# Introduction to Stochastic Differential Equations

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## Aims

• Introduce the basic theoretical concepts related to Stochastic Differential Equations (SDEs) of the form

$$\begin{aligned} dX_t &= b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 &= \xi, \end{aligned}$$

and the fundamental tools for proving the existence and the uniqueness of a solution to this equation.

• Review some examples of SDEs arising in Physics, Finance, Biology, ...

• Present the link between the theory of SDEs and the theory of Partial Differential Equations.

• Discuss more complex SDEs and their recent applications to statistical mechanics, mean field games, ...

**Overview:**

$I$. **Elements of stochastic processes and stochastic calculus.**

$II$. **Strong solution to a Stochastic Differential Equations and related properties.**

$III$. **Weak solution to a Stochastic Differential Equations and related properties.**

$IV$. **Martingale problems.**

$V$. **Links with Partial Differential Equations.**

$VI$. **Some advanced models: SDEs of McKean-Vlasov type, interacting particle systems and their applications in physics and economy; Optimal Stochastic Control Problems.**

## Bibliography

(F06) A. Friedman (2006): *Stochastic Differential Equations and Applications*.

(IK81) N. Ikeda and S. Watanabe (1981): *Stochastic Differential Equations and Diffusion Processes*.

(KS88) I. Karatzas and S. E. Shreve (1988): *Brownian motion and Stochastic Calculus*.

(O00) B. Oksendal (2000): *Stochastic Differential Equations*.

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(SV79) D. W. Stroock and S. R. S. Varadhan (1979): *Multidimensional Diffusion Processes*.

## Elements of Stochastic Processes and Stochastic Calculus.

## Basic definitions

From now on, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote an abstract probability space, equipped with some $\sigma$-algebra (or $\sigma$-field) $\mathcal{F}$ and some probability measure $\mathbb{P}$.

### Definition 1.1

Given $I \subset [0, \infty)$, a $\mathbb{R}^d$-valued stochastic process $(X_t;\ t \in I)$ is a collection of random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$, equipped with its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d)$.

If $I = \mathbb{N}$, $(X_t;\ t \in I)$, simply denoted by $(X_n;\ n \in \mathbb{N})$, is a discrete-time stochastic process. If $I = [0, \infty)$, $(X_t;\ t \geq 0)$ is a continuous-time stochastic process.

Assuming that $d = 1$,
∘ Given $t \geq 0$, $X_t : \omega \in \Omega \to X_t(\omega) \in \mathbb{R}$ is a random variable (i.e., for all $A \in \mathcal{B}(\mathbb{R})$, $X_t^{-1}(A) \in \mathcal{F}$).

∘ Given $\omega \in \mathcal{F}$, $X_\cdot(\omega) : t \in [0, \infty) \to X_t(\omega) \in \mathbb{R}$ is a Borel measurable function which describes a sample path (a possible trajectory) of the process $(X_t;\ t \geq 0)$.

The index $t$ refer to a time parameter, and a continuous-time stochastic process $(X_t;\ t \geq 0)$ models the time evolution of a random system (for example: the evolution of a fluid particle, a stock price, a population, ...).

## Basic definitions

**Sample path of a stochastic process**:

• $(X_t;\ t \geq 0)$ is said to have continuous paths (or simply to be continuous) if and only if, for almost all $\omega \in \Omega$, $t \mapsto X_t$ is continuous. Equivalently,

$$\mathbb{P}\left(\forall\, t_0 \geq 0, \lim_{t \to t_0} X_t = X_{t_0}\right) = 1.$$

• $(X_t;\ t \geq 0)$ is said to be càdlàg (*continue à droite et limité à gauche*) if and only if, for almost all $\omega \in \Omega$, $t \mapsto X_t$ is càdlàg. Equivalently,

$$\mathbb{P}\left(\forall\, t_0 \geq 0, \lim_{t \to t_0,\, t > t_0} X_t = X_{t_0}, \lim_{t \to t_0,\, t < t_0} |X_t| < \infty\right) = 1.$$

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## Filtration

### Definition 1.2

A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of $\sigma$-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ such that

$$\forall\, t,\ \mathcal{F}_t \subset \mathcal{F}$$

and $\{\mathcal{F}_t\}_{t\geq 0}$ is non-decreasing in the sense:

$$\forall\, t_1 \leq t_2,\ \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}.$$

The quadruplet $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ defines a filtered probability space.

**Example**: To any given stochastic process $(X_t;\ t \geq 0)$, we can associate a natural filtration $(\mathcal{F}^X_t;\ t \geq 0)$ defined by: for all $t \geq 0$,

$$\mathcal{F}^X_t = \sigma(X_r;\ 0 \leq r \leq t).$$

For each $t$, $\mathcal{F}^X_t$ is the small $\sigma$-algebra for which, for all $0 \leq s \leq t$, $X_s$ is measurable. $\mathcal{F}^X_t$ models the past and current events related to the sample paths of $X$ up to time $t$.

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**Example**: To any given stochastic process $(X_t;\, t \geq 0)$, we can associate a natural filtration $(\mathcal{F}^X_t;\, t \geq 0)$ defined by: for all $t \geq 0$,

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## Filtration

### Definition 1.3

A continuous-time stochastic process $(X_t;\ t \geq 0)$ is said to be $\mathcal{F}_t$-adapted if for all $t$, $X_t$ is measurable with respect to $\mathcal{F}_t$ i.e. for all $A \in \mathcal{B}(\mathbb{R}^d)$,

$$X_t^{-1}(A) \in \mathcal{F}_t.$$

**Example**: Any given stochastic process $(X_t;\ t \geq 0)$ is $\mathcal{F}_t^X$-adapted.

### Definition 1.4

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ a filtered probability space, a continuous-time stochastic process $(X_t;\ t \geq 0)$ is said to be progressively measurable if for all $t$, the mapping $X : (s, \omega) \in [0, t] \times \Omega \mapsto X_s(\omega)$ is measurable with respect to $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ i.e. for all $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\{(s, \omega) \in [0, t] \times \Omega \,|\, X_s(\omega) \in A\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t.$$

### Proposition 1.5

*Every $\mathcal{F}_t$-adapted stochastic process is progressively measurable. Reciprocally, if $(X_t;\ t \geq 0)$ is $\mathcal{F}_t$-adapted and càdlàg then $(X_t;\ t \geq 0)$ is progressively measurable.*

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**Definition 1.6**

*A filtration $(\mathcal{F}_t;\, t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy the usual conditions i.f.f. the following properties hold true:*
- *$(\mathcal{F}_t;\, t \geq 0)$ is right-continuous: $\mathcal{F}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$,*
- *$\mathcal{F}_0$ contains the collection of $\mathbb{P}$-null sets of $\mathcal{F}_\infty$; i.e. contains all $A \in \mathcal{F}_\infty$ such that $\mathbb{P}(A) = 0$.*

**Augmentation of the filtration generated by a stochastic process:** Let $(\mathcal{F}^X_t;\, t \geq 0)$ be the filtration on $(\Omega, \mathcal{F}, \mathbb{P}^\mu)$ related to a stochastic process $(X_t;\, t \geq 0)$ where $\mu = \mathbb{P} \circ (X_0)^{-1}$. Define the set of all $\mathbb{P}^\mu$-negligible events: The augmentation of $(\mathcal{F}^X_t;\, t \geq 0)$ is the filtration $(\overline{\mathcal{F}}^X_t;\, t \geq 0)$ defined by

$$\overline{\mathcal{F}}^X_t = \sigma\left(\mathcal{F}^X_t \cup \mathcal{N}^\mu\right),$$

where $\mathcal{N}^\mu$ is the set of $\mathbb{P}$-null events given by

$$\mathcal{N}^\mu = \left\{F \subset \Omega \,|\, \exists G \in \mathcal{F}^X_t \text{ for some } 0 \leq t < \infty \text{ such that } F \subset G,\, \mathbb{P}^\mu(G) = 0\right\}.$$

For a large class of stochastic processes (notably the strong Markov processes that will considered later), the augmentation and completion filtration of $(\mathcal{F}^X_t;\, t \geq 0)$ satisfies the usual conditions (see e.g. (KS88), chapter 2).
From now on, we will implicitly assume that the filtration considered in our statements satisfy the usual conditions.

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## Examples

**The Poisson process:**

### Definition 1.7

*A Poisson process $(N_t;\ t \geq 0)$ with intensity $\lambda > 0$ is a $\mathbb{N}$-valued $\mathcal{F}_t$-adapted continuous time process satisfying the following properties:*
- $N_0 = 0$ *a.s.,*
- *for almost all $\omega \in \Omega$, $t \mapsto N_t(\omega)$ is càdlàg,*
- *for $0 < s < t < \infty$, the increment $N_{t+s} - N_s$ is independent to $\mathcal{F}_s$*
- *and, for all $0 < s < t < \infty$, $N_{t+s} - N_s \sim \mathcal{P}(\lambda t)$:*

$$\mathbb{P}(N_{t+s} - N_s = n) = \exp\{-\lambda t\} \frac{(\lambda t)^n}{n!},\ n \in \mathbb{N}.$$

Commonly used to describe queue or rare events models, the Poisson process is one of the most fundamental and simplest example of time continuous stochastic process with stationary and independent increments; namely for

$$0 = t_0 < t_1 < t_2 < \cdots < t_N < \infty, \text{ such that } t_1 - t_0 = t_2 - t_1 = \cdots = t_N - t_{N-1},$$

the random variables

$$N_{t_1} - N_{t_0},\ N_{t_2} - N_{t_1},\ \cdots,\ N_{t_N} - N_{t_{N-1}},$$

are (mutually) independent and identically distributed.

## Examples

**Example**: Let $T_1, T_2, \cdots, T_n, \cdots$ be a sequence of i.i.d. r.v's defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $T_1 \sim \mathcal{E}(\lambda)$:

$$f_{T_1}(x) = \lambda e^{-\lambda x} \mathbb{1}_{\{x \geq 0\}},$$

and define

$$S_n = T_1 + T_2 + \cdots + T_n = \sum_{k=1}^{n} T_k,\; n \geq 1.$$

The process $(N_t;\, t \geq 0)$ defined by

$$N_t = \sum_{n} \mathbb{1}_{\{S_n \leq t\}}$$

is a Poisson process with intensity $\lambda > 0$ for the filtration $(\mathcal{F}_t^N;\, t \geq 0)$ generated by $(N_t;\, t \geq 0)$.

## Examples

$(N_t;\ t \geq 0)$ is a counting process modeling the number of arrival (e.g. the number of customers arriving at an office) or the number of occurrence of random events ($\lambda$ corresponds then to the averaged frequency of these events). In particular, since

$$\{N_t \geq n\} = \{S_n \leq t\}$$

the sequence $\{S_n\}_n$ corresponding to jump times of $(N_t;\ t \geq 0)$ can be interpreted as the arrival or occurrence times in a time interval $[0, t]$.

**Some properties**:
◦ $\mathbb{E}[N_t] = \lambda t$ and $\mathrm{Var}(N_t) = \lambda t$.
◦ For all $n$, $S_n \sim \Gamma(n, \lambda)$:

$$f_{S_n}(x) = \frac{\lambda}{(n-1)!} e^{-\lambda x} (\lambda x)^{n-1} \mathbb{1}_{\{x \geq 0\}}.$$

## Examples

**The Brownian motion**

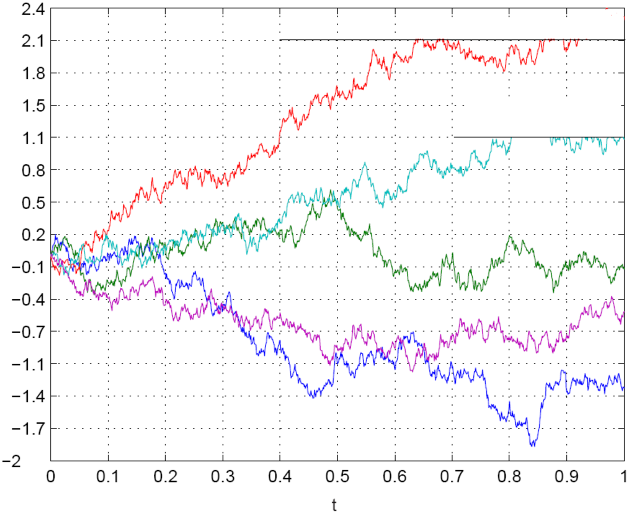
### Definition 1.8

*A (standard) Brownian motion* $(W_t;\ t \geq 0)$ *is a* $\mathbb{R}$*-valued* $\mathcal{F}_t$*-adapted continuous-time stochastic process such that*
- ∘ $W_0 = 0$ *a.s.*,
- ∘ $(W_t;\ t \geq 0)$ *has continuous paths*,
- ∘ *for all* $0 \leq s \leq t < \infty$, $W_t - W_s$ *is independent to* $\mathcal{F}_s$
- ∘ *and, for all* $0 \leq s < t < \infty$,

$$W_t - W_s \stackrel{\mathcal{D}}{=} W_{t-s} \sim \mathcal{N}(0, t-s).$$

Introduced by R. Brown in 1828 to describe the evolution of pollen particle, and later by A. Einstein in 1902 and P. Langevin in 1908 for modeling the motion of fluid particles, the Brownian motion represents a prototypical example of a continuous-time stochastic process modeling the time evolution of a system whose motions are erratic, with rapid change of directions and without memory of its previous states.

## The Brownian motion



**The Brownian motion as a limit of a random walk:**

- **Symmetric random walk**:

$$\sum_{k=1}^{n} \xi_k,\ t \in \mathbb{N},$$

where $\xi_k,\ k \in \mathbb{N}$ is a family of i.i.d. random variables such that

$$\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}.$$

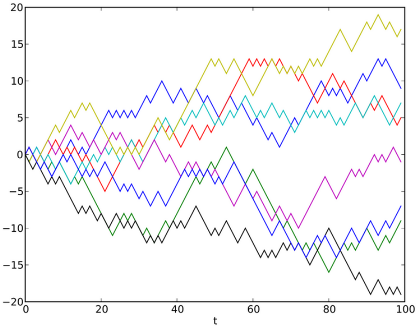



Figure: Simulation of sample paths of a random walk

- **Re-scaled symmetric random walk:**

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{nt} \xi_k,$$

such that $nt$ is an integer.

**Properties**:

∘ For $0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m < \infty$ such that $nt_j$ is an integer,

$$S_n(t_1) - S_n(t_0),\ S_n(t_2) - S_n(t_1),\ \cdots,\ S_n(t_m) - S_n(t_{m-1}),$$

are independent.

∘ $\mathbb{E}[S_n(t) - S_n(s)] = 0$ and $\mathrm{Var}(S_n(t) - S_n(s)) = t - s$.

We have, for all $0 \leq t < \infty$,

$$S_n(t) \xrightarrow{\mathcal{D}} W_t \sim \mathcal{N}(0, t),$$

by the Central Limit Theorem.

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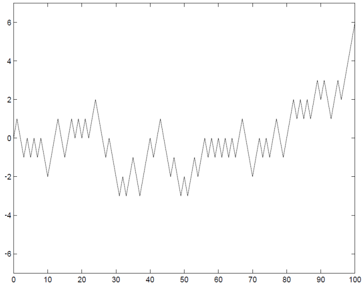
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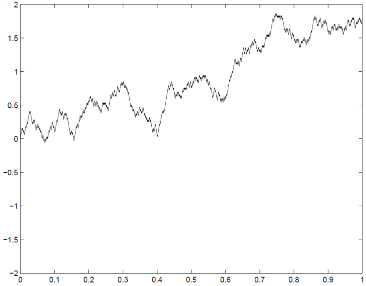
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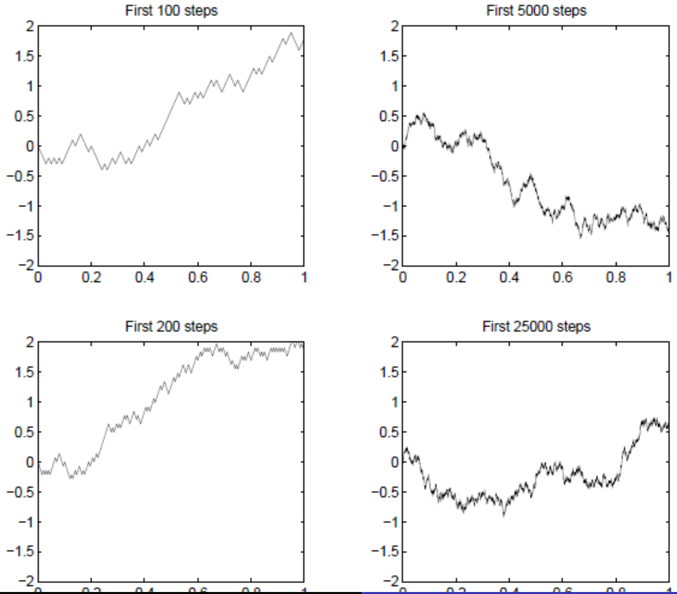
First 100 steps
First 5000 steps
First 200 steps
First 25000 steps

## Construction of a Brownian motion on the sample space $\mathcal{C}([0,\infty);\mathbb{R})$

**Wiener (1923-1924):** For $\Omega = \mathcal{C}([0,\infty);\mathbb{R})$ equipped with the metric

$$d_{\mathcal{C}}(f,g) = \sum_{n\geq 1} \frac{1}{2^n} \max_{0\leq t\leq n} |f(t) - g(t)| \wedge 1,$$

$\mathcal{F} = \mathcal{B}(\mathcal{C}([0,\infty);\mathbb{R}))$ the $\sigma$-algebra generated by all open sets of $\mathcal{C}([0,\infty);\mathbb{R})$, there exists a probability measure $\mathbb{P}_W$ on $(\Omega,\mathcal{F})$ such that under $(\Omega,\mathcal{F},\mathbb{P})$, the canonical process $(W_t;\ t\geq 0)$ defined by

$$W_t(\omega) = \omega(t),\ \omega\in\Omega,$$

is a Brownian motion.

**Donsker (1951):** Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space under which is defined a sequence of i.i.d. r.v. $\{\xi_n\}_{n\in\mathbb{N}}$ such that

$$\mathbb{E}\left[\xi_1\right] = 0,\ \mathbb{E}\left[\xi_1^2\right] = \sigma^2(\neq 0).$$

Let $\{(X_t^n;\ t\geq 0)\}_{n\in\mathbb{N}}$ be defined as

$$X_t^n = \frac{1}{\sigma\sqrt{n}} Y_{nt}^n,\ Y_t^n = \sum_{k=1}^{\lfloor t\rfloor} \xi_k + (t - \lfloor t\rfloor) \sum_{k=1}^{\lfloor t\rfloor +1} \xi_k,\ Y_0 = 0$$

where $\lfloor t\rfloor$ is the integer part of $t$. Then $(X_t^n;\ t\geq 0) \xrightarrow{\mathcal{D}} (W_t;\ t\geq 0)$.
For details on the constructions, see (KS88) chapter 2.

## Construction of a Brownian motion on the sample space $\mathcal{C}([0,\infty);\mathbb{R})$

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**Donsker (1951):** Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space under which is defined a sequence of i.i.d. r.v. $\{\xi_n\}_{n\in\mathbb{N}}$ such that

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$$X_t^n = \frac{1}{\sigma\sqrt{n}} Y_{nt}^n,\; Y_t^n = \sum_{k=1}^{\lfloor t\rfloor} \xi_k + (t-\lfloor t\rfloor) \sum_{k=1}^{\lfloor t\rfloor+1} \xi_k,\; Y_0 = 0$$

where $\lfloor t\rfloor$ is the integer part of $t$. Then $(X_t^n;\, t\geq 0) \xrightarrow{\mathcal{D}} (W_t;\, t\geq 0)$.
For details on the constructions, see (KS88) chapter 2.

**Some properties:** Let $(X_t;\, t \geq 0)$ be a $\mathcal{F}_t$-Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \geq 0), \mathbb{P})$. Then we have

◦ **Symmetry:** $(-X_t;\, t \geq 0)$ is a $\mathcal{F}_t$-Brownian motion,

◦ **Scaling property:** For all $c > 0$, the process $Y_t = \frac{1}{c} X_{c^2 t}$ is a $\mathcal{G}_t$-Brownian motion for the filtration $\mathcal{G}_t = \mathcal{F}_{c^2 t}$.

◦ **Time-reversal:** Given $0 < T < \infty$, the process

$$Y_t = X_T - X_{T-t},\, 0 \leq t \leq T,$$

is a $\mathcal{F}^Y_t$-Brownian motion.

**Exercise**: Prove the above two first statements.

## The Brownian motion

∘ $\lim_{t\to\infty} \frac{W_t}{t} = 0$ a.s..
∘ For all $0 \leq s < t < \infty$, $n \in \mathbb{N}$, there exists a constant $0 < C_n < \infty$ such that

$$\mathbb{E}\left[|W_t - W_s|^{2n}\right] \leq C_n(t-s)^n.$$

∘ Kolmogorov-Centsov: Almost surely $t \mapsto W_t$ is Hölder continuous; that is there exists $0 < \alpha < 1$ such that

$$\mathbb{P}\left(\limsup_{0\leq s<t<\infty} \frac{|W_t - W_s|}{|t-s|^{\alpha}} < \infty\right) = 1.$$

∘ Almost surely $(W_t;\ t \geq 0)$ is nowhere differentiable:

$$\mathbb{P}\left(\text{For all } 0 \leq t < \infty \limsup_{h\to 0} \frac{W_{t+h} - W_t}{h} = \infty\right) = 1,$$

$$\mathbb{P}\left(\text{For all } 0 \leq t < \infty \liminf_{h\to 0} \frac{W_{t+h} - W_t}{h} = -\infty\right) = 1.$$

## The Brownian motion

**Multidimensional Brownian motion**

### Definition 1.9

*A stochastic process $(X_t;\ t \geq 0)$ is said to be a $\mathbb{R}^d$-valued $\mathcal{F}_t$-Brownian motion i.f.f.*
- $X_0 = 0$,
- $(X_t;\ t \geq 0)$ *has continuous paths in* $\mathbb{R}^d$,
- *for all* $0 \leq s \leq t < \infty$, $X_t - X_s$ *is independent of* $\mathcal{F}_s$,
- *and*

$$X_t - X_s \sim \mathcal{N}(0, (t-s)I_d) \text{ for } I_d \text{ the identity matrix of } \mathbb{R}^{d\times d}.$$

Equivalently, a continuous-time stochastic process $(X_t = (X_t^{(1)}, \cdots, X_t^{(d)});\ t \geq 0)$ is a $\mathbb{R}^d$-Brownian motion i.f.f. the component $(X_t^{(i)};\ t \geq 0)$, $i = 1, \cdots, d$ are independent and each component $(X_t^{(i)};\ t \geq 0)$ is a standard Brownian motion.

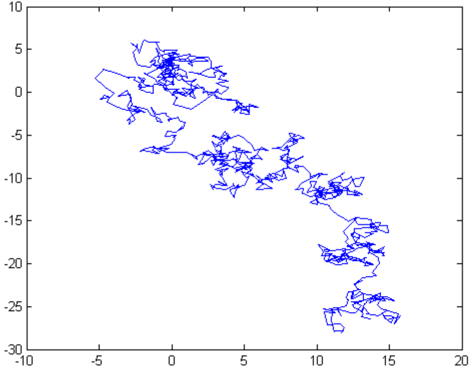


Figure: Simulation of a sample path of a two dimensional Brownian motion

## Square-integrable martingales

### Definition 1.10

*A $\mathcal{F}_t$-adapted stochastic process $(X_t;\, t \geq 0)$ is said to be a $\mathcal{F}_t$-martingale iff $\forall\, 0 \leq s \leq t$*

$$\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] = X_s.$$

*A $\mathcal{F}_t$-adapted stochastic process $(X_t;\, t \geq 0)$ is said to be a square-integrable $\mathcal{F}_t$-martingale is $(X_t;\, t \geq 0)$ is a $\mathcal{F}_t$-martingale and for all $t \geq 0$, $\mathbb{E}[(X_t)^2] < \infty$.*

**Examples**:

∘ A $\mathcal{F}_t$-Brownian motion $(W_t;\, t \geq 0)$ is a square-integrable martingale. Indeed

$$\mathbb{E}[W_t^2] = \frac{1}{\sqrt{2\pi t}} \int x^2 e^{-x^2/2t}\, dx = t,$$

and, by independence of the increment of $(W_t;\, t \geq 0)$,

$$\mathbb{E}[W_t \mid \mathcal{F}_s] = \mathbb{E}[W_t - W_s + W_s \mid \mathcal{F}_s] = \mathbb{E}[W_t - W_s] + W_s = W_s.$$

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## Square-integrable martingales

∘ Given $(N_t;\, t \geq 0)$ a Poisson process with intensity $\lambda > 0$, $(N_t - \lambda t;\, t \geq 0)$ is a square-integrable martingale. Indeed

$$\mathbb{E}[(N_t)^2] = e^{-\lambda t} \sum_n n^2 \frac{(\lambda t)^n}{n!} = \lambda t(1 + \lambda t),$$

and, since $\mathbb{E}[N_t] = \lambda t$ and, by independence of the increment of $(N_t;\, t \geq 0)$,

$$\begin{aligned}
\mathbb{E}[N_t - \lambda t \,|\, \mathcal{F}_s] &= \mathbb{E}[N_t - N_s \,|\, \mathcal{F}_s] + N_s - \lambda t \\
&= \mathbb{E}[N_t - N_s] + N_s - \lambda t \\
&= \lambda(t - s) + N_s - \lambda t = N_s - \lambda s.
\end{aligned}$$

**Exercise**: Check if the following processes are (or not) square integrable martingales:

$$(W_t)^2 - t;\ \exp\{W_t - \frac{t^2}{2}\};\ (N_t - \lambda t)^2.$$

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**Exercise**: Check if the following processes are (or not) square integrable martingales:

$$(W_t)^2 - t;\ \exp\{W_t - \frac{t^2}{2}\};\ (N_t - \lambda t)^2.$$

## Square integrable martingales

**Supermartingale and submartingale:**

### Definition 1.11

*A stochastic process* $(X_t;\ t \geq 0)$ *defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$, $\mathcal{F}_t$*-adapted is said to be a square integrable* $\mathcal{F}_t$*-supermartingale i.f.f. for all* $t$, $\mathbb{E}[X_t^2] < \infty$ *and*

$$\forall\, 0 \leq s \leq t,\ \mathbb{E}\left[X_t \mid \mathcal{F}_s\right] \leq X_s.$$

*A stochastic process* $(X_t;\ t \in I)$ *defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$, $\mathcal{F}_t$*-adapted is said to be a* $\mathcal{F}_t$*-submartingale i.f.f. for all* $t$, $\mathbb{E}[X_t^2] < \infty$ *and*

$$\forall\, 0 \leq s \leq t,\ \mathbb{E}\left[X_t \mid \mathcal{F}_s\right] \geq X_s.$$

## Stopping time

**Stopping time:** A random variable $\tau : \Omega \to [0, \infty]$ is said to be a $\mathcal{F}_t$-stopping time i.f.f., for all $0 < t < \infty$, the event $\{\tau < t\}$ is $\mathcal{F}_t$-measurable.

**Example**: For $(X_t;\, t \geq 0)$ a $\mathcal{F}_t$-adapted process, the first time $(X_t;\, t \geq 0)$ hits $a \in \mathbb{R}$,

$$\tau = \begin{cases} \inf\{t \geq 0 \,|\, X_t = a\} \text{ if } \{t \geq 0 \,|\, X_t = a\} \neq \emptyset, \\ \infty \text{ if } \{t \geq 0 \,|\, X_t = a\} = \emptyset. \end{cases}$$

is a stopping time.

**Proposition 1.12**

*Let $\tau$ and $\beta$ be two $\mathcal{F}_t$-stopping times. Then*

$$\tau \wedge \beta = \min(\tau, \beta),\ \tau \vee \beta = \max(\tau, \beta),\ \tau + \beta$$

*are also $\mathcal{F}_t$-stopping times.*

**Definition 1.13**

*Let $\tau$ be a $\mathcal{F}_t$-stopping time. The $\sigma$-field $\mathcal{F}_\tau$ of events determined prior to $\tau$ consists of all events $A \in \mathcal{F}$ such that, for all $0 \leq t < \infty$,*

$$A \cap \{\tau < t\} \in \mathcal{F}_t.$$

## Stopping time

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### Proposition 1.12

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### Definition 1.13

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## Stopping time

### Proposition 1.14

*Let $\tau$ and $\beta$ be two $\mathcal{F}_t$-stopping times such that $\tau \leq \beta$ a.s.. Then $\mathcal{F}_\tau \subset \mathcal{F}_\beta$.*

### Proposition 1.15

*Let $(X_t;\ t \geq 0)$ be a $\mathcal{F}_t$-adapted process and let $\tau$ be a $\mathcal{F}_t$-stopping time such that $\tau < \infty$ a.s.. The random variable*

$$\omega \in \Omega \mapsto X_{\tau(\omega)}(\omega)$$

*is $\mathcal{F}_\tau$-measurable. In addition the stopped process $(X_{\tau\wedge t};\ t \geq 0)$ given by*

$$X_{\tau\wedge t} = \begin{cases} X_t \; if \, t \leq \tau, \\ X_\tau \; if \, t > \tau, \end{cases}$$

*is $\mathcal{F}_t$-progressively measurable.*

### Proposition 1.16

*Let $(W_t;\ t \geq 0)$ be a $\mathcal{F}_t$-Brownian motion and let $\tau$ be a $\mathcal{F}_t$-stopping time. Then, for all $0 < t < \infty$ $W_{t+\tau} - W_\tau$ is independent to $\mathcal{F}_\tau$.*

## Martingale inequalities

**Probabilistic inequalities for continuous time martingales** Let $(X_t;\ t \geq 0)$ be a $\mathcal{F}_t$-martingale with continuous paths and values in $\mathbb{R}$. Then, for all (deterministic) $0 \leq \tau \leq \beta < \infty$:

∘ For all $t$, $\lambda > 0$,

$$\mathbb{P}\left(\max_{\tau \leq t \leq \beta} X_t \geq \lambda\right) \leq \mathbb{E}\left[\left|X_\beta\right|\right]/\lambda$$

and

$$\mathbb{P}\left(\min_{\tau \leq t \leq \beta} X_t \leq -\lambda\right) \leq \left(\mathbb{E}\left[\max(X_\beta, 0)\right] - \mathbb{E}\left[X_\tau\right]\right)/\lambda.$$

∘ **Doob's maximal inequality:** For all $1 < p < \infty$,

$$\mathbb{E}\left[\max_{\tau \leq t \leq \beta} |X_t|^p\right] \leq \frac{p}{p-1}\mathbb{E}\left[\left|X_\beta\right|^p\right].$$

∘ **Doob's Optional Sampling theorem**: Let $(X_t;\ t \geq 0)$ be a $\mathcal{F}_t$-submartingale and let $\tau$ and $\beta$ be two bounded $\mathcal{F}_t$-stopping times such that $\tau \leq \beta$. Then, a.s.,

$$\mathbb{E}[X_\beta \,|\, \mathcal{F}_\tau] \geq X_\tau.$$

If $(X_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale then $\mathbb{E}[X_\beta \,|\, \mathcal{F}_\tau] = X_\tau$.

