Postgraduate Workshop in Stochastics

Lecture 24

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by

Assoc. Prof. John van der Hoek

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26 Lévy Processes

26.1 Lévy Processes, basic definitions

Section 26 will deal with general Lévy Process matters. Here we start only with definitions. Separate section numbers will be given for each Lévy process. Section 27 will deal with the Poisson Process.

We let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space.

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \quad \text{for} \quad 0 < s < t$$

We will later impose conditions on $\mathcal{F}_t$ as they are needed.

A process $X = \{X_t : t \geq 0\}$ is called Lévy process if [Sato, page 3]

(1) For any $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$,

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$$

are independent. We say then that X has independent increments.

(2) $X_0 = 0$ a.s. (almost surely) (this is the same as wp1, which means: with probability 1)

(3) For any $s \geq 0, t > 0$

$$X_{s+t} - X_s \stackrel{d}{=} X_t - X_0$$

that is, they are equal in distribution. We then say that X has stationary increments.

(4) The process is stochastically continuous (or continuous in probability). This means that for any $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{s \to t,\ s \geq 0} \mathrm{P}(|X_s - X_t| > \varepsilon) = 0$$

Of course, in the case $t = 0$ this is right stochastic continuity.

For each $\omega \in \Omega$, the map
$$t \to X_t(\omega)$$
defines a **sample path** of X. For any $t \geq 0$ almost every sample path is continuous at t.

(5) There exists $\Omega_0 \subset \mathcal{F}, P(\Omega_0) = 1$ s.t. for all $\omega \in \Omega_0$ the sample path

$$t \to X_t(\omega)$$

is càdlàg (French 'continue à droite, limitée à gauche') which in English is RCLL ('right continuous with left limits'), or CORLOL ('continuous on (the) right, limit on (the) left').

Remarks:

(i) In fact, conditions (1)–(4) are all that are required for a Lévy process. This is because a process satisfying (1)–(4) will always have a càdlàg version.

We say that $\{Y_t\}$ is a version of $\{X_t\}$ if $P(X_t = Y_t) = 1$ for all $t \geq 0$.

This claim follows from a similar result for martingales. We will treat this at a future time.

Once this result is proved we can redefine $X_t(\omega) = 0$ for all $t \geq 0$ when $\omega \notin \Omega_0$ and then all the sample paths of X are càdlàg. So we will always assume that we have such a version.

(ii) Lévy processes are usually described in terms of their characteristic function
$$f_t(u) = \mathbf{E}[\, e^{iuX_t} \,]$$
Lévy and Khintchine independently gave an expression for $f_t(u)$ which is called the Lévy–Khintchine formula. Later we will study some alternative ways to derive it.

Examples:

There are many examples: Brownian motion, Poisson processes, Compound Poisson Processes, Gamma Processes, Stable processes, Inverse Gaussian Processes, and many more. An interesting and important class processes are the subordinators which are Lévy processes which have non-decreasing sample paths. They can be used to make stochastic time changes to other processes.

Some References

We will often refer to the following texts and others which we will add when required.

Applebaum, D., Lévy Processes and Stochastic Calculus, CUP 2004.

Bertoin, J. Lévy Processes, CUP 1996.

Çinlar, E., Introduction to Stochastic Processes, Prentice Hall 1975.

Cont, R. and **Tankov**, T., Financial Modeling with Jump Processes, Chapman and Hall/CRC 2004.

Elliott, R. J., Stochastic Calculus and Applications, Springer Verlag 1982.

Feller, W. An Introduction to Probability Theory and Its Applications. Volume I and II, Wiley 1971.

Protter, Ph., Stochastic Integration and Differential Equations, 2/e., Springer Verlag 2004.

Sato, K., Lévy Processes and Infinitely Divisible Distributions, CUP 1999.

Skorohod, A. V., Random Processes with Independent Increments, Kluwer 1991.

Skorohod, A. V., Lectures on the Theory of Stochastic Processes, VSP 1996.

27. Poisson Process

We will use the set on integers

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

and we will work in a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathrm{P})$$

We follow **Protter**, section 1.3 with some extra discussion.

