Probabilistic Machine Learning with Julia

Lecture 4 | Probabilistic Models for Time Series

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Today's Topics

Today we will have an overview of parametric probabilistic models for time series data:

- Bayesian Neural Networks
- Autoregressive models and Dynamic Bayesian Networks
- Time series modeling with Bayesian Differential Equations
- Motivation for nonparametric models

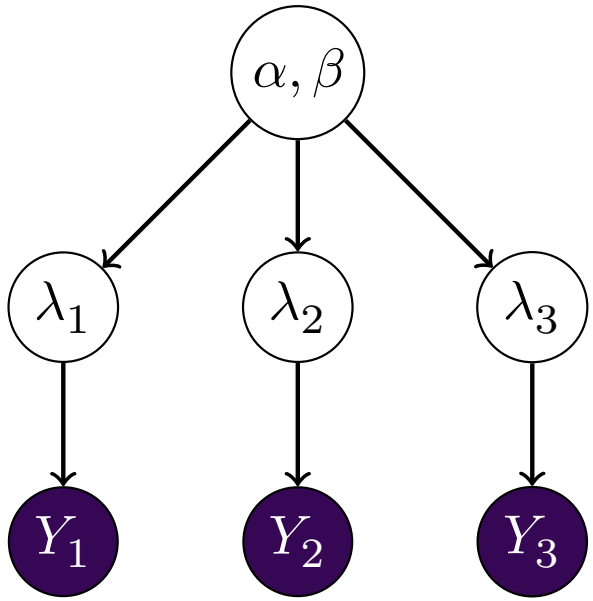
Poisson Regression

In the previous lecture, we discussed a simple form of Poisson Regression:

$$
\begin{aligned}
Y_\tau &\sim \text{Poisson}(\lambda_\tau) \\
\log \lambda_\tau &= \alpha\tau + \beta \\
\alpha, \beta &\sim \text{Normal}(0, 1)
\end{aligned}
$$

```
@model function poisson_linear_model(y)
    α ~ Normal(0.0, 1.0)
    β ~ Normal(0.0, 1.0)

    for τ ∈ 1:length(y)
        log_λ = α*τ + β
        y[τ] ~ Poisson(exp(log_λ))
    end
end
```



Poisson Regression

$$p(\alpha, \beta) \xrightarrow{\text{Bayesian Inference}} p(\alpha, \beta | y)$$

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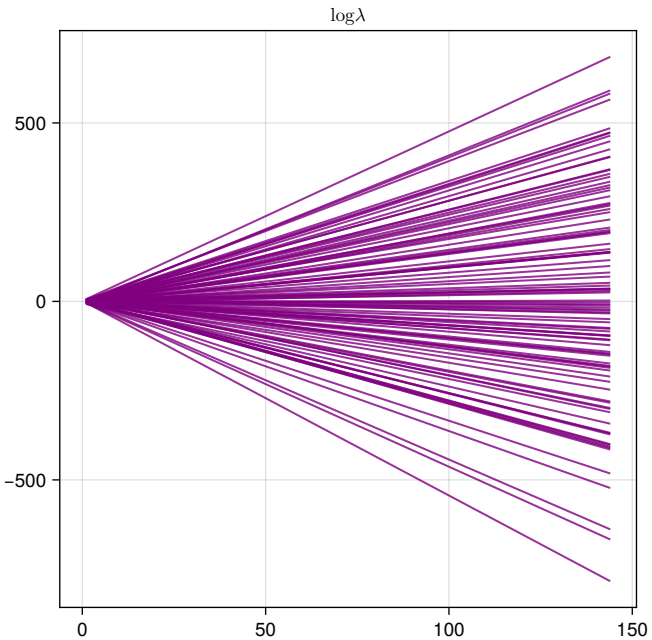

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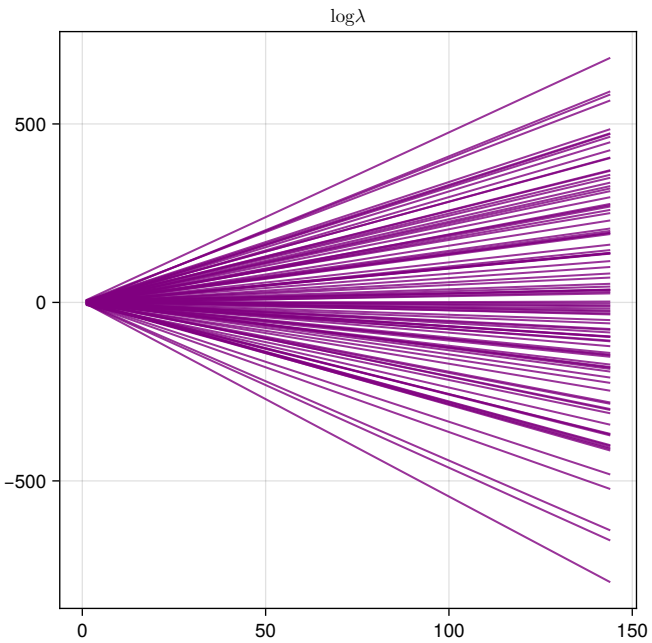
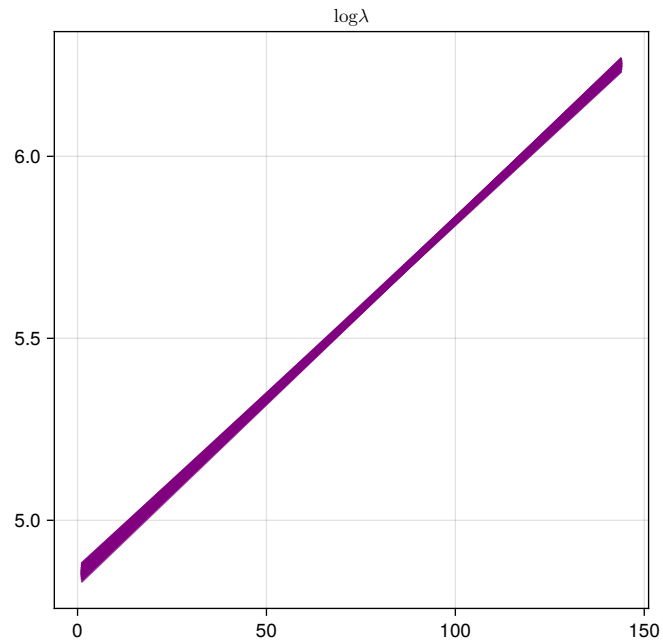
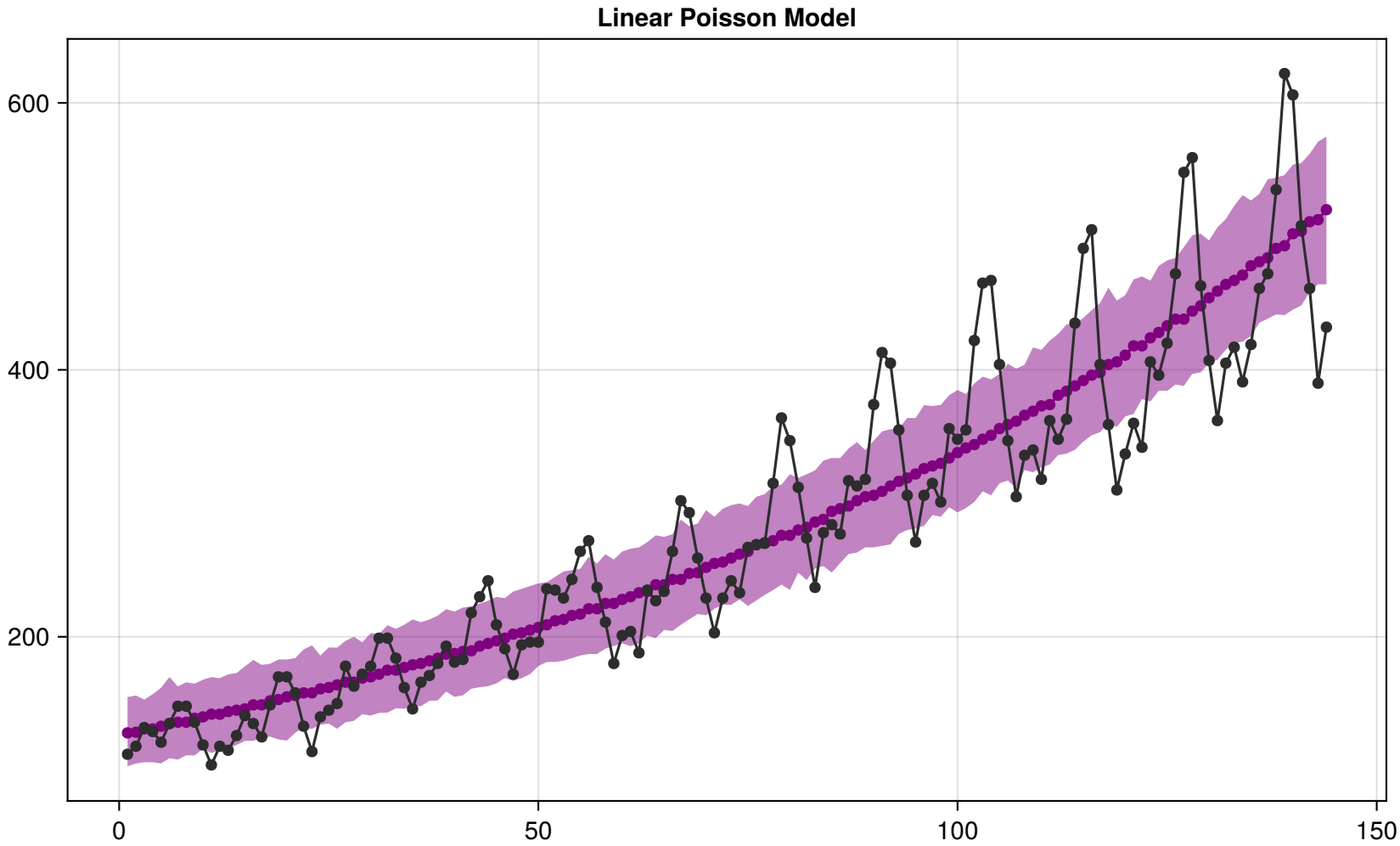




Poisson Regression



Poisson Regression

It is easy to generalize this model by parametrizing $\log \lambda_\tau$ using a nonlinear function:

$$\log \lambda_\tau = f(.; w)$$

Poisson Regression

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For example in the previous session, we added a nonlinear term for handling seasonality:

$$\log \lambda_\tau = \alpha\tau + \nu \cos(\omega\pi(\tau - \phi)) + \beta$$

Poisson Regression

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$$\log \lambda_\tau = f(.; w)$$

For example in the previous session, we added a nonlinear term for handling seasonality:

$$\log \lambda_\tau = \alpha\tau + \nu \cos(\omega\pi(\tau - \phi)) + \beta$$

We can also use more expressive functions like a neural network if necessary.
Let's discuss **Bayesian Neural Network** first for a classification problem and then for our time series example.

