Exact Primal-Dual Certificates for a Two-Bidder, Two-Item Auction

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Abstract

Multidimensional incentives couple allocations across reports and across bidders. For two additive bidders and two items with independent uniform values, we combine complete joint menu changes with a common capacity-price certificate. An explicit randomized, pointwise DSIC and ex-post IR mechanism has exactly integrated algebraic revenue $0.8764641644718\ldots$. An unrestricted upper bound of $0.8829230532587\ldots$ leaves a certified gap below 0.006459, hence below 0.01, and guarantees more than 99.2684% of optimal revenue. On the primal side, a monotone exchange threshold with a jump and an inverse plateau changes ownership on a region of positive measure. Occupation constraints and convex-utility traces solve full randomized conditional screening on 64.1627% of opponent-report space. On the dual side, a reusable conditional support and four sparse long-range IC cycles strictly reduce the total common capacity price while preserving validity for both bidders. An exact slack identity separates information rents, unused capacity and incentive losses from numerical enclosure remainders. Independent implementations reconstruct the entire new revenue and the stream integral's continuous majorant; the conditional subtraction has a full exact calculation and separate bounded cross-checks. The final measure has strictly positive slack against the selected mechanism. The unrestricted optimum and a matching auction remain open.

1 Introduction

Optimal auction design is completely understood for a single item under the classical regularity conditions of Myerson [6]. With several items, bidder types become multidimensional and dominant-strategy incentive compatibility imposes global cross-type restrictions. Even the canonical case of two additive bidders, two items, and independent uniform values has resisted an exact characterization [4, 8].

Computational work has produced strong feasible mechanisms and rigorous upper bounds. Affine-maximizer searches provide transparent deterministic benchmarks [8], while GemNet constructs fully strategyproof menu mechanisms after a compatibility transformation [9]. Jiang, Parkes, and Wang report approximately 0.876 revenue for that GemNet benchmark on the present instance. On the dual side, the same paper develops flow-conserving DSIC kernels and a continuous lifting theorem; its strict 50×50 uniform continuous certificate is 0.8919 [4]. These are external benchmarks. The GemNet value is a reported computational revenue, whereas 0.8919 is a rigorous upper certificate. We improve that upper benchmark and provide an explicitly specified mechanism with independently integrated exact revenue. Its value exceeds the printed 0.876, but that rounded external report does not prove a ranking against GemNet's unrounded revenue.

Our approach treats feasibility and information rents together. Fixing one bidder leaves a full multidimensional screening problem under the residual capacity supplied by its allocation. Convex utility traces turn some occupied regions into restrictions on all admissible conditional menus. We use these restrictions to prove full randomized screening bounds and to identify which changes must involve both bidders. A jump in a monotone exchange threshold and the corresponding inverse plateau then implement a positive-volume ownership transfer with exact revenue accounting.

A conditional certificate alone does not bound the whole auction unless its capacity price also controls the opposing bidder. We construct such a lower-cost common price by combining a universal single-buyer support with a continuous sign certificate for the opposing stream field. Four nonnegative long-range IC cycles preserve flow balance and decrease the integrated virtual maximum. Exact tensor-Bernstein integration supplies a rational upper bound for the resulting continuous measure. The original stream maximum already gives zero charged screening values for both bidders. The advance is to reduce its total price while preserving those inequalities; an equality allocation remains missing.

Contributions. The paper develops four connected components:

1. a complete jointly feasible functional-exchange mechanism with a jump/inverse-plateau pair, an explicit rule for all reports and ties, and an independently reconstructed algebraic revenue;

2. full randomized screening certificates on specified actual residuals, including an occupied-wing identity that charges a boundary information rent over the strip on which incentives force it to persist;

3. a reusable conditional support and sparse long-range IC cycles that strictly decrease a common capacity measure's total mass while preserving both unrestricted charged inequalities, and an exact separation of mechanism slack from numerical remainder; and

4. certified endpoints of $0.87646416447\ldots$ and $0.88292305326\ldots$, gap below 0.006459, and a revenue guarantee exceeding 99.2684% of unrestricted optimal revenue.

The retained stream weak-duality proof, deterministic predecessors and earlier residual-support constructions explain the infrastructure. The active lower mechanism is specified in Section 6.7, and the common upper certificate in Section 7.5. Finite menus describe our constructions and equality cases only. No finite-range restriction is imposed on competing mechanisms, and conditional optimality is not promoted to global optimality.

2 Relation to prior dual formulations

Rochet's cyclic-monotonicity characterization turns DSIC into convexity of truthful utility and identifies allocation vectors with subgradients [7]. Continuous duality for the one-bidder multi-item problem has been developed through transport and partial-derivative formulations [2, 3]. These results motivate integration-by-parts certificates but do not directly enforce item supply across several bidders.

Multi-bidder dual formulations also predate the present work. Finite-type flow-induced virtual values arise in unified Bayesian mechanism-design duality [1]; Zuo considers both Bayesian and dominant- strategy formulations [10]; and Beckmann-style transport gives a continuous strong-duality formulation for multi-item, multi-bidder BIC auctions in interim space [5]. The latter problem uses weaker incentive and individual-rationality requirements than the pointwise DSIC and ex-post IR requirements studied here.

Most directly, Jiang, Parkes, and Wang construct nonnegative, flow-conserving DSIC deviation kernels, derive continuous weak duality, and lift coarse uniform-grid certificates to continuous type spaces [4]. Our proof shares the same high-level weak-duality architecture: conservation eliminates payments, the dual object induces virtual coefficients, and ex-post feasibility bounds virtual surplus itemwise. The representation and certification are different. We use a signed vector field with a distributional radial source plus a divergence-free tangent stream correction, and prove its validity directly against Lipschitz convex utilities. We do not assert that every such stream field is generated by a nonnegative deviation kernel, or conversely. The contribution is an explicit rational witness for this canonical DSIC instance, an exact Bernstein certificate for its continuous objective, and explicit conditional capacity supports that guide feasible two-bidder reallocations. The convex-mixture concavity of residual value is standard infrastructure; the new theorem content is the displayed residual-specific measures, their full-class slack identities, and the resulting complete primal variations. We make no first-in-literature priority claim for the general use of convex potentials, transport, or complementary slackness.

3 Setting and proved bounds

Write a report profile as

$$(x, y, z, w) = (v_{11}, v_{12}, v_{21}, v_{22}) \in [0, 1]^4.$$

The two bidders are risk-neutral and additive; the two items are indivisible. The four coordinates are independent and uniform. A direct mechanism specifies jointly measurable expected allocation probabilities $x_{ij} : [0, 1]^4 \to [0, 1]$ and expected payments $p_i : [0, 1]^4 \to \mathbb{R}$, with $p_i \in L^1([0, 1]^4)$. At every report profile, itemwise feasibility requires $x_{1j} + x_{2j} \leq 1$. These marginals can be realized by assigning each item to bidder 1, bidder 2, or the seller with the corresponding probabilities. DSIC and ex-post IR are required at every type and opponent report, with utilities averaged over the mechanism's internal randomization. Let OPT be the supremum of expected revenue over this class. Our constructed mechanisms are Borel. The upper argument also covers joint Lebesgue measurability with integrable payments: on every opponent fiber, DSIC supplies a convex Lipschitz utility and the selected allocation equals its gradient off an own-type null set; Fubini supplies the integrated inequality. No pointwise Borel-representative repair is asserted. For the coordinate-line supports below, tangential selections are monotone and their line integrals are well-defined. Residual functions are kept pointwise, not identified modulo area-null sets.

Theorem 1 (Audited coordinated bracket)**.** *The complete mechanism in Proposition 10 has exact revenue R_* in* (12)*, with*

$$0.876464164471798049944906113027 < R_* < 0.876464164471798049944906113028.$$

For the unrestricted randomized class,

$$R_* \leq \mathrm{OPT} \leq U_* < 0.882923053259,$$

where the rational U_* is specified in (19)*. In particular,*

$$0.006458888786918919315720686827 < U_* - R_* < 0.006458888786918919315720686828 < \frac{1}{100},$$

$$\frac{R_*}{\mathrm{OPT}} \geq \frac{R_*}{U_*} > 0.992684.$$

Thus the mechanism earns more than 99.2684% *of unrestricted optimal revenue. These bounds do not identify* OPT *or decide its attainment.*

Current claim	Proof location and role
Complete mechanism	Proposition 10 specifies all menus, exceptional reports and jointly feasible tie selection.
Exact revenue	Equation (12) and Section 6.7 integrate the final mechanism in full.
Conditional screening	Section 6.8 proves full randomized optimality on Q/E/W under the actual residual capacities.
Unrestricted upper	Theorem 15 and (19) give the common measure and rational endpoint.

Table 1: Proof roadmap for the active bracket. Earlier constructions are retained as infrastructure; their revenue layers are not the definition of the current R_*.

Conditional screening fixes the other bidder's allocation. Its optimality does not rule out joint changes that also alter the residual capacity.

The lower follows from the complete pointwise construction and exact continuous integration; the upper follows from the common measure, sign and cycle proofs and the componentwise checks described below. The historical stream endpoint remains

$$U_{\text{stream}} = \frac{3715139591287203}{4194304000000000} = 0.8857583025186545848846435546875,$$

but the active U_* is strictly smaller. Relative to the external continuous upper 0.8919 reported by Jiang, Parkes and Wang [4], U_* is lower by more than 0.008976946741. This compares exact local certificates with their printed upper decimal, not a replay of their dual array. The same source's GemNet revenue ≈ 0.876 remains a rounded computational benchmark; the new exact revenue exceeds that printed number but cannot by itself establish superiority to the unrounded mechanism.

4 An exact base affine maximizer

Set

$$a = \frac{159}{250}, \qquad b = \frac{91}{100}, \qquad s = \frac{1137}{1000}.$$

For each feasible outcome A, let c_A and the affine score be as follows. The row number is also the tie-breaking priority.

id	allocation	score	c_A
0	seller keeps both	0	0
1	$1 \to 1$	$x - a$	a
2	$2 \to 1$	$y - a$	a
3	$\{1,2\} \to 1$	$x + y - b$	b
4	$1 \to 2$	$z - a$	a
5	$2 \to 2$	$w - a$	a
6	$\{1,2\} \to 2$	$z + w - b$	b
7	$1 \to 1,\ 2 \to 2$	$x + w - s$	s
8	$2 \to 1,\ 1 \to 2$	$y + z - s$	s

The base outcome $A^0(v)$ is the smallest-id maximizer of the nine scores. This defines the outcome on every equality surface.

Let

$$H_1(z,w) = \max\{0, z-a, w-a, z+w-b\}, \quad H_2(x,y) = \max\{0, x-a, y-a, x+y-b\}.$$

If $V_i(A)$ is bidder i's reported value for the items assigned in A, the base payments are

$$p_1^0 = H_1 - V_2(A^0) + c_{A^0}, \qquad p_2^0 = H_2 - V_1(A^0) + c_{A^0}.$$

Lemma 2. *The base mechanism is feasible, pointwise DSIC, and pointwise ex-post IR.*

Proof. Every listed outcome is feasible. Fix bidder 2's report. Apart from the constant H_1, bidder 1's utility from an induced outcome A, evaluated at the true type, is

$$V_1(A) + V_2(A) - c_A - H_1.$$

Truthful reporting selects a maximizer of the expression preceding H_1; no report can give greater utility. The same argument applies to bidder 2. It remains valid on ties because the selected row is still a maximizer. The outcomes excluding bidder 1 attain H_1, so the winning score is at least H_1, and bidder 1's truthful utility is nonnegative. The symmetric argument proves bidder 2's IR. □

For a bidder facing opponent report $q = (q_1, q_2)$, define

$$H(q) = \max\{0, q_1 - a, q_2 - a, q_1 + q_2 - b\}.$$

The base taxation menu has prices

$$A_0(q) = H(q) + \min\{a, s - q_2\}, \quad B_0(q) = H(q) + \min\{a, s - q_1\}, \quad C_0(q) = H(q) + b \tag{1}$$

for item 1, item 2, and the bundle, respectively; opt-out costs zero. These formulas follow directly by minimizing the outcome cost net of the opponent's value among outcomes assigning the indicated bundle.

5 The deterministic entry-surcharge predecessor

Opponent-conditioned taxation menus ensure each bidder's DSIC, but their separate choices need not be simultaneously feasible. The following containment principle supplies that link.

Lemma 3 (Deletion containment)**.** *Let a deterministic predecessor mechanism have feasible allocations $S_i^0(v)$. For each bidder and opponent report, fix a finite taxation menu containing opt-out at price zero. Suppose its declared tie rule selects a utility-maximizing bundle $S_i(v) \subseteq S_i^0(v)$ at every report profile. Then the resulting mechanism is pointwise DSIC, ex-post IR, and ex-post feasible.*

Proof. Fixing the opponent report fixes all available bundles and prices. A truthful utility maximizer cannot gain by reporting another type, and opt-out guarantees nonnegative utility. Containment preserves the predecessor's disjoint item allocations. The assumptions concern the selected bundles, including ties, so all three conclusions hold pointwise. □

For an opponent report $q = (q_1, q_2)$, set

$$t = \max\{q_1, q_2\}, \qquad \rho = \min\{q_1, q_2\}.$$

The retained deterministic predecessor is the finite rational rule frozen in

`certificate/refined_item_containment_bundle_pivot_lower_bound/manifest.json`.

