Postgraduate Workshop in Stochastics

Lecture 28

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by

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28. Martingales

In this section we collect results that are of general nature about martingales.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be filtered probability space.

We say that $\{M_t\}$ is a $\{\mathcal{F}_t\}$ **martingale** if the following three conditions hold:

a) For each t, M_t is $\mathcal{F}_t$ measurable;

b) For each t, $\mathbf{E}[|M_t|] < \infty$;

c) For each $s \geq t$, $\mathbf{E}[M_s|\mathcal{F}_t] = M_t$.

We say that $\{M_t\}$ is a $\{\mathcal{F}_t\}$ **sub-martingale** if instead of (c) we have for $s \geq t$, $\mathbf{E}[M_s|\mathcal{F}_t] \geq M_t$, and $\{M_t\}$ is a $\{\mathcal{F}_t\}$ **super-martingale** if instead of (c) we have for $s \geq t$, $\mathbf{E}[M_s|\mathcal{F}_t] \leq M_t$.

Usually we have two situations:

(i) $t \in [0, \infty)$

(ii) $t \in \{0, 1, 2, 3, \ldots\}$

In (ii) we can replace (c) by $\mathrm{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ for all $n \geq 1$.

For if $m > n$:

$$\begin{aligned}\mathbf{E}[M_m|\mathcal{F}_n] &= \mathbf{E}[\mathbf{E}[M_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] \\ &= \mathbf{E}[M_{m-1}|\mathcal{F}_n] \\ &= \ldots \\ &= M_n\end{aligned}$$

Example 1 :

Let $\{X_t\}$ be a Lévy process and define for $\theta \in \mathbb{R}$

$$M_t^\theta = \frac{e^{i\theta X_t}}{\mathbf{E}[e^{i\theta X_t}]} \equiv \frac{e^{i\theta X_t}}{f_t(\theta)}$$

We note that $f_t(\theta)$ is always defined and $|f_t(\theta)| \le 1$. Furthermore, $f_{s+t}(\theta) = f_s(\theta)f_t(\theta)$ for all $s, t \ge 0$. This follows because

$$\begin{aligned} f_{s+t}(\theta) &= \mathbf{E}[e^{i\theta X_{s+t}}] \\ &= \mathbf{E}[e^{i\theta(X_{s+t}-X_t+X_t)}] \\ &= \mathbf{E}[e^{i\theta(X_{s+t}-X_t)}e^{i\theta X_t}] \\ &= \mathbf{E}[e^{i\theta(X_{s+t}-X_t)}]\,\mathbf{E}[e^{i\theta X_t}] \\ &= \mathbf{E}[e^{i\theta X_s}]\,\mathbf{E}[e^{i\theta X_t}] \\ &= f_s(\theta)f_t(\theta) \end{aligned}$$

where we have used $X_{s+t} - X_t$ and X_t are independent and $X_{s+t} - X_t \stackrel{d}{=} X_s$. As Lévy processes are stochastically continuous, $t \to f_t(\theta)$ is continuous. Now using the arguments of Lecture 24, Lemma B and setting $f_1(\theta) = e^{-\psi(\theta)}$ we can show that

$$f_t(\theta) = e^{-t\psi(\theta)}$$

for all $t \ge 0$. The exact expression for $\psi(\theta)$ here is given by the so-called Lévy-Khinchine formula. We now show that $\{M_t^\theta\}$ is a $\{\mathcal{F}_t\}$ martingale, where $\mathcal{F}_t = \sigma\{X_u \,|\, 0 \le u \le t\}$. This is because (a) of the definition holds,

$$\mathbf{E}\left[|M_t^\theta|\right] = |e^{t\psi(\theta)}| < \infty$$

for all t and if $s > t$,

$$\begin{aligned} \mathbf{E}\left[M_s^\theta \,|\mathcal{F}_t\right] &= \mathbf{E}\left[\frac{e^{i\theta X_s}}{f_s(\theta)}\,\Big|\mathcal{F}_t\right] \\ &= \frac{e^{i\theta X_t}}{f_t(\theta)}\,\mathbf{E}\left[\frac{e^{i\theta(X_s-X_t)}}{f_{s-t}(\theta)}\,\Big|\mathcal{F}_t\right] \\ &= \frac{e^{i\theta X_t}}{f_t(\theta)}\,\mathbf{E}\left[\frac{e^{i\theta(X_s-X_t)}}{f_{s-t}(\theta)}\right] \\ &= \frac{e^{i\theta X_t}}{f_t(\theta)}\,\mathbf{E}\left[\frac{e^{i\theta X_{s-t}}}{f_{s-t}(\theta)}\right] \\ &= M_t^\theta \end{aligned}$$

and so (c) holds. Thus $\{M_t^\theta\}$ is an $\{\mathcal{F}_t\}$ martingale.

