A Comparison of the Heston and Heston-Jump Models in Pricing Oil Options During the World Oil Crisis Parameter Estimation using MCMC

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1 Introduction

The recent oil crisis has sent shock waves through financial markets with damaging effects for related financial instruments. Oil price data depicts high volatility resulting from the persistent slump in oil prices. Motivated by this, we are interested in considering the possibility of modelling oil contracts in the best possible way at this time of uncertainty. Oil futures option prices are affected not only by the randomness of the underlying asset but also by that of the volatility of the asset's return. Volatility is therefore not a traded asset and makes the market incomplete which have many implications on option prices. The Heston Model [10] is one of the most widely used stochastic volatility models today. Financial institutions employ this model in the pricing of complex instruments to an acceptable degree of accuracy to produce the right returns for investors. The model's attractiveness lies in the powerful duality of its tractability and robustness relative to other stochastic models. It describes a joint process between an asset's price and its variance, assuming that the volatility of the asset is not constant, nor even deterministic but follows a random process[17]. It is also known to fit price data very well when there are no significant spikes or jumps in stock prices. Its robustness however allows for the incorporation of jumps into the modelling framework. Our plan is to add a jump diffusion term to the model according to Bates [1] and compare this with the Heston model. In view of this, our research is threefold: We investigate how including the jump term in the Heston model affects parameter estimation in the model. In this regard, we will consider how the jump term affects the distributions of the parameters in the model. Finally, we measure the accuracy of each of the models in pricing options.

In the next section of this paper we talk about the reasons behind the fall of crude oil prices. In section 3 we discuss the mathematical formulation of the Heston and Heston-Jumps models. In section 4 we summarized the procedures for computation and we used market data of oil futures prices to estimate parameters and price futures options.

2 The World Oil Crisis

After a fall of 29% in 2015 and 44% in 2014 in oil prices these prices continue to plunge further in 2016. While markets took these drops in stride in the past few years, something seems different about the volatile plunge so far in 2016. Why are oil prices so low? And why are the world's asset markets so worked up about it? Oil prices are low because both demand and supply forces are conspiring to make it so.

Here is a summary of reasons:

- **Declining Demand**

  The demand for oil is highly correlated to economic activity. In good times, consumers typically have growing income and thus have a higher demand for goods. Companies, ever eager to supply these goods, have to run factories longer or faster, and demand more energy to do so. More goods get produced, more get transported, and more people drive to buy them or to deliver them. Currently the economies of the top 5 oil-consuming countries in the world - U.S., China, Japan, India, and Russia are weakening. At the same time, vehicles are becoming more efficient, these factors contribute to the declining demand in oil.

- **Oversupply**

  In 2014 the International Energy Agency reported that the U.S. had become the largest oil and natural gas producer, surpassing both Russia and Saudi Arabia. At almost 500 million barrels, U.S. crude oil inventories are at the highest level in at least the last 80 years. This addition to the global oil market was enough to induce oil prices to fall from their highs above $100 per barrel to levels around $70. Actually the total oil production by year-end 2015 rose to over 9.35 million barrels per day, higher than the 9.3 million barrels per day forecast in February 2015. This shows that not only is the market oversupplied, but supply is actually increasing.

  Typically when prices fall, OPEC, the Organization of Petroleum Exporting Countries, has historically controlled the majority of the oil's supply, and tended to increase production when prices were

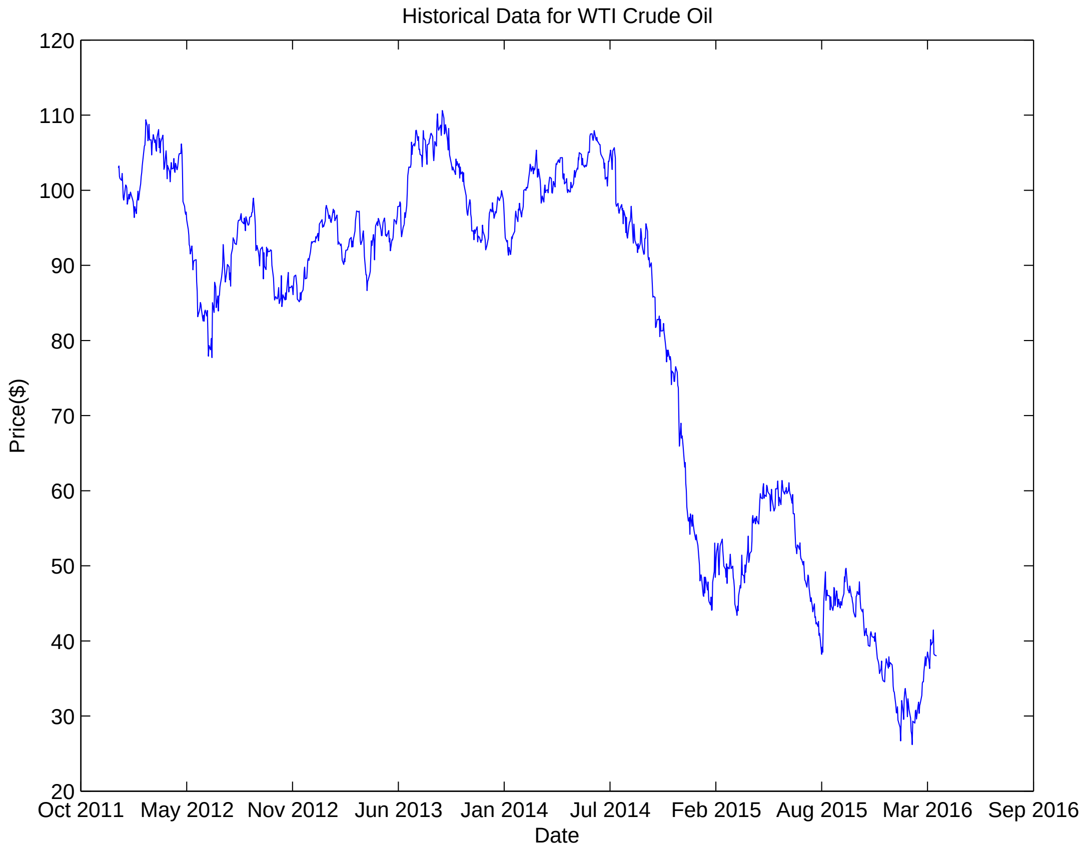


Figure 1: Crude oil prices from 2011- March 2016

too high and cut production with prices fell too much. However, the November 2014 OPEC meeting delivered a significant shock to the world; instead of cutting production to prop up prices, Saudi Arabia in particular showed it was prepared to keep pumping and even lose money, in order to try to force the new higher cost producers out of the market. Prices of OPEC's benchmark crude oil have fallen 50% since the organization decided against cutting production at the 2014 OPEC meeting. It resulted in a further oversupply of oil, placing downward pressure on crude oil prices for the long term.

- **The Strong US Dollar**

  The strong U.S. dollar has been the main driver for the price decline of crude oil over the last few years. This puts the market under a lot of pressure, because when the value of the dollar is strong, the value of commodities fall. Global commodity prices are usually quoted in dollars and fall when the U.S. dollar is strong.

- **Iran Nuclear Deal**

  The Iran nuclear deal is a preliminary framework agreement reached between Iran and a group of world powers. The framework seeks to redesign, convert and reduce Iran's nuclear facilities. The U.S. nuclear deal with Iran allows more Iranian oil exports. The deal removes Western sanctions against Iran, and investors fear it will add to the world's oversupply of oil. Markets have already reacted to this news by decreasing the price of crude oil.+

The implications of the foregoing discussion in financial markets is apparent. The declining oil industry has put pressure on investors and created an uncertain environment in their decision making regarding what investments to make. Though some deem the crisis as an opportunity to benefit one cannot overemphasize the economic implications should an investor make a bad decision. We now introduce the mathematical framework for our analysis.

3 Mathematization

Asset pricing is volatility dependent[10] since volatility captures the market's assessment of future uncertainty in the prices of derivatives. In general, the evolution of an asset price (spot price) depends on the growth rate of the asset's price and the volatility associated with the asset itself.

3.1 Heston's Stochastic Volatility Model

The Heston model[10] assumes the spot price at time, t follows the geometric brownian motion process;

$$dS(t) = \mu S_t dt + \sqrt{v(t)} S_t dW_1(t) \tag{1}$$

where $W_1(t)$ is a Wiener process. Given the assumption of stochastic volatility, the volatility of the option over time is given by the following stochastic process:

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma_v \sqrt{v(t)} dW_2(t) \tag{2}$$

where κ is a mean-reversion speed parameter, θ is the long-run mean of the variance and σ_v is the volatility of the variance process. A correlation parameter, ρ is used to reflect the relationship between the asset return and volatility. Thus

$$\rho = \mathrm{Corr}(W_1(t), W_2(t))$$

For simplicity, Heston assumes the interest rate is constant. Asset pricing depends on the state-price density[1] of the asset [11] and since the Heston model has two sources of randomness, the bivariate Ito's lemma is used to derive the fundamental partial differential equation associated with (1) and 2).

Ito's Lemma Ito derived his lemma from the idea of the Taylor differential expansion but defines ∂S to follow an Ito drift process. Hence for a twice differentiable function $f(S,t)$ where $dS(t) = \mu dt + \sigma dW_1$, we get the Ito's equation as

$$\begin{aligned} \partial f &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + ... \\ \implies \partial f &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}(\mu dt + \sigma dW_1) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\mu dt + \sigma dW_1)^2 + ... \\ &= \Big(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial S^2}\Big)dt + \sigma dW_1, \end{aligned} \tag{3}$$

where we follow the rule of setting dt^2 and $dtdW_1$ terms to zero and substituting dt for $(dW_1)^2$ with the higher order of the differential going to zero.

The steps involved in the derivation of the Heston option pricing formula are the same as those in the no-arbitrage derivation for Black-Scholes formula [3] except that two derivative assets are required to obtain a risk-neutral portfolio. Now lets consider shorting a call option C together with long position in δ units of the underlying asset and γ units of a second call option C_1 written on the same underlying. C_1 differs from C by its maturity or strike price. Let us rewrite, using shorter notations, the system of equations (1) and (2) as

$$dS = \mu_s dt + \sigma_s dW_1$$

[1] This is the price of an asset in a particular state(time)

$$dv = \mu_v dt + \sigma_v dW_2$$

Then from the definition of Ito's lemma[2], we obtain that the dynamics C may be written as

$$dC = \Big[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C}{dS^2} + \rho\sigma_S\sigma_v \frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 C}{\partial v^2} + \mu_S \frac{\partial C}{\partial S} + \mu_v \frac{\partial C}{\partial v} + \frac{\partial C}{\partial t}\Big] dt + \sigma_S \frac{\partial C}{\partial v} dW_1 + \sigma_v \frac{\partial C}{\partial v} dW_2. \quad (4)$$

By the same lemma the dynamics of a portfolio value, $W = C - \delta S - \gamma C_1$, is

$$\begin{aligned} dW &= dC - \delta dS - \gamma C_1 \\ &= \Big[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C}{dS^2} + \rho\sigma_S\sigma_v \frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 C}{\partial v^2} + \mu_S \frac{\partial C}{\partial S} + \mu_v \frac{\partial C}{\partial v} + \frac{\partial C}{\partial t} - \delta\mu_s\Big] dt \\ &\quad - \gamma\Big[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C_1}{dS^2} + \rho\sigma_S\sigma_v \frac{\partial^2 C_1}{\partial S\partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2 C_1}{\partial S^2} + \mu_S \frac{\partial C_1}{\partial S} + \mu_v \frac{\partial C_1}{\partial S} + \frac{\partial C_1}{\partial t}\Big] dt \\ &\quad + \Big[\sigma_S \frac{\partial C}{\partial S} - \delta\sigma_S - \gamma\sigma_S \frac{\partial C_1}{\partial S}\Big] dW_1 + \Big[\sigma_v \frac{\partial C}{\partial v} - \gamma\sigma_v \frac{\partial C_1}{\partial v}\Big] dW_2. \end{aligned} \quad (5)$$

To obtain risk neutrality, the coefficients of dW_1 and dW_2 must be zero, so that the two sources of uncertainty no longer play a role in the portfolio value dynamic. Hence

$$\frac{\partial C}{\partial S} = \delta + \gamma \frac{\partial C_1}{\partial S} \quad (6)$$

$$\frac{\partial C}{\partial v} = \gamma \frac{\partial C_1}{\partial v} \quad (7)$$

With these two conditions, the instantaneous change of value of the fully hedged portfolio must be equal to the return on a risk-free investment otherwise there will be an arbitrage opportunity. Hence

$$dW = r[C - \delta S - \gamma C_1] \quad (8)$$

If we equate (5) and (8), and substitute the values of δ and γ from (6) and (7) respectively, we obtain

$$\begin{aligned} &\Big[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C}{dS^2} + \rho\sigma_S\sigma_v \frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 C}{\partial v^2} + rS \frac{\partial C}{\partial S} + \mu_v \frac{\partial C}{\partial v} + \frac{\partial C}{\partial t} - rC\Big] \Big/ \frac{\partial C}{\partial v} \\ &= \Big[\frac{1}{2}\sigma_S^2 \frac{\partial^2 C_1}{\partial S^2} + \rho\sigma_S\sigma_v \frac{\partial^2 C_1}{\partial S\partial v} + \frac{1}{2}\sigma_v^2 \frac{\partial^2 C_1}{\partial v^2} + rS \frac{\partial C_1}{\partial S} + \mu_v \frac{\partial C_1}{\partial v} + \frac{\partial C_1}{\partial t} - rC_1\Big] \Big/ \frac{\partial C_1}{\partial v} \end{aligned} \quad (9)$$

Notice that the two sides of 9 are the same in their expression, differing only in the option to which they apply. Given that the same equation must hold for any type of call option of any maturity and strike price, then each side of the equality must be independent from the type of option that one considers. This suggests that each side will be equal to some function, say $\lambda(S, v, t)$ that depends on S and v. This function may be interpreted as a volatility risk premium[11]. Replacing the parameters $\mu_v, \mu_s, \sigma_v, \sigma_s$ by their actual units as defined in (1) and (2), we see that the fundamental partial differential equation is now

$$0 = \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma v S \frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 C}{\partial V^2} + rS \frac{\partial C}{\partial S} + [\kappa(\theta - v) - \lambda(S, v, t)] \frac{\partial C}{\partial v} - rC + \frac{\partial C}{\partial t} \quad (10)$$

This is the well-known Heston pde. Heston further makes the important assumption that the volatility risk premium is a linear function of v_t such that $\lambda(S, v, t) = \lambda v$. Moreover, letting $x = \ln S$ or $S = e^x$ then by (10), we have

$$0 = \frac{1}{2} v_t \frac{\partial^2 C}{\partial S^2} + \rho\sigma v_t \frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 C}{\partial v^2} + [\kappa(\theta - v_t) - \lambda v_t] \frac{\partial C}{\partial v} + \Big(r - \frac{1}{2} v_t\Big) \frac{\partial C}{\partial x} - rC + \frac{\partial C}{\partial t} \quad (11)$$

[2]The bivariate version

This PDE is easy to solve since it does not contain coefficients in S as in equation (10). In the case of a European Call option we have the following boundary conditions

$$\begin{aligned} C(S,v,T) &= \max(0, S-K) \\ C(0,v,T) &= 0, \\ \frac{\partial C}{dS}(\infty,v,t) &= 1, \\ rS\frac{\partial C}{dS}(S,0,t) - rC(S,0,t) + C_t(S,0,t) &= 0, \\ C(S,\infty,t) &= S. \end{aligned} \tag{12}$$

