Introduction to Stochastic Differential Equations: Quiz I

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Course questions (4 pts).
• Formulate the definition of a standard Brownian motion.
• Formulate the main properties (linearity, martingale, isometry) of the stochastic integral $\int_0^t \beta_s \, dW_s$, for $(W_t;\, t \geq 0)$ a $\mathcal{F}_t$-Brownian motion and $(\beta_t,\, t \geq 0)$ a real valued $\mathcal{F}_t$-adapted stochastic process such that,

$$\mathbb{E}[\int_0^t |\beta_s|^2 \, ds] < \infty,\, t \geq 0.$$

• Formulate the complete definition of a strong solution to a Stochastic Differential Equation of the form:

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t) dW_t, \\ X_0 = \xi, \end{cases}$$

for $b : [0, \infty) \times \mathbb{R}$ and $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$.

PROBLEM 1: Brownian motion and Itô's formula (3 pts).

Exercise 1. Let $(W_t;\, t \geq 0)$ be a $\mathcal{F}_t$-Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \geq 0), \mathbb{P})$. Justify that the following processes are also a Brownian motion:

- $(-W_t;\, t \geq 0)$;
- $(\alpha W_t + (1 - \alpha)\tilde{W}_t;\, t \geq 0)$ for $0 < \alpha < 1$ a deterministic constant and $(\tilde{W}_t;\, t \geq 0)$ another $\mathcal{F}_t$-Brownian motion independent to $(W_t;\, t \geq 0)$;
- $(\int_0^t \mathbb{1}_{\{W_s \geq 0\}} \, dW_s;\, t \geq 0)$.

Exercise 2.

Help: **Itô formula:** For $(X_t;\, t \geq 0)$, solution to

$$X_t = \xi + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s,\, t \geq 0,$$

we have, for $f : \mathbb{R} \to \mathbb{R}$ a smooth function,

$$f(X_t) = f(\xi) + \int_0^t f'(X_s) b(s, X_s)\, ds + \int_0^t f'(X_s)\sigma(s, X_s)\, dW_s$$
$$+ \frac{1}{2}\int_0^t f''(X_s)\sigma^2(s, X_s)\, ds.$$

Let $(W_t;\, t \geq 0)$ be a $\mathcal{F}_t$-Brownian motion. Apply the Itô formula in the three following cases:

$a)$ $M_t^1 = (W_t)^2$; $b)$ $M_t^2 = \cos(W_t)$; $c)$ $M_t^3 = \exp(\sigma W_t + \mu t)$, $\sigma, \mu \in \mathbb{R}$ deterministic constants.

In the case (c), for which values of σ and μ the process is a martingale ?
Bonus: In each of the preceding cases, compute the covariation $\langle M^i \rangle_t$ of M^1, M^2 and M^3.
Exercise 3. For $(W_t;\, t \geq 0)$ be a $\mathcal{F}_t$-Brownian motion, show that the process

$$M_t = \int_0^t \exp\{\frac{|W_s|^2}{4}\}\, dW_s,\ \ 0 \leq t \leq T,$$

is a continuous square integrable martingale up to any arbitrary horizon time $T < 1$.

PROBLEM 2: Existence and uniqueness of a strong solution to a SDE (2 pts). In this exercise, we implicitly assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space under which are defined a standard Brownian motion $(W_t;\, t \geq 0)$ and an independent $\mathbb{R}$-valued random variable ξ. $(\mathcal{F}_t;\, t \geq 0)$ is the augmented filtration generated by $(\xi, (W_t;\, t \geq 0))$. Assuming that

$$\mathbb{E}[\xi^2] < \infty,$$

Justify that the following SDE:

$$\begin{cases} dX_t = \dfrac{X_t}{1 + |X_t|^2}\, dt + 2(1 + \dfrac{1}{1 + |X_t|^2}) dW_t, \\ X_0 = \xi, \end{cases} \tag{1}$$

admits a strong solution and justify that this solution is unique.