Bayesian Neural Network

Let $f(x; \boldsymbol{w})$ be a neural network with three layers and sigmoid activation:

$$\boldsymbol{w} = [\boldsymbol{w}^{L1}, \boldsymbol{w}^{L2}, \boldsymbol{w}^{L3}]$$

$$f(x; \boldsymbol{w}) = \sigma(\sigma(\sigma(x\boldsymbol{w}^{L1})\boldsymbol{w}^{L2})\boldsymbol{w}^{L3})$$

Bayesian Neural Network

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We define a Multivariavte Normal prior over $\boldsymbol{w}$

$$\boldsymbol{w} \sim \text{Normal}(0, I)$$

Bayesian Neural Network

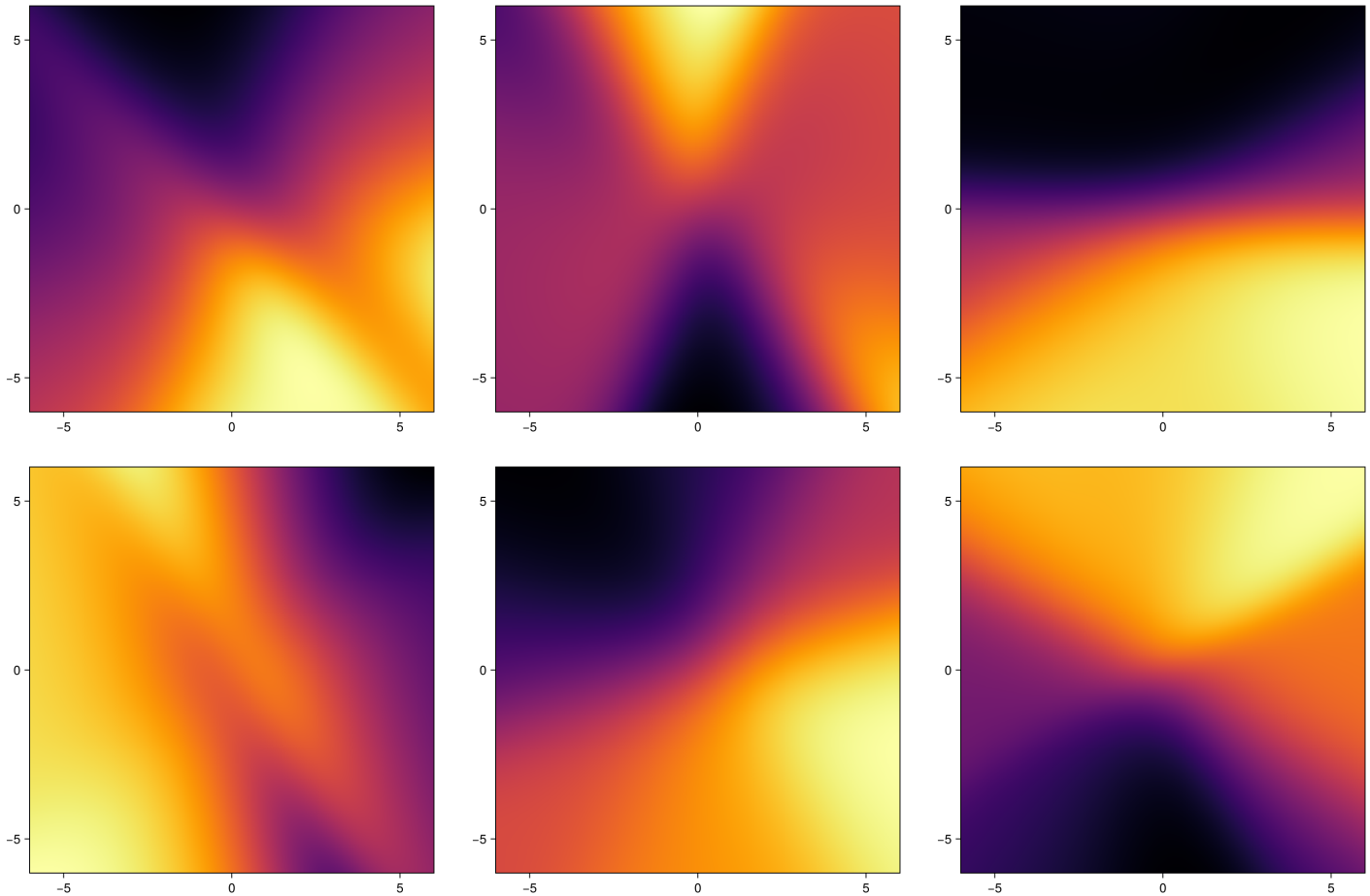
```julia
function neural_network(X, L1, L2, L3, H_dim)
    H = X * reshape(L1, (2, H_dim))
    H = sigmoid.(H)

    H = H * reshape(L2, (H_dim, H_dim))
    H = sigmoid.(H)

    O = H * reshape(L3, (H_dim, 1))
    O = sigmoid.(O)

    return O
end
```

Bayesian Neural Network

```julia
@model function bayesian_neural_network(X, y, σ, H_dim)
    n_L1 = 2 * H_dim
    n_L2 = H_dim * H_dim
    n_L3 = H_dim

    Σ(n) = Diagonal(abs2.(σ .* ones(n)))

    L1 ~ MvNormal(zeros(n_L1), Σ(n_L1))
    L2 ~ MvNormal(zeros(n_L2), Σ(n_L2))
    L3 ~ MvNormal(zeros(n_L3), Σ(n_L3))

    O = neural_network(X, L1, L2, L3, H_dim)

    for i in eachindex(y)
        y[i] ~ Bernoulli(O[:][i])
    end
end
```

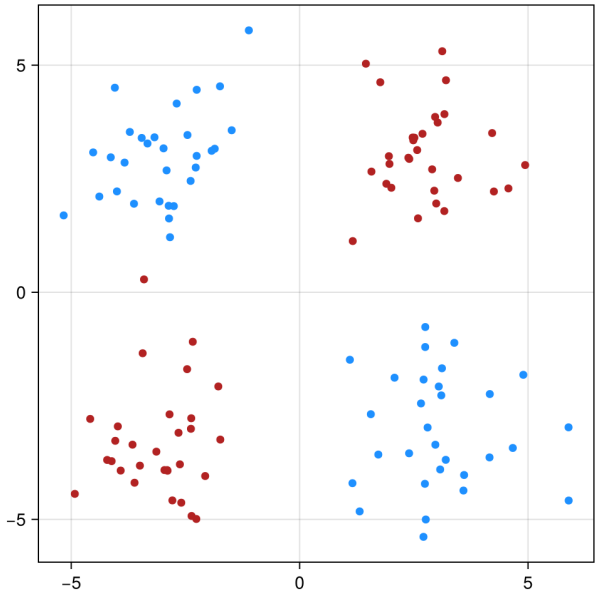
Bayesian Neural Network

Some samples from $p(\boldsymbol{w})$



Bayesian Neural Network

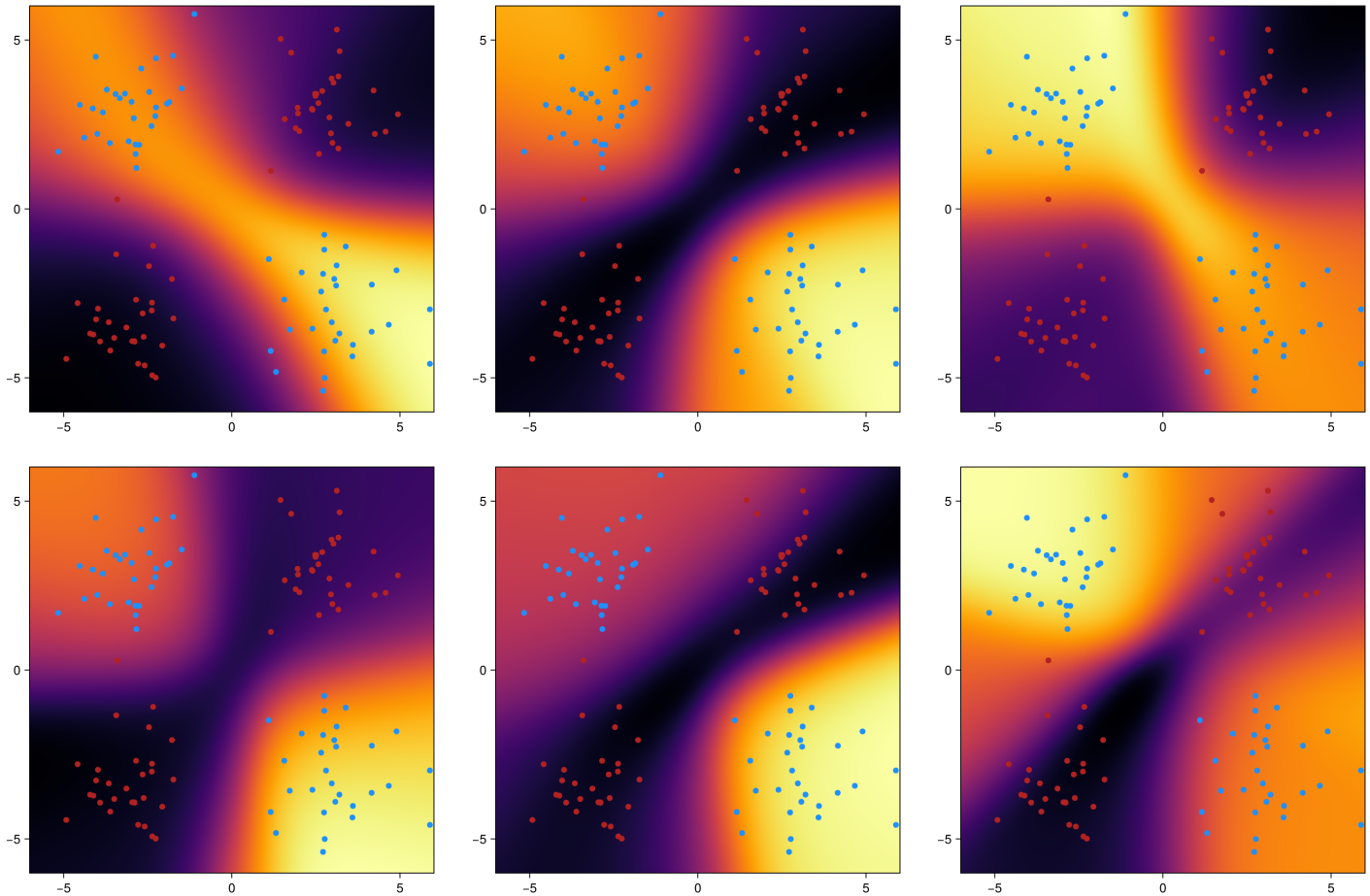
Now we update our belief about network weights using the below dataset