## Doob-Meyer decomposition

**Theorem 1.17 (Doob-Meyer decomposition for non-negative submartingale)**

*Let $(X_t;\ t \geq 0)$ be a continuous $\mathcal{F}_t$-submartingale such that $X_t \geq 0$. Then there exists a $\mathcal{F}_t$-martingale $(M_t;\ t \geq 0)$ and an increasing process $(A_t;\ t \geq 0)$ such that*

$$X_t = X_0 + M_t + A_t,\ t \geq 0.$$

*This decomposition is unique.*

**Particular case**: Let $(X_t;\ t \geq 0)$ be a continuous square-integrable $\mathcal{F}_t$-martingale. Then $(X_t^2;\ t \geq 0)$ is a continuous $\mathcal{F}_t$-submartingale and so admits the Doob-Meyer decomposition

$$X_t^2 = X_0^2 + M_t + A_t,\ t \geq 0.$$

## Quadratic variations

### Definition 1.18

*When $(X_t;\ t \geq 0)$ is a square integrable martingale, the quadratic variation $(\langle X \rangle_t;\ t \geq 0)$ of $(X_t;\ t \geq 0)$ is the nondecreasing process such that $\langle X \rangle_0 = 0$ and*

$$(X_t^2 - \langle X \rangle_t;\ t \geq 0)$$

*is a martingale.*

**Example**:
∘ The quadratic variation of a Brownian motion $(W_t;\ t \geq 0)$ is given by $\langle W \rangle_t = t$.
∘ The quadratic variation of a Brownian motion $(\tilde{N}_t := N_t - \lambda t;\ t \geq 0)$ is given by $\langle \tilde{N} \rangle_t = \lambda t$.

## Quadratic variations

For $0 < t < \infty$, let $P(N)$ be a $N$-partition of $[0, t]$ of the form

$$P(N) : 0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N = t.$$

and define the mesh of the partition as $|P(N)| = \max_i |t_{i+1} - t_i|$. The 2th variation of a stochastic process $(X_t;\ t \geq 0)$ over $P(N)$ is given by

$$\sum_{k=0}^{N-1} \left(X_{t_{k+1}} - X_{t_k}\right)^2.$$

**Theorem 1.19**

*Let $(X_t;\ t \geq 0)$ be a continuous square-integrable $\mathcal{F}_t$-martingale. Then*

$$\langle X \rangle_t = \limsup_{P(N);\, |P(N)| \to 0} \sum_{k=0}^{N-1} \left(X_{t_{k+1}} - X_{t_k}\right)^2,$$

*where the limit is taken in probability, i.e. for all $\epsilon > 0$,*

$$\limsup_{P(N);\, |P(N)| \to 0} \mathbb{P}\left(\left|\langle X \rangle_t - \sum_{k=0}^{N-1} \left(X_{t_{k+1}} - X_{t_k}\right)^2\right| > \epsilon\right) = 0.$$

## Quadratic variations

**The Burkholder-Davis-Gundy inequalities:** Let $(X_t;\ t \geq 0)$ be a continuous $\mathcal{F}_t$-martingale. Then, for all $0 < p < \infty$, there exist $0 < k < K < \infty$ depending only on $p$ such that

$$k\mathbb{E}\left[\langle X\rangle_T^p\right] \leq \mathbb{E}\left[\max_{0\leq t\leq T} |X_t|^{2p}\right] \leq K\mathbb{E}\left[\langle X\rangle_T^p\right]$$

**Cross-variations:** The cross-variation (or quadratic covariation) of two $\mathbb{R}$-valued continuous processes $X$ and $Y$ is a non-decreasing continuous process, denoted $(\langle X, Y\rangle_t;\ t \geq 0)$ defined as

$$\langle X, Y\rangle_t = \frac{1}{4}\left(\langle X+Y\rangle_t - \langle X-Y\rangle_t\right),$$

and such that $(X_t Y_t - \langle X, Y\rangle_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale.

## Quadratic variations

**Properties**:
∘ $\langle\, , \,\rangle$ is a symmetric bilinear operator:

$$\langle X, Y\rangle_t = \langle Y, X\rangle_t,$$

$$\langle X + \lambda Z, Y\rangle_t = \langle X, Y\rangle_t + \lambda\langle Z, Y\rangle_t.$$

∘ $|\langle X, Y\rangle_t|^2 \leq \langle X\rangle_t\langle Y\rangle_t$.
∘ Link with variations:

$$\langle X, Y\rangle_t = \limsup_{P(N);\,|P(N)|\to 0} \sum_{k=0}^{N-1} \left(X_{t_{k+1}} - X_{t_k}\right)\left(Y_{t_{k+1}} - Y_{t_k}\right)$$

where the limit is taken in the sense of the convergence in probability.

## Stochastic integration

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ be a filtered probability space under which is defined a Brownian motion $(W_t;\ t \geq 0)$. We aim to construct the integral

$$\int_0^t b_s\, dW_s,\ t \geq 0,$$

for a large class of progressively measurable process $(b_t;\ t \geq 0)$.

For simplicity, the constructions below will be set on finite time interval $[0, T]$ for $T$ an arbitrary time horizon.

## Stochastic integration and Itô's formula

**Remark**: Naive stochastic integration doesn't work.

∘ **Lebesgue-Stieltjes integral**: Generalization of the Riemann and Lebesgue integral.

• A function $A : [0, \infty) \to \mathbb{R}$ is said to have bounded variations if, for all $0 < T < \infty$,

$$\sup_{P(N);\, N \in \mathbb{N}} \left\{ \sum_{i=0}^{N-1} |A(t_{i+1}) - A(t_i)| \right\} < \infty$$

where the supremum is taken over all possible $N$-partitions

$$P(N): \; 0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$$

of the interval $[0, T]$.

• Let $A$ be a function with bounded variations and let $f$ be a real valued continuous function on a finite interval $[0, T]$. The Lebesgue-Stieltjes

$$\int_0^T f(s) dA(s)$$

is defined as the limit of

$$\limsup_{P(N)\,;\,|P(N)| \to 0} \sum_{i=0}^{N-1} f(t_i) \left( A(t_{i+1}) - A(t_i) \right).$$

## Stochastic integration and Itô's formula

◦ Almost surely, a Brownian motion doesn't have bounded variations. Indeed, recalling that on any finite interval $[0, t]$,

$$\langle W\rangle_t = \limsup_{P(N);\,|P(N)|\to 0} \sum_{k=0}^{N-1} \left(W_{t_{k+1}} - W_{t_k}\right)^2,$$

where the limit is in the sense of the convergence in probability over all any partition $P(N) = \{0 = t_0 < t_1 < \cdots < t_N = t\}$.

If we assume that Brownian motion had bounded variations, we would have the contradiction

$$\begin{aligned}\langle W\rangle_t &= \limsup_{P(N);\,|P(N)|\to 0} \sum_{k=0}^{N-1} \left(W_{t_{k+1}} - W_{t_k}\right)^2\\ &\leq C \limsup_{P(N);\,|P(N)|\to 0} \left|W_{t_{k+1}} - W_{t_k}\right| = 0,\end{aligned}$$

for

$$C = \sup_{P(N);\,N\in\mathbb{N}} \left\{\sum_{i=0}^{N-1} \left|W_{t_{i+1}} - W_{t_i}\right|\right\}.$$

## Stochastic integration and Itô's formula

**Simple processes**: A progressively measurable process $(b_t;\ t \geq 0)$ is said to be a simple process if

$$b_t = \sum_{i=0}^{N-1} b_{i+1}\, \mathbb{1}_{\{t_i \leq t < t_{i+1}\}}$$

for $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N < \infty$ and $b_1, b_2, \cdots, b_N$ a family of real random variables such that $b_i$ is $\mathcal{F}_{t_i}$-measurable.

Whenever $(b_t;\ t \geq 0)$ is a simple process, the integral

$$\int_0^t b_s\, dW_s$$

is defined by

$$\sum_{i=0}^{n-2} b_{i+1} \left(W_{t_{i+1}} - W_{t_i}\right) \mathbb{1}_{\{t_i \leq t < t_{i+1}\}} + b_n \left(W_t - W_{t_{n-1}}\right)$$

whenever $t_{n-1} < t < t_n$.

## Stochastic integration

Given $0 < T < \infty$, define the space

$$L^2((0,T)\times\Omega) = \left\{ (b_t;\, 0 \leq t \leq T) \text{ progressively measurable} \,\Big|\, \int_0^T \mathbb{E}[b_t^2]\,dt < \infty \right\}.$$

**Theorem 1.20**

*For all $0 < T < \infty$, the class of simple processes is dense in $L^2((0,T)\times\Omega)$; that is for all $(b_t;\, 0 \leq t \leq T)$ in $L^2((0,T)\times\Omega)$, there exists a sequence of simple processes $\{(b_t^{(n)};\, 0 \leq t \leq T)\}_{n\in\mathbb{N}}$ such that*

$$\lim_n \int_0^T \mathbb{E}\left[\left(b_t^{(n)} - b_t\right)^2\right] dt = 0.$$

Using the preceding theorem, the integral $\int_0^t b_s\,dW_s$ can be extended from the class of simple processes to the elements of $L^2((0,T)\times\Omega)$ as

$$\int_0^t b_s\,dW_s = \lim_n \int_0^t b_s^{(n)}\,dW_s,\ 0 \leq t \leq T.$$

## Stochastic integration

More precisely, defining

$$M_t^{(n)} = \int_0^t b_s^{(n)}\, dW_s,\ 0 \leq t \leq T,$$

we have

$$\mathbb{E}\left[\max_{0\leq t\leq T}\left|M_t^{(n)} - M_t^{(n-1)}\right|\right] \leq C\sqrt{\mathbb{E}\left[\int_0^T \left|b_t^{(n)} - b_t^{(n-1)}\right|^2\, dt\right]}$$

Since $(b_t^n;\ 0 \leq t \leq T)$ converges in $L^2((0,T)\times\Omega)$,

$$\lim_n \mathbb{E}\left[\int_0^T \left|b_t^{(n)} - b_t^{(n-1)}\right|^2\, dt\right] = 0$$

The space

$$L_T^2 = \left\{(X_t;\ 0 \leq t \leq T)\,\mathcal{F}_t - \text{adapted} \mid \mathbb{E}[\max_{0\leq t\leq T} X_t^2] < \infty\right\}$$

equipped with the distance $d_{L_T^2}(X,Y) = \mathbb{E}\left[\max_{0\leq t\leq T}|X_t - Y_t|\right]$ being complete, $\{(\int_0^t b_s^{(n)}\, dW_s,\ 0 \leq t \leq T)\}_n$ converges in $L_T^2$ and we take

$$\int_0^t b_s\, dW_s = \lim_n \int_0^t b_s^{(n)}\, dW_s,\ 0 \leq t \leq T.$$

## Stochastic integration

The preceding construction formulates mathematically as:

**Definition 1.21**

*The Itô integral of* $(b_t;\, 0 \leq t \leq T) \in L^2((0,T) \times \Omega)$ *with respect to* $(W_t;\, 0 \leq t \leq T)$ *is the unique continuous process denoted* $(\int_0^t b_s\, dW_s;\, 0 \leq t \leq T)$ *such that*

$$\lim_{n \to \infty} d_{L^2_T}\Big(\int_0^{\cdot} b_s\, dW_s, \int_0^{\cdot} b_s^{(n)}\, dW_s\Big) = 0,$$

*for* $(b_t^{(n)};\, 0 \leq t \leq T)$ *the sequence of simple processes converging to* $(b_t;\, 0 \leq t \leq T)$ *in* $L^2((0,T) \times \Omega)$ *is the*

## Proposition 1.22 (Properties of the Itô integral)

• **Linearity**: *Let* $\alpha \in \mathbb{R}$, $(b^1_t;\, 0 \leq t \leq T)$ *and* $(b^2_t;\, 0 \leq t \leq T)$ *in* $L^2((0,T)\times\Omega)$,

$$\int_0^t \left(b^1_s + \lambda b^2_s\right)\, dW_s = \int_0^t b^1_s\, dW_s + \lambda \int_0^t b^2_s\, dW_s,\ 0 \leq t \leq T.$$

• **Martingale property**: $t \in [0,T] \mapsto \int_0^t b_s\, dW_s$ *is a continuous* $\mathcal{F}_t$*-martingale, in the sense*

$$\mathbb{E}\left[\int_0^t b_r\, dW_r \,|\, \mathcal{F}_s\right] = \int_0^s b_r\, dW_r,\ \ 0 \leq s \leq t \leq T.$$

• **Itô's isometry:** *For all* $(b_t;\, 0 \leq t \leq T)$ *in* $L^2((0,T)\times\Omega)$,

$$\mathbb{E}\left[\left(\int_0^T b_s\, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^T (b_s)^2\, ds\right].$$

• **Cross variation property:** *For all* $(b^1_t;\, 0 \leq t \leq T)$ *and* $(b^2_t;\, 0 \leq t \leq T)$ *in* $L^2((0,T)\times\Omega)$,

$$\langle \int_0^\cdot b^1_s\, dW_s, \int_0^\cdot b^2_s\, dW_s \rangle_t = \int_0^t b^1_s b^2_s\, ds.$$

## Stochastic integration and Itô's formula

**Stochastic integral for martingales**: Let $(M_t;\ t \geq 0)$ be a continuous square integrable $\mathcal{F}_t$-martingale. Given $(b_t;\ 0 \leq t \leq T)$ a progressively measurable process such that

$$\mathbb{E}[\int_0^T (b_s)^2 \, d\langle M\rangle_s] < \infty.$$

there exists a sequence of simple processes $\{(b_t^{(n)};\ 0 \leq t \leq T)\}_n$ such that, for all $0 \leq t \leq T$,

$$\mathbb{E}[\int_0^T |b_s - b_s^{(n)}|^2 \, d\langle M\rangle_s] \to 0.$$

The integral

$$(\int_0^t b_s \, dM_s;\ t \geq 0)$$

is constructed as the limit of

$$(\int_0^t b_s^{(n)} \, dM_s;\ t \geq 0)$$

in $L^2_T$.

## Stochastic integration and Itô's formula

### Proposition 1.23 (Properties of the Itô integral)

- **Linearity**: *Let* $\alpha \in \mathbb{R}$, $(b^1_t;\, 0 \leq t \leq T)$ *and* $(b^2_t;\, 0 \leq t \leq T)$ *in* $L^2((0, T) \times \Omega)$,

$$\int_0^t \left(b^1_s + \lambda b^2_s\right)\, dM_s = \int_0^t b^1_s\, dM_s + \lambda \int_0^t b^2_s\, dM_s,\ 0 \leq t \leq T.$$

- **Martingale property**: *For all* $(b_t;\, 0 \leq t \leq T)$ *in* $L^2((0, T) \times \Omega)$,

$$t \in [0, T] \mapsto \int_0^t b_s\, dM_s$$

*is a continuous* $\mathcal{F}_t$*-martingale, in the sense*

$$\mathbb{E}\left[\int_0^t b_r\, dM_r \,\middle|\, \mathcal{F}_s\right] = \int_0^s b_r\, dM_r,\ \ 0 \leq s \leq t \leq T.$$

- **Itô's isometry:**

$$\mathbb{E}\left[\left(\int_0^T b_s\, d\langle M\rangle_s\right)^2\right] = \mathbb{E}\left[\int_0^T (b_s)^2\, d\langle M\rangle_s\right].$$

## Stochastic integration and Itô's formula

### Proposition 1.24 (Cross variation property)

*Let $(M^1_t;\, 0 \leq t \leq T)$ and $(M^2_t;\, 0 \leq t \leq T)$ be two continuous square integrable $\mathcal{F}_t$-martingale. Then, for all $(b^1_t;\, 0 \leq t \leq T)$ and $(b^2_t;\, 0 \leq t \leq T)$ in $L^2((0,T) \times \Omega)$,*

$$\langle \int_0^{\cdot} b^1_s \, dM^1_s, \int_0^{\cdot} b^2_s \, dM^2_s \rangle_t = \int_0^t b^1_s b^2_s \, d\langle M^1, M^2 \rangle_s.$$

**Note:** The definition of the stochastic integral

$$\int_0^t b_s \, dM_s,\ t \geq 0,$$

can be extended to the situation where $(M_t;\, t \geq 0)$ is a continuous $\mathcal{F}_t$-local martingale, and if $(b_t;\, t \geq 0)$ is a continuous $\mathcal{F}_t$-adapted stochastic process such that

$$\mathbb{P}\left(\int_0^T (b_s)^2 \, d\langle M \rangle_s < \infty\right) = 1,\ \forall\, 0 < T < \infty.$$

### Definition 1.25 (Local martingale)

*An $\mathcal{F}_t$-adapted continuous process $(X_t;\, t \geq 0)$ is a local martingale if there exists a sequence of increasing stopping times $\{\tau_n\}_{n \in \mathbb{N}}$ such that a.s. $\lim_{n \to \infty} \tau_n = \infty$ and such that $(X_{t \wedge \tau_n};\, t \geq 0)$ is a $\mathcal{F}_t$-martingale.*

## Stochastic integration and Itô's formula

**Itô's formula for square integrable martingales:**

### Theorem 1.26

*Let $(X_t;\, t \geq 0)$ be a continuous square integrable $\mathcal{F}_t$-martingale and let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ on $\mathbb{R}$ (i.e. $f$ admits continuous derivatives up to order $2$). Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s,\ t \geq 0.$$

**Comparison with the classical chain rule**: Let $(A(t);\, t \geq 0)$ be a real valued function defined on $\mathbb{R}$ with bounded variations. Then, for all $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2$ on $\mathbb{R}$,

$$f(A(t)) = f(A(0)) + \int_0^t f'(A(s))\, dA(s),\ t \geq 0.$$

## Stochastic integration and Itô's formula

**Elements of proof:** Assume that $f:\mathbb{R}\to\mathbb{R}$ is of class $\mathcal{C}^3$ on $\mathbb{R}$ with bounded derivatives, and, for a fixed $t\geq 0$, let $P(N)$ be a partition of $[0,t]$ of the form:

$$0=t_0<t_1<\cdots<t_N=t,$$

such that $|t_{i+1}-t_i|\leq 1/N$. Then we have

$$\begin{aligned}
f(X_t)-f(X_0)&=\sum_{i=0}^{N-1}\left(f(X_{t_{i+1}})-f(X_{t_i})\right)\\
&=\sum_{i=0}^{N-1}f'(X_{t_i})\left(X_{t_{i+1}}-X_{t_i}\right)+\sum_{i=0}^{N-1}f''(X_{t_i})\frac{\left(X_{t_{i+1}}-X_{t_i}\right)^2}{2}\\
&\quad+\sum_{i=0}^{N-1}R(f''',\left|X_{t_{i+1}}-X_{t_i}\right|)\\
&=:(I_1^N)+(I_2^N)+(I_3^N),
\end{aligned}$$

where the third equality follows by applying a Taylor expansion to $f$, with a residual term $R$ such that $\lim_{h\to 0}R(f''',h)/h^2=0$.

It remains to apply the limit $N\to\infty$ to the above expression.

- $\lim_N(I_1^N)$: By construction of the stochastic integral $\int_0^t b_s\,dX_s$,

$$\lim_N(I_1^N) = \lim_N \sum_{i=0}^{N-1} f'(X_{t_i})\left(X_{t_{i+1}} - X_{t_i}\right) = \int_0^t f'(X_s)\,dX_s.$$

- $\lim_N(I_2^N)$: By definition of the quadratic variation:

$$\lim_N(I_2^N) = \lim_N \frac{1}{2}\sum_{i=0}^{N-1} f''(X_{t_i})\left(X_{t_{i+1}} - X_{t_i}\right)^2 = \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s.$$

- $\lim_N(I_3^N)$: By definition of the quadratic variation:

$$\begin{aligned}\lim_N\left|(I_3^N)\right| &= \lim_N\left|\sum_{i=0}^{N-1} R(f''', |X_{t_{i+1}} - X_{t_i}|)\right|\\ &\leq \lim_N\left|\sum_{i=0}^{N-1}\frac{R(f''', |X_{t_{i+1}} - X_{t_i}|)}{|X_{t_{i+1}} - X_{t_i}|^2}\left|X_{t_{i+1}} - X_{t_i}\right|^2\right| = 0.\end{aligned}$$

Therefore, for all function $f:\mathbb{R}\to\mathbb{R}$ of class $\mathcal{C}^3$, with bounded derivatives, we have, a.s.,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s,\ t\geq 0.$$

As, for all $f$ of class $\mathcal{C}^2$, there exists a sequence $\{f^{(n)}\}_n$ of class $\mathcal{C}^3$ with bounded derivatives such that $\lim_n f^{(n)} = f$ on every finite interval of $\mathbb{R}$, by a truncating

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- $\lim_N(I_3^N)$: By definition of the quadratic variation:

$$\begin{aligned}\lim_N \left|(I_3^N)\right| &= \lim_N \left|\sum_{i=0}^{N-1} R(f''', |X_{t_{i+1}} - X_{t_i}|)\right| \\ &\leq \lim_N \left|\sum_{i=0}^{N-1} \frac{R(f''', |X_{t_{i+1}} - X_{t_i}|)}{|X_{t_{i+1}} - X_{t_i}|^2}\,|X_{t_{i+1}} - X_{t_i}|^2\right| = 0.\end{aligned}$$

Therefore, for all function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^3$, with bounded derivatives, we have, a.s.,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s,\ t \geq 0.$$

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- $\lim_N (I_3^N)$: By definition of the quadratic variation:

$$\begin{aligned}\lim_N \left|(I_3^N)\right| &= \lim_N \left| \sum_{i=0}^{N-1} R(f''', |X_{t_{i+1}} - X_{t_i}|) \right| \\ &\leq \lim_N \left| \sum_{i=0}^{N-1} \frac{R(f''', |X_{t_{i+1}} - X_{t_i}|)}{\left|X_{t_{i+1}} - X_{t_i}\right|^2} \left|X_{t_{i+1}} - X_{t_i}\right|^2 \right| = 0.\end{aligned}$$

Therefore, for all function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^3$, with bounded derivatives, we have, a.s.,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s,\ t \geq 0.$$

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## Stochastic integration and Itô's formula

**Applications:**

∘ Let $(W_t;\, t \geq 0)$ be a Brownian motion and let $X_0$ be a real r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$ independent to $(W_t;\, t \geq 0)$. Then, for

$$X_t = X_0 + W_t,\ t \geq 0,$$

and by applying the Itô formula to

$$f : x \in \mathbb{R} \mapsto f(x) = x^2,$$

we have

$$\begin{aligned}(X_t)^2 &= (X_0)^2 + 2\int_0^t X_s\, dX_s + \frac{2}{2}\int_0^t d\langle X\rangle_s \\ &= (X_0)^2 + 2X_0 W_t + 2\int_0^t W_s\, dW_s + t.\end{aligned}$$

In the same way, by applying the Itô formula to

$$f : x \in \mathbb{R} \mapsto f(x) = \exp\{rx\},\ r > 0,$$

we have

$$\begin{aligned}\exp\{rX_t\} &= \exp\{rX_0\} + r\int_0^t \exp\{rX_s\}\, dX_s + \frac{r^2}{2}\int_0^t \exp\{rX_s\}\, d\langle X\rangle_s \\ &= \exp\{rX_0\} + r\int_0^t \exp\{rX_s\}\, dW_s + \frac{r^2}{2}\int_0^t \exp\{rX_s\}\, ds.\end{aligned}$$

## Stochastic integration and Itô's formula

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## Stochastic integration and Itô's formula

**Extension**: **Notation**: A function $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is said to be of class $\mathcal{C}^{1,2}([0, \infty) \times \mathbb{R})$ if the partial derivatives $\partial_t f$, $\partial_x f$ and $\partial_x^2 f$ exist and are continuous on $[0, \infty) \times \mathbb{R}$.

### Theorem 1.27

*Let $(X_t;\ t \geq 0)$ be a continuous square integrable $\mathcal{F}_t$-martingale. Then, for all scalar function $f$ in $\mathcal{C}^{1,2}([0, \infty) \times \mathbb{R})$,*

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_s f(s, X_s)\, ds + \int_0^t \partial_x f(s, X_s)\, dX_s + \frac{1}{2} \int_0^t \partial_x^2 f(s, X_s)\, d\langle X \rangle_s.$$

**Exercise**: Given $X_t = X_0 + W_t$ and

$$f : (t, x) \in [0, \infty) \times \mathbb{R} \mapsto f(t, x) = \exp\{rx - \frac{r^2 t}{2}\},\ r > 0,$$

apply the Itô formula to $f(t, X_t)$ and show that $f(t, X_t)$ is a continuous square integrable $\mathcal{F}_t^X$-martingale.

## Application of the Itô formula

**Application: Levy's Characterization of a standard Brownian motion**

**Theorem 1.28**

*A standard $\mathcal{F}_t$-Brownian motion is the unique square integrable $\mathcal{F}_t$-martingale with values in $\mathbb{R}$ such that $M_0 = 0$ and $\langle M \rangle_t = t$.*

**Elements of proof:** $\Rightarrow$ Immediate.

$\Leftarrow$ [(P90), p. 89] Taking $F(t,x) = \exp\left\{iux + \frac{u^2}{2}t\right\}$ for $u \in \mathbb{R}$, applying Itô's formula to $F(t, M_t)$,

$$F(t, M_t) = 1 + iu \int_0^t F(s, M_s)\, dM_s.$$

$t \mapsto F(t, M_t)$ is a continuous $\mathcal{F}_t$-martingale and

$$\mathbb{E}\left[\exp\{iu(X_t - X_s)\} \,|\, \mathcal{F}_s\right] = \exp\left\{-\frac{u^2}{2}(t-s)\right\}.$$

$\square$

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□

## Application of the Itô formula

### The Itô integration by part formula

**Theorem 1.29**

*Let $(X_t;\ t \geq 0)$ and $(Y_t;\ t \geq 0)$ be two continuous square-integrable $\mathcal{F}_t$-martingale. Then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y \rangle_t.$$

**Note**: In the case where $(X_t;\ t \geq 0)$ and $(K_t;\ t \geq 0)$, a continuous $\mathcal{F}_t$-adapted stochastic process of the form

$$K_t = \int_0^t b_s\, ds,$$

then

$$\langle K, X \rangle_t = 0.$$

In particular, the integration by part formula reduces in this case to

$$X_t K_t = \int_0^t X_s\, dK_s + \int_0^t K_s\, dX_s.$$

## Stochastic integration and Itô's formula

**The Itô formula (multi-dimensional case):** Let $(X_t;\ t \geq 0)$ be a $\mathbb{R}^d$-valued continuous stochastic process of the form

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t b_s\, ds,$$

for $(b_t;\ t \geq 0)$ and $(\sigma_t;\ t \geq 0)$ two progressively measurable processes with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$ respectively and such that, a.s., for all $0 < T < \infty$,

$$\int_0^T \sum_i |b_s^{(i)}|\, ds + \int_0^T \sum_{k,l} |\sigma_s^{k,l}|^2\, ds < \infty$$

In this multi-dimension setting, the Itô formula writes as

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \partial_s f(s, X_s)\, ds + \int_0^t \nabla_x f(s, X_s) \cdot b_s\, ds + \int_0^t \nabla_x f(s, X_s) \cdot \sigma_s\, dW_s \\ &+ \sum_{i,j=1}^d \frac{1}{2} \int_0^t (\sigma_s \sigma_s^*)_{i,j}\, \partial^2_{x_i x_j} f(s, X_s)\, ds \end{aligned}$$

for $\sigma^*$ the transpose of $\sigma$ and $\nabla_x$ the gradient operator.

# *Strong solution to a SDE and related properties*

Preliminaries on Stochastic Differential Equations.

## Some preliminaries on SDEs

**Generic form of a SDE:** A stochastic process $(X_t;\ t \geq 0)$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$, is said to satisfy the stochastic differential equation:

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = \xi, \end{cases} \tag{1}$$

i.f.f.
∘ $\mathbb{P}$-a.s. $X_0 = \xi$,
∘ $\mathbb{P}$-a.s., for all $t > 0$,

$$X_t^i = \xi^i + \int_0^t b^i(s, X_s)\,ds + \sum_{j=1}^m \int_0^t \sigma^{i,j}(s, X_s)\,dW_s^j,\ i = 1, \cdots, d. \tag{2}$$

**Inputs of the equation:**
∘ $(W_t;\ t \geq 0)$ is a $\mathbb{R}^m$-Brownian motion,
∘ $\xi$ is a given $\mathbb{R}^d$-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$, independent to $(W_t;\ t \geq 0)$,
∘ $b$ is a given Borel measurable vector field defined on $(0, \infty) \times \mathbb{R}^d$ with values in $\mathbb{R}^d$,
∘ $\sigma$ is a given Borel measurable defined on $(0, \infty) \times \mathbb{R}^d$ with (matrix) values in $\mathbb{R}^{d \times m}$.

Stochastic Differential Equation of the form (1) corresponds to a particular class of semi-martingales in continuous time where

$$X_t = X_0 + \underbrace{\int_0^t b(s, X_s)\, ds}_{\text{Finite variation part}} + \underbrace{\int_0^t \sigma(s, X_s)\, dW_s}_{\text{Finite quadratic variation part}} ,$$

which have several applications in

• Physics (describing additional microscopic phenomenons in the evolution of a system),

• Finance (describing the source of randomness -such as market risk- in the evolution of shares, options, interest rate, ...),

• Biology, Economy, ...

**Heuristic properties:**

• Initial state: As

$$X_{t=0} = \xi,$$

$\xi$ describes the initial state of the solution (which can be deterministic, namely $\xi = x$ for $x$ given in $\mathbb{R}^d$).

• Mean evolution:

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[\xi\right] + \int_0^t \mathbb{E}\left[b(s, X_s)\right]\, ds.$$

$b$ is often called the derive vector or the drift of the equation.

• Quadratic variations:

$$\langle X^{(i)}, X^{(j)} \rangle_t = \int_0^t a_{i,j}(s, X_s)\, ds$$

for $a : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ defined by $a(t, x) = \sigma\sigma^*(t, x)$ for $\sigma^*(t, x)$ the transpose matrix of $\sigma(t, x)$. Equivalently

$$a_{i,j}(t, x) = (\sigma\sigma^*)_{i,j}(t, x) = \sum_{k=1}^{m} \sigma_{i,k}(t, x)\sigma_{j,k}(t, x),\ 1 \le i, j \le d.$$

$\sigma$ or $a$ is called the diffusion or dispersion matrix.

**Time approximation of a SDE on a arbitrary interval $[0, T]$:** Conversely a natural, and computationally feasible, approximation, of a SDE of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t,\ X_0 = \xi,$$

on a finite interval $[0, T]$, is given by the time series $\{X_{t_i}\}_i$ given by:

$$X_{t_{i+1}} - X_{t_i} = b(X_{t_i})\triangle t + \sigma(X_{t_i})\sqrt{\triangle t}\zeta_i,\ X_{t_0} = \xi,$$

for $t_k = kT/N$ defining a partition

$$0 = t_0 \leq t_1 \leq \cdots \leq t_N = T,$$

of $[0, T]$ into $N$ elements.
Whenever $b$ and $\sigma$ are regular, the time series converges to the solution of the SDE.
The ARMA (Auto-Regressive Moving Average) model

$$X_{n+1} = \beta X_n + \sigma\zeta_{n+1},\ X_0 = \xi,$$

for $\beta < 1$ and $\{\zeta_n;\ n \in \mathbb{N}\}$ a family of i.i.d. $\mathcal{N}(0, \sigma^2)$-distributed r.v., is a discrete version of

$$dX_t = (\beta - 1)X_t\,dt + \sigma dW_t,\ X_0 = \xi.$$

## ODE and SDE

**Comparison with ODE:** Given $x \in \mathbb{R}^d$, $b : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, consider the following Ordinary Differential Equation

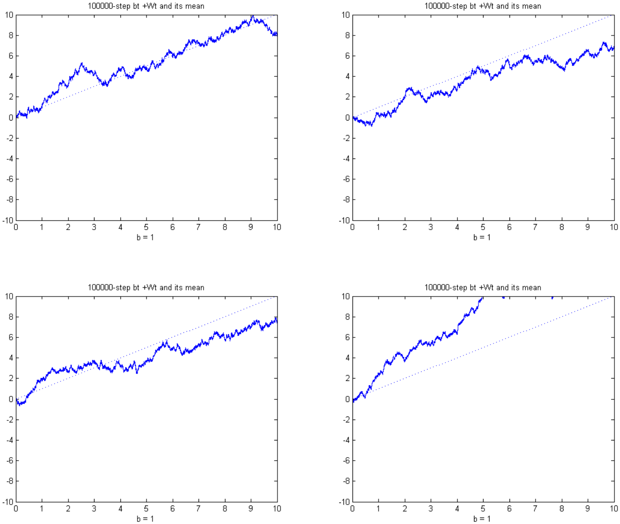
$$dX(t) = b(t, X(t))\, dt,\ X(t = 0) = x,$$

where, at each time $t$, $X(t)$ describes the state of a system.