Let $\{\mathrm{T}_n\}_{n\geq 0}$ strictly increasing sequence of non-negative random variables on this probability space. We thus assume that $\mathrm{T}_0 = 0$ (almost surely) and $\mathrm{T}_{n+1} > \mathrm{T}_n$ holds almost surely. This means

$$\mathrm{P}\left(\{\omega \in \Omega \,|\, \mathrm{T}_{n+1}(\omega) > \mathrm{T}_n(\omega)\}\right) = 1 \ .$$

We write

$$\mathbf{I}[\, t \geq \mathrm{T}_n](\omega) = \begin{cases} 1, & t \geq \mathrm{T}_n(\omega) \\ 0, & t < \mathrm{T}_n(\omega) \end{cases}$$

and

$$\mathrm{N}_t = \sum_{n=1}^{\infty} \mathbf{I}[\, t \geq \mathrm{T}_n\,]$$

The process $\mathrm{N} = \{\mathrm{N}_t \,:\, t \geq 0\}$ defined this way, takes values in $\mathbb{N} \cup \{\infty\}$ and is called the counting process of the sequence $\{\mathrm{T}_n\}_{n\geq 0}$

This is a good name as $\mathrm{N}_t(\omega) = \#\{n : \ \mathrm{T}_n(\omega) \leq t\}$.

Notation:

Let

$$\mathrm{T} = \sup_n \mathrm{T}_n$$

which means

$$\mathrm{T}(\omega) = \sup_n \mathrm{T}_n(\omega)$$

$$[\mathrm{T}_n, \infty) = \{\mathrm{N} \geq n\} = \{(t, \omega) \,|\, \mathrm{N}_t(\omega) \geq n\}$$

$$[\mathrm{T}_n, \mathrm{T}_{n+1}) = \{\mathrm{N} = n\} = \{(t, \omega) \,|\, \mathrm{N}_t(\omega) = n\}$$

$$[\mathrm{T}, \infty) = \{\mathrm{N} = \infty\}$$

T is called the **explosion time** of N.

If $\mathrm{T} = \infty$ almost surely, then N is counting process without explosions (this means $\mathrm{N}_t(\omega) < \infty$ almost surely for $\omega \in \Omega$ and for all $t \geq 0$).

Note that N is càdlàg by definition and if $0 \leq s < t < \infty$,

$$\mathrm{N}_t - \mathrm{N}_s = \sum_{n=1}^{\infty} \mathbf{I}[s < \mathrm{T}_n \leq t]$$

counts **events** in $(s, t]$

Lemma A

$\{\mathrm{N}_t\}$ is adapted to $\{\mathcal{F}_t\}$ if and only if $\{\mathrm{T}_n\}_{n\geq 1}$ are $\{\mathcal{F}_t\}$ stopping times.

Remarks:

(i) We say that $\{\mathrm{N}_t\}$ is adapted to $\{\mathcal{F}_t\}$ if N_t is $\mathcal{F}_t$ measurable for each $t \geq 0$.

(ii) We call T a stopping time, (with respect to $\{\mathcal{F}_t\}$), if

1. $T : \Omega \to [0, \infty)$
2. $\{\omega \in \Omega \,|\, T(\omega) \leq t\} \in \mathcal{F}_t \quad \text{for all} \quad t \geq 0$

Proof:

First assume that $\{\mathrm{T}_n\}_{n\geq 1}$ are $\{\mathcal{F}_t\}$ stopping times. Then for integers $k = 0, 1, 2, ...$

$$\{\omega \in \Omega \,:\, \mathrm{N}_t(\omega) \leq k\} = \{\omega \in \Omega \,:\, \mathrm{T}_k(\omega) \geq t\} \ \in \mathcal{F}_t$$

and for any $a \geq 0$, find k with $k \leq a < k + 1$, and then

$$\{\omega : \mathrm{N}_t(\omega) \leq a\} = \{\omega : \mathrm{N}_t(\omega) \leq k\} \ \in \mathcal{F}_t$$

We conclude that $\{\mathrm{N}_t\}$ is adapted to $\{\mathcal{F}_t\}$.