The complete evaluable tables and endpoint conventions are specified by this manifest and its hash-bound predecessor manifests. That predecessor manifest contains the eight item-containment rows. The twenty Z/S rows and 41 bundle-pivot rows are specified, respectively, in

`certificate/piecewise_surcharge_twenty_band_lower_bound/manifest.json`
`certificate/piecewise_surcharge_bundle_pivot_lower_bound/manifest.json`.

We summarize the three-layer construction here.

First, twenty rational Z/S half-bands add a common fee $F(q)$ to all three nonempty prices in (1). If none matches, set

$$u = t + \rho, \qquad c = b - a = 137/500, \qquad d = s - a = 501/1000,$$

and first test the bundle-pivot region

$$91/100 \le u \le 1, \qquad c \le \rho \le u - d.$$

Only inside this region define $r = (\rho - c)/(u - c - d)$, whose denominator satisfies $u - c - d \ge 27/200 > 0$, and test the 41 positive cells in (u, r). Outside the region no bundle-pivot fee is applied and r need not be evaluated. Those cells also add one common fee to all nonempty prices. The Z/S rows have priority on shared boundaries. Their interiors are disjoint from the bundle cells: Z support has $u \le b$ and S support has $\rho \le c$, whereas a bundle-cell interior has $u > b$ and $\rho > c$.

Finally, eight rows inside the two S5 half-bands add $\delta(q)$ to the singleton corresponding to the opponent's low coordinate and to the grand bundle, leaving the other singleton unchanged. This coordinate is unique: on every item row, $t > 139/200 > 137/500 \ge \rho$. Each row is left-open and right-closed in t; at $t = 139/200$, $\delta = 0$. If maximum utility is zero, choose opt-out. At a positive tie, retain the predecessor-selected outcome whenever it remains maximizing; otherwise choose the maximizing itemwise subset with smallest id under the fixed predecessor order.

Proposition 4. *The refined surcharge mechanism is deterministic, measurable, ex-post feasible, pointwise DSIC, and pointwise ex-post IR on the entire cube, including all menu, band, cell, and item-surcharge boundaries.*

Proof. Every common fee preserves the ranking among nonempty bundles, so its selected bundle is the base bundle or opt-out. On an item row, orient the predecessor prices as

$$(A, B, C) = (d + F, t + F, t + c + F).$$

The row changes them to $(A + \delta, B, C + \delta)$, and its exact verifier checks

$$0 < \delta < A + B - C = d - c + F.$$

If the changed singleton was selected, its utility lead over the unchanged singleton is at least $A + B - C$, and hence remains positive after the change; its ranking against the bundle is unchanged. The unchanged singleton remains selected when it was selected, a bundle buyer can move only to a subset, and an opt-out cannot enter because no nonempty price falls. Thus every final bundle is an itemwise subset of its predecessor bundle, which is itself a subset of the feasible base allocation.

For each fixed opponent report the declared rule selects a utility maximizer from a taxation menu independent of the bidder's report, with opt-out at price zero. Lemma 3 therefore proves DSIC, ex-post IR, and simultaneous feasibility at each layer. The finite rational semialgebraic tables and literal endpoint priority are Borel measurable. Numerical searches selected rational rows and fees, but no discovery file is a verifier input. □

6 Exact revenue and residual screening

Lemma 5 (Revenue of a four-option menu)**.** *Suppose*

$$0 \le A, B \le C \le 1, \qquad C \le A + B.$$

Let an additive type be uniform on $[0,1]^2$. The expected payment from the four-option menu is

$$r(A,B,C) = A(1-A)(C-A) + B(1-B)(C-B) \\ + C\left[(1-C+B)(1-C+A) - \frac{(A+B-C)^2}{2}\right].$$

Proof. The strict item-1 region has area $(1-A)(C-A)$, and the strict item-2 region has area $(1-B)(C-B)$. The bundle region starts with the rectangle $[C-B,1]\times[C-A,1]$ and removes a right triangle of side $A+B-C$. Multiplying the three areas by their prices proves the formula. Ties have area zero for this integral; their pointwise allocation was already fixed in Proposition 4. □

Suppose both (A,B,C) and $(A+e,B+e,C+e)$ satisfy Lemma 5. For the nonnegative common surcharges used here, the original price conditions together with $C+e\le 1$ ensure this. Direct expansion then gives the actual revenue change

$$r(A+e,B+e,C+e) - r(A,B,C) = eG - \frac{3C}{2}e^2 - \frac{1}{2}e^3, \tag{2}$$

where

$$G = (C-A)(1-2A) + (C-B)(1-2B) + (1-C+A)(1-C+B) \\ - \frac{(A+B-C)^2}{2} - C(A+B-C).$$

For a Z-band orientation the prices and cross-section length are

$$(A,B,C) = (s-t,a,b), \qquad b-t,$$

and for an S-band orientation they are

$$(A,B,C) = (t,s-a,t+b-a), \qquad b-a.$$

Exact integration of the twenty frozen half-bands gives

$$G_Z = \frac{113291698497241117}{1000000000000000000000}, \qquad G_S = \frac{2400553481931223}{31250000000000000000},$$

and therefore the common-fee gain

$$G_Z + G_S = \frac{190109409919040253}{1000000000000000000000}.$$

On a bundle-pivot cell the prices are

$$(A,B,C) = (u-c,\rho+227/1000,u).$$

The normalized map has Jacobian $u-31/40$; including both orientations and both bidders gives symmetry factor four. Direct bivariate integration of the 41 positive cells gives

$$G_B = \frac{70946101529751}{10240000000000000000}.$$

The eight item-containment rows give

$$G_I = \frac{3963982653557301}{800000000000000000000}.$$

The base mechanism's exact expected revenue is

$$R_0 = \frac{26232089810531183}{30000000000000000}.$$

The exact verifier in `certificate/ama_lower_bound` reconstructs the $9 \cdot 4 \cdot 4 = 144$ affine/pivot cells using rational arithmetic, finds 39 full-dimensional cells, triangulates them into 16,336 rational four-simplices, checks total volume one, and integrates the affine payments.

The added gains are integrated layer by layer. The Z/S and bundle-pivot supports have disjoint interiors. Item-containment rows lie within the S5 support, but G_I is the change relative to the already surcharged predecessor menu, not relative to the base. Thus the revenue sum telescopes without double counting.

The deterministic predecessor checker binds every input by SHA-256, directly reconstructs the eight item-containment price regimes, checks their deletion slack and coverage, and integrates their exact symbolic polynomials. Its non-importing replay intersects the linear utility-comparison half-planes with $[0, 1]^2$. At rational prices, the resulting demand polygons have rational vertices, so their areas and conditional revenues are computed exactly. On each item row, the prices are affine in t and the price-regime inequalities hold throughout the interval by exact endpoint checks. Lemma 5 therefore makes each conditional revenue a polynomial of degree at most three in t. Boole's rule, which integrates polynomials of degree at most five exactly, then gives the row integral in rational arithmetic. The common-fee rows remain within that degree range after multiplication by their affine cross-section lengths. On bundle cells, the prices are affine in each of u, r; including the Jacobian gives degree at most four in u and three in r, so tensor Boole quadrature is exact. Finite differences at the sampled rational points are consistency checks, not proofs of a polynomial degree bound on the whole interval. The hash-bound sealed predecessors separately recompute the 20 common-fee rows and 41 positive bundle-pivot cells; the bundle predecessor has its own direct bivariate integration and independent scalar tensor-quadrature replay. These replays independently recompute the added gains, but share the base value R_0 certified by the single polytope verifier. Table 2 summarizes the dependencies. SHA-256 bindings establish the identity of these shared inputs.

Component	Revenue term	Exact integration and replay
Affine base	R_0	One rational polytope verifier
Twenty common-fee rows	$G_Z + G_S$	Polynomial integral; scalar replay
41 bundle-pivot cells	G_B	Bivariate integral; tensor replay
Eight item rows	G_I	Polynomial integral; polygon replay

Table 2: Revenue decomposition. Each added term is measured against its predecessor; all added-gain replays share the same certified base revenue.

Therefore

$$R_0 + G_Z + G_S + G_B + G_I = L_{\text{det}} = \frac{83962078694672281756033}{96000000000000000000000},$$

certifying the retained deterministic predecessor.

The bundle-pivot, twenty-band, ten-band, and former two-rectangle certificates are retained as reproducible predecessors in their separate directories. The construction below is the retained V4.6 lower mechanism and its residual-screening infrastructure. The active mechanism follows in Section 6.7.

6.1 Residual capacity as a screening constraint

Deletion containment gives a convenient sufficient condition for feasibility, but excludes profitable changes in ownership. We instead fix a complete mechanism for bidder 1 and optimize bidder 2 against the residual allocation. For a jointly measurable capacity function

$r : [0,1]^4 \to [0,1]^2$, let

$$\mathcal{V}(r) = \sup\{\mathrm{Rev}_2(M_2) : M_2 \text{ is pointwise DSIC/IR and } x_2 \le r\}.$$

The mechanisms in this definition remain jointly measurable, with integrable payments. Grouping feasible pairs by their first component gives exactly

$$\mathrm{OPT} = \sup_{M_1}\{\mathrm{Rev}_1(M_1) + \mathcal{V}(1 - x_1)\}.$$

This identity does not assert existence of a best response or invoke a measurable selection of pointwise optimizers.

Proposition 6 (Residual value and supporting measures)**.** *The functional $\mathcal{V}$ is nondecreasing and concave. On a fixed opponent fiber, suppose a finite nonnegative capacity-price measure π satisfies $R(u,a) \le \langle \pi, a\rangle$ for every admissible conditional mechanism, and a feasible candidate at r^* attains $R^* = \langle \pi, r^*\rangle$. Then that candidate solves the full randomized conditional problem and, for every residual for which the pairing is defined,*

$$V(r) \le V(r^*) + \langle \pi, r - r^*\rangle.$$

Proof. Monotonicity is immediate. Mix any two admissible mechanisms using one exogenous coin with constant probability θ. All pointwise incentive, IR and capacity inequalities are preserved, as are measurability and integrability. Apply this to arbitrarily near-optimal competitors and pass to the supremum. For the second assertion, $R \le \langle \pi, a\rangle \le \langle \pi, r\rangle$, and equality at r^* gives the claim. No own-report-dependent mixture is used. □

Concavity is the standard convex-mixture observation. The substantive step below is an explicit support for residuals with binding report-line constraints. These supports price information rents as well as item supply. They are conditional certificates. A common measure is constructed in Section 7.5; requiring that measure to match a jointly feasible candidate remains a separate, unresolved condition.

A report line has zero probability under the uniform distribution, but DSIC links utility at that line to utility at all other reports. Its capacity constraint can therefore restrict implementable allocations on a region of positive probability. A line price values this incentive restriction; it does not represent positive-probability sales on the line itself. All residuals in the two conditional theorems below are Borel, so their singular pairings are well-defined. The DSIC comparison class remains that of Section 3.

A sufficient common-certificate interface. On the full report space $\Omega = [0,1]^4$, suppose nonnegative finite measures Π_j, one per physical item, have well-defined pairings with every admissible allocation and satisfy, for each bidder's complete DSIC/IR mechanism,

$$R_i(M_i) \le \sum_{j=1}^{2} \int_\Omega x_{ij}\, d\Pi_j.$$

If both inequalities are equalities at a joint candidate and $\sum_j \int (1 - x^*_{1j} - x^*_{2j})\, d\Pi_j = 0$, summing and using capacity gives

$$R_1(M_1) + R_2(M_2) \le \sum_j \Pi_j(\Omega) = R^*_1 + R^*_2.$$

This is a sufficient interface, not an existence, completeness or strong-duality claim. Conditional supports must first be assembled on the same full report space, with both revenue inequalities and their pairing domains justified. Throughout the following historical construction, the final mechanism means the V4.6 mechanism M_J. The active mechanism M_* is specified later in Section 6.7. The strict joint gain below rules out a matching global certificate for its reference before the release. A matching certificate for the final mechanism is still unknown.