$$y \sim \text{Bernoulli}(f(x; \boldsymbol{w}))$$

Bayesian Neural Network

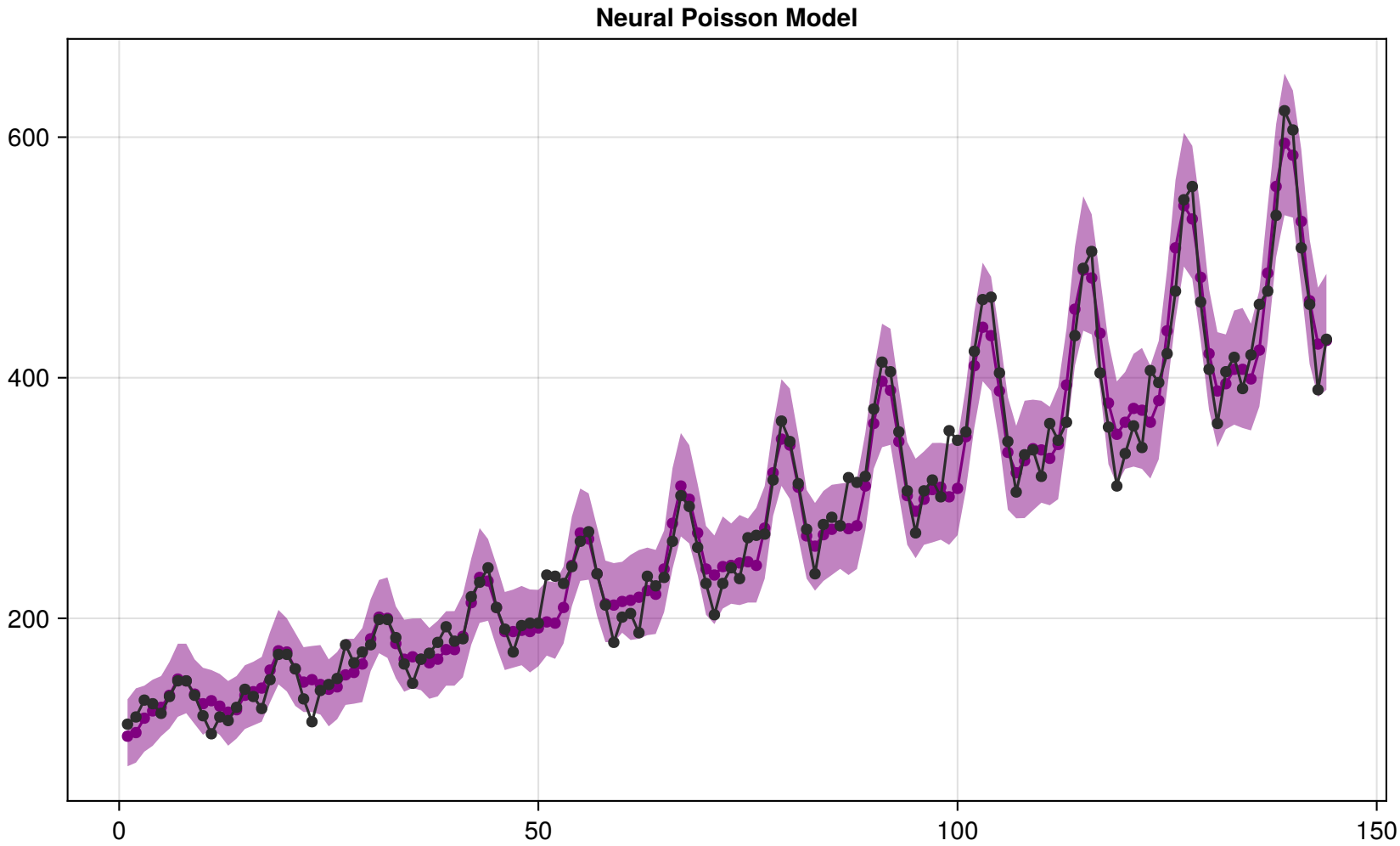
Some samples from $p(\boldsymbol{w}|y)$



Neural Poisson Regression

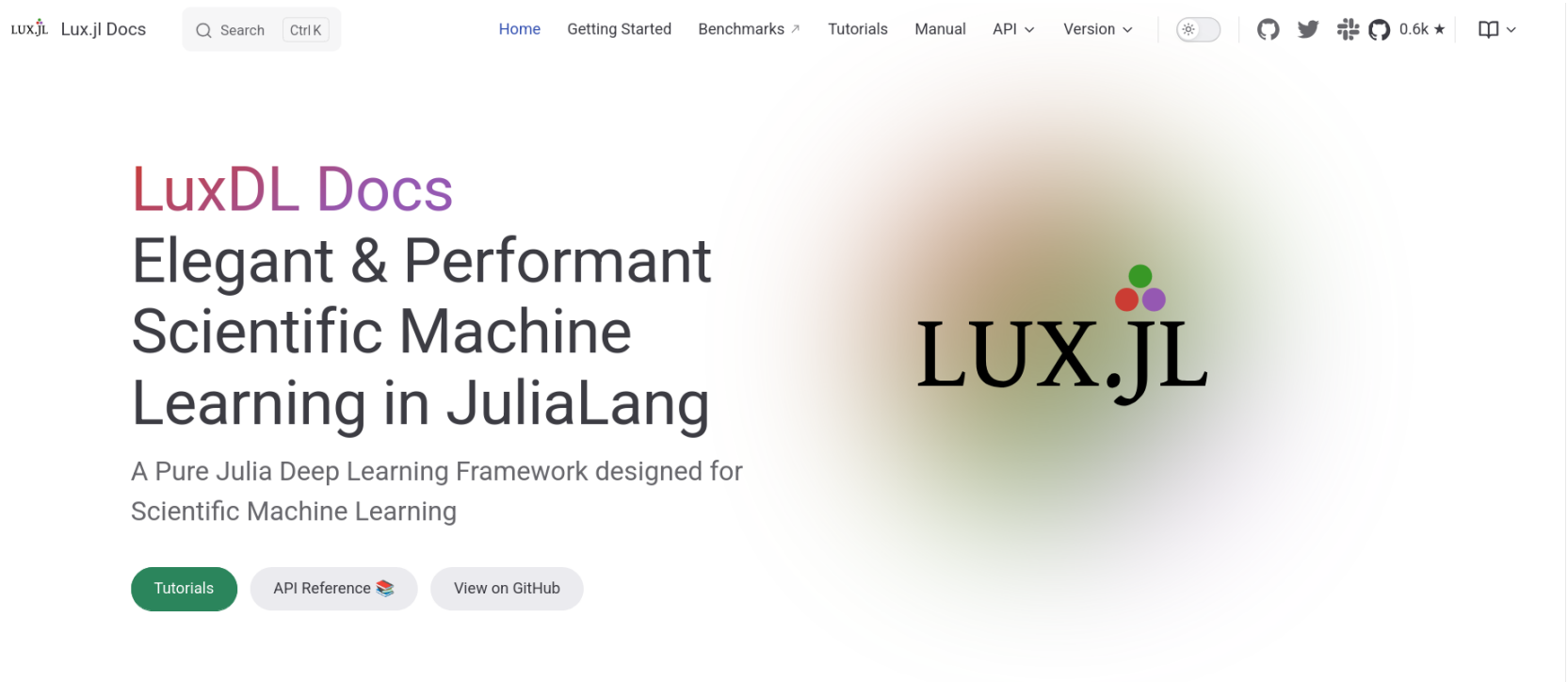
```julia
@model function neural_poisson_model(y)
  w ~ MvNormal(zeros(12), I)
  L1 = reshape(w[1:8], (4,2))
  L2 = reshape(w[9:12], (1,4))
  b = w[12:12]

  ϕ ~ Normal(0.0, 2.0)
  ω ~ Normal(0.0, 2.0)

  for t in 1:length(y)
    x = [t, cos(ω*(t-ϕ))]
    log_λ = ANN(L1, L2, b, x)[1]
    y[t] ~ Poisson(exp(log_λ))
  end
end
```