By the analogy used in obtaining the Black-Scholes formula, the guessed solution of (11) is of the form

$$\begin{aligned} C(S,v,T) &= SP_1 - e^{r\tau}KP_2 \\ &= e^x P_1 - e^{rt-rT}KP_2 \end{aligned} \tag{13}$$

where $\tau = (T-t)$, P_1 and P_2 are cumulative density functions in relation to the moneyness of the option at maturity. From (13), we obtain the following partial derivatives:

$$\begin{aligned} \frac{\partial C}{\partial x} &= e^x P_1 + e^{rt-rT}K\frac{\partial P_2}{\partial x} \\ \frac{\partial^2 C}{\partial x^2} &= e^x P_1 + 2e^x\frac{\partial P_1}{\partial x} + e^x\frac{\partial^2 P_1}{\partial X^2} - e^{rt-rT}K\frac{\partial^2 P_2}{\partial x^2} \\ \frac{\partial C}{\partial x^2} &= e^x\frac{\partial P_1}{\partial v} - e^{rt-rT}K\frac{\partial P_2}{\partial v} \\ \frac{\partial^2 C}{\partial v^2} &= e^x\frac{\partial^2 P_1}{\partial v} - e^{rt-rT}K\frac{\partial^2 P_2}{\partial v^2} \\ \frac{\partial^2 C}{\partial x \partial v} &= e^x\frac{\partial P_1}{\partial v} + e^x\frac{\partial P_1}{\partial x \partial v} - e^{rt-rT}K\frac{\partial^2 P_2}{\partial x \partial v} \\ \frac{\partial C}{\partial t} &= e^x\frac{\partial P_1}{\partial t} - re^{rt-rT}KP_2 - e^{rt-rT}K\frac{\partial P_2}{\partial t} \end{aligned}$$

Substituting these into (11) gives

$$e^x[f(P_1)] - e^{rt-rT}K[f(P_2)] = 0 \tag{14}$$

where $f(P_i), i = 1, 2$ are polynomial functions of P_i. A solution for (14) is

$$f(P_1) = f(P_2) = 0$$

which means

$$\begin{aligned} f(P_1) = \frac{1}{2}v_t\frac{\partial^2 P_1}{\partial x^2} + \rho\sigma v_t\frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 P_1}{\partial v^2} + \left(r + \frac{1}{2}v_t\right)\frac{\partial P_1}{\partial x} \\ + [\kappa(\theta - v_t) - \lambda v_t + \rho\sigma v_t]\frac{\partial P_1}{\partial v} + +\frac{\partial P_1}{\partial t} \end{aligned}$$

and

$$\begin{aligned} f(P_2) = \frac{1}{2}v_t\frac{\partial^2 P_2}{\partial x^2} + \rho\sigma v_t\frac{\partial^2 P_2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 P_2}{\partial v^2} + \left(r - \frac{1}{2}v_t\right)\frac{\partial P_2}{\partial x} \\ + [\kappa(\theta - v_t) - \lambda v_t]\frac{\partial P_2}{\partial v} + +\frac{\partial P_2}{\partial t} \end{aligned}$$

which is the same as Heston presented in his paper. The cumulative probability function P_j in [10] are not immediately available in closed form but it can be shown that their characteristic functions $f_j(x, v, T; \phi)$

must satisfy the same PDEs in (11) subject to the terminal condition

$$f_j(x, v, T; \phi) = e^{i\phi x}$$

Assuming that the characteristic function has the solution

$$f_j(x, v, T; \phi) = e^{C+Dv+i\phi x} \tag{15}$$

where C and D are functions of $\tau = (T - t)$. Hence, the task now is to find suitable C and D such that $f_j(x, v, T; \phi)$ satisfies the Fokker-Planck forward equation stated in [10]:

$$0 = \frac{1}{2}v_t\frac{\partial^2 f}{\partial x^2} + \rho\sigma v_t\frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 f}{\partial v^2} + (r + \mu_j v_t)\frac{\partial f}{\partial x} + [a - b_j v_t]\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} \tag{16}$$

From (15)

$$\frac{\partial f}{\partial x} = i\phi f, \quad \frac{\partial^2 f}{\partial x^2} = -\phi^2 f, \quad \frac{\partial f}{\partial v} = Df, \quad \frac{\partial^2 f}{\partial v^2} = D^2 f, \quad \frac{\partial^2 f}{\partial x \partial v} = i\phi Df, \quad \frac{\partial f}{\partial t} = \Big(\frac{\partial C}{\partial t} + v\frac{\partial D}{\partial t}\Big)f \tag{17}$$

Substitute (17) into (16) we obtain

$$\frac{-1}{2}v_t\phi^2 f + \rho\sigma v_t i\phi Df + \frac{1}{2}\sigma^2 v_t D^2 f + (r + \mu_j v_t)i\phi f + [a - b_j v_t]Df + \Big(\frac{\partial C}{\partial t} + v_t\frac{\partial D}{\partial t}\Big)f = 0$$

Rearranging we get

$$\Big(\frac{-1}{2}\phi^2 + \rho\sigma i\phi D + \frac{1}{2}\sigma^2 D^2 + \mu_j i\phi - b_j D + \frac{\partial D}{\partial t}\Big)v + \Big(ri\phi + aD + \frac{\partial C}{\partial t}\Big) = 0$$

This gives rise to a system of two Ricatti equations:

$$\frac{-1}{2}\phi^2 + \rho\sigma i\phi D + \frac{1}{2}\sigma^2 D^2 + \mu_j i\phi - b_j D + \frac{\partial D}{\partial t} = 0$$

$$ri\phi + aD + \frac{\partial C}{\partial t} = 0$$

Solving these pair of Ricatti equations, using Mathematica, gives the solution for equation(15) with

$$C(\tau; \phi) = ri\phi\tau + \frac{a}{\sigma^2}\Big[(b_j - \rho\sigma i\phi + d)r - 2\ln\Big(\frac{1 - ge^{d\tau}}{1 - g}\Big)\Big]$$

$$D(\tau; \phi) = \frac{b_j - \rho\sigma i\phi + d}{\sigma^2}\Big[\frac{1 - ge^{d\tau}}{1 - ge^{d\tau}}\Big]$$

and

$$g = \frac{b_j - \rho\sigma i\phi + d}{b_j - \rho\sigma i\phi - d}$$

$$d = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2\mu_j i\phi - \phi^2)}$$

Using the Fourier inversion theorem, we invert the characteristic function to get the desired probabilities

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\Big[\frac{e^{-i\phi x \ln K} f_j(x_t, v_t, t; \phi)}{i\phi}\Big]d\phi$$

with j=1,2. In [6] it has been shown how to use the fast Fourier method to evaluate this integral.

3.2 Jump Processes

In recent time, crude oil prices have exhibited sharp spikes in their returns. These spikes (or jumps) cannot be accounted for in a diffusion model with continuous paths. In the Heston model, the price behaves locally like a Brownian motion thus the probability that the stock moves by a large amount over a short period of time is very small, unless one fixes an unrealistically high value of volatility[16]. The basic building block for jump models is the Poisson process[3]. The most widely used stochastic process for modelling jumps in finance is the Compound Poisson process([16],[5],[14]). Given a Poisson Process N_t with intensity, λ the compound Poisson Process is given as

$$Q_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{18}$$

where Y_i are independent and identically distributed random variables. In [5] and [13] the $Y_i s$ are allowed to follow a log-normal distribution.

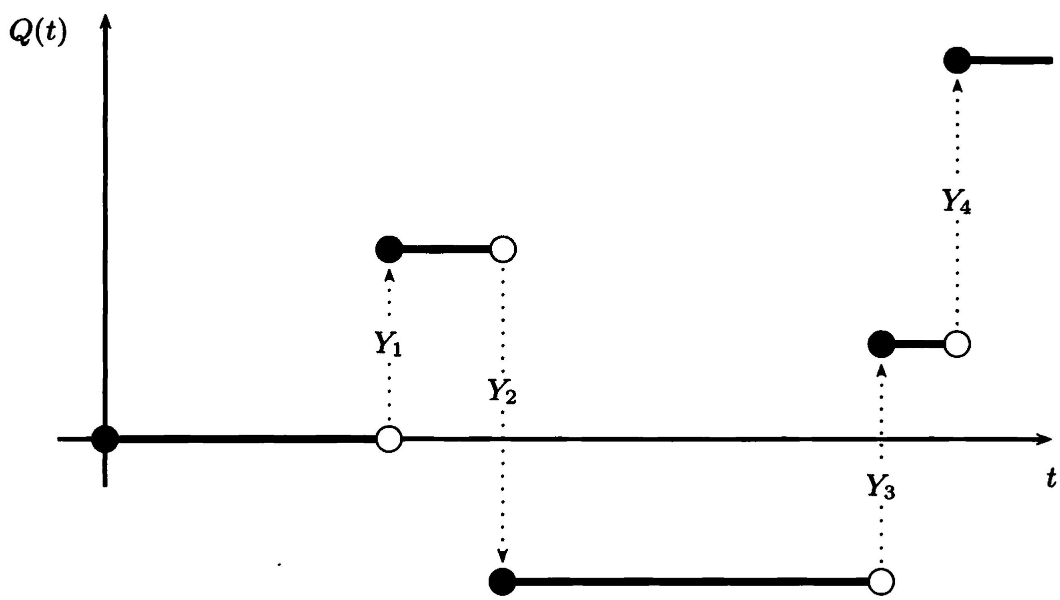


Figure 3.1: A typical Compound Poison Process with independent jumps Y_i

We now introduce the Heston-jump model.

3.3 Heston model with jumps

Given the dynamics described in section3.1 a new model according to [5] is given as

$$\begin{aligned} \frac{dS_t}{S_t} &= (\mu - \lambda \bar{J})dt + \sqrt{v_t}dW_{t_1} + J_t dZ_t \\ dv_t &= \kappa(\theta - v_t)dt + \sigma_{v_t}\sqrt{v_t}dW_{t_2}, \\ \rho dt &= \ \mathrm{Corr}(dW_{t_1}, dW_{t_2}), \end{aligned} \tag{19}$$

[3]This is a sequence of independent and indentically distributed exponential random variables $T_1, T_2, T_3, ...T_n$, where T_i is the time before the next jump after time T_{i-1} with arrival time expressed as $S_n = \sum_{i=1}^{n} T_i$

where Z_t is the compound Poisson process with intensity λ and independent jumps J_t with

$$\ln(1+J_t) \sim N\Big(\ln(1+\bar{J}) - \frac{1}{2}\delta^2, \delta^2\Big), \tag{20}$$

where the parameters $\bar{J}$ and δ determine the distribution of the jump and the Poisson process is assumed to be independent of the Wiener processes. Also $\mathbb{P}(dZ_t = 1) = \lambda dt$
Now we change measure such that $\mathbb{P} \rightarrow \mathbb{Q}$ (from normal measure to risk-neutral measure).
We then get

$$\begin{aligned} \frac{dS_t}{S_t} &= (\mu - \lambda^* \bar{J}^*)dt + \sqrt{v_t} dW_{t_1}^{\mathbb{Q}} + J_t^* dZ_t^* \\ dv_t &= \kappa^*(\theta^* - v_t)dt + \sigma_{v_t}\sqrt{v_t} dW_{t_2}^{\mathbb{Q}}, \\ \rho dt &= E^{\mathbb{Q}}[dW_{t_1}^{\mathbb{Q}} dW_{t_2}^{\mathbb{Q}}], \end{aligned} \tag{21}$$

where

$$\begin{aligned} \kappa^* &= \kappa + \epsilon \\ \theta^* &= \frac{\kappa\theta}{\kappa + \epsilon} \end{aligned}$$

such that ϵ is the volatility market price and

$$\begin{aligned} J^* &= J + JE^{\mathbb{P}}\Big[\frac{\Delta J_w}{J_w}\Big] \\ \bar{J}^* &= \bar{J} + \frac{Cov\Big(J, \frac{\Delta J_w}{J_w}\Big)}{1 + E^{\mathbb{P}}\Big[\frac{\Delta J_w}{J_w}\Big]} \end{aligned} \tag{22}$$

where $\frac{\Delta J_w}{J_w}$ is a random percentage jump conditional on a jump occurring and $\frac{dJ_w}{J_w}$ is the percentage shock in the absence of a jump. We note that when $\epsilon = 0$ we have $\kappa^* = \kappa$ and $\theta^* = \theta$. We set $\epsilon = 0$, because when we estimate the risk-nuetral parameters to price options we do not ned to estimate ϵ [1]. Also when $\Delta J_w / J_w \rightarrow 0$ we have $J^* = J$ and $\bar{J}^* = \bar{J}$.
Applying the Black-Scholes no-arbitrage argument[3] and proceeding as was done in section 3.1, we have

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 v_t S_t^2 \frac{\partial^2 C}{\partial v^2} + \rho\sigma v_t S_t \frac{\partial^2 C}{\partial S \partial v} + (\mu - \lambda\bar{J})S_t \frac{\partial C}{\partial S} + \kappa(\theta - v_t)\frac{\partial C}{\partial v} - rC + I_C = 0 \tag{23}$$

where

$$\begin{aligned} I_C =& \lambda \int_0^\infty [C(S\epsilon, v, t) - C(S, v, t)] g(\epsilon) d\epsilon \\ g(\epsilon) &= \frac{1}{\sqrt{2\pi}\alpha\epsilon} e^{-\frac{1}{2\alpha^2}(\ln\epsilon - m)^2} \\ m &= \ln(1 + J) - \frac{1}{2\alpha^2} \end{aligned} \tag{24}$$

Let $x_t = \ln S_t$, then

$$C(t, S_t, v_t, J, K, T) = SP_1 - Ke^{-r\tau} P_2$$

where

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big[\frac{e^{-i\phi \ln K} f_j(x_t, v_t, t; \phi)}{i\phi}\Big] d\phi$$

for j=1,2

$$f_j(x, v, t; \phi) \equiv E[e^{i\phi X_T}] = \exp\Big(A_j(\tau, \phi) + B_j(\tau, \phi)v_t + i\phi S + \lambda^*\tau(1 + J*)^{\mu_j + \frac{1}{2}} * [(1 + J^*)^{i\phi} e^{\delta^2(\mu_j i\phi - \frac{1}{2}\phi^2) - 1}\Big)$$

and

$$B_j(\tau,\phi) = (r - \lambda^* J^*)i\phi\tau - \frac{\alpha\tau}{\eta^2}(\rho\sigma i\phi - \beta_j - \gamma_j)\frac{2\alpha}{\sigma^2}\log\Big[1 + \frac{1}{2}(\rho\sigma i\phi - \beta_j - \gamma_j)\frac{1 - e^{\gamma_j\tau}}{\gamma_j}\Big]$$

$$A_j(\tau;\phi) = -2\frac{\mu_j\phi - \frac{1}{2}\phi^2}{\rho\sigma i\phi - \beta_j + \gamma_j(1 + e^{\gamma_j\tau})/(1 - e^{\gamma_j\tau})}$$

$$\gamma_j = \sqrt{(\rho\sigma\phi - \beta_j)^2 - 2\sigma^2\Big(\mu_j i\phi - \frac{1}{2}\phi^2\Big)},$$

$$u_1 = \frac{1}{2} \quad u_2 = \frac{-1}{2}, \quad \beta_1 = \beta^* - \rho\sigma \text{ and } \quad \beta_2 = \beta^* \tag{25}$$

4 Planned computations

In our project we plan to:

1. calibrate both models using MCMC simulation techniques to historical market data. Parameters in both models will be estimated and compared.

2. compare the distributions of each of the parameters in the models in an attempt to determine how much of an influence the jump term has on the model parameters.

3. finally use the estimated parameters to price oil options and compare our results to actual market prices. In view of results in [15] that suggest that the Heston-jump model is better suited to short-term options while the plain Heston model does well for medium and long-term-options we plan to:

   - price long to medium-term oil options with a 21-day maturity period.
   - price short-term oil options with a 9-day maturity period.

4.1 Data

In view of our objective to price oil derivatives under the Heston-jump and Heston models, and because of the high volume of trades associated with these particular options per day we chose WTI crude oil futures contracts CLM16 and CLJ16 as the underlying assets.
WTI: West Texas Intermediate (WTI), also known as Texas light sweet, is a grade of crude oil used as a benchmark in oil pricing. This grade is described as light because of its relatively low density, and sweet because of its low sulfur content.

4.1.1 Data for Parameter Estimation

- CLM16: CLM16 is a futures contract that obligates the buyer to purchase 1000 barrels of crude oil on the 20th May 2016 and designed price.