6.2 A diagonal-hole capacity certificate

Use aligned own coordinates (x, y), where item 1 is scarce. Put $A = 2/3$, $k = C - B < 1/2$, $0 \le k$, and

$$B = C - k \le A, \qquad C_0 = 5/6 + 3k^2/4, \qquad C \ge C_0.$$

The candidate menu has prices $(0, A, B, C)$ and utility $u^* = \max(0, x - A, y - B, x + y - C)$. Assume its complete allocation is feasible against a Borel residual r^* and that (the first two conditions are needed only when $m = 2(C - C_0) > 0$)

$$\begin{aligned} r_1^*(x, 0) &= 0 \quad (0 < x < 1/2), \\ r_2^*(1/2, y) &= 0 \quad (0 < y < C - 1/2), \\ r_1^*(x, 1) &= 0 \quad (0 < x < k) \quad \text{if } B < A. \end{aligned} \tag{3}$$

Define $\ell(x) = B$ below k and $\ell(x) = C - x$ on (k, A); let $D_0 = \{x < A, y < \ell(x)\}$. Set $f = 3\ell - 2$ below A and $f = 1$ above A. Direct integration gives

$$m = \int_0^1 f = 2(C - C_0) \ge 0, \qquad 3|D_0| = m + 1.$$

The function $F_m(x) = \int_x^1 f(s)\, ds - m\mathbf{1}_{x<1/2}$ is nonnegative. Indeed, below the anchor it equals $-\int_0^x f$, while above the anchor its tail is nonnegative because $f \le 0$ below A and $f = 1$ above A.

Theorem 7 (Anchored full screening support)**.** *Under these hypotheses the candidate solves the complete randomized conditional problem. A supporting capacity measure is*

$$\begin{aligned} \pi_1 &= 3(x - A)_+\, dx\, dy + F_m(x)\, dx\, \delta_1(dy) + m\mathbf{1}_{0<x<1/2}\, dx\, \delta_0(dy), \\ \pi_2 &= 3(y - \ell(x))_+ \mathbf{1}_{x<A}\, dx\, dy + m\delta_{1/2}(dx)\, dy. \end{aligned}$$

For every Borel residual r and randomized DSIC/IR competitor with selected allocation $a \le r$,

$$\langle \pi, r \rangle - R = \langle \pi, r - a \rangle + 3 \int_{D_0} u - mu(0, 0) \ge 0. \tag{4}$$

Proof. Uniform revenue is $R = \int u(1, y)\, dy + \int u(x, 1)\, dx - 3 \int u$. Horizontal envelopes on $x > A$ and vertical envelopes above $\ell(x)$ give the displayed volume prices and the residual top pairing $\int f(x)u(x, 1)\, dx$. Anchoring it at $1/2$ gives

$$\int f(x)u(x, 1)\, dx = mu(1/2, 1) + \int F_m(x)a_1(x, 1)\, dx.$$

Expand $u(1/2, 1)$ along the bottom edge and then the vertical line at $1/2$. This produces the remaining two prices and (4). Monotonicity and IR imply $u \ge u(0, 0) \ge 0$, so the non-capacity slack is at least $u(0, 0)$ by $3|D_0| = m + 1$.

At the candidate this slack vanishes. Every positive volume price charges a unit marginal. The top price below k is zero if $B = A$ and is saturated by the last condition in (3) otherwise. The other two conditions saturate the bottom and vertical charges. All line endpoints have zero price mass, but retain their complete allocation rules. □

This proof uses the actual tangential allocation on each priced line. Pointwise DSIC makes that component monotone and equal to the convex trace's derivative almost everywhere on the line. It therefore covers arbitrary selected subgradients and allocations with an infinite range. It neither normalizes $u(0, 0)$ to zero nor restricts competitors to a finite menu. At $B = A$, $C = b_0 = (4 - \sqrt{2})/3$, the mass m is zero and no occupied trace is required. This recovers the full-capacity single-buyer support.

6.3 A lottery-junction capacity certificate

Set $c = 137/500$, $q = 113/500$, $T = 613/750$, and $A < t < T$. Write

$$\alpha = 3t - 2, \quad \delta = q + 3q^2/4 - 3qt/2 + \alpha^2/16, \quad \beta = \frac{\alpha}{\alpha + 2\delta},$$
$$B = 1/2 + \delta, \quad C = t + c + \delta, \quad k = t - q, \quad \ell_* = \delta + \alpha/2, \quad Y = c + \ell_*, \quad j = 1 - t/2.$$

Here $0 < k < j < A < t < 1$, $c < Y < B < A$ and $0 < \beta < 1$. Offer, in order,

$$(0, 0; 0),\ (0, 1; B),\ (1, 0; t),\ (1, 1; C),\ (1, \beta; t + \beta c),$$

choosing the first utility maximizer. Assume feasibility at r^* and

$$r_1^*(x, 1) = 0\ (x < k), \qquad r_1^*(x, c) = 0\ (A < x < t). \tag{5}$$

Define $z(y)$ by t below c, $t - \beta(y - c)$ on (c, Y), and A above Y. Define $e(x)$ by B below k, $C - x$ on (k, j), and Y on (j, A). Horizontal integration on $G = \{x > z(y)\}$ and vertical integration on $H = \{x < A, y > e(x)\}$ partition the positive-utility region up to volume-zero faces. Put $D_0 = [0, 1]^2 \setminus (G \cup H)$; the candidate has zero utility *almost everywhere* on D_0. This statement about a volume integral does not discard pointwise capacity or line prices.

Let $f = 3e - 2$ below A and $f = 1$ above A. The displayed formula for δ implies $\int f = 0$, and $F(x) = \int_x^1 f \geq 0$. Set $\lambda = \alpha(c + \ell_*/4)$ and let $w(x)$ equal zero below A, $3t(c + \ell_*/4)$ on (A, t), and λ above t.

Theorem 8 (Full randomized lottery support)**.** *Under these hypotheses the five-option candidate is optimal over all randomized conditional mechanisms. A supporting measure is*

$$\pi_1 = 3(x - z(y))_+ \, dx\, dy + F(x)\, dx\, \delta_1(dy) + w(x)\, dx\, \delta_c(dy),$$
$$\pi_2 = 3(y - e(x))_+ \mathbf{1}_{x<A}\, dx\, dy.$$

For every Borel residual r and competitor with $a \leq r$,

$$\langle \pi, r \rangle - R = \langle \pi, r - a \rangle + T_g + \lambda M_a + S_u \geq 0. \tag{6}$$

Proof. Let $g(y) = u(1, y)$ and let ν be its nonnegative second-derivative measure. The horizontal and vertical envelopes leave top coefficient f and right coefficient $\sigma(y)$, equal to α below c, $\alpha - 3\beta(y - c)$ on (c, Y) and zero above Y. The top coefficient integrates to the stated top-line price. Convexity on the whole lottery interval gives

$$\begin{aligned} T_g &= \lambda g(c) - \int \sigma(y) g(y)\, dy \\ &= \alpha \int_0^c [g(c) - g(y)]\, dy + \frac{\beta}{2} \int_0^{\ell_*} s(\ell_* - s)^2\, d\nu(c + s) \geq 0. \end{aligned}$$

This follows from the exact hinge coefficient $\int_s^{\ell_*} (\alpha - 3\beta z)(z - s)\, dz = -\beta s(\ell_* - s)^2/2$ and $\beta \ell_* = \alpha/2$. Next, monotonicity of $a_1(\cdot, c)$ gives

$$M_a = \frac{t}{t - A} \int_A^t a_1(x, c)\, dx - \int_0^t a_1(x, c)\, dx \geq 0.$$

Expand $g(c)$ along this line to obtain its price w. Finally, $S_u = 3 \int_{D_0} u - \lambda u(0, c) \geq 0$ because D_0 contains $[0, k] \times [c, B]$ up to volume-zero faces and

$$3k(B - c) - \lambda \geq q(1 - 3q/2) > 0.$$

These identities yield (6) without removing origin utility. At the candidate the trace is constant below c, affine on (c, Y), and all three non-capacity slacks vanish. Feasibility and (5) saturate the prices. The safe-item price is zero on the genuine lottery cell, consistent with its fractional allocation. □

The full slack identity rules out higher conditional revenue from additional lotteries or more general randomized allocations, including scarce marginals below one. Its hinge term controls nonlinear convex trace changes over the whole lottery interval. The finite menu is an equality case of an unrestricted bound; neither uniqueness nor a classification of all equality cases is asserted.

6.4 Specification of the retained V4.6 mechanism and its revenue

The retained V4.6 lower-bound mechanism is M_J, including the final joint price release. Its reference before the new modifications is $M_{4.5}$. This specification fixes their parameters, order of composition and exceptional reports. All source paths below are relative to `certificate/joint_residual_screening_lower_bound/source/` in the archive. A profile is ordered as $\omega = (w_1, w_2, v_1, v_2) \in [0,1]^4$: w is bidder 1's report and v is bidder 2's report, each in physical item order. A menu option $(x; p)$ gives expected allocation $x \in [0,1]^2$, payment p, and utility $w \cdot x - p$ or $v \cdot x - p$ for the respective bidder. Masks $0, 1, 2, 3$ mean empty, item 1, item 2 and both items.

Constants and retained tariffs. Put

$$a = \frac{159}{250}, \quad b = \frac{91}{100}, \quad c = b - a = \frac{137}{500},$$
$$s_0 = \frac{1137}{1000}, \quad s = \frac{142}{125}, \quad d = s - a = \frac{1}{2}, \quad q = s - b = \frac{113}{500}.$$

The earlier deterministic construction uses s_0, hence $d_0 = 501/1000$ and $q_0 = 227/1000$. This historical mechanism changes the shared allocation cost to $s = s_0 - 1/1000$ and retains the V3 price increments as functions of the opponent report. In particular, the constants in those frozen functions are not replaced by d, q. For an opponent report u, define

$$H(u) = \max\{0, u_1 - a, u_2 - a, u_1 + u_2 - b\},$$
$$m^r(u) = (0, H(u) + \min\{a, r - u_2\}, H(u) + \min\{a, r - u_1\}, H(u) + b).$$

Let $P^0(u)$ be the four prices returned by `V3/verifier/joined_threshold.py:menu(u)`, using `V3/verifier/baseline_mechanism.py:menu(u)['prices']` when the first function returns `None`. These are explicit piecewise algebraic functions: the joined rule applies on $d_0 \le t \le 1$, $0 \le \rho \le b - \min(t, a)$, where $t = \max(u)$, $\rho = \min(u)$; its successive branches use the tests $t \le a$, $(t - q_0)^2 \le r_0$, $(t - q_0)^2 < r_1$, and $t \le 2/3$, with $r_0 = b^2 - (2/3 + c^2)$ and $r_1 = 4c/3 - 2/9$ exactly as in that function. The remaining affine branch is `V3/verifier/stationary_s.py:prices_for_t(t)`. The finite tariff data are `V3/certificate/baseline_mechanism.json`: 20 common rows, 41 bundle rows and 8 item rows, in their stored order. The literal first-match and endpoint rules are `baseline_mechanism.py:fee_rows` and `baseline_mechanism.py:menu` in the same verifier directory. Thus the retained prices used by both bidders are exactly

$$P^s(u) = m^s(u) + P^0(u) - m^{s_0}(u). \tag{7}$$

This is `V4_02/verifier/split_cost_candidate.py:retained_menu` at `CANDIDATE_THETA`$= -1/1000$.

Shared selection and the reference $M_{4.5}$. First maximize reported total value minus cost over the ordered mask pairs

$$(0,0), (1,0), (2,0), (3,0), (0,1), (0,2), (0,3), (1,2), (2,1),$$

with respective costs $0, a, a, b, a, a, b, s, s$, choosing the first maximizer. This is the retained source's literal `base_outcome_order`, including $(3,0)$ before $(0,1)$; it fixes the tie convention for

this construction. For each bidder maximize utility at $P^s(u)$: choose empty if the maximum is zero; otherwise retain that bidder's shared mask if it maximizes utility; otherwise choose the smallest maximizing mask contained in the shared mask. This selection rule is part of the mechanism.

Next override bidder 2's menu when $w \in Q = \{\max(w) \le A,\ w_1 + w_2 \le b\}$, where $A = 2/3$ and $b_0 = (4 - \sqrt{2})/3$. If $t = \max(w) \le d$, its singleton and bundle prices are (A, A, b_0), with mask priority $(0, 1, 2, 3)$. If $t > d$, put $k = t - q$, $C_0 = 5/6 + 3k^2/4$ and use aligned prices $(A, C_0 - k, C_0)$ for the scarce singleton, safe singleton and bundle, with priority (empty, safe, scarce, bundle). The scarce item is the item at the opponent's larger coordinate. The source predicate is `u[0] > u[1]` for scarce item 1 and item 2 otherwise; equal coordinates cannot occur in a constrained Q row because $t > d$ and $\sum u \le b < 2d$. These closed-Q rules are `split_cost_candidate.py:screening_menu`; the resulting evaluator is `V4_02/verifier/split_cost_candidate.py:candidate`.