Conversely, if $\{\mathrm{N}_t\}$ is adapted to $\{\mathcal{F}_t\}$, then

$$\{\omega \in \Omega \,:\, \mathrm{T}_k(\omega) \leq t\} = \{\omega \in \Omega \,:\, \mathrm{N}_t(\omega) \geq k\} \in \mathcal{F}_t$$

and so $\{\mathrm{T}_n\}_{n\geq 1}$ are $\{\mathcal{F}_t\}$ stopping times. QED.

Remark:

We say X is $\mathcal{F}$ measurable if $\{\omega \in \Omega \,|\, X(\omega) \leq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$. Then

$$\{\omega \in \Omega \,|\, X(\omega) \geq a\} = \Omega \backslash \{\omega \in \Omega \,|\, X(\omega) < a\}$$

and

$$\{\omega \in \Omega \,|\, X(\omega) < a\} = \bigcup_{n=1}^{\infty} \{\omega \in \Omega \,|\, X(\omega) \leq a - \frac{1}{n}\} \in \mathcal{F}$$

and so

$$\{\omega \in \Omega \,|\, X(\omega) \geq a\} \in \mathcal{F}$$

This type of fact is usually covered in courses on measure theory.

Remark:

We could take $\mathcal{F}_t = \sigma\{\mathrm{N}_u : u \leq t\}$ then $\mathrm{T}_n(\omega) = \inf\{t \geq 0, \mathrm{N}_t(\omega) \geq n\}$ will be stopping time for $\{\mathcal{F}_t\}$.

T_n are called arrival times, jump times, event times, depending on the application. The first term come from queuing theory.

There are several equivalent definitions of a Poisson process. We will now select one and discuss its equivalence to some other definitions.

Definition: [Protter following Çinlar]

An adapted counting process N without explosions is a Poisson process if

(1) for any $0 \leq s < t < \infty$, $\mathrm{N}_t - \mathrm{N}_s$ is independent of $\mathcal{F}_s$.

(2) for $0 \leq s < t < \infty$ and $0 \leq u < v < \infty$ with $t - s = v - u$ then

$$\mathrm{N}_t - \mathrm{N}_s \stackrel{d}{=} \mathrm{N}_v - \mathrm{N}_u$$

The condition (1) is equivalent to saying N has independent increments and (2) means that N has stationary increments.

We claim that this N is a Lévy. We will need to check that (4) holds. We have already seen that the counting process is càdlàg and so condition (5) holds.

Theorem:

For some $\lambda \geq 0$

$$P(\mathrm{N}_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \ \ n = 0, 1, 2, 3, ...$$

We will use the following Lemma several times.

Lemma B - version 1 (in Feller Vol 1, page 459)

Suppose that $f : (0, \infty) \to \mathbb{R}$ with

$$f(s+t) = f(s)\, f(t) \quad \text{for all} \quad s, t > 0 \tag{1}$$

and f is bounded on some interval $I = (u, v)$ with $0 \leq u < v$.

Then either:

(a) $f(t) = 0$ for all $t > 0$

or

(b) for some $\lambda \in \mathbb{R}$, $f(t) = e^{-\lambda t}$ for all $t > 0$.

Proof:

For any $a > 0$,

$$f(a) = f\left(\frac{a}{2}\right)^2 \tag{$*$}$$

Case 1: Suppose $f(a) = 0$ for some $a > 0$

We observe that $f(s) = 0$ with $s > 0$ implies $f(t) = 0$ for all $t > s$. This is true because

$$f(t) = f(s)\ f(t-s) = 0$$

As $f(a) = 0$, $(*)$ implies that $f\left(2^{-n}a\right) = 0$ for all $n = 1, 2, 3, ...$

Let $t > 0$ be arbitrary. There is an $n \geq 1$ so that $0 < 2^{-n}a < t$. Thus $f(t) = 0$ from the first observation.