- CLJ16: CLJ16 is a future contract that obligating the buyer to purchase 1000 barrels crude oil at 21st March 2016 and designed price.

From `http://www.barchart.com/detailedquote/futures/CLM16` we downloaded the historical data for CLM16 oil futures. We also got the CLJ16 price data from `http://www.barchart.com/detailedquote/futures/CLJ16`. For each of the futures the historical data was collected from 28th March 2014 to 25th Feb 2016. Since our models make use of log returns we calculate the log-return of each of the futures prices as

$$Y_t = \ln S_t - \ln S_{t-1}$$

4.1.2 Data for Option Pricing

For the oil options, we chose CLM16 and CLJ16 call options whose underlying assets are CLM16 and CLJ16 futures respectively.

- For the CLM16 options the contract's expiry date is 17th May 2016 with a time-to-maturity of 21 business days.
- The CLJ16 options expired on 16th March, 2016 with a maturity period of 9 days.

The benchmark date for both option prices was 7th March 2016 at 6:00pm. Also, we chose options with different strike prices, ranging from $25 to $55.

4.2 Parameter Estimation/Calibration

Parameter estimation using MCMC techniques typically involves deriving posterior (real) distributions from prior(assumed) distributions of the variables we wish to estimate. After finding a likelihood function the the Bayesian method used to derive the posterior distribution is

$$\text{Posterior distribution } = \text{ Likelihood function } \times \text{ Prior distribution.[5]}$$

Then using MCMC, we simulate the posterior distributions for the parameters. The Heston model parameters we need to estimate are: $\mu, \kappa, \theta, \sigma_V$ and ρ. For the variance process V_t, we only need a point estimate of the distribution.The estimate normally used is the mean([5],[15]), serving as a proxy for the variance when we price the options. In the next subsection we summarize the computation involved in calculating the posterior distributions for eac parameter. Detailed derivations can be found in the Appendix of this paper.

4.2.1 Posterior Distributions

The prior distributions we choose are chosen based on distributions we have realised were frequently used in research that involved parameter estimation for the Heston model([5],[14][16]).

1. Posterior Distribution of μ.
   The prior distribution of μ is chosen as $\mu \sim N(\mu_0, \sigma_0^2)$. The posterior distribution of μ is thus

$$\begin{aligned} P(\mu|\, Y, V, \kappa, \theta, \psi, \Omega) &= P(Y, V|\mu, \kappa, \theta, \psi, \Omega) \cdot P(\mu) \\ &= \exp\left(-\frac{1}{2\Omega}\Sigma_{t=1}^{T}[(\Omega+\psi^2)(\epsilon_t^S)^2 - 2\psi\epsilon_t^V\epsilon_t^S]\right) \cdot \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) \end{aligned} \tag{26}$$

   Here, $P(Y, V|\mu, \kappa, \theta, \psi, \Omega)$ is the likelihood function that will be described in more detail in our work. After little more computation we derive that the posterior distribution for μ is $\mu \sim N(\mu^*, \sigma^{*2})$ where

$$\mu^* = \frac{\begin{array}{c}\Sigma_{t=1}^{T}((\Omega+\psi^2)(Y_t + \frac{1}{2}V_{t-1}\Delta t)/\Omega V_{t-1}) \\ -\Sigma_{t=1}^{T}(\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})/+\Omega V_{t-1})\mu_0/\sigma_0^2\end{array}}{\Delta t\sum_{t=1}^{T}((\Omega+\psi^2)/\Omega V_{t-1} + 1/\sigma_0^2}$$

   and

$$\sigma^{*2} = \frac{1}{\Delta t\sum_{t=1}^{T}((\Omega+\psi^2)/\Omega V_{t-1} + 1/\sigma_0^2}$$

2. Posterior Distribution of (κ, θ).
   The prior is chosen as $\theta \sim N(\theta_0, \sigma_\theta^2)$. The posterior distribution is thus $\theta \sim N(\theta^*, \sigma_\theta^{*2})$ where

$$\theta^* = \frac{\begin{array}{c}\Sigma_{t=1}^T((\kappa)(V_t - (1-\kappa\Delta t)V_{t-1})/\Omega V_{t-1}) \\ -\Sigma_{t=1}^T(\psi(Y_t - \mu\Delta t + \frac{1}{2}V_{t-1}\Delta t)\kappa/\Omega V_{t-1}) + \theta_0/\sigma_\theta^2\end{array}}{\Delta t \sum_{t=1}^T(\kappa^2/\Omega V_{t-1}) + 1/\sigma_\theta^2}$$

   and

$$\sigma_\theta^{*2} = \frac{1}{\Delta t \sum_{t=1}^T(\kappa^2/\Omega V_{t-1}) + 1/\sigma_\theta^2}.$$

   The prior of κ is chosen as $\kappa \sim N(\kappa_0, \sigma_\kappa^2)$. The posterior distribution is given as $\kappa \sim N(\kappa^*, \sigma_\kappa^{*2})$ where

$$\kappa^* = \frac{\begin{array}{c}\Sigma_{t=1}^T((\theta - V_{t-1})(V_t - V_{t-1})/\Omega V_{t-1}) \\ -\Sigma_{t=1}^T(\psi(Y_t - \mu\Delta t + \frac{1}{2}V_{t-1}\Delta t)(\theta - V_{t-1})/\Omega V_{t-1}) + \kappa_0/\sigma_\theta^2\end{array}}{\Delta t \sum_{t=1}^T((V_{t-1} - \theta)^2/\Omega V_{t-1}) + 1/\sigma_\kappa^2}$$

   and

$$\sigma_\kappa^{*2} = \frac{1}{\Delta t \sum_{t=1}^T((V_{t-1} - \theta)^2/\Omega V_{t-1}) + 1/\sigma_\kappa^2}.$$

3. Posterior Distribution of the variance, V_t.
   This is given by

$$\begin{aligned}P(V_t) = \frac{1}{V_t\Delta t}\exp\Bigg(& -\frac{1}{2\Omega}\frac{(\Omega+\psi^2)(\frac{1}{2}V_t\Delta t + Y_{t+1} - \mu\Delta t)^2}{V_t\Delta t} \\ & -\frac{1}{2\Omega}\frac{-2\psi(\frac{1}{2}V_t\Delta t + Y_{t+1} - \mu\Delta t)(-(1-\kappa\Delta t)V_t - \kappa\theta\Delta t + V_{t+1}}{V_t\Delta t} \\ & -\frac{1}{2\Omega}\frac{(-(1-\kappa\Delta t)V_t - \kappa\theta\Delta t + V_{t+1})^2}{V_t\Delta t}\Bigg) \\ \times\exp\Bigg(& -\frac{1}{2\Omega}\frac{-2\psi(Y_t - \mu\Delta t + \frac{1}{2}V_{t-1}\Delta t)(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})}{V_{t-1}\Delta t} \\ & -\frac{1}{2\Omega}\frac{(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})^2}{V_{t-1}\Delta t}\Bigg).\end{aligned}$$

4.2.2 Parameter Estmation in the Heston-Jump model

Due to the traction in the Heston model estimation of most of the parameters is quite similar to the methodology already employed even after additional parameters are included in the model [10]. As a result, estimating most of the paramters in the Heston-jump model is fairly routine with the exception of the jump parameters Z_t, B_t, μ_s and σ_s^2. We give a brief description of the methodology employed for estimating these. Again, the priors are chosen here based on what we realised were frequently chosen in most jump models([16], [5])

1. Posterior distribution of Z_t, the magnitude of the jump.
   Assuming the prior is chosen as $Z_t \sim \mathcal{N}(\mu_S, \sigma_S^2)$. The by the same methodology described earlier the posterior is derived as $Z_t \sim \mathcal{N}(\mu_S^*, \sigma_S^{*2})$ where

$$\mu_S^* = \frac{\begin{array}{c}((\Omega+\psi^2)(Y_t + (1/2)V_{t-1}\Delta t - \bar{\mu}\Delta t)/\Omega V_{t-1}\Delta t) \\ -(\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})/\Omega V_{t-1}\Delta t) + \mu_S/\sigma_S^2\end{array}}{(\Omega+\psi^2)/\Omega V_{t-1}\Delta t + 1/\sigma_S^2}$$

and

$$\sigma_S^{*2} = \frac{1}{(\Omega+\psi^2)/\Omega V_{t-1}\Delta t + 1/\sigma_S^2}$$

2. Posterior distribution of λ_D. The prior distribution for λ_D is assigned to be $\lambda_D \sim \text{Beta}(\alpha', \beta')$. We have therefore

$$\begin{aligned} P(\lambda_D \mid B) &= P(B \mid \lambda_D) \,.\, P(\lambda_D) \\ &= \binom{T}{\sum_{t=1}^T B_t} \lambda^{\sum_{B_{t=1}}^T} (1-\lambda_D)^{T-\sum_{t=1}^T B_t} \,.\, P(\lambda_D). \end{aligned}$$

Plugging in the density of the Beta function we get

$$P(\lambda_D \mid B) = \binom{T}{\sum_{t=1}^T B_t} \lambda^{\Sigma_{B_{t=1}}^T} (1-\lambda_D)^{T-\sum_{t=1}^T B_t} \,.\, \frac{\lambda_D^{\alpha'-1}(1-\lambda_D)^{\beta'-1}}{B(\alpha',\beta')}$$

The posterior distribution of λ_D is therefore $\lambda_D \sim \text{Beta}(\alpha^*_{\lambda D}, \beta^*_\lambda)$, where

$$\begin{aligned} \alpha^*_\lambda &= \alpha' + \Sigma_{t=1}^T B_t, \\ \beta^*_\lambda &= \beta' + T - \Sigma_{t=1}^T B_t. \end{aligned}$$

3. Posterior distributions of $\mu_S|\sigma_S^2$ and $\sigma_S^2|\mu_S$. The prior distributions are assigned so that $\mu_S \sim \mathcal{N}(0, S_0)$ and $\sigma_S^2 \sim \mathcal{IG}(\alpha_S, \beta_S)$.

$$\begin{aligned} P(\mu_S|\sigma_S^2, Z) &= P(Z \mid \mu_S, \sigma_S^2) \,.\, P(\mu_S) \\ &= (\sigma_S)^{-T} \exp\left(-\frac{1}{2}\sum_{t=1}^T \frac{(Z_t-\mu_S)^2}{\sigma_S^2} \right) \,.\, \exp\left(-\frac{\mu_S^2}{2S_0} \right), \end{aligned}$$

while

$$\begin{aligned} P(\sigma_S^2|\mu_S, Z) &= P(Z|\mu_S, \sigma_S^2) \,.\, P(\sigma_S^2) \\ &= (\sigma_S)^{-T} \exp\left(-\frac{1}{2}\sum_{t=1}^T \frac{(Z_t-\mu_S)^2}{\sigma_S^2} \right) \,.\, \frac{\beta_S^{\alpha_S}}{\Gamma(\alpha_S)} (\sigma_S^2)^{-\alpha_S-1} \exp\left(-\frac{\beta_S}{\sigma_S^2} \right) \end{aligned}$$

So the posterior distribution of $\mu_S|\sigma_S^2$ is $\mu_S \sim \mathcal{N}(\mu^*_{\mu_S}, \sigma^{*2}_{\mu_S})$, where

$$\mu^*_{\mu_S} = \frac{\Sigma_{t=1}^T (Z_t/\sigma_S^2)}{1/S_0 + \Sigma_{t=1}^T(1/\sigma_S^2)} \text{ and } \sigma^{*2}_{\mu_S} = \frac{1}{1/S_0 + \Sigma_{t=1}^T(1/\sigma_S^2)}$$

and the posterior distribution of $\sigma_S^2|\mu_S$ is $\sigma_S^2 \sim \mathcal{IG}(\alpha_S^*, \beta_S^*)$, where

$$\alpha_S^* = \alpha_S + T/2 \text{ and } \beta_S^* = \beta_S + \frac{1}{2}\sum_{t=1}^T (Z_t-\mu_S)^2.$$

4.2.3 Simulating from the posterior distributions

At this point we utilize tools in MCMC to obtain draws from the posterior distributions just calculated. The process is summarized in the following steps:

- For the parameters $\{\mu, \kappa, \theta, Z_t, \lambda_D, \mu_S, \sigma_S\}$ a Gibbs sampler is used [5]. The process involves mainly

  1. Starting from a set $\{\mu^{(0)}, \kappa^{(0)}, \theta^{(0)}, V_0^{(0)}, V_1^{(0)}, ..., V_{T+1}^{(0)}\}$ of initial values.

2. Finding the distribution of μ conditional upon these values to obtain a draw $\mu^{(1)}$ thus updating our initial state to $\{\mu^{(1)}, \kappa^{(0)}, \theta^{(0)}, V_0^{(0)}, V_1^{(0)}, ..., V_{T+1}^{(0)}\}$.
3. Applying step 2 to obtain draws for $\kappa^{(1)}$ and $\theta^{(1)}$ and updating the current state of the Markov chain in step 1 with each draw.
4. The process in step 3 is repeated for $1, 2, ..., n$ iterations.

- For the state space $\{V_0, ..., V_T\}$ a Metropolis- Hastings approach is used[5],[16]. This involves
  1. Starting with the initial values for the 0th step;
  $$\{V_0^{(0)}, ..., V_{T+1}^{(0)}\}.$$
  2. Running the algorithm for n steps, to get $\{V_0^{(g)}, V_1^{(g)}, ..., V_{T+1}^{(g)}\}$, $g \in \{1, ..., n\}$.
  3. For a fixed gth step, draw from the proposal density for $V_t^{(g)}$ for $t \in \{1, ..., T\}$ by the following:
  $$V_t^{*(g)} = V_t^{(g-1)} + \mathcal{N}_t, \quad \text{where } \mathcal{N}_t \sim \mathcal{N}(0, \sigma_N^2)$$
  where $t \in \{1, ..., T\}$ and $V_t^{*(g)}$ is our proposal for $V_t^{(g)}$.

5 Parameter Estimation & Option Pricing

5.1 Model Parameters

Using the oil future prices data described in section 4.1, we estimate $\mu, \mu_s, \sigma_S, Z, \lambda, \kappa, \theta, \sigma_V$ and V.

Tables 5.1 and 5.2 below provide summary statistics of the continuously compounded returns for the CLM16 and CLJ16 futures.

Table 5.1: Summary of returns of CLM16 Oil futures.

	Mean	Volatility	Skewness	Kurtosis	Min	Max
CLM16	-0.17753	2.002827	27.11884	381.9969	-8.31248	9.244653

Table 5.2: Summary of returns of CLJ16 Oil futures.

	Mean	Volatility	Skewness	Kurtosis	Min	Max
CLJ16	-0.16824	2.182542	34.01354	337.2352	-8.54654	10.15029

The following were chosen as the prior distribution parameters, modelling our selections off of [5].

$$\begin{aligned}
\bar{\mu} &\sim N(0, 1), \\
\kappa &\sim N(0, 1), \\
\theta &\sim N(0, 1), \\
\psi &\sim N(0, \frac{\Omega}{2}), \\
\Omega &\sim IG(2, \frac{1}{200}), \\
\lambda_D &\sim \text{Beta}(2, 40), \\
\mu_S &\sim N(0, 1), \\
\sigma_S^2 &\sim IG(5.0, 0.2).
\end{aligned}$$

Initial values for the MCMC algorithm were chosen based off the observed data when possible or a random assignment when more educated estimates were not possible[5]. As a result, the following initials were chosen:

$$\begin{aligned}
\bar{\mu}^{(0)} &= 0.1, \\
\kappa^{(0)} &= 5, \\
\theta^{(0)} &= 0.0225, \\
\Omega &= 0.02, \\
\psi^{(0)} &\sim N\left(0, \frac{\Omega^{(0)}}{2}\right), \\
\lambda_D^{(0)} &\sim \text{Beta}(2, 40), \\
\mu_S^{(0)} &= 0, \\
\sigma_S^{2(0)} &= 0.1, \\
V_t^{(0)} &\sim N(0.0225, 0.005)1_{V_t>0}, \\
B_t^{(0)} &\sim \ \text{Bernoulli}(\lambda_D^{(0)}), \\
Z_t^{(0)} &= 0.
\end{aligned}$$

After our simulations, in the Heston model, we discarded the first 3000 runs as 'burn-in' period and used the last 8,000 iterations to estimate model parameters. Means of the draws from the posterior distributions of each parameter are reported as well as the standard deviation of the means for the distribution.For the Heston-Jump model the same procedure was applied to estimate parameters. The algorithm was run 10 times, recording the parameter values after each run after which the means were calculated from the ten runs. Each run took about 20 minutes and was done completely in the statistical language of R, utilizing pre-defined routines for random number generation. Table 5.3 provides a summary of the results obtained from the MCMC simulations. Values have been converted to percentages for easy reading.