**Perturbation by a "white noise"**

$$dX(t) = b(t, X(t))\, dt + dW_t,\ X(t = 0) = x.$$

## A simple example: The perturbed line $d = r = 1$, $X_t = X_0 + bt + \sigma W_t$, $\forall\, t \geq 0$.



## ODE and SDE

**Evolution of a scalar quantity along the solution of the ODE:** For any function $f:\mathbb{R}^m\to\mathbb{R}$ of class $\mathcal{C}^1$ on $\mathbb{R}^d$,

$$f(X_t)=f(X_0)+\int_0^t \nabla f(X_s)\cdot b(s,X_s)\,ds.$$

**Evolution of a scalar quantity along the solution of the SDE $X_t$:** By Itô's formula, for any function $f:\mathbb{R}^d\to\mathbb{R}$ of class $\mathcal{C}^2$ with compact support,

$$\begin{aligned}
f(X_t)&=f(X_0)+\sum_i\int_0^t \partial_{x_i}f(X_s)\cdot b^{(i)}(s,X_s)\,ds+\sum_{i,j}\int_0^t \partial_{x_i}f(X_s)\cdot\sigma_{i,j}(s,X_s)\,dW_s\\
&\quad+\frac{1}{2}\sum_{i,j}\int_0^t\left(\partial^2_{x_ix_j}f(X_s)a_{i,j}(s,X_s)\right)\,ds\\
&=f(X_0)+\int_0^t\nabla f(X_s)\cdot b(s,X_s)\,ds+\int_0^t\nabla f(X_s)\cdot\sigma(s,X_s)\,dW_s\\
&\quad+\frac{1}{2}\int_0^t\mathrm{Trace}\left(\nabla^2 f(X_s)a(s,X_s)\right)\,ds.
\end{aligned}$$

## ODE and SDE

**Classical mechanics:** The second law of Newton of the motion of a body:

$$m\frac{d^2q(t)}{dt^2} = F(t, q(t))$$

where $q(t)$ is the position of the body, $dq(t)/dt$ its velocity, $d^2q(t)/dt^2$ its acceleration, $m$ is the mass of the body and $F$ the force field (external and possibly internal) acting on the evolution of the body.
**Stochastic Mechanics:** Langevin 1908 (following Einstein 1905)

$$m\frac{d^2q(t)}{dt^2} = F(t, q(t)) + \zeta(t)$$

where $(\zeta(t);\ t \geq 0)$ model fluctuations due to microscopic thermal agitations.
**Second Order Stochastic Differential Equation:**

$$\begin{aligned} dq_t &= v_t\,dt, \\ dv_t &= F(t, q_t)\,dt + dW_t. \end{aligned}$$

## Strong solution to a SDE

**Setting:** From now on, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ under which are defined

- $\circ$ a r.v. $\xi : \Omega \to \mathbb{R}^d$, $F_0$-measurable distributed according to $\mu_0$,
- $\circ$ a $d$-dimensional $\mathcal{F}_t$-Brownian motion $(W_t;\ t \geq 0)$.

**Construction:** Set $\Omega := \mathcal{C}(\mathbb{R}^+; \mathbb{R}^d) \times \mathbb{R}^d$, $(W_t;\ t \geq 0)$ is the canonical process on $\mathcal{C}(\mathbb{R}^+; \mathbb{R}^d)$ $(W_t(\omega) = \omega(t),\ \omega \in \Omega, t \geq 0)$, $\mathcal{F}_t = \sigma(\mathcal{F}_t^W \cup \sigma(\xi))$ and $\mathbb{P} := \mathbb{P}^W \times \mu_0$ where $\mathbb{P}^W$ is the Wiener measure on $\Omega$.
To satisfy the usual condition, the filtration is then completed by

$$\mathcal{F}_t = \sigma\left(\mathcal{F}_t^W \cup \mathcal{N}\right)$$

where $\mathcal{N}$ is the $\sigma$-algebra of the $\mathbb{P}$-negligible sets:

$$\mathcal{N} = \{N \subset \Omega \,|\, \exists 0 \leq t < \infty \,\text{and}\, G \in F_t \,\text{such that}\, N \subset G \,\text{and}\, \mathbb{P}(G) = 0\}\,.$$

Under $\mathbb{P}$, given the canonical process of $\Omega$

$$x_t : \omega = (\omega_1, \omega_2) \in \Omega \mapsto (\omega_1(t), \omega_2)$$

defines the pair $((W_t;\ t \geq 0), \xi)$.

The first fundamental problem related to a SDE of the form

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \\ X_0 = \xi, \end{cases}$$

is to know, in which situations, there exists a stochastic process $(X_t;\ t \geq 0)$ satisfying the equation, how can we construct such process, and to know if this solution is unique.

Two possible way to solve this problem through:

○ the notion of strong solution or pathwise solution where, conceptually, the classical theory of ODEs is extended to a probabilistic framework.

○ the notion of weak solution or solution in distribution which take advantages of the probabilistic setting of the equation.

**Strong solution to a SDE**:

### Definition 2.1

*Let $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be two Borel-measurable functions. A stochastic process $(X_t;\ t \geq 0)$ is a strong solution to the SDE*

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = \xi, \end{cases} \tag{3}$$

*i.f.f. $(X_t;\ t \geq 0)$ satisfies the following properties:*
*(i) $(X_t;\ t \geq 0)$ is continuous and $\mathcal{F}_t$-adapted,*
*(ii) $\mathbb{P}(X_0 = \xi) = 1$,*
*(iii) $\mathbb{P}$-a.s., for all $t \geq 0$,*

$$\int_0^t \sum_i \left| b^{(i)}(s, X_s) \right| ds + \int_0^t \sum_{i,j} \left| \sigma_{i,j}(s, X_s) \right|^2 ds < +\infty,$$

*(iv) $\mathbb{P}$-a.s., for all $t \geq 0$,*

$$X_t = \xi + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s. \tag{4}$$

**On the conditions $(i)$ and $(iii)$:**
$(iii)$ is a sufficient to ensure that the integral formulation (4) is well defined.
The $\mathcal{F}_t$-adaptedness in $(i)$ of $(X_t;\ t \geq 0)$ is fundamental in the notion of strong solution and formulates the causality principe (*A same cause produces a same effect*) of the equation:

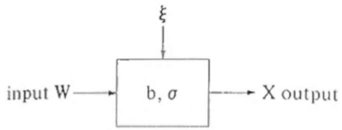



Figure: Illustration of the causality principe.

- **Strong uniqueness**:

**Definition 2.2**

*The SDE* (3) *admits a unique strong solution i.f.f., given two strong solutions,* $(X^1_t;\ t \geq 0)$ *and* $(X^2_t;\ t \geq 0)$, *to* (3), *that is: for* $i = 1, 2$,

(*i*) $(X^i_t;\ t \geq 0)$ *is continuous and* $\mathcal{F}_t$*-adapted,*

(*ii*) $\mathbb{P}(X^i_0 = \xi) = 1$,

(*iii*) $\mathbb{P}$*-a.s., for all* $t \geq 0$,

$$\int_0^t \sum_j \left| b^{(j)}(s, X^i_s) \right| ds + \int_0^t \sum_{k,l} \left| \sigma_{k,l}(s, X^i_s) \right|^2 ds < +\infty,$$

(*iv*) $\mathbb{P}$*-a.s., for all* $t \geq 0$,

$$X^i_t = \xi + \int_0^t b(s, X^i_s)\, ds + \int_0^t \sigma(s, X^i_s)\, dW_s,$$

*then* $(X^1_t;\ t \geq 0)$ *and* $(X^2_t;\ t \geq 0)$ *are indistinguishable, in the sense that*

$$\mathbb{P}\left(X^1_t = X^2_t,\ t \geq 0\right) = 1.$$

- **Existence result**:

**Theorem 2.3**

*Assume that*
$(H_0)$ $\mathbb{E}\left[|\xi|^2\right] < +\infty$.
$(H_1)$ **Lipschitz continuity:** *There exists $K > 0$ such that*

$$\|\sigma(t,x) - \sigma(t,y)\| + |b(t,x) - b(t,y)| \leq K|x-y|, \forall t,x,y.$$

$(H_2)$ **Growth condition:** *There exists $\kappa > 0$ such that*

$$\|\sigma(t,x)\| + |b(t,x)| \leq \kappa\left(1 + |x|\right), \forall t,x.$$

*Then the SDE (3) admits, at least, one strong solution, and, for this solution, we have: for all $0 < T < \infty$, there exists $0 < C < \infty$ such that*

$$\mathbb{E}\left[\max_{t\in[0,T]} |X_t|^2\right] < C\left(1 + \mathbb{E}\left[|\xi|^2\right]\right)e^{CT}.$$

**Notations**: $|\,|$ is the Euclidean norm, $\|\,\|$ is the matrix norm defined by

$$\|A\| = \sup_{x\in\mathbb{R}^n\,;\,|x|=1} |Ax|.$$

**Proof:** Picard-Lindelöf approximation of the solution to (3): We construct the family of stochastic processes $(X^n_t;\, t \geq 0)$, $n \in \mathbb{N}$, defined by:

◦ For $n = 0$:

$$X^0_t = \xi,\ \forall\, t \geq 0;$$

◦ For $n > 0$: given $(X^{n-1}_t;\, t \geq 0)$,

$$X^n_t = \xi + \int_0^t b(s, X^{n-1}_s)\, ds + \int_0^t \sigma(s, X^{n-1}_s)\, dW_s,\ t \geq 0.$$

The idea for constructing a solution to (3) is to show that $(X^n_t;\, t \geq 0)$, $n \in \mathbb{N}$, admits a limit $(X_t,\ t \geq 0)$, as $n$ tends to $\infty$, in an appropriate space so that, due to the smoothness of $b$ and $\sigma$, $(X_t,\ t \geq 0)$ is such that

$$X_t = \xi + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s,\ t \geq 0.$$

At each step $n$, $(X^n_t;\, t \geq 0)$ is a $\mathcal{F}_t$-adapted continuous process, such that $\mathbb{P}(X^n_0 = \xi)$, for all $0 < T < \infty$, a.s., for all $0 \leq t < \infty$,

$$\int_0^t |b(s, X^n_s)|\, ds + \int_0^t \|\sigma(s, X^n_s)\|^2\, ds < +\infty.$$

In addition, by assumptions $(H_0)$ and $(H_2)$, for all $n \geq 1$, there exists some constant $C$ depending only on $T, \kappa$ and $d$ such that

$$\begin{aligned}
\mathbb{E}\left[\max_{t\in[0,T]} |X^n_t|\right] &\leq \mathbb{E}\left[\max_{0\leq t\leq T}\left|\xi + \int_0^t b(s, X^{n-1}_s)\, ds + \int_0^t \sigma(s, X^{n-1}_s)\, dW_s\right|^2\right] \\
&\leq C\left(\mathbb{E}\left[|\xi|^2 + \int_0^T \left|b(s, X^{n-1}_s)\right|^2 + \|\sigma(s, X^{n-1}_s)\|^2\, ds\right]\right) \\
&\leq C\mathbb{E}\left[|\xi|^2\right] + 3C\kappa^2\left(1 + \mathbb{E}\left[\int_0^T \left|X^{n-1}_t\right|^2 dt\right]\right) \\
&\leq C(\mathbb{E}\left[|\xi|^2\right] + 3\kappa^2) + 3C\kappa^2\int_0^T \mathbb{E}\left[\max_{0\leq s\leq t}\left|X^{n-1}_s\right|^2\right] dt
\end{aligned}$$

The first inequality is obtained by using the Itô isometry property, and the second inequality follows from the growth condition of $b$ and $\sigma$.

**Exercise**: Fill the details in the above inequalities.

At each step $n$, $(X^n_t;\, t \geq 0)$ is a $\mathcal{F}_t$-adapted continuous process, such that $\mathbb{P}(X^n_0 = \xi)$, for all $0 < T < \infty$, a.s., for all $0 \leq t < \infty$,

$$\int_0^t |b(s, X^n_s)|\; ds + \int_0^t \|\sigma(s, X^n_s)\|^2\, ds < +\infty.$$

In addition, by assumptions $(H_0)$ and $(H_2)$, for all $n \geq 1$, there exists some constant $C$ depending only on $T, \kappa$ and $d$ such that

$$\begin{aligned}
\mathbb{E}\left[\max_{t\in[0,T]} |X^n_t|\right] &\leq \mathbb{E}\left[\max_{0\leq t\leq T}\left|\xi + \int_0^t b(s, X^{n-1}_s)\, ds + \int_0^t \sigma(s, X^{n-1}_s)\, dW_s\right|^2\right]\\
&\leq C\left(\mathbb{E}\left[|\xi|^2 + \int_0^T \left|b(s, X^{n-1}_s)\right|^2 + \|\sigma(s, X^{n-1}_s)\|^2\, ds\right]\right)\\
&\leq C\mathbb{E}\left[|\xi|^2\right] + 3C\kappa^2\left(1 + \mathbb{E}\left[\int_0^T \left|X^{n-1}_t\right|^2\, dt\right]\right)\\
&\leq C(\mathbb{E}\left[|\xi|^2\right] + 3\kappa^2) + 3C\kappa^2\int_0^T \mathbb{E}\left[\max_{0\leq s\leq t}\left|X^{n-1}_s\right|^2\right]\, dt
\end{aligned}$$

The first inequality is obtained by using the Itô isometry property, and the second inequality follows from the growth condition of $b$ and $\sigma$.
**Exercise**: Fill the details in the above inequalities.

Let us show that the preceding inequality gives

$$\mathbb{E}\left[\max_{t\in[0,T]}|X_t^n|^2\right] < C\left(1+\mathbb{E}\left[|\xi|^2\right]\right)e^{CT}$$

with $C$ independent to $n$. By Iterating the preceding inequality and by denoting $0 < C < \infty$ some constant which may change from line to line, we observe that

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[0,T]}|X_t^n|\right]\\
&\leq C(\mathbb{E}\left[|\xi|^2\right]+3\kappa^2)+3C\kappa^2\int_0^T\mathbb{E}\left[\max_{0\leq s\leq t}|X_s^{n-1}|^2\right]dt\\
&\leq C(\mathbb{E}\left[|\xi|^2\right]+1)+C\int_0^T\mathbb{E}\left[\max_{0\leq s\leq t}|X_s^{n-1}|^2\right]dt\\
&\leq C(\mathbb{E}\left[|\xi|^2\right]+1)+C\int_0^T\left(C(\mathbb{E}\left[|\xi|^2\right]+1)+C\int_0^t\mathbb{E}\left[\max_{0\leq r\leq s}|X_r^{n-2}|^2\right]ds\right)dt\\
&\leq C(\mathbb{E}\left[|\xi|^2\right]+1)+(C)^2T(\mathbb{E}\left[|\xi|^2\right]+1)+(C)^2\int_0^T\int_0^t\mathbb{E}\left[\max_{0\leq r\leq s}|X_r^{n-2}|^2\right]ds\,dt\\
&\vdots\\
&\leq C(\mathbb{E}\left[|\xi|^2\right]+1)\left(1+CT+\frac{(CT)^2}{2}+\cdots+\frac{(CT)^n}{n!}\right)\leq C(\mathbb{E}\left[|\xi|^2\right]+1)e^{CT}.
\end{aligned}$$

Next, observe that

$$(X_t^{n+1} - X_t^n) = \int_0^t \left(b(s,X_s^n) - b(s,X_s^{n-1})\right)\, ds + \int_0^t \left(\sigma(s,X_s^n) - \sigma(s,X_s^{n-1})\right)\, dW_s,\ t \geq 0,$$

and that, by Doob's maximal inequality and assumption $(H_1)$, for all $0 \leq T < \infty$

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[0,T]} \left|X_t^{n+1} - X_t^n\right|^2\right] \\
&= \mathbb{E}\left[\max_{t\in[0,T]} \left|\int_0^t b(s,X_s^n) - b(s,X_s^{n-1})\, ds + \int_0^t \sigma(s,X_s^n) - \sigma(s,X_s^{n-1})\, dW_s\right|^2\right] \\
&\leq C\mathbb{E}\left[\int_0^T \left|b(s,X_s^n) - b(s,X_s^{n-1})\right|^2\, ds + \int_0^T \left\|\sigma(s,X_s^n) - \sigma(s,X_s^{n-1})\right\|^2 ds\right] \\
&\leq CK^2\mathbb{E}\left[\int_0^T \left|X_s^n - X_s^{n-1}\right|^2\, ds\right] \leq CK^2\mathbb{E}\left[\int_0^T \max_{r\in[0,t]} \left|X_r^n - X_r^{n-1}\right|^2\, dt\right],
\end{aligned}$$

where $C$ doesn't depend on $n$ or $T$.

Then, iterating the preceding inequality, we deduce that

$$\mathbb{E}\left[\max_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right|^2\right] \leq \left(CK^2\right)^n \mathbb{E}\left[\int_0^T\left(\int_0^{t_1}\cdots\int_0^{t_{n-1}}\max_{r\in[0,t]}\left|X_r^1-\xi\right|^2 dt_n\cdots\right)dt_1\right]$$
$$\leq \frac{\left(CK^2T\right)^n}{n!}\mathbb{E}\left[1+|\xi|^2\right],$$

and conclude that $\lim_n \mathbb{E}\left[\max_{t\in[0,T]}\left|X_t^{n+1}-X_t^n\right|^2\right]=0$. For all $0\leq T<\infty$, $(X_t^n;\, t\geq 0)$ is then a Cauchy sequence on the Banach space (i.e. complete metric space)

$$L_T^2=\left\{(Y_t;\, t\geq 0),\ \text{continuous, } \mathcal{F}_t-\text{adapted such that } \|Y\|_{L_T^2}<+\infty\right\}$$

for the norm $\|Y\|_{L_T^2}=\mathbb{E}[\max_{t\in[0,T]}|Y_t|^2]$.
Denoting $(X_t;\, 0\leq t\leq T)$ the limit of $(X_t^n;\, 0\leq t\leq T)$ as $n$ tends $\infty$ and since $(H_1)$ ensures that $x\mapsto b(t,x),\sigma(t,x)$ are continuous, we can check that, for all $0\leq T<\infty$, a.s.,

$$X_t=\lim_{n\to\infty}X_t^n=\xi+\int_0^t b(s,X_s)\,ds+\int_0^t\sigma(s,X_s)\,dW_s,\ 0\leq t\leq T.$$

Then, iterating the preceding inequality, we deduce that

$$\begin{aligned}\mathbb{E}\left[\max_{t\in[0,T]}\left|X^{n+1}_t - X^n_t\right|^2\right] &\leq \left(CK^2\right)^n \mathbb{E}\left[\int_0^T \left(\int_0^{t_1}\cdots\int_0^{t_{n-1}} \max_{r\in[0,t]}\left|X^1_r-\xi\right|^2 dt_n\cdots\right) dt_1\right]\\ &\leq \frac{\left(CK^2T\right)^n}{n!}\mathbb{E}\left[1+|\xi|^2\right],\end{aligned}$$

and conclude that $\lim_n \mathbb{E}\left[\max_{t\in[0,T]}\left|X^{n+1}_t - X^n_t\right|^2\right] = 0$. For all $0\leq T<\infty$, $(X^n_t;\, t\geq 0)$ is then a Cauchy sequence on the Banach space (i.e. complete metric space)

$$L^2_T = \left\{(Y_t;\, t\geq 0),\ \text{continuous, } \mathcal{F}_t - \text{adapted such that } \|Y\|_{L^2_T} < +\infty\right\}$$

for the norm $\|Y\|_{L^2_T} = \mathbb{E}[\max_{t\in[0,T]}|Y_t|^2]$.
Denoting $(X_t;\, 0\leq t\leq T)$ the limit of $(X^n_t;\, 0\leq t\leq T)$ as $n$ tends $\infty$ and since $(H_1)$ ensures that $x\mapsto b(t,x),\sigma(t,x)$ are continuous, we can check that, for all $0\leq T<\infty$, a.s.,

$$X_t = \lim_{n\to\infty} X^n_t = \xi + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s,\ 0\leq t\leq T.$$

## Uniqueness of a strong solution.

Since the time horizon $T$ considered preciously is arbitrary, we have constructed a stochastic process $(X_t;\ t \geq 0)$ solution to (3) and such, for all $0 < T < \infty$,

$$\lim_n \mathbb{E}[\max_{0\leq t\leq T} |X^n_t - X_t|^2] = 0.$$

Therefore,

$$\begin{aligned}\mathbb{E}[\max_{0\leq t\leq T} |X_t|^2] &\leq \liminf_n \mathbb{E}[\max_{0\leq t\leq T} |X^n_t|^2]\\ &\leq C(1+\mathbb{E}[|\xi|^2])\exp\{CT\},\end{aligned}$$

by applying the Fatou lemma

**Lemma 2.4 (Fatou lemma)**

*Let $\{f_n\}_n$ be a sequence of non-negative Borel functions defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then*

$$\int \liminf_n f_n \leq \liminf_n \int f_n.$$

## Uniqueness of a strong solution

### Theorem 2.5

*If $\xi$, $b$ and $\sigma$ satisfy the assumptions $(H_0)$, $(H_1)$ and $(H_2)$ then the SDE*

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = \xi, \end{cases}$$

*admits at most one solution such that, for all $0 \leq t \leq T$, $\mathbb{E}[\max_{0\leq t\leq T} |X_t|^2] < \infty$.*

**Proof:** Let $(X^1_t;\, t \geq 0)$ and $(X^2_t;\, t \geq 0)$ be two solutions of the SDE. It is enough to show that, for any arbitrary $0 < T < \infty$, $(X^1_t;\, 0 \leq t \leq T) = (X^2_t;\, 0 \leq t \leq T)$ a.s.

Fixing $0 < T < \infty$, we observe that

$$X^1_t - X^2_t = \int_0^t b(s, X^1_s) - b(s, X^1_s)\,ds + \int_0^t \sigma(s, X^1_s) - \sigma(s, X^1_s)\,dW_s,\ t \in [0, T],$$

with $(X^1_t;\, t \in [0, T])$ and $(X^2_t;\, t \in [0, T])$ in $L^2_T$.

In particular, for all $0 \leq T_0 \leq T$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\max_{t\in[0,T_0]}\left|X^1_t - X^2_t\right|^2\right]\\
&\leq 2\mathbb{E}\left[\max_{t\in[0,T_0]}\left|\int_0^t b(s,X^1_s) - b(s,X^1_s)\,ds\right|^2\right] + 2\mathbb{E}\left[\max_{t\in[0,T_0]}\left|\int_0^t \sigma(s,X^1_s) - \sigma(s,X^1_s)\,dW_s\right|^2\right]\\
&\leq 2T_0\mathbb{E}\left[\max_{t\in[0,T_0]}\int_0^t \left|b(s,X^1_s) - b(s,X^1_s)\right|^2\,ds\right] + 2\mathbb{E}\left[\max_{t\in[0,T_0]}\int_0^t \|\sigma(s,X^1_s) - \sigma(s,X^1_s)\|^2\,ds\right]\\
&\leq 2K(1+T_0)\int_0^{T_0}\mathbb{E}\left[\max_{0\leq r\leq t}\left|X^1_r - X^2_r\right|^2\right]\,dt,
\end{aligned}
$$

using the Cauchy-Schwarz inequality and the assumption $(H_1)$ for the last inequality. This estimate ensures that $t \mapsto \mathbb{E}\left[\max_{t\in[0,t]}\left|X^1_r - X^2_r\right|^2\right]$ satisfies an inequality of the form

$$f(t) \leq \beta\int_0^t f(s)\,ds,\ \forall t \in [0,T].$$

### Lemma 2.6 (Gronwall Lemma)

*Let $f : [0, T] \to \mathbb{R}$ be a non-negative continuous function such that, for some $\beta \geq 0$ and $\alpha$ a non-negative function, continuous and integrable on $(0, T)$, we have: for all $0 \leq t \leq T$,*

$$f(t) \leq \alpha(t) + \beta \int_0^t f(s)\, ds.$$

*Then, for all $0 \leq t \leq T$, $f(t) \leq \alpha(t) + \beta \int_0^t \alpha(s) e^{\beta(t-s)}\, ds$.*

Applying the Gronwall Lemma, we obtain: for all $0 \leq T_0 \leq T$,

$$\mathbb{E}\left[\max_{t\in[0,T_0]} \left|X_t^1 - X_t^2\right|^2\right] = 0$$

from which we conclude that

$$X_t^1 = X_t^2, \text{ for all } 0 \leq t \leq T.$$

The preceding uniqueness result can be slightly extended replacing the assumption $(H_1)$ by the following assumption of local Lipschitz continuity:

$$(H_1') \quad \begin{array}{l} \text{For all } 0 < T < \infty \text{ and for all closed ball } \overline{B(0,N)} = \{x \in \mathbb{R}^d \,|\, |x| \leq N\}, \\ \text{there exists } K_N > 0 \text{ such that, for all } t \in [0,T],\ x, y \in B(0,N), \\ \|\sigma(t,x) - \sigma(t,y)\| + |b(t,x) - b(t,y)| \leq K_N |x - y|. \end{array}$$

**Theorem 2.7**

*Under the assumptions $(H_0)$, $(H_1')$ and $(H_2)$, the SDE*

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = \xi, \end{cases}$$

*admits at most one strong solution.*

**Elements of proof:** The core of the proof consists in replicating the proof of Theorem 2.5 using a "truncation-localization" argument: Let $(X^1_t;\, t \geq 0)$ and $(X^2_t;\, t \geq 0)$ be two strong solution to the SDE. For $i = 1, 2$, define

$$\tau^i_N := \inf\{t > 0 \,|\, |X^i_t| \geq N\},\ \ \tau_N = \tau^1_N \wedge \tau^2_N.$$

Then $(H_0)$ and $(H_2)$ ensure that, a.s., $\tau^1_N$ and $\tau^2_N$ (and, by extension, $\tau_N$) tend to $+\infty$ as $N$ grows to $\infty$.
Next, computing the difference between $X^1$ and $X^2$ on a interval $[0, T_0 \wedge \tau_N]$ for $0 < T_0 \leq T < +\infty$, we get

$$\begin{aligned}
&\mathbb{E}\left[\max_{t \leq T_0 \wedge \tau_N} \left|X^1_t - X^2_t\right|^2\right] \\
&\leq 2\mathbb{E}\left[\max_{t \in [0, T_0 \wedge \tau_N]} \left|\int_0^t b(s, X^1_s) - b(s, X^2_s)\, ds\right|^2\right] \\
&\quad + 2\mathbb{E}\left[\max_{t \in [0, T_0 \wedge \tau_N]} \left|\int_0^t \sigma(s, X^1_s) - \sigma(s, X^2_s)\, dW_s\right|^2\right] \\
&\leq K_N \mathbb{E}\left[\int_0^{T_0 \wedge \tau_N} \left|X^1_s - X^2_s\right|^2\, ds\right] \leq K_N \mathbb{E}\left[\int_0^{T_0} \max_{0 \leq r \leq s \wedge \tau_N} \left|X^1_r - X^2_r\right|^2\, ds\right].
\end{aligned}$$

where the second inequality follows for the Itô isometry and Doob optional sampling theorem. Applying the Gronwall lemma, we conclude that $X^1_t = X^2_t$, $t \leq T \wedge \tau_N$. Taking the limit $N \to \infty$ gives the result.

## Comments on the assumptions $(H_0)$-$(H_2)$:

• The assumptions $(H_0)$ and $(H_2)$ enable to control the growth in time of the moments of any strong solution to (3). In particular, the $p^{th}$-moment of $(X_t;\, t \geq 0)$ is controlled by the $p^{th}$-moment of the initial condition $\xi$. Hence combining $(H_0)$ and $(H_2)$ ensure that the second (and, by extension, all non-negative moments of order less than 2) moments of any strong solution to (3) are finite. Without this kind of control, the solution can have a finite explosion time. For instance, a solution to the SDE:

$$dX_t = (1 + |X_t|^2)\, dB_t,\; X_0 = 1,\; d = m = 1,$$

*explodes* or *blows up* to $\infty$ at finite time, a.s..

## Comments on the assumptions $(H_0)$-$(H_2)$:

• $(H_1)$ has a key role in the demonstrations of Theorems 2.3 and 2.5 as it enables to control the distance between two components in the iterative construction of the Picard-Lindelöf approximation. Let us further notice that in the case $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$, $(H_1)$ implies $(H_2)$ with

$$\|\sigma(t,x)\| + |b(t,x)| = \|\sigma(x)\| + |b(x)| \leq K(|x|) + |\sigma(0)| + |b(0)|,$$

for $\kappa = \max(1, \frac{\|\sigma(0)\|+|b(0)|}{K})$.

If, for some $x_0$, $t \to b(t,x_0)$ and $t \to \sigma(t,x_0)$ are continuous or locally bounded, then $(H_1)$ still implies $(H_2)$ in the sense that $(H_1)$ ensures that

$$\begin{aligned}\|\sigma(t,x)\| + |b(t,x)| &\leq K(|x-x_0|) + \|\sigma(t,x_0)\| + |b(t,x_0)| \\ &\leq K|x| + K|x_0| + \|\sigma(t,x_0)\| + |b(t,x_0)|,\end{aligned}$$

and so $(H_2)$ with $\kappa = \max(1, K|x_0| + \frac{\|\sigma(t,x_0)\|+|b(t,x_0)|}{K})$.

## Alternative construction of a strong solution on a finite time interval

**Discrete time approximation:** For simplicity, assume that $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$. Given $0 < T < \infty$, define the sequence $I_N = (t_0^N, t_1^N, \cdots, t_N^N)$, $N \in \mathbb{N}$, as

$$t_k^N = kT/N,\ k = 0, 1, \cdots, N.$$

For each $N$, $I_N$ provides a decomposition of the interval $[0, T]$ into $N$-subinterval $[t_k^N, t_{k+1}^N]$, such that

$$\lim_{N\to\infty} \sup_k |t_k^N - t_{k+1}^N| = 0$$

Then, for any $N \in \mathbb{N}$ we can construct the process $(X_t^N;\ t \geq 0)$ as

$$X_t^N = \xi + \int_0^t b(X_{\eta_N(s)}^N)\, ds + \int_0^t \sigma(X_{\eta_N(s)}^N)\, dW_s$$

for

$$\eta_N(t) = t_i^N \text{ whenever, for some i , } t_i^N \leq t < t_{i+1}^N,\ \eta_N(T) = T.$$

Whenever $t$ is in the interval $[t_k^N, t_{k+1}^N]$, the coefficient $b$ and $\sigma$ are *frozen* at the state $X_{t_k^N}$ so that

$$X_t^N = X_{t_k}^N + \int_{t_k^N}^t b(X_{t_k}^N)\, ds + \int_{t_k^N}^t \sigma(X_{t_k}^N)\, dW_s,\ t_k^N < t \leq t_{k+1}^N.$$

Hence, for any $N$, $(X^N_t;\ t \geq 0)$ is well-defined, and we have

### Theorem 2.8

*Under the assumptions* $(H_0)$, $(H_1)$ *and* $(H_2)$, $(X^N_t;\ t \in [0, T])$ *converges to* $(X_t;\ t \in [0, T])$, *the strong solution to* (3), *and*

$$\mathbb{E}\left[\max_{t\in[0,T]} |X^N_t - X_t|^2\right] \leq C/N,\ C = C(T, d, K).$$

The interest of introducing $(X^N_t;\ t \in [0, T])$ is the construction of a numerical simulation of $(X_t;\ t \in [0, T])$ as follows: Given $\xi$, $N \in \mathbb{N}$, and $\zeta_k, k = 1, \cdots, N$, a family of $N$ i.i.d. $\mathcal{N}(0, I_d)$-distributed random variables. Then we can sample the path of $t \mapsto \tilde{X}^N_t$ with: $\tilde{X}^N_k = X^N_{t_k}$, and

$$X^N_t = \tilde{X}^N_k + b(\tilde{X}^N_k)\left(t - t^N_k\right) + \sigma(\tilde{X}^N_k)\sqrt{t - t^N_k}\zeta^N_k.$$

According to Theorem 2.8, $(X^N_t;\ t \geq 0)$ converges as $N \to \infty$ to the solution to (3) endowing the Brownian motion obtained from

$$\sum_{k=0}^{N} \sqrt{t - t^N_k}\zeta^N_k \mathbb{1}_{\{t^N_k \leq t < t^N_{k+1}\}} \xrightarrow{\text{a.s.}} W_t.$$

**Application:** Numerical simulation of the solution to a SDE and theoretical rate of convergence.

## SDE as a transformation of the Brownian motion

Assume that $d = m = 1$, and that $b(t,x) = b(x)$, $\sigma(t,x) = \sigma(x)$. Assume further $(H_0)$, $(H_1)$ and $(H_2)$ hold true. Then consider the mapping $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\partial_\beta h(\alpha,\beta) = \sigma(h(\alpha,\beta)),\ h(\alpha,0) = \alpha.$$

Since $\sigma$ is Lipschitz continuous, $h$ is well defined and smooth with respect to $\alpha$ and $\beta$.