Example 2 :

Let $\{X_t\}$ be a Lévy process and suppose for some $\theta \in \mathbb{R}$ that

$$g_t(\theta) = \mathbf{E}[e^{\theta X_t}]$$

is finite for all $t \geq 0$, then

$$M_t^\theta = \frac{e^{\theta X_t}}{g_t(\theta)}$$

defines a non-negative $\{\mathcal{F}_t\}$ martingale. Again, as in Example 1,

$$g_{s+t}(\theta) = g_s(\theta) g_t(\theta)$$

and

$$g_t(\theta) = e^{t\eta(\theta)}$$

where $g_1(\theta) = e^{\eta(\theta)}$. Condition (b) holds with $\mathbf{E}[|M_t^\theta|] = 1$ for all t.

This martingale is often used for subordinator Lévy processes where $\theta \leq 0$ is used.

Example 1A :

If X is a Brownian motion $B = \{B_t\}$ in Example 1, $f_t(\theta) = e^{-\frac{1}{2}\theta^2 t}$ and so

$$M_t^\theta = e^{i\theta B_t + \frac{1}{2}\theta^2 t}$$

If X is Poisson process $N = \{N_t\}$ in Example 1,

$$f_t(\theta) = e^{\lambda t \left(e^{i\theta} - 1\right)}$$

and so

$$M_t^\theta = e^{i\theta N_t - \lambda t \left(e^{i\theta} - 1\right)}$$

Example 2A :

If X is a Brownian motion $B = \{B_t\}$ in Example 2, $g_t(\theta) = e^{\frac{1}{2}\theta^2 t}$ and so

$$M_t^\theta = e^{\theta B_t - \frac{1}{2}\theta^2 t}$$

If X is Poisson process $N = \{N_t\}$ in Example 2,

$$g_t(\theta) = e^{\lambda t\left(e^\theta - 1\right)}$$

and so

$$M_t^\theta = e^{\theta N_t - \lambda t\left(e^\theta - 1\right)}$$

Both these martingales are defined for any $\theta \in \mathbb{R}$.

Stopping times

We say that $\mathrm{T} : \Omega \to [0, \infty)$ is an $\{\mathcal{F}_t\}$ stopping time if and only if

$$\{\omega \in \Omega | \mathrm{T}(\omega) \leq t\} \in \mathcal{F}_t\,.$$

Similarly, $\mathrm{T} : \Omega \to \{0, 1, 2, \ldots\}$ is an $\{\mathcal{F}_n\}$ stopping time if and only if

$$\{\omega \in \Omega | \mathrm{T}(\omega) \leq n\} \in \mathcal{F}_n\,.$$

In the second case, we note that $\mathrm{T} : \Omega \to \{0, 1, 2, \ldots\}$ is an $\{\mathcal{F}_n\}$ stopping time if and only if

$$\{\omega \in \Omega | \mathrm{T}(\omega) = n\} \in \mathcal{F}_n\,.$$

To show this equivalence we use the following observations:

$$\{\mathrm{T} \leq n\} = \bigcup_{k=0}^{n} \{\mathrm{T} = k\}$$

and

$$\{\mathrm{T} = n\} = \{\mathrm{T} \leq n\} \setminus \{\mathrm{T} \leq n - 1\}$$

Remark

If we allow $\mathrm{T}(\omega) = \infty$, then some authors (like Shiryaev) require that $P(\mathrm{T} < \infty) = 1$. Otherwise T is called a Markov-time. However, this does not seem to be a universal convention. The definition in each text should be looked at carefully.

Lemma A

Let T be an $\{\mathcal{F}_t\}$ stopping time. Then for each t,

$$\{\mathrm{T} < t\} \in \mathcal{F}_t$$

Proof

$$\{\mathrm{T} < t\} = \bigcup_{n=1}^{\infty} \left\{ \mathrm{T} \leq t - \frac{1}{n} \right\} \in \mathcal{F}_t$$

Remark:

Converse is not true. However it will be true when

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$$

for then

$$\{\mathrm{T} \leq t\} = \bigcap_{n=1}^{\infty} \left\{ \mathrm{T} < t + \frac{1}{n} \right\} \in \mathcal{F}_{t+}$$

implies $\{\mathrm{T} \leq t\} \in \mathcal{F}_t$.

