query, result_set.worstDist(), root)
```

If an octant is not overlapping with query ball, skip

If query ball is inside an octant, stop



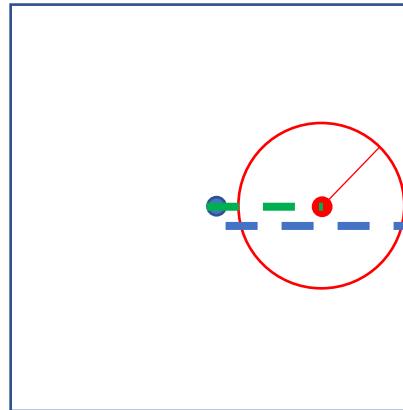
## Function *inside*

```
def inside(query: np.ndarray, radius: float, octant:Octant):
    """
    Determines if the query ball is inside the octant
    :param query:
    :param radius:
    :param octant:
    :return:
    """
    query_offset = query - octant.center
    query_offset_abs = np.fabs(query_offset)
    possible_space = query_offset_abs + radius
    return np.all(possible_space < octant.extent)
```

Green dash line

Blue dash line

Red line





# Function *overlaps*

```
def overlaps(query: np.ndarray, radius: float, octant:Octant):
    """
    Determines if the query ball overlaps with the octant
    :param query:
    :param radius:
    :param octant:
    :return:
    """
    query_offset = query - octant.center
    query_offset_abs = np.fabs(query_offset)
```

```
# completely outside, since query is outside the relevant area
max_dist = radius + octant.extent
if np.any(query_offset_abs > max_dist):
    return False
```

Case 1

```
# if pass the above check, consider the case that the ball is contacting the face of the octant
if np.sum((query_offset_abs < octant.extent).astype(np.int)) >= 2:
    return True
```

Case 2

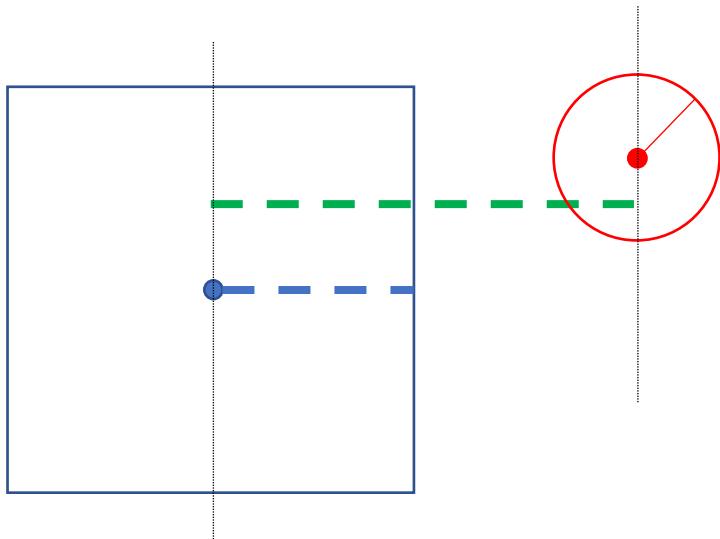
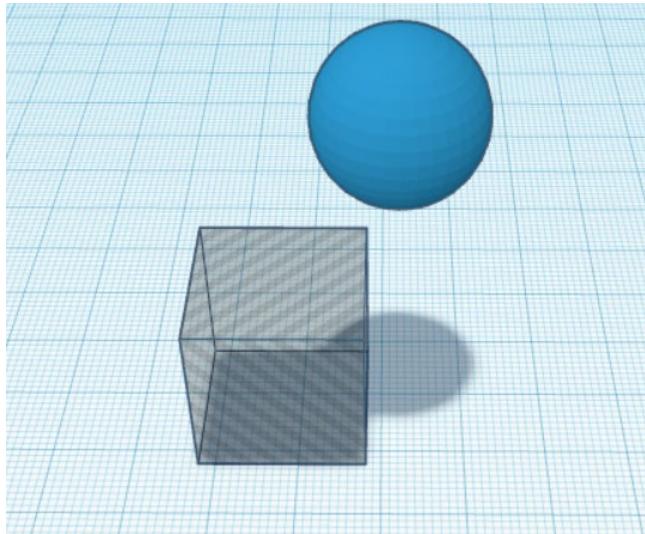
```
# consider the case that the ball is contacting the edge or corner of the octant
# since the case of the ball center (query) inside octant has been considered,
# we only consider the ball center (query) outside octant
x_diff = max(query_offset_abs[0] - octant.extent, 0)
y_diff = max(query_offset_abs[1] - octant.extent, 0)
z_diff = max(query_offset_abs[2] - octant.extent, 0)