**Theorem 2.9 (Doss 1977)**

*Assume $(H_0)$, $(H_1)$ and $(H_2)$ and that $\sigma$ is of class $\mathcal{C}^1$ on $\mathbb{R}$ with bounded derivative. Then the solution to (3) is given by*

$$X_t = h(D_t, W_t)$$

*where*

$$D_t = \xi + \int_0^t \exp\left\{-\int_0^{W_s} \sigma'(h(D_r,W_r))\,dr\right\}\left(b(h(D_s,W_s)) - \frac{1}{2}\left(\sigma'\sigma\right)\left(h(D_s,W_s)\right)\right) ds.$$

### Lemma 2.10 (Comparison lemma)

*Let $(W_t;\, t \geq 0)$ be a standard Brownian motion and $\xi$ be a r.v. independent to $(W_t;\, t \geq 0)$, both defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \geq 0), \mathbb{P})$. Let $\sigma \in \mathbb{R}$, $b_1, b_2 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and let $(X^1_t;\, t \geq 0)$ and $(X^2_t;\, t \geq 0)$ be two strong solutions to*

$$X_t = \xi + \int_0^t b_i(s, X_s)\, ds + \sigma W_t,\ \forall t \geq 0.$$

*If $b_1(t,x) \leq b_2(t,x)$, $\forall (t,x) \in (0,\infty) \times \mathbb{R}$, and if one of the two following properties is satisfied*

- *$(i)$ $b_1$ and $b_2$ do not depend on $x$ ( $b_1(t,x) = b_1(t), b_2(t,x) = b_2(t)$),*
- *$(ii)$ $x \mapsto b_1(t,x)$ and $x \mapsto b_2(t,x)$ are Lipschitz continuous,*

*then $\mathbb{P}$-a.s., $X^1_t \leq X^2_t$, $\forall t \geq 0$.*

**Proof:** Define $Y_t := X^1_t - X^2_t = \int_0^t b_1(s, X^1_s) - b_2(s, X^2_s)\, ds$. Since $x \mapsto |(x \vee 0)|^2$ is $\mathcal{C}^1$ on $\mathbb{R}$, applying Itô's formula gives that, $\mathbb{P}$-a.s.,

$$|Y_t \vee 0|^2 = 2 \int_0^t \left(b_1(s, X^1_s) - b_2(s, X^2_s)\right) \left((X^1_s - X^2_s) \vee 0\right)\, ds \leq 0,\ \forall 0 \leq t < \infty.$$

and therefore $\mathbb{P}$-a.s., $Y_t$ is non-positive, for all $t \geq 0$.

## Flow of solution and Markov property

In this part, we consider the link between the strong uniqueness of a SDE with time-invariant coefficients and the Markov property of the solution.
**Recall:**

**Definition 2.11**

*A stochastic process* $(X_t;\, t \geq 0)$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \geq 0), \mathbb{P})$. $(X_t;\, t \geq 0)$ *is said to have the Markov property i.f.f.,* $\forall f \in \mathcal{C}_b$,

$$\mathbb{E}\left[f(X_t) \,|\, \mathcal{F}_s\right] = \mathbb{E}\left[f(X_t) \,|\, X_s\right], \text{ for all } s \leq t.$$

$(X_t;\, t \geq 0)$ *is further to be a homogeneous Markov process i.f.f.,* $\forall f \in \mathcal{C}_b$,

$$\mathbb{E}\left[f(X_t) \,|\, \mathcal{F}_s\right] = \mathbb{E}\left[f(X_{t-s}) \,|\, X_0\right], \text{ for all } s \leq t.$$

- The notion of flow of solutions to a SDE of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \tag{5}$$

**Definition 2.12**

*Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ be a filtered probability space, under which are defined a $\mathcal{F}_t$-Brownian motion $(B_t;\ t \geq 0)$ with values in $\mathbb{R}^m$ and a r.v. $\xi$ distributed according to $\mu_0$. The flows of solution of (5) is the family of stochastic processes $\{(X_t^{s,x};\ t \geq s);\ (t,x) \in [0,\infty) \times \mathbb{R}^d\}$ defined in $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$, continuous and such that, for all $(s,x) \in [0,\infty) \times \mathbb{R}^m$,*

$$\text{a.s.},\ X_t^{s,x} = x + \int_s^t b(X_r^{s,x})\,dr + \int_s^t \sigma(X_r^{s,x})\,dB_r,\ s \leq t < \infty.$$

In other words, the flow of solutions of (5) corresponds to a parametrization of the SDE given the initial time at a time $s$ in $[0,\infty)$ and the initial position $x$ in $\mathbb{R}^d$. Given $(s,x)$, $(X_t^{s,x};\ 0 \leq t)$ satisfies the SDE

$$\begin{aligned} dX_t^{s,x} &= b(X_t^{s,x})\,dt + \sigma(X_t^{s,x})\,dB_t, \\ X_s^{s,x} &= x, \end{aligned} \tag{6}$$

for $(B_t;\ t \geq 0)$ a $\mathbb{R}^d$-Brownian motion.

Suppose that, for all $(s,x)$ in $[0,\infty)\times\mathbb{R}^d$, there exists a unique strong solution to (6) in $(s,\infty)$ equipped with the initial condition $x$. Therefore, for all $0\leq s<\infty$, $x$ in $\mathbb{R}^d$,

$$X_t^{0,x}=X_t^{s,X_s^{0,x}}\ t\geq 0.$$

Indeed, if $t\leq s$ then $X_t^{s,X_s^{0,x}}=X_s^{0,x}$. If $t\geq s$ then we can observe that $(X_t^{0,X_s^{0,x}};\,t\geq s\geq 0)$ is the unique solution to

$$X_t^{s,X_s^{0,x}}=X_s^{0,x}+\int_s^t b(X_r^{s,X_s^{0,x}})\,dr+\int_s^t\sigma(X_r^{s,X_s^{0,x}})\,dB_r,\ t\geq s.$$

Meanwhile, for all $t\geq s$,

$$\begin{aligned}
&X_t^{0,x}\\
&=x+\int_0^s b(X_r^{0,x})\,dr+\int_s^t\sigma(X_r^{0,x})\,dB_r+\int_s^t b(X_r^{0,x})\,dr+\int_s^t\sigma(X_r^{0,x})\,dB_r\\
&=X_s^{0,x}+\int_s^t b(X_r^{0,x})\,dr+\int_s^t\sigma(X_r^{0,x})\,dB_r.
\end{aligned}$$

By uniqueness of the SDE, $(X_t^{0,x};\,t\geq s)$ and $(X_t^{s,X_s^{0,x}};\,t\geq s)$ are hence indistinguishable. This result asserts that the dependency of $(X_t^{0,x};\,t\geq s)$ with respect to $\mathcal{F}_s$ is reduced to $\sigma(X_s^{0,x})$.

In particular, we can show the following causality principle (see (F06), Chapter 5, page 109):

**Theorem 2.13**

*Assume that $(H_0)$, $(H_1)$ and $(H_2)$. Then*

$$X_t = X_t^{0,\xi} = \Phi(\xi, (W_r;\, 0 \leq r \leq t) = X_t^{s, X_s^{0,x}} = \Phi(X_s^{0,x}, (W_r - W_s;\, s \leq r \leq t),$$

*for $0 \leq s \leq t < \infty$ and for some measurable function*

$$\Phi : \mathbb{R}^d \times \mathcal{C}([0,\infty), \mathbb{R}^m) \to \mathcal{C}([0,\infty), \mathbb{R}^d).$$

Due to the property of the increment of the Brownian motion, $X_t^{0,\xi}$ depends of $\mathcal{F}_s$ only though $X_s^{0,\xi}$ and thus is a Markov homogeneous process.

## Wellposedness for a one-dimensional SDE

Assuming $d = m = 1$, let us relax the assumption on the regularity of $\sigma$.

### Theorem 2.14 (Uniqueness result for non-Lipschitz diffusion)

*Suppose that $b$ and $\sigma$ are such that*

$(h1)$ *There exists* $K > 0$ *such that* $|b(t,x) - b(t,y)| \leq K|x-y|, \forall\, t,x,y,$
$(h2)$ *There exists* $\alpha : [0,\infty) \to [0,\infty)$ *such that* $|\sigma(t,x) - \sigma(t,y)| \leq \alpha(|x-y|) \,\forall\, t,x,y,$

*for* $\alpha : [0,\infty) \to [0,\infty)$ *(strictly) increasing and such that:* $\alpha(0) = 0$ *and*

$$\int_0^\epsilon \frac{1}{\alpha^2(r)}\,dr = +\infty,\ \forall\,\epsilon > 0.$$

*Then there exists at most one solution to the SDE:*

$$\begin{aligned} dX_t &= b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t, \\ X_{t=0} &= \xi. \end{aligned} \tag{7}$$

The preceding theorem enables to deal with diffusion component of the form $\sigma(x) = |x|^p$ with $p \geq 1/2$.

## Further results for one-dimensional SDE

**Elements of proof:** Under $(h2)$, there exist (see details in (KS88), chapter 5) a sequence of real valued function $\{\psi_n\}_n$ satisfying the following properties:

- $(p1)$ $\{\psi_n;\ n \in \mathbb{N}\}$ is non-increasing,
- $(p2)$ For all $n$, $x \mapsto \psi_n(x)$ is an even $\mathcal{C}^2$-function such that $|\psi_n'(x)| \leq 1$ and $|\psi_n''(x)| \leq \frac{2}{n\alpha^2(x)}$,
- $(p3)$ $\lim_{n\to+\infty} \psi_n(x) = |x|$.

The sequence $\{\psi_n,\ n \in \mathbb{N}\}$ defines a $\mathcal{C}^2$-approximation, positive and increasing, of the function $x \mapsto |x|$ such that its derivative of first order is bounded, uniformly in $n$, and the *increasingness* of the derivative of second order is inferior to $\frac{2}{n\alpha^2(x)}$. Let $(X_t^1;\ t \geq 0)$ and $(X_t^2;\ t \geq 0)$ be two solutions to (7) such that, for $i = 1, 2$,

$$(iii')\ \mathbb{E}\left[\int_0^t |\sigma(s, X_s^i)|^2\, ds\right] < +\infty,\ \forall\, 0 \leq t < \infty.$$

Then

$$Y_t := X_t^1 - X_t^2 \left( = \int_0^t b(s, X_s^1) - b(s, X_s^2)\, ds + \int_0^t \sigma(s, X_s^1) - \sigma(s, X_s^2)\, dW_s \right)$$

and, applying Itô's formula to $t \to \psi_n(Y_t)$, we get

$$\begin{aligned}\psi_n(Y_t) = &\int_0^t \left(b(s, X_s^1) - b(s, X_s^2)\right) \psi_n'(Y_s)\, ds + \int_0^t \left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right) \psi_n'(Y_s)\, dW_s \\ &+ \frac{1}{2} \int_0^t \left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right)^2 \psi_n''(Y_s)\, ds.\end{aligned}$$

Since $|\psi_n'| \leq 1$, taking the expectation on both sides of the equality gives

$$\begin{aligned}&\mathbb{E}\left[\psi_n(Y_t)\right] \\ &= \mathbb{E}\left[\int_0^t \left(b(s, X_s^1) - b(s, X_s^2)\right) \psi_n'(Y_s)\, ds\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right)^2 \psi_n''(Y_s)\, ds\right].\end{aligned}$$

Next, using $(h2)$ and $(p2)$, we observe that

$$\frac{1}{2}\mathbb{E}\left[\int_0^t \left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right)^2 \psi_n''(Y_s)\, ds\right] \leq \frac{1}{2}\mathbb{E}\left[\int_0^t \psi_n''(Y_s)\alpha^2(Y_s)\, ds\right] \leq \frac{t}{n}.$$

Then, using $(h1)$ and the properties of $\psi_n$, we obtain that

$$\mathbb{E}\left[\psi_n(Y_t)\right] \leq K \int_0^t \mathbb{E}\left[|Y_s|\right] \, ds + \frac{t}{n}.$$

Taking the limit $n \to +\infty$, it follows that

$$\mathbb{E}\left[|Y_t|\right] \leq K \int_0^t \mathbb{E}\left[|Y_s|\right] \, ds,$$

and, next that, for all $0 \leq t < \infty$, $\mathbb{E}\left[|Y_t|\right] = 0$. Since $(Y_t;\ t \geq 0)$ is continuous, we conclude that $\mathbb{E}\left[\max_{0 \leq t \leq T} |Y_t|\right] = 0$ for all $0 \leq T < \infty$ and that $(X^1;\ t \geq 0)$ and $(X^2_t;\ t \geq 0)$ are undistinguishable.

## Explosion time and local solution

**Estimating the explosion/blow up time of a SDE** ((KS88) chapter 5, Rogers and Williams (2000), p.297-299).
Consider the following one-dimensional SDE:

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \\ X_{t=0} = x, \end{cases}$$

where $x \in \mathbb{R}$, $b$ and $\sigma$ are continuous functions with $\sigma^2(x) > 0$ for all $x$ in $\mathbb{R}$, admits a solution. Finite moment condition on $\xi$ and the growth condition $(H_1)$ in Theorem 2.8 enables to have, a-priori, a control on a moment of the solution to the SDE. If the growth condition is removed, a solution to the SDE may still exists but only up to the explosion time

$$e = \lim_{n\to\infty} \tau_N = \sup_N \tau_N,$$

$$\tau_N = \inf\{t > 0 \,|\, |X_t| > N\}.$$

## Theorem 2.15 (One-dimensional case)

*Define*

$$\kappa(x) = 2\int_1^x s(\theta)\left(\int_1^\theta \frac{1}{s(y)\sigma^2(y)}dy\right)d\theta,$$
$$s(x) = \exp\left\{-\int_1^x \frac{2b(z)}{\sigma^2(z)}\,dz\right\}$$

*and assume that*

$$X_t = x + \int_0^t b(X_r)\,dr + \int_0^t \sigma(X_r)\,dB_r,$$

*where* $(B_t;\, t \geq 0)$ *is a one-dimensional Brownian motion,* $x \in \mathbb{R}$*,* $b$ *and* $\sigma$ *are continuous functions on* $[1,\infty)$ *and* $\sigma^2(x) > 0$ *for all* $x$ *in* $\mathbb{R}$*. Then, for* $\{\beta_N\}_{N\in\mathbb{N}}$ *the sequence of stopping times defined by*

$$\beta_N = \inf\{t > 0 \,|\, X_t > N\}$$

*it holds*

$$\mathbb{P}\left(\sup_N \beta_N = \infty\right) = 1 \Leftrightarrow \lim_{x\to\infty}\kappa(x) = \infty.$$

**Elements of proof (for $\Leftarrow$):** Find a increasing (strictly) positive $\mathcal{C}^2$-function $u:\mathbb{R}\to(0,\infty)$ such that $\lim_{x\to\infty} u(x)=\infty$ and

$$b(x)u'(x)+\frac{\sigma^2(x)}{2}u''(x)\leq u(x)\,\text{in}\,\mathbb{R}.$$

Then, applying Itô's formula,

$$e^{-t}u(X_t)\leq u(X_0)+\int_0^t u'(X_s)\sigma(X_s)\,dB_s.$$

$t\mapsto u(X_t)e^{-t}$ is thus a nonnegative supermartingale so that

$$u(x)=u(X_0)\geq\mathbb{E}\left[u(X_{\beta_N})e^{-\beta_N}\right]=u(N)\mathbb{E}\left[e^{-\beta_N}\right]$$

where the right hand side equality follows from the continuity of $t\mapsto X_t$. Then, we deduce that, for all $x\in\mathbb{R}$,

$$\lim_{N\to\infty}\mathbb{E}\left[e^{-\beta_N}\right]\leq\lim_{N\to\infty}\frac{u(x)}{u(N)}=0,$$

from which we conclude that $\beta_N\to\infty$ a.s..

Method for construing $u$: Define the sequence $\{u_n\}_{n\in\mathbb{N}}$ with: $u_0(x) = 1$, and

$$u_n(x) = \begin{cases} 2\int_1^x s(y)\left(\int_1^\theta \dfrac{u_{n-1}}{s(y)\sigma^2(y)}dy\right)\, d\theta \,\text{if}\, x \geq 1, \\ 1 \,\text{if}\, x < 1. \end{cases}$$

For all $n$, $u_n$ is a positive increasing $\mathcal{C}^2$-function satisfying

$$u_n(x) \leq u_1(x)/n!.$$

Taking

$$u(x) = \sum_{n\in\mathbb{N}} u_n(x),\ x \geq 1,$$

$u$ is also is a positive increasing $\mathcal{C}^2$-function which further satisfies:

$$1 + u_1(x) \leq u(x) \leq \exp\{u_1(x)\}$$

Since $u_1(x) = \kappa(x)$ for $x \geq 1$, $\lim_{x\to\infty} u(x) = \infty$.

In addition, since, for all $x \geq 1$

$$
\begin{aligned}
& b(x)u_n'(x) + \frac{\sigma^2(x)}{2} u_n''(x) \\
& = 2b(x)s(x)\sigma(x) \int_1^x \frac{u_{n-1}(y)}{s(y)\sigma^2(y)}\, dy + \sigma^2(x)s'(x) \int_1^x \frac{u_{n-1}(y)}{\sigma^2(y)s(y)}\, dy + u_{n-1}(x) \\
& = u_{n-1}(x),
\end{aligned}
$$

we have

$$b(x)u'(x) + \frac{\sigma^2(x)}{2} u''(x) = u(x).$$

**Example of "blowing" SDE:** Any solution to $dX_t = (1 + |X_t|^2)\, dt + dB_t$, $dX_t = (1 + |X_t|^2)\, dB_t$ explodes at finite time. More generally, if $b$ is non-decreasing, non-negative and

$$\int_c^\infty \frac{1}{b(r)} < \infty,$$

$$0 < \sigma^2(x) \leq cb(x), \forall\, x \in \mathbb{R}.$$

any solution to the SDE blows up at finite time.

### Theorem 2.16 (Feller's test for one-dimensional SDE)

*Suppose that* $(X_t;\, t \geq 0)$ *solves the SDE*

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t,\ X_{t=0} = x,$$

*where* $(B_t;\, t \geq 0)$ *is a standard Brownian motion,* $x \in \mathbb{R}$*,* $b$ *and* $\sigma$ *are continuous functions in* $\mathbb{R}$ *with* $\sigma^2(x) > 0$*. Then, for any interval*

$$I = (r, l),\ -\infty \leq r < l \leq \infty,$$

$$u(x) = 2\int_c^x s(\theta)\left(\int_c^\theta \frac{1}{s(y)\sigma^2(y)}dy\right)d\theta,\ s(x) = \exp\left\{-\int_c^x \frac{2b(z)}{\sigma^2(z)}\,dz\right\}$$

*for some* $c \in I$*, we have*

$$\mathbb{P}(S = \infty) = 1 \Leftrightarrow \lim_{x \to l^+} u(x) = \lim_{x \to r^-} u(x) = \infty,$$

*where* $S = \inf\{t > 0 \,|\, X_t \notin I\}$*.*

For the proof details, see (KS88) chapter 5.

A similar principle holds true for multidimensional SDE: **Khasminskii's test for multi-dimensional SDE**: Suppose that $(X_t;\, t \geq 0)$ solves the SDE

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \\ X_{t=0} = x, \end{cases}$$

where $(B_t;\, t \geq 0)$ is a $\mathbb{R}^d$-Brownian motion, $x \in \mathbb{R}^d$, $b$ and $\sigma$ are "smooth" functions in $\mathbb{R}^d$. Then

$$\mathbb{P}\left(sup_N \tau_N = \infty\right) = 1 \;\Leftrightarrow\; \lim_{r \to \infty} \kappa(r) = \infty,$$

for $\tau_N = \inf\{t > 0 \,|\, |X_t| > N\}$,

$$\kappa(r) = 2\int_{x_0}^{x} s(\theta)\left(\int_{x_0}^{\theta} \frac{1}{s(y)\Theta^2(y)} dy\right)\, d\theta,$$

$$s(x) = \exp\left\{-\int_{x_0}^{x} \frac{2\mu(z)}{\Theta^2(z)}\, dz\right\},$$

where $x_0$ is close to 0,

$$\mu(r) = sup\left\{\mathrm{Trace}(a(x)) + 2x \cdot b(x) \,|\, x \in \mathbb{R}^d \text{ s.t. } |x|^2 = r\right\},$$

$$\Theta(r) = sup\left\{2x \cdot a(x)x \,|\, x \in \mathbb{R}^d \text{ s.t. } |x|^2 = r\right\}.$$

**Elements of proof:** Defining $R_t = |X_t|^2$,

$$dR_t = \left(2X_t \cdot b(X_t) + X_t a(X_t) X_t\right)\, dt + 2X_t \cdot \sigma(X_t)\, dB_t$$

Taking $u$ as in the proof of Theorem 2.15, it follows that

$$\begin{aligned} d(e^t u(R_t)) = e^{-t} &\left(\left(2X_t \cdot b(X_t) + X_t a(X_t) X_t\right) u'(R_t) + \left(2X_t \cdot a(X_t) X_t\right) u''(R_t) - u(R_t)\right)\, dt \\ &+ 2e^{-t} u'(R_t) X_t \cdot \sigma(X_t)\, dB_t \end{aligned}$$

By property of $u$,

$$\begin{aligned} &\left(2X_t \cdot b(X_t) + X_t a(X_t) X_t\right) u'(R_t) + \left(2X_t \cdot a(X_t) X_t\right) u''(R_t) - u(R_t) \\ &\leq \mu(R_t) u'(R_t) + \Theta(R_t) u''(R_t) - u(R_t) = 0 \end{aligned}$$

from which we deduce that $(e^{-t} u(R_t))$ is a non-negative supermartingale. Getting $\mathbb{P}(\sup_N \tau_N = \infty) = 1$ follows from the same argument as in the one-dimensional case.

# *Weak solution to a SDE and related properties*

## Definition 3.1

*Let $\mu_0$ be a probability measure defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let $b : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ be two given Borel measurable functions. A weak solution or a solution in law of the SDE*

$$\begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t,\ t \geq 0, \\ X_{t=0} = \xi \text{ with } \xi \sim \mu_0, \end{cases} \tag{8}$$

*is a triplet $(\xi', (X_t, W_t;\ t \geq 0)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t;\ t \geq 0)$, where*
*(i) $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ is a filtered probability where $(\mathcal{F}_t;\ t \geq 0)$ satisfies the usual conditions,*
*(ii) $(X_t;\ t \geq 0)$ is a continuous $\mathcal{F}_t$-adapted $\mathbb{R}^d$-valued process, $(W_t;\ t \geq 0)$ is a $\mathcal{F}_t$-Brownian motion with values in $\mathbb{R}^m$, and $\xi'$ is a r.v., defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of $(W_t;\ t \geq 0)$,*
*(iii)*

$$\mathbb{P}(\xi' \in A) = \mu_0(A),\ \forall\, A \in \mathcal{B}(\mathbb{R}^d),$$

$$a.s.,\ \int_0^t |b(s,X_s)|\,ds + \int_0^t \|\sigma(s,X_s)\|^2\,ds < \infty,\ \forall\, t \geq 0;$$

*(iv) $\mathbb{P}$-a.s.*

$$X_t = \xi' + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s,\ \forall\, t \geq 0.$$

**Comments:** Compared to the definition of a strong solution to the SDE (8), the filtered probability space under which is defined the solution, the Brownian motion and the initial condition which govern the dynamics of the SDE are no more *inputs* but are now part of the solution, and have to be constructed adequately to find a solution to 8.

In particular:

• the filtration $(\mathcal{F}_t;\, t \geq 0)$ is not required to be the filtration related to $(W_t;\, t \geq 0)$ and $\xi'$. Therefore, the solution $(X_t;\, t \geq 0)$ to (8) may depend on others information than the Brownian motion and the initial condition $\xi$;

• the existence of a weak solution does not guarantee that, for all initial condition $\xi$ and all Brownian motion $(W_t;\, t \geq 0)$, we are able to construct a process satisfying the dynamic

$$X_t = \xi + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \,\forall\, t \geq 0.$$

In this way, the existence of a strong solution to (8) implies the existence of a weak solution to (8) but the reciprocal is not, in general, true.

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## Definition 3.2

*A SDE of the form*

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t,\ t \geq 0 \\ X_{t=0} = \xi \sim \mu_0, \end{cases} \tag{9}$$

*is said to have a unique weak solution i.f.f., for any couple of weak solutions to* (9)*:*

$$(\xi, (X_t, W_t;\ t \geq 0)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t;\ t \geq 0),$$

*and*

$$(\tilde{\xi}, (\tilde{X}_t, \tilde{W}_t;\ t \geq 0)), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), (\tilde{\mathcal{F}}_t;\ t \geq 0),$$

*then* $(X_t;\ t \geq 0)$ *and* $(\tilde{X}_t;\ t \geq 0)$ *have the same distribution on* $\mathcal{C}([0, \infty); \mathbb{R}^d)$*; that is, for all* $A \in \mathcal{B}(\mathcal{C}([0, \infty); \mathbb{R}^d))$*,*

$$\mathbb{P}((X_t;\ t \geq 0) \in A) = \tilde{\mathbb{P}}((\tilde{X}_t;\ t \geq 0) \in A).$$

**Comment:** Compared to the notion of uniqueness for strong solution to (9), the notion of uniqueness in law only relies on the law generated by the paths of the solution rather than the path itself. Again strong uniqueness implies weak uniqueness. For instance if any solution $(X_t;\ t\geq 0)$ to (9) is of the form $(\Phi(\xi,(W_r;\ 0\leq r\leq t);\ t\geq 0)$ then naturally two weak solutions generate the same probability measure on $\mathcal{C}([0,\infty);\mathbb{R}^d)$.
Conversely, starting from the notion of weak solution, (9) admits a unique strong solution i.f.f.

$$(\xi',(X_t,W_t;\ t\geq 0)),(\Omega,\mathcal{F},\mathbb{P}),(\mathcal{F}_t;\ t\geq 0),$$
$$(\tilde{\xi}',(\tilde{X}_t,\tilde{W}_t;\ t\geq 0)),(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}),(\tilde{\mathcal{F}}_t;\ t\geq 0),$$

with

$$\left((\xi',(W_t;\ t\geq 0)),(\Omega,\mathcal{F},\mathbb{P})\right)=\left((\tilde{\xi}',(\tilde{W}_t;\ t\geq 0)),(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})\right),$$

then

$$\mathbb{P}(X_t=\tilde{X}_t;\ t\geq 0)=1.$$

**A very simple example:** Given $\xi\sim\mu_0$, then $(\xi+W_t;\ t\geq 0)$ and $(\xi-W_t;\ t\geq 0)$ have different path, but, by symmetry of the Brownian motion, generated the same law on $\mathcal{C}([0,\infty);\mathbb{R}^m)$.

**An example of a SDE which admits a weak solution but not a strong solution**: Consider the SDE

$$dX_t = sign(X_t)\, dW_t,\ X_0 = 0. \tag{10}$$

where $(W_t;\ t \geq 0)$ is a standard Brownian motion and

$$sign(x) = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ if } x \leq 0. \end{cases}$$

• **Uniqueness of a weak solution to** (11): Observe, that if

$$(\xi, (X_t, W_t;\ t \geq 0)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t;\ t \geq 0),$$

is a weak solution to (11), then $(X_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale, and, for all $t \geq 0$,

$$\langle X \rangle_t = \int_0^t |sign(X_s)|^2\, ds = t.$$

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• **Existence of a weak solution to** (11): Let $(X_t;\ t \geq 0)$ be a standard Brownian motion defined on a space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}^X_t;\ t \geq 0)$ be the (augmented) filtration generated by $(X_t;\ t \geq 0)$ and define $(W_t;\ t \geq 0)$ as

$$W_t = \int_0^t \operatorname{sign}(X_s)\, dX_s,\ \ t \geq 0.$$

Then, one can check that

$$(W_t;\ t \geq 0) \text{ is a standard Brownian motion,}$$

and that

$$\begin{aligned}\int_0^t \operatorname{sign}(X_s)\, dW_s &= \int_0^t \operatorname{sign}(X_s)\, (\operatorname{sign}(X_s)\, dX_s)\\ &= \int_0^t (\operatorname{sign}(X_s))^2\ dX_s = X_t.\end{aligned}$$

We can conclude that

$$(0, (X_t, W_t;\ t \geq 0), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}^X_t;\ t \geq 0)$$

is a weak solution to (11).

**Uniqueness of strong solution to SDE** (11) **doesn't hold true**: Let $(X_t;\, t \geq 0)$ be a strong solution to (11) generated by a given standard Brownian motion $(W_t;\, t \geq 0)$. One can observe that $(-X_t;\, t \geq 0)$ is also a solution to (11) generated by $(W_t;\, t \geq 0)$. Since the solution of (11) cannot be 0 on $[0, \infty)$, this contradicts the property of the uniqueness of a strong solution to (11).

**The SDE** (11) **doesn't admit a strong solution**:
*Preliminary*:

**Theorem 3.3 (Tanaka's formula for the one-dimensional Brownian motion (see (KS88), Chapter 3))**

*Let $(W_t;\, t \geq 0)$ be a standard Brownian motion, $z \in \mathbb{R}$ and let $x \mapsto (x)^- := (-x) \vee 0$ be the negative part function. Then, for all $a \in \mathbb{R}$, $(X_t = W_t + z;\, t \geq 0)$ there exists a continuous non-decreasing $\mathcal{F}^X_t$-adapted process $(L^X_t(a);\, t \geq 0)$ such that*

$$(X_t - a)^- = (z - a)^- - \int_0^t \mathbb{1}_{\{X_s \leq a\}}\, dW_s + L^X_t(a)$$

*and*

$$|X_t - a| = |z - a| + \int_0^t sign(X_s - a)\, dW_s + 2L^X_t(a),\ t \geq 0.$$

The Tanaka formula extends the classical Itô formula

$$f(X_t - a) = f(z - a) + \int_0^t f'(X_s - a)\, dW_s + \frac{1}{2}\int_0^t f''(X_s - a)\, ds$$

to the case of $f(x) = |x - a|$. In particular, the process $(L^X_t(a);\, t \geq 0)$ describes the amount of time that $(X_t;\, t \geq 0)$ spend at the level $x = a$ and is referred to as the local time of $(X_t;\, t \geq 0)$ at $a$.