Table 5.3: MCMC estimates of model parameters for CLM16(21-day maturity) and CLJ16(9-day maturity) oil futures.

	CLM16		CLJ16	
	Heston-Jumps	Heston	Heston-Jumps	Heston
μ	-15.13%		-20.00%	
	(12.08%)		(11.43%)	
μ_s	0.50%		-0.01%	
	(4.17%)		(2.46%)	
σ_s	14.73%		11.87%	
	(9.00%)		(1.86%)	
Z	0.33%		-0.12%	
	(14.02%)		(11.41%)	
λ	3.20%		5.70%	
	(1.14%)		(2.63%)	
κ	49.57%	45.29%	65.17%	56.23%
	(45.70%)	(41.42%)	(47.03%)	(43.82%)
θ	26.99%	40.02%	17.91%	29.26%
	(33.48%)	(39.92%)	(25.53%)	(33.12%)
σ_V	14.44%	31.71%	15.06%	24.26%
	(7.55%)	(17.24%)	(16.54%)	(3.08%)
ρ	-63.17%	-58.35%	-78.13%	-73.17%
	(12.17%)	(10.91%)	(8.33%)	(10.10%)
V	4.27%	6.89%	4.24%	7.36%
	(2.65%)	(4.99%)	(2.51%)	(5.10)

From Table 5.3 we observe that:

1. Both of the estimates for the average return in the models (given by μ) are negative reflecting the negative expected return of the oil futures due to the continual decline in oil prices.

2. The estimates for V, the volatility under the Heston-Jump model are smaller than the estimates for the Heston model since part of the volatility is accounted for by the jumps.

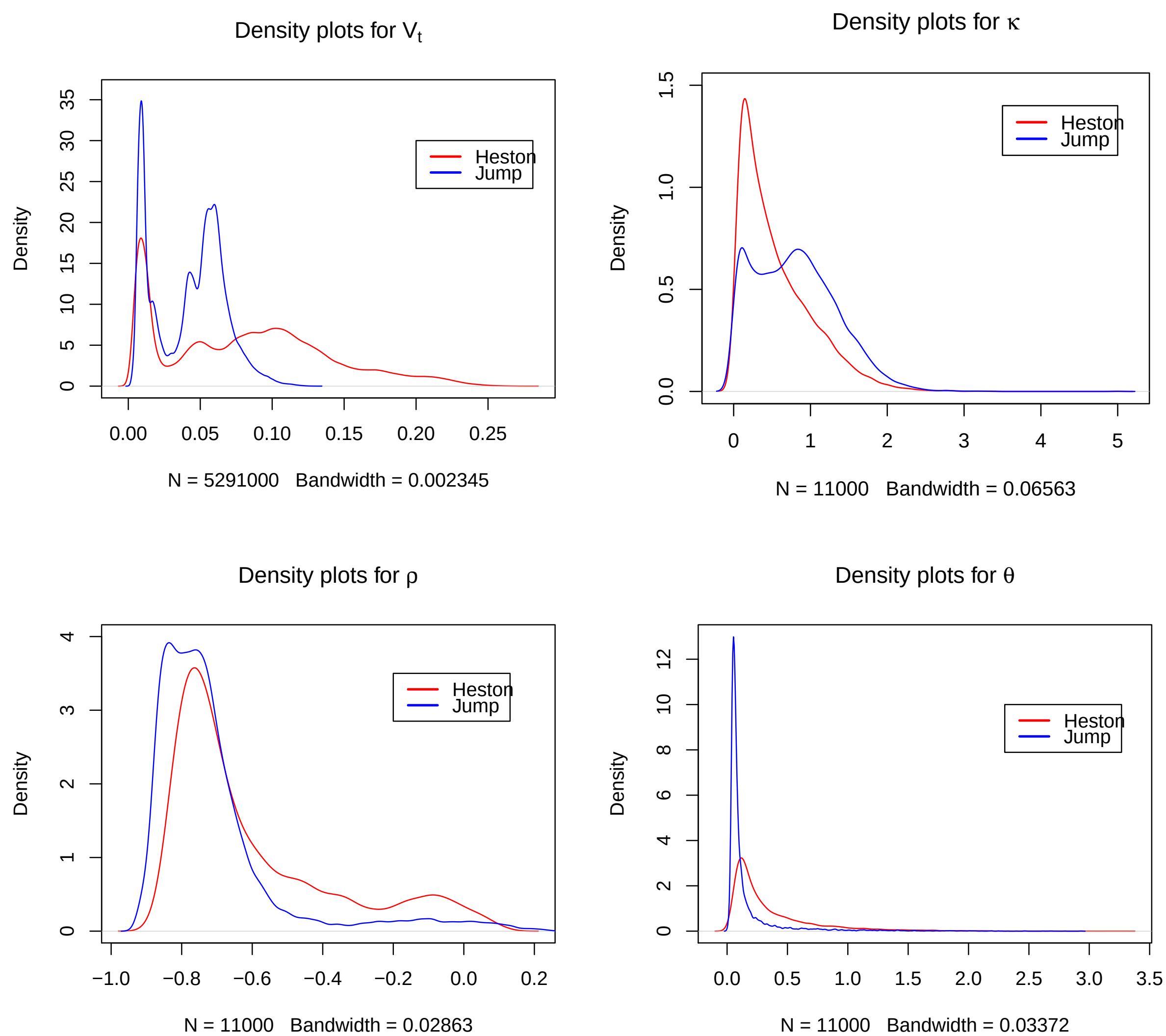
3. The parameter ρ which measures the correlation between instantaneous volatility and returns indicates a strong negative correlation in the Heston-Jump model. The same can be said about this result for the Heston model however we realize that the correlation is even stronger in the Heston-Jump model.

4. The estimates of κ, the mean reversion indicates that jumps are a very infrequent event, with roughly three to five jumps observed per year.

5.1.1 Comparing Parameter Distributions

We now compare the distributions of each of the parameters in the models to ascertain whether or not the jump process has an effect on the estimated parameters given the available data. Figure 5.1 is a plot of the densities of parameters that feature in both of the models. We notice significant differences in the densities of V, the variance and θ.

Figure 5.1: Plots of the distributions of paramters in both models



Despite these considerable differences observed in the plots we use the Kolmogorov-Smirnov test to ascertain whether the differences are statistically significantly[4] The test we use is the Kolmogorov-Smirnov test.

5.2 Kolmogorov-Smirnov(K-S) Test

The K-S test is a non-parametric test to determine whether or not two sampling distributions are statistically different. We chose this test since it does not require that the sampling data follow a normal distribution and since some of our parameters do not have normal distributions the test is good. Again, the test compares the overall shape of the distribution, not specifically measures of central tendency, dispersion or other parameters. A summary of how the test is used is as follows:

[4]The reason for this is because we noticed that λ in Table5.3 is quite small meaning there are very few jumps observed in the time period. As a result, one would expect the distributions to be similar in shape.

1. Set up hypotheses: So given two sampling distributions $f(X)$ and $f(Y)$ we set up a null and alternative hypothesis. i.e

$$\mathrm{H}_0 : F(X) = F(Y)$$
$$\mathrm{H}_\mathrm{A} : F(X) \neq F(Y)$$

   Here $F(X)$ and $F(Y)$ are the cumulative distribution functions of $f(X)$ and $f(Y)$ respectively[5].

2. Calculate the maximum difference between the distributions at each sample point called D. i.e $D_{KS} = \max\{F(X) - F(Y)\}$. This value could be positive or negative.

3. Determine a significance value, p that determines whether or not we reject or accept the null hypothesis, H_0. To calculate p all possible permutations of the data are found and D, calculated. The we have $p = \frac{\#\text{ of D}\geq D_{KS}}{\text{total }\#\text{ of permutations}}$

4. Finally, reject H_0 if $p < 0.05$ else fail to reject H_0

For each of the common parameters in our models the test was carried out in the statistical language of R. Result are given below:

Table 5.4: Kolmogorov-Smirnov Test results for common parameters

	V_t	ρ	κ	θ
D_{KS}	0.46633	0.21909	0.24664	0.49491
p value ($p < x$)	2.2e-16	2.2e-16	2.2e-16	2.2e-16

From table 5.4 we may conclude that observed differences between distributions of the common parameters in the models are statistically significant. The K-S test thus reveals that the jump term has an effect on parameter distributions. Despite this, the differences observed in the common parameters may not lead to statistically significant differences in the entire models. The reasons for this are as follows:

- The estimated value of the jump intensity parameter λ (ranging from 3.20% to 5.70% in table 5.3) is very small indicating that jumps were very infrequent during the chosen time period.

- One of the motivating factors for us in adding jumps to the model was because of the high volatility[6] we noticed in the data set. High volatility within a data set of prices does not necessarily imply that jumps are frequent in the prices. From tables 5.1 and 5.2 we notice that there is a high volatility in the oil futures prices, even more in CLJ16 oil futures. This manifests in the fact that prices have been on a steady decline within the period under consideration with insignificant improvements at irregular intervals. As such, one would expect that the standard value of deviations from the mean price (the standard deviation) would be large. This does not necessarily translate into spikes in the prices themselves. Volatility may therefore be very high with little or no spikes whatsoever. Frequent spikes on the other hand always imply high volatility.

- In most of our research we notice that the periods over which the studies are conducted to estimate parameters in both models actually has a lot of spikes in price data resulting from a number of adverse socio-economic occurrences [7]. As such, parameter estimates (especially the parameter λ) are significantly different. The period under consideration in our work is a period of sustained economic downturn with very few occurrences that have the ability to send prices shooting up or down even momentarily.

[5] The cumulative distribution function gives the probability up to a certain value relative to an underlying distribution.

[6] This is measured as the standard deviation of the data.

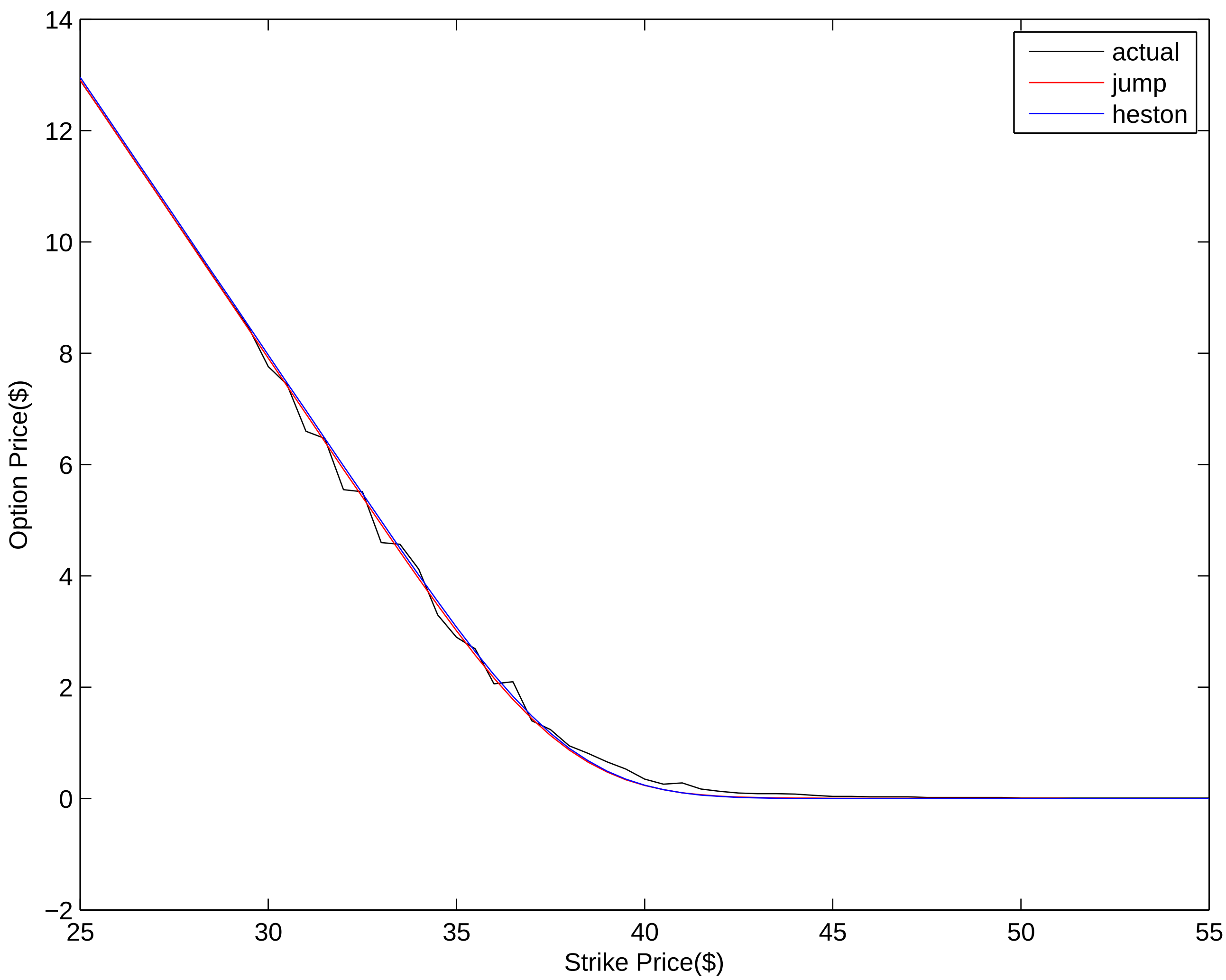
[7] The periods are usually chosen to conveniently include either the stock market crash of 1987, the bursting of the 'dot-com' bubble in the 2000s, a brief recovery in the mid 2000s or the onset of the great recession in 2008

5.3 Option Pricing

In this part, we want to use the estimated mean of parameters for both models to get estimated option prices under different strikes.

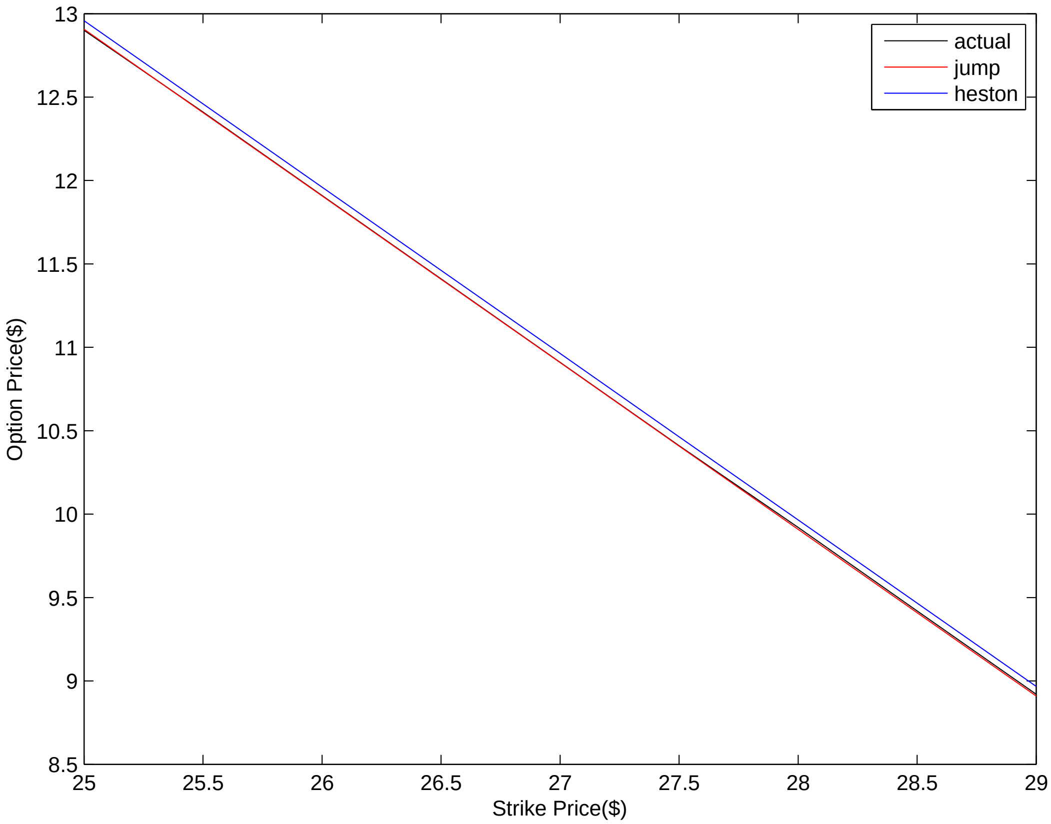
Based on the pricing models in section(2), we get the corresponding option prices under the strike prices between \$25 and \$55. The following figures are plots of prices gotten under the Heston and Heston-Jump models as well as the actual market prices. We would like to point out that our focus was on pricing call options since the put option value can easily be found using the put-call parity relation[8]

Figure 5.2: CLJ16 Option Prices under Different Strikes



[8]Given a call option price, $c(T, x)$ the value of a put is $p(T, x) = c(T, x) - S_0 + Ke^{-r(T-t)}$.