Case 2: $f(t) > 0$ for all $t > 0$.

Of course $(*)$ rules out $f(t) < 0$.

For any $m = 1, 2, 3, ...$ we have $f(m) = f(1)^m$ by repeated use of (1). Also

$$f(1) = f\left(\frac{m}{m}\right) = f\left(\frac{1}{m}\right)^m$$

and so

$$f\left(\frac{1}{m}\right) = f(1)^{\frac{1}{m}}$$

and so for $m, n = 1, 2, 3, ...$

$$f\left(\frac{m}{n}\right) = f\left(\frac{1}{n}\right)^m = f\,(1)^{\frac{m}{n}}$$

So if we write

$$f(1) = e^{-\lambda}$$

then from what we have shown

$$f(t) = e^{-\lambda t}$$

for all $t \in \mathbb{Q}^+$ (the positive rational numbers).

We now want to show that $f(t) = e^{-\lambda t}$ holds for all reals $t > 0$. To this end set

$$g(t) = e^{\lambda t} f(t)$$

Then for $s, t > 0$

$$g(s + t) = g(s)\, g(t)$$

and $g(t) = 1$ for all $t \in \mathbb{Q}^+$.

We now observe that if $s, t > 0$ and $s - t \in \mathbb{Q}^+$ or $t - s \in \mathbb{Q}^+$ then $g(s) = g(t)$. For example, in the first case

$$g(s) = g(t)\, g(s - t) = g(t)$$

This observation implies that for any $t > 0$, there is $t' \in I$ and $g(t) = g(t')$.

Suppose that there is $a > 0$ so that $g(a) = c$ and $c \neq 1$. If $c > 1$, define $t_n = a^n$, then

$$g(t_n) = g(a^n) = c^n \to \infty$$

as $n \to \infty$. For each $n = 1, 2, 3, ...$ let $t'_n \in I$ with $g(t'_n) = g(t_n)$. So g and hence f is unbounded on I, which is a contradiction to our assumption.

If $c < 1$ define $t_n = a^{-n}$ and so

$$g(t_n) = g(a^{-n}) = c^{-n} \to \infty$$

as $n \to \infty$. For each $n = 1, 2, 3, ...$ again, let $t'_n \in I$ with $g(t'_n) = g(t_n)$. So g and hence f is unbounded on I, which is again a contradiction to our assumption.

We conclude from these contradictions that $g(t) = 1$ for all $t > 0$ and hence that $f(t) = e^{-\lambda t}$ for all $t > 0$. QED.

Lemma B - version 2

Suppose that $f : (0, \infty) \to \mathbb{R}$ with

$$f(s + t) = f(s)\, f(t) \quad \text{for all} \quad s, t > 0 \tag{1}$$

and f is a non-increasing function.

Then either:

(a) $f(t) = 0$ for all $t > 0$

or

(b) for some $\lambda \geq 0$, $f(t) = e^{-\lambda t}$ for all $t > 0$.

Proof:

We only need to modify the discussion of case 2 in the previous proof.

Let $f(1) = e^{-\lambda}$ in that case. Now $f(1) \geq f(2) = f(1)^2$ implies

$$e^{-\lambda} = f(1) \geq f(1)^2 = e^{-2\lambda}$$

and $\lambda \geq 0$ follows.

We also have $f(t) = e^{-\lambda t}$ for $t \in \mathbb{Q}^+$.

Let $t > 0$, then there exists two sequences: $\{t_n\} \subset \mathbb{Q}^+$ is strictly increasing and $t_n \to t$ as $n \to \infty$; $\{s_n\} \subset \mathbb{Q}^+$ is strictly decreasing and $s_n \to t$ as $n \to \infty$. But

$$f(s_n) \leq f(t) \leq f(t_n)$$

for $n = 1, 2, 3, \ldots$ Now both $f(t_n) \to e^{-\lambda t}$ and $f(s_n) \to e^{-\lambda t}$ as $n \to \infty$ shows that $f(t) = e^{-\lambda t}$ and we are done. QED.