Suppose that there exists a strong solution $(X_t;\ t \geq 0)$ generated by a given $\mathbb{R}$-Brownian motion $(W_t;\ t \geq 0)$. This solution must be $\mathcal{F}_t^W$-adapted and we have $\mathcal{F}_t^X \subset \mathcal{F}_t^W$. On one hand, observe, that

$$\int_0^t sign(X_s)\, dX_s = \int_0^t sign^2(X_s)\, dW_s = \int_0^t dW_s = W_t.$$

On the other hand, applying the Tanaka formula

$$|X_t| = \int_0^t sign(X_s)\, dX_s + 2L_t^X(0) = W_t + 2L_t^X(0)$$

where $(L_t^X(0);\ t \geq 0)$ is the local time of $X$ in 0 given by

$$L_t^X(0) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{|X_s| \leq \epsilon\}}\, ds.$$

Denoting $(\mathcal{F}_t^{|X|};\ t \geq 0)$, the augmented filtration of $(|X_t|;\ t \geq 0)$, $L_t^X(0)$ is $\mathcal{F}_t^{|X|}$-adapted. Consequently, $(W_t;\ t \geq 0)$ is $\mathcal{F}_t^{|X|}$-adapted which contradicts the fact that $\mathcal{F}_t^X \subset \mathcal{F}_t^W$.

## Yamada-Watanabe result on the link between weak and strong solution

Yamada and Watanabe (1971) investigated the link between weak and strong solution whenever a SDE satisfies a strong uniqueness condition

**Proposition 3.4 (Yamada-Watanabe 1971)**

*Suppose that there exists a weak solution to*

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \\ X_{t=0} = \xi \sim \mu, \end{cases}$$

*If, furthermore strong uniqueness holds true, then weak uniqueness holds true.*

**Corollary 3.5**

*If there exists a weak solution to* (3.4) *and if* (3.4) *admits at most one strong solution then there exists a unique strong solution to* (3.4).

**Preliminary: Regular conditional probability:**

**Definition 3.6**

*A complete metric space $(E, d_E)$ is said to be separable if there exists a countable space $F$ which is dense in $E$; that is, for all*

**Examples:** $E = \mathbb{R}$ $(F = \mathbb{Q})$; $E = \mathbb{R}^d$ $(F = \mathbb{Q}^d)$; $E = \mathcal{C}([0, T]; \mathbb{R})$ equipped with $d_E(f) = \max_{0\leq t\leq T} |f(t)|$ ($F =$space of polynomial functions with rational coefficients).

**Definition 3.7**

*Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ be a $\sigma$-algebra. A function $\mathbb{Q} : (\omega, A) \in \Omega \times \mathcal{F} \mapsto \mathbb{Q}(\omega, A) \in [0, 1]$ is a (regular conditional probability) for $\mathcal{F}$ given $\mathcal{G}$ if the following properties holds true:*
*(1) For all $\omega \in \Omega$, $\mathbb{Q}(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$;*
*(2) For all $A \in \mathcal{F}$, $\omega \mapsto \mathbb{Q}(\omega, A)$ is $\mathcal{G}$-measurable;*
*(3) For all $A \in \mathcal{F}$ and for $\mathbb{P}$-a.a. $\omega \in \Omega$, $\mathbb{Q}(\omega, A) = \mathbb{P}(A \,|\, \mathcal{G})(\omega)$ where $\mathbb{P}(\cdot \,|\, \mathcal{G})$ is the conditional probability of $\mathbb{P}$ given $\mathcal{G}$.*

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(3) *For all $A \in \mathcal{F}$ and for $\mathbb{P}$-a.a. $\omega \in \Omega$, $\mathbb{Q}(\omega, A) = \mathbb{P}(A \,|\, \mathcal{G})(\omega)$ where $\mathbb{P}(\cdot \,|\, \mathcal{G})$ is the conditional probability of $\mathbb{P}$ given $\mathcal{G}$.*

## Theorem 3.8 (Construction of regular conditional distribution)

*Let $(E,\mathcal{E})$ be a measurable space where $E$ is a complete separable metric space and $E = \mathcal{B}(E)$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space where $\Omega$ is a complete separable (that is $\Omega$ has a numerable basis) metric space and $\mathcal{F} = \mathcal{B}(\Omega)$. For any r.v. $Y : (\Omega,\mathcal{F}) \to (E,\mathcal{E})$, there exists a function*

$$Q : (x, A) \in E \times \mathcal{F} \mapsto \mathbb{Q}(\omega; A) \in [0,1]$$

*such that*
(1) *For all $x \in E$, $\mathbb{Q}(x;\cdot)$ is a probability measure on $(\Omega,\mathcal{F})$;*
(2) *For all $A \in \mathcal{F}$, $x \mapsto \mathbb{Q}(x; A)$ is $\mathcal{E}$-measurable;*
(3) *Given $A \in \mathcal{F}$, for $\mathbb{P} \circ Y^{-1}$-a.e. $x \in E$, $\mathbb{Q}(x; A) = \mathbb{P}(A \,|\, Y = x)$.*
*Such function is called the regular conditional probability of $\mathcal{F}$ given $Y$ and is unique in the sense that if there exists an other function $\tilde{\mathbb{Q}}$ which satisfies $(1)-(3)$ then*

$$\text{for}\, \mathbb{P} \circ Y^{-1}\text{-a.e. } x \in E,\ \mathbb{Q}(x; A) = \tilde{\mathbb{Q}}(x; A),\ \forall A \in \mathcal{E}.$$

*If, additionally, $E$ is also a complete separable metric space and if $\mathcal{E} = \mathcal{B}(E)$ then, for $\mathbb{P} \circ Y^{-1}$-a.e. $x \in E$,*

$$\mathbb{Q}(x; \{\omega \in \Omega \,|\, Y(\omega) \in B\}) = \mathbb{1}_{\{x \in B\}},\ \forall B \in \mathcal{E}.$$

**Proof of Proposition 3.4:** Let

$$(\xi^1,(X^1_t,W^1_t;\ t\geq 0)),(\Omega^1,\mathcal{F}^1,\mathbb{Q}^1),(\mathcal{F}^1_t;\ t\geq 0),$$

and

$$(\xi^2,(X^2_t,W^2_t;\ t\geq 0)),(\Omega^2,\mathcal{F}^2,\mathbb{Q}^2),(\mathcal{F}^2_t;\ t\geq 0),$$

be two weak solutions to (3.4). It is enough to show that, for all $0\leq T<\infty$, $(X^1_t;\ 0\leq t\leq T)$ and $(X^2_t;\ 0\leq t\leq T)$ have the same distributions:

$$\mathbb{Q}^1\circ(X^1_t;\ 0\leq t\leq T)^{-1}=\mathbb{Q}^2\circ(X^2_t;\ 0\leq t\leq T)^{-1}.$$

Set $Q^1$ and $Q^2$ as

$$Q^i:=\mathbb{Q}^i\circ(\xi^i,(W^i_t;\ 0\leq t\leq T),(X^i_t-\xi^i;\ 0\leq t\leq T))^{-1},\ i=1,2.$$

$Q^1$ and $Q^2$ generate two probability measures on the sample space of $(\xi,(W_t;\ 0\leq t\leq T),\ (X_t-\xi;\ 0\leq t\leq T))$:

$$\Omega:=\mathbb{R}^d\times\mathcal{C}_0([0,T];\mathbb{R}^m)\times\mathcal{C}_0([0,T];\mathbb{R}^d),\ \ \mathcal{C}_0([0,T];\mathbb{R}^d)=\{f\in\mathcal{C}([0,T];\mathbb{R}^d)\,|\,f(0)=0\},$$

equipped by the $\sigma$-algebra $\mathcal{F}:=\mathcal{B}(\Omega)$. We equip $(\Omega,\mathcal{F})$ with the augmented filtration generated by the canonical process on $\Omega$:

$$(x_0,(w(t);\ 0\leq t\leq T),(y(t);\ 0\leq t\leq T)).$$

For $i = 1, 2$, under $Q^i$, $(w(t);\, 0 \leq t \leq T)$ is a $\mathbb{R}^m$-Brownian motion,

$$Q^i - \text{a.s., } y(t) = \int_0^t b(s, y(s) + x_0)\, ds + \int_0^t \sigma(s, y(s) + x_0)\, dw(s),\ 0 \leq t \leq T,$$

and, for all $A \in \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d))$,

$$Q^i((y(t) + x_0;\, 0 \leq t \leq T) \in A) = \mathbb{Q}^i((X_t^i;\, 0 \leq t \leq T) \in A).$$

In addition, set

$$E := \mathbb{R}^d \times \mathcal{C}_0([0, T]; \mathbb{R}^m),\ \mathcal{E} := \mathcal{B}(E),$$

and $\mathbb{P}_W$, the Wiener probability measure on $\mathcal{C}([0, T]; \mathbb{R}^m)$.

Since $E$ is a complete separable metric space, we can define

$$Q^i : (x_0, w, A) \in E \times \mathcal{F} \mapsto Q^i(x_0, w; A) \in [0,1],\ i = 1, 2,$$

the regular conditional probability measure of $\mathcal{F}$ given $(x_0, (w(t);\, 0 \leq t \leq T))$ and the probability measure

$$P(A \times B) = \int_{(x_0,w)\in E} Q^1(x_0, w; A) Q^2(x_0, w; B) \mathbb{P}_W(dw)\mu(dx_0)$$

defined on $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), (\tilde{\mathcal{F}}_t;\, 0 \leq t \leq T))$ where

$$\tilde{\Omega} := \mathcal{C}([0, T]; \mathbb{R}^d) \times \mathcal{C}([0, T]; \mathbb{R}^d),$$

and

$$\tilde{\mathcal{F}}_t = \tilde{\mathcal{G}}_{t^+},\ \tilde{\mathcal{G}}_t := \sigma(\mathcal{G}_t \cup \mathcal{N}),\ \mathcal{G}_t := \sigma(\{x_0, w(r), y^1(r), y^2(r),\, 0 \leq r \leq t\}),$$

for

$$(x_0, (w(t);\, 0 \leq t \leq T), (y^1(t);\, 0 \leq t \leq T), (y^2(t);\, 0 \leq t \leq T))$$

the canonical process of $\tilde{\Omega}$.

We can observe that, under $P$, $(w(t);\, 0 \leq t \leq T)$ is a $\mathbb{R}^d$-Brownian motion and

$$y^i(t) = \int_0^t b(s, y^i(s) + x_0)\, ds + \int_0^t \sigma(s, y^i(s) + x_0)\, dw(s),\, 0 \leq t \leq T,\, i = 1, 2.$$

By assumption, the SDE
admits at most one strong solution. Therefore, we have

$$P-\text{a.s.}, y^1(t) = y^2(t), \forall\, 0 \leq t \leq T,$$

which implies that $\forall A \in \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d))$,

$$\begin{aligned}
Q^1((y(t) + x_0;\, 0 \leq t \leq T) \in A) &= P(\{(y^1(t) + x_0;\, 0 \leq t \leq T) \in A\} \times \mathcal{C}([0, T]; \mathbb{R}^d)) \\
&= P(\mathcal{C}([0, T]; \mathbb{R}^d) \times \{(y^2(t) + x_0;\, 0 \leq t \leq T) \in A\}) \\
&= Q^2((y(t) + x_0;\, 0 \leq t \leq T) \in A).
\end{aligned}$$

Then, $\forall A \in \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d))$,

$$\begin{aligned}\mathbb{Q}^1((X^1_t;\, 0\leq t\leq T)\in A) &= Q^1((y(t)+x_0;\, 0\leq t\leq T)\in A)\\ &= Q^2((y(t)+x_0;\, 0\leq t\leq T)\in A)\\ &= \mathbb{Q}^2((X^2_t;\, 0\leq t\leq T)\in A).\end{aligned}$$

which implies the weak uniqueness of (3.4).

**Elements of proof for Corollary 3.5:** The main idea of the demonstration is to show the existence of a Borel measurable function

$$h : \mathbb{R}^d \times \mathcal{C}([0,T];\mathbb{R}^d) \to \mathcal{C}([0,T];\mathbb{R}^d)$$

which, for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ under which are defined a $\mathbb{R}^d$-Brownian motion $(W_t;\, 0 \leq t \leq T)$ and an independent r.v. $\xi$, the process

$$(X_t;\, 0 \leq t \leq T) = h(\xi, (W_t;\, 0 \leq t \leq T))$$

is solution to the SDE (3.4). The existence of such function follows the same arguments as in Proposition 3.4.

We define

$$E := \mathbb{R}^d \times \mathcal{C}_0([0,T];\mathbb{R}^m),\, \mathcal{E} := \mathcal{B}(E),$$

and we endow $(E, \mathcal{B}(E))$ with product measure $\mu \otimes \mathbb{P}_W$ for $\mathbb{P}_W$ the Wiener measure on $\mathcal{C}_0([0,T];\mathbb{R}^d)$.

Since $E$ is a complete separable metric space, there exists

$$Q^i : (x_0, w, A) \in E \times \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d)) \mapsto Q^i(x_0, w; A) \in [0,1],\ i = 1, 2,$$

the regular conditional probability of $\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d))$ given $(x_0, (w(t);\, 0 \leq t \leq T))$. Due to the strong uniqueness assumption, $Q^1(x_0, w; \cdot)$ and $Q^2(x_0, w; \cdot)$ have full support on a single point $k_T$ of $\mathcal{C}([0,T];\mathbb{R}^d)$ in such a way (see details in (KS88) p. 391) that

$$Q^i(x_0, w; A) = \mathbb{1}_{\{k_T(x_0, w) \in A\}},$$

$k$ is measurable (by property of $Q^i$, $i = 1, 2$,) and

$$P\left(\left\{\omega \in \tilde{\Omega} \,|\, x^1_\cdot = x^2_\cdot = x_0 + k_T(x_0, w_\cdot)\right\}\right) = 1.$$

In particular, the process

$$(x_t;\, 0 \leq t \leq T) = x_0 + k_T(x_0, (w_t;\, 0 \leq t \leq T))$$

is a strong solution to the SDE (3.4).

**Remark**: The function $k$ exhibited in the proof of Corollary 3.5 characterize the strong solution of (3.4) in the sense that, for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ under which are defined $\xi \sim \mu$ and an independent $\mathbb{R}^m$-valued Brownian motion $(W_t;\, t \geq 0)$, and setting the augmented filtration $(\mathcal{F}_t;\, t \geq 0)$ as $(\mathcal{G}_t = \sigma(\xi, (W_r;\, 0 \leq r \leq t);\, t \geq 0)$, the process $(X_t;\, t \geq 0)$ defined by

$$(X_t;\, 0 \leq t \leq T) = h(\xi, (W_t;\, 0 \leq t \leq T))$$

is the strong solution to the SDE (3.4).

The notion of weak solution enable to introduce particular probabilistic techniques which enable to transform and simplify a SDE of the form

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,\ t \geq 0, \\ X_{t=0} = \xi \sim \mu_0, \end{cases} \tag{11}$$

Hereafter, we review two particular techniques:

- the method of change of probability measure (Girsanov transformation) which enables to remove the drift component in the equation,
- the method of change of time which applies for one-dimensional SDE ($d = m = 1$) and which takes advantage of the scaling property of the Brownian motion

$$(\frac{1}{c} W_{c^2 t};\ t \geq 0) \overset{\mathcal{D}}{=} (W_t;\ t \geq 0),\ c > 0,$$

to simplify the diffusion component.

## The Girsanov transformation

### Theorem 3.9 (Cameron and Martin 1944, Girsanov 1960)

*Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ be a filtered probability space under which are defined a $\mathbb{R}^m$-Brownian motion $(W_t;\ t \geq 0)$ and $(\beta_t;\ t \geq 0)$, a $\mathcal{F}_t$-adapted stochastic process with values in $\mathbb{R}^m$, such that, for all $0 < T < \infty$,*

$$\mathbb{P} - a.s.\ \int_0^T |\beta_s|^2\, ds < \infty.$$

*If the process*

$$Z_t := \exp\left\{\int_0^t \beta_s\, dW_s - \frac{1}{2}\int_0^t |\beta_s|^2\, ds\right\},\ t \geq 0,$$

*is a $\mathcal{F}_t$-martingale under $\mathbb{P}$, then the process*

$$\tilde{W}_t := -\int_0^t \beta_s\, ds + W_t,\ \ t \geq 0,$$

*is a $\mathcal{F}_t$-Brownian motion under the probability $\mathbb{Q}$ defined, in $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0))$, by*

$$\mathbb{Q}\,(A) = \mathbb{E}\left[\mathbb{1}_A Z_T\right],\ A \in \mathcal{F}_T,$$

*or equivalently* $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = Z_T$.

**Preliminary:**

**Definition 3.10**

*Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures defined on the same measurable space $(\Omega, \mathcal{F})$. $\mathbb{Q}$ is said to be absolutely continuous with respect to $\mathbb{P}$ ($\mathbb{Q} \ll \mathbb{P}$) i.f.f., for all $A \in \mathcal{F}$,*

$$\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0.$$

*$\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent ($\mathbb{P} \sim \mathbb{Q}$) i.f.f. $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$.*

**Theorem 3.11 (Radon-Nikodym theorem)**

*If $\mathbb{P} \ll \mathbb{Q}$ then there exists a non-negative function $Y : \Omega \to [0, \infty)$, $\mathcal{F}$-measurable and $\mathbb{P}$-integrable (i.e. $\mathbb{E}_{\mathbb{P}}[Y] < \infty$) such that*

$$\mathbb{Q}(A) = \mathbb{E}\left[\mathbb{1}_A Y\right], \; A \in \mathcal{F}.$$

*In this case, $Y$ is called the density of $\mathbb{Q}$ with respect to $\mathbb{P}$.*

**Notation**: The density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is usually denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

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**Notation**: The density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is usually denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

**A simple case:** Assume that $m = 1$, $(W_t;\ t \geq 0)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $X_t = \lambda \in \mathbb{R}$.

Define

$$\tilde{W}_t := -\lambda t + W_t,\ 0 \leq t \leq T, \qquad Z_t = \exp\left\{\lambda W_t - \frac{|\lambda|^2 t}{2}\right\},\ 0 \leq t \leq T.$$

Then one can easily check that,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[Z_T\right] &= \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\lambda W_T - \frac{|\lambda|^2 T}{2}\right\}\right] \\ &= \frac{1}{(2\pi T)^{\frac{d}{2}}}\int_{\mathbb{R}^m} e^{\lambda x - \frac{|\lambda|^2 T}{2}} e^{-\frac{|x|^2}{2T}}\, dx = 1.\end{aligned}$$

**Exercise**: Show that $(Z_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale.

Defining $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ with

$$d\mathbb{Q}/d\mathbb{P}\big|_{\mathcal{F}_T} = Z_T.$$

$(\tilde{W}_t;\ t \in [0, T])$ is still a stochastic process, $\mathcal{F}_t$-adapted, and since $\mathbb{Q} \ll \mathbb{P}$, $(\tilde{W}_t;\ t \geq 0)$ has continuous paths and starts at 0 for $t = 0$.
In addition, observing that, for all $u \in \mathbb{R}$, $0 \leq s \leq t < \infty$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\exp\{i(\tilde{W}_t - \tilde{W}_s)u\} \,|\, \mathcal{F}_s\right] \\
&= \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}\left[Z_t \exp\{i(\tilde{W}_t - \tilde{W}_s)u\} \,|\, \mathcal{F}_s\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{Z_t}{Z_s}\exp\{i(\tilde{W}_t - \tilde{W}_s)u\} \,|\, \mathcal{F}_s\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\lambda(W_t - W_s) - \frac{|\lambda|^2(t-s)}{2}\right\}\exp\{i(W_t - W_s - \lambda(t-s))u\} \,|\, \mathcal{F}_s\right] \\
&= \exp\left\{\frac{-|\lambda|^2(t-s)}{2} + iu\lambda(t-s)\right\}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{(\lambda + iu)\left(W_t - W_s\right)\right\} \,|\, \mathcal{F}_s\right] = e^{\frac{-(t-s)u^2}{2}},
\end{aligned}$$

one can check that $(\tilde{W}_t;\ t \geq 0)$ is a $\mathcal{F}_t$-Brownian motion under $\mathbb{Q}$.

Defining $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ with

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$(\tilde{W}_t;\ t \in [0, T])$ is still a stochastic process, $\mathcal{F}_t$-adapted, and since $\mathbb{Q} \ll \mathbb{P}$, $(\tilde{W}_t;\ t \geq 0)$ has continuous paths and starts at 0 for $t = 0$.
In addition, observing that, for all $u \in \mathbb{R}$, $0 \leq s \leq t < \infty$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\exp\{i(\tilde{W}_t - \tilde{W}_s)u\} \,|\, \mathcal{F}_s\right] \\
&= \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}\left[Z_t \exp\{i(\tilde{W}_t - \tilde{W}_s)u\} \,|\, \mathcal{F}_s\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{Z_t}{Z_s}\exp\{i(\tilde{W}_t - \tilde{W}_s)u\} \,|\, \mathcal{F}_s\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\lambda(W_t - W_s) - \frac{|\lambda|^2(t-s)}{2}\right\}\exp\{i(W_t - W_s - \lambda(t-s))u\} \,|\, \mathcal{F}_s\right] \\
&= \exp\left\{\frac{-|\lambda|^2(t-s)}{2} + iu\lambda(t-s)\right\}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{(\lambda + iu)\left(W_t - W_s\right)\right\} \,|\, \mathcal{F}_s\right] = e^{\frac{-(t-s)u^2}{2}},
\end{aligned}$$

one can check that $(\tilde{W}_t;\ t \geq 0)$ is a $\mathcal{F}_t$-Brownian motion under $\mathbb{Q}$.

• The general idea of the Girsanov transformation is concentrate in the fact that a stochastic process of the form

$$\tilde{W}_t := -\int_0^t \beta_s \, ds + W_t, \ t \geq 0,$$

can be transformed into a standard Brownian motion providing that we apply a change of probability from $\mathbb{P}$ to $\mathbb{Q}$ for

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\left\{\int_0^T \beta_s \, dW_s - \frac{1}{2}\int_0^T |\beta_s|^2 \, ds\right\}$$

This transformation doesn't change neither the initial measurable space $(\Omega, \mathcal{F})$ nor the filtration $(\mathcal{F}_t;\ t \geq 0)$ under which is defined

$$\int_0^t \beta_s \, ds, \ t \geq 0.$$

Additionally all $\mathbb{P}$-negligible sets are preserved by $\mathbb{Q}$.

• On the condition that $(Z_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale: On the one hand, this condition is a sufficient condition to ensure that $\mathbb{Q}$ is a probability measure since it guarantees that $\mathbb{E}_{\mathbb{P}}[Z_T] = \mathbb{E}_{\mathbb{P}}[Z_0] = 1$ and that

$$\mathbb{Q}(\Omega) = \mathbb{E}_{\mathbb{P}}[Z_T](= 1).$$

• The general idea of the Girsanov transformation is concentrate in the fact that a stochastic process of the form

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can be transformed into a standard Brownian motion providing that we apply a change of probability from $\mathbb{P}$ to $\mathbb{Q}$ for

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• On the condition that $(Z_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale: On the one hand, this condition is a sufficient condition to ensure that $\mathbb{Q}$ is a probability measure since it guarantees that $\mathbb{E}_{\mathbb{P}}[Z_T] = \mathbb{E}_{\mathbb{P}}[Z_0] = 1$ and that

$$\mathbb{Q}(\Omega) = \mathbb{E}_{\mathbb{P}}[Z_T](= 1).$$

On the other hand, the Itô formula gives

$$Z_t = 1 + \int_0^t Z_s \beta_s \, dW_s, \ t \geq 0,$$

so that $(Z_t;\ t \in [0, T])$ is a continuous $\mathcal{F}_t$-local martingale. Taking

$$\tau_M := \inf\{t \geq 0 \,|\, \int_0^t (Z_s)^2 |X_s|^2 \, ds = M\},\ M > 0,$$

we can observe that, for all $0 \leq s \leq t \leq T$

$$\mathbb{E}_{\mathbb{P}}\left[Z_{t \wedge \tau_M} \,|\, \mathcal{F}_s\right] = Z_{s \wedge \tau_M}.$$

Since, $\mathbb{P}$-a.s, $(Z_t)^2$ and $\int_0^t |\beta_s|^2 \, ds$ are finite, $\lim_{M \to \infty} \tau_M = \infty$. By continuity of $t \mapsto Z_t$ and by Fatou's Lemma, it follows that

$$\mathbb{E}_{\mathbb{P}}\left[Z_t \,|\, \mathcal{F}_s\right] \leq Z_s$$

and that $(Z_t;\ t \geq 0)$ is a $\mathcal{F}_t$-super-martingale. Since $Z_t$ is non-negative, the martingale condition of $(Z_t;\ t \geq 0)$ holds true i.f.f.

$$1 = \mathbb{E}_{\mathbb{P}}\left[Z_T\right] \leq \mathbb{E}_{\mathbb{P}}\left[Z_t\right] \leq \mathbb{E}[Z_0] = 1,\ \forall\, t \geq 0.$$

## Criterion to apply the Girsanov transformation

### Lemma 3.12 (Novikov's Criterion (1972))

*Let $(Z_t;\ t \geq 0)$ be as in Theorem 3.9. If*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T |\beta_s|^2\, ds\right\}\right] < \infty,\ \forall 0 < T < \infty,$$

*then, for all $0 \leq t < \infty$, $\mathbb{E}_{\mathbb{P}}[Z_t] = 1$ and $(Z_t;\ t \geq 0)$ is a $\mathcal{F}_t$-martingale under $\mathbb{P}$.*

### Corollary 3.13

*If there exists a real sequence $\{t_n;\ n \in \mathbb{N}\}$, (strictly) increasing such that $\lim_{n\to\infty} t_n = \infty$ and*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_{t_n}^{t_{n+1}} |\beta_s|^2\, ds\right\}\right] < \infty,\ \forall n \geq 0,$$

*then the same conclusion of Lemma 3.12 hold true.*

For the proof details, see (KS88), Chapter 3.

## Criterion to apply the Girsanov transformation

**Lemma 3.14 (Beneš's Criterion (1971))**

*Let $(Z_t;\, t \geq 0)$ be as in Theorem 3.9 and suppose that $(\beta_t;\, t \geq 0)$ is $\mathcal{F}_t$-adapted such that*

$$\beta_t = F(t, (W_r;\, 0 \leq r \leq t))$$

*If, for each $0 \leq T < \infty$, there exists some constant $K_T$ which depends only on $T$, such that $\mathbb{P}$-a.s.,*

$$|F(t, (W_r;\, 0 \leq r \leq t))| \leq K_T \left(1 + \max_{0 \leq r \leq t} |W_t|\right)$$

*then*

$$\mathbb{E}_{\mathbb{P}}\left[Z_t\right] = 1,\, \forall\, t \geq 0,$$

*and $(Z_t;\, t \geq 0)$ is a $\mathcal{F}_t$-martingale under $\mathbb{P}$.*

For the proof details, see (KS88), Chapter 3, page 200.

**Application to the wellposedness of weak solution**

**Proposition 3.15**

*For $0 \leq T < \infty$, consider the SDE*

$$\begin{cases} dX_t = b(t, X_t)\, ds + dW_t,\ t \in [0, T] \\ X_0 = \xi, \end{cases} \tag{12}$$

*If $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel measurable function then there exists a unique weak solution to the SDE.*

**Proof: Existence part:** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ be a filtered probability space under which are defined, independently, a $\mathcal{F}_t$-Brownian motion $(W_t;\ t \geq 0)$ and $\xi \sim \mu_0$. Define then $X_t = \xi + W_t$ and

$$Z_t = \exp\left\{\int_0^t b(s, X_s)\, dW_s - \frac{1}{2}\int_0^t |b(s, X_s)|^2\, ds\right\}$$

Since $b$ is bounded, the Novikov criterion is satisfied and, according to Theorem 3.9,

$$\tilde{W}_t = -\int_0^t b(s, X_s)\, ds + W_t,\ 0 \leq t \leq T$$

is a $\mathcal{F}_t$-Brownian motion under $\mathbb{Q}$ defined with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T.$$

Since $\mathbb{Q}$-a.s.,

$$X_t = \xi + \int_0^t b(s, X_s)\, ds - \int_0^t b(s, X_s)\, ds + W_t = \xi + \int_0^t b(s, X_s)\, ds + \tilde{W}_t,\ 0 \leq t \leq T$$

the triplet

$$(\xi, (X_t, \tilde{W}_t;\ 0 \leq t \leq T)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t;\ 0 \leq t \leq T),$$

is a weak solution to the SDE.

**Uniqueness part:** Let

$$(\xi, (X_t, W_t;\ 0 \leq t \leq T)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t;\ 0 \leq t \leq T),$$

and

$$(\tilde{\xi}, (\tilde{X}_t, \tilde{W}_t;\ 0 \leq t \leq T)), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), (\tilde{\mathcal{F}}_t;\ 0 \leq t \leq T),$$

be two weak solution to (12). Define $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T = \exp\left\{ -\int_0^t b(s, X_s)\, dW_s - \frac{1}{2}\int_0^t |b(s, X_s)|^2\, ds \right\}$$

and

$$\frac{d\tilde{\mathbb{Q}}}{d\tilde{\mathbb{P}}} = \tilde{Z}_T = \exp\left\{ -\int_0^t b(s, \tilde{X}_s)\, d\tilde{W}_s - \frac{1}{2}\int_0^t |b(s, \tilde{X}_s)|^2\, ds \right\},$$

Then,

$$(\xi, (X_t - \xi, Z_t,\ 0 \leq t \leq T)) \text{ and } \left(\tilde{\xi}, \tilde{X}_t - \tilde{\xi}, \tilde{Z}_t,\ 0 \leq t \leq T\right),$$

generate the same distribution on $\mathcal{C}([0, T]; \mathbb{R}^d)$. Since the Novikov criterion is satisfied in both cases, for all $A \in \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d))$,

$$\mathbb{P}(X \in A) = \mathbb{E}_{\mathbb{Q}}\left[ Z_T^{-1} \mathbb{1}_{\{X \in A\}} \right] = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[ \tilde{Z}_T^{-1} \mathbb{1}_{\{\tilde{X} \in A\}} \right] = \tilde{\mathbb{P}}(\tilde{X} \in A),$$

and uniqueness in law holds true for (12).

The existence and uniqueness result in Proposition 3.15 can be extended in the following ways.