Finally, for each bidder whose opponent satisfies $A < t < T = 613/750$ and $u_1 + u_2 \le b$, replace its menu by

$$(0, 0; 0),\ (0, 1; d + \delta),\ (1, 0; t),\ (1, 1; t + c + \delta),\ (1, \beta; t + \beta c), \tag{8}$$

in the displayed priority, where

$$\delta = q + \frac{3q^2}{4} - \frac{3qt}{2} + \frac{(3t-2)^2}{16}, \qquad \beta = \frac{3t-2}{3t-2+2\delta}.$$

The first aligned coordinate is the scarce item; reverse allocation coordinates when $u_1 < u_2$. This gives $M_{4.5}$, evaluated by `V4_5/verifier/residual_lottery.py:mechanism`. The endpoints $t = A, T$ retain the preceding rule; the face $\sum u = b$ is included in the lottery rule.

Ordered modifications defining M_J. Apply the following operations to $M_{4.5}$, retaining every earlier row unless it is explicitly replaced. First, for both bidders extend (8) to $A < t < T$, $b - t < \rho < c$; retain the preceding rule on $\rho = c$. Second, for bidder 1's opponents in $F = \{\max(v) \le d,\ \sum v \le b_0\}$ use prices (A, A, b_0). For opponents in $G = \{\max(v) \le d,\ b_0 < \sum v < b\}$ use $(a, a, \sum v)$. Both use mask priority $(0, 1, 2, 3)$. Third, on bidder 2's Q rows replace the bundle price by $\max(b_0, \sum w)$ if $\max(w) \le d$. On a constrained row, if $\sum w > C_0$, replace the aligned prices by $(A, \sum w - k, \sum w)$; otherwise retain its preceding Q menu. Keep the Q priorities above. The boundaries $\sum v = b_0, b$ retain their preceding bidder-1 rules, and equality $\sum w = C_0$ retains the earlier constrained row. These operations, including the intermediate bundle-only response before the coupled response, are the successive `mechanism` functions in `V4_6/verifier/outer_lottery_strip.py`, `free_bidder_one.py`, `outer_bundle_exchange.py` and `outer_bundle_reoptimized.py`, all in that verifier directory. Call the resulting mechanism M_6.

Set $\varepsilon = 9/10000$ and use the closed rectangles

$$W = [7/10, 71/100] \times [0, 1/5], \qquad S = [7/10 - 4\varepsilon, 71/100] \times [c - 2\varepsilon, c + 4\varepsilon].$$

When $v \in S$, charge bidder 1 an additional ε on every nonempty option of its M_6 menu: choose empty when its previous maximum utility is at most ε, and otherwise retain its selected option. When $w \in W$, offer bidder 2 the five options in (8), lowering only the lottery payment by ε, with the same first-maximizer priority. This last operation uses the displayed physical orientation only. The final evaluator is `V4_6/verifier/price_joint_reallocation.py:mechanism(profile)`. Its `allocations` and `payments` fields define M_J. The real mechanism interprets every displayed and source-defined branch over the reals; the executable evaluates rational reports using exact rational and quadratic comparisons. No floating-point extension or numerical tie tolerance defines the mechanism.

Exact revenue. For each specified evaluator E, let $p_i^E(w,v)$ be its selected payment. With four independent uniform values, define the exact number

$$\mathcal{R}(E) = \int_0^1\int_0^1\int_0^1\int_0^1 [p_1^E(w_1,w_2,v_1,v_2) + p_2^E(w_1,w_2,v_1,v_2)]\, dv_2\, dv_1\, dw_2\, dw_1. \tag{9}$$

In particular $R_{4.5} = \mathcal{R}(M_{4.5})$ and $R_J = \mathcal{R}(M_J)$. The following table fixes the reduced-integral entrypoints; it does not define revenue through a saved decimal.

Quantity	Exact definition and reduction in the source directory
Historical base R_{02}	$\mathcal{R}$(`split_cost_candidate.candidate`). `V4_02/verifier/split_cost_revenue.py:calculate` reduces its change from V3.1 using `BASE_D`, `base_Q`, `inner_Q` and `moments`; V3.1 is `V3_1/verifier/constrained_candidate.py:mechanism`.
$R_{4.5} - R_{02}$	$4\int_A^{t_o}(b-t)g_1(t)\,dt + 4\int_{t_o}^T (b-t)g_2(t)\,dt$; `V4_5/verifier/independent_lottery.py:reconstruct`.
$R_6 - R_{4.5}$	$\int_{[0,1]^4}\sum_i (p_i^{M_6} - p_i^{M_{4.5}})$; `V4_6/verifier/independent_revenue.py:calculate` returns `refined_addition_coefficients` in the basis $(1,\sqrt{2},\sqrt{23})$.
$J = R_J - R_6$	$\int_{[0,1]^4}\sum_i (p_i^{M_J} - p_i^{M_6})$; `V4_6/verifier/price_joint_revenue.py:calculate` reduces this integral to a rational constant and three logarithms.

Here $\delta_o(t) = 1/2 - c + 3q_0^2/4 - 3q_0 t/2$, $t_o = (1/2 - c + 3q_0^2/4)/(3q_0/2)$, $g_1 = (\delta-\delta_o)^2 + \delta_o(3t-2)^2/8$ and $g_2 = \delta^2$. The V4.02 reduction uses its explicitly integrated polynomial on $[s_0, s]$ and the fixed tariff moments; its historical base ultimately shares the earlier affine-polytope revenue calculation. `V4_6_archive_audit/revenue/fresh_exact_gap.py:algebraic_increment` and `joint_increment` independently reconstruct the two new V4.6 additions. They retain the V4.5 base enclosure and therefore do not constitute a second integration of the entire historical base. The rational radical and logarithm bounds in that file and `V4_6/verifier/consolidated_bounds.py:verify` give

$$0.8758198541484224553460 < R_J < 0.8758198541484224553461.$$

This definition and enclosure concern the constructed mechanism. They do not assert unrestricted optimality or a matching full-auction certificate.

6.5 Complete joint reallocations

Throughout this subsection let $\mathbf{w} = (w_1, w_2)$ be bidder 1's report and $\mathbf{v} = (v_1, v_2)$ bidder 2's, in physical item order. In the formulas below w and v abbreviate these vectors, replacing the scalar-coordinate notation of Section 3. A set of opponent reports indexes a complete own-type menu. Lottery rows put the opponent's high item first and are rotated back to physical items after the menu is defined; their equal-coordinate convention uses item 1 first. The constrained Q rule retains its specified scarce-item selection. The final rectangles W and S use physical item 1 as the high item and are not rotated. Thus $((.45,.45),(.44,.44))$ means $(\mathbf{w},\mathbf{v})$, whereas a final lottery change is indexed by $\mathbf{w} \in W$ and may require a fee in the opposing menu indexed by $\mathbf{v} \in S$.

The reference $M_{4.5}$ is specified in Section 6.4, with its finite menu rules, rational tables and algebraic tests in the source-bound certificate `joint_residual_screening_lower_bound`. Its complete predecessor proofs and exact revenue integrals are included in that package. The following modifications are applied in order. All new menu selections have the stated fixed priority; unmodified rows retain their complete predecessor maximizer selection, which need

not be independent of the own report. Selecting any utility-maximizing option from a fixed opponent menu is sufficient for pointwise DSIC.

First extend both lottery menus to $E^+ = \{A < \max(v) < T, \min(v) < c\}$. Retain the predecessor on the $\min(v) = c$ face. The retained competing bidder cannot take its low item there, and the two occupied traces in (5) remain in place. Theorem 8 therefore solves both corresponding inner problems at this reference residual.

Second, bidder 2 receives nothing at every opponent report whenever its own type lies in the closed set

$$F = \{\max(v) \le 1/2,\ v_1 + v_2 \le b_0\}, \qquad b_0 = (4 - \sqrt{2})/3.$$

On opponents in F, replace bidder 1's constant menu (a, a, b) by the full single-buyer optimal menu (A, A, b_0). Here $a = 159/250$, $b = 91/100$. Capacity is entirely free on these fibers; zero-utility ties select empty.

Third, on $G = \{\max(v) \le 1/2, b_0 < v_1 + v_2 < b\}$, bidder 1 offers bundle price $v_1 + v_2$, retaining singleton prices (a, a). Bidder 2's complete response changes on $Q = \{\max(w) \le A, \sum w \le b\}$. For $\max(w) \le 1/2$ its prices are $(A, A, \max(b_0, \sum w))$. In a constrained row write $t = \max(w) > 1/2$, $k = t - q$, $C_0 = 5/6 + 3k^2/4$, and $z = \sum w$. If $z > C_0$, the aligned prices are $(A, z - k, z)$; otherwise retain its earlier Q menu. Moving the safe and bundle prices together preserves $C - B = k$. Relative to the earlier feasible Q menu this only discards allocations. A new bidder-1 bundle requires $\sum w > \sum v$, and the opposing bundle threshold prevents double allocation. At $(w, v) = ((.45, .45), (.44, .44))$, ownership transfers strictly from bidder 2 to bidder 1, who pays .88.

On the reopened Q rows, F/G bundle purchases supply the bottom and vertical conditions in (3); the top hole is inherited from the previous capacity rule. Theorem 7 recertifies the whole response. This includes all Q after combining unchanged and changed rows. It also shows why increasing only the opposing bundle price was insufficient. The complete constrained conditional revenue change is exactly $-(z - C_0)^2$ at the coupled response.

Finally, the conditional lottery certificate identifies a consumed junction that admits a profitable joint capacity release. Use one physical orientation and set

$$\varepsilon = 9/10000, \quad W = [.7, .71] \times [0, 1/5], \quad S = [.7 - 4\varepsilon, .71] \times [c - 2\varepsilon, c + 4\varepsilon].$$

The rectangles are closed. On bidder-2 menus indexed by $w \in W$, lower only the lottery payment by ε. On bidder-1 menus indexed by $v \in S$, add ε to every nonempty option. Choose empty whenever the old maximal utility is at most ε; otherwise retain its old maximizer and charge the additional fee. All other rows are unchanged.

Proposition 9 (Pointwise joint release)**.** *The composed mechanism is pointwise DSIC, ex-post IR and jointly feasible. Its allocations are jointly measurable and its payments are integrable. The final release has strictly positive exact revenue increment J given below.*

Proof. Each change specifies a complete opponent-dependent menu before maximizing; the empty option and full predecessor rules handle every exceptional report. For the final change only a newly chosen lottery can introduce a capacity conflict, since the fee contracts bidder 1's allocation. Writing $v = (x, y)$, its comparisons with singleton, bundle and empty force

$$c - \varepsilon/\beta \le y \le Y + \varepsilon/(1 - \beta), \qquad x + \beta(y - c) - t + \varepsilon > 0.$$

Throughout W, $3/5 < \beta < 3/4$ and $x \ge 1 - t/2 - 4\varepsilon > 1/2$. The F/G modifications are therefore absent at every newly chosen lottery report. Bidder 1 can only take the scarce singleton, at price $P \ge \max(x, x + y - c)$. If $t > P$, the displayed lottery inequalities force $(x, y) \in S$ and

$$t - P \le t - \max(x, x + y - c) < \varepsilon.$$

The entry fee selects empty, releasing that item. Its low item is already free because its value is at most $1/5 < c$. Equalities use the explicit empty priority. The earlier splices have the full same-orientation, opposite- orientation and reverse-Q capacity checks in the certificate proof notes; these checks include the excluded $\rho = c$ and $\sum v = b$ faces. Finite menus with Borel parameters yield joint measurability and bounded payments. Feasible marginals can be realized itemwise. □

This release preserves the full F and Q certificates. It changes some Eplus menus and their binding residuals. A sufficient preserved Eplus scope excludes S for bidder 1 and excludes $\max(w) \in [.7 - 4\varepsilon, .71 + \varepsilon]$ for bidder 2. The wider second exclusion includes releases outside W. No conditional optimality is asserted on excluded fibers, and no common auction support is obtained.

Opponent set	Bidder	Complete menu change	Full conditional support after release
F	Both	Single-buyer menu (A, A, b_0)	Yes, at full residual capacity
Q	2	Retained or coupled diagonal response	Yes, Theorem 7 and retained rows
G	1	Bundle price equal to opponent total	No new inner-optimality claim
E^+	Both	Five-option lottery	Theorem 8 only outside the exclusions above
W	2	Lottery payment minus ε	No matching inner claim on the changed rows
S	1	Fee ε on every nonempty option	No matching inner claim on the changed rows

Table 3: Report roles and final certificate scope. Sets index opponent menus; F is contained in Q, and W is contained in one orientation of E^+. Entries state certified scope, not a claim that the other fibers are suboptimal.

