BDA3 Chapter 2 - Q19 - Conjugate Prior

$$
\begin{aligned}
p(\theta | y_{i:n}, \gamma, \alpha_0, \beta_0) &\propto \left[\prod_{i=1}^{n} \theta e^{-\theta y_i} \right] \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{\alpha_0 - 1} e^{-\beta_0 \theta} \\
&\propto \theta^n e^{-\theta \sum_{i=1}^{n} y_i} \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{\alpha_0 - 1} e^{-\beta_0 \theta} \\
&\propto \theta^{n + \alpha_0 - 1} e^{-\theta \left(\sum_{i=1}^{n} y_i + \beta_0 \right)}
\end{aligned}
$$

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$$p(\theta|y_{i:n}, \gamma, \alpha_0, \beta_0) \propto \theta^{n+\alpha_0-1} e^{-\theta\left(\sum_{i=1}^{n} y_i + \beta_0\right)}$$

Is another gamma distribution with parameters:

$$\alpha' = n + \alpha_0$$
$$\beta' = \sum_{i=1}^{n} y_i + \beta_0$$

From which we can also compute the appropriate normalisation constants.

Let

$$\phi = \frac{1}{\theta}$$

We find the distribution of the transformed parameter through application of the rule:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| .$$

In this case functions g, and g^{-1} just compute the reciprocal of their input.

$$
\begin{aligned}
p(\phi) &= p(\theta) \left| \frac{d}{d\phi} \theta \right| \\
&= p(\theta) \frac{1}{\phi^2} \\
&= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{\alpha_0 - 1} e^{-\beta_0 \theta} \frac{1}{\phi^2} \\
&= C \cdot \frac{1}{\phi^{\alpha_0 - 1}} e^{-\frac{\beta_0}{\phi}} \frac{1}{\phi^2} \\
&= C \cdot \phi^{-\alpha_0 - 1} e^{-\frac{\beta_0}{\phi}}
\end{aligned}
$$

The length of life of a light bulb manufactured by a certain process is exponentially distributed with unknown rate θ:

$$y \sim \mathit{exponential}(\theta)$$

Additionally, assume that the prior for θ is a gamma distribution with **coefficient of variation**[1] equal to a half:

$$y \sim \textit{exponential}(\theta)$$
$$\theta \sim \textit{gamma}(\alpha_0, \beta_0)$$

such that:

$$C_v = \frac{\sqrt{\mathbb{V}[\theta]}}{\mathbb{E}[\theta]} = 0.5$$

[1] Standard deviation divided by mean

How many light bulbs must be tested to reduce C_V to 0.1?

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- We saw from the chapter that the average posterior variation is less that the prior.
- This question is about how many observations are needed to get a desired reduction in posterior uncertainty.

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Since the posterior has the same form of the prior, only need to consider the properties of the gamma distribution.

$$\sqrt{\mathbb{V}[\theta]} = \frac{\sqrt{\alpha_0}}{\beta_0}$$
$$\mathbb{E}[\theta] = \frac{\alpha_0}{\beta_0}$$

$$C_v = \frac{\beta_0\sqrt{\alpha_0}}{\beta_0\alpha_0} = 0.5$$

Our coefficient of variation for the prior is equal to:

$$C_v = \alpha_0^{-1/2} = 0.5$$

which has:

$$\alpha_0 = 4$$

Similarly for the posterior the coefficient of variation is:

$$\begin{aligned} 0.1 &= (n + \alpha_0)^{-1/2} \\ &= (n + 4)^{-1/2} \end{aligned}$$

so

$$n = 96$$

For the inverse gamma distribution our model is

$$\propto \phi^{-n-\alpha_0-1} e^{\frac{1}{\phi}\sum y_i+\beta_0}$$

Our coefficient of variation is equal to

$$C_v = (\alpha - 2)^{-1/2}$$

For the prior we get

$$\alpha_0 = 6$$

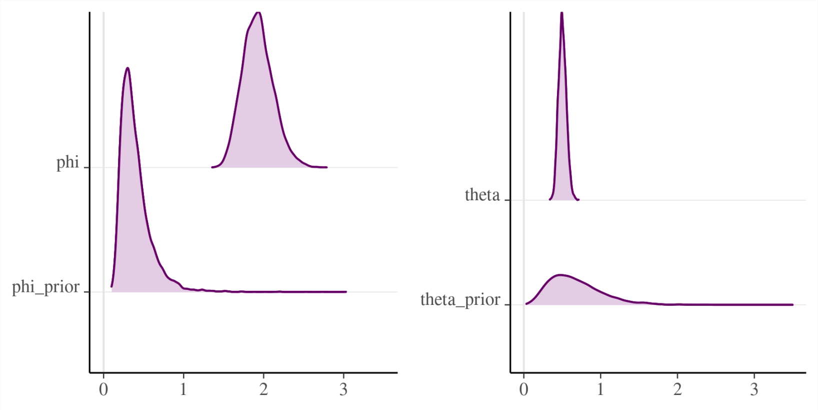
for the posterior

$$C_v = 0.1 = (\alpha_0 + n - 2)^{-1/2} = (n + 4)^{-1/2}$$

Which is the same as what we observed when using the standard gamma distribution.

- The gamma and inverse gamma have an equivalent effect on their respective posteriors.
- Any reduction in the coefficient of variation is independent of the choice of β_0

Simulation



$$C_V^{\theta} \approx 0.102$$
$$C_V^{\phi} \approx 0.099$$