### Proposition 3.16

*Given* $0 \leq T < \infty$, *let*

$$(\xi^i, (X^i_t, W^i_t; 0 \leq t \leq T)), (\Omega^i, \mathcal{F}^i, \mathbb{P}^i), (\mathcal{F}^i_t; 0 \leq t \leq T), \, i = 1, 2,$$

*be two weak solutions to*

$$\begin{cases} dX_t = b(t, X_t)\, dt + dW_t, \; t \in [0, T], \\ X_0 = \xi, \end{cases}$$

*If*

$$\mathbb{P}^i \left( \int_0^T |b(s, X^i_s)|^2 \, ds < \infty \right) = 1$$

*then* $(X^1_t, \xi^1, W^1_t; 0 \leq t \leq T)$ *and* $(X^2_t, \xi^2, W^2_t; 0 \leq t \leq T)$ *have the same distribution.*

## Proposition 3.17

*Given $0 < T < \infty$, let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded Borel measurable function and let $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be such that*

$$\underline{m}|V|^2 \leq (V \cdot \sigma(t,x)V) \leq \overline{M}|V|^2, \forall (t,x) \in [0,T] \times \mathbb{R}^d,\ V \in \mathbb{R}^d,$$

*for $0 < \underline{m} < \overline{M} < \infty$. In addition, suppose that there exists a unique weak solution to the SDE*

$$\begin{cases} dX_t = \sigma(t, X_t)\, dW_t,\ t \in [0, T], \\ X_0 = \xi \sim \mu_0. \end{cases}$$

*Then there exists a unique weak solution to*

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,\ t \in [0, T], \\ X_0 = \xi \sim \mu_0. \end{cases}$$

## The method of time change (Reference: (IK81) pages 197-202)

**Preliminary:** Time change for martingales:

**Theorem 3.18 (Dambis 1965, Dubins & Schwarz 1965)**

*Let $(M_t;\, t \geq 0)$ be a continuous $\mathcal{F}_t$-local martingale, with values in $\mathbb{R}$ and such that*

$$\lim_{t\to\infty} \langle M\rangle_t = \infty,\ \mathbb{P}\text{-a.s.}.$$

*Therefore, given the family of stopping time*

$$\beta(s) := \inf\{\theta > 0 \,|\, \langle M\rangle_\theta > s\},\ s \geq 0,$$

*and the filtration $(\mathcal{G}_s := \mathcal{F}_{\beta(s)},\ s \geq 0)$ (which satisfies the usual assumptions), the process*

$$B_s := M_{\beta(s)},\ 0 \leq s < \infty$$

*is a $\mathcal{G}_s$-adapted standard Brownian motion.*

**Note**: Reciprocally, we have

$$M_t = B_{\langle M\rangle_t},\ \forall t \geq 0.$$

**Note:** The main idea of Theorem 3.18 is that a martingale can be transformed into a standard Brownian motion using an appropriate change of time: Although $t \mapsto \langle M \rangle_t$ may not be invertible, the random time $s \mapsto \beta(s)$ define the generalized inverse of $t \mapsto \langle M \rangle_t$ in the sense that (see (KS88), Problems 4.5 (*ii*) and (*iii*)), we have

$$\langle M \rangle_{\beta(s)} = s,$$

and

$$\beta(\langle M \rangle_t) = \sup\{\theta \geq t \,|\, \langle M \rangle_\theta = \langle M \rangle_t\}.$$

Heuristically, since

$$B_s = \int_0^{\beta(s)} dM_r = \int_0^{\infty} \mathbb{1}_{\{r \leq \beta(s)\}} dM_r,$$

$$\langle B \rangle_s = \int_0^{\infty} \left(\mathbb{1}_{\{r \leq \beta(s)\}}\right)^2 d\langle M \rangle_r = \langle M \rangle_{\beta(s)} = s$$

and, since the martingale property of $(M_t, \mathcal{F}_t;\, t \geq 0)$ is preserved with $(M_{\beta(s)}, \mathcal{F}_{\beta(s)};\, s \geq 0)$, $(M_{\beta(s)};\, s \geq 0)$ is a $\mathcal{F}_{\beta(s)}$-Brownian motion, by the Levy characterization of a Brownian motion.

We can also observe that

$$\{\langle M\rangle_t \leq s\} = \{\beta(s) \geq t\} \text{ and } \{\langle M\rangle_t > s\} = \{\beta(s) < t\}.$$

Then $\{\beta(s);\ s \geq 0\}$ are random times and, since $(\mathcal{F}_t;\ t \geq 0)$ satisfies the usual assumptions and since $\beta(0) = 0$ and $s \mapsto \beta(s)$ is strictly increasing and right-continuous, $(\mathcal{G}_s;\ s \geq 0)$ also satisfies the usual assumptions. Next, considering $0 \leq s_1 < s_2 < \infty$ and

$$\tilde{M}_t := M_{\beta(s_2)\wedge t},\ t \geq 0,$$

in such a way that, for all $t \geq 0$,

$$\langle \tilde{M}\rangle_t = \langle M\rangle_{t\wedge\beta(s_2)} \leq \langle M\rangle_{\beta(s_2)} = s_2.$$

It follows that $(\tilde{M}_t;\ t \geq 0)$ and $(\tilde{M}_t^2 - \langle \tilde{M}\rangle_t;\ t \geq 0)$ are uniformly integrable.

## Definition 3.19

(a) *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ *be a filtered probability space. A process of time change is a stochastic process* $(\phi_t;\ t \geq 0)$, $\mathcal{F}_t$*-adapted, with values in* $[0, \infty)$ *such that,* $\mathbb{P}$*-a.s.*

- $\phi_0 = 0$,
- $t \mapsto \phi_t$ *is continuous and (strictly) increasing,*
- $\lim_{t\to+\infty} \phi_t = +\infty$.

(b) *To any process of time change, we can associate its inverse* $\left(\phi_t^{-1};\ t \geq 0\right)$ *defined as*

$$\phi_t^{-1} = \inf\{s \geq 0 \,|\, \phi_s > t\}$$

*which is a* $\mathcal{F}_t$*-stopping time, for all* $t$.

(c) *Given* $(X_t;\ t \geq 0)$ *a continuous,* $\mathcal{F}_t$*-adapted process, and* $(\phi_t;\ t \geq 0)$ *a process of time change,*

$$\mathbf{T}_t^{\phi}(X) = X_{\phi_t^{-1}},\ t \geq 0,$$

*is the process* $X$ *changed in time by* $\phi$.

## Theorem 3.20

*Let $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous positive function such that*

$$C_1 \leq \sigma(t, x) \leq C_2, \forall (t, x) \in [0, \infty) \times \mathbb{R},$$

*for some constants $0 < C_1 < C_2 < \infty$. In addition, let $(\Omega, \mathcal{G}, (\mathcal{G}_t;\, t \geq 0), \mathbb{P})$ be a filtered probability space under which are defined a $\mathbb{R}$-Brownian motion $(B_t;\, t \geq 0)$ and a r.v. $\xi$ independent of each other. Defining $Y_t = \xi + B_t$ and assuming that there exists a process of time change $(\phi_t;\, t \geq 0)$ such that, $\mathbb{P}$-a.s.,*

$$\phi_t = \int_0^t \frac{1}{\sigma^2(\phi_s, Y_s)}\, ds, \forall\, t \geq 0. \tag{13}$$

*Therefore, for $\mathcal{F}_t := \mathcal{G}_{\phi_t^{-1}}$, the process $X_t := Y_{\phi_t^{-1}}$ satisfies the equation*

$$X_t = \xi + \int_0^t \sigma(s, X_s)\, dW_s$$

*for some $\mathcal{G}_t$-Brownian motion $(W_t;\, t \geq 0)$.*

**Theorem 3.21**

*If there exists a process of time change satisfying* (13) *in Theorem 3.20, then the SDE*

$$dX_t = \sigma(t, X_t)\, dW_t,\ X_0 = \xi,$$

*admits at most a unique solution in law.*

Uniqueness of the time change process have to be understood in the sense that if there exists an other process of time change $(\psi_t;\ t \geq 0)$ satisfying (13) then

$$\mathbb{P} - \text{a.s.}, \phi_t = \psi_t, \forall t \geq 0.$$

**Application:** Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function such that

$$\sigma(x) \geq C > 0, \ \forall x \in \mathbb{R}.$$

Then there exists a unique weak solution to

$$\begin{cases} dX_t = \sigma(X_t)\, dW_t, \ t \geq 0, \\ X_0 = \xi \sim \mu_0. \end{cases}$$

**Elements of proof**: According to Theorem 3.20, the problem of construing a solution to the SDE reduces to establish the existence of a process of time change such that

$$\phi_t = \int_0^t \frac{1}{\sigma^2(Y_s)}\, ds,$$

for $Y_t = \xi + B_t$ defined on a filtered probability space $(\Omega, \mathbb{F}, (\mathcal{G}_t;\ t \geq 0), \mathbb{P})$. Indeed, taking since $\sigma$ is (strictly) bounded and positive, $(\phi_t;\ t \geq 0)$ is a process of time change. Then, setting

$$\mathcal{F}_t = \mathcal{G}_{\phi_t^{-1}},\ X_t = Y_{\phi_t^{-1}},\ t \geq 0,$$

and applying Theorem 3.20, the triplet

$$(\xi, (X_t,\ Y_t - \xi)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t,\ t \geq 0)$$

is the weak solution to the SDE. Uniqueness in law holds true applying Theorem 3.21.

# *Some Stochastic Differential equations in Physics and Finance*

## The Ornstein-Uhlenbeck process

- **The one-dimensional Ornstein-Uhlenbeck process**

$$(O.U.)\ \begin{cases} dX_t = -\beta(X_t - a)\,dt + \sigma dW_t, \\ X_0 = \xi \end{cases} \tag{14}$$

where $(W_t;\ t \geq 0)$ is a $\mathbb{R}$-Brownian motion, $\beta > 0, a \in \mathbb{R}$ and $\sigma \neq 0$.

**Deterministic case**: If $\sigma = 0$ and $\xi = x_0$ is deterministic:

$$X_t = e^{-\beta t}x_0 + a(1 - e^{-\beta t}) \text{ and } \lim_{t \to +\infty} X_t = a.$$

In the equation $(O.U.)$, the drift component model a friction and attractive force where

- the term $-X_t$ describes a friction force,
- the term $a$ corresponds to a mean of attraction,
- and the term $\frac{1}{\beta}$ defines a characteristic time of return toward $a$.

According to Theorems 1 and 2, the SDE $(O.U.)$ admits a unique strong solution. Furthermore, applying Theorem 4, we can observe that the solution of the SDE is explicitly given by

$$X_t = e^{-\beta t}\xi + a(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)}\, dW_s.$$

Owing to its explicit formulation, for all $t \geq 0$, $X_t$ is a Gaussian random variable with mean

$$\mathbb{E}\left[X_t\right] = e^{-\beta t}\mathbb{E}\left[\xi\right] + a(1 - e^{-\beta t})$$

and variance

$$\mathrm{Var}\left[X_t\right] = e^{-2\beta t}\mathrm{Var}\left[\xi\right] + \sigma^2 \left(\frac{1 - e^{-2\beta t}}{2\beta}\right).$$

The Ornstein-Uhlenbeck process defines a prototypical example of a Gaussian dynamic which admits a stationary distribution, that is, there exists a nonnegative measure $\nu$ such that, if for any $\xi$, $\mathcal{L}(X_t) = \nu$ as $t \to \infty$, and if $\xi \sim \nu$ then $\mathcal{L}(X_t) = \nu$ for all $t > 0$.

The Ornstein-Uhlenbeck process is used in Finance to model rate of interest and in physics for Lagrangian model.

**The multidimensional Ornstein-Uhlenbeck process:** Given $B$ a matrix of size $d \times d$, $\sigma$ a matrix of size $m \times d$, consider

$$(O.U.m.) \begin{cases} dX_t = -BX_t \, dt + \sigma dW_t, \\ X_0 = \xi. \end{cases} \tag{15}$$

where $(W_t;\, t \geq 0)$ is a $\mathbb{R}^d$-Brownian motion. Then, as in the one-dimensional case, there exists a unique strong solution to the SDE, which is given by

$$X_t = e^{-Bt}\xi + \int_0^t e^{-B(t-s)}\sigma \, dW_s.$$

where

$$e^{-Bt} = \sum_{k=0}^{\infty} \frac{(-t)^k (B)^k}{k!},\ B^k = B \times B \cdots B \,(\text{product of matrices}).$$

## The Langevin model

**The Langevin model:** Introduced by P. Langevin in 1908, the Langevin model was one of the first probabilistic dynamic introduced in Physics and is defined by

$$(L)\ \begin{cases} dX_t = U_t dt, \\ dU_t = -\beta U_t dt + \sigma dW_t, \\ X_0 = \xi_1,\ U_0 = \xi_2. \end{cases} \tag{16}$$

The model describes, at each time $t$, the position $X_t$ and the velocity $U_t$ of a particle submitted to a friction force $-\beta U_t$ and a "continuous white noise" modeling microscopic changes of the particle. The SDE provides a prototypical form of stochastic mechanics based on a "probabilistic" analogous of the Newton law:

$$x''_t = F(x_t, x''_t) + \frac{dW_t}{dt},$$
$$F(t, x, x') = -\beta x',\ \frac{dW_t}{dt} = \text{white noise.}$$

$(L)$ admits a unique strong solution given by

$$U_t = e^{-\beta t}\xi_2 + \sigma \int_0^t e^{-\beta(t-s)}\, dW_s \text{ and}$$

$$X_t = \xi_1 + \int_0^t U_s\, ds = \xi_1 + \xi_2 \left( \frac{1 - e^{-\beta t}}{\beta} \right) + \sigma \int_0^t e^{-\beta s} \int_0^s e^{\beta r} dW_r\, ds.$$

At each time $t$, $Z_t := (X_t, U_t)$ is a Gaussian vector such that

$$\mathbb{E}[Z_t] = \left( \mathbb{E}_{\mathbb{P}}[\xi_1] + \mathbb{E}[\xi_2] \left( \frac{1 - e^{-\beta t}}{\beta} \right), e^{-\beta t}\mathbb{E}[\xi_2] \right) \text{ and}$$

$$\mathrm{Cov}(Z_t) = \sigma^2 \times$$

$$\begin{pmatrix} \mathbb{E}\left[ \left( \int_0^t e^{-\beta s} \int_0^s e^{\beta r} dW_r\, ds \right)^2 \right] & \mathbb{E}\left[ \int_0^t e^{-\beta(t-s)}\, dW_s \cdot \int_0^t e^{-\beta s} \int_0^s e^{\beta r} dW_r\, ds \right] \\ \mathbb{E}\left[ \int_0^t e^{-\beta(t-s)}\, dW_s \cdot \int_0^t e^{-\beta s} \int_0^s e^{\beta r} dW_r\, ds \right] & \mathbb{E}\left[ \left( \int_0^t e^{-\beta(t-s)}\, dW_s \right)^2 \right] \end{pmatrix}$$

$$= \frac{\sigma^2}{\beta^2} \begin{pmatrix} 2\sinh(\beta t) & 2e^{-\beta t/2}\sinh(\beta t/2) + e^{-\beta t}\sinh(\beta t) \\ 2e^{-\beta t/2}\sinh(\beta t/2) + e^{-\beta t}\sinh(\beta t) & 2e^{-\beta t}\sinh(\beta t) \end{pmatrix}.$$

**An extended Langevin model:** Let $(X_t, U_t;\ t \geq 0)$ be a $\mathbb{R} \times \mathbb{R}$-valued stochastic process such that

$$Z_t = (X_t, U_t),\ t \geq 0,$$

is solution to the SDE

$$(L_{ext})\ \ dZ_t = AZ_t\,dt + KdB_t,\ Z_0 = \xi,$$

for $(B_t;\ t \geq 0)$ a $\mathbb{R}^2$-Brownian motion

$$A = \begin{pmatrix} 0 & 1 \\ \lambda & \beta \end{pmatrix}, K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe, that, by the Itô formula,

$$d\left(e^{-At}Z_t\right) = -Ae^{At}Z_t\,dt + e^{At}dZ_t = e^{At}K\,dB_t = (Ke^{At})^*dB_t.$$

Therefore, the unique solution to $(L_{ext})$ is $Z_t = e^{At}\xi + \int_0^t e^{A(t-s)}K\,dB_s,\ t \geq 0$.
**Application: The oscillating Lanvegin model:** (see (KS88) page 362)

$$(L.o)\ \begin{cases} dX_t = U_t dt, \\ dU_t = -\alpha X_t\,dt - \beta U_t dt + \sigma dW_t, \\ X_0 = \xi_1,\ U_0 = \xi_2, \end{cases} \tag{17}$$

for $\alpha > 0, \beta > 0$ and $\sigma \neq 0$.

**Methods for simplifying the coefficients of a SDE: The Lamperti transformation**. (See (O00), Chapter 5).

**General idea**: Let $(X_t;\, t \geq 0)$ be a solution (weak or strong) to the one-dimensional SDE:

$$X_t = \xi + \int_0^t b(X_s)\, ds, + \int_0^t \sigma(X_s)\, dW_s,\ t \geq 0,$$

where $\xi \sim \mu_0$ and $(W_t;\, t \geq 0)$ is a standard Brownian motion. For $\psi : \mathbb{R} \to \mathbb{R}$of class $\mathcal{C}^2$ on $\mathbb{R}$, by Itô's formula, we have

$$\begin{aligned}\psi(X_t) &= \psi(X_0) + \int_0^t \psi'(X_s)\, dX_s + \frac{1}{2}\int_0^t \psi''(X_s)\, d\langle X\rangle_s \\ &= \psi(\xi) + \int_0^t \psi'(X_s)\, b(X_s) ds + \int_0^t \psi'(X_s)\, \sigma(X_s) dW_s + \frac{1}{2}\int_0^t \psi''(X_s)\sigma^2(X_s)\, ds\end{aligned}$$

Therefore, the dynamic of $(Z_t = \psi(X_t);\, t \geq 0)$ is characterized by

$$dZ_t = \left(\psi'(X_t)\, b(X_t) + \frac{1}{2}\psi''(X_t)\sigma^2(X_t)\right) dt + \psi'(X_t)\, \sigma(X_t)\, dW_t,\ Z_0 = \psi(\xi).$$

**Note**: Remark that the dynamic of $(Z_t;\, t \geq 0)$ depends only on the derivatives of $\psi$.

**Methods for simplifying the coefficients of a SDE: The Lamperti transformation**. (See (O00), Chapter 5).

**General idea**: Let $(X_t;\ t \geq 0)$ be a solution (weak or strong) to the one-dimensional SDE:

$$X_t = \xi + \int_0^t b(X_s)\,ds, + \int_0^t \sigma(X_s)\,dW_s,\ t \geq 0,$$

where $\xi \sim \mu_0$ and $(W_t;\ t \geq 0)$ is a standard Brownian motion. For $\psi : \mathbb{R} \to \mathbb{R}$of class $\mathcal{C}^2$ on $\mathbb{R}$, by Itô's formula, we have

$$\psi(X_t) = \psi(X_0) + \int_0^t \psi'(X_s)\,dX_s + \frac{1}{2}\int_0^t \psi''(X_s)\,d\langle X\rangle_s$$

$$= \psi(\xi) + \int_0^t \psi'(X_s)\,b(X_s)ds + \int_0^t \psi'(X_s)\,\sigma(X_s)dW_s + \frac{1}{2}\int_0^t \psi''(X_s)\sigma^2(X_s)\,ds$$

Therefore, the dynamic of $(Z_t = \psi(X_t);\ t \geq 0)$ is characterized by

$$dZ_t = \left(\psi'(X_t)\,b(X_t) + \frac{1}{2}\psi''(X_t)\sigma^2(X_t)\right)dt + \psi'(X_t)\,\sigma(X_t)\,dW_t,\ Z_0 = \psi(\xi).$$

**Note**: Remark that the dynamic of $(Z_t;\ t \geq 0)$ depends only on the derivatives of $\psi$.

**Methods for simplifying the coefficients of a SDE: The Lamperti transformation**. (See (O00), Chapter 5).

**General idea**: Let $(X_t;\ t \geq 0)$ be a solution (weak or strong) to the one-dimensional SDE:

$$X_t = \xi + \int_0^t b(X_s)\,ds, + \int_0^t \sigma(X_s)\,dW_s,\ t \geq 0,$$

where $\xi \sim \mu_0$ and $(W_t;\ t \geq 0)$ is a standard Brownian motion. For $\psi : \mathbb{R} \to \mathbb{R}$of class $\mathcal{C}^2$ on $\mathbb{R}$, by Itô's formula, we have

$$\begin{aligned}\psi(X_t) &= \psi(X_0) + \int_0^t \psi'(X_s)\,dX_s + \frac{1}{2}\int_0^t \psi''(X_s)\,d\langle X\rangle_s\\ &= \psi(\xi) + \int_0^t \psi'(X_s)\,b(X_s)ds + \int_0^t \psi'(X_s)\,\sigma(X_s)dW_s + \frac{1}{2}\int_0^t \psi''(X_s)\sigma^2(X_s)\,ds\end{aligned}$$

Therefore, the dynamic of $(Z_t = \psi(X_t);\ t \geq 0)$ is characterized by

$$dZ_t = \left(\psi'(X_t)\,b(X_t) + \frac{1}{2}\psi''(X_t)\sigma^2(X_t)\right)\,dt + \psi'(X_t)\,\sigma(X_t)\,dW_t,\ Z_0 = \psi(\xi).$$

**Note**: Remark that the dynamic of $(Z_t;\ t \geq 0)$ depends only on the derivatives of $\psi$.

Assuming that $\sigma(x) \neq 0$, $\psi : \mathbb{R} \to \mathbb{R}$ is also invertible and that $\psi$ is a primitive function of $1/\sigma(x)$ $\psi'(x) = \sigma^{-1}(x)$, $x \in \mathbb{R}$, then

$$\begin{aligned} dZ_t &= \left( \psi'(X_t)\, b(X_t) + \frac{1}{2}\psi''(X_t)\sigma^2(X_t) \right) dt + \psi'(X_t)\,\sigma(X_t)\, dW_t \\ &= \left( \frac{b(X_t)}{\sigma(X_t)} - \frac{\sigma''(X_t)}{2} \right) dt + dW_t \\ &= \left( \frac{b(\psi^{-1}(Z_t))}{\sigma(\psi^{-1}(Z_t))} - \frac{\sigma'(\psi^{-1}(Z_t))}{2} \right) dt + dW_t \end{aligned}$$

for $\psi^{-1}$, the inverse function of $\psi$. In this situation, there is an explicit connection between the solution (weak or strong) of the SDEs:

$$dX_t = b(X_t)\, dt, +\sigma(X_t)\, dW_t,\ X_0 = \xi,$$

$$dZ_t = \left( \frac{b(\psi^{-1}(Z_t))}{\sigma(\psi^{-1}(Z_t))} - \frac{\sigma'(\psi^{-1}(Z_t))}{2} \right) dt + dW_t,\ Z_0 = \psi(X_0).$$

## Proposition 3.22

*Assume that $(X_t;\, t \geq 0)$ is a strong solution to the SDE:*

$$dX_t = b(X_t)\, dt, +\sigma(X_t)\, dW_t,\;\; X_0 = x \in \mathbb{R},$$

*and assume that $x \mapsto \sigma(t,x)$ is continuously differentiable on $\mathbb{R}$, and that, a.s. $\sigma(X_t) > 0$, for all $t \geq 0$. Then, for any given primitive $\psi : \mathbb{R} \to \mathbb{R}$ of $x \mapsto 1/\sigma(x)$, the process*

$$Z_t = \psi(X_t),\; t \geq 0,$$

*is a strong solution to*

$$\begin{aligned} dZ_t &= \left( \frac{b(t, \psi^{-1}(Z_t))}{\sigma(\psi^{-1}(Z_t))} - \frac{\sigma'(\psi^{-1}(Z_t))}{2} \right) dt + dW_t, \\ Z_0 &= \psi_0(x). \end{aligned}$$

*where for $x \mapsto \psi^{-1}(x)$ is the inverse function of $x \mapsto \psi(t,x)$.*

**Application**: The geometric Brownian motion: Consider the following SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \;\; S_0 = x > 0,$$

for $\mu, \sigma > 0$. Since $b(t,x) = \mu x$, $\sigma(t,x) = \sigma x$ are Lipschitz continuous, there exists a unique solution to the SDE. In addition, defining

$$v(x) = \int_1^x \rho'(y) \left( \int_1^y \frac{2}{\rho'(r)\sigma^2(r)} \, dr \right) \, dz, \; \rho(x) = \int_1^x \exp\{-2 \int_1^y \frac{b(r)}{\sigma^2(r)} \, dr\} \, dy,$$

which gives

$$v(x) = \int_1^x \exp\{-\frac{2\mu}{\sigma}(y-1)\} \int_1^y \frac{2\exp\{\frac{2\mu}{\sigma} z\}}{\sigma^2 z^2} \, dz \, dy,$$

the Feller explosion test ensures that

$$\mathbb{P}(\tau = \infty) = 1, \;\; \tau = \inf\{t > 0 \,|\, S_t = 0\},$$

and that the solution to the SDE is a.s. positive on $(0, \infty)$.

Setting

$$\psi(t, X_t) = \int_1^{S_t} \frac{1}{\sigma(t, y)}\, dy = \frac{1}{\sigma} \ln(S_t)$$

and applying the Lamperti transformation, we get, for $Z_t = \psi(t, X_t)$,

$$dZ_t = \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\, dt + \, dW_t,\ Z_0 = \ln(x).$$

Therefore, the unique solution to

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,\ \ S_0 = x > 0,$$

is given by

$$S_t = \exp\{\sigma Z_t\} = x \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\},\ t \geq 0.$$

**Case of time-dependent coefficients:** Assume that $(X_t;\ t \geq 0)$ is a strong solution to the SDE:

$$dX_t = b(t, X_t)\, dt, + \sigma(t, X_t)\, dW_t,\ \ X_0 = x \in \mathbb{R},$$

and assume that, $(t, x) \mapsto \sigma(t, x)$ is continuously differentiable on $[0, \infty) \times \mathbb{R}$, and that, a.s. $\sigma(t, X_t) > 0$, for all $t \geq 0$. Then, for any given $\psi(t, x) = \int \frac{1}{\sigma(t,y)}\, dy$, the process

$$Z_t = \psi(t, X_t),\ t \geq 0,$$

is a strong solution to

$$dZ_t = \left( \partial_t \psi(t, \psi^{-1}(t, X_t)) + \frac{b(t, \psi^{-1}(t, Z_t))}{\sigma(t, \psi^{-1}(t, Z_t))} - \frac{\sigma''(t, \psi^{-1}(t, Z_t))}{2} \right) dt + dW_t,$$
$$Z_0 = \psi_0(x).$$

where for $x \mapsto \psi^{-1}(t, x)$ is the inverse function of $x \mapsto \psi(t, x)$.

**Model of population growth in a stochastic environment:** (See (O00), Chapter 5) A stochastic model for the time evolution of the size $X_t$ of a population is given by the SDE

$$\begin{cases} dX_t = rX_t\left(K - X_t\right)\, dt + \beta X_t dW_t, \\ X_0 = x > 0. \end{cases}$$

where $\beta, r \in \mathbb{R}$, $K > 0$ and $(W_t;\ t \geq 0)$ is a $\mathbb{R}$-Brownian motion. The solution is then given by

$$X_t = \frac{\exp\left((rK - \frac{1}{2}\beta^2)t + \beta W_t\right)}{\frac{1}{x} + r\int_0^t \exp\left((rK - \frac{1}{2}\beta^2)s + \beta W_s\right)\, ds}$$

**The Brownian motion on the disc:** Let $(B_t;\, t \geq 0)$ be a $\mathbb{R}$-Brownian motion. We define the $\mathbb{R}^2$-valued SDE

$$\begin{cases} dY_t^1 = -\dfrac{1}{2} Y_t^1 dt - Y_t^2 dW_t, \\ dY_t^2 = -\dfrac{1}{2} Y_t^2 dt + Y_t^1 dW_t, \\ Y_0 = \xi, \text{ with } |\xi_1|^2 + |\xi_2|^2 = 1. \end{cases}$$

The solution of this equation is the Brownian motion on the disc

$$Y_t = (Y_t^1, Y_t^2) = (\cos(\xi_2 + B_t), \sin(\xi_2 + B_t))$$

## The Brownian bridge

**The Brownian bridge (or the pinned Brownian motion):** (see (IK81) page 243). The Brownian bridge models the situation of Brownian motion whose position at a given time $0 < T < \infty$ is fixed. This particular dynamic is given by the following SDE: Given $x, y \in \mathbb{R}$, consider the one-dimensional SDE

$$\begin{aligned} dX_t &= \frac{X_t - y}{T - t}\, dt + dW_t,\ t \in (0, T), \\ X_0 &= x \in \mathbb{R}. \end{aligned}$$

The solution of this equation is characterized by the fact that $\mathbb{P}$-a.s. $X_T = y$. For any $0 \leq t < T$, the solution is uniquely determined. Hence strong uniqueness of the solution holds true to the SDE. Noticing that

$$(T-t)d\left(\frac{X_t - y}{T - t}\right) = -\frac{X_t - y}{T - t}\, dt + dX_t = -\frac{X_t - y}{T - t}\, dt + \frac{X_t - y}{T - t}\, dt + dW_t = dW_t$$

the solution is given by

$$X_t = x + \frac{t}{T}(y - x) - (T - t)\int_0^t \frac{dW_s}{T - s},\ t < T,\ X_T = y.$$

**Norm of the Brownian motion:** Let $x \in \mathbb{R}^m$ and $(B_t;\, t \geq 0)$ be a $\mathbb{R}^m$-Brownian motion and define $Y_t = |x + B_t|^2$ ($|\cdot|$ being the Euclidean norm of $\mathbb{R}^m$). Since, whenever $x \neq 0$,

$$\partial_{x_i}|x|^2 = x_i,\ \partial^2_{x_i x_j}|x|^2 = \delta_{i,j}$$

for $\delta_{i,j}$ the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Applying the Itô formula, we observe that

$$\begin{aligned} Y_t &= |x|^2 + 2\sum_{i=1}^{m}\int_0^t (x^i + B^i_s)\, dB^i_s + \sum_{i,j=1}^{m}\int_0^t \delta_{i,j}\, d\langle B^i, B^j\rangle_s \\ &= |x|^2 + 2\sum_{i=1}^{m}\int_0^t \sqrt{Y_s}\frac{(x^i + B^i_s)}{\sqrt{Y_s}}\, dB^i_s + mt. \end{aligned}$$

(Since, for all $t \geq 0$, $\mathbb{P}(B_t = x) = 0$, we have $\mathbb{P}(Y_t = 0) = 0$ and, according to the Levy characterization of a Brownian motion, $t \mapsto \sum_{i=1}^m \int_0^t \frac{B^i_t}{\sqrt{Y_t}}\, dB^i_s$ is a standard Brownian motion. Therefore $(Y_t;\, t \geq 0)$ is a (weak) solution to

$$dY_t = m\, dt + 2\sqrt{Y_t}\, dW_t,\ t \geq 0,\ Y_0 = |x|^2.$$

## SDEs with square root diffusion:

**SDE with diffusion of the form $\sqrt{x}$:** (see (IK81) page 235). Given $\sigma > 0$ and $\beta, \alpha$ in $\mathbb{R}$, consider the one-dimensional SDE

$$(*) \begin{array}{l} dX_t = (\beta - \alpha X_t)\, dt + \sigma \sqrt{X_t} dW_t, \\ X_0 = \xi > 0,\ \mathbb{P}\text{-a.s.}. \end{array}$$

Necessary, any solution to this equation must have non-negative paths.
To handle the well-posedness of the SDE $(*)$, we rewrite the equation into

$$(**) \begin{array}{l} dX_t = (\beta - \alpha X_t)\, dt + \sigma \sqrt{(X_t) \vee 0} dW_t, \\ X_0 = \xi > 0,\ \mathbb{P}\text{-a.s.}. \end{array}$$

Then any solution to $(*)$ is a solution of (2) and any non-negative solution to $(**)$ is solution to $(*)$ up to their first hitting time at $x = 0$.
According to Theorem 2.14, there is at most one strong solution to $(**)$. (Note: If $\beta = 0$ and $\mathbb{P}$-a.s. $\xi = 0$, then this solution is the trivial solution $X_t = 0$, $t \geq 0$).

Finally, note that if $\beta > 0$ (and $\xi \geq 0$) therefore $X_t \geq 0$ for all $t \geq 0$. Indeed, defining

$$\tau_{-\epsilon} = \inf\{t > 0 \,|\, X_t = -\epsilon\},\ \epsilon \geq 0,$$

if $\mathbb{P}$-a.s. $\tau_{-\epsilon} < \infty$ for all $\epsilon > 0$, then, for all $r \leq \tau_{-\epsilon}$ the SDE reduces to

$$dX_t = (\beta - \alpha X_t)\, dt,\ \text{en}\,(r, \tau_{-\epsilon}).$$

Since $\beta$ and $\alpha$ are positive, the path $t \mapsto X_t$ is (strictly) increasing, and so is positive, in $(r, \tau_{-\epsilon})$.
To construct a solution to $(**)$, we need to define the times where the solution hits 0. Using again the Feller explosion test, we get

**Lemma 3.23 ((IK88))**

*For $(X_t;\, t \geq 0)$ solution to $(**)$, define*

$$\tau_0 = \inf\{t > 0 \,|\, X_t = 0\}$$

*Then we have the following situations:*

- *If $2\beta \geq \sigma^2$ then $\mathbb{P}$-a.s., $\tau_0 = \infty$.*
- *If $0 \leq 2\beta < \sigma^2$ and $\alpha \geq 0$ then $\mathbb{P}$-a.s., $\tau_0 < \infty$.*
- *If $0 \leq 2\beta < \sigma^2$ and $\alpha < 0$ then $\mathbb{P}(\tau_0 < \infty) > 0$.*

**Application:**

○ The Cox-Ingersoll-Ross model for interest rates (for $\beta \geq \sigma$): For an introduction on this topic, see the book **Introduction to Stochastic Calculus applied to Finance** D. Lamberton and D. Lapeyre.