Neural Poisson Regression



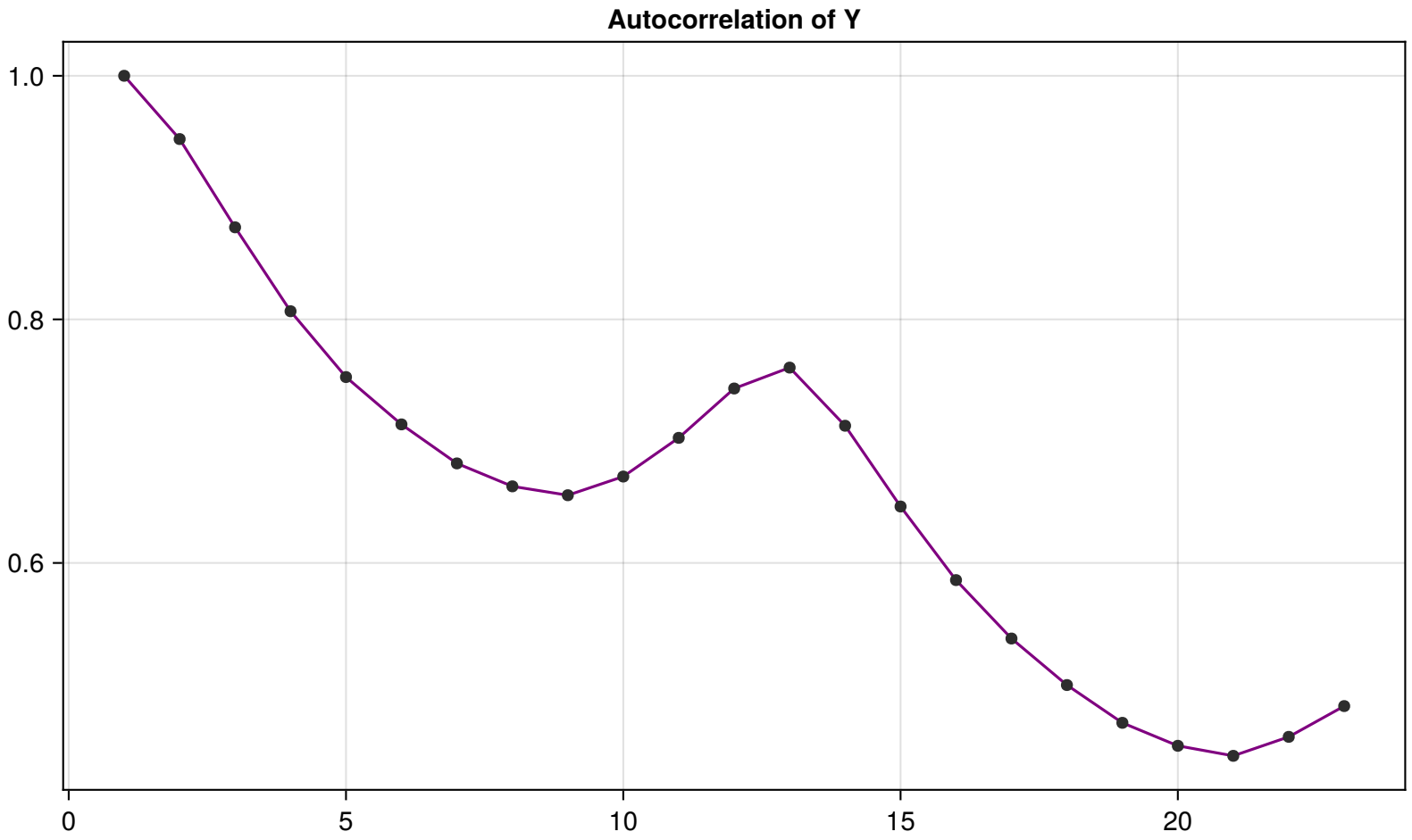
Neural Poisson Regression

Of course, we don't have to manually define neural networks



Autoregressive Models

A popular assumption in modeling time series data is **Autoregression**

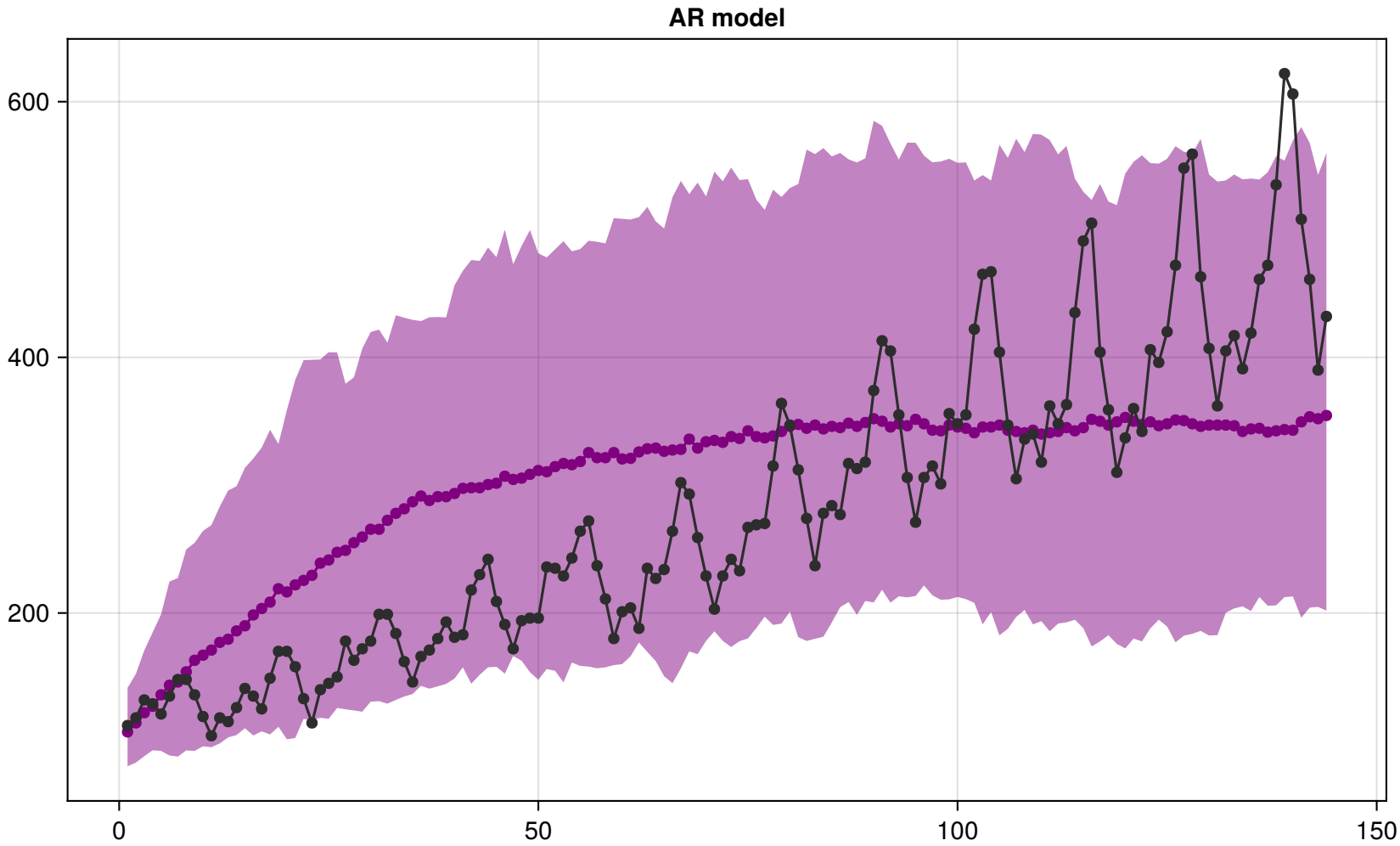


Autoregressive Models

$$
\begin{aligned}
Y_\tau &\sim \text{Poisson}(\lambda_\tau) \\
\log \lambda_\tau &= \alpha \log \lambda_{\tau-1} + \beta \\
\alpha, \beta &\sim \text{Normal}(0, 1) \\
\lambda_0 &\sim \text{Poisson}(100.0)
\end{aligned}
$$

```
@model function AR(y)
  α ~ Normal(0.0, 2.0)
  β ~ Normal(0.0, 2.0)
  y0 ~ filldist(Poisson(100.0), 1)

  for t in 1:length(y)
    log_λ = α * log(lag(y, y0, t-1)) + β
    λ = clamp(exp(log_λ), 100.0, 1000.0)
    y[t] ~ Poisson(λ)
  end
end
```

Autoregressive Models

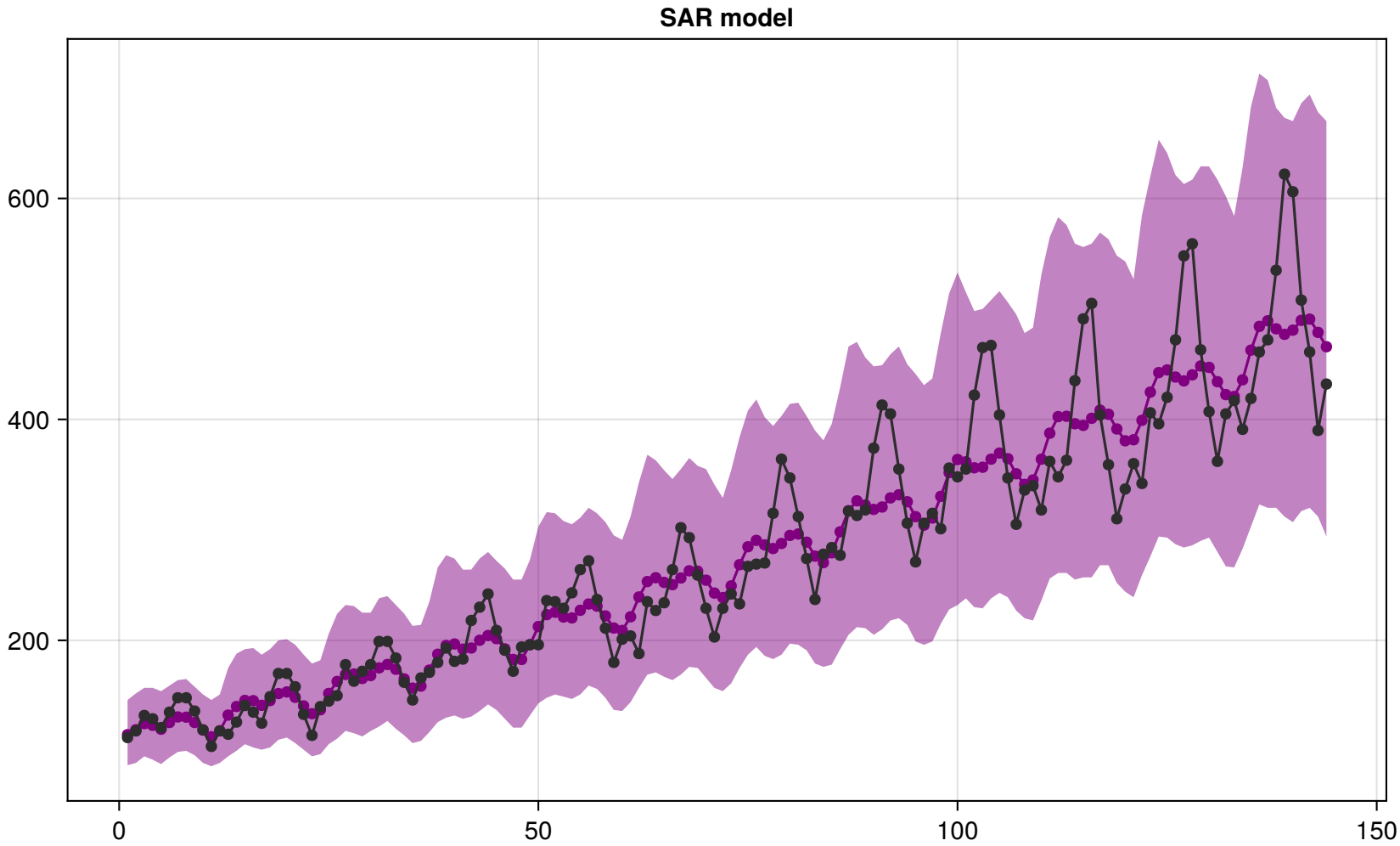


Autoregressive Models

$$Y_\tau \sim \text{Poisson}(\lambda_\tau)$$
$$\log \lambda_\tau = \alpha_1 \log \lambda_{\tau-1} + \alpha_2 \log \lambda_{\tau-2} + \alpha_3 \log \lambda_{\tau-12} + \beta$$

```
@model function SAR(y)
  α1 ~ Normal(0.0, 2.0)
  α2 ~ Normal(0.0, 2.0)
  α3 ~ Normal(0.0, 2.0)
  β ~ Normal(0.0, 2.0)
  y0 ~ filldist(Poisson(100.0), 12)

  for t in 1:length(y)
    log_λ = α1 * log(lag(y, y0, t-1)) +
            α2 * log(lag(y, y0, t-2)) +
            α3 * log(lag(y, y0, t-12)) + β
    λ = clamp(exp(log_λ), 100.0, 1000.0)
    y[t] ~ Poisson(λ)
  end
end
```

Autoregressive Models



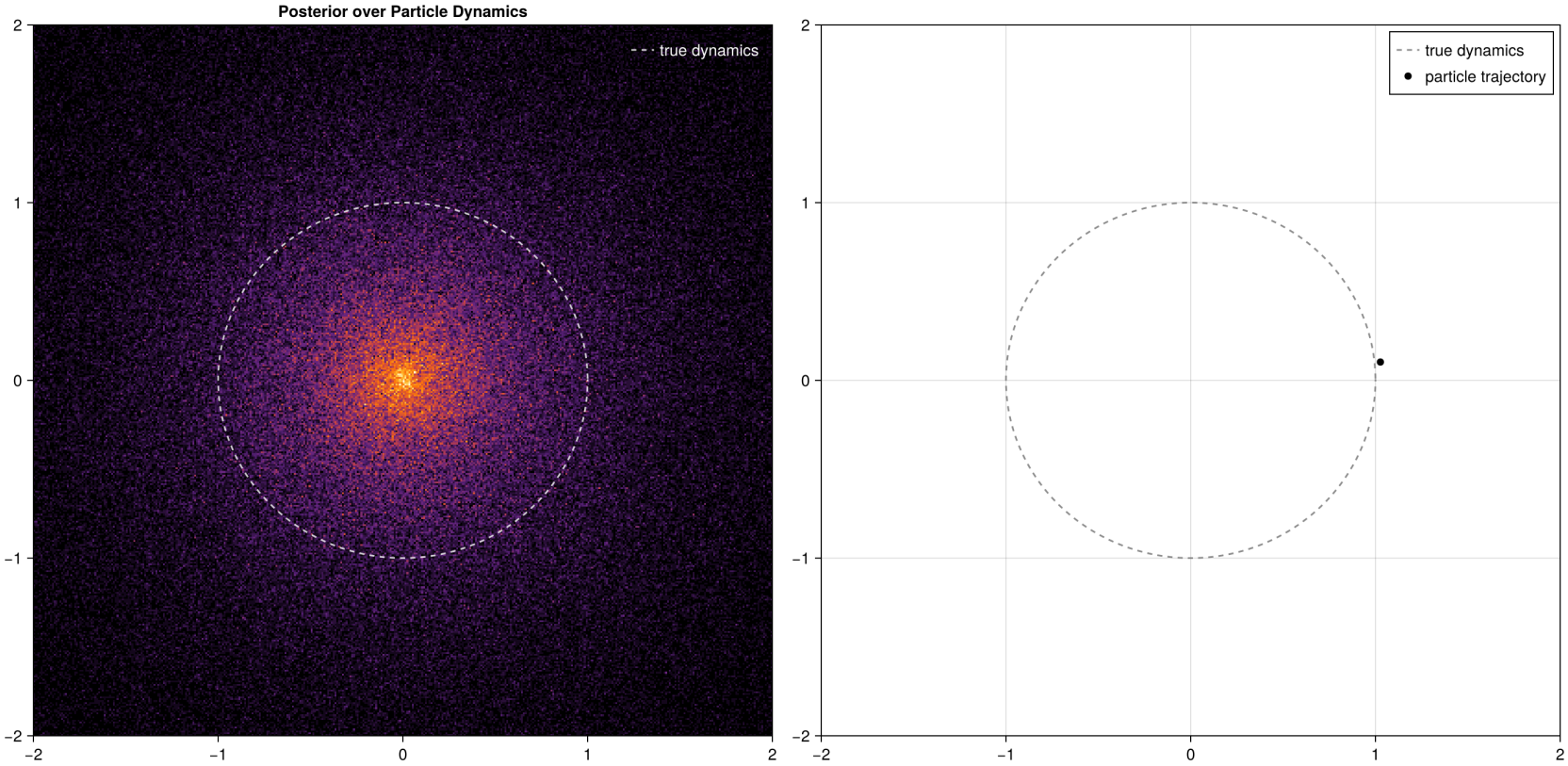
Autoregressive Models

Generalizing to a multivariate scenarios is straightforward:

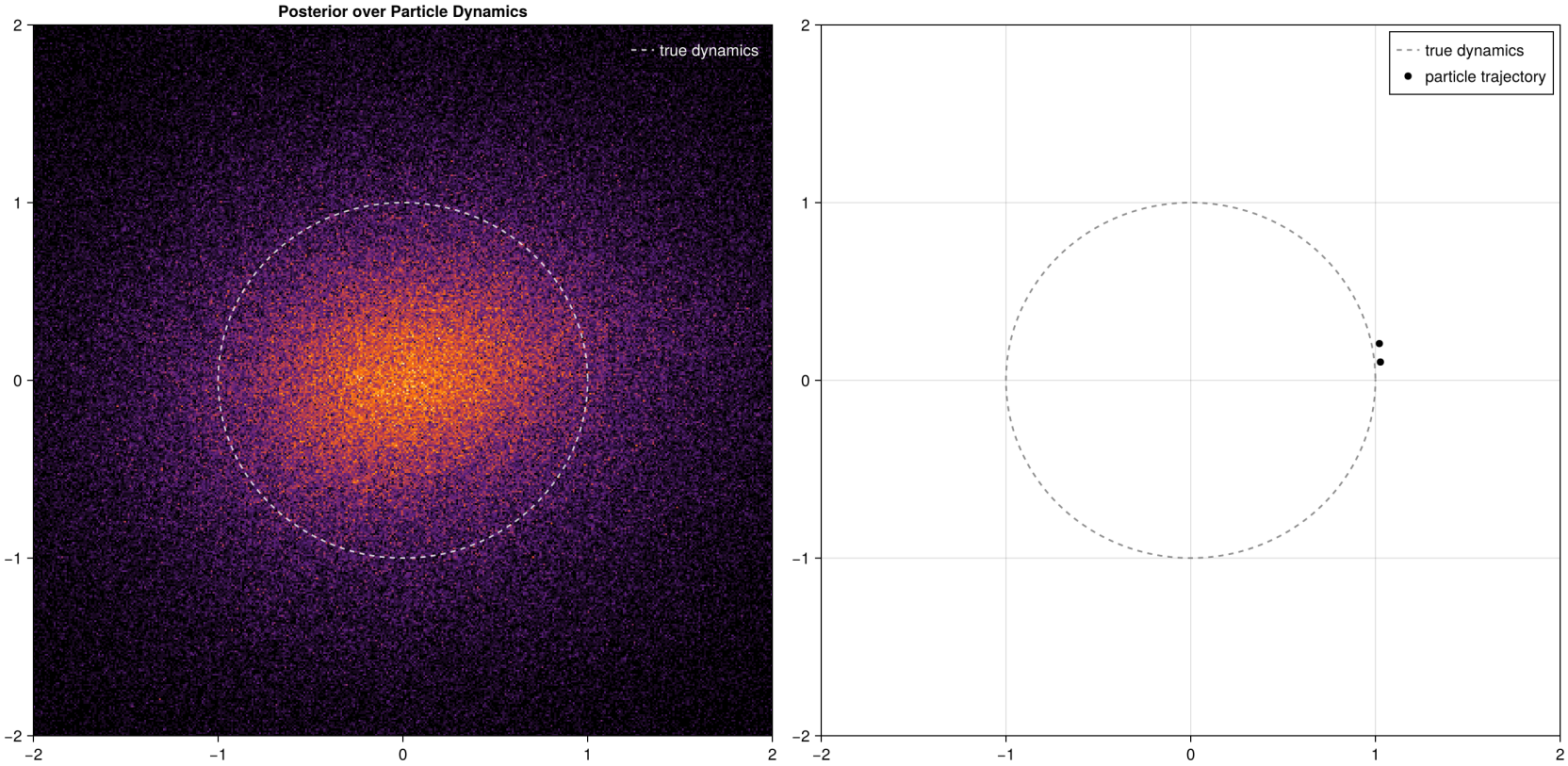
$$X_t \sim \text{MvNormal}(AX_{t-1}, \sigma I)$$
$$A \sim \text{MvNormal}(0, I)$$

```
@model function VAR(x)
    A ~ MvNormal(zeros(4), I)
    σ ~ Exponential(1.0)

    for t in 2:length(x)
        x[t] ~ MvNormal(reshape(A, (2,2)) * x[t-1], σ * I(2))
    end
end
```

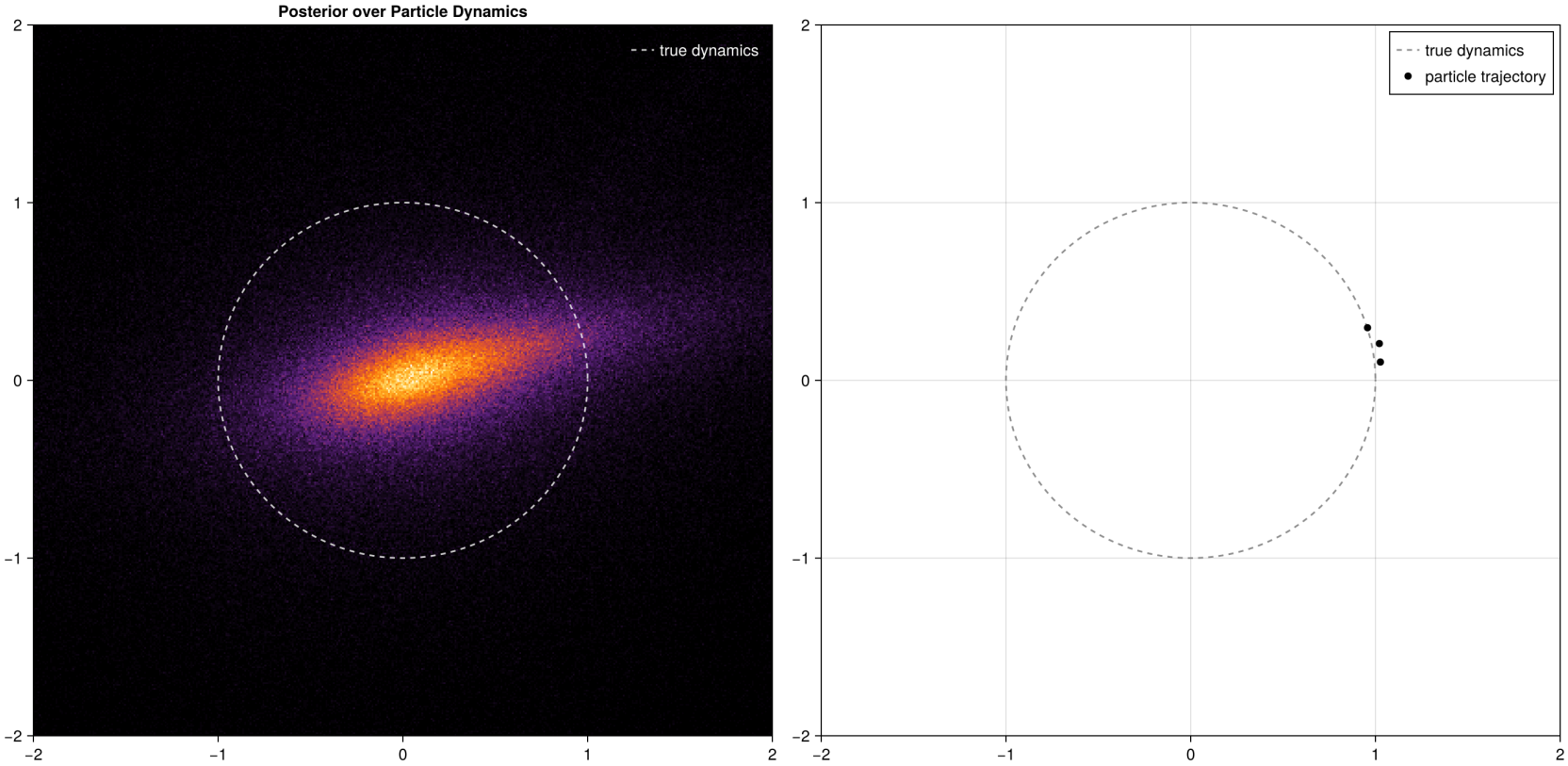
Autoregressive Models



Autoregressive Models



Autoregressive Models



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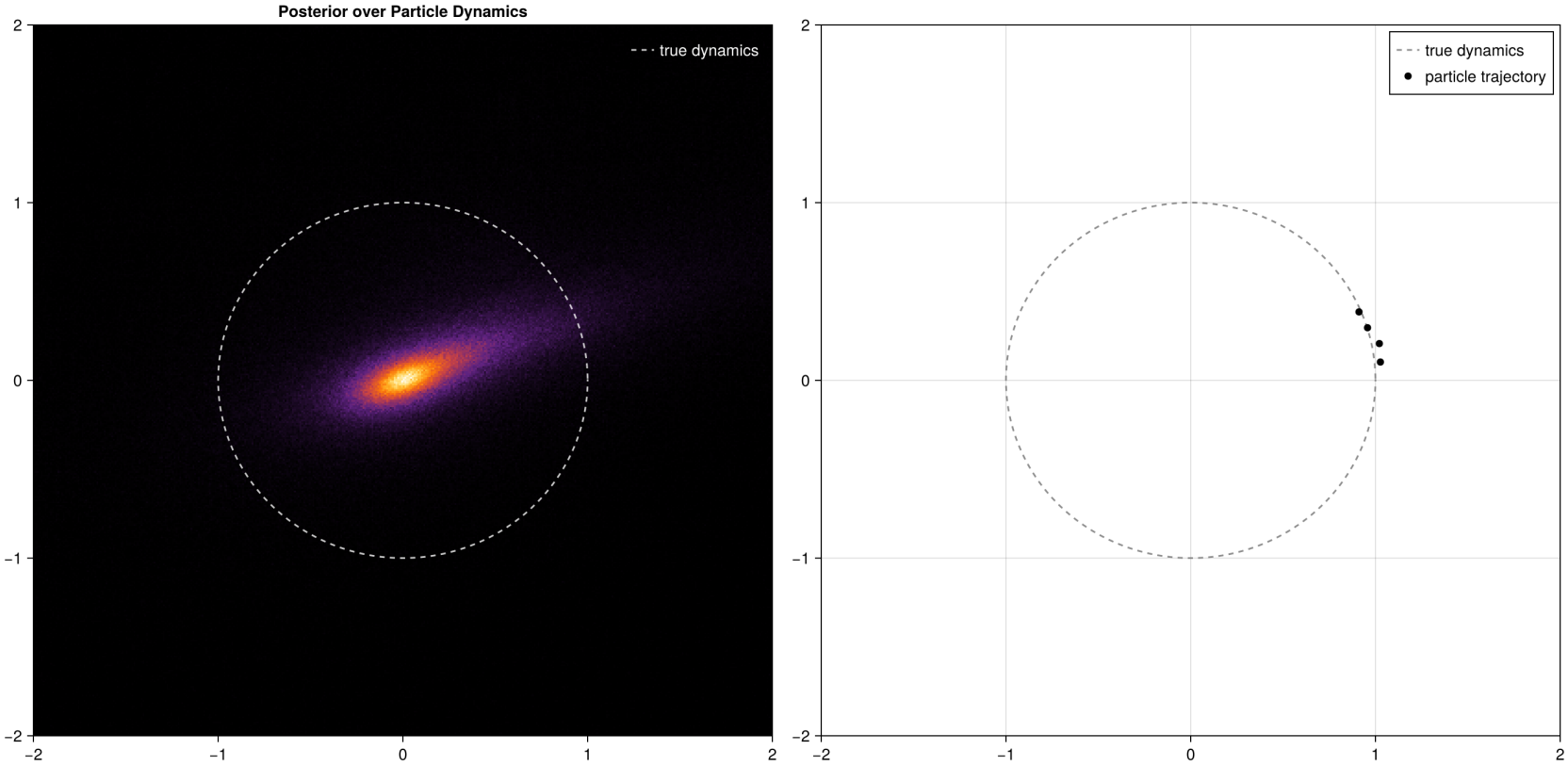