return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```

Case 3



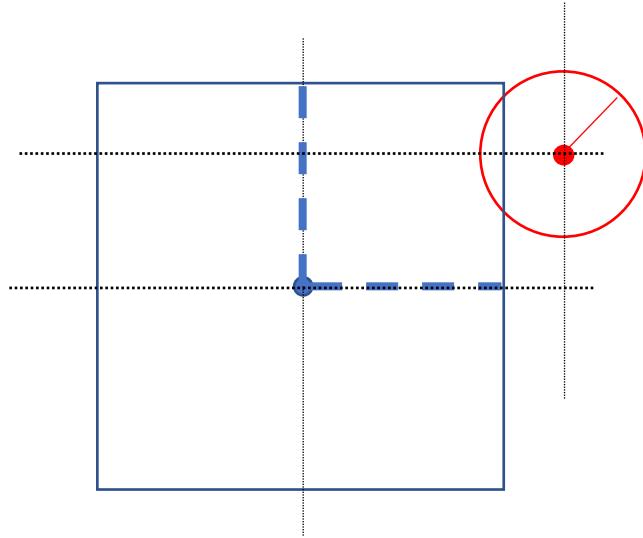
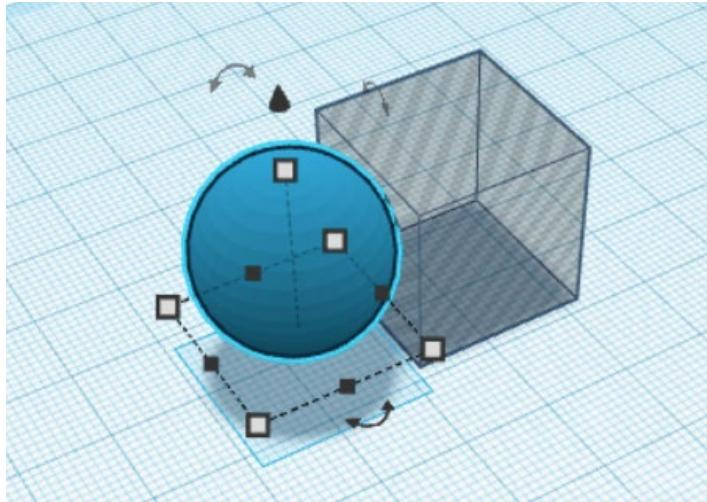
## Function *overlaps* – Case 1



```
# completely outside, since query is outside the relevant area
max_dist = radius + octant.extent
if np.any(query_offset_abs > max_dist):
    return False
```



