A primer for lattice QCD simulations

1 Quenched gauge field configurations

The gauge action is ($\xi_4 = 1$, but showing it makes the expression look nicer)

$$S_G[U] = \sum_{x,\mu,\nu>\mu} \frac{\beta}{u_\mu^2 u_\nu^2} \frac{\xi_1\xi_2\xi_3\xi_4}{\xi_\mu^2\xi_\nu^2} \left[P_{\mu\nu}(x) + \frac{g_\mu}{3}\left(P_{\mu\nu}(x) - \frac{R_{\mu\nu}(x)}{4u_\mu^2}\right) + \frac{g_\nu}{3}\left(P_{\mu\nu}(x) - \frac{R_{\nu\mu}(x)}{4u_\nu^2}\right)\right]$$

with

$$P_{\mu\nu}(x) = 1 - \frac{1}{3}\mathrm{ReTr}\left[U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)\right]$$

is the $\mu \times \nu = 1 \times 1$ plaquette and

$$R_{\mu\nu}(x) = 1 - \frac{1}{3}\mathrm{ReTr}\left[U_\mu(x)U_\mu(x+\mu)U_\nu(x+2\mu)U_\mu^\dagger(x+\mu+\nu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)\right]$$

is the $\mu \times \nu = 2 \times 1$ rectangle (not 1×2 rectangle).

The user-defined parameters are:

$$\begin{aligned}
\beta &= \text{bare gauge field coupling} \\
g_\mu &= \begin{cases} 0, & \text{for no improvement in the } \mu \text{ direction} \\ 1, & \text{for improvement in the } \mu \text{ direction} \end{cases} \\
\xi_\mu &= \text{lattice spacings in units of temporal spacing} \equiv a_\mu/a_t \\
u_\mu &= \text{tadpole factor in } \mu \text{ direction}
\end{aligned}$$

2 Fermion propagation

The Wilson+clover fermion action is

$$\begin{aligned}
S_F[\bar\psi,\psi,U] &= \frac{1}{2\kappa}\sum_{x,y}\bar\psi(x)\left[A(x,y) - \kappa B(x,y)\right]\psi(y) \\
A(x,y) &= \delta_{x,y}\left[1 + \frac{\kappa c_{SW}}{2}\sum_{\mu,\nu}\frac{r}{\xi_\mu\xi_\nu} i\sigma_{\mu\nu}F_{\mu\nu}(x)\right] \\
B(x,y) &= \sum_\mu \frac{1}{\xi_\mu^2 u_\mu}\left[(r-\xi_\mu\gamma_\mu)U_\mu(x)\delta_{x+\mu,y} + (r+\xi_\mu\gamma_\mu)U_\mu^\dagger(y)\delta_{x-\mu,y}\right] \\
F_{\mu\nu}(x) &= \frac{1}{8u_\mu^2u_\nu^2}\left[Q_{\mu\nu}(x) - Q_{\mu\nu}^\dagger(x)\right]
\end{aligned}$$

$$
\begin{aligned}
Q_{\mu\nu}(x) &= U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) \\
&+U_\nu(x)U_\mu^\dagger(x-\mu+\nu)U_\nu^\dagger(x-\mu)U_\mu(x-\mu) \\
&+U_\mu^\dagger(x-\mu)U_\nu^\dagger(x-\mu-\nu)U_\mu(x-\mu-\nu)U_\nu(x-\nu) \\
&+U_\nu^\dagger(x-\nu)U_\mu(x-\nu)U_\nu(x+\mu-\nu)U_\mu^\dagger(x)
\end{aligned}
$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = -\sigma_{\nu\mu}$$

The definition agrees with the isotropic result of Luscher, Sint, Sommer and Weisz, hep-lat/9605038, who carefully record their conventions for Dirac matrices and $\sigma_{\mu\nu}$.

For the anisotropic action, Alford, Klassen and Lepage NPB496, 377 (1997) used $r = 1$ but Groote and Shigemitsu PRD62, 014508 (2000) used $r = a_s/a_t$ (for a spatially isotropic action). See Aoki et al hep-lat/0107009 for a discussion of both options; they conclude in favour of Groote and Shigemitsu. However, Harada, Kronfeld, Matsufuru, Nakajima and Onogi, hep-lat/0103026 seem to make the opposite choice. It is not immediately obvious how to generalize Groote and Shigemitsu, to a spatially-anisotropic lattice.

In my codes, the convention for Dirac algebra is

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

which leads to

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

A useful comparison for the isotropic version of this action is Luscher, Sint, Sommer and Weisz, hep-lat/9605038.