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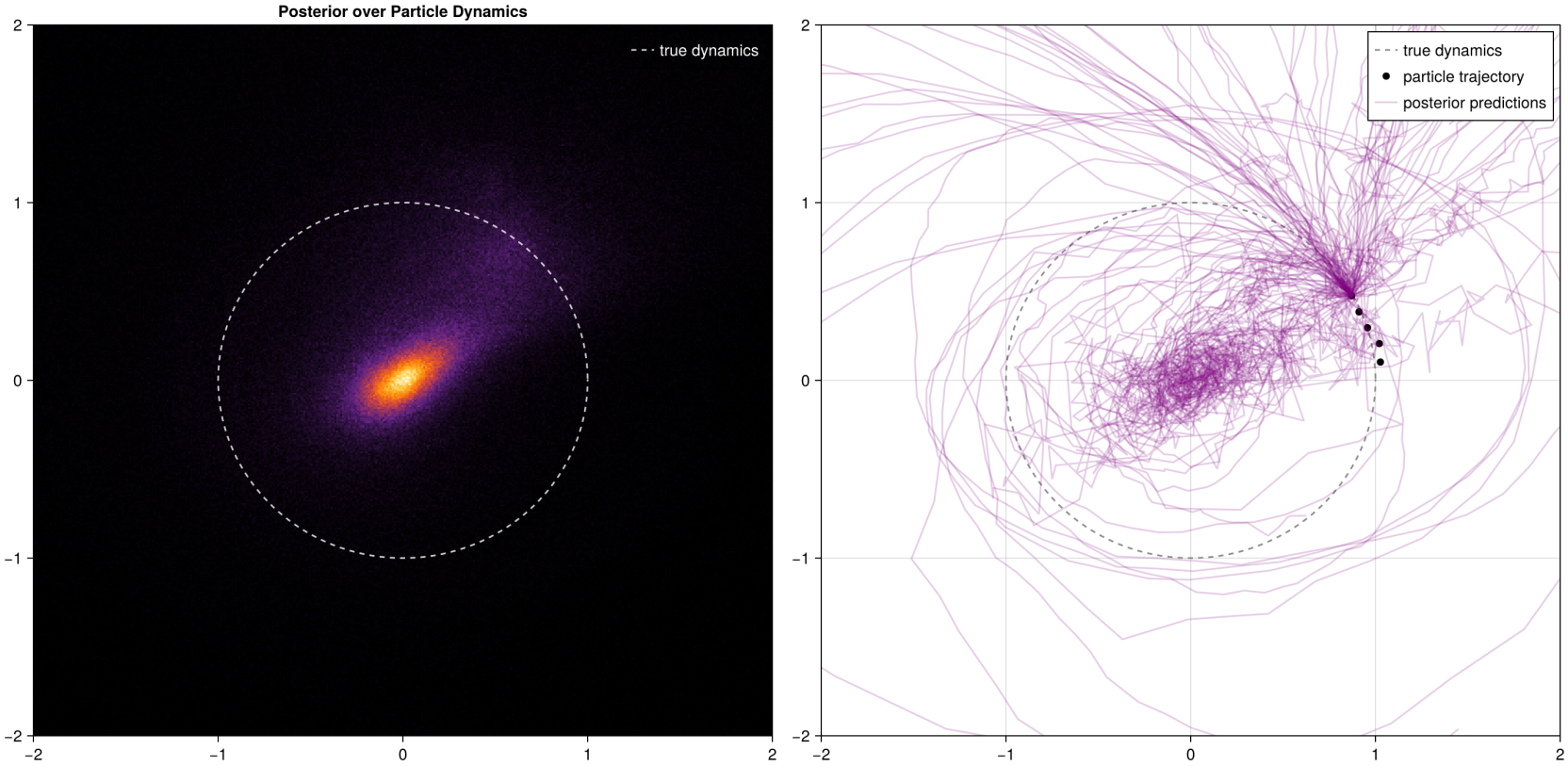

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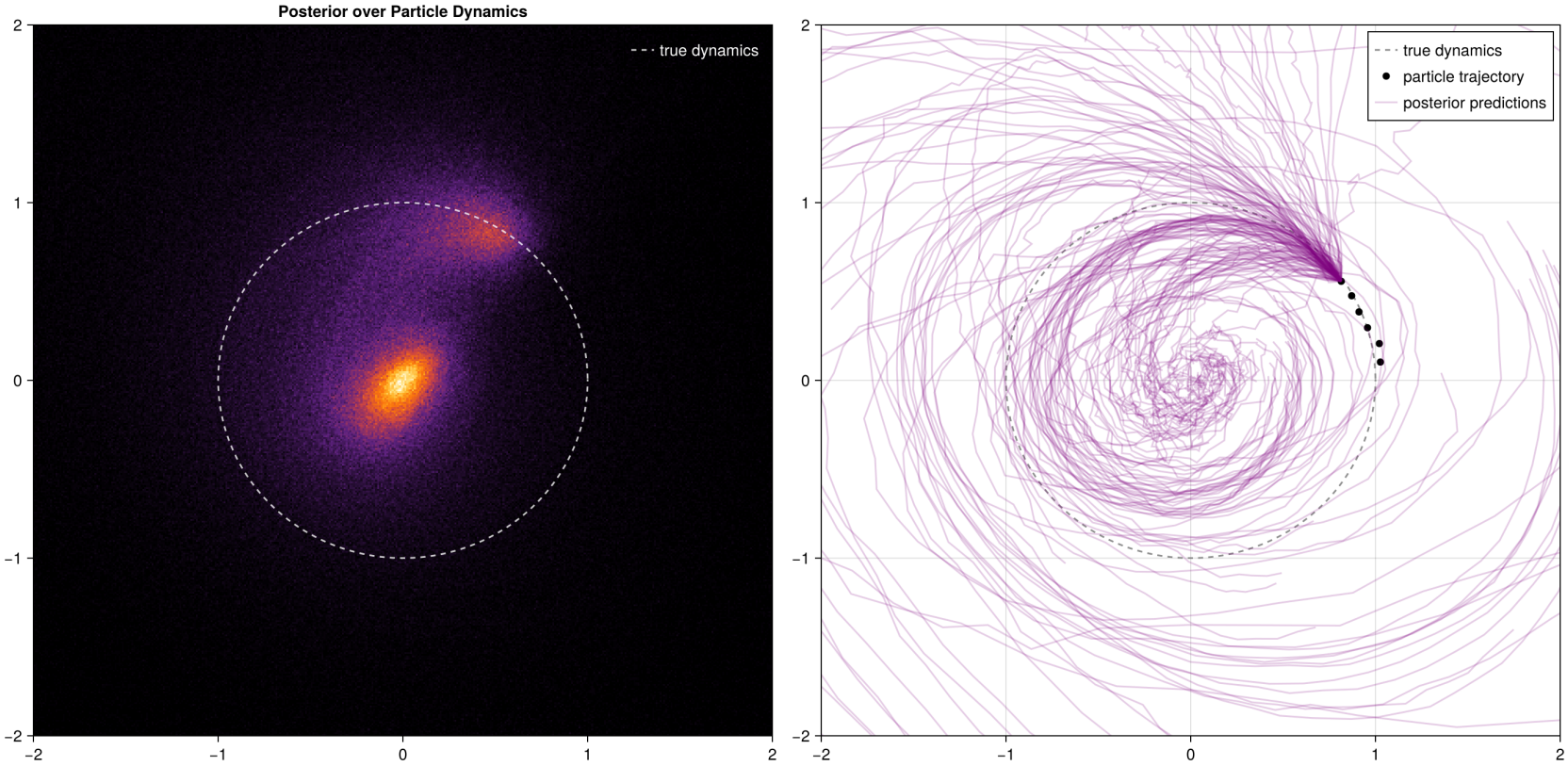

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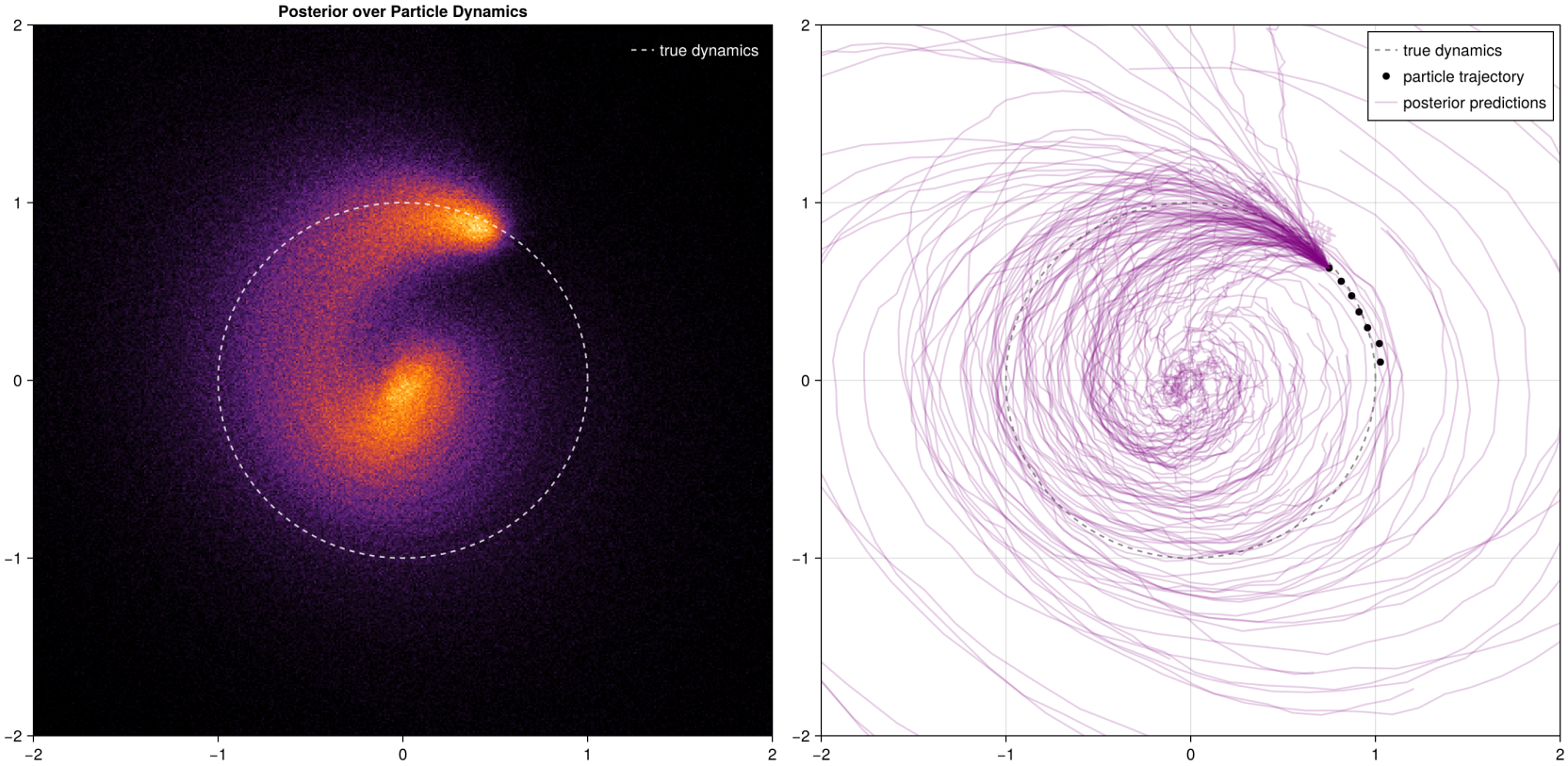