6.6 Exact new revenue and a sub-0.01 bracket

Let $R_{4.5} = \mathcal{R}(M_{4.5})$ be the exact payment integral (9). Its displayed reduction equals the V4.02 reference integral plus 4726177077987584601086 53/10106279712000000000000000000. The baseline is an exact mathematical integral, not its printed decimal. Exact continuous-cell integration of all subsequent complete menus gives

$$R_J = R_{4.5} + a_0 + a_2\sqrt{2} + a_{23}\sqrt{23} + J, \tag{10}$$

where

$$a_0 = -\frac{160954137966823166124954703825 1}{1591739054640000000000000000000000}, \qquad a_2 = \frac{57567793}{7593750000},$$
$$a_{23} = -\frac{12167}{1328906250},$$

and

$$J = \frac{6578986777818169914603}{1065267568750000000000000000000000} + \frac{6248637}{282500000000000}\log\frac{161}{176} - \frac{61575471}{282500000000000}\log\frac{613}{628} - \frac{9308601}{5000000000000000000}\log\frac{13}{10}.$$

Here $J = 0.0000000039161279421422536668970261 4\ldots$ and an independently proved rational lower bound is $J \geq 26902077489/12500000000000000000 > 0$.

All induced information-rent changes are included. In a lottery row,

$$\Delta R_2 = \frac{\alpha\ell_*}{4}\varepsilon - \frac{\alpha}{4\beta(1-\beta)}\varepsilon^2 - \frac{1}{2\beta(1-\beta)^2}\varepsilon^3.$$

The coefficient $\lambda = \alpha(c + \ell_*/4)$ in the certificate's bound on $g(c)$ is not this row's revenue derivative. On the fixed lottery interval, the price cut changes the convex-trace slack by exactly

$$\Delta T_g = \alpha\left(c\varepsilon - \frac{\varepsilon^2}{2\beta}\right), \qquad \left.\frac{d\Delta R_2}{d\varepsilon}\right|_{0+} = \lambda - \alpha c = \frac{\alpha\ell_*}{4}.$$

Indeed, below c the right-trace increase is $(\varepsilon + \beta(y-c))_+$, whose integral is $\varepsilon^2/(2\beta)$; on (c, Y) it is constant, so the hinge-curvature term is unchanged. Thus even a saturated conditional support must be combined with its changing incentive slacks before predicting the revenue from a complete menu deformation.

For a complete entry-fee row with no-sale area D and bundle price C,

$$\Delta R_1 = (1-3D)\varepsilon - \frac{3}{2}C\varepsilon^2 - \frac{1}{2}\varepsilon^3.$$

This does not use the earlier four-option lemma's assumption $C \le 1$. For the actual rows and $0 \le h \le \varepsilon$, the internal menu junctions keep their order, the no-sale area is $D(h) = D + Ch + h^2/2$, and the top and right utility traces stay positive after the fee. The uniform revenue identity then gives $R'(h) = -2 + 3(1 - D(h)) = 1 - 3D(h)$; integration proves the formula. For example, the lottery row $t = 7/10$ has $\delta = 1727/62500$ and $C = 31301/31250 > 1$, while $t + \varepsilon < 1$ and $B + \varepsilon < 1$ keep the boundary traces positive. Its no-sale polygon satisfies the same area identity.

For the selected $\varepsilon = 9/10000$, integrating both complete menu changes gives

$$J_1 = -\frac{24631898853429}{1250000000000000000000000} < 0,$$
$$J_2 = J - J_1 = 0.000000004113183132969685 6\ldots > 0.$$

The gain therefore pays the actual negative revenue change of bidder 1. A direct strict certificate, using $|W| = 1/500$ and the complete fee-strip bounds, is

$$\begin{aligned} J &\ge \frac{1}{500}\left(\frac{1213}{625000}\varepsilon - \frac{13}{60}\varepsilon^2 - \frac{40}{3}\varepsilon^3\right) \\ &\quad - (1/100 + 4\varepsilon)\left(\frac{2}{25}\varepsilon^2 + \frac{903}{100}\varepsilon^3 + 3\varepsilon^4\right) \\ &= \frac{26902077489}{12500000000000000000} > 0. \end{aligned}$$

The whole lottery row contributes a linear term, whereas the compensating fee is integrated over a strip whose width shrinks with the perturbation. These formulas explain the certified fixed choice. We do not infer a uniformly admissible perturbation family, an optimal step size, or an unrestricted stationarity condition from this comparison.

The literal retained tariff chambers, including the distinct $y = c$ face, are checked independently by rational demand polygons. Integration over opponent reports gives the expression above. Radical bounds use integer square-root enclosures; logarithm bounds use signed rational series remainders. An independent reconstruction uses direct bivariate menu integration and local pole expansions instead of the primary partial- fraction linear system.

The resulting rational enclosure is

$$0.8758198541484224553460 < R_J < 0.8758198541484224553461.$$

Consequently

$$0 < U_{\text{stream}} - R_J < 0.0099384483702321295386 < \frac{1}{100}.$$

This establishes the historical V4.6 lower bound. The active lower claim in Theorem 1 follows from Proposition 10. The historical mechanism improves the deterministic predecessor by $0.001214867745586187\ldots$ and the V4.5 reference by $0.000565326708609134\ldots$. These are internal certified improvements, not comparisons with the reported GemNet revenue.

6.7 A functional exchange with an inverse plateau

The selected lower mechanism replaces the preceding construction. Its parameters are selected jointly and verified exactly, so its revenue is integrated afresh rather than obtained by adding gains from incompatible candidates. The exchange threshold moves complete conditional menus. An upward jump in that threshold induces a plateau in its generalized inverse, hence changes a positive-measure interval of opposing reports.

Put

$$A = \frac{2}{3}, \quad d = \frac{1}{2}, \quad c = \frac{157}{500}, \quad q = \frac{93}{500}, \quad \nu = \frac{q^2}{2}, \quad a = A + \nu,$$
$$b = a + c = \frac{1496947}{1500000}, \qquad s = a + d = \frac{1775947}{1500000}, \quad \lambda = \frac{4}{5}, \quad \bar{u} = \frac{3421}{10000}.$$

The symbol $\bar{u}$ denotes a threshold parameter, not utility. Define

$$g(r) = \lambda(\bar{u} - r)\mathbf{1}_{[c,\bar{u}]}(r), \qquad h(r) = r + q + g(r), \qquad J = g(c) = \frac{281}{12500}. \tag{11}$$

The increasing, right-continuous h jumps upward at c. Its generalized inverse on the needed reports is

$$k(t) = \begin{cases} t - q, & t \le d, \\ c, & d < t \le d + J, \\ (t - q - \lambda\bar{u})/(1 - \lambda), & d + J < t < \bar{u} + q, \\ t - q, & t \ge \bar{u} + q. \end{cases}$$

In particular, the proof uses $x < k(t) \Rightarrow h(x) < t$ and $x \ge k(t) \Rightarrow h(x) \ge t$. The plateau does not satisfy $h(k(t)) = t$ at all its interior points.

Complete menus. For either bidder, let the opposing report be w, with $t = \max(w)$, $\rho = \min(w)$ and $z = w_1 + w_2$. In oriented menus, "scarce" means the opponent's high-valued physical item and "safe" means the other item. If the coordinates tie, designate item 2 as high. Put $b_0 = (4 - \sqrt{2})/3$, $T = 1 - 2c/3$ and $T' = 5/3 - 2c$. Every menu includes empty allocation at zero payment.

1. On the closed region $Q = \{t \le A, z \le b\}$, if $t \le d$, offer both singletons at A and the bundle at $\max(b_0, z)$. If $t > d$, set

   $$C_0(k) = \frac{5}{6} + \frac{3}{4}k^2, \quad C = \max\{C_0(k(t)), z, k(t) + h(\rho)\}, \quad B = \min\{A, C - k(t)\}.$$

   Offer safe at B, scarce at A, and bundle at C.

2. On the open region $E = \{A < t < T, \rho < c\}$, put

   $$\delta = \frac{9}{16}(T - t)(T' - t), \quad \alpha = 3t - 2, \quad \beta = \frac{\alpha}{\alpha + 2\delta}.$$

   Offer safe at $d + \delta$, scarce at t, bundle at $t + c + \delta$, and allocation $(1, \beta)$ in scarce/safe order at price $t + \beta c$.

3. For every remaining opponent report, put

$$H = \max(0, w_1 - a, w_2 - a, z - b), \qquad P_j = H + \min(a, s - w_{3-j}), \qquad P_B = H + b.$$

Offer both singletons and bundle at these prices, adding $g(\rho)$ to the safe singleton and bundle. The face $\rho = c$ receives the positive fee in (11).

At zero maximal utility, retain empty only. At positive maximal utility, retain every maximizing option. From the Cartesian product of the two retained sets choose the first jointly feasible pair in lexicographic order, bidder 1 first. Menu orders are empty/item 1/item 2/bundle for free Q and base, empty/safe/scarce/bundle for constrained Q, and empty/safe/scarce/bundle/lottery for E.

Proposition 10 (Complete functional-exchange mechanism)**.** *This selection is defined at every real report profile. It is Borel, pointwise feasible, DSIC and ex-post IR. Its expected revenue is*

$$\begin{aligned} R_* = {} & \frac{357914042523416218525277357478616 37}{4252500000000000000000000000000000} + \frac{31}{1215}\sqrt{2} \\ & - \frac{15059524650320123}{8305664062500000000000}\sqrt{493894}. \end{aligned} \tag{12}$$

Proof. The feasibility argument partitions opponent-menu pairs into base/base, Q/base, Q/Q, E/base, E/Q and E/E. The full all-real inequalities are supplied in the source-bound structural proof identified below; the following are the additional compatibility conditions created by the changed parameters. The base tariffs have strict bundle discount at least q. Raising the safe and bundle prices by the same nonnegative fee therefore admits a maximizing subset of an old base maximizer, including positive ties. The base/base and base-facing comparisons can use that containment; the complete new mechanism does not have predecessor containment.

In constrained Q, the bound $k+h(\rho)$ is redundant. On the fee support its minimum sufficient slack is

$$[C_0(A - q) - (A - q)] - h(b - A) = \frac{37}{5000000} > 0;$$

outside the support the sum z provides the required bound. Thus $C = \max(C_0(k), z)$ everywhere, including reports with $C_0(k) > 1$. The cap at A controls the bundle upgrade $\max(C - A, k)$. The two generalized-inverse implications above establish the complementary threshold comparisons at Q/base and Q/Q junctions.

The E/Q lottery comparison requires extra care because $a > A$. A Q type's lottery utility is at most $A - t + \beta(a - A)$. On the open E interval, $(t - A)/\beta = t - A + 2\delta/3$ is strictly increasing with infimum $q^2/2$. Since $a - A = q^2/2$, this utility is strictly negative. The singleton and bundle comparisons and the E/base and E/E cases then leave a feasible maximizing pair, as detailed in the finite case proof. The closed Q, open E and remaining base rule cover every boundary. In particular, the argument does not assume that independently chosen tie priorities preserve supply.

The finite jointly feasible maximizer rule is Borel. For a fixed opponent, every selected option maximizes the same menu; any deviation returns another option of that menu. This proves DSIC for every report, and empty proves IR. Payments are nonnegative and at most two by IR. For each item, disjoint intervals of lengths x_{1j}, x_{2j} under a uniform seed implement samplewise feasible supply. DSIC concerns expected utility; truthfulness for each fixed seed is not asserted.

Revenue is computed by integrating each conditional menu over its own unit square and then the full opponent square. Two implementations agree on all three algebraic coefficients in (12). One uses rational polynomial dictionaries and simplex moments, the other uses a direct Q decomposition and Green boundary integrals over base polygons. A third calculation in the present audit, importing no proof calculator, independently reconstructs the full integral

by vertical slicing and recovers the same three coefficients. All include the negative base-to-E correction on $A < t < a$ and the extra fee strip $c \le \rho \le b - A$, $A < t < b - \rho$. The inverse plateau has positive opponent measure and is included. Ties do not alter the integrals, while the selection argument above covers them pointwise. □

The new revenue exceeds the V4.6.1 predecessor by more than 17/2000000. The exact difference lies in

$$(0.0000085506354721977987 5,\ 0.0000085506354721977987 6).$$

There is an ownership change throughout the open box of coordinate radius 10^{-5} around

$$(v_1, v_2) = ((103/200, 1/10), (63/200, 99/100)).$$

The predecessor splits the goods; the new mechanism gives bidder 2 the bundle. Uniform strict menu comparisons establish the entire box; its sixteen corners alone would not prove a continuum claim.

Why these deformations help. In an auxiliary family with $\widetilde{c} = 47/150$, $a = A = 2/3$, $d = 1/2$ and $V = 69/200$, integrating both bidders' full menu changes produces a one-dimensional kernel $\int_{\widetilde{c}}^{V} [\mathsf{A}(r)g(r) - \mathsf{B}(r)g(r)^2]\, dr$. This calculation includes reports forced to move by the inverse plateau. A second complete-menu variation explains the cap: in a proper four-option menu with scarce price A, safe price B and bundle price C, $\partial_B \mathcal{R} = (2 - 3B)(C - B)$. If $C - k = A + \eta > A$, capping the safe price at $A = 2/3$ gains $3k\eta^2/2 + \eta^3/2 > 0$. These effects motivate globally admissible changes. They do not establish unrestricted stationarity or optimality of the selected parameters.