(a) In-the-Money CLJ16 Option Prices



(b) Out of the Money CLJ16 Option Prices

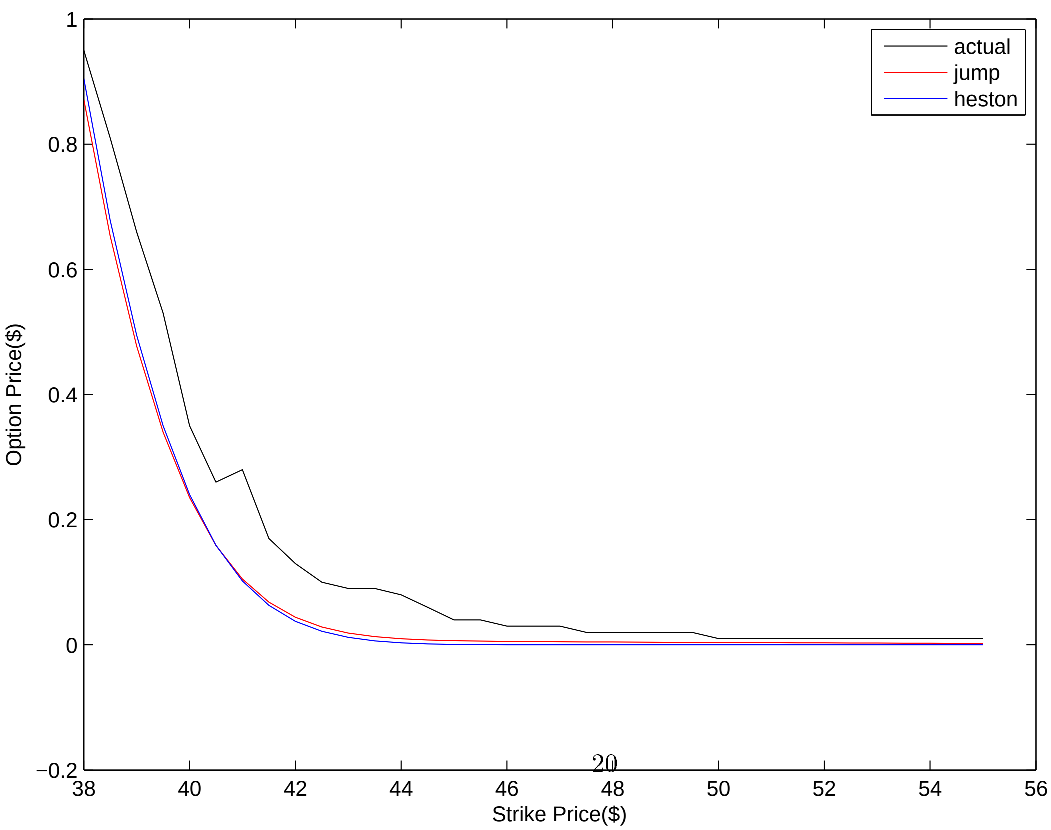
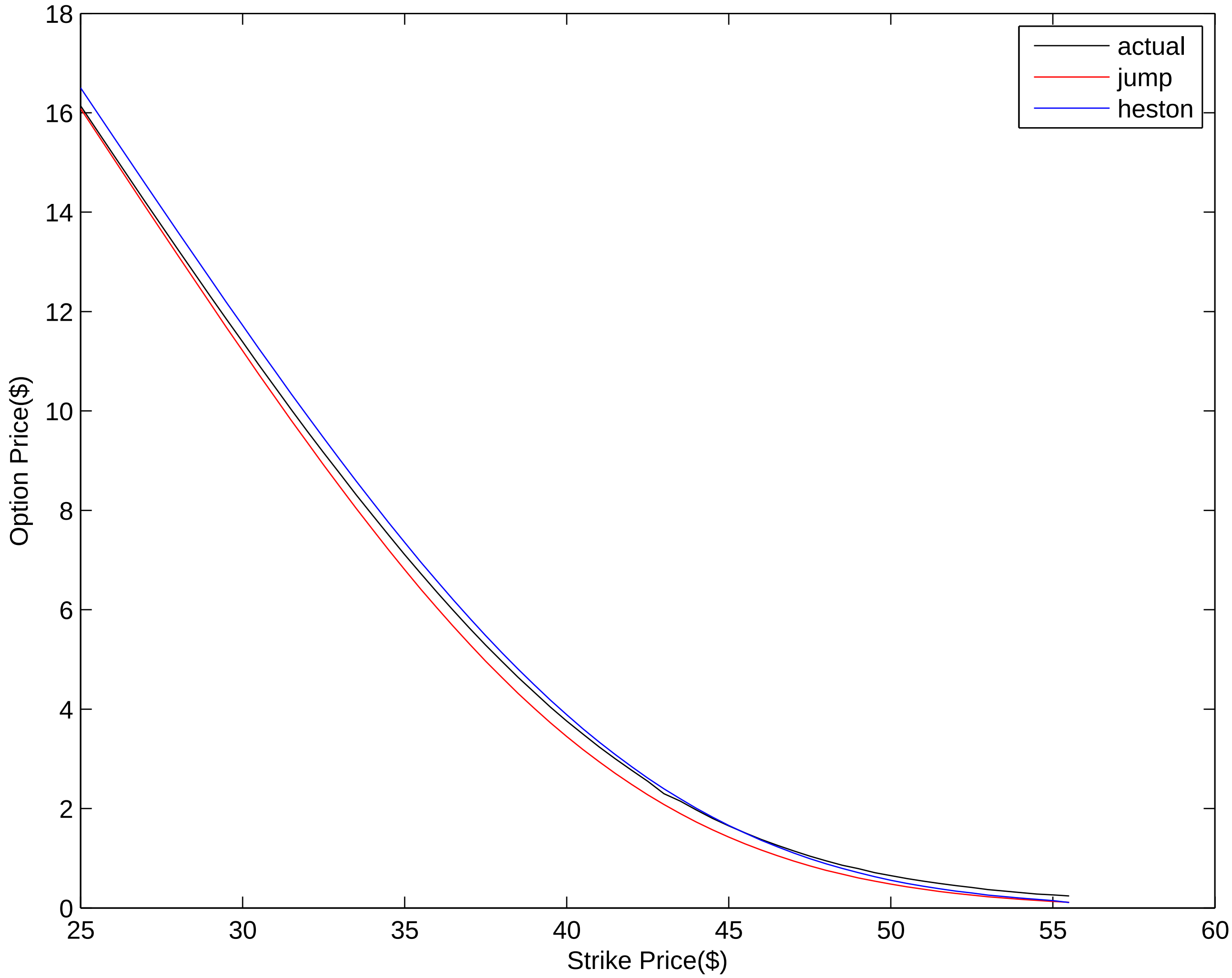
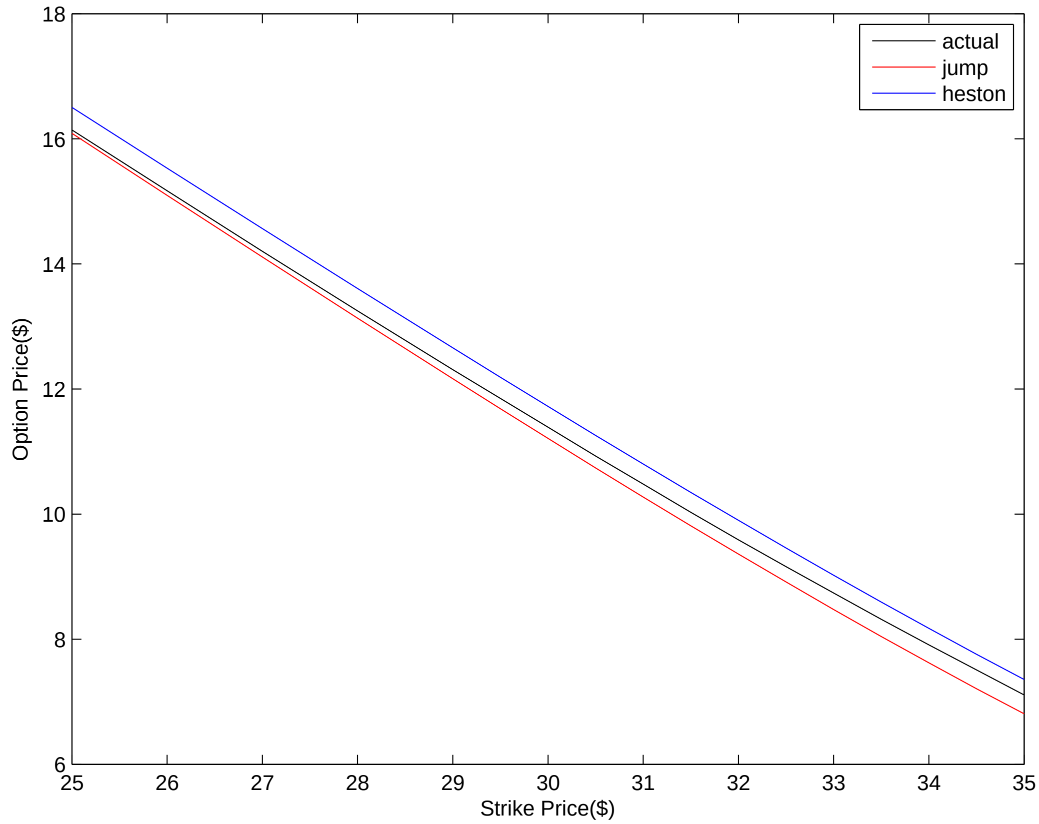


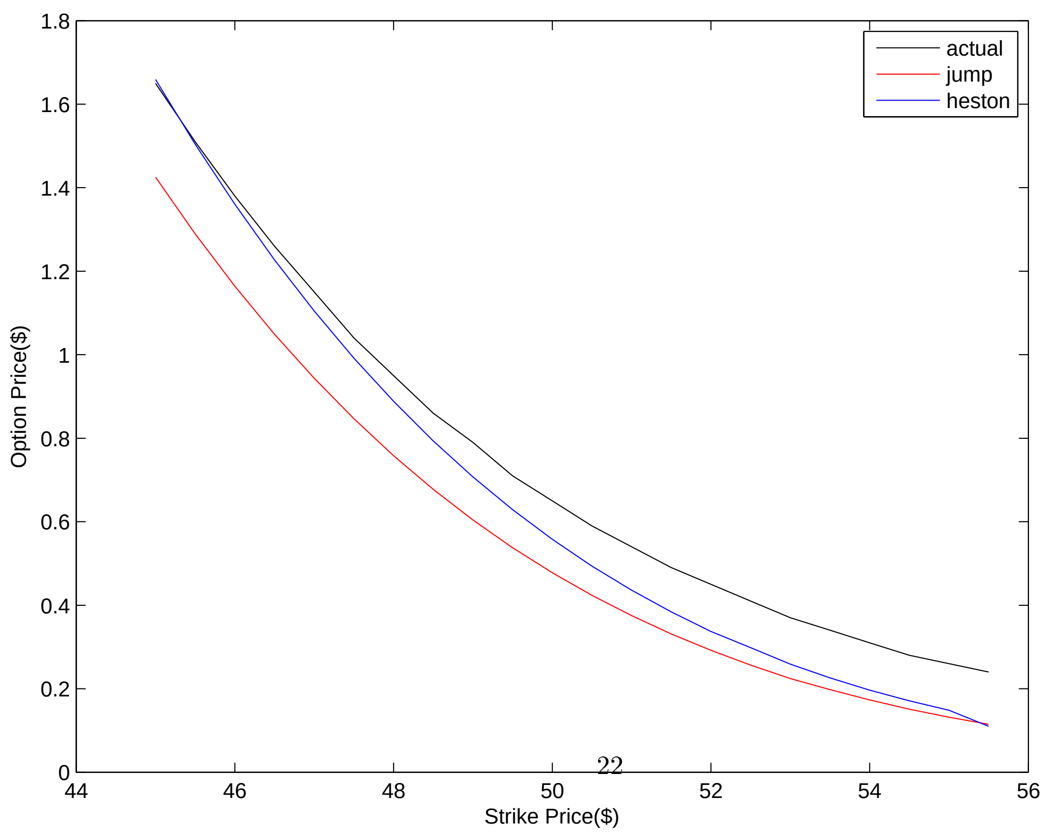
Figure 5.4: CLM16 Option Prices under Different Strikes



(a) In-the-Money CLM16 Option Prices



(b) Out of Money CLM16 Option Prices



Discussion of Figures Comparing Figure(5.2) with Figure (5.4) we notice that the Heston-jump model seems to perform better for short-term maturities options than for medium-term options. This result is in line with similar results in [15]. Also, figures (5.2)(a)(b) and (5.4)(a)(b)the jump model seems to perform better than Heston's model in In-The -Money and Out-of-The-Money cases, while Heston's model performs better for the At-The-Money cases. [9] For CLJ16 options the prevailing price as at March 7, 2016 was \$37.9. The in-the-money period is thus \$25 to \$29 while the out-of-the-money period is \$40 to \$55. Also for CLM16 options the spot price was \$39.8. The in-the-money period is thus \$25 to \$35 while the out-of-the- money period is \$45 to \$55.

It is well known that short-term options have market implied volatilities that exhibit a significant skew across strikes. Models with jumps generically lead to significant skews for short-term maturities. More generally, adding jumps to returns in diffusion based stochastic volatility model, the so-obtained model can generate sufficient variability and asymmetry in the short-term returns to match implied volatility skews for short-term maturities. These factors show the importance of adding a jump component in modelling asset price dynamics when we are dealing with the short-term maturities option pricing problems. [13] showed that the Black-Scholes model tends to undervalue deep ITM and OTM options while overvalues the ATM options. Also, in [7] it is shown that the presence of jumps inflates OTM or ITM option values relatively to the ATM values, which means that using jump diffusion models may be required when valuing options struck away from the current forward rate.

5.3.1 The Chi-square test

Finally, a robust measure of testing the accuracy of models is the Chi-square test. The chi-square goodness-of-fit test is used to determine whether sampled data are consistent with a hypothesized distribution, in this case the distribution of actual market prices. We chose this test because it is the most widely used test by researchers to determine how close estimated values are to actual values. Aside that, the chi-square statistic is easily computable and available in numerous software packages. The test is as follows:

1. Set up hypothesis:

$$\begin{aligned} \mathrm{H_0} &: \text{ Estimated prices have a good fit} \\ \mathrm{H_A} &: \text{ Estimated prices do not have a good fit} \end{aligned}$$

2. Formulate an analysis plan i.e We reject H_0 if the p-value is less than 0.05 in each case else we fail to reject H_0.