Many books assume $\mathcal{F}_{t+} = \mathcal{F}_t$ and this is probably one of the reasons why.

Theorem 1 :

Let $\{M_n\}$ be a $\{\mathcal{F}_n\}$ martingale, and let T be a bounded stopping time. Then

$$\mathbf{E}[M_{\mathrm{T}}] = \mathbf{E}[M_0]$$

Remark:

To say that T is a bounded stopping time means that for some integer K, $\mathrm{T}(\omega) \leq K$ for (almost) all $\omega \in \Omega$.

Proof : (Bass)

Let us first note that

$$\begin{aligned}\mathbf{E}[M_n] &= \mathbf{E}[\mathbf{E}[M_n|\mathcal{F}_{n-1}]] \\ &= \mathbf{E}[M_{n-1}] \\ &= \dots \\ &= \mathbf{E}[M_0]\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}[M_\mathrm{T}] &= \mathbf{E}\left[M_\mathrm{T}\sum_{j=0}^{K}\mathrm{I}[\mathrm{T}=j]\right] \\ &= \sum_{j=0}^{K}\mathbf{E}[M_\mathrm{T}\,\mathrm{I}[\mathrm{T}=j]] \\ &= \sum_{j=0}^{K}\mathbf{E}[M_j\,\mathrm{I}[\mathrm{T}=j]] \qquad (*) \\ &= \sum_{j=0}^{K}\mathbf{E}[M_K\,\mathrm{I}[\mathrm{T}=j]] \\ &= \mathbf{E}\left[M_K\sum_{j=0}^{K}\mathrm{I}[\mathrm{T}=j]\right] \\ &= \mathbf{E}[M_K] \\ &= \mathbf{E}[M_0]\end{aligned}$$

At $(*)$ we used the fact that $\mathbf{E}[M_K|\mathcal{F}_j] = M_j$ holds if and only if $\mathbf{E}[M_K\mathrm{I}(\mathrm{A})] = \mathbf{E}[M_j\mathrm{I}(\mathrm{A})]$ for every $\mathrm{A} \in \mathcal{F}_j$. We choose $\mathrm{A} = \{\mathrm{T} = j\} \in \mathcal{F}_j$ at this step.

Theorem 2 :

Let $\{M_n\}$ be a $\{\mathcal{F}_n\}$ sub-martingale and T is a stopping time bounded by K. Then

$$\mathbf{E}[M_\mathrm{T}] \leq \mathbf{E}[M_\mathrm{K}]$$

Proof : (Bass)

The proof is the same as for Theorem 1, but as

$$\mathbf{E}[M_K|\mathcal{F}_j] \geq M_j$$

for $j \leq K$, we use

$$\mathbf{E}[M_j\,\mathrm{I}[\mathrm{T}=j]] \leq \mathbf{E}[M_K\,\mathrm{I}[\mathrm{T}=j]]$$

and proceed as above.

Theorem 3 :

Let $\{M_t\}$ be a $\{\mathcal{F}_t\}$ martingale. Assume that $\{M_t\}$ is right continuous. Let T be a stopping time bounded above by $\mathrm{K} \geq 0$. Then

$$\mathbf{E}[M_{\mathrm{T}}] = \mathbf{E}[M_{\mathrm{K}}] = \mathbf{E}[M_0]$$

Remark:

We say the process $\{Y_t\}$ is a **modification** of the process $\{X_t\}$ if for all t, $P(Y_t = X_t) = 1$. We say that $\{Y_t\}$ and $\{X_t\}$ are **indistinguishable** if for almost all $\omega \in \Omega$, $Y_t(\omega) = X_t(\omega)$ for all t. If $\{Y_t\}$ is a modification of the process $\{X_t\}$ and both processes are right continuous, then $\{Y_t\}$ and $\{X_t\}$ are indistinguishable (see Elliott (1982), page 13). It is known that any martingale process has a càdlàg modification, and it is always possible to assume that our martingales are right continuous. This fact is a consequence of Doob's up-crossing inequalities, which we may study at another time. We say all this to indicate that the assumption of the theorem is not very severe.