6.8 Conditional optimality on the actual residual

The selected mechanism has full randomized conditional screening certificates for either bidder on three disjoint opponent regions, Q, E and

$$W_{\text{wing}} = \{\max(w) \ge 1 - q,\ w_1 + w_2 \ge 1 + \bar{u}\}.$$

Their areas are respectively

$$\frac{1746937679191}{4500000000000},\quad \frac{4867}{62500},\quad \frac{438867}{2500000},$$

with union 2887322279191/4500000000000, approximately 64.1627% of the opponent square. This is conditional coverage, not a proportion of the global revenue gap closed. The remaining area is unresolved, not certified to have positive conditional regret.

The Q theorem applies the preceding unrestricted envelope with the capped price and verified actual top and anchor capacities. For E, the earlier literal zero-capacity assertion on a junction face may fail when $a > A$. Instead, zero scarce capacity in the adjacent open rectangle $A < x < t$, $0 < y < c$ forces every feasible competitor's utility to be horizontally constant. Continuity and the two horizontal IC comparisons force its selected scarce marginal to vanish on the required face. The contribution on $A < x < t$ of the singular line $y = c$ vanishes by incentives; its $x > t$ term remains. This is a restriction imposed by utility compatibility on that face.

Theorem 11 (Screening on an occupied wing)**.** *Let $0 < p \le 1$, $1/3 \le k \le 1$, $z = k + p \ge 4/3$, and $(3k + 1)p \ge k + 1$. Suppose an actual pointwise residual admits the menu $u^*(x, y) = \max(0, y - p, x + y - z)$, has first marginal zero on the open strip $x < k$ and in the no-sale interior $D_0 = \{y < p, x + y < z\}$, and has second marginal zero on $\{0\} \times [0, p)$. Then that menu is optimal over all randomized DSIC/IR conditional mechanisms within the residual, with value*

$$V(k, p) = pk(1 - p) + (k + p)\left[(1 - k)(1 - p) + \frac{(1 - k)^2}{2}\right].$$

Proof. Let u be the utility of an arbitrary complete measurable DSIC/IR competitor, a its selected allocation and $u_0 = u(0,0) \geq 0$. The occupied left edge and horizontal strip force $u = u_0$ throughout D_0 and $u(x,y) = u(0,y)$ on $x < k$. These are pointwise convex-utility statements derived from chosen subgradients and Lipschitz continuity. Averaging horizontal and vertical envelopes yields

$$R = \int \left[\frac{3x-1}{2}a_1 + \frac{3y-1}{2}a_2\right] - \frac{1}{2}\int u(0,y)\,dy - \frac{1}{2}\int u(x,0)\,dx.$$

The bottom trace is u_0, and $\int u(0,y)dy = u_0 + k^{-1}\int_{x<k}(1-y)a_2$. Since $a = 0$ almost everywhere on D_0, substitution gives

$$\begin{aligned} R + u_0 = &\int_G \left[\frac{3x-1}{2}a_1 + \frac{3y-1}{2}a_2\right] dxdy \\ &+ \int_H \frac{(3k+1)y-(k+1)}{2k}a_2\,dxdy, \end{aligned} \tag{13}$$

where $G = \{x > k, x+y > z\}$ and $H = \{x < k, y > p\}$. All displayed coefficients are nonnegative by the hypotheses. The candidate saturates them, proving the value and equality. □

For $w = (t,\rho)$ in W_{wing}, orient the scarce item first and set $p = \min(\rho+q, 1)$, $k = t+\rho-p$. The actual menu reduces to safe at p and bundle at $t+\rho$; an unaffordable scarce singleton has no positive-utility demand. The opposite bidder occupies the scarce item on the required strip and no-sale interior and both items on the left-edge trace. These are strict comparisons for every maximizing option in the required regions, across base, Q and E menus, including $p = 1$. The uniform sign margin is

$$(4-3q)(\bar{u}+q) - (2-q) = \frac{18601}{5000000} > 0.$$

Thus Theorem 11 applies to the actual residual for both bidders. The source-bound proof includes the exceptional-face and orientation cases. The theorem eliminates replacements with that residual held fixed. Its rent elimination uses occupied sets and does not assert that the remaining volume density alone supports arbitrary residual perturbations.

Stable source specification. The complete all-real case proof, independent integral derivation and actual wing occupation proof are exported under

`certificate/coordinated_primal_dual/source/V4_6_1_1_lower_bound/`

The proof notes in `research_log/` are `refined_structure_audit.md`, `refined_revenue_audit.md` and `residual_wing_screening.md`.

The exact implementation is `verifier/refined_candidate.py`. The selected revenue is the single expression (12); earlier functional or reserve-only experiments are not added to it.

7 Exact upper bounds for all randomized mechanisms

The proof has three layers: an analytic weak-duality theorem, an explicit rational witness satisfying its hypotheses, and a finite exact enclosure of the witness objective. The analytic theorem does not depend on polynomial degree, subdivision depth, or arithmetic scale.

7.1 Analytic weak duality

For own type $v_i = (x, y)$ and $r = \max(x, y)$, define the radial field

$$F^0(x, y) = \frac{3r^2 - 1}{2r^2}(x, y), \qquad F^0(0, 0) = 0.$$

Its value at the origin is immaterial to integration. Since $|F^0| = O(1/r)$, the field is integrable on the square.

Theorem 12 (Stream weak duality)**.** *For each bidder i, let G_i be a jointly C^1 vector field on $[0, 1]^4$. Suppose that, for each opponent report, its own-type divergence is zero and its normal component vanishes on every side of the own-type square. Put $F_i(v) = F^0(v_i) + G_i(v)$. Every admissible mechanism M satisfies*

$$\mathrm{Rev}(M) \le \mathcal{E}(F) := \int_{[0,1]^4} \sum_{j=1}^{2} \max\{0, F_{1j}(v), F_{2j}(v)\}\, dv. \tag{14}$$

Consequently $\mathrm{OPT} \le \mathcal{E}(F)$.

Proof. Fix bidder i and an opponent report on a full-measure set for which the conditional expected-payment slice is integrable. Write truthful utility as $u(v)$, expected allocation as $X(v) \in [0, 1]^2$, and expected payment as $p(v)$. Pointwise DSIC gives

$$u(v) \ge u(r) + (v - r) \cdot X(r) \qquad \text{for every } v, r.$$

Thus u is convex and $X(r) \in \partial u(r)$. These inequalities and $0 \le X_j \le 1$ also make u Lipschitz. At every differentiability point, $X = \nabla u$. At nonsmooth points we retain the mechanism's given jointly measurable subgradient selection; its pointwise feasibility is part of the model, not inferred from an almost-everywhere gradient.

Joint measurability and the assumed integrability of the payment, together with boundedness of $v \cdot X$, make $u = v \cdot X - p$ measurable and integrable. Thus the conditional identities below apply on this full-measure set of opponent reports and may then be integrated by Fubini.

We retain the origin utility explicitly. The Lipschitz bound gives $|u(v) - u(0)| \le 2$, so the opponent-dependent value $u(0)$ is integrable as well. Ex-post IR gives $u(0) \ge 0$.

Partition the square into the two radial charts

$$v = s(1, t) \quad \text{and} \quad v = s(t, 1), \qquad 0 \le s, t \le 1,$$

each with Jacobian s. On a fixed ray set $q = (1, t)$ or $(t, 1)$ and $g(s) = u(sq)$. Convexity makes g absolutely continuous and $g'(s) = q \cdot X(sq)$ almost everywhere and $g(0) = u(0)$. Since $p(sq) = sg'(s) - g(s)$,

$$\int_0^1 p(sq)s\, ds = \int_0^1 \frac{3s^2 - 1}{2}\, g'(s)\, ds - \frac{u(0)}{2}.$$

Combining the two charts gives the conditional revenue identity

$$\int_{[0,1]^2} p(v)\, dv = \int_{[0,1]^2} F^0(v) \cdot X(v)\, dv - u(0).$$

The correction is bounded, and $u \in W^{1,\infty}$. Weak Green integration, zero own-type divergence, and the zero normal trace therefore give

$$\int_{[0,1]^2} G_i \cdot \nabla u = 0.$$

Thus replacing F^0 by F_i preserves the conditional identity.

Apply this identity to both bidders and then integrate over opponent reports. Fubini is justified by the integrability established above. Dropping the nonpositive origin terms leaves the sum of the field-allocation pairings. For every profile and item j, feasibility gives $X_{1j}+X_{2j} \le 1$, with both allocations nonnegative. Hence

$$\sum_i F_{ij}X_{ij} \le \max\{0, F_{1j}, F_{2j}\}.$$

This pointwise relaxation is valid for arbitrary expected allocations and therefore for arbitrary internal randomization. Integrating proves (14); the maximizing allocation in its integrand need not itself be implementable by a DSIC mechanism. □

Boundary interpretation. The radial identity can also be checked by integration by parts. The classical divergence of F^0 is 3 in each open triangle $x > y > 0$ and $y > x > 0$, hence almost everywhere. Continuity across the diagonal away from the origin makes the interface fluxes cancel. The outer boundary flux is 2, while the flux around a puncture at the corner $(0,0)$ tends to 1, with the normal pointing into the removed square. Applying Green's formula to the two triangles outside $[0,\varepsilon]^2$ and letting $\varepsilon \downarrow 0$ gives

$$\int F^0 \cdot \nabla u = \int_{x=1} u + \int_{y=1} u - 3\int_{[0,1]^2} u + u(0).$$

Lipschitz continuity replaces the trace at the puncture by $u(0)$ with vanishing error. This accounts for the corner term without treating a boundary corner as an interior distributional source.

7.2 An uncorrected radial benchmark

Taking $G_i = 0$ gives a useful analytic benchmark. In the radial charts, $F^0(sq) = \phi(s)q$, where $\phi(s) = (3s^2-1)/(2s)$.

For completeness, the remaining scalar expectation can be evaluated exactly. Under the two radial charts, a uniform type can be generated by taking S with density $2s$, an independent fair choice of which coordinate is maximal, and an independent $T \sim U[0,1]$. For one fixed item its radial coordinate therefore has distribution

$$Q \sim \frac{1}{2}\delta_1 + \frac{1}{2}U[0,1].$$

Set $Y = \max\{0, \phi(S)\}$, independently of Q. For two bidders let $a = \max(Y_1, Y_2)$ and $b = \min(Y_1, Y_2)$. Conditioning on the four atom/uniform cases for Q_1, Q_2 gives

$$\mathbb{E}_{Q_1,Q_2} \max\{Y_1Q_1, Y_2Q_2\} = \frac{3}{4}a + \frac{b^2}{6a},$$

where b^2/a is defined as zero when $a = 0$. Since the two items are symmetric,

$$U_{\text{rad}} = \frac{3}{2}\mathbb{E}[a] + \frac{1}{3}\mathbb{E}\left[\frac{b^2}{a}\right]. \tag{15}$$

Put $r = 1/\sqrt{3}$. The function ϕ is increasing from zero to one on $[r,1]$, while $Y = 0$ below r. The larger of two independent S's has density $4s^3$, hence

$$\mathbb{E}[a] = \int_r^1 4s^3\phi(s)\,ds = \frac{8}{15} + \frac{4}{45\sqrt{3}}.$$

Ordering the two S's gives the second moment as the explicit integral

$$\mathbb{E}\left[\frac{b^2}{a}\right] = 8\int_r^1 \frac{s}{\phi(s)}\left(\int_r^s t\phi(t)^2\,dt\right)ds.$$

The inner integral is

$$J(s) = \frac{9}{16}s^4 - \frac{3}{4}s^2 + \frac{1}{4}\log s + \frac{3}{16} + \frac{1}{8}\log 3,$$

so the last display equals

$$16\int_r^1 \frac{s^2 J(s)}{3s^2-1}\,ds.$$

Set $x = \sqrt{3}s$. The cancellation of the $\log 3$ terms in J reduces this integral to

$$\frac{1}{3\sqrt{3}}\int_1^{\sqrt{3}} x^2(x^2-3)\,dx + \frac{4}{3\sqrt{3}}\int_1^{\sqrt{3}} \frac{x^2\log x}{x^2-1}\,dx.$$

Use $x^2/(x^2-1) = 1 + 1/(x^2-1)$. For the only non-elementary term, the substitution $y = (x-1)/(x+1)$ has upper endpoint $d = 2-\sqrt{3}$ and gives

$$\int_1^{\sqrt{3}} \frac{\log x}{x^2-1}\,dx = \frac{1}{2}\int_0^d \frac{\log(1+y)-\log(1-y)}{y}\,dy = \frac{1}{2}[\mathrm{Li}_2(d) - \mathrm{Li}_2(-d)],$$

where

$$\frac{d}{dz}\mathrm{Li}_2(z) = -\frac{\log(1-z)}{z}$$

was used in the last equality. The remaining polynomial and $\int \log x\,dx$ terms then give

$$-\frac{26}{15} + \frac{8}{5\sqrt{3}} + \frac{4}{3}\log\sqrt{3} + \frac{2}{3\sqrt{3}}\big[\mathrm{Li}_2(d) - \mathrm{Li}_2(-d)\big].$$

Substitution in (15) gives

$$U_{\mathrm{rad}} = \frac{2}{9} + \frac{2}{3\sqrt{3}} + \frac{4}{9}\log\sqrt{3} + \frac{2}{9\sqrt{3}}\left[\mathrm{Li}_2(2-\sqrt{3}) - \mathrm{Li}_2(\sqrt{3}-2)\right].$$

This gives $\mathrm{OPT} \le U_{\mathrm{rad}} = 0.920577474278749\ldots$. We next improve this benchmark with a rational stream correction.