The chi-square test was performed in the statistical language of R. Results are in table 5.5 below.

Table 5.5: χ^2 Test for CLJ16 and CLJ16

p-value	CLJ16	CLM16
Heston-Jump	0.246	0.238
Heston	0.0001	0.241

From table(5.5), we conclude that the Heston-Jump model is a better fit for CLJ16 options while both models can be good fits for CLM16 options.

6 Conclusion

The MCMC algorithm was successful in obtaining draws from $P(\mu, \psi, \Omega, \kappa, \theta, \lambda, \mu_S, \sigma_S^2, V, Z, B|Y)$, the posterior distribution of interest. As expected, the estimated value of variance under the jump model are

[9] A call option will be referred to as In-The-Money (ITM), At-The-Money (ATM),or Out-of-The-Money (OTM) if the strike price is less than, approximately equal to, or greater than the forward price on the underlying asset.

smaller than in Heston's model as part of the volatility is accounted for using by jumping. Like other published results, we found a strong negative correlation between instantaneous volatility and returns. The estimates of λ indicate that jumps were a very infrequent event over the time period we considered.

While we are comfortable in the recognition that our estimates for the jump parameters are similar to other published works, we urge caution when assigning priors to these parameters, especially σ_S. We noticed that perturbations to this value led to significant differences in estimated parameters after each run. While our results are consistent with those in [5] and [12] they also differ in concrete ways as well. First, our technique incorporates Ito's lemma, which is often neglected in the derivations.

As another check for the algorithm, we also looked at returns for which the program indicated an elevated probability of having a jump. We found that these proposed jump dates matched very closely with the dates of greatest positive and greatest negative return. In addition, at a time of relatively low volatility, returns were more likely to be considered as a result of a jump if the return had a large magnitude. On the other hand, there were relatively fewer jumps detected than would be expected at times of high volatility, such as during the oil price drop in late 2014. As a result, difficulty in detecting jumps during times of high volatility may be viewed as a shortcoming of the Heston-jump model, although this may also be viewed as an overall issue of identifiability.

7 Source of the Code

For MCMC Estimation The code used for parameter estimation in the Heston model was made available to us by Jeffrey Liebner, one of the authors of [5]. He also provided a few ideas on the implementation of the code. We editted the code to include the jump parameters in the jump model, later returning our modifications to him for checking. After we agreed it was fine we used it for the jump model and used his code for the Heston model. The code can be found in the attached Jupyter Notebook.

For option pricing Coding for the option pricing was done by ourselves based on the mathematical formulation of the pricing formulas in both models.

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Appendices

In this section we describe the process of deriving each of the posterior distributions in the models.

A Heston model Parameters

Starting with the Heston model;

$$\begin{aligned} dS &= \mu S dt + \sqrt{V} S dW_1, \\ dV &= \kappa(\theta - V)dt + \sigma_V \sqrt{V} dW_2, \\ dW_1 dW_2 &= \rho dt, \end{aligned} \tag{27}$$

Applying Ito's lemma we have;

$$\begin{aligned} \mathrm{d}\ln S &= (\mu - \frac{1}{2}V)dt + \sqrt{V}dW_1, \\ \mathrm{d}V(t) &= \kappa[\theta - V]dt + \sqrt{V}\sigma_V dW_2, \\ \rho\,\mathrm{d}t &= dW_1 dW_2. \end{aligned}$$

Discretizing and letting $Y_t = lnS_t - lnS_{t-1}$ we have,

$$\begin{aligned} Y_t &= (\mu - \frac{1}{2}V_{t-1})\Delta t + \sqrt{V_{t-1}}\sqrt{\Delta t}\epsilon_t^S, \\ V_t - V_{t-1} &= \kappa(\theta - V_{t-1})\Delta t + \sqrt{V_{t-1}}\sqrt{\Delta t}\epsilon_t^V \\ \epsilon_t^S &\sim N(0,1), \\ \epsilon_t^V &\sim N(0, \sigma_V^2), \\ \mathrm{Corr}(\epsilon_t^S, \epsilon_t^V) &= \rho, \end{aligned}$$

The following assumptions are made about ψ and Ω; where $\psi = \rho\sigma_V,\ \Omega = \sigma_V^2(1-\rho^2)$[5]. We also assume Feller's condition; $2\kappa\theta \geq \sigma_V^2$. Simplifying the above equations;

$$\begin{aligned} Y_t &= (\mu - \frac{1}{2}V_{t-1})\Delta t + \sqrt{V_{t-1}\Delta t}\epsilon_t^S, \\ V_t &= \kappa\theta\Delta t + (1 - \kappa\Delta t)V_{t-1} + \sqrt{V_{t-1}\Delta t}\epsilon_t^V. \end{aligned}$$

Now let

$$\epsilon_t^S = \frac{Y_t - \mu\Delta t - \frac{1}{2}V_{t-1})\Delta t - Z_t B_t}{\sqrt{V_{t-1}}\sqrt{\Delta t}} \tag{28}$$

$$\epsilon_t^V = \frac{V_t - \kappa\theta\Delta t - (1 - \kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}}\sqrt{\Delta t}}, \tag{29}$$

$$(\epsilon_t^S, \epsilon_t^V) \sim N\left((0,0), \begin{pmatrix} 1 & \rho\sigma_V \\ \rho\sigma_V & \sigma_V^2 \end{pmatrix}\right) = N\left((0,0), \begin{pmatrix} 1 & \psi \\ \psi & \psi^2 + \Omega \end{pmatrix}\right)$$

A.1 Estimating the likelihood function

We now estimate $\{\mu, \kappa, \theta, \psi, \Omega\}$ and a state space $\{V_0, ...V_T\}$ using the likelihood function $P(Y, V \mid \mu, \kappa, \theta, \psi, \Omega)$. Since $(\epsilon_t^S, \epsilon_t^V)$ has a bivariate normal distribution we calculate the likelihood function using the property that for a bivariate normal random variable $(X, Y) \sim N((\mu_X, \mu_Y), \Sigma^*)$ the joint density function is given by

$$f(x,y) \propto \frac{1}{|\Sigma^*|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\text{trace}\left({\Sigma^*}^{-1}\begin{pmatrix} X-\mu_X \\ Y-\mu_Y\end{pmatrix}\right)(X-\mu_X, Y-\mu_Y)\right) [5].$$

Transforming the variables involves the determinant of the Jacobian Matrix $\begin{pmatrix} \frac{\partial \epsilon_t^S}{\partial Y_t} & \frac{\partial \epsilon_t^V}{\partial Y_t} \\ \frac{\partial \epsilon_t^S}{\partial V_t} & \frac{\partial \epsilon_t^V}{\partial V_t}\end{pmatrix}$ which yields $\frac{1}{V_{t-1}\Delta t}$. The likelihood function is thus calculated as follows:

$$\begin{aligned} P(Y,V|\mu,\kappa,\theta,\psi,\Omega) &\propto \prod_{t=1}^{T} P(Y_t, V_t|V_{t-1},\mu,\kappa,\theta,\psi,\Omega) \\ &= \prod_{t=1}^{T}\frac{1}{V_{t-1}\Delta t}\left|\begin{pmatrix} 1 & \psi \\ \psi & \psi^2+\Omega\end{pmatrix}\right|^{-1/2} \times \exp\left(-\frac{1}{2}\ \text{trace}\left(\begin{pmatrix} 1 & \psi \\ \psi & \Omega+\psi^2\end{pmatrix}^{-1}\begin{pmatrix}\epsilon_t^S \\ \epsilon_t^V\end{pmatrix}\begin{pmatrix}\epsilon_t^S & \epsilon_t^V\end{pmatrix}\right)\right). \end{aligned}$$

Calculating the trace,

$$\begin{aligned} &\text{trace}\left(\begin{pmatrix} 1 & \psi \\ \psi & \Omega+\psi^2\end{pmatrix}^{-1}\begin{pmatrix}\epsilon_t^S \\ \epsilon_t^V\end{pmatrix}\begin{pmatrix}\epsilon_t^S & \epsilon_t^V\end{pmatrix}\right) \\ &= \text{trace}\left(\frac{1}{\Omega}\begin{pmatrix}\Omega+\psi^2 & -\psi \\ -\psi & 1\end{pmatrix}\begin{pmatrix}{\epsilon_t^S}^2 & \epsilon_t^S\epsilon_t^V \\ \epsilon_t^V\epsilon_t^S & {\epsilon_t^V}^2\end{pmatrix}\right) \\ &= \text{trace}\begin{pmatrix}\frac{\Omega+\psi^2}{\Omega}{\epsilon_t^S}^2 - \frac{\psi}{\Omega}\epsilon_t^S\epsilon_t^V & \frac{\Omega+\psi^2}{\Omega}\epsilon_t^S\epsilon_t^V - \frac{\psi}{\Omega}{\epsilon_t^V}^2 \\ -\frac{psi}{\Omega}{\epsilon_t^S}^2 + \frac{1}{\Omega}\epsilon_t^V\epsilon_t^S & -\frac{\psi}{\Omega}\epsilon_t^S\epsilon_t^V + \frac{1}{\Omega}{\epsilon_t^V}^2\end{pmatrix} \\ &= \frac{\Omega+\psi^2}{\Omega}(\epsilon_t^S)^2 - \frac{2\psi}{\Omega}\epsilon_t^S\epsilon_t^V + \frac{1}{\Omega}(\epsilon_t^V)^2. \end{aligned}$$

Since $\left|\begin{pmatrix} 1 & \psi \\ \psi & \psi^2+\Omega\end{pmatrix}\right|^{-1/2} = \Omega^{-1/2}$, the likelihood function is

$$\begin{aligned} P(Y,V|\mu,\kappa,\theta,\psi,\Omega) &= \prod_{t=1}^{T}\frac{1}{V_{t-1}\triangle t}\Omega^{-1/2}\exp\left(-\frac{1}{2}\left[\frac{\Omega+\psi^2}{\Omega}(\epsilon_t^S)^2 - \frac{2\psi}{\Omega}\epsilon_t^S\epsilon_t^V + \frac{1}{\Omega}(\epsilon_t^V)^2\right]\right) \\ &= \Omega^{-T/2}\prod_{t=1}^{T}\frac{1}{V_{t-1}\triangle t}\prod_{t=1}^{T}\exp\left(-\frac{1}{2\Omega}[(\Omega+\psi^2)(\epsilon_t^S)^2 - 2\psi\epsilon_t^S\epsilon_t^V + (\epsilon_t^V)^2]\right) \\ &= \Omega^{\frac{-T}{2}}\left(\prod_{t=1}^{T}\frac{1}{V_{t-1}\Delta t}\right)\exp\left(\frac{-1}{2\Omega}\Sigma_{t=1}^{T}[(\Omega+\psi^2(\epsilon_t^S)^2 - 2\psi\epsilon_t^S\epsilon_t^V + (\epsilon_t^V)^2]\right) \end{aligned}$$

where ϵ_t^S and ϵ_t^V are defined as in (28) and (29) respectively. We now estimate the posterior distributions of each state variable noting that

$$\text{Posterior distribution} \ \propto \ \text{Likelihood function} \ \times \ \text{Prior distribution.}$$

A.2 Posterior Distributions

1. Posterior Distribution of μ.
   The prior distribution of μ is chosen as $\mu \sim N(\mu_0, \sigma_0^2)$. The posterior distribution of μ is thus

$$
\begin{aligned}
P(\mu|\, Y, V, \kappa, \theta, \psi, \Omega) &\propto P(Y, V|\mu, \kappa, \theta, \psi, \Omega) \centerdot P(\mu) \\
&\propto \exp\left(-\frac{1}{2\Omega}\Sigma_{t=1}^{T}[(\Omega+\psi^2)(\epsilon_t^S)^2 - 2\psi\epsilon_t^V\epsilon_t^S]\right) \centerdot \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) \\
&\propto \exp\left(-\frac{1}{2}\Sigma_{t=1}^{T}\left[\frac{\Omega+\psi^2}{\Omega}\left(\frac{Y_t - \mu\Delta t + \frac{1}{2}V_{t-1}\Delta t}{\sqrt{V_{t-1}\Delta t}}\right)^2 \right.\right. \\
&\left.\left. - \frac{2\psi}{\Omega}\left(\frac{V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta t}}\right)\left(-\frac{\mu\Delta t}{\sqrt{V_{t-1}\Delta t}}\right)\right]\right) \exp\left(-\frac{(\mu^2 - 2\mu_0\mu)}{2\sigma_0^2}\right) \\
&\propto \exp\left(-\frac{1}{2}\left[\left(\Sigma_{t=1}^{T}\frac{\Omega+\psi^2}{\Omega V_{t-1}}\right)\mu^2\Delta t \right.\right. \\
&\left.\left. - 2\sum_{t=1}^{T}\left(\frac{(\Omega+\psi^2)(Y_t + \frac{1}{2}V_{t-1}\Delta t)}{\Omega V_{t-1}} - \frac{\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})}{\Omega V_{t-1}}\right)\mu\right]\right) \\
&\centerdot \exp\left(-\frac{1}{2}\left[\left(\frac{1}{\sigma_0^2}\right)\mu^2 - 2\left(\frac{\mu_0}{\sigma_0^2}\right)\mu\right]\right).
\end{aligned}
$$

   By completing the square in μ; $\mu \sim N(\mu^*, \sigma^{*2})$ where

$$
\mu^* = \frac{\begin{array}{c}\Sigma_{t=1}^{T}((\Omega+\psi^2)(Y_t + \frac{1}{2}V_{t-1}\Delta t)/\Omega V_{t-1}) \\ -\Sigma_{t=1}^{T}(\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})/ + \Omega V_{t-1})\mu_0/\sigma_0^2\end{array}}{\Delta t \sum_{t=1}^{T}((\Omega+\psi^2)/\Omega V_{t-1} + 1/\sigma_0^2}
$$

   and

$$
\sigma^{*2} = \frac{1}{\Delta t \sum_{t=1}^{T}((\Omega+\psi^2)/\Omega V_{t-1} + 1/\sigma_0^2}
$$

2. Posterior Distribution of ψ and Ω.
   Priors of ψ and Ω are chosen as $\Omega \sim IG(\tilde{\alpha}, \tilde{\beta})$ (inverse gamma distribution) and $\psi|_\Omega \sim N(\psi_0, \Omega/p_0)$

so

$$
\begin{aligned}
P(\psi,\Omega|Y,V,\kappa,\theta,\mu) &\propto P(Y,V\mid\psi,\Omega,\kappa,\theta,\mu)\cdot P(\psi\mid\Omega)\cdot P(\Omega)\\
&\propto \Omega^{-T/2}\exp\left(-\frac{1}{2\Omega}\sum_{t=1}^{T}[(\Omega+\psi^2)(\epsilon_t^S)^2+(\epsilon_t^V)^2-2\psi\epsilon_t^S\epsilon_t^V]\right)\\
&\quad\cdot\sqrt{\frac{p_0}{\Omega}}\exp\left(-\frac{(\psi-\psi_0)^2}{2\Omega/p_0}\right)\cdot\frac{\tilde{\beta}^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha})}\Omega^{-\tilde{\alpha}-1}\exp\left(-\frac{\tilde{\beta}}{\Omega}\right)\\
&\propto \Omega^{-T/2-\tilde{\alpha}-1}\frac{1}{\Omega^{1/2}}\cdot\exp\left(-\frac{1}{2\Omega}\sum_{t=1}^{T}[(\Omega+\psi^2)(\epsilon_t^S)^2+(\epsilon_t^V)^2-2\psi\epsilon_t^S\epsilon_t^V]\right.\\
&\quad\left.-\frac{1}{2}\frac{(\psi-\psi_0)^2}{\Omega/p_0}-\frac{\tilde{\beta}}{\Omega}\right)\\
&\propto \Omega^{-T/2-\tilde{\alpha}-1}\exp\left(-\frac{1}{\Omega}\left[\tilde{\beta}+\frac{1}{2}\sum_{t=1}^{T}(\epsilon_t^V)^2\right]\right)\cdot\exp\left(-\frac{1}{2\Omega}p_0\psi_0^2\right)\cdot\frac{1}{\Omega^{1/2}}\\
&\quad\exp\left(-\frac{1}{2\Omega}\left[\left(p_0+\sum_{t=1}^{T}(\epsilon_t^S)^2\right)\psi^2-2\left(p_0\psi_0+\sum_{t=1}^{T}\epsilon_t^S\epsilon_t^V\right)\psi\right.\right.\\
&\quad\left.\left.+\frac{(p_0\psi_0+\sum_{t=1}^{T}\epsilon_t^S\epsilon_t^V)^2}{p_0+\sum_{t=1}^{T}(\epsilon_t^S)^2}-\frac{(p_0\psi_0+\sum_{t=1}^{T}\epsilon_t^S\epsilon_t^V)^2}{p_0+\sum_{t=1}^{T}(\epsilon_t^S)^2}\right]\right)
\end{aligned}
$$