Remark:

For version 1, we cannot conclude that $\lambda \geq 0$.

Historical Remarks:

Clearly if case 1 does not hold, then $f(t) > 0$ for all $t > 0$. We then set $h(t) = \log f(t)$ and get Cauchy's functional equation

$$h(s + t) = h(s) + h(t)$$

which has been widely studied. Cauchy in 1821 showed that $h(t) = ct$ if h is assumed continuous; Darboux in 1875 showed that continuity at one point is all that is needed, and in 1880 Darboux showed that the same result holds when $h(t) \geq 0$ for t sufficiently small and positive. Later it was shown that monotonicity suffices, but monotonicity on some interval also suffices (I guess we can use arguments from both proofs above), h bounded on an interval suffices as we have seen. If no assumptions are made on h, Hamel in 1905 showed how to construct (infinitely many) non–continuous solutions.

Surprisingly, h need only be bounded on a Lebesgue measurable set C with positive Lebesgue measure (for then h is bounded on the set $C + C$. A theorem of Steinhaus says that $C + C$ contains a nonempty open set and so this case reduces to the case we have already studied).

Proof of theorem:

Step 1. We claim for some $\lambda \geq 0$ that

$$P(\mathrm{N}_t = 0) = e^{-\lambda t} \quad \text{for} \quad t \geq 0$$

We only need to show this for $t > 0$.

Now for $t, s > 0$

$$\{\mathrm{N}_{s+t} = 0\} = \{\mathrm{N}_s = 0\} \cap \{\mathrm{N}_{s+t} - N_s = 0\}$$

implies

$$\begin{aligned} P(\mathrm{N}_{s+t} = 0) &= P(\mathrm{N}_s = 0) P(\mathrm{N}_{s+t} - \mathrm{N}_s = 0) \\ &= P(\mathrm{N}_s = 0) P(\mathrm{N}_t = 0) \end{aligned}$$

and if we define

$$f(t) = P(N_t = 0)$$

then

$$f(s + t) = f(s)\, f(t)$$

But from its definition, f is non-increasing function. We then apply Lemma B - version 2 to conclude that $f(t) \equiv 0$ or for some $\lambda \geq 0$ that $f(t) = e^{-\lambda t}$ for all $t \geq 0$.

Suppose that $P(\mathrm{N}_t = 0) = 0$ for all $t > 0$. We derive a contradiction. Now this assumption implies

$$P(\mathrm{N}_t > 0) = 1 \quad \text{for all} \quad t > 0$$

and as N has jumps of size 1, this is the same as

$$P(\mathrm{N}_t \geq 1) = 1 \quad \text{for all} \quad t > 0$$

Let $n \geq 1$ be arbitrary

$$\mathrm{N}_t = \mathrm{N}_t - \mathrm{N}_{\frac{n-1}{n}t} + \mathrm{N}_{\frac{n-1}{n}t} - \mathrm{N}_{\frac{n-2}{n}t} + ... + \mathrm{N}_{\frac{t}{n}} - \mathrm{N}_0 \geq n$$

with probability 1, since

$$P(\mathrm{N}_{\frac{t}{n}} \geq 1) = 1$$

for each n. But then

$$\mathrm{N}_t(\omega) = \infty$$

holds for almost all $\omega \in \Omega$. But this implies that

$$t \geq \mathrm{T}_n(\omega)$$

for all n and for almost all $\omega \in \Omega$. So

$$\mathrm{T}(\omega) = \sup_n \mathrm{T}_n(\omega) \leq t$$

for almost all $\omega \in \Omega$. This contradicts the assumption that $T = \infty$ almost surely.

This contradiction implies that for some $\lambda \geq 0$ that

$$P(\mathrm{N}_t = 0) = e^{-\lambda t} \quad \text{for} \quad t \geq 0$$

Remark:

If $\lambda = 0$ then the process has no jumps as $\mathrm{N}_t = 0$ almost surely for all $t > 0$. But this also contradicts the fact that $\mathrm{T}_n < \infty$ also surely for each n. We conclude that $\lambda > 0$ should hold.