## Function *overlaps* – Case 2

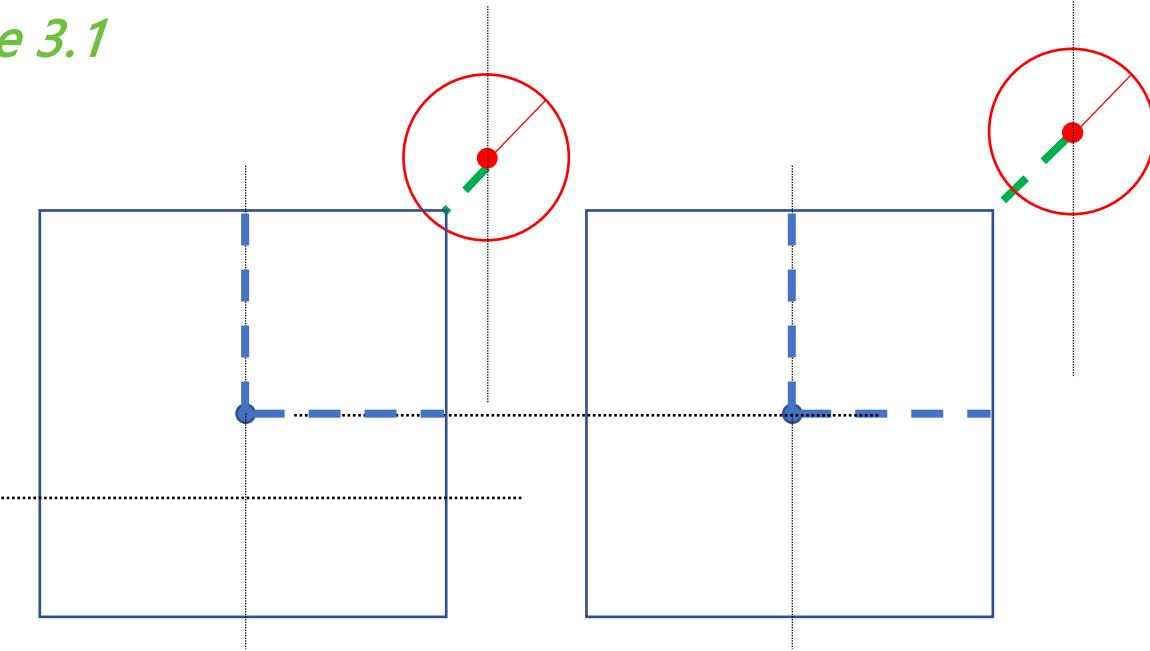
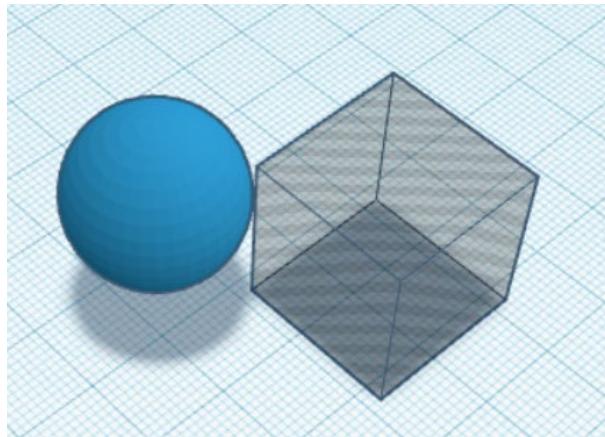


Check if the ball is contacting the face of the octant

```
if np.sum((query_offset_abs < octant.extent).astype(np.int)) >= 2:  
    return True
```



## Function *overlaps* – Case 3.1



```
# consider the case that the ball is contacting the edge or corner of the octant
# since the case of the ball center (query) inside octant has been considered,
# we only consider the ball center (query) outside octant
x_diff = max(query_offset_abs[0] - octant.extent, 0)
y_diff = max(query_offset_abs[1] - octant.extent, 0)
z_diff = max(query_offset_abs[2] - octant.extent, 0)

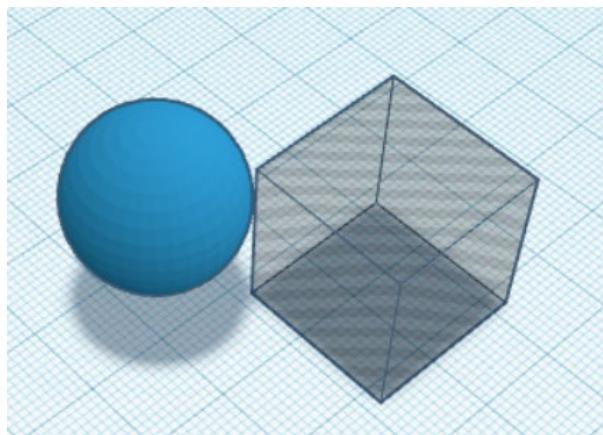
return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```



## Function *overlaps* – Case 3.2

In 3D, there is the case that the cube's edge cut into the query ball

```
| # consider the case that the ball is contacting the edge or corner of the octant
| # since the case of the ball center (query) inside octant has been considered,
| # we only consider the ball center (query) outside octant
| x_diff = max(query_offset_abs[0] - octant.extent, 0)
| y_diff = max(query_offset_abs[1] - octant.extent, 0)
| z_diff = max(query_offset_abs[2] - octant.extent, 0)
|
| return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```