Completing squares for all the terms in ψ and combining the terms containing Ω we get $\Omega\sim IG(\alpha_*,\beta_*)$ where

$$
\alpha_*=\frac{T}{2}+\tilde{\alpha}\text{ and }\beta_*=\tilde{\beta}+\frac{1}{2}\sum_{t=1}^{T}(\epsilon_t^V)^2+\frac{1}{2}p_0\psi_0^2-\frac{1}{2}p_0\psi_0^2-\frac{(p_0\psi_0+\sum_{t=1}^{T}\epsilon_t^S\epsilon_t^V)^2}{p_0+\sum_{t=1}^{T}(\epsilon_t^S)^2},
$$

and $\psi|_{\Omega}\sim N(\psi^*,\sigma_{\psi}^{*2})$ where

$$
\psi^*=\frac{p_0\psi_0+\sum_{t=1}^{T}\epsilon_t^S\epsilon_t^V}{p_0+\sum_{t=1}^{T}(\epsilon_t^S)^2}\text{ and }\sigma_{\psi}^{*2}=\frac{\Omega}{p_0+\sum_{t=1}^{T}(\epsilon_t^S)^2}.
$$

3. Posterior Distribution of (κ,θ). The prior of θ is chosen as $\theta\sim N(\theta_0,\sigma_\theta^2)$. The posterior distribution

is therefore

$$P(\theta \,|Y, V, \mu, \kappa, \psi, \Omega) \propto P(Y, V \mid \mu, \kappa, \theta, \psi, \Omega) \,.\, P(\theta)$$
$$\propto \exp\Bigg(- \frac{1}{2\Omega} \sum_{t=1}^{T} [(\epsilon_t^V)^2 - 2\psi \epsilon_t^V \epsilon_t^S] \Bigg) \,.\, \exp\Bigg(- \frac{(\theta - \theta_0)^2}{2\sigma_\theta^2} \Bigg)$$
$$\propto \exp\Bigg(- \frac{1}{2} \sum_{t=1}^{T} \Bigg[\frac{1}{\Omega} \Bigg(\frac{V_t - \kappa\theta\Delta t - (1 - \kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta t}} \Bigg)^2$$
$$- \frac{2\psi}{\Omega} \Bigg(\frac{Y_t - \mu\Delta t + \frac{1}{2} V_{t-1}\Delta t}{\sqrt{V_{t-1}\Delta t}} \Bigg) \Bigg(- \frac{\kappa\theta\Delta t}{\sqrt{V_{t-1}\Delta t}} \Bigg) \Bigg] \Bigg)$$
$$\,.\, \exp\Bigg(- \frac{(\theta^2 - 2\theta_0\theta)}{2\sigma_\theta^2} \Bigg) \propto \exp\Bigg(- \frac{1}{2} \Bigg[\Bigg(\sum_{t=1}^{T} \frac{\kappa^2}{\Omega V_{t-1}} \Bigg) \theta^2 \Delta t$$
$$- 2 \sum_{t=1}^{T} \Bigg(\frac{(\kappa)(V_t - (1 - \kappa\Delta t)V_{t-1})}{\Omega V_{t-1}}$$
$$- \frac{\psi(Y_t - \mu\Delta t + \frac{1}{2} V_{t-1}\Delta t)\kappa}{\Omega V_{t-1}} \Bigg) \theta \Bigg] \Bigg) \,.\, \exp\Bigg(- \frac{1}{2} \Bigg[\Bigg(\frac{1}{\sigma_\theta^2} \theta^2 - 2 \Bigg(\frac{\theta_0}{\sigma_\theta^2} \Bigg) \theta \Bigg] \Bigg)$$

By completing the square for terms in θ we get the posterior as $\theta \sim N(\theta^*, \sigma_\theta^{*2})$ where

$$\theta^* = \frac{\begin{array}{c} \Sigma_{t=1}^{T}((\kappa)(V_t - (1 - \kappa\Delta t)V_{t-1})/\Omega V_{t-1}) \\ -\Sigma_{t=1}^{T}(\psi(Y_t - \mu\Delta t + \dfrac{1}{2} V_{t-1}\Delta t)\kappa/\Omega V_{t-1}) + \theta_0/\sigma_\theta^2 \end{array}}{\Delta t \sum_{t=1}^{T}(\kappa^2/\Omega V_{t-1}) + 1/\sigma_\theta^2}$$

and

$$\sigma_\theta^{*2} = \frac{1}{\Delta t \sum_{t=1}^{T}(\kappa^2/\Omega V_{t-1}) + 1/\sigma_\theta^2}.$$

The prior of κ is chosen as $\kappa \sim N(\kappa_0, \sigma_\kappa^2)$. So the posterior distribution is calculated as

$$P(\kappa \,|Y, V, \mu, \theta, \psi, \Omega) \propto P(Y, V \mid \mu, \kappa, \theta, \psi, \Omega) \,.\, P(\kappa)$$
$$\propto \exp\Bigg(- \frac{1}{2\Omega} \sum_{t=1}^{T} [(\epsilon_t^V)^2 - 2\psi \epsilon_t^V \epsilon_t^S \Bigg) \,.\, \exp\Bigg(- \frac{(\kappa - \kappa_0)^2}{2\sigma_\kappa^2} \Bigg)$$
$$\propto \exp\Bigg(- \frac{1}{2} \sum_{t=1}^{T} \Bigg[\frac{1}{\Omega} \Bigg(\frac{V_t - \kappa\theta\Delta t - (1 - \kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta t}} \Bigg)^2$$
$$- \frac{2\psi}{\Omega} \Bigg(- \frac{1}{2} \Bigg[\Bigg(\sum_{t=1}^{T} \frac{(V_{t-1} - \theta)^2}{\Omega V_{t-1}} \Bigg) \kappa^2 \Delta t - 2 \sum_{t=1}^{T} \Bigg(\frac{(\theta - V_{t-1})(V_t - V_{t-1})}{\Omega V_{t-1}}$$
$$- \frac{\psi(Y_t - \mu\Delta t + \frac{1}{2} V_{t-1}\Delta t)(\theta - V_{t-1})}{\Omega V_{t-1}} \Bigg) \kappa \Bigg] \Bigg) \,.\, \exp\Bigg(- \frac{1}{2} \Bigg[\Bigg(\frac{1}{\sigma_\kappa^2} \Bigg) \kappa^2 - 2 \Bigg(\frac{\kappa_0}{\sigma_\kappa^2} \Bigg) \kappa \Bigg] \Bigg).$$

Completing the square in κ we get $\kappa \sim N(\kappa^*, \sigma_\kappa^{*2})$ where

$$\kappa^* = \frac{\begin{array}{c} \Sigma_{t=1}^{T}((\theta - V_{t-1})(V_t - V_{t-1})/\Omega V_{t-1}) \\ -\Sigma_{t=1}^{T}(\psi(Y_t - \mu\Delta t + \dfrac{1}{2} V_{t-1}\Delta t)(\theta - V_{t-1})/\Omega V_{t-1}) + \kappa_0/\sigma_\theta^2 \end{array}}{\Delta t \sum_{t=1}^{T}((V_{t-1} - \theta)^2/\Omega V_{t-1}) + 1/\sigma_\kappa^2}$$

and

$$\sigma_\kappa^{*2} = \frac{1}{\Delta t \sum_{t=1}^{T}((V_{t-1}-\theta)^2/\Omega V_{t-1}) + 1/\sigma_\kappa^2}.$$

4. Posterior Distribution of the state variable of variance, V_t.
   This is given by

$$\begin{aligned}
&P(V_t \mid Y, V_{t+1}, V_{t-1}, \kappa, \theta, \psi, \Omega, \mu)\\
&\quad= P(Y, V_{t+1}, V_t \mid V_{t-1}, \kappa, \theta, \psi, \Omega, \mu)\frac{P(V_{t-1} \mid \kappa, \theta, \psi, \Omega, \mu)}{P(Y, V_{t+1}, V_{t-1} \mid \kappa, \theta, \psi, \Omega, \mu)}\\
&\quad\propto P(Y, V_{t+1}, V_t \mid V_{t-1}, \kappa, \theta, \psi, \Omega, \mu)\\
&\quad\propto \frac{1}{V_t\Delta t}\exp\Bigg(-\frac{1}{2\Omega}[(\Omega+\psi^2)(\epsilon_{t+1}^S)^2 - 2\psi(\epsilon_{t+1}^S)^2 - 2\psi\epsilon_{t+1}^S\epsilon_{t+1}^V + (\epsilon_{t+1}^V)^2]\\
&\qquad - \frac{1}{2\Omega}[(\Omega+\psi^2)(\epsilon_t^S)^2 - 2\psi\epsilon_t^S\epsilon_t^V + (\epsilon_t^V)^2]\Bigg)\\
&\quad= \frac{1}{V_t\Delta t}\exp\Bigg(-\frac{1}{2\Omega}\frac{(\Omega+\psi^2)(\frac{1}{2}V_t\Delta t + Y_{t+1} - \mu\Delta t)^2}{V_t\Delta t}\\
&\qquad - \frac{1}{2\Omega}\frac{-2\psi(\frac{1}{2}V_t\Delta t + Y_{t+1} - \mu\Delta t)(-(1-\kappa\Delta t)V_t - \kappa\theta\Delta t + V_{t+1}}{V_t\Delta t}\\
&\qquad - \frac{1}{2\Omega}\frac{(-(1-\kappa\Delta t)V_t - \kappa\theta\Delta t + V_{t+1})^2}{V_t\Delta t}\Bigg)\\
&\qquad \times\exp\Bigg(-\frac{1}{2\Omega}\frac{-2\psi(Y_t - \mu\Delta t + \frac{1}{2}V_{t-1}\Delta t)(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})}{V_{t-1}\Delta t}\\
&\qquad - \frac{1}{2\Omega}\frac{(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})^2}{V_{t-1}\Delta t}\Bigg).
\end{aligned}$$

B Parameter Estimation in the Heston-jump diffusion Model

We may rewrite the model in (19) as:

$$\begin{aligned}
dS(t) &= \mu S dt + \sqrt{V} S dW_1 + d\left(\sum_{j=1}^{N(t)} S_{\tau_j} - [\mathrm{e}^{Z_j} - 1]\right) - \lambda(\mathrm{e}^{\mu_S + (1/2)\sigma_S^2} - 1)S dt,\\
dV(t) &= \kappa(\theta - V)dt + \sigma_V\sqrt{V}dW_2,\\
\rho dt &= dW_1 dW_2,
\end{aligned}$$

where all parameters are as described in the Heston model with the inclusion of

- The jump size, e^{Z_j} which has a log-normal distribution with $Z_j \sim N(\mu_S, \sigma_S^2)$,
- τ_j is a stopping time that represents the time of the jth jump.
- $\mathrm{e}^{Z_j} - 1$ is the percentage jump size which lies between -100% and ∞

The number of jumps in each period $[0, t]$ is defined as $N(t)$ with $N(t) \sim$ Poisson(λt).

Applying Ito's lemma;

$$\begin{aligned}
\mathrm{d}\ln S &= \left(\mu - \lambda(\mathrm{e}^{\mu_S+(1/2)\sigma_S^2} - 1) - \frac{1}{2}V\right)dt + \sqrt{V}dW_1 + d\left(\sum_{j=1}^{N(t)} Z_j\right), \\
\mathrm{d}V(t) &= \kappa[\theta - V(t)]dt + \sqrt{V(t)}\sigma_V dW_2, \\
\rho\,\mathrm{d}t &= dW_1 dW_2.
\end{aligned}$$

Discretizing,

$$\begin{aligned}
Y_t &= \left(\mu - \lambda(\mathrm{e}^{\mu_S+(1/2)\sigma_S^2} - 1) - \frac{1}{2}V_{t-1}\right)\Delta t + \sqrt{V_{t-1}}\sqrt{\Delta t}\epsilon_t^S + Z_t B_t, \\
V_t - V_{t-1} &= \kappa(\theta - V_{t-1})\Delta t + \sqrt{V_{t-1}}\sqrt{\Delta t}\epsilon_t^V \\
\epsilon_t^S &\sim N(0,1), \\
\epsilon_t^V &\sim N(0,\sigma_V^2), \\
\mathrm{Corr}(\epsilon_t^S, \epsilon_t^V) &= \rho, \\
Z_t &\sim N(\mu_S, \sigma_V^2), \\
B_t &\sim \mathrm{Bernoulli}(\lambda\Delta t)
\end{aligned}$$

The following assumptions are made;
$\psi = \rho\sigma_V$, $\Omega = \sigma_V^2(1-\rho^2), \bar{\mu} = \mu - (\mathrm{e}^{\mu_S+(1/2)\sigma_S^2} - 1)$ and $\lambda\Delta t = \lambda_D$[5], λ_D being the daily intensity rate for the poisson process. Plugging into the above equations;

$$\begin{aligned}
Y_t &= (\bar{\mu} - \frac{1}{2}V_{t-1})\Delta t + \sqrt{V_{t-1}\Delta t}\epsilon_t^S + Z_t B_t, \\
V_t &= \kappa\theta\Delta t + (1-\kappa\Delta t)V_{t-1} + \sqrt{V_{t-1}\Delta t}\epsilon_t^V.
\end{aligned}$$

Now let

$$\epsilon_t^S = \frac{Y_t - \bar{\mu}\Delta t - \frac{1}{2}V_{t-1})\Delta t - Z_t B_t}{\sqrt{V_{t-1}}\sqrt{\Delta t}} \tag{30}$$

$$\epsilon_t^V = \frac{V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}}\sqrt{\Delta t}}, \tag{31}$$

$$\begin{aligned}
(\epsilon_t^S, \epsilon_t^V) \sim N\left((0,0), \begin{pmatrix} 1 & \rho\sigma_V \\ \rho\sigma_V & \sigma_V^2 \end{pmatrix}\right) &= N\left((0,0), \begin{pmatrix} 1 & \psi \\ \psi & \psi^2+\Omega \end{pmatrix}\right) \\
Z_t &\sim N(\mu_S, \sigma_S^2), \\
B_t &\sim \mathrm{Bernoulli}(\lambda_D)
\end{aligned}$$

B.1 Estimating the likelihood function

The same method used in the Heston model is employed. The likelihood function is therefore

$$\begin{aligned}
P(Y,V|\bar{\mu},\kappa,\theta,\psi,\Omega,B,Z) &= \prod_{t=1}^{T} \frac{1}{V_{t-1}\triangle t}\Omega^{-1/2}\exp\left(-\frac{1}{2}\left[\frac{\Omega+\psi^2}{\Omega}(\epsilon_t^S)^2 - \frac{2\psi}{\Omega}\epsilon_t^S\epsilon_t^V + \frac{1}{\Omega}(\epsilon_t^V)^2\right]\right) \\
&= \Omega^{-T/2}\prod_{t=1}^{T}\frac{1}{V_{t-1}\triangle t}\prod_{t=1}^{T}\exp\left(-\frac{1}{2\Omega}[(\Omega+\psi^2)(\epsilon_t^S)^2 - 2\psi\epsilon_t^S\epsilon_t^V + (\epsilon_t^V)^2]\right) \\
&= \Omega^{\frac{-T}{2}}\left(\prod_{t=1}^{T}\frac{1}{V_{t-1}\Delta t}\right)\exp\left(\frac{-1}{2\Omega}\Sigma_{t=1}^{T}[(\Omega+\psi^2(\epsilon_t^S)^2 - 2\psi\epsilon_t^S\epsilon_t^V + (\epsilon_t^V)^2]\right)
\end{aligned}$$

where ϵ_t^S and ϵ_t^V are as defined as (30) and (31) respectively.