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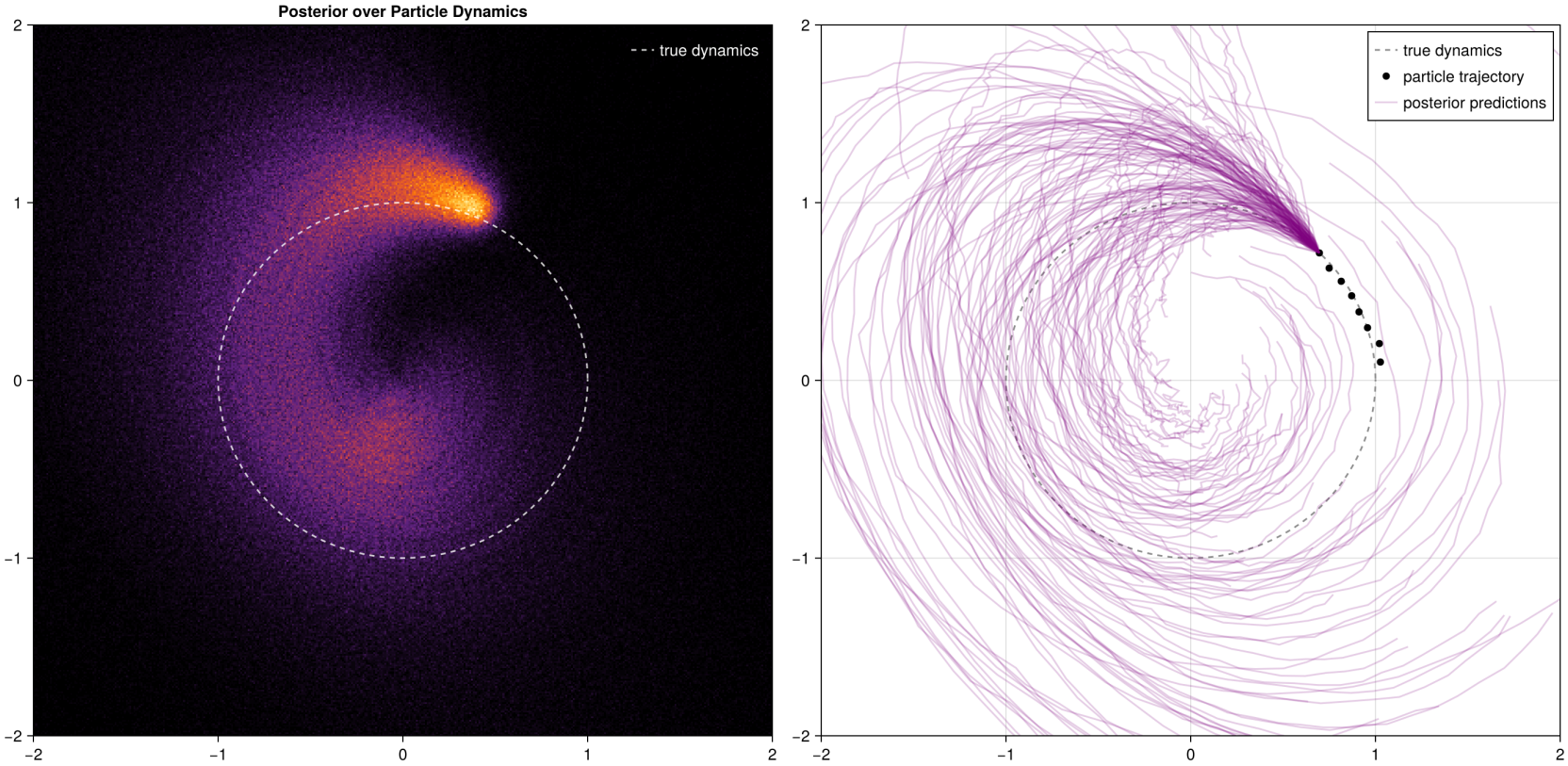

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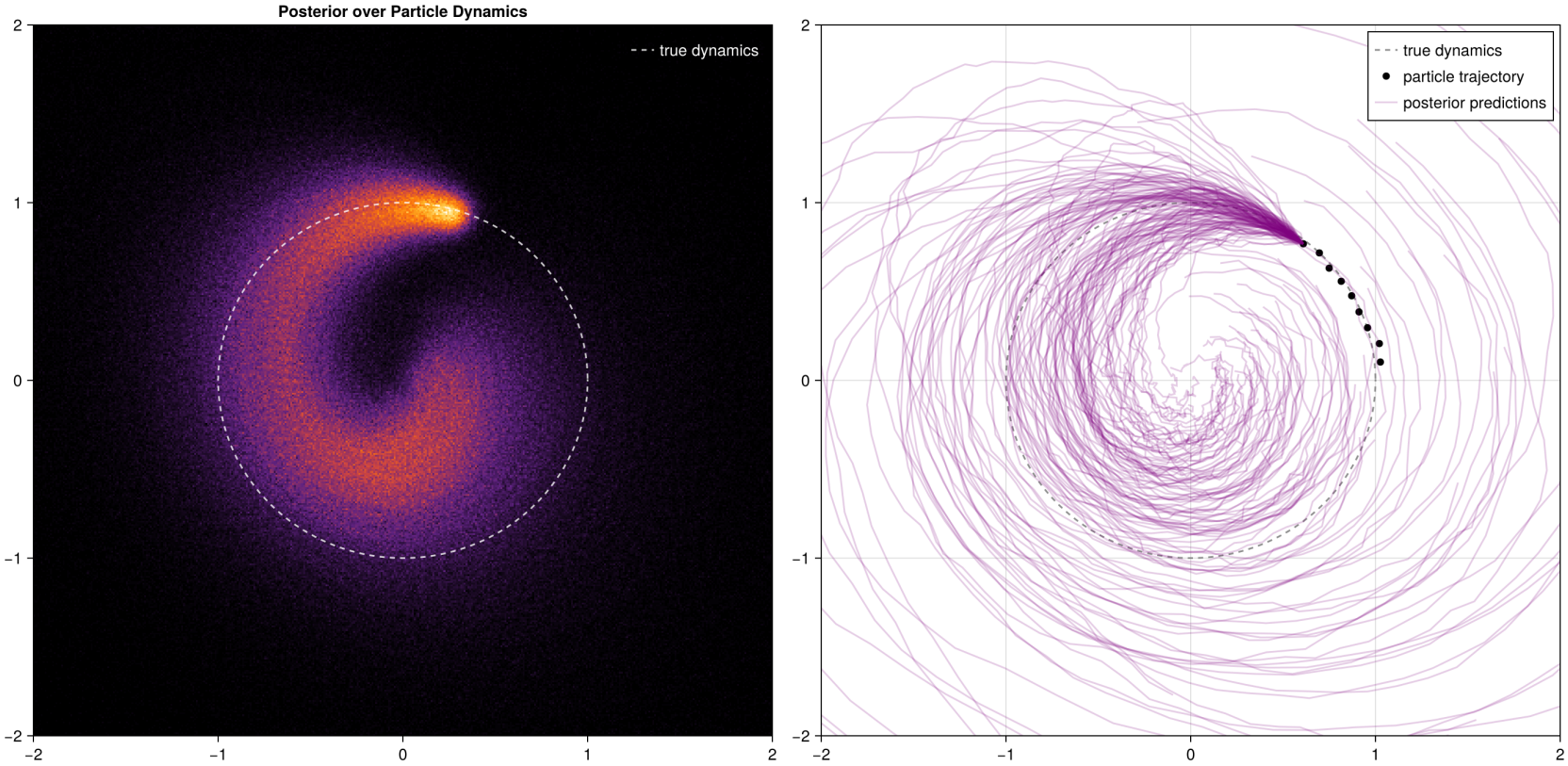

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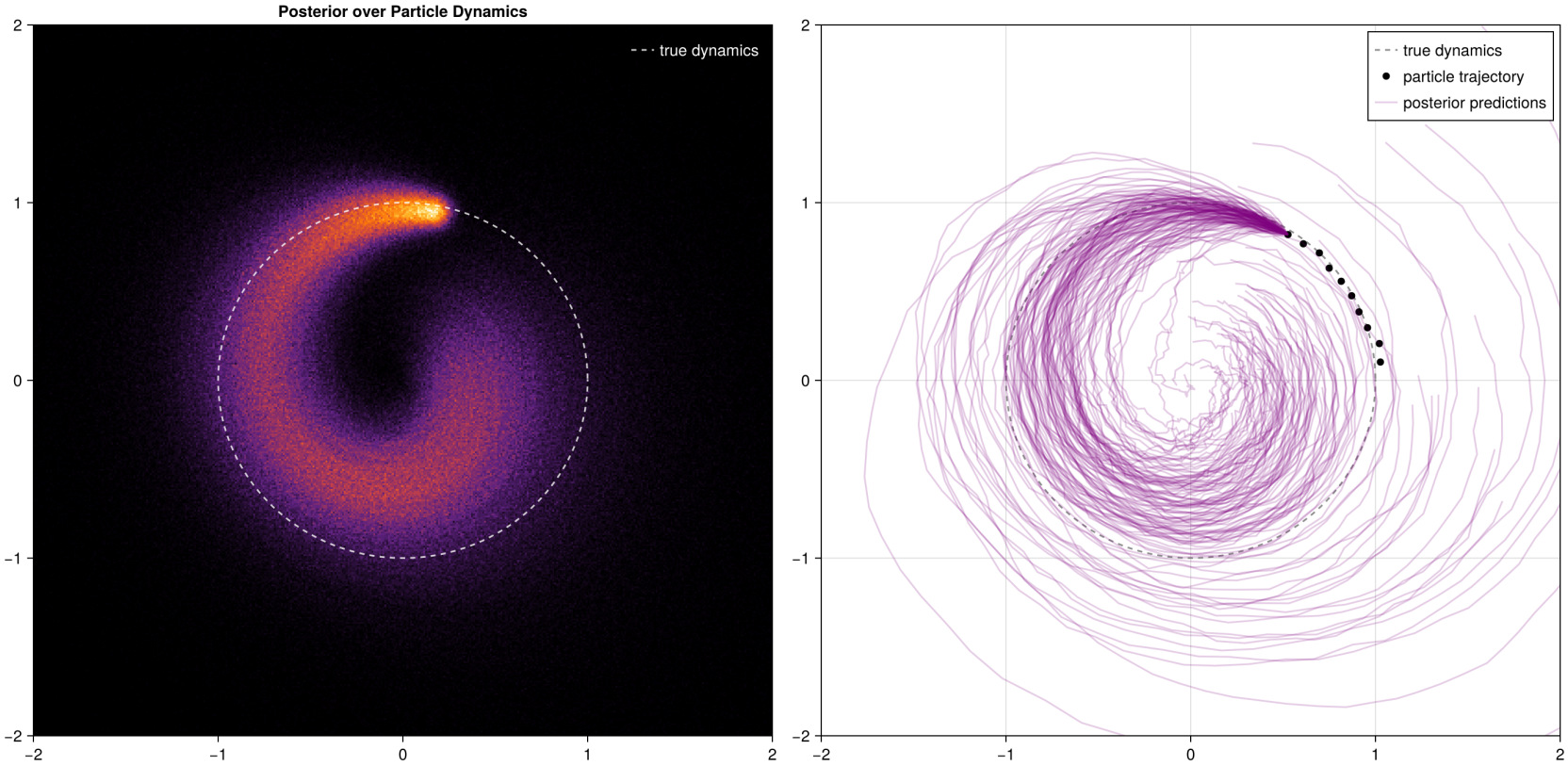