7.3 The rational stream witness

We now add an opponent-dependent curl. For an exponent $e = (e_0, e_1, e_2, e_3)$, put

$$m_e = x^{e_0}y^{e_1}z^{e_2}w^{e_3}, \qquad \sigma(e) = (e_1, e_0, e_3, e_2).$$

Let E be the lexicographically smaller member of every nontrivial σ-pair of total degree at most four. Define

$$h(x,y,z,w) = \sum_{e\in E}\theta_e(m_e - m_{\sigma(e)}), \quad \psi = x(1-x)y(1-y)h, \quad G = (\partial_y\psi, -\partial_x\psi).$$

The thirty-two rational coefficients are listed below in the exact basis order used by the verifier. A convex quasi-Monte Carlo search was used only to discover them; the proof defines the field

by the following rational numbers and does not trust the numerical optimization.

e	θ_e	e	θ_e
$(0,0,0,1)$	$-2995385/993184$	$(0,0,0,2)$	$2337131/676164$
$(0,0,0,3)$	$5724621/964727$	$(0,0,0,4)$	$-4591777/714341$
$(0,0,1,2)$	$-834567/846478$	$(0,0,1,3)$	$6291553/970203$
$(0,1,0,0)$	$2762675/446201$	$(0,1,0,1)$	$206964/961681$
$(0,1,0,2)$	$-6090207/979156$	$(0,1,0,3)$	$6862090/913753$
$(0,1,1,0)$	$-5206994/854575$	$(0,1,1,1)$	$515889/117356$
$(0,1,1,2)$	$-3346363/843816$	$(0,1,2,0)$	$-125263/465506$
$(0,1,2,1)$	$3005651/895053$	$(0,1,3,0)$	$-1043238/468269$
$(0,2,0,0)$	$1660919/994155$	$(0,2,0,1)$	$1276309/943222$
$(0,2,0,2)$	$-2075136/255347$	$(0,2,1,0)$	$404816/59269$
$(0,2,1,1)$	$250988/352269$	$(0,2,2,0)$	$3825425/994154$
$(0,3,0,0)$	$-5383501/354845$	$(0,3,0,1)$	$3312081/940771$
$(0,3,1,0)$	$-4885471/759831$	$(0,4,0,0)$	$7648954/974695$
$(1,1,0,1)$	$-603823/508896$	$(1,1,0,2)$	$417332/188831$
$(1,2,0,0)$	$17884893/955196$	$(1,2,0,1)$	$992961/785407$
$(1,2,1,0)$	$2761171/712822$	$(1,3,0,0)$	$-3841999/611370$

Give bidder 2 the same construction with (x, y) and (z, w) swapped, and put $F_i = F_i^0 + G_i$.

Lemma 13 (Rational witness feasibility)**.** *These fields satisfy Theorem 12. They are equivariant under bidder exchange and simultaneous item exchange.*

Proof. The corrections are polynomials. Equality of mixed partial derivatives gives $\operatorname{div}_{x,y} G = 0$. Moreover $G \cdot n = 0$ on each side of the own-type square: on a vertical side $\partial_y \psi = 0$, and on a horizontal side $-\partial_x \psi = 0$, because of the boundary factors in ψ. Bidder symmetry follows from the definition of G_2. Finally, $h(y, x, w, z) = -h(x, y, z, w)$, hence the same antisymmetry holds for ψ. Differentiation gives

$$G(y, x, w, z) = (G_y(x, y, z, w), G_x(x, y, z, w)),$$

where G_x, G_y denote the two components. The radial field has the same item-exchange property, proving the claim. □

7.4 Finite exact enclosure

Proposition 14 (Certified envelope)**.** *For the fields in Lemma 13, $\mathcal{E}(F) \le U_{\text{stream}}$.*

Computer-assisted proof. It remains to bound the integral in (14) exactly. Split each bidder square into (s, st) and (st, s). In each of the four chart pairs, multiplication by the Jacobian $s_1 s_2$ turns both item-1 competitors into exact rational polynomials in (s_1, t_1, s_2, t_2). The antisymmetry of h under $(x, y, z, w) \mapsto (y, x, w, z)$ exchanges the two field components; invariance of the uniform distribution therefore makes the item-2 integral identical. The two charts overlap only on the diagonal and at the origin, which are null sets.

The certificate converts each polynomial and their exact difference to tensor Bernstein form. A polynomial lies in the convex hull of its Bernstein control coefficients, and its box integral is the box volume times their mean. Every exact initial control c is stored as an integer $m = \lfloor 10^9 c \rfloor$ with error radius one. A degree-n de Casteljau half-box split makes n floor averages, so the rigorous radius update is $E \leftarrow E + n$. Lower bounds for a competitor and for the exact competitor difference certify a fixed winner; otherwise the maximum upper control gives a valid supremum charge.

The tree uses the maximum-variation axis above the final four levels of a depth-21 base partition and exact one-step child-bound lookahead in those final four levels. Each unresolved depth-21 leaf is split once along the best exact child-bound axis if and only if that split saves at least one integer accumulator unit; otherwise its depth-21 charge is retained. Each unresolved child of an accepted depth-22 split is then tested independently: its best depth-23 split is accepted if and only if it saves at least one further unit, and otherwise its depth-22 charge is retained. Taking the minimum of several valid upper bounds is valid. All contributions are accumulated over the common depth-23 dyadic denominator. Each accepted split replaces its parent charge by charges on two children with disjoint interiors; a retained box is charged once. The coverage check sums the leaf volumes over all four chart pairs to $4 \cdot 2^{23} = 33554432$ common-denominator units.

The manifest-driven verifier reconstructs the polynomials and reports

$$\begin{aligned}
\text{boxes fixed before unresolved depth-21 leaves}: &\quad 404804,\\
\text{unresolved depth-21 leaves}: &\quad 462796,\\
\text{leaves refined to depth 22}: &\quad 391618,\\
\text{depth-21 leaves retained}: &\quad 71178,\\
\text{fixed depth-22 children}: &\quad 173198,\\
\text{unresolved depth-22 children}: &\quad 610038,\\
\text{depth-22 children refined}: &\quad 609951,\\
\text{depth-22 children retained}: &\quad 87,\\
\text{fixed depth-23 children}: &\quad 249438,\\
\text{unresolved depth-23 children}: &\quad 970464,\\
\text{integer accumulator}: &\quad 3715139591287203.
\end{aligned}$$

After the four chart pairs and the item-symmetry factor two, this gives

$$\int \sum_j \max\{0, F_{1j}, F_{2j}\} \le \frac{2 \cdot 3715139591287203}{10^9\, 2^{23}} = \boxed{\frac{3715139591287203}{4194304000000000}}.$$

□

Together with Theorem 12 and Lemma 13, this proves the retained stream endpoint. Section 7.5 strengthens this to the active upper half of Theorem 1. The verifier and its clean transcript are in the following certificate directory:

`certificate/continuous_stream_degree4_two_level_nonuniform_upper_bound/`

7.5 A common capacity measure from conditional supports and IC cycles

The stronger upper certificate retains the rational stream coefficients but changes their integration and their use in the whole-auction bound. A full-capacity conditional support can replace stream prices on a region where the other bidder's virtual values have a verified sign. Sparse long-range incentive cycles make a further compatible reduction. The common price is specified independently of any finite allocation range or finite report domain.

A reusable conditional inequality. Set $A = 2/3$, $b_0 = (4 - \sqrt{2})/3$, $k_0 = b_0 - A$, $Z = 3/4$ and

$$\ell(x) = \begin{cases} A, & x < k_0, \\ b_0 - x, & k_0 < x < A, \end{cases} \qquad D_0 = \{x < A,\ y < \ell(x)\}.$$

Let $f(x) = 3\ell(x) - 2$ below A and $f(x) = 1$ above A, $F(x) = \int_x^1 f(s)ds$ and $H_Z(y) = 4(y - Z)_+$. Define

$$\begin{aligned}
\Psi_1(x, y) &= 3(x - A)_+ + 4F(x)\mathbf{1}_{y>Z}, \\
\Psi_2(x, y) &= 3(y - \ell(x))_+\mathbf{1}_{x<A} + f(x)H_Z(y).
\end{aligned} \tag{16}$$

The first term of Ψ_2 is zero above A. Integration-boundary values may be chosen Borel. Both densities are bounded and nonnegative. For example, above Z and below A the second density is $(1-H_Z)3(y-\ell)+H_Z(3y-2)\geq 0$. The edge-transfer envelope gives the universal identity

$$\langle \Psi, a\rangle - R = 3\int_{D_0} u \geq 0 \tag{17}$$

for every full randomized DSIC/IR conditional mechanism, including unnormalized utility. Equivalently, $\operatorname{div}\Psi = 3$ outside D_0 and zero inside, with the required top/right boundary flux. The menu $\max(0, x-A, y-A, x+y-b_0)$ attains equality and saturates every positive price, so $\int(\Psi_1+\Psi_2) = R_{\mathrm{SB}} = 4/9 + 2\sqrt{2}/27$. Unlike the occupied-wing identity, (17) is valid without freezing the residual's occupation constraints.

Compatibility with the opposite bidder. Write $W_0 = [0, 43/100]^2$, distinct from W_{wing} in the lower construction. The support Ψ vanishes on W_0. For the inherited stream field ϕ_i, exact rational Bernstein subdivision on both radial charts proves

$$\phi_{ij}(v_i, v_{-i}) \leq 0 \quad \text{if } v_i \in W_0 \quad (j=1,2),$$

for every opposing report. The field value at its integrable radial origin can be set to zero; the origin IR trace remains in the envelope. This sign condition permits replacing prices according to the opponent's report while retaining an upper bound on its own complete screening problem.

Add four symmetric, disjoint two-way IC flows to obtain ϕ_i'. One flow acts for a fixed opposing box centered at $(3/5, 1/5)$, joining own-report boxes centered at $(2/5, 39/40)$ and $(7/20, 23/40)$. All coordinate halfwidths are $3/200$ and the translated-pair flow density is $1/20$. Forward and reverse flows have equal mass. Bidder exchange and simultaneous item exchange produce four copies. Their eight profile boxes avoid every profile with either bidder in W_0. Each weighted inequality is an actual DSIC deviation with the opponent held fixed; mass balance cancels utilities and introduces no interior sink. The virtual correction is opposite on the paired boxes.

Define a common nonnegative capacity density, in physical item coordinates,

$$\Pi_j'(w,v) = \begin{cases} 0, & w, v \in W_0, \\ \Psi_j(v), & w \in W_0, v \notin W_0, \\ \Psi_j(w), & v \in W_0, w \notin W_0, \\ \max(0, \phi_{1j}', \phi_{2j}'), & w, v \notin W_0. \end{cases} \tag{18}$$

Theorem 15 (A common unrestricted charged support)**.** *For the full class of complete single-bidder DSIC/IR mechanisms with marginal allocations in* $[0,1]^2$*, define* $H_i(\Pi) = \sup_{M_i}\{R_i - \langle \Pi, a_i\rangle\}$*. Then* $H_1(\Pi') = H_2(\Pi') = 0$ *and* $\mathrm{OPT} \leq \int \sum_j \Pi_j'$*.*

Proof. When the opponent belongs to W_0, use (17). Outside that opponent region, the stream envelope and the added nonnegative IC inequalities bound revenue by the ϕ_i' allocation pairing minus nonnegative IC/IR slack. If the own report is in W_0, its virtual coefficients are nonpositive and the cycle correction is zero; otherwise (18) dominates them. Thus $R_i - \langle \Pi', a_i\rangle \leq 0$ for every complete M_i. The empty mechanism attains zero. Integrating joint capacity gives the auction bound. This argument applies directly to jointly measurable mechanisms with integrable payments and bounded allocations; it never interchanges an uncountable supremum with an integral. Positive field parts are integrable, and the absolutely continuous pairings cover the Lebesgue-measurable class stated in Section 3. □

What is improved relative to the stream measure. The original field $\phi_i = F_i$ of Lemma 13 already defines a common density $\Pi^0_j = \max(0, \phi_{1j}, \phi_{2j})$. The stream identity gives

$$R_i = \int \sum_j \phi_{ij} a_{ij} - \int u_i(0; v_{-i}) \le \langle \Pi^0, a_i \rangle,$$

and the empty mechanism attains equality. Hence $H_1(\Pi^0) = H_2(\Pi^0) = 0$ as well. Existence of a common measure and zero charged values are already consequences of the retained stream argument. The conditional replacements and compatible IC cycles below instead prove a strict decrease in its total mass, $\int \Pi' < \int \Pi^0$, while preserving validity for both unrestricted conditional problems. This integrated comparison does not assert a pointwise price decrease.