B.2 Posterior Distributions

With the exception of the jump term, the derivations for the posterior distributions of $\bar{\mu}, \kappa, \theta, \psi, \Omega, B, Z$ for the Heston-Jump model are identical to the derivations in the Heston model([5], [17], [14]). We therefore describe these briefly.

1. Posterior distribution of $\bar{\mu}$.
   If the prior distribution of $\bar{\mu}$ is $\bar{\mu} \sim N(\mu_0, \sigma_0^2)$ then the posterior distribution is given by

$$P(\bar{\mu} \mid Y, V, \kappa, \theta, \psi, \Omega) \propto P(Y, V \mid \mu\kappa, \theta, \psi, \Omega) \cdot P(\mu)$$

   giving that the prior distribution is $\bar{\mu} \sim N(\mu^*, \sigma^{*2})$ where

$$\mu^* = \frac{\begin{array}{l}\Sigma_{t=1}^T((\Omega + \psi^2)(Y_t + \frac{1}{2}V_{t-1}\Delta t - Z_t B_t)/\Omega V_{t-1}) \\ -\Sigma_{t=1}^T(\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}) + \mu_0/\sigma_0^2\end{array}}{\Delta t \sum_{t=1}^T((\Omega + \psi^2)/\Omega V_{t-1} + 1/\sigma_0^2}$$

   and

$$\sigma^{*2} = \frac{1}{\Delta t \sum_{t=1}^T((\Omega + \psi^2)/\Omega V_{t-1} + 1/\sigma_0^2}$$

2. Posterior distribution of ψ and Ω.
   The prior distributions of ψ and Ω are chosen as $\Omega \sim \mathcal{IG}(\bar{\alpha}, \bar{\beta})$ and $\psi|_\Omega \sim N(\psi_0, \Omega/p_0)$ respectively. By the same method in the previous section we have that $\Omega \sim \mathcal{IG}(\bar{\alpha}_*, \bar{\beta}_*)$ where

$$\alpha_* = \frac{T}{2} + \bar{\alpha} \quad \text{and} \quad \beta_* = \bar{\beta} + \frac{1}{2}\sum_{t=1}^T(\epsilon_t^V)^2 + \frac{1}{2}p_0\psi_0^2 - \frac{1}{2}\frac{(p_0\psi_0 + \sum_{t=1}^T \epsilon_t^S \epsilon_t^V)^2}{p_0 + \sum_{t=1}^T(\epsilon_t^S)^2}$$

   and $\psi|_\Omega \sim N(\psi^*, \sigma_\psi^{*2})$ where

$$\psi^* = \frac{p_0\psi_0 + \sum_{t=1}^T \epsilon_t^S \epsilon_t^V}{p_0 + \sum_{t=1}^T(\epsilon_t^S)^2} \quad \text{and} \quad \sigma_\psi^{*2} = \frac{\Omega}{p_0 + \sum_{t=1}^T(\epsilon_t^S}\,)^2.$$

3. Posterior distribution of (κ, θ)
   If the prior of θ is $\theta \sim N(\theta_0, \sigma_\theta^2)$, the posterior distribution of θ is $\theta \sim N(\theta^*, \sigma_\theta^{*2})$ where

$$\theta^* = \frac{\begin{array}{l}\Sigma_{t=1}^T((\kappa)(V_t - (1-\kappa\Delta t)V_{t-1})/\Omega V_{t-1}) \\ -\Sigma_{t=1}^T(\psi(Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t - Z_t B_t)\kappa/\Omega V_{t-1}) + \theta_0/\sigma V_{t-1}) + \theta_0/\sigma_0^2\end{array}}{\Delta t \sum_{t=1}^T(\kappa^2/\Omega V_{t-1}) + 1/\sigma_\theta^2}$$

   and

$$\sigma_\theta^{*2} = \frac{1}{\Delta t \sum_{t=1}^T((\kappa^2/\Omega V_{t-1} + 1/\sigma_0^2}$$

   If the prior of κ is $\kappa \sim N(\kappa_0, \sigma_\kappa^2)$, the the posterior of κ is $\kappa \sim N(\kappa^*, \sigma_\kappa^{*2})$ where

$$\kappa^* = \frac{\begin{array}{l}\Sigma_{t=1}^T((\theta - V_{t-1})(V_t - V_{t-1})/\Omega V_{t-1}) \\ -\Sigma_{t=1}^T(\psi(Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t - Z_t B_t)(\theta - V_{t-1}) + \kappa_0/\sigma_\kappa^2\end{array}}{\Delta t \sum_{t=1}^T(\kappa^2/\Omega V_{t-1}) + 1/\sigma_\theta^2}$$

   and

$$\sigma_\kappa^{*2} = \frac{1}{\Delta t \sum_{t=1}^T((V_{t-1} - \theta)^2/\Omega V_{t-1} + 1/\sigma_\kappa^2}$$

4. Posterior Distribution of V_t, the state variable of variance.

$$
\begin{aligned}
&P(V_t|\, Y, V_{t+1}, V_{t-1}, \kappa, \theta, \psi, \Omega, \mu, Z_t, B_t, Z_{t+1}, B_{t+1}) = \\
&\frac{1}{V_t \Delta t} \exp\Bigg(- \frac{1}{2\Omega} \frac{(\Omega+\psi^2)(\frac{1}{2}V_t\Delta t + Y_{t+1} - Z_{t+1}B_{t+1} - \bar{\mu}\Delta t)^2}{V_t \Delta t} \\
&\qquad - \frac{1}{2\Omega} \frac{-2\psi(\frac{1}{2}V_t\Delta t + Y_{t+1} - Z_{t+1}B_{t+1} - \bar{\mu}\Delta t)(-(1-\kappa\Delta t)V_t - \kappa\theta\Delta t + V_{t+1})}{V_t\Delta t} \\
&\qquad - \frac{1}{2\Omega} \frac{(-(1-\kappa\Delta t)V_t - \kappa\theta\Delta t + V_{t+1})^2}{V_t \Delta t} \\
&\qquad - \frac{1}{2\Omega} \frac{-2\psi(Y_t - Z_tB_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t)(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})}{V_{t-1}\Delta t} \\
&\qquad - \frac{1}{2\Omega} \frac{(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})^2}{V_{t-1}\Delta t} \Bigg)
\end{aligned}
$$

5. Posterior distribution of Z_t, the magnitude of the jump.
Assuming the prior is chosen as $Z_t \sim \mathcal{N}(\mu_S, \sigma_S^2)$. We consider first the case where $B_t = 1$.

$$
\begin{aligned}
&P(Z_t \mid \bar{\mu}, \kappa, \theta, \psi, \Omega, Y, V, B_t = 1) \propto P(Y_t, V_t \mid \bar{\mu}, \kappa, \theta, \psi, \Omega, Z_t, V/\{V_t\}, Y/\{Y_t\}, B_t = 1) \cdot P(Z_t) \\
&\propto \exp\left(\frac{1}{2} \left[\frac{\Omega+\psi^2}{\Omega} \left(\frac{Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t - Z_t}{\sqrt{V_{t-1}\Delta t}} \right)^2 \right.\right. \\
&\quad \left.\left. - \frac{2\psi}{\Omega} \left(\frac{V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta t}} \right) \left(\frac{Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t - Z_t}{\sqrt{V_{t-1}\Delta t}} \right) \right] \right) \\
&\quad \cdot \exp\left(- \frac{(Z_t^2 - 2\mu_S Z_t)}{2\sigma_S^2} \right) \propto \exp\left(- \frac{1}{2} \left[\left(\frac{\Omega+\psi^2}{\Omega V_{t-1}\Delta t} \right) Z_t^2 \right.\right. \\
&\quad \left.\left. - 2\left(\frac{(\Omega+\psi^2)(Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t)}{\Omega V_{t-1}\Delta t} - \frac{\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})}{\Omega V_{t-1}\Delta t} Z_t \right] \right) \\
&\quad \exp\left(- \frac{1}{2} \left[\left(\frac{1}{\sigma_S^2} \right) Z_t^2 - 2\left(\frac{\mu_S}{\sigma_S^2} \right) Z_t \right] \right).
\end{aligned}
$$

By completing the square in Z_t we have $Z_t \sim \mathcal{N}(\mu_S^*, \sigma_S^{*2})$ where

$$
\mu_S^* = \frac{\begin{array}{c} ((\Omega+\psi^2)(Y_t + (1/2)V_{t-1}\Delta t - \bar{\mu}\Delta t)/\Omega V_{t-1}\Delta t) \\ -\, (\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})/\Omega V_{t-1}\Delta t) + \mu_S/\sigma_S^2 \end{array}}{(\Omega+\psi^2)/\Omega V_{t-1}\Delta t + 1/\sigma_S^2}
$$

and

$$
\sigma_S^{*2} = \frac{1}{(\Omega+\psi^2)/\Omega V_{t-1}\Delta t + 1/\sigma_S^2}
$$

6. Posterior distribution of B_t

A prior distribution for B_t is chosen as $B_t \sim \text{Bernoulli}(\lambda_D)$. The posterior distribution of B_t is

$$
\begin{aligned}
&P(B_t \mid \bar{\mu}, \kappa, \theta, \psi, \Omega, Y, V, Z_t, \lambda_D) \propto P(Y_t, V_t \mid \bar{\mu}, \kappa, \theta, \psi, \Omega, V/V_t, Y/Y_t, Z_t, B_t, \lambda_D) \cdot P(B_t) \\
&\propto \exp\Bigg(-\frac{1}{2}\Bigg[\frac{\Omega+\psi^2}{\Omega}\Bigg(\frac{Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t - Z_t B_t}{\sqrt{V_{t-1}}\Delta t}\Bigg)^2 \\
&\quad - \frac{2\psi}{\Omega}\Bigg(\frac{V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta t}}\Bigg)\Bigg(\frac{Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t - Z_t B_t}{\sqrt{V_{t-1}\Delta t}}\Bigg) \\
&\quad + \frac{1}{\Omega}\Bigg(\frac{V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta t}}\Bigg)^2\Bigg]\Bigg) \cdot (\lambda_D)^{B_t}(1-\lambda_D)^{1-B_t}
\end{aligned}
$$

$$
\begin{aligned}
&\propto \exp\Bigg(-\frac{1}{2}\Bigg[\frac{\Omega+\psi^2}{\Omega}\Bigg(\frac{Z_t^2 B_t^2 - 2(Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t)Z_t B_t}{V_{t-1}\Delta}\Bigg) \\
&\quad - \frac{2\psi}{\Omega}\Bigg(\frac{V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1}}{\sqrt{V_{t-1}\Delta}}\Bigg)\Bigg(\frac{-Z_t B_t}{\sqrt{V_{t-1}\Delta t}}\Bigg)\Bigg]\Bigg) \cdot \Bigg(\frac{\lambda_D}{1-\lambda_D}\Bigg)^{B_t}
\end{aligned}
$$

$$
\propto \exp\Bigg(-\frac{1}{2}AB_t\Bigg) \cdot \Bigg(\frac{\lambda_D}{1-\lambda_D}\Bigg)^{B_t} \propto \Bigg(\exp\Bigg(-\frac{1}{2}A\Bigg) \cdot \frac{\lambda_D}{1-\lambda_D}\Bigg)^{B_t} \propto \Bigg(\frac{P^*}{1-P^*}\Bigg)^{B_t}
$$

So B_t has a Bernoulli distribution with probability

$$
P^* = \frac{\lambda_D \exp(-\frac{1}{2}A)/(1-\lambda_D)}{1+\lambda_D \exp(-\frac{1}{2}A)/(1-\lambda_D)} = \frac{1}{(1-\lambda_D)\exp(\frac{1}{2}A)/(\lambda_D+1)},
$$

where

$$
A = \frac{(\Omega+\psi^2)(Z_t^2 - 2Z_t(Y_t - \bar{\mu}\Delta t + \frac{1}{2}V_{t-1}\Delta t)) + 2\psi(V_t - \kappa\theta\Delta t - (1-\kappa\Delta t)V_{t-1})Z_t}{\Omega V_{t-1}\Delta t}
$$

7. Posterior distribution of λ_D. The prior distribution for λ_D is assigned to be $\lambda_D \sim \text{Beta}(\alpha', \beta')$. We have therefore

$$
\begin{aligned}
P(\lambda_D \mid B) &\propto P(B \mid \lambda_D) \cdot P(\lambda_D) \\
&\propto \binom{T}{\sum_{t=1}^T B_t} \lambda^{\Sigma_{B_{t=1}}^T}(1-\lambda_D)^{T-\Sigma_{t=1}^T B_t} \cdot P(\lambda_D).
\end{aligned}
$$

Plugging in the density of the Beta function,

$$
\propto \binom{T}{\sum_{t=1}^T B_t} \lambda^{\Sigma_{B_{t=1}}^T}(1-\lambda_D)^{T-\Sigma_{t=1}^T B_t} \cdot \frac{\lambda_D^{\alpha'-1}(1-\lambda_D)^{\beta'-1}}{B(\alpha', \beta')}
$$

The posterior distribution of λ_D is therefore $\lambda_D \sim \text{Beta}(\alpha^*_{\lambda D}, \beta^*_\lambda)$, where

$$
\begin{aligned}
\alpha^*_\lambda &= \alpha' + \Sigma_{t=1}^T B_t, \\
\beta^*_\lambda &= \beta' + T - \Sigma_{t=1}^T B_t.
\end{aligned}
$$

8. Posterior distributions of $\mu_S|\sigma_S^2$ and $\sigma_S^2|\mu_S$. The prior distributions are assigned so that $\mu_S \sim \mathcal{N}(0, S_0)$ and $\sigma_S^2 \sim \mathcal{IG}(\alpha_S, \beta_S)$.

$$\begin{aligned} P(\mu_S|\sigma_S^2, Z) &\propto P(Z \mid \mu_S, \sigma_S^2) \cdot P(\mu_S) \\ &\propto (\sigma_S)^{-T} \exp\left(-\frac{1}{2} \sum_{t=1}^{T} \frac{(Z_t - \mu_S)^2}{\sigma_S^2} \right) \cdot \exp\left(-\frac{\mu_S^2}{2S_0} \right), \end{aligned}$$

$$\begin{aligned} P(\sigma_S^2|\mu_S, Z) &\propto P(Z|\mu_S, \sigma_S^2) \cdot P(\sigma_S^2) \\ &\propto (\sigma_S)^{-T} \exp\left(-\frac{1}{2} \sum_{t=1}^{T} \frac{(Z_t - \mu_S)^2}{\sigma_S^2} \right) \cdot \frac{\beta_S^{\alpha_S}}{\Gamma(\alpha_S)} (\sigma_S^2)^{-\alpha_S - 1} \exp\left(-\frac{\beta_S}{\sigma_S^2} \right) \end{aligned}$$

So the posterior distribution of $\mu_S|\sigma_S^2$ is $\mu_S \sim \mathcal{N}(\mu^*_{\mu_S}, \sigma^{*2}_{\mu_S})$, where

$$\mu^*_{\mu_S} = \frac{\Sigma_{t=1}^T (Z_t/\sigma_S^2)}{1/S_0 + \Sigma_{t=1}^T (1/\sigma_S^2)} \text{ and } \sigma^{*2}_{\mu_S} = \frac{1}{1/S_0 + \Sigma_{t=1}^T (1/\sigma_S^2)}$$

and the posterior distribution of $\sigma_S^2|\mu_S$ is $\sigma_S^2 \sim \mathcal{IG}(\alpha_S^*, \beta_S^*)$, where

$$\alpha_S^* = \alpha_S + T/2 \text{ and } \beta_S^* = \beta_S + \frac{1}{2} \sum_{t=1}^{T} (Z_t - \mu_S)^2.$$