Prove: $y = u_4$ 是该问题最优解

$y' = u_4 + v$

问题描述: $\min_y \|Dy\|_2^2$, s.t. $\|y\| = 1$

$$D^TD = \sum_{i=1}^{4} \sigma_i^2 u_i u_i^T$$

$$D^TD = [u_1, u_2, u_3, u_4] \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ & & & \sigma_4^2 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \\ u_4^T \end{bmatrix}$$

$$u_4^T D^T D u_4 = u_4^T [u_1, u_2, u_3, u_4] \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ & & & \sigma_4^2 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \\ u_4^T \end{bmatrix} u_4$$

$$= \sigma_4^2$$

Let $u' = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4$ $\quad \|u'\| = \|u_4\| = 1$

$$u'^T D^T D u' = \sum_{i=1}^{4} \alpha_i^2 \sigma_i^2 = \sum_{i=1}^{3} \alpha_i^2 \sigma_i^2 + \alpha_4^2 \sigma_4^2 \geq (1 - \alpha_4^2)\sigma_4^2 + \alpha_4^2 \sigma_4^2$$

$\therefore$ $u'^T D^T D u' \geq u_4 D^T D u_4$