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i) defferentiable and strictly convex

$\nabla \varphi(x(i)) = 1 + \ln x(i) \rightarrow$

$\nabla \varphi(x) = (1 + \ln x(1), \dots, 1 + \ln x(d))$ first derivative

$\nabla^2 \varphi(x) = \left(\frac{1}{x(1)}, \frac{1}{x(2)}, \dots, \frac{1}{x(d)}\right)$ second derivative

a function is strictly convex if its second derivative is positiv $\square$

ii) $\|\nabla \Phi(x)\| = \|1 + \ln x(1), \dots, 1 + \ln x(d)\| \longrightarrow \infty$ when $x \longrightarrow 0$

$\|\nabla \Phi(x(t))\| = \|1 + \ln x(t)\| \longrightarrow +\infty$ $\square$

iii) For any $y \in \mathbb{R}^d$ there are $y = \nabla \varphi(x)$; $x(i) = e^{y(i)-1}$

$\longrightarrow$ surjective $\square$

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$\boxed{2}$ $D_\phi(x\|y) = f(x) - f(y) - \nabla\phi(y)^T(x-y)$

$$= \sum_{i=1}^{d} x_i \ln x_i - \sum_{i=1}^{d} y_i \ln y_i - \sum_{i=1}^{d} (\ln y_i + 1)(x_i - y_i)$$

$$= \sum_{i=1}^{d} x_i \ln x_i - \sum_{i=1}^{d} y_i \ln y_i - \sum_{i=1}^{d} x_i + \sum_{i=1}^{d} y_i$$

$$= \sum_{i=1}^{d} x_i \ln\left(\frac{x_i}{y_i}\right) - \sum_{i=1}^{d} x_i + \sum_{i=1}^{d} y_i$$

$$= \sum_{i=1}^{d} \left\{ x_i \ln\left(\frac{x_i}{y_i}\right) - x_i + y_i \right\} \quad \square$$

$\boxed{3}$ $D_\phi(x\|y) = \sum_{i=1}^{d} \left\{ x_i \ln \frac{x_i}{y_i} - x_i + y_i \right\}$

$$\pi(y) = \underset{x \in \Delta_d \cap D}{\arg\min}\, D_\phi(x\|y) = \underset{x \in \Delta_d \cap D}{\arg\min} \left\{ x_i \ln \frac{x_i}{y_i} - x_i + y_i \right\}$$

$$= \underset{x \in \Delta_d \cap D}{\arg\min} \left\{ x_i \ln \frac{x_i}{y_i} \right\}$$

Since $\sum_{i=1}^{d} x_i = 1$ for all $x \in \Delta_d$, Letting $f(x) = x \ln x$, which is convex Jensen's enequality yield

$$\sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i} = \sum_{i=1}^{d} y_i \ln \frac{x_i}{y_i} \geqslant f\left(\frac{\sum_{i=1}^{d} x_i}{\|y\|_1} \right) \|y\|_1 = \ln\left(\frac{1}{\|y\|_1}\right)$$

and enequality holds iff all x_i/y_i are equal $x_i = \frac{y_i}{\|y\|_1}$

Thus, $\frac{y}{\|y\|_1} = \underset{x \in \Delta_d \cap D}{\arg\min}\, D_\phi(x\|y) = \pi(y) \quad \square$

$\boxed{4}$

$$\min_{x \in \Delta_d \cap D} \varphi(x)$$

$$\sum_{i=1}^{d} x_i \ln x_i \geqslant \sum_{i=1}^{d} 1 \cdot f\left(\frac{\sum_{i=1}^{d} x_i}{\sum_{i=1}^{d} 1}\right) =$$

Jensen's enequality

$$= d \cdot f\left(\frac{1}{d}\sum_{i=1}^{d} x_i\right) \quad ; \quad f(x) = x \ln x$$

$$= d \cdot \frac{1}{d}\sum_{i=1}^{d} x_i \cdot \ln \frac{1}{d}\sum x_i = \boxed{\ln \frac{1}{d}} \quad ; \quad \sum_{i=1}^{d} x_i = 1$$

For $x(1) = \cdots = x(d)$ & $\sum_{i=1}^{d} x_i = 1 \Longrightarrow x(1) = \cdots = x(d) = \frac{1}{d}$

$$\varphi(x_1) = \sum_{i=1}^{d} x_i \ln x_i = \sum_{i=1}^{d} \frac{1}{d} \ln \frac{1}{d} = \frac{1}{d}\left(\ln \frac{1}{d}\right)^{d}$$

$$\boxed{= \ln \frac{1}{d}}$$

$\square$

$\boxed{5}$

$$x_{t+1}(i) = \pi\left(x_{t+1}(i)\right)$$

$$= \pi\left[\nabla\varphi^{-1}\left(\nabla\varphi\left(x_t(i) - \eta_t \nabla_t(i)\right)\right)\right]$$

$$= \pi\left[\nabla\varphi^{-1}\left(1 + \ln x_t(i) - \eta_t \nabla_t(i)\right)\right]$$

$$= \pi\left[\nabla\varphi^{-1}\left(1 + \ln x_t(i) + \ln\left(\exp(-\eta_t \nabla_t(i))\right)\right)\right]$$

$$= \pi\left[\nabla\varphi^{-1}\underbrace{\left(\ln\left(x_t(i) + \exp(-\eta_t \nabla_t(i))\right) + 1\right)}_{\nabla\varphi\left(x_t(i)\exp(-\eta_t \nabla_t(i))\right)}\right]$$

$$= \pi\left(x_t(i)\exp(-\eta_t \nabla_t(i))\right)$$

$$= \frac{x_t(i)\exp(-\eta_t \nabla_t(i))}{\left\| x_t(i)\exp(-\eta_t \nabla_t(i)) \right\|}$$

$$= \frac{x_t(i)\exp(-\eta_t \nabla_t(i))}{\sum_{j=1}^{d} x_t(j)\exp(-\eta_t \nabla_t(i))} \quad \square$$