Solving both charged problems does not produce an equality auction. For example, empty leaves all positively priced capacity unused. A matching mechanism must satisfy the capacity and incentive equalities at the same time.

A rational endpoint with componentwise replay. Let B be the exact integral of the original stream maximum. The new continuous majorant uses

$$\max\left(0, \sum_k B_k a_k, \sum_k B_k b_k\right) \le \sum_k B_k \max(0, a_k, b_k)$$

for nonnegative tensor Bernstein basis functions B_k summing to one. Integrating the right-hand side retains within-box variation. Exact rational construction, directed fixed-point error radii and complete dyadic coverage give

$$B \le B_{20} = \frac{231831916327659047}{262144000000000000}.$$

The primary and independently reconstructed traversals each visit 1,351,220 tree nodes and agree on the five recorded depth bounds and all chart coverages.

Let D be the reduction obtained by the two conditional replacements before the cycles. The overlap $W_0 \times W_0$ has zero old and new price, so the two reductions add. At the single-buyer menu a^*,

$$D = 2 \int_{w \in W_0} \int_v \sum_j \left[(\phi_{2j}(v, w))_+ - a^*_j(v) \phi_{2j}(v, w) \right].$$

The integrand is nonnegative. Integrate opponent variables exactly and partition each own radial chart into rational rectangles wholly inside a single menu region. On each such rectangle K use $\int_K (f)_+ \ge (\int_K f)_+$. Boundary-crossing rectangles are omitted. In particular, a bundle rectangle must meet both the total-price and lower-coordinate upgrade thresholds. Requiring total value alone would incorrectly include some singleton buyers. Seven nested partitions give increasing rational lower bounds $\Delta_{16}, \ldots, \Delta_{1024} \le D$. These partitions certify integrals over continuous types; they do not replace the auction domain by a grid. The complete 1024-partition accumulation is performed by the primary verifier. The independent check reconstructs all averaged coefficients, cross-checks classifications at sizes up to 64 and selected cell integrals, and reuses the primary classifier and integral tables. It does not perform a second complete 1024-partition accumulation.

On every sparse-cycle box, exact polynomial bounds keep the old and new virtual winners strict. Integration then gives the exact combined reduction $81/10^{10}$. Disjointness from the conditional replacement regions makes the two improvements compatible. Thus the active endpoint is

$$U_* = B_{20} - \Delta_{1024} - \frac{81}{10^{10}}, \qquad \mathrm{OPT} \le U_*. \tag{19}$$

Here Δ_{1024} is the final exact rational `delta` in `V4_6_2_upper/certificate/conditional_global_splice.json`; its full numerator and denominator and those of U_* are exported in `certificate/coordinated_primal_dual/manifest.json`. Directed rational arithmetic gives

$$0.8829230532587169692606 < U_* < 0.8829230532587169692607.$$

The sixteen-stage ledger is strictly decreasing. It is not a proof that this fixed certificate architecture converges to the optimum.

Certified reduction	Approximate size	Source of improvement
$U_{\text{stream}} - B_{20}$	0.00138972521893	Tighter enclosure of the same stream integral
Δ_{1024}	0.00144551594101	Compatible conditional support replacement
$81/10^{10}$	0.0000000081	Four nonlocal balanced IC cycles

Table 4: The three contributions to the active upper-bound decrease. The first improves numerical enclosure; the other two certify decreases in common price mass. The cycles expose additional usable IC directions, but their present numerical contribution is small.

Which equality conditions remain unsatisfied. Put $O_i = \{v_{-i} \in W_0\}$. The exact gap for the measure is the sum of four nonnegative mechanism slacks,

$$\begin{aligned}
S_{\text{screen}} &= \sum_i 3 \int_{O_i} \int_{D_0} u_i, \\
S_{\text{capacity}} &= \sum_j \int \Pi'_j (1 - a_{1j} - a_{2j}), \\
S_{\text{virtual}} &= \sum_{i,j} \int_{O_i^c} (\Pi'_j - \phi'_{ij}) a_{ij}, \\
S_{\text{IC}} &= \sum_i \int_{O_i^c} u_i(0; v_{-i}) + \sum_{\text{cycle edges}} \int \text{weighted truthful IC residual.}
\end{aligned}$$

They satisfy $\int \Pi' - R = \sum S$. The origin IR trace is explicit; there is no positive interior sink at every type. Screening's internal utility terms are not counted again in S_{IC}. The reported rational endpoint additionally includes

$$E_{\text{cert}} = (B_{20} - B) + (D - \Delta_{1024}) \geq 0, \qquad U_* - R = \sum S + E_{\text{cert}}.$$

Numerical majorization slack is thus separated from mechanism equality. The archived lottery-box tests show positive slack against the historical V4.6 and V4.6.1 candidates. Such a test diagnoses the tested certificate; it does not establish an admissible primal improvement or suboptimality of a candidate. The fresh audit also checks the selected mechanism on the box centered at $(151/200, 3/40, 9/10, 63/200)$ with halfwidth 10^{-4}. Its base fee is active, but bidder 1 remains uniquely empty and bidder 2 uniquely chooses $(1, \beta)$, with $9/10 < \beta < 49/50$. An independently reconstructed whole-box bound makes its safe virtual value strictly negative. Consequently the safe capacity-plus-virtual slack is at least $-\beta\phi'_{2,\text{safe}} > 0$ there. The exact positive integrated rational lower bound is sealed in the fresh audit certificate. This establishes that the common measure does not match the latest mechanism, without inferring its primal suboptimality.

The exported combined proof, sign certificate, sparse-cycle proof and independent exact checks are under `V4_6_2_upper/` in the coordinated source package. A residual-specific theorem

alone need not provide a price for arbitrary capacity changes. Here the reusable identity (17) and the opposing sign certificate are precisely the additional conditions that make the support common.

7.6 Reproducibility and trusted computations

The publication archive separates the proof inputs from the numerical search that found them. The trusted upper-bound directory contains the rational coefficient manifest, polynomial constructor, exact verifier, non-importing replay, and captured transcripts; the root SHA-256 manifest covers these files. The verifier checks all manifest expectations and fails closed on any mismatch. Before fixed-point conversion, every polynomial and Bernstein coefficient is constructed with `fractions.Fraction`. The largest axis degree is eight, the maximum initial absolute control is 1475346733, and the maximum observed directed-rounding radius is 181 fixed-point units. Signed array operations are checked against $2^{63} - 1$; child charges and the global accumulator use arbitrary-precision Python integers.

The non-importing iterative replay independently reconstructs the basis, chart maps, polynomial arithmetic, Bernstein conversion, subdivision, winner tests, selective-refinement rule, and coverage accounting. Both implementations visit 3,738,334 retained-tree nodes and reproduce all refinement counts, with

$$\text{accumulator} = 3715139591287203, \qquad \text{coverage} = 33554432/33554432.$$

The independence is at the implementation level: both programs share the rational input manifest and the Bernstein enclosure principle. They audit the finite arithmetic; Theorem 12 supplies the analytic weak-duality argument. The lower endpoint and remaining-gap fields printed inside those unchanged upper manifests and transcripts are historical comparisons. The active bracket uses the new majorant, conditional subtraction and cycles in Section 7.5, paired with R_*.

The lower-bound chain reconstructs the 144 affine/pivot cells with one exact base verifier. Separate replays recompute the added gains from twenty common-fee rows, 41 positive bundle-pivot cells, and eight item-containment rows, while retaining the shared base revenue. They use scalar rational arithmetic, finite-difference consistency checks, exact Boole quadrature, and, for the final non-common perturbation, exact demand-polygon clipping. The degree bounds needed for exact quadrature follow from the analytic menu formula and the verified price regions, not from finite sampling. The retained V4.6 source-bound lower package separately replays the 29 declared mathematical entry points and independent audits of the final menu, coordinate-line supports, and exact radical/logarithmic increments. It retains the earlier certified reference integral explicitly. This is not a claim of a second from-scratch integration of every historical base layer. The active coordinated package separately invokes 46 mathematical entry-points, including the new mechanism, two source revenue integrations, the present third reconstruction, conditional identities, both complete upper traversals, and a latest-candidate slack audit. It materializes a sealed temporary mirror and needs no original research workspace. The old slower upper traversal is retained for historical reproduction. For the active upper, B_{20} has two complete independently implemented traversals. The subtraction Δ_{1024} has one complete partition accumulation, plus independent reconstruction of every averaged polynomial coefficient, classification cross-checks for partition sizes up to 64 and selected exact cell-integral checks. That second check reuses the primary classifier and integral-table functions; it is not a second complete 1024-partition accumulation. The four IC cycles have separate whole-box strict-sign and exact decrement proofs. The supported release entry point is `python -B verification/reproduce_all.py`. It reruns all publication-facing checks, compares their structured outputs with the theorem endpoints, and verifies the archive hash manifest. The computational trusted base comprises Python's integer semantics, `fractions.Fraction`, the checked NumPy integer operations, JSON parsing, SHA-256, and file I/O. The entry point

rejects optimized Python execution with an explicit runtime check and a nonzero exit status; this protects the executable assertions retained in predecessor kernels. Direct kernel invocations also require non-optimized Python. No optimizer output is read on the trusted proof path.

8 Discussion and open problem

The coordinated search produces a certified gap of 0.00645888887869189193157..., below 0.01, and a mechanism earning more than 99.2684% of unrestricted optimal revenue. The numerical advance has two distinct sources. Complete joint menu changes increase the lower revenue, while reusable conditional screening and long-range IC constraints reduce the global upper. Exact continuous calculations, with the componentwise independence specified above, support both endpoints; no finite type model substitutes for the auction.

The theoretical link is the information-rent effect of capacity. In the occupied-wing theorem, a marginal unavailable on a strip makes utility constant in one coordinate. A left-edge rent can then be integrated over all reports forced by incentives to share it. This solves a full randomized conditional problem even where a pointwise scarcity price alone would miss the constraint. On the primal side, an inverse plateau moves a positive-volume set of menus together. Its revenue effect must include every induced switch and payment change. These statements explain how a joint reallocation can improve a mechanism even where isolated conditional replacements cannot.

A different conditional identity is reusable at arbitrary residuals. Together with an opposing sign condition, it decreases the total common capacity price; compatible IC cycles reduce it further. The stream maximum already has both unrestricted charged values zero, so their solvability alone is not a new conclusion. The smaller common price still falls short of an equality auction. The explicit gap decomposition separates four mechanism slacks from the Bernstein and omitted-cell remainders. A strict lottery-box slack against the selected mechanism proves that this particular measure does not match it; it does not prove the mechanism suboptimal.

Convex potentials, transport duality and complementary slackness are established ideas. The instance-specific contributions are the displayed complete exchange, the occupied-region screening identities, the compatible reductions in common price mass and its exact arithmetic certification. We make no claim of first use of the underlying principles. The formulas depend on the uniform distribution and the stated capacity geometry; general success on other distributions or larger auctions has not been established. The external reported GemNet revenue is rounded, so the numerical comparison alone does not certify a ranking.

The remaining problem is to make allocations, information rents and common capacity prices satisfy equality simultaneously. The conditional coverage map identifies frozen-residual directions that cannot help, while the slack identities identify what a prospective joint change or dual representation must alter. Neither finite-family stationarity, a fixed decreasing sequence of bounds nor failure of one certificate proves unrestricted optimality. The optimal auction and a matching global upper bound remain open.

Data, code, and materials availability

The project repository is publicly accessible at

`https://github.com/Baymax-ray/two-bidder-two-item-dsic-certificates`.

The accompanying release archive contains all exact certificate data, deterministic verification code, transcripts, environment information, and SHA-256 manifests needed to reproduce the stated bounds. Manuscript, certificate data, and other textual materials are released under CC BY 4.0. Verification software is separately released under the MIT License. Discovery-only experiments are segregated from the trusted proof artifacts.

AI-assisted tools declaration

This manuscript was developed with the assistance of OpenAI GPT-5.6 Sol and OpenAI Codex. The tool assisted with mathematical exploration, numerical candidate generation, implementation and debugging of certificate and verification code, and drafting and revision of the manuscript. Separately scoped AI-assisted audits checked the new proofs, source implementations and exact arithmetic; these are internal checks, not external peer review. AI-generated suggestions and numerical search outputs were not treated as proofs; the stated results are supported by the analytic arguments and exact certificates described here. The author retains responsibility for the content, correctness, and final presentation of this work.

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