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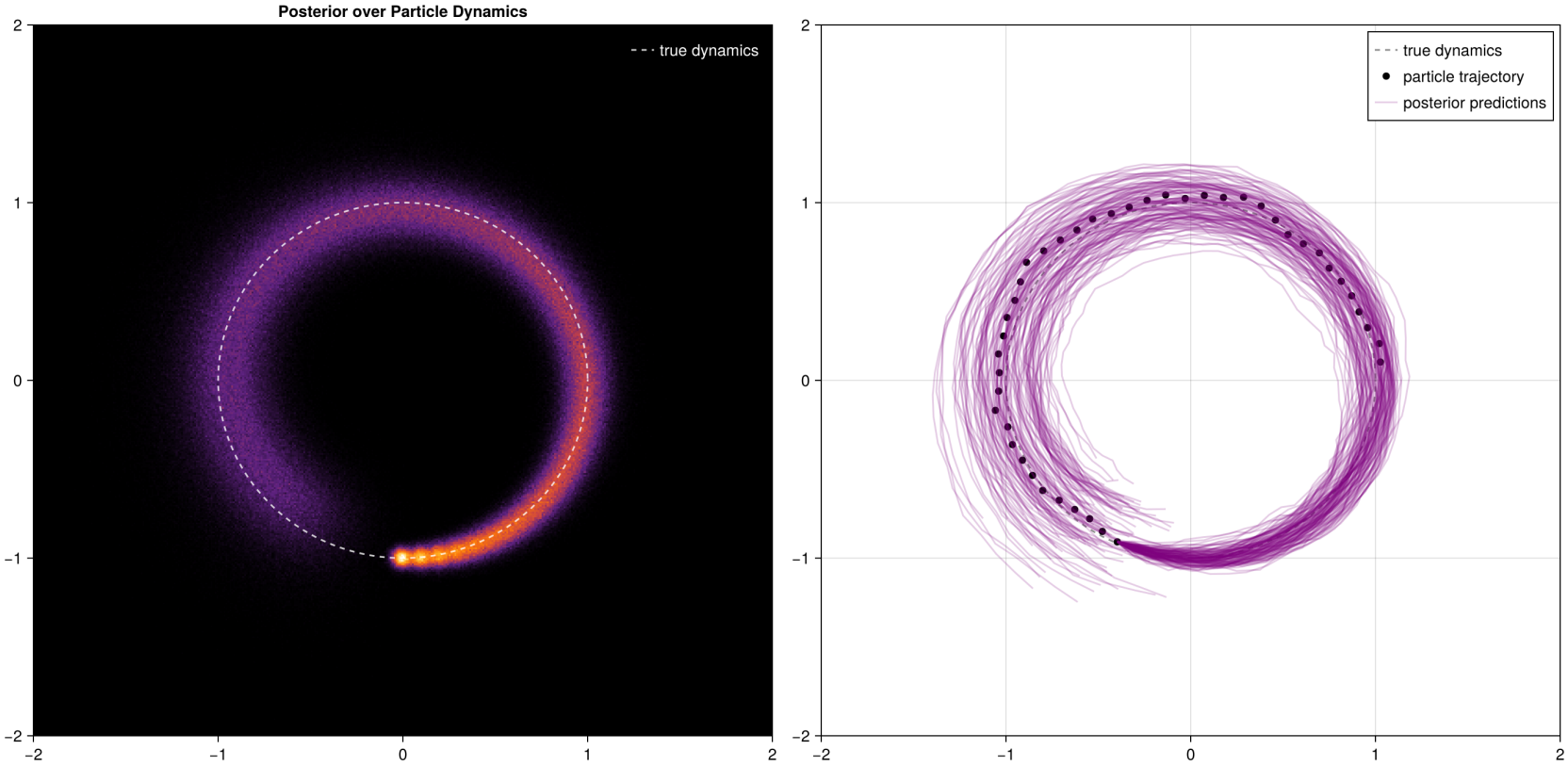


Latent ODE

Autoregressive models and in general Dynamic Bayesian Networks are discrete-time models.

Can we find a continuous-time model which is computationally tractable?

Bayesian Differential Equations

Latent ODE

Remember our linear Poisson model $\log \lambda_\tau = \alpha\tau + \beta$. A similar model can be described by the following ODE with an unknown parameter and initial value:

$$\begin{aligned}
\frac{d \log \lambda}{d\tau} &= \alpha \\
\log \lambda(1) &= \beta \\
\alpha, \beta &\sim \text{Normal}(0, 1)
\end{aligned}$$

We can solve ODEs in a probabilistic program and infer the unknown parameters!

Latent ODE

SciML: Open Source Software for Scientific Machine Learning

- Advanced differential equation solvers
- Differentiable ODE solvers compatible with Turing

Latent ODE

Install DifferentialEquations.jl:

```
Pkg.add("DifferentialEquations.jl")
```

Define the rate dynamics:

```
function rate_dynamics(dlogλ, logλ, p, t)
    α = p[1]
    dlogλ[1] = α
end
```

Latent ODE

Now we can use the ODE in a probabilistic program:

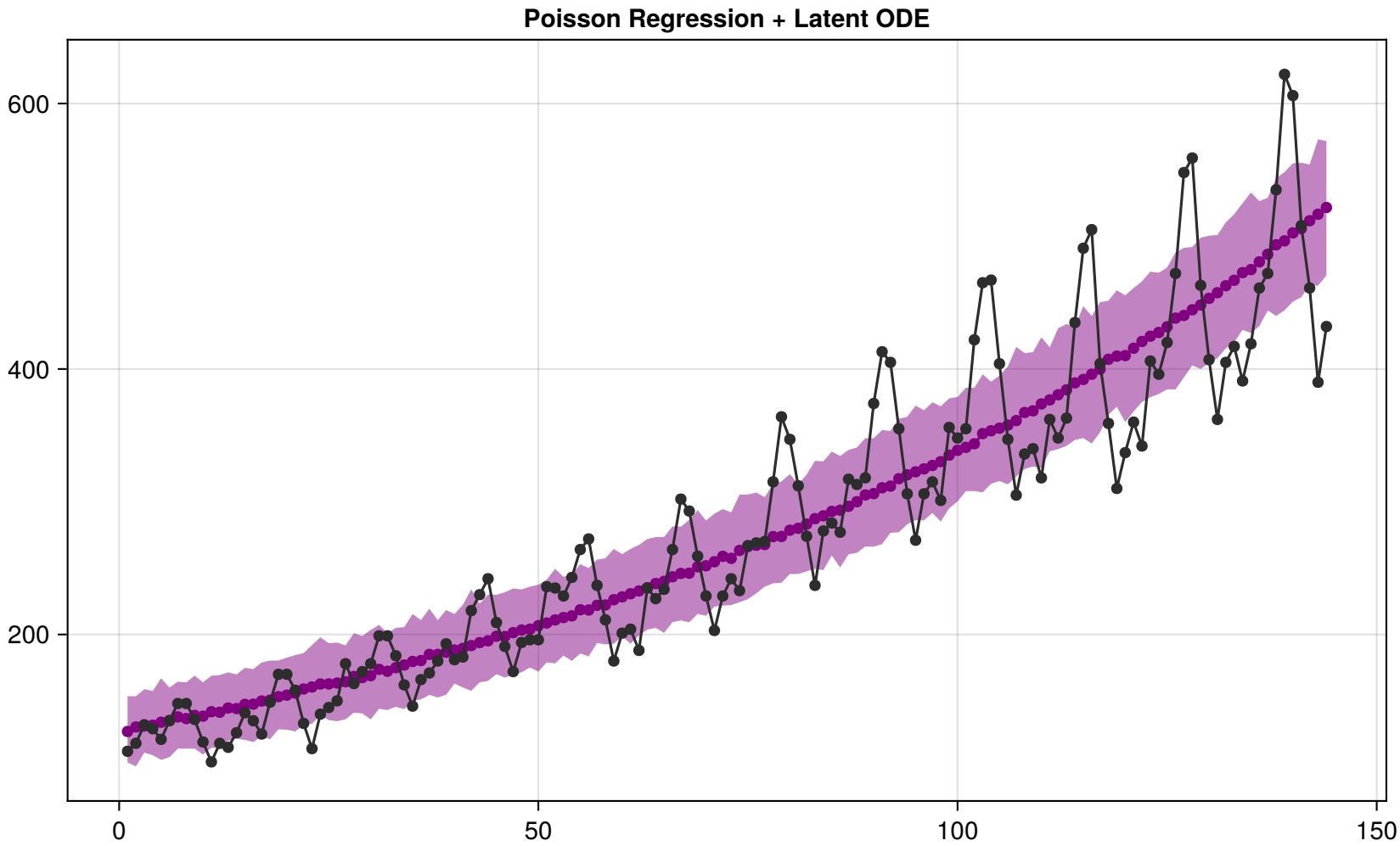
```
@model function latent_ODE_poisson(y)
    α ~ Normal(0.0, 2.0)
    β ~ Normal(0.0, 2.0)

    # solve the ODE for λ
    logλ0 = [β]
    tspan = (1.0, length(y))
    p = [α]
    prob = ODEProblem(rate_dynamics, logλ0, tspan, p)

    logλ = solve(prob, Tsit5(); saveat=1.0)

    for τ in 1:length(y)
        λ = clamp(exp(logλ[τ][1]), 100.0, 1000.0)
        y[τ] ~ Poisson(λ)
    end
end
```

Latent ODE



Latent ODE

Next we try a system of ODEs with latent parameters

$$\frac{du}{d\tau} = \alpha_1$$
$$\frac{dz_1}{d\tau} = \alpha_2 z_2(\tau) + \alpha_3$$
$$\frac{dz_2}{d\tau} = \alpha_4 z_1(\tau) + \alpha_5$$
$$\lambda(\tau) = e^{u(\tau)} + e^{z_1(\tau)}$$

```
function rate_dynamics(dlogλ, logλ, p, t)
    α1, α2, α3, α4, α5 = p
    dlogλ[1] = α1
    dlogλ[2] = α2 * logλ[3] + α3
    dlogλ[3] = α4 * logλ[2] + α5
end
```

Latent ODE