Proof of Theorem 3 : (Bass)

Define $\{\mathrm{T}_n\}$ by

$$\mathrm{T}_n(\omega) = \begin{cases} \frac{j+1}{2^n}\mathrm{K} & \text{if } \frac{j\mathrm{K}}{2^n} \leq \mathrm{T}(\omega) < \frac{j+1}{2^n}\mathrm{K} \quad \text{where} \quad j = 0, 1, \ldots 2^n - 1 \\ \mathrm{K} & \text{otherwise} \end{cases}$$

Then T_n is stopping time with respect to $\left\{\mathcal{F}_{\frac{j\mathrm{K}}{2^n}},\ j = 0, 1, \ldots 2^n\right\}$

This follows as

$$\left\{\mathrm{T}_n = \frac{j+1}{2^n}\mathrm{K}\right\} = \left\{\mathrm{T} < \frac{j+1}{2^n}\mathrm{K}\right\} \setminus \left\{\mathrm{T} < \frac{j}{2^n}\mathrm{K}\right\} \in \mathcal{F}_{\frac{j+1}{2^n}\mathrm{K}}$$

for $j = 0, \ldots, 2^n - 1$ and

$$\{\mathrm{T}_n = \mathrm{K}\} = \left\{\mathrm{T} \geq \frac{2^n - 1}{2^n}\mathrm{K}\right\} = \Omega \setminus \left\{\mathrm{T} < \frac{2^n - 1}{2^n}\mathrm{K}\right\} \in \mathcal{F}_{\mathrm{K}}$$

We have used Lemma A in these arguments.

Further, $\left\{M_{\frac{j}{2^n}K}\right\}$ is a $\left\{\mathcal{F}_{\frac{j\mathrm{K}}{2^n}}\right\}$ martingale.

By Theorem 1,

$$\mathbf{E}[M_{\mathrm{T}_n}] = \mathbf{E}[M_{\mathrm{K}}] = \mathbf{E}[M_0]\,.$$

We now let $n \to \infty$. We note that $\mathrm{T}_n(\omega) \downarrow \mathrm{T}(\omega)$ for all $\omega \in \Omega$ as $n \to \infty$. Using the right continuity of $\{M_t\}$,

$$M_{\mathrm{T}_n(\omega)}(\omega) \to M_{\mathrm{T}(\omega)}(\omega) = (M_T)(\omega)$$

We now show that

$$\lim_{n\to\infty} \mathrm{E}[M_{\mathrm{T}_n}] = \mathrm{E}[M_{\mathrm{T}}]$$

by establishing that the collection $\{M_{\mathrm{T}_n}\}$ are uniformly integrable, and using the limit Theorem 2A on page 385 of Lecture 27.

In fact, the singleton set $\{M_{\mathrm{K}}\}$ is uniformly integrable. Then by the Theorem of de la Valée Poussin (page 369, Lecture 26), there exists a function G defined on $[0, \infty)$ which is non-decreasing, convex, $G(0) = 0$, $\frac{\mathrm{G}(x)}{x} \to \infty$ as $x \to \infty$ and

$$\mathbf{E}[\mathrm{G}(|M_{\mathrm{K}}|)] < \infty\,.$$

Now $x \to \mathrm{G}(|x|)$ is convex, for if $x, y \in [0, \infty)$ and $0 \leq \lambda \leq 1$ set $z = \lambda x + (1-\lambda)y$. Then

$$|z| \leq \lambda|x| + (1-\lambda)|y|$$

implies, as G is non decreasing

$$\begin{aligned} \mathrm{G}(|z|) &\leq \mathrm{G}(\lambda|x| + (1-\lambda)|y|) \\ &\leq \lambda \mathrm{G}(|x|) + (1-\lambda)\mathrm{G}(|y|) \end{aligned}$$

Then $\{\mathrm{G}(|M_t|)\}$ is sub-martingale. For when $s \leq t$, we use Jensen's inequality to obtain

$$\mathbf{E}[\mathrm{G}(|M_t|) \,|\, \mathcal{F}_s] \geq \mathrm{G}(|\mathbf{E}[M_t \,|\, \mathcal{F}_s]|) = \mathrm{G}(|M_s|) \tag{$*$}$$