That's why there is a “*max*” to reduce this case into 3.1



## Octree Radius NN Search

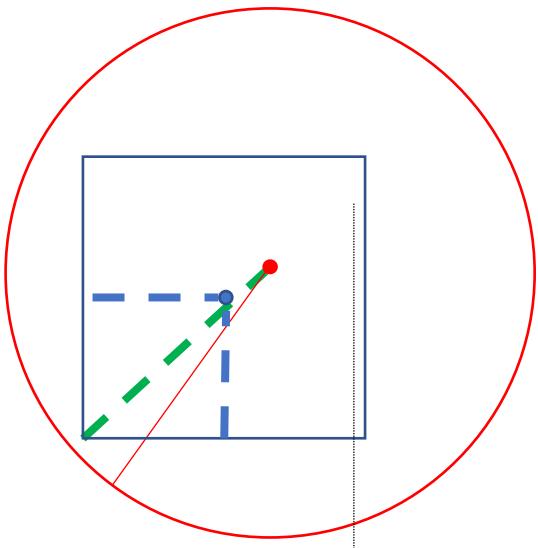
Simple one: replace KNNResultSet with RadiusNNResultSet

Better one:

- If the query ball *contains* the octant, just compare the query with all point
- No need to go into children of that octant



## Function *contains*



```
def contains(query: np.ndarray, radius: float, octant:Octant):
    """
    Determine if the query ball contains the octant
    :param query:
    :param radius:
    :param octant:
    :return:
    """

    query_offset = query - octant.center
    query_offset_abs = np.fabs(query_offset)

    query_offset_to_farthest_corner = query_offset_abs + octant.extent
    return np.linalg.norm(query_offset_to_farthest_corner) < radius
```

Green dash line

Red line



# Octree Search Complexity

- ➊ 1NN search is  $O(\log n)$
  
- ➋ kNN/radiusNN complexity is hard to analyze
  - Depends on the distribution of points
  - Depends on  $k$  or  $r$
  - Varies from  $O(\log n)$  to  $O(n)$

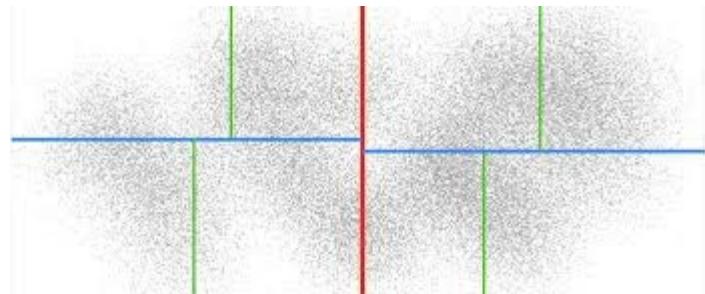
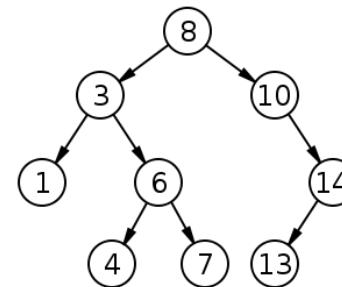


# BST / Kd-tree / Octree



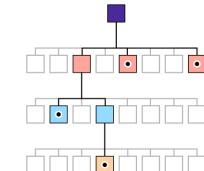
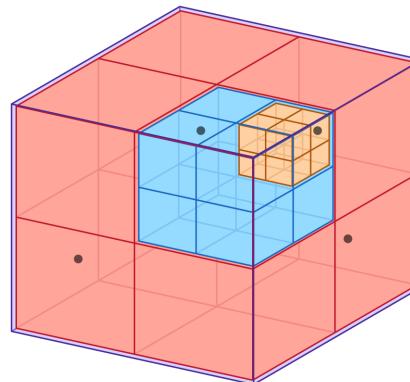
## Dimension

- BST for one dimension
- Kd-tree works for any dimension
- Octree is optimized for 3D



## Idea

- Same – space partition





## Summary

-  Space partition
-  Find a method to skip some partitions
-  Pythons codes:  
<https://github.com/lijx10/NN-Trees>



# Homework

- We provide one  $N \times 3$  point cloud
- 8-NN search for each point to the point cloud
- Implement 3 NN algorithms
  1. Numpy brute-force search
  2. `scipy.spatial.KDTree`
  3. Your own kd-tree/octree in python or C++
- Report timing using method 1 as baseline
- This is a competition!
  - Timing of method 3 determine your grade