○ **The Bessel processes:** (See (IK81) page 237). Define

$$Z_t = \sqrt{Y_t}\,(= |x + B_t|)\,,\ t \geq 0.$$

Applying (formally) the Itô formula, we have

$$\begin{aligned} Z_t &= |x| + \int_0^t \frac{1}{2\sqrt{Y_s}}\, dY_s - \frac{1}{8}\int_0^t \frac{1}{Y_s^{3/2}}\, d\langle Y\rangle_s \\ &= |x| + \int_0^t \frac{m}{2\sqrt{Y_s}}\, ds + W_t - \frac{1}{8}\int_0^t \frac{4Y_s}{Y_s^{3/2}}\, ds \\ &= |x| + \int_0^t \frac{m-1}{2\sqrt{Y_s}}\, ds + W_t. \end{aligned}$$

In a more general the strong solution to a SDE of the form is a called the square of the $m$-dimensional Bessel process (for an extended discussion on the Bessel processes see Revuz-Yor [RY05], Chapter 11).

## The Geometric Brownian motion and its applications in Finance

Consider

$$(G.B.m)\ \begin{cases} dX_t = rX_t dt + \sigma X_t dW_t, \\ X_0 = \xi \end{cases} \tag{18}$$

where $r$ and $\sigma$ two given real constant. The unique solution to $(G.B.m.)$ is given by

$$X_t = \xi \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

This process has a log-normal distribution in the sense

$$\log(X_t) \sim \mathcal{N}(rt - \frac{\sigma^2}{2} t, \sigma^2 t),$$

and, if $\mathbb{P}$-a.s., $\xi \geq 0$, $X_t \geq 0$, for all $t \geq 0$.
The geometric Brownian motion is a important stochastic model in Financial Mathematics and was at the basis of the Black and Scholes theory of asset pricing.

## Mathematical Finance

**Historical:**

• Louis Bachelier (1900) *Théorie de la spéculation*;

• Harry Markowitz (1952): *Portfolio selection*. Economy Nobel Prize in 1990;

• Fisher Black and Myron Scholes (1973): *The Pricing of Options and Corporate Liabilities*. Economy Nobel Prize, with Robert Merton in 1997.

• Robert Merton (1973): *Theory of Rational Option Pricing*.

• John C. Cox and Stephen A. Ross (1976): *The Valuation of Options for Alternative Stochastic Processes*,

**The Black & Scholes Financial model:** Probabilistic model for the valorization and hedging of option contract of European type:
*Call European Option*: The owner of the option have the right (but not the obligation) to sell an asset to the dealer (agent) of the option at a fixed price (strike price) $K$ at a given date (maturity date) $T$.

*Put European Option*: The owner of the option have the right (but not the obligation) to buy an asset to the dealer (agent) of the option at a fixed price $K$ at a given date $T$.

The Black and Scholes model aims to give a fair price (for the owner and the dealer) to the option contract-

**Mathematical model:** Let $T$ be the maturity of the option, $K$ the strike price and $(S_t;\ t \in [0,T])$ the price of the underlying asset of the asset between the present date $t=0$ and the future termination time $T$. We adopt the point of view of the dealer of the option: At time $T$, we have to cover the price of the contract which is
in the case of a *Call* option: $(S_T - K)_+$;

in the case of a *Put* option: $(K - S_T)_+$.

We face two problems
◦ The *pricing* problem: At which fair price can we sell the option to the buyer of the option ?

◦ The *hedging* problem: Can we build a portfolio (a position in the market) to cover the cost of the option ?

**Asset and portfolio model:** Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the market model is given by:
∘ a risky asset whose price per unit $(S_t;\, 0 \leq t \leq T)$ evolves according to the SDE

$$\begin{cases} dS_t = S_t\left(\mu dt + \sigma dW_t\right), \\ S_{t=0} = s_0 \end{cases}$$

where $\mu \in \mathbb{R}$, $\sigma \neq 0$, $s_0 > 0$ (price at the starting of the contract) and $(W_t;\, t \geq 0)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
∘ a non-risky asset whose price per unit $(S^0_t;\, 0 \leq t \leq T)$ is given by

$$S^0_t = s^0_0 e^{rt},$$

where $r > 0$ represents a constant rate of interest (for instance: the rate of interest of a bank deposit).
We denote by $(\mathcal{F}^W_t;\, t \geq 0)$ the filtration generated by $(s_0, s^0_0, (W_t;\, t \geq 0))$.

**Portfolio model:** The wealth of the dealer is modeled, at each time $0 \leq t \leq T$, is given by

$$V_t = H_t S_t + H_t^0 S_t^0,\ t \in [0, T],$$

where $V_0$ is the initial investment in the market and $(H_t,\ H_t^0;\ t \in [0, T])$ is the strategy of investment with

∘ $H_t$ represents the amount of risky assets owned at time $t$,

∘ and $H_t^0$ represents the amount of non-risky assets owned at time $t$.

**Discounted values:** We define

$$\tilde{S}_t = \frac{S_t}{S_t^0} \text{ and } \tilde{V}_t = \frac{V_t}{S_t^0}$$

which implies that

$$d\tilde{S}_t = d(e^{-r\cdot} S)_t = \tilde{S}_t\,((\mu - r)\,dt + \sigma\,dW_t)$$

and

$$d\tilde{V}_t = H_t\,d\tilde{S}_t.$$

**Portfolio model:** The wealth of the dealer is modeled, at each time $0 \leq t \leq T$, is given by

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**Discounted values:** We define

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which implies that

$$d\tilde{S}_t = d(e^{-r\cdot} S)_t = \tilde{S}_t \left((\mu - r)\, dt + \sigma\, dW_t\right)$$

and

$$d\tilde{V}_t = H_t\, d\tilde{S}_t.$$

**Admissible strategy:** A strategy is said to be admissible i.f.f. $\tilde{V}_t \geq 0$, $t \in [0, T]$ (non-negative position),

$$\mathbb{E}_{\mathbb{P}^*}\left[\max_{t\in[0,T]} |\tilde{V}_t|^2\right] < \infty.$$

and that

$$dV_t = H_t\, dS_t + H_t^0\, dS_t^0 \text{ (auto-financing strategy)},$$

or equivalently, $\mathbb{P}$-a.s.,

$$\int_0^T |H_s^0|\, ds + \int_0^T \sigma^2(S_t)^2(H_t)^2\, dt < \infty.$$

**The hypothesis of absence of arbitrage opportunity:** To model a fair market, we need to assume that there is no admissible strategy which allows a sure gain starting from an zero initial investment; that is, there is no admissible strategy $(H_t,\ H_t^0;\ t \in [0, T])$ such that

$$V_0 = 0 \text{ and } \mathbb{P}(\tilde{V}_T > 0) > 0.$$

**The first fundamental theorem of asset pricing :**

### Proposition 3.24

*If there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$, such that, under $\mathbb{P}^*$, $(V_t;\, t \in [0, T])$ is a $\mathcal{F}_t$-martingale then the hypothesis of absence of arbitrage opportunity is satisfied. Such probability measure $\mathbb{P}^*$ que is called a risk neutral probability or an equivalent martingale measure.*

**Construction of a risk neutral probability:** Define $\mathbb{P}^*$ with

$$\begin{aligned}\frac{d\mathbb{P}^*}{d\mathbb{P}} &= \exp\left\{-\int_0^T \frac{r-\mu}{\sigma}\, dW_t - \frac{1}{2}\int_0^T \frac{(r-\mu)^2}{\sigma^2}\, dt\right\}\\ &= \exp\left\{\frac{(r-\mu)W_T}{\sigma} - \frac{(r-\mu)^2 T}{2\sigma^2}\right\}\end{aligned}$$

Since the Novikov condition is satisfied, the Girsanov transformation holds true and, under $\mathbb{P}^*$, the process $\left(\hat{W}_t := \frac{(r-\mu)t}{\sigma} + W_t\right)$ is a standard Brownian motion and $(\tilde{S}_t;\, t \in [0, T])$ is a $\mathcal{F}_t$-martingale. Since $d\tilde{V}_t = H_t\, d\tilde{S}_t$, it follows that $(\tilde{V}_t;\, t \in [0, T])$ is a $\mathcal{F}_t$-martingale for any admissible strategy. Therefore, if $V_0 = 0$ then $\mathbb{E}_{\mathbb{P}^*}[\tilde{V}_t] = 0$, for all $0 \leq t \leq T$ and $\tilde{V}_t = 0$, for all $0 \leq t \leq T$.

**The Black and Scholes formula: Pricing of an European option at time $t = 0$:** Let us consider the case of a European Call contract: In this situation the wealth $V_T$ of the dealer must cover the cost of the option at the maturity date; that is:

$$V_T = (S_T - K)_+ \text{, or equivalently } \tilde{V}_T = e^{-rT}(S_T - K)_+ .$$

Using the risk-neutral probability,

$$\tilde{V}_t = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(S_T - K)_+ \mid \mathcal{F}_t\right].$$

Observe then that, under $\mathbb{P}^*$, we have

$$\begin{aligned}\tilde{V}_t &= \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(S_T - K)_+ \mid S_t\right] \\ &= \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(K - S_{T-t})_+\right].\end{aligned}$$

Consequently, the price $C_t$ at time $t$ of the Call option is given by

$$C_t = e^{rt}\tilde{V}_t = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(T-t)}(S_T - K)_+ \mid S_t\right].$$

Given the definition of $\mathbb{P}^*$, we deduce the **Black and Scholes formula** for the European Call Option:

$$V_t = F(t, S_t),\ F(t,x) := \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(T-t)}(S_T - K)_+ \mid S_t = x\right] = xN(d_1(x)) - Ke^{r(T-t)}N(d_2(x)$$

where $N$ is the cumulative distribution function if the normal distribution $\mathcal{N}(0,1)$:

$$N(x) = \int_{-\infty}^{x} e^{-x^2/2}\,dx/\sqrt{2\pi},$$

and where

$$d_1(x) = \frac{\log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\ d_2(x) = d_1(x) - \sigma\sqrt{T-t}.$$

Following the same reasoning, and using the Call-Put formula:

$$(K - S_T)_+ = (S_T - K)_+ - (S_T - K)$$

the instantaneous price of a Put option is given by

$$P_t = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(T-t)}(K - S_T)_+ \mid S_0\right]$$

**Hedging of an European option:**

**Theorem 3.25 (Martingale representation (see KS88, Chapter 3))**

*Let $(M_t;\, t \geq 0)$ be a real continuous local square-integrable martingale defined under $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \geq 0), \mathbb{P})$ such that $\langle M \rangle_t$ is absolutely continuous. Then there exists an extension $(\Omega', \mathcal{F}', (\mathcal{F}'_t;\, t \geq 0), \mathbb{P}')$ under which are defined a $\mathcal{F}'_t$-Brownian motion $(W_t;\, t \geq 0)$ and a $\mathcal{F}_t$-adapted process $(\phi_t;\, t \geq 0)$ such that*

$$\forall\, 0 \leq t < \infty,\, \mathbb{E}_{\mathbb{P}'}[\int_0^t |\phi_s|^2\, ds] < \infty,$$

*and $M_t = \int_0^t \phi_s\, dW_s$.*

Since under, $\mathbb{P}^*$, $(\tilde{V}_t;\, t \in [0, T])$ is a martingale, the Martingale representation theorem implies that there exists $(\psi_t;\, t \in [0, T])$ en $L^2(dt \times d\mathbb{P}^*)$ such that, $\mathbb{P}^*$-a.s.,

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \psi_s\, d\hat{W}_s,\, \forall\, t \in [0, T].$$

Taking

$$H_t = \frac{\psi_t}{\sigma S_t}, H_t^0 = \tilde{V}_t - H_t \tilde{S}_t,$$

we obtain the admissible strategy (see e.g. [LL00] Chapters 4 and 5).

$$H_T S_T + H_T^0 S_T^0 = (S_T - K)_+.$$

**The Black and Scholes formula**: Explicating the hedging strategy: Taking back

$$F(t,x) = \mathbb{E}_{\mathbb{P}'}\left[\exp -r(T-t)f(S_T)\,|\,S_t = x\right],\ f(x) = (x-K)_+,$$

and let us define

$$\tilde{F}(t,x) = e^{-rt}F(t,xe^{rt}).$$

so that $\tilde{F}(t,\tilde{S}_t) = e^{-rt}F(t,S_t) = e^{-rt}V_t$. Observing that $(t,x)\mapsto F(t,x)$ (and by extension $(t,x)\mapsto \tilde{F}(t,x)$) is of class $\mathcal{C}^{1,2}$, and that under $\mathbb{P}'$,

$$d\tilde{S}_t = \sigma\tilde{S}_t d\hat{W}_t,\ \tilde{S}_0 = S_0,$$

the Itô formula yields

$$\tilde{F}(t,\tilde{S}_t) = \tilde{F}(0,S_0) + \int_0^t \left(\partial_r\tilde{F}(r,\tilde{S}_r) + \frac{\sigma^2(\tilde{S}_r)^2}{2}\partial_x^2\tilde{F}(r,\tilde{S}_r)\right)dr + \sigma\int_0^t \tilde{S}_r\partial_x\tilde{F}(r,\tilde{S}_r)\,d\hat{W}_r,$$

that we can compare with the expression

$$d\tilde{V}_t = \sigma\tilde{S}_t H_t d\hat{W}_t$$

Observing that $\tilde{F}(t,\tilde{S}_t)$ is a martingale then, necessarily

$$\partial_r\tilde{F}(r,\tilde{S}_r) + \frac{\sigma^2(\tilde{S}_r)^2}{2}\partial_x^2\tilde{F}(r,\tilde{S}_r) = 0,\ 0\leq t\leq T,$$

and

$$H_t = \partial_x\tilde{F}(t,\tilde{S}_t) = \partial_x F(t,\tilde{S}_t) = \mathcal{N}(d_1(S_t)),\ 0\leq t\leq T.$$

**Characteristic components of the Black and Scholes formula**:

$$
\begin{aligned}
\partial_x F(t,x) &= \text{Delta of the option} \\
&= \text{Variation of the option price with respect to the variations of the underlying asset,} \\
\partial_x^2 F(t,x) &= \text{Gamma of the option,} \\
\partial_t F(t,x) &= \text{Theta of the option,} \\
\partial_\sigma F(t,x) &= \text{Vega of the option} \\
&= \text{Variation of the option price with respect to the variations of the volatility.}
\end{aligned}
$$

**Computational methods for the Black and Scholes**: Probabilistic and statistical techniques (Monte-Carlo methods; time-discretization scheme for the geometric Brownian motion; Statistics on SDEs for estimating $\mu$ and $\sigma$); Deterministic techniques (Feynman-Kac representation formula numerical pde methods)

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The Black and Scholes model was celebrated for its use to advanced stochastic calculus to exhibit a suitable method for the pricing and hedging of financial contract. Although the underlying financial assumption of the model (complete liquid market with the possibility to buy or sell any amount of asset, no portfolio cost, volatility $\sigma$ of the market constant, ...), the work of Black and Scholes (and Merton) was the beginning of various development between stochastic analysis and mathematical financial. For further discussion on this topic, see

**Stochastic Calculus Models for Finance** (Volumes I and II), S.E. Shreve.

**Options, Futures and Other Derivatives**, by J. Hull.

**Introduction to Stochastic Calculus Applied to Finance**, by D. Lamberton and B. Lapeyre.

**Arbitrage Theory in Continuous Time** by T. Björk.

**Mathematical Methods for Financial Markets**, by M. Chesney, M. Jean-Blanc and M. Yor.

# The martingale problem related to a SDE.

**Preliminary:**

**Lemma 4.1**

*Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \geq 0), \mathbb{P})$ be a filtered probability space and let $(X_t;\, t \geq 0)$, be a continuous $\mathcal{F}_t$-adapted process with values in $\mathbb{R}$ such that $X_0 = 0$. Therefore $(X_t;\, t \geq 0)$ is a $\mathcal{F}_t$-Brownian motion i.f.f., for all $f \in \mathcal{C}^2(\mathbb{R})$, the process $(M^f_t;\, t \geq 0))$ defined as*

$$M^f_t = f(X_t) - f(0) - \frac{1}{2}\int_0^t \partial^2_x f(X_s)\, ds,\ t \geq 0,$$

*is a continuous $\mathcal{F}_t$-local martingale.*

**Proof:**

$\Leftarrow$ Immediate from Itô's formula.

$\Rightarrow$ According to the Levy characterization of a Brownian motion, we only need to show that $(X_t;\, t \geq 0)$ is a continuous $\mathcal{F}_t$-local martingale starting from 0 and such that $\langle X \rangle_t = t$, for all $t \geq 0$.

This assertion follows from an appropriate choice of $f$:
• Taking $f(x) = x$, we get that

$$f(X_t) - \frac{1}{2}\int_0^t \partial_x^2 f(X_s)\, ds = X_t,\ t \geq 0,$$

is a $\mathcal{F}_t$-local martingale such that $X_0 = 0$.
• Taking $f(x) = x^2$,

$$f(X_t) - \frac{1}{2}\int_0^t \partial_x^2 f(X_s)\, ds = (X_t)^2 - t,\ t \geq 0,$$

is a $\mathcal{F}_t$-local martingale.

**Proof:**

$\Leftarrow$ Immediate from Itô's formula.

$\Rightarrow$ According to the Levy characterization of a Brownian motion, we only need to show that $(X_t;\, t \geq 0)$ is a continuous $\mathcal{F}_t$-local martingale starting from 0 and such that $\langle X \rangle_t = t$, for all $t \geq 0$.

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• Taking $f(x) = x^2$,

$$f(X_t) - \frac{1}{2}\int_0^t \partial_x^2 f(X_s)\, ds = (X_t)^2 - t,\ t \geq 0,$$

is a $\mathcal{F}_t$-local martingale.

**Notation**: $\triangle_x$ denote the Laplace differential operator (Laplacian): For $f : \mathbb{R}^d \to \mathbb{R}$,

$$\triangle_x f(x) = \sum_{i=1}^{d} \partial^2_{x_i} f(x).$$

**Lemma 4.2**

*Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$ be a filtered probability space and let $(X_t;\ t \geq 0)$ be a continuous $\mathcal{F}_t$-adapted process with values in $\mathbb{R}^m$ such that $X_0 = 0$. Then $(X_t;\ t \leq 0)$ is a $\mathbb{R}^m$-Brownian motion i.f.f., for all $f \in \mathcal{C}^2(\mathbb{R}^m; \mathbb{R})$,*

$$M^f_t := f(X_t) - f(0) - \frac{1}{2}\int_0^t \triangle_x f(X_s)\, ds$$

*is a continuous $\mathcal{F}_t$-local martingale.*

**Proof:**Same arguments as before:
$\Leftarrow$ relies on the Itô formula. For $\Rightarrow$: we need the **Levy's Characterization of a $\mathbb{R}^d$-Brownian motion**

**Theorem 4.3**

*A $\mathbb{R}^d$-Brownian motion is the unique square integrable $\mathcal{F}_t$-martingale $(M_t;\, t \geq 0)$ with values in $\mathbb{R}^d$ such that $M_0 = 0$ and*

$$\langle M^i, M^j \rangle_t = \begin{cases} t \; if \, i = j, \\ 0 \; otherwise \end{cases}$$

• Taking $f(x) = x_i, i = 1, \cdots, d$ implies that each component $(X_t^i;\ t \geq 0)$, $1 \leq i \leq d$ is a $\mathcal{F}_t$-local martingale.

• Taking $f(x) = |x_i|^2$ ensures that, for each $1 \leq i \leq d$,

$$\langle X^i \rangle_t - t,\ t \geq 0,$$

is a local martingale.

• Taking $f(x) = x_i x_j, i, j = 1, \cdots, d,\ i \neq j$, we observe that

$$X_t^i X_t^j;\ t \geq 0,$$

is a local martingale so that $\langle X^i, X^j \rangle_t = 0$ whenever $i \neq j$.

- Taking $f(x) = x_i, i = 1, \cdots, d$ implies that each component $(X^i_t;\ t \geq 0)$, $1 \leq i \leq d$ is a $\mathcal{F}_t$-local martingale.

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- Taking $f(x) = x_i x_j, i, j = 1, \cdots, d,\ i \neq j$, we observe that

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• Taking $f(x) = x_i, i = 1, \cdots, d$ implies that each component $(X^i_t;\, t \geq 0)$, $1 \leq i \leq d$ is a $\mathcal{F}_t$-local martingale.

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• Taking $f(x) = x_i x_j, i, j = 1, \cdots, d,\; i \neq j$, we observe that

$$X^i_t X^j_t;\; t \geq 0,$$

is a local martingale so that $\langle X^i, X^j \rangle_t = 0$ whenever $i \neq j$.

## Link with SDE

**Generalization:** At each SDE of the form

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t,\ t \geq 0, \\ X_0 = \xi \sim \mu_0, \end{cases}$$

we can associate a (local) martingale problem defined as follows: For $\Omega := \mathcal{C}([0, \infty); \mathbb{R}^d)$ equipped with its canonical process $(x(t);\ t \geq 0)$ given by

$$x(t) : \omega \in \mathcal{C}([0, \infty); \mathbb{R}^d) \mapsto x(t)(\omega) = \omega(t),$$

and the filtration $(\mathcal{F}_t;\ t \geq 0)$ given by

$$\mathcal{F}_t = \sigma(x(s);\ 0 \leq s \leq t)$$

find a probability measure $\mathbb{P}$ defined on $\Omega$ such that, for all function $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, the process

$$\begin{aligned} M_t^f &:= f(X_t) - f(X_0) - \int_0^t b(s, X_s) \cdot \nabla f(X_s)\,ds - \frac{1}{2}\int_0^t \mathrm{Trace}\left((\sigma\sigma^*)(s, X_s)\nabla^2 f(X_s)\right)\,ds \\ &= f(X_t) - f(X_0) - \sum_{i=1}^d \int_0^t b^{(i)}(s, X_s)\partial_{x_i} f(X_s)\,ds - \frac{1}{2}\sum_{i,j=1}^d \int_0^t (\sigma\sigma^*)_{i,j}(s, X_s)\partial^2_{x_i x_j} f(X_s)\,ds \end{aligned}$$

is a $\mathcal{F}_t$-local martingale under $\mathbb{P}$.

## Link with SDE

For simplicity, we will only restrict the study hereafter to the link between a SDE of the form

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,\ t \in [0, T], \\ X_0 = \xi \sim \mu_0, \end{cases}$$

for $0 < T < \infty$, an arbitrary time horizon, and martingale problems up to time $T$: For $\Omega := \mathcal{C}([0, T]; \mathbb{R}^d)$ equipped with its canonical process $(x(t);\ 0 \leq t \leq T)$ given by

$$x(t) : \omega \in \mathcal{C}([0, T]; \mathbb{R}^d) \mapsto x(t)(\omega) = \omega(t),$$

and the filtration $(\mathcal{F}_t;\ t \geq 0)$ given by

$$\mathcal{F}_t = \sigma(x(s);\ 0 \leq s \leq t)$$

find a probability measure $\mathbb{P}$ defined on $\Omega$ such that, for all function $f \in \mathcal{C}^2_b(\mathbb{R}^d)$, the process

$$M^f_t := f(X_t) - f(X_0) - \int_0^t b(s, X_s) \cdot \nabla f(X_s)\, ds - \frac{1}{2} \int_0^t \mathrm{Trace}\left((\sigma\sigma^*)(s, X_s) \nabla^2 f(X_s)\right)\, ds$$

is a $\mathcal{F}_t$-local martingale under $\mathbb{P}$.

## Definition 4.4

*Let $\mu_0$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be two Borel measurable functions and let $\{\mathcal{A}_t\}_{t\in[0,T]}$ be the family of differential operators:*

$$\mathcal{A}_t(f)(x) = b(t,x) \cdot \nabla_x f(x) + \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} f(x)$$

*where $a = \sigma\sigma^*$.*
*A probability measure $\mathbb{P}$ defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$ is a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{t\in[0,T]})$ i.f.f. the following properties hold true*
(1) $\mathbb{P} \circ (x(0))^{-1} = \mu_0$ *and*
(2) *For all $f$ in $\mathcal{C}^2(\mathbb{R}^d)$, the*

$$M^f_t := f(x(t)) - f(x(0)) - \int_0^t \mathcal{A}_s(f)(x(s))\, ds,\ 0 \leq t \leq T,$$

*is a $\mathcal{F}_t$-(local) martingale under $\mathbb{P}$.*

• Notion of existence for a martingale problem (local or not) related to $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$ on a finite time interval $[0, T]$:

◦ There exists a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$ i.f.f. there exists a probability measure $\mathbb{P}$ defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$ satisfying the properties (1) and (2) of Definition 4.4.

◦ Uniqueness of a martingale problem holds true i.f.f., for any couple of solutions $\mathbb{P}$ and $\tilde{\mathbb{P}}$ to the martingale problem, we have

$$\mathbb{P} = \tilde{\mathbb{P}},$$

where the equality is in the sense of probability measures defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$.

• Notion of existence for a martingale problem (local or not) related to $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$ on a finite time interval $[0, T]$:

∘ There exists a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$ i.f.f. there exists a probability measure $\mathbb{P}$ defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$ satisfying the properties (1) and (2) of Definition 4.4.

∘ Uniqueness of a martingale problem holds true i.f.f., for any couple of solutions $\mathbb{P}$ and $\tilde{\mathbb{P}}$ to the martingale problem, we have

$$\mathbb{P} = \tilde{\mathbb{P}},$$

where the equality is in the sense of probability measures defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$.

• Notion of existence for a martingale problem (local or not) related to $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$ on a finite time interval $[0, T]$:

◦ There exists a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$ i.f.f. there exists a probability measure $\mathbb{P}$ defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$ satisfying the properties (1) and (2) of Definition 4.4.

◦ Uniqueness of a martingale problem holds true i.f.f., for any couple of solutions $\mathbb{P}$ and $\tilde{\mathbb{P}}$ to the martingale problem, we have

$$\mathbb{P} = \tilde{\mathbb{P}},$$

where the equality is in the sense of probability measures defined on $(\mathcal{C}([0, T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d)))$.

- If there exists a weak solution

$$(\xi, (X_t, W_t;\, 0 \leq t \leq T)), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t;\, 0 \leq t \leq T),$$

to

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \\ X_0 = \xi \sim \mu_0 \end{cases} \tag{19}$$

then $\mathbb{Q} = \mathbb{P} \circ (X_t;\, 0 \leq t \leq T)^{-1}$, the probability measure generated by $(X_t;\, 0 \leq t \leq T)$ on $(\mathcal{C}([0,T];\mathbb{R}^d), \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d)))$ is a solution to the local martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{0 \leq t \leq T})$ with

$$\mathcal{A}_t(f)(x) = b(t,x) \cdot \nabla_x f(x) + \frac{1}{2} \sum_{i,j=1}^{d} a^{i,j}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} f(x),\, 0 \leq t \leq T,$$

for $a = \sigma\sigma^*$.

- Reciprocally,

### Proposition 4.5

*If there exists a solution $\mathbb{Q}$ to the local martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{t\in[0,T]})$ for*

$$\mathcal{A}_t(f)(x) = b(t,x)\cdot\nabla_x f(x) + \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} f(x),\ 0 \leq t \leq T,$$

*then there exists a weak solution to the SDE*

$$\begin{aligned} dX_t &= b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t,\ t \geq 0,\\ X_0 &= \xi \sim \mu_0. \end{aligned}$$

The main idea of the demonstration is to build a $\mathbb{R}^d$-Brownian motion $(w(t);\, 0 \leq t \leq T)$ (defined on $(\mathcal{C}([0,T];\mathbb{R}^d), \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d)), \mathbb{Q})$) such that

$$x(t) = x(0) + \int_0^t b(s, x(s))\, ds + \int_0^t \sigma(s, x(s))\, dw(s).$$

For this construction, we need the following representation theorem (see the preceding lecture for the one-dimensional case):

**Theorem 4.6 ((KS88) Theorem 4.2, Chapter 3)**

*Let $(M_t;\, 0 \leq t \leq T)$ be a continuous $\mathcal{F}_t$-local martingale defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$. Suppose that, for all $i, j$ $t \mapsto \langle M^i, M^j \rangle_t(\omega)$ is an absolutely continuous function, for $\mathbb{P}$-almost all $\omega$ in $\Omega$. Then there exists an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t;\, 0 \leq t \leq T), \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0 \leq t \leq T), \mathbb{P})$ on which is defined a $\tilde{\mathcal{F}}_t$-Brownian motion, valued in $\mathbb{R}^d$, and a matrix valued $(\Psi_t = \{\Psi_t^{i,j}\}_{1 \leq i,j \leq d};\, 0 \leq t \leq T)$ progressively measurable process such that, for all $0 < T < \infty$, $1 \leq i, j \leq d$, $\mathbb{P}\left( \int_0^T |\Psi_s^{i,j}|^2\, ds < \infty \right) = 1$, and $\tilde{\mathbb{P}}$-a.s., and consider the martingale in the case $f(x) = x$ and $f(x) = x^2$.*

$$M_t^i = \sum_{j=1}^d \int_0^t \Psi_s^{i,j}\, dW_s^j,\ \langle M^i, M^j \rangle_t = \sum_{k=1}^d \int_0^t \Psi_s^{i,k} \Psi_s^{j,k}\, ds,\ 0 \leq t \leq T.$$

The main idea of the demonstration is to build a $\mathbb{R}^d$-Brownian motion $(w(t);\, 0 \leq t \leq T)$ (defined on $(\mathcal{C}([0,T];\mathbb{R}^d), \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d)), \mathbb{Q})$) such that

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$$M_t^i = \sum_{j=1}^d \int_0^t \Psi_s^{i,j}\, dW_s^j,\ \langle M^i, M^j \rangle_t = \sum_{k=1}^d \int_0^t \Psi_s^{i,k} \Psi_s^{j,k}\, ds,\ 0 \leq t \leq T.$$

**Proof of Proposition 4.5 in the one dimensional case:** Since $\mathbb{Q}$ is solution to the martingale problem, taking $f(x) = x$ implies that

$$M_t := x(t) - x(0) - \int_0^t b(s, x(s))\,ds,\ 0 \leq t \leq T,$$

is a local martingale under $\mathbb{Q}$. To compute the quadratic variation of $(M_t;\, 0 \leq t \leq T)$, let us observe that

$$\begin{aligned}
M_t^2 &= \left(x(t) - x(0) - \int_0^t b(s, x(s))\,ds\right)^2 \\
&= x^2(t) + x^2(0) + \left(\int_0^t b(s, x(s))\,ds\right)^2 \\
&\quad - 2x(t)x(0) - 2x(t)\int_0^t b(s, x(s))\,ds + 2x(0)\int_0^t b(s, x(s))\,ds \\
&= x^2(t) + x^2(0) + \left(\int_0^t b(s, x(s))\,ds\right)^2 \\
&\quad - 2x(0)M_t - 2x(t)\int_0^t b(s, x(s))\,ds.
\end{aligned}$$

Coming back to the martingale problem, taking $f(x) = x^2$ ensures that

$$\tilde{M}_t := x^2(t) - x^2(0) - 2\int_0^t x(s)b(s,x(s))\,ds - \int_0^t \sigma^2(s,x(s))\,ds,\ 0 \leq t \leq T,$$

is a local martingale under $\mathbb{Q}$. Then, we have

$$\begin{aligned} M_t^2 = \tilde{M}_t + 2x^2(0) - 2x(0)M_t + \int_0^t \sigma^2(s,x(s))\,ds \\ - 2x(t)\int_0^t b(s,x(s))\,ds + 2\int_0^t x(s)b(s,x(s))\,ds + \left(\int_0^t b(s,x(s))\,ds\right)^2. \end{aligned}$$

Observing that

$$
\begin{aligned}
&2x(t)\int_0^t b(s,x(s))\,ds + 2\int_0^t x(s)b(s,x(s))\,ds + \left(\int_0^t b(s,x(s))\,ds\right)^2\\
&= 2\int_0^t \left(x(t)-x(s)-\int_0^s b(r,x(r))\,dr\right) b(s,x(s))\,ds\\
&= 2\int_0^t \left(M_t - M_s\right) b(s,x(s))\,ds\\
&= 2\int_0^t \left(\int_0^s b(r,x(r))\,dr\right)\,dM_s,
\end{aligned}
$$

applying Itô's formula. It follows that

$$
M_t^2 = \tilde{M}_t + 2x^2(0) - 2x(0)M_t + \int_0^t \sigma^2(s,x(s))\,ds - 2\int_0^t \left(\int_0^s b(r,x(r))\,dr\right)\,dM_s
$$

and $M_t^2 - \int_0^t \sigma^2(s,x(s))\,ds$ is a local martingale.