To verify the other conditions for a sub-martingale, we need to show that $\mathbf{E}[\mathrm{G}(|M_t|) < \infty$ for all t. In order to apply Theorem 2, we only need the integrability to hold for $t \leq K$ and this follows from $(*)$. By Theorem 2,

$$\mathbf{E}[\mathrm{G}(|M_{\mathrm{K}}|)] \geq \mathbf{E}[\mathrm{G}(|M_{\mathrm{T}_n}|)]$$

for each n. This implies

$$\sup_n \mathbf{E}[\mathrm{G}(|M_{\mathrm{T}_n}|)] \leq \mathbf{E}[\mathrm{G}(|M_{\mathrm{K}}|)] < \infty$$

Then by the Theorem of de la Valée Poussin, $\{M_{\mathrm{T}_n}\}$ are uniformly integrable, and therefore,

$$\mathbf{E}[M_{\mathrm{T}}] = \mathbf{E}[M_{\mathrm{K}}] = \mathbf{E}[M_0]\,.$$

Theorem 4: (Doob's optional stopping theorem)

Let $S \leq \mathrm{T}$ are two stopping times bounded by K and let $\{M_t\}$ be a right continuous $\{\mathcal{F}_t\}$ martingale. Then

$$\mathbf{E}[M_T \,|\, \mathcal{F}_S] = M_S$$

Remark :

Recall that $\mathrm{A} \in \mathcal{F}_T$ means $\mathrm{A} \in \mathcal{F}$ and $\mathrm{A} \cap (T \leq t) \in \mathcal{F}_t$ for all $t \geq 0$.

Proof :

Let $\mathrm{A} \in \mathcal{F}_S$ be arbitrary. We need to show that

$$\mathbf{E}[M_{\mathrm{T}}\,\mathrm{I}(\mathrm{A})] = \mathbf{E}[M_S\,\mathrm{I}(\mathrm{A})]\,.$$

Actually since $\{|M_t|\}$ is an $\{\mathcal{F}_t\}$ sub-martingale, we can use Theorem 2 to show $\mathbf{E}[|M_{\mathrm{T}}|]$ and $\mathbf{E}[|M_{\mathrm{S}}|]$ are bounded above by $\mathbf{E}[|M_{\mathrm{K}}|] < \infty$.

Define

$$\mathrm{U}(\omega) = \begin{cases} S(\omega) & \omega \in \mathrm{A} \\ \mathrm{T}(\omega) & \omega \notin \mathrm{A} \end{cases}$$

Then U is a stopping time bounded by K, for

$$\begin{aligned} \{\mathrm{U} \leq t\} &= [\{\mathrm{U} \leq t\} \cap \mathrm{A}] \cup [\{\mathrm{U} \leq t\} \cap \mathrm{A}^c] \\ &= [\{S \leq t\} \cap \mathrm{A}] \cup [\{T \leq t\} \cap \mathrm{A}^c] \\ &\in \mathcal{F}_t \end{aligned}$$

We used $A \in \mathcal{F}_T$ (and hence $A^c \in \mathcal{F}_T$). This is true because

$$\begin{aligned} \mathrm{A} \cap \{T \leq t\} &= (\mathrm{A} \cap \{T \leq t\} \cap \{S \leq t\}) \cup (\mathrm{A} \cap \{T \leq t\} \cap \{S > t\}) \\ &= (\mathrm{A} \cap \{S \leq t\}) \cap \{T \leq t\} \in \mathcal{F}_t \end{aligned}$$

By Theorem 3,

$$\mathbf{E}[M_{\mathrm{U}}] = \mathbf{E}[M_{\mathrm{K}}] = \mathbf{E}[M_{\mathrm{T}}]$$

$$\mathbf{E}[M_{\mathrm{U}}\mathrm{I}(\mathrm{A})] + \mathbf{E}[M_{\mathrm{U}}\mathrm{I}(\mathrm{A}^c)] = \mathbf{E}[M_{\mathrm{T}}\mathrm{I}(\mathrm{A})] + \mathbf{E}[M_{\mathrm{T}}\mathrm{I}(\mathrm{A}^c)]$$

The left hand side is

$$\mathbf{E}[M_S\mathrm{I}(\mathrm{A})] + \mathrm{E}[M_{\mathrm{T}}\mathrm{I}(\mathrm{A}^c)]$$

and the right hand side is

$$\mathbf{E}[M_{\mathrm{T}}\mathrm{I}(\mathrm{A})] + \mathrm{E}[M_{\mathrm{T}}\mathrm{I}(\mathrm{A}^c)]$$

We conclude that

$$\mathbf{E}[M_s\mathrm{I}(\mathrm{A})] = \mathbf{E}[M_{\mathrm{T}}\mathrm{I}(\mathrm{A})]$$

and the theorem is proved.

