PnP Problem

$$g(\cdot) = \arg\min_{R,t} \frac{1}{N} \| \sum_{i=1}^{N} \| P_i - (R q_i + t) \|^2$$

Set $P_i' = P_i - u_p$

$q_i' = q_i - u_q$

Then $g(R,t) = \arg\min_{R,t} \frac{1}{N} \sum_{i=1}^{N} \| P_i' + u_p - (R(q_i' + u_q) + t) \|^2$

$$= \arg\min_{R,t} \frac{1}{N} \sum_{i=1}^{N} \| P_i' - Rq_i' + u_p - Ru_q - t \|^2$$

$$= \arg\min_{R,t} \frac{1}{N} \sum_{i=1}^{N} \left(\| P_i' - Rq_i' \|^2 + \| u_p - Ru_q - t \|^2 + 2(P_i' - Rq_i')(u_p - Ru_q - t) \right)$$

$$= \arg\min_{R,t} \frac{1}{N} \sum_{i=1}^{N} \| P_i' - Rq_i' \|^2 + \| u_p - Ru_q - t \|^2 + \boxed{\frac{2}{N} \sum_{i=1}^{N} (P_i' - Rq_i')(u_p - Ru_q - t)} \quad ①$$

$$\to \frac{2}{N} \left(\sum_{i=1}^{N} (P_i') - R \sum_{i=1}^{N} q_i' \right) (u_p - Ru_q - t)$$

$$\to 0 \times (u_p - Ru_q - t) \Rightarrow 0$$

$$g(R,t) = \arg\min_{R,t} \frac{1}{N} \sum_{i=1}^{N} \| P_i' - Rq_i' \|^2 + \boxed{\arg\min_{R,t} \| u_p - Ru_q - t \|^2}$$

Assume we had the optimal R, we can get t to make ② minimum easily.

So. The target function now is

$$\arg\min_R \frac{1}{N} \sum_{i=1}^{N} \| P_i' - Rq_i' \|^2$$

$$\Rightarrow \arg\min_R \sum_{i=1}^{N} \| P_i' - Rq_i' \|^2$$

$$= \arg\min_R \sum_{i=1}^{N} \left((P_i')^2 + (Rq_i')^2 - 2P_i'^T (Rq_i') \right)$$

$$= \arg\min_R (Rq_i')^T (Rq_i) - 2 \sum_{i=1}^{N} P_i'^T (Rq_i)$$

$$R = \arg\min_R (q_i'^T (R^T R) q_i - 2 \arg\min_R \sum_{i=1}^{N} P_i'^T (Rq_i')$$

$$R = \arg\max_R \sum_{i=1}^{N} P_i'^T (Rq_i') \Rightarrow \text{It is a scalar!}$$

$$\sum_{i=1}^{N} P_i'^T (Rq_i') = \sum_{i=1}^{N} \mathrm{Trace}((Rq_i') P_i'^T) = \mathrm{Trace}\left(R \sum_{i=1}^{N} q_i' P_i'^T\right)$$

Set $H = \sum_{i=1}^{N} q_i' P_i'^T$

$$R = \arg\max_R \mathrm{Trace}(RH)$$

Assume there exises the best B then

$$\mathrm{Trace}(BH) > \mathrm{Trace}(B'BH) \quad \text{for all } B'$$

Also, the lemma: For Positive Definitive Matrix AA^T.

For any orthonormal Matrix B, $\mathrm{Trace}(AA^T) \geq \mathrm{Trace}(BAA^T)$

Transfer $BH \longrightarrow AA^T$

SVD: $H = U\Sigma V^T$

Set $B = VU^T$ Then $BH \rightarrow (V\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}^T}V^T)$

Then. BH could be represented by AA^T

when $B = VU^T$

$\therefore B = VU^T$

$t = U_x - BU_y$