1. $\{Z,+\}$ 是否为群？

1. 封闭性，$\forall a_1,a_2 \in Z$，$a_1+a_2 \in Z$

2. 结合律，$\forall a_1,a_2,a_3 \in Z$，$(a_1+a_2)+a_3 = a_1+(a_2+a_3)$

3. 幺元，$0 \in Z$，$\forall a \in Z$，$0+a = a+0 = a$

4. 逆，$\forall a \in Z$，$\exists a^{-1} = -a \in Z$，$a-a=0 \in Z$

2. $\{N,+\}$ 为群，原因同上

3. 验证向量叉乘的李代数性质

1. 封闭性 $\forall X,Y \in V$，$X \times Y \in V$

$\because$ $V=R^3$, $F=R$

$$X \times Y = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -x_3y_2+x_2y_3 \\ x_3y_1-x_1y_3 \\ -x_2y_1+x_1y_2 \end{bmatrix}$$

满足封闭性

2. 双线性：$(aX+bY)\times Z = a(X\times Z)+b(Y\times Z)$

(根据矩阵叉乘性质)

$[Z, aX+bY] = a[Z,X]+b[Z,Y]$

3. 自反性：$\forall X \in V$

$$\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3x_2+x_2x_3 \\ x_3x_1-x_1x_3 \\ -x_2x_1+x_1x_2 \end{bmatrix} = 0$$

4. 雅可比等价：$[\vec{X},[\vec{Y},\vec{Z}]]+[\vec{Y},[\vec{Z},\vec{X}]]+[\vec{Z},[\vec{X},\vec{Y}]]$

$= \vec{X}\times\vec{Y}\times\vec{Z}+\vec{Y}\times\vec{Z}\times\vec{X}+\vec{Z}\times\vec{X}\times\vec{Y}$

$$\vec{X}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{Y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \vec{Z}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

4. 推导SE(3)的指数映射

设 $\xi = [\rho\ \phi]^T \in se(3)$ 它的指数映射为

$$\exp(\xi^\wedge) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}(\phi^\wedge)^n & \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi^\wedge)^n \rho \\ 0^T & 1 \end{bmatrix}$$

$\because \exp(\xi^\wedge) = \exp\left(\begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix}\right)$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix}^n$$

$$\begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix}^2 = \begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix}\begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix} = \begin{bmatrix} \theta^2 a^\wedge a^\wedge & \theta \rho a^\wedge \\ 0^T & 0 \end{bmatrix}$$

$$\begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix}^3 = \begin{bmatrix} \theta^2 a^\wedge a^\wedge & \theta \rho a^\wedge \\ 0^T & 0 \end{bmatrix}\begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix} = \begin{bmatrix} \theta^3 a^\wedge a^\wedge a^\wedge & \theta^2 \rho a^\wedge a^\wedge \\ 0^T & 0 \end{bmatrix}$$

$\vdots$

$$\begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix}^n = \begin{bmatrix} \theta^n (a^\wedge)^n & \theta^{n-1} \rho (a^\wedge)^{n-1} \\ 0^T & 0 \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \theta a^\wedge & \rho \\ 0^T & 0 \end{bmatrix}^n = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}(\theta a^\wedge)^n & \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \rho (\theta a^\wedge)^n \\ 0^T & 0 \end{bmatrix}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}(\theta a^\wedge)^n = I + \theta a^\wedge + \frac{1}{2!}\theta^2 a^\wedge a^\wedge + \frac{1}{3!}\theta^3 a^\wedge a^\wedge a^\wedge \cdots$$

$$= a^\wedge a^\wedge + I + \sin\theta a^\wedge - \cos\theta a^\wedge a^\wedge$$

$$= (1 - \cos\theta) a^\wedge a^\wedge + I + \sin\theta a^\wedge$$

$$= \cos\theta I + (1-\cos\theta) a a^T + \sin\theta a^\wedge$$

易知 这就是~~旋转矩阵的表达~~ $\triangleq R$

且题目已经提供 $\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\rho(\theta a^\wedge)^n \triangleq J\rho$

$\therefore\ \exp(\xi^\wedge) = \begin{bmatrix} R & J\rho \\ 0^T & 1 \end{bmatrix} = T$

5. ① Prove: $R p^\wedge R^T = (Rp)^\wedge$

$\Rightarrow\ Rp^\wedge = (Rp)^\wedge R$

$\Rightarrow \forall\, u \in \mathbb{R}^3,\ Rp^\wedge u = (Rp)^\wedge R u$

$\Rightarrow\ R(p \times u) = (Rp) \times (Ru)$

(向量叉乘旋转不变性

①* 思路二

$\forall v \in \mathbb{R}^3\quad (Ra)^\wedge v = (Ra) \times v = (Ra) \times (RR^{-1}v)$

$= R[a \times (R^{-1}v)]$

$= Ra^\wedge R^{-1} v$

② Prove $R\exp(p^\wedge)R^T = \exp((Rp)^\wedge)$

$\exp((Rp)^\wedge) = \exp(Ra^\wedge R^T)$

$= \sum_{n=0}^{\infty}\frac{1}{n!}(Ra^\wedge R^T)^n$

$= \sum_{n=0}^{\infty}\frac{1}{n!}R(a^\wedge)^n R^T$

$= R\sum_{n=0}^{\infty}\frac{1}{n!}(a^\wedge)^n R^T$

$= R\exp(p^\wedge)R^T$