Applying Theorem 4.6, there exists an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{Q}})$ of $(\mathcal{C}([0,T];\mathbb{R}^d), \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d)), \mathbb{Q})$ under which is defined a standard Brownian motion (up to $T$) $(\tilde{w}(t);\, 0 \leq t \leq T)$, a filtration $(\tilde{\mathcal{F}}_t;\, 0 \leq t \leq T)$ and a $\mathcal{F}_t$-adapted process $(\Psi_t;\, 0 \leq t \leq T)$ such that, $\tilde{Q}$-a.s.,

$$M_t = \int_0^t \Psi_s \, d\tilde{w}(s), \, \forall 0 \leq t \leq T,$$

$$\tilde{\mathbb{Q}}\left(\int_0^t |\Psi_s|^2 \, ds < \infty, \, 0 \leq t \leq T\right) = 1,$$

and

$$M_t^2 = \int_0^t \psi_s^2 \, ds, \forall 0 \leq t \leq T.$$

Now it remains to construct a Brownian motion $(w(t);\, 0 \leq t \leq T)$ such that

$$\int_0^t \Psi_s \, d\tilde{w}(s) = \int_0^t \sigma(s, x(s)) \, dw(s), \, \forall\, 0 \leq t \leq T.$$

Since

$$\tilde{\mathbb{Q}}\left(\int_0^t \sigma^2(s,x(s))\,ds = \int_0^t \Psi_s^2\,ds,\,\forall\, 0 \leq t \leq T\right) = 1,$$

we have

$$\tilde{\mathbb{Q}}\left(\sigma^2(t,x(t)) = \Psi_t^2, \text{ for a.e. } 0 \leq t \leq T\right) = 1,$$

and $\Psi_t = \pm\sigma(t,x(t))$. Define

$$w(t) = \int_0^t \frac{\Psi_s}{\sigma(s,x(s))}\,d\tilde{w}(s),$$

in such a way that

$$\int_0^t \sigma(s,x(s))\,dw(s) = \int_0^t \Psi_s\,d\tilde{w}(s).$$

Observing that $(w(t);\,0 \leq t \leq T)$ is a local martingale satisfying

$$\langle w\rangle_t = \int_0^t \left(\frac{\Psi_s^2}{\sigma^2(s,x(s))}\right)^2 d\langle\tilde{w}\rangle_s = \langle\tilde{w}\rangle_t = t,$$

$(w(t);\,0 \leq t \leq T)$ is a $\mathbb{R}$-Brownian motion. This concludes the one-dimensional case.

## An existence result

### Theorem 4.7

*Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ two bounded and continuous functions and let $\mu_0$ be a probability measure defined in $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} |x|^2 \, \mu_0(dx) < \infty.$$

*Then there exists a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{t\in[0,T]})$ where*

$$\mathcal{A}_t(f)(x) = b(t,x) \cdot \nabla_x f(x) + \frac{1}{2} Trace(a(t,x)\nabla^2 f(x)), \; a = \sigma\sigma^*.$$

**Applications:**

- $dX_t = \; dt + \min\left(|X_t|^\alpha, 1\right) dB_t, \; X_0 = 1$ for $0 < \alpha < 1$ and $(B_t)$ a standard Brownian motion;
- $dX_t = -X_t \, dt + \sqrt{(1 + \sqrt{|X_t|})} \, dW_t, \; X_0 = 0$ for $0 < \alpha < 1$ and $(W_t)$ a standard Brownian motion.

**Proof of Theorem 4.6:**

The idea of the proof consists in a smoothing technique, that is: approximating the drift vector $b$ and the diffusion matrix $a$ by sequences of smooth functions $b_n$ and $a_n$ for which we can construct a solution to the martingale problem for these coefficients and show that at least one limit point of $P^n$ is solution to the martingale problem for $(\mu_0, \{\mathcal{A}_t\}_{0\leq t\leq T})$.

**Preliminary:**

**Notation**: A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be in $\mathcal{C}_c^\infty(\mathbb{R}^m)$ if $f$ admits continuous derivatives at all orders and that $f = 0$ outside a compact (i.e. bounded closed) subset of $\mathbb{R}^m$ (for instance outside $\overline{B}(0, R) = \{z \in \mathbb{R}^m \mid |z| \leq R\}$.)

**Lemma 4.8**

*For all function $f \in \mathcal{C}_b(\mathbb{R}^m)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that, for $n$, $f_n \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, $\|f_n\|_\infty \leq \|f\|_\infty$ and, for any compact subset $K \subset \mathbb{R}^d$,*

$$\lim_{n\to\infty} \max_{x\in K} |f_n(x) - f(x)| = 0.$$

Applying the previous lemma, there exists $\{b_n\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ (and by extension $\{a_n\}_{n\in\mathbb{N}}$), $\mathcal{C}_c^\infty$-approximations of $b$ and $a$ respectively. In particular, for all $n$, $b_n$ and $\sigma_n$ are Lipschitz functions.

For all $n$, there exists a unique process $(X^n_t; 0 \leq t \leq T)$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowing a $\mathbb{R}^d$-Brownian motion $(W_t; 0 \leq t \leq T)$ and an independent r.v. $\xi$ distributed according to $\mu_0$, such that

$$\begin{cases} X^n_t = \xi + \int_0^t b_n(s, X^n_s)\, ds + \int_0^t \sigma_n(s, X^n_s)\, dW_s,\ 0 \leq t \leq T, \\ \xi \sim \mu_0. \end{cases}$$

By extension, there exists a unique solution $P^n$ to the martingale problem related to $(\mu_0, \{\mathcal{A}^n_t\}_{t \geq 0})$ where

$$\mathcal{A}^n_t(f)(x) = b_n(t,x) \cdot \nabla f(x) + \frac{1}{2}\text{Trace}\left(a_n(t,x)\nabla^2 f(x)\right).$$

The sequence $\{P^n\}_{n \in \mathbb{N}}$ can be constructed on $(\mathcal{C}([0,T]; \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0,T]; \mathbb{R}^d)))$. Now our aim is to show that we can extract from $\{P^n\}_{n \in \mathbb{N}}$ a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{t \geq 0})$ for

$$\mathcal{A}_t(f)(x) = b(t,x) \cdot \nabla f(x) + \frac{1}{2}\text{Trace}\left(a(t,x)\nabla^2 f(x)\right).$$

**Convergence of a sequence of probability measures defined on $\mathcal{C}([0,T];\mathbb{R}^d)$**
(Reference: **Convergence of Probability Measures**, P. Billingsley, 1999)
Let $E$ be a general separable metric space and $\mathcal{E}=\mathcal{B}(E)$.

• A sequence of probability measures $\{P^N\}_{N\in\mathbb{N}}$ defined on a metric space $(E,\mathcal{E})$ is said to be tight i.f.f., for all $\epsilon>0$, there exists a compact subset $K_\epsilon\subset E$ such that $\sup_{N\in\mathbb{N}} P^N(K_\epsilon^c)\leq\epsilon$ for $K_\epsilon^c$ the complementary set of $K_\epsilon$.

• $\{P^N\}_{N\in\mathbb{N}}$ is said to be relatively compact i.f.f. for any subsequence $\{P^{N_k}\}_{k\in\mathbb{N}}$ such that we can extract a further subsequence $\{P^{N_{k_m}}\}_{m\in\mathbb{N}}$ such that $P^{N_{k_m}}$ converges weakly to some probability measure $P^\infty$ defined on $(E,\mathcal{E})$ i.e.

$$P^{N_{k_m}} \overset{\text{in distribution}}{\longrightarrow} P^\infty,$$

or equivalently: for all $F:E\to\mathbb{R}$ continuous and bounded, we have

$$\lim_m \int_E F(z)P^{N_{k_m}}(dz)=\int_E F(z)P^\infty(dz).$$

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• $\{P^N\}_{N\in\mathbb{N}}$ is said to be relatively compact i.f.f. for any subsequence $\{P^{N_k}\}_{k\in\mathbb{N}}$ such that we can extract a further subsequence $\{P^{N_{k_m}}\}_{m\in\mathbb{N}}$ such that $P^{N_{k_m}}$ converges weakly to some probability measure $P^\infty$ defined on $(E,\mathcal{E})$ i.e.

$$P^{N_{k_m}} \overset{\text{in distribution}}{\longrightarrow} P^\infty,$$

or equivalently: for all $F : E \to \mathbb{R}$ continuous and bounded, we have

$$\lim_m \int_E F(z) P^{N_{k_m}}(dz) = \int_E F(z) P^\infty(dz).$$

## Lemma 4.9 (Billingsley (1999), p. 59)

*If $\{P^N\}_{N\in\mathbb{N}}$ is tight then $\{P^N\}_{N\in\mathbb{N}}$ is relatively compact. In particular if $\{P^N\}_{N\in\mathbb{N}}$ is tight and if any subsequence $\{P^{N_k}\}_{k\in\mathbb{N}}$ converges weakly towards the same probability measure $P^\infty$ then $\{P^N\}_{N\in\mathbb{N}}$ converges weakly towards $P^\infty$.*

In the case $E = \mathcal{C}([0,T];\mathbb{R}^d)$ equipped with its uniform metric

$$\|\omega\|_\infty = \max_{0\leq t\leq T} |\omega(t)|,$$

the compact sets of $\mathcal{C}([0,T];\mathbb{R}^d)$ are characterized by the following criterion: A subset $K$ of $\mathcal{C}([0,T];\mathbb{R}^d)$ is relatively compact i.f.f.

$$\sup_{f\in A} |f(0)| < \infty,$$

and

$$\lim_{\delta\to\infty} \sup_{f\in A} |w_f(\delta)| < \infty,$$

where $w_f(\delta)$ is the modulus of continuity of $f$:

$$w_f(\delta) = \max_{0\leq s,t\leq T,\, |t-s|\leq\delta} |f(t)-f(s)|.$$

**Theorem 4.10 (Billingsley (1999), p. 59)**

*A sequence of probability measure* $\{P^N\}_{N\in\mathbb{N}}$ *defined on* $(\mathcal{C}([0,T];\mathbb{R}^d),\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^d)))$ *is tight i.f.f. the two following conditions are satisfied:*
*(i) For all* $\epsilon > 0$, *there exists* $\kappa(=\kappa(\epsilon)) > 0$ *and a rank* $N_0(=N_0(\epsilon))$ *such that, for all* $N \geq N_0$,

$$P^N(x(0) \geq \kappa) \leq \epsilon$$

*(ii) For all* $\epsilon, K > 0$, *there exists* $\kappa > 0$, $0 < \delta < 1$ *and a rank* $N_0 = N_0(\epsilon)$ *(all depending on* $\epsilon$, $K$*) such that, for all* $N \geq N_0$,

$$P^N\left(\max_{0\leq s,t\leq T,\,|t-s|\leq\delta}|x(t)-x(s)|\geq\kappa\right)\leq\epsilon.$$

In the particular case where $\{P^N\}_{N\in\mathbb{N}}$ are the laws generated by a sequence of continuous process $\{(X^N_t;\,0\leq t\leq T)\}_{N\in\mathbb{N}}$, we have the following criterion

**Lemma 4.11**

*Let $\{(X^N_t;\,0\leq t\leq T)\}_{N\in\mathbb{N}}$ be a sequence of continuous processes defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the conditions:*

$$(i)\quad \sup_N \mathbb{E}_{\mathbb{P}}\left[|X^N_0|^\nu\right]<\infty,$$

*and, for all $0\leq s,t\leq T$, there exits $0<C_T<\infty$ such that*

$$(ii)\quad \sup_N \mathbb{E}_{\mathbb{P}}\left[\left|X^N_t-X^N_s\right|^\alpha\right]\leq C_T|t-s|^{1+\beta},$$

*for some positive finite real numbers $\nu,\beta,\alpha$. Then the sequence of probability measures $\{P^N\}_{N\in\mathbb{N}}$ defined by*

$$P^N=\mathbb{P}\circ(X^N;\,0\leq t\leq T)^{-1}$$

*is tight on $\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))$.*

Coming back to our case, and observing that, since $b$ and $a$ are bounded $b^n$ and $a^n$ are also bounded, we get that

$$\mathbb{E}_{\mathbb{P}}\left[|X^n_0|^2\right] = \mathbb{E}_{\mathbb{P}}\left[|X_0|^2\right] = \int |x|^2 \mu_0(dx) < \infty,$$

by assumption, and that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[|X^n_t - X^n_s|^2\right] &= \mathbb{E}_{\mathbb{P}}\left[|\int_s^t b_n(r, X^n_r)\,dr + \int_s^t \sigma_n(r, X^n_r)\,dW_r|^2\right] \\
&\leq \mathbb{E}_{\mathbb{P}}\left[2\int_s^t |b_n(r, X^n_r)|^2\,dr + \int_s^t a_n(r, X^n_r)\,dr\right] \leq C\left(\|b\|_\infty + \|a\|_\infty\right)(t-s)
\end{aligned}$$

therefore the sequence $\{P^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{C}([0,T];\mathbb{R}^d)$.

Let us consider $\{P^{n_{k_m}}\}_{m\in\mathbb{N}}$ a converging subsequence of $\{P^n\}_{n\in\mathbb{N}}$ and $P^\infty$ its limit. Our aim is now to show that $P^\infty$ is a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{t\geq 0})$; that is $P^\infty$ satisfies the following conditions
(1) $P^\infty \circ x(0)^{-1} = \mu_0$ and

(2) For all function $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, the process

$$M_t^f = f(x(t)) - f(x(0)) - \int_0^t b(s, x(s)) \cdot \nabla f(x(s))\, ds - \frac{1}{2}\int_0^t \mathrm{Trace}(a(s, x(s)) \nabla^2 f(s, x(s)))\, ds$$

is a martingale under $P^\infty$.

**For the condition (1):** By weak convergence of $P^{n_{k_m}}$, for all $f \in \mathcal{C}_b(\mathbb{R}^d)$,

$$\mathbb{E}_{P^\infty}[f(x(0))] = \lim_{m\to\infty} \mathbb{E}_{P^{n_{k_m}}}[f(x(0))] = \int f(x)\mu_0(dx),$$

which implies that $P^\infty \circ x(0)^{-1} = \mu_0$.

Let us consider $\{P^{n_{k_m}}\}_{m\in\mathbb{N}}$ a converging subsequence of $\{P^n\}_{n\in\mathbb{N}}$ and $P^\infty$ its limit. Our aim is now to show that $P^\infty$ is a solution to the martingale problem related to $(\mu_0, \{\mathcal{A}_t\}_{t\geq 0})$; that is $P^\infty$ satisfies the following conditions
(1) $P^\infty \circ x(0)^{-1} = \mu_0$ and

(2) For all function $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, the process

$$M_t^f = f(x(t)) - f(x(0)) - \int_0^t b(s, x(s))\cdot\nabla f(x(s))\,ds - \frac{1}{2}\int_0^t \mathrm{Trace}(a(s, x(s))\nabla^2 f(s, x(s)))\,ds$$

is a martingale under $P^\infty$.

**For the condition (1):** By weak convergence of $P^{n_{k_m}}$, for all $f \in \mathcal{C}_b(\mathbb{R}^d)$,

$$\mathbb{E}_{P^\infty}[f(x(0))] = \lim_{m\to\infty} \mathbb{E}_{P^{n_{k_m}}}[f(x(0))] = \int f(x)\mu_0(dx),$$

which implies that $P^\infty \circ x(0)^{-1} = \mu_0$.

**For the condition** (2)**:** It is enough to show that, for all $0 \leq s \leq t \leq T$ and $F \in \mathcal{C}_b(\mathcal{C}([0,s];\mathbb{R}^d))$,

$$\mathbb{E}_{P^\infty}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^f - M_s^f\right)\right] = 0.$$

Since $\omega \in \mathcal{C}([0,T];\mathbb{R}^d) \mapsto F(x(r)(\omega);\, 0 \leq r \leq s)\left(M_t^f(\omega) - M_s^f(\omega)\right)$ is bounded and continuous, by the weak convergence $P^{n_{k_m}} \to P$, we have

$$\begin{aligned}&\mathbb{E}_{P^\infty}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^f - M_s^f\right)\right]\\ &= \lim_{m\to\infty} \mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^f - M_s^f\right)\right].\end{aligned}$$

Since, for all $m$, $P^{n_{k_m}}$ is a solution to the martingale related to $(\mu_0, \{\mathcal{A}_t^{n_k}\}_{t\geq 0})$,

$$\mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^{f,n_{k_m}} - M_s^{f,n_{k_m}}\right)\right] = 0,$$

where

$$M_t^{f,n_{k_m}} = f(x(t)) - f(x(0)) - \int_0^t \mathcal{A}_{n_{k_m}}(f)(s,x(s))\,ds.$$

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$$\begin{aligned}
&\mathbb{E}_{P^\infty}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^f - M_s^f\right)\right] \\
&= \lim_{m\to\infty} \mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^f - M_s^f\right)\right].
\end{aligned}$$

Since, for all $m$, $P^{n_{k_m}}$ is a solution to the martingale related to $(\mu_0, \{\mathcal{A}_t^{n_k}\}_{t\geq 0})$,

$$\mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\, 0 \leq r \leq s)\left(M_t^{f,n_{k_m}} - M_s^{f,n_{k_m}}\right)\right] = 0,$$

where

$$M_t^{f,n_{k_m}} = f(x(t)) - f(x(0)) - \int_0^t \mathcal{A}_{n_{k_m}}(f)(s, x(s))\,ds.$$

Therefore to show that

$$\mathbb{E}_{P^\infty}\left[F(x(r);\,0\leq r\leq s)\left(M^f_t-M^f_s\right)\right]=0,$$

it is enough to show that

$$\lim_{m\to\infty}\Big|\mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\,0\leq r\leq s)\left(M^f_t-M^f_s\right)\right] - \mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\,0\leq r\leq s)\left(M^{f,n_{k_m}}_t-M^{f,n_{k_m}}_s\right)\right]\Big|=0.$$

Observing that

$$
\begin{aligned}
&\Big|\mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\,0\leq r\leq s)\left(M^f_t-M^f_s\right)\right]\\
&\qquad-\mathbb{E}_{P^{n_{k_m}}}\left[F(x(r);\,0\leq r\leq s)\left(M^{f,n_k}_t-M^{f,n_k}_s\right)\right]\Big|\\
&\leq\|F\|_\infty\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t\left(b_{n_k}(r,x(r))-b(r,x(r))\right)\cdot\nabla f(r,x(r))\,dr\right|\right]\\
&\quad+\|F\|_\infty\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t\mathrm{Trace}\left(\left(a_{n_k}(r,x(r))-a(r,x(r))\right)\nabla^2 f(r,x(r))\right)\,dr\right|\right]\\
&\leq\|F\|_\infty\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t\mathbb{1}_{|x(r)|\leq\kappa}\left(b_{n_k}(r,x(r))-b(r,x(r))\right)\cdot\nabla f(r,x(r))\,dr\right|\right]\\
&\quad+\|F\|_\infty\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t\mathbb{1}_{|x(r)|\leq\kappa}\mathrm{Trace}\left(\left(a_{n_k}(r,x(r))-a(r,x(r))\right)\nabla^2 f(r,x(r))\right)\,dr\right|\right]\\
&\quad+\|F\|_\infty\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t\mathbb{1}_{|x(r)|>\kappa}\left(b_{n_k}(r,x(r))-b(r,x(r))\right)\cdot\nabla f(r,x(r))\,dr\right|\right]\\
&\quad+\|F\|_\infty\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t\mathbb{1}_{|x(r)|>\kappa}\mathrm{Trace}\left(\left(a_{n_k}(r,x(r))-a(r,x(r))\right)\nabla^2 f(r,x(r))\right)\,dr\right|\right]
\end{aligned}
$$

for some $\kappa>0$.

Since, for all fixed $\kappa > 0$,

$$(b_{n_k}, a_{n_k}) \to (b, a) \text{ uniformly on } \{x \in \mathbb{R}^d \mid |x| \leq \kappa\},$$

we have

$$\lim_{m\to\infty} \|F\|_\infty \mathbb{E}_{P^{n_{k_m}}} \left[\left|\int_s^t \mathbb{1}_{|x(r)|\leq\kappa} \text{Trace}\left(\left(a_{n_k}(r,x(r)) - a(r,x(r))\right) \nabla^2 f(r,x(r))\right)\, dr\right|\right]$$
$$+ \lim_{m\to\infty} \|F\|_\infty \mathbb{E}_{P^{n_{k_m}}} \left[\left|\int_s^t \mathbb{1}_{|x(r)|\leq\kappa} \left(b_{n_k}(r,x(r)) - b(r,x(r))\right) \cdot \nabla f(r,x(r))\, dr\right|\right] = 0.$$

For the remaining terms, since

$$\mathbb{E}_{P^{n_{k_m}}} \left[\left|\int_s^t \mathbb{1}_{|x(r)|>\kappa} \left(b_{n_k}(r,x(r)) - b(r,x(r))\right) \cdot \nabla f(r,x(r))\, dr\right|\right]$$
$$\leq C\|F\|_\infty \|\nabla_x f\|_\infty \|b\|_\infty \int_s^t \mathbb{E}_{P^{n_{k_m}}} \left(|x(r)| > \kappa\right)\, dr$$
$$\leq C\|F\|_\infty \|\nabla_x f\|_\infty \|b\|_\infty \sup_n \max_{0\leq r\leq t} \mathbb{P}\left(|X^n_r| \geq \kappa\right)$$

Since

$$\max_{0\leq r\leq t}\mathbb{P}\left(|X^n_r|\geq\kappa\right)\leq\frac{\mathbb{E}_{\mathbb{P}}\left[\max_{0\leq r\leq t}|X^n_r|^2\right]}{\kappa^2}\leq\frac{C'}{\kappa^2},$$

for $C'$ depending only on $t, d, \|b\|_\infty$ and $\|a\|_\infty$, we get that

$$\lim_{\kappa\to\infty}\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t \mathbb{1}_{|x(r)|>\kappa}\left(b_{n_k}(r,x(r))-b(r,x(r))\right)\cdot\nabla f(r,x(r))\,dr\right|\right]=0$$

In the same way,

$$\lim_{\kappa\to\infty}\mathbb{E}_{P^{n_{k_m}}}\left[\left|\int_s^t \mathbb{1}_{|x(r)|>\kappa}\mathrm{Trace}\left(\left(a_{n_k}(r,x(r))-a(r,x(r))\right)\nabla^2 f(r,x(r))\right)\,dr\right|\right]=0.$$

This ensures that

$$\begin{aligned}\lim_{k\to\infty}\Big(&\mathbb{E}_{P^{n_k}}\left[F(x(r);\,0\leq r\leq s)\left(M^f_t-M^f_s\right)\right]\\&-\mathbb{E}_{P^{n_k}}\left[F(x(r);\,0\leq r\leq s)\left(M^{f,n_k}_t-M^{f,n_k}_s\right)\right]\Big)=0,\end{aligned}$$

from which we can conclude that

$$\mathbb{E}_{P^\infty}\left[F(x(r);\,0\leq r\leq s)\left(M^f_t-M^f_s\right)\right]=0.$$

## A uniqueness result

### Theorem 4.12

*Suppose that, for all $f \in \mathcal{C}^2_c(\mathbb{R}^d)$ and for all $T_0 \in [0, T]$, there exists a function $u^f : [0, T_0] \times \mathbb{R}^d \to \mathbb{R}$ in $\mathcal{C}^1([0, T]; \mathcal{C}^2(\mathbb{R}^d))$ such that*

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}(u)(t,x) = 0, \ (t,x) \in [0, T_0) \times \mathbb{R}^d, \\ u(T_0, x) = f(x), \ x \in \mathbb{R}^d. \end{cases} \tag{20}$$

*Suppose also that $b$ and $\sigma$ are locally bounded in $\mathbb{R}^d$.*
*Then, for all $x \in \mathbb{R}^d$ and for all solution $\mathbb{P}^x$ and $\tilde{\mathbb{P}}^x$ to the martingale problem related to $(\delta_{\{x\}}, \{\mathcal{A}_t\}_{t\geq 0})$, we have*

$$\mathbb{E}_{\mathbb{P}^x}\left[f(x(T_0))\right] = \mathbb{E}_{\tilde{\mathbb{P}}^x}\left[f(x(T_0))\right].$$

**Proof of Theorem 4.12:** Under either $\mathbb{P}^x$ or $\tilde{\mathbb{P}}^x$, the canonical process $(x(t);\ t \in [0, T])$ satisfies the SDE

$$dx(t) = b(x(t))\,dt + \sigma(x(t))dw(t),\ x(0) \sim \mu_0,$$

where $(w(t);\ t \in [0, T])$ is some Brownian motion.

Since $u$ is in $\mathcal{C}^1((0, T_0); \mathcal{C}^2(\mathbb{R}^d)) \cap \mathcal{C}([0, T_0] \times \mathbb{R}^d)$, Itô's formula implies that, for all $\tilde{T} \leq T_0$,

$$u(\tilde{T}, x(\tilde{T})) = u(0, x(0)) + \int_0^{\tilde{T}} (\partial_s u + \mathcal{A}(u))\,(s, x(s))\,ds + \int_0^{\tilde{T}} \nabla_x u(s, x(s))\sigma(x(s))\,dw(s).$$

Since

$$\partial_t u + \mathcal{A}(u) = 0 \,\text{en}\, [0, \tilde{T}) \times \mathbb{R}^d,$$

the preceding equality reduces to

$$u(\tilde{T}, x(\tilde{T})) = u(0, x(0)) + \int_0^{\tilde{T}} \nabla_x u(s, x(s))\sigma(x(s))\,dw(s).$$

where $u(0, x(0)) = u(0, x)$ $\mathbb{P}^x$-a.s. and $\tilde{\mathbb{P}}^x$-a.s..

**Proof of Theorem 4.12:** Under either $\mathbb{P}^x$ or $\tilde{\mathbb{P}}^x$, the canonical process $(x(t);\ t \in [0, T])$ satisfies the SDE

$$dx(t) = b(x(t))\,dt + \sigma(x(t))dw(t),\ x(0) \sim \mu_0,$$

where $(w(t);\ t \in [0, T])$ is some Brownian motion.
Since $u$ is in $\mathcal{C}^1((0, T_0); \mathcal{C}^2(\mathbb{R}^d)) \cap \mathcal{C}([0, T_0] \times \mathbb{R}^d)$, Itô's formula implies that, for all $\tilde{T} \leq T_0$,

$$\begin{aligned} u(\tilde{T}, x(\tilde{T})) = u(0, x(0)) &+ \int_0^{\tilde{T}} \left(\partial_s u + \mathcal{A}(u)\right)(s, x(s))\,ds \\ &+ \int_0^{\tilde{T}} \nabla_x u(s, x(s))\sigma(x(s))\,dw(s). \end{aligned}$$

Since

$$\partial_t u + \mathcal{A}(u) = 0 \,\text{en}\, [0, \tilde{T}) \times \mathbb{R}^d,$$

the preceding equality reduces to

$$u(\tilde{T}, x(\tilde{T})) = u(0, x(0)) + \int_0^{\tilde{T}} \nabla_x u(s, x(s))\sigma(x(s))\,dw(s).$$

where $u(0, x(0)) = u(0, x)$ $\mathbb{P}^x$-a.s. and $\tilde{\mathbb{P}}^x$-a.s..

Taking $\tilde{T} = \tau_M \wedge T_0$ where

$$\tau_M = \inf\left\{ t > 0 \,|\, \int_0^t |\nabla_x u(s, x(s))\sigma(s, x(s))|^2 \, ds > M \right\},$$

and using Doob's inequality, we deduce that

$$\mathbb{E}_{\mathbb{P}^x}\left[u(\tau_M \wedge T_0, x(\tau_M \wedge T_0))\right] = u(0, x) = \mathbb{E}_{\mathbb{P}^x}\left[u(\tau_M \wedge T_0, x(\tau_M \wedge T_0))\right].$$

Then, observing that

$$\mathbb{P}^x(\lim_{M \to \infty} \tau_M = T_0) = 1 = \tilde{\mathbb{P}}^x(\lim_{M \to \infty} \tau_M = T_0),$$

and since $x \to u(T_0, x)$, we have

$$\mathbb{E}_{\mathbb{P}^x}\left[f(x(T_0))\right] = \mathbb{E}_{\tilde{\mathbb{P}}^x}\left[u(0, x(0))\right] = u(0, x).$$

## Theorem 4.13 (Existence result, (F06) Theorem 4.5, Chapter 6)

*Suppose that $a$ is bounded and definite positive in the sense that there exist $0 < \lambda < \Lambda < \infty$ such that*

$$\lambda|\xi|^2 \leq (\xi \cdot a(x)\xi) \leq \Lambda|\xi|^2, \forall (t,x), \forall \xi \in \mathbb{R}^d,$$

*and suppose that $b$ and $a$ are $\mathcal{C}^\alpha$-continuous $(0 < \alpha < 1)$:*

$$|b(x) - b(y)| + |a(x) - a(y)| \leq C|x - y|^\alpha, \forall x, y \in \mathbb{R}^d.$$

*Suppose also that $f$ is continuous with compact support. Therefore, there exists a unique solution of class $\mathcal{C}([0,T] \times \mathbb{R}^d) \cap \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^d))$ to the Cauchy problem (20). This solution further admits the representation*

$$u(t,x) = \int \Gamma(x,t;y,T) f(y)\, dy$$

*where $\Gamma$ is such that*

$$|D_x^m \Gamma(x,t;y,T)| \leq \left(\frac{C}{T-t}\right)^{\frac{|m|-d}{2}} \exp\left(-c\frac{|x-y|^2}{T-t}\right)$$

*for $D_x^m = \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \cdots \partial_{x_d}^{m_d}$, $|m| = 0, 1$ and $C, c$ some finite positive constants.*

**Extension**: One can show the existence of a solution to the martingale problem related to $\mu$ and

$$\mathcal{A}_t(f)(x) = b(t,x)\cdot\nabla_x f(x) + \frac{1}{2}\mathrm{Trace}(a(t,x)\nabla_x^2 f(t,x))$$

under more general assumptions

**Theorem 4.14 ((SV79), Chapter 7)**

*Suppose that $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ are two Borel measurable functions such that:*
(1) *$b$ is bounded (that is: there exists $0 < K < \infty$ such that, for all $(t,x)$, $|b(t,x)| \le K$);*
(2) *There exist $0 < \lambda < \Lambda < \infty$ such that,*

$$\lambda|\xi|^2 \le (\xi, \sigma\sigma^*(t,x)\xi) \le \Lambda|\xi|^2,\ \forall\,\xi\in\mathbb{R}^d,\ (t,x)\in[0,T]\times\mathbb{R}^d,$$

(3) *There exists $\delta : (0,\infty)\times(0,\infty)$ increasing such that $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$ and*

$$|(\sigma\sigma^*)(t,x) - (\sigma\sigma^*)(t,y)| \le \delta(|x-y|).$$

*Then there exists a unique solution to the martingale problem related to $(\delta_{\{x\}}, \{\mathcal{A}_t\}_{t\in[0,T]})$ for all $x$ en $\mathbb{R}^d$.*