$\boxed{1}$

$$R(\hat{\theta}_n) - \min_{\theta \in \mathcal{M}} R(\theta) \leq \max\left(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta)\right)$$

$$R(\hat{\theta}_n) - \min R(\theta) = R(\hat{\theta}_n) - R(\theta)$$

$$= \left[R(\hat{\theta}_n) - (1+c)R_n(\hat{\theta}_n)\right] + \left[(1+c)R_n(\hat{\theta}_n) - (1+c)R_n(\bar{\theta})\right] + \left[(1+c)R_n(\bar{\theta}) - R(\bar{\theta})\right]$$

$$= \left[R(\hat{\theta}_n) - (1+c)R_n(\hat{\theta}_n)\right] + (1+c)\underbrace{\left[R_n(\hat{\theta}_n) - R_n(\bar{\theta})\right]}_{\leq 0 \text{ by definition of } \hat{\theta}_n} + \left[(1+c)R_n(\bar{\theta}) - R(\bar{\theta})\right]$$

$$\leq \left[R(\hat{\theta}_n) - (1+c)R_n(\hat{\theta}_n)\right] + \left[(1+c)R_n(\bar{\theta}) - R(\bar{\theta})\right]$$

Reorder $$= \left[R(\hat{\theta}_n) - R(\bar{\theta})\right] - (1+c)\left[R_n(\hat{\theta}_n) - R_n(\bar{\theta})\right]$$

$$= \bar{R}(\hat{\theta}_n) - (1+c)\bar{R}_n(\hat{\theta}_n)$$

$$\leq \max_{\theta \in \mathcal{M}}\left(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta_n)\right) \quad \square$$

2

$$\mathbb{P}(R(\hat{\theta}_n) - \min R(\theta) > t) \leq |M| \max \mathbb{P}(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) > t)$$

Proof:

From question 1 we found that:

$$R(\hat{\theta}_n) - \min R(\theta) \leq \max(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta)), \text{ so,}$$

$$\mathbb{P}(\max(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta)) > t)$$

$$= \mathbb{P}(\cup\{\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) > t\})$$

$$\leq \sum_{\theta \in M} \mathbb{P}(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) > t\})$$

$$\leq \underbrace{|M|}_{\text{number of elements}} \max_{\theta \in M} \mathbb{P}(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) > t) \quad \square$$

$$\mathbb{P}(\bar{R}(\theta) - \bar{R}_n(\theta) > \varepsilon) \leq \frac{Var(\ell(\theta, z_1) - \ell(\bar{\theta}, z_1))}{n\varepsilon}$$ $\boxed{3}$

$E[X] = \frac{1}{n}\sum x_i$

proof:

$$\bar{R}(\theta) = \mathbb{E}[\ell(\theta, z_1) - \ell(\bar{\theta}, z_1)]$$

$$\bar{R}_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{(\ell(\theta, z_i) - \ell(\bar{\theta}, z_i))}_{x_i}$$

by using Chebichev's inequality:

$$\mathbb{P}(\bar{R}(\theta) - \bar{R}_n(\theta) > \varepsilon) \leq \frac{Var(\ell(\theta, z_1) - \ell(\bar{\theta}, z_1))}{n\varepsilon^2}$$ $\square$

$$\mathbb{P}(\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) > t) \leq \frac{\beta}{tn} \frac{(1+c)^2}{4c}$$

$$\mathbb{P}\{\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) > t\}$$

$$= \mathbb{P}\{(1+c)\bar{R}(\theta) - (1+c)\bar{R}_n(\theta) - c\bar{R}(\theta) > t\}$$

$$= \mathbb{P}\{(1+c)(\bar{R}(\theta) - \bar{R}_n(\theta)) > t + c\bar{R}(\theta)\}$$

$$= \mathbb{P}\{\bar{R}(\theta) - \bar{R}_n(\theta) > \underbrace{\frac{t + c\bar{R}(\theta)}{(1+c)}}_{\varepsilon}\}$$

$$\leq \frac{1}{n} \cdot \frac{(1+c)^2}{(t + c\bar{R}(\theta))^2} \cdot \mathrm{Var}[\ell(\theta, z_1) - \ell(\bar{\theta}, z_1)]$$

from question 3

$$\leq \frac{1}{n} \cdot \frac{(1+c)^2}{(t + c\bar{R}(\theta))^2} \beta \underbrace{\mathbb{E}[\ell(\theta, z_1) - \ell(\bar{\theta}, z_1)]}_{\bar{R}(\theta)} \quad (L)$$

by Hint

$$(t + c\bar{R}(\theta))^2 \geq 4tc\bar{R}(\theta) \Rightarrow \frac{1}{(t + c\bar{R}(\theta))^2} \leq \frac{1}{4tc\bar{R}(\theta)}$$

$$(L) = \frac{1}{n} \cdot \frac{(1+c)^2}{4tc\cancel{\bar{R}(\theta)}} \beta \cancel{\bar{R}(\theta)} = \frac{\beta}{tn} \cdot \frac{(1+c)^2}{4c}$$

5

$$\mathbb{P}\Big(\underbrace{R(\hat{\theta}_n) - \min_{\theta \in M} R(\theta)}_{EE} \leq \frac{\beta |M|}{\delta n}\Big) \geq 1 - \delta$$

$$\mathbb{P}(EE > t) \overset{\text{from step 2}}{\leq} |M| \max_{\theta \in M} \mathbb{P}\big(\bar{R}(\theta) - (1+C)\bar{R}_n(\theta) > t\big)$$

$$\overset{\text{from step 4}}{\leq} |M| \frac{\beta}{tn} \cdot \frac{(1+C)^2}{4C}$$

Setting $\delta = |M| \frac{\beta}{tn} \cdot \frac{(1+C)^2}{4C}$ suppos $C=1$

$$\delta = |M| \frac{\beta}{tn}$$

$$\Longleftrightarrow t = \frac{\beta |M|}{\delta n}$$

$$\mathbb{P}\Big(EE > \frac{\beta |M|}{\delta n}\Big) < \delta$$

$$\Longrightarrow \mathbb{P}\Big(EE \leq \frac{\beta |M|}{\delta n}\Big) \geq 1 - \delta$$

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Done!