Remark:

Theorem 4 can also be proved without assuming that the stopping times are bounded. See Elliott (1982), page 36 (Theorem 4.12). In many cases when we have an unbounded stopping time T, we approximate it with bounded ones: $\mathrm{T}_n = \mathrm{T} \wedge n = \min(\mathrm{T}, n)$. We then prove a result with T_n and then justify letting $n \to \infty$.

Theorem 5: (Doob)

Let $\{M_n\}$ be an $\{\mathcal{F}_n\}$ martingale. Define

$$M_n^* = \max_{m \leq n} |M_n|$$

then

$$P(M_n^* \geq a) \leq \frac{1}{a}\,\mathbf{E}[|M_n|\,]$$

Proof:

Let $\mathrm{T} = \inf\{j : |M_j| \geq a\}$. This means that

$$\mathrm{T}(\omega) = \inf\{j : |M_j(\omega)| \geq a\}$$

This infimum is put to ∞ if $|M_j(\omega)| < a$ for all j.

T is a $\{\mathcal{F}_n\}$ stopping time since

$$\{\mathrm{T} = n\} = \bigcup_{j=1}^{n} \{|M_j| \geq a\} \in \mathcal{F}_n$$

Furthermore, $\{|M_n|\}$ is sub-martingale, since $x \to |x|$ is a convex function.

$$\begin{aligned} P(M_n^* \geq a) &= P(\mathrm{T} \leq n) \\ &\leq \mathbf{E}\left[\frac{|M_\mathrm{T}|}{a}\,\mathrm{I}\,[\mathrm{T} \leq n]\right] \quad \text{as } |M_\mathrm{T}| \geq a \text{ when } T < \infty \\ &= \mathbf{E}\left[\frac{|M_{\mathrm{T}\wedge n}|}{a}\,\mathrm{I}\,[\mathrm{T} \leq n]\right] \\ &\leq \frac{1}{a}\mathrm{E}[\,|M_{\mathrm{T}\wedge n}|\,] \leq \frac{1}{a}\mathrm{E}[\,|M_n|] \end{aligned}$$

since $\mathrm{T} \wedge n$ is a stopping time bounded by n as

$$\{\mathrm{T} \wedge n \leq t\} = \{\mathrm{T} \leq t\} \cup \{n \leq t\} \in \mathcal{F}_t$$

to which we apply Theorem 2.

Theorem 6:

Let $\{M_t\}$ be a $\{\mathcal{F}_t\}$ martingale and assume that $\{M_t\}$ is right continuous. Define

$$M_t^* = \sup_{s \leq t} |M_s|$$

Then for $a \geq 0$

$$P(M_t^* \geq a) \leq \frac{1}{a}\mathbf{E}\left[|M_t|\right]$$

Proof :

Define the stopping time

$$\mathrm{T} = \inf\{t \geq 0; |M_t| \geq a\}$$

and proceed as in Theorem 5.

Theorem 7:

Let $\{X_n\}$ be real valued process and adapted to $\{\mathcal{F}_n\}$ and let T be an $\{\mathcal{F}_n\}$ stopping time. Then X_{T} is $\mathcal{F}_{\mathrm{T}}$ measurable.

Proof:

Let $a \in \mathbb{R}$, then

$$\{X_{\mathrm{T}} \leq a\} \cap \{\mathrm{T} = n\} = \{X_n \leq a\} \cap \{\mathrm{T} = n\} \in \mathcal{F}_n$$

for each $n = 0, 1, 2, ...$ and so $\{X_{\mathrm{T}} \leq a\} \in \mathcal{F}_{\mathrm{T}}$. The result follows.

Remark:

This result often suffices, but for continuous time processes we need to impose extra conditions on X. Without going into too many details here, if $\{X_t\}$ with respect to $\{\mathcal{F}_t\}$ is progressively measurable (or just progressive), then X_{T} is $\mathcal{F}_{\mathrm{T}}$ measurable.

The process $\{X_t\}$ with respect to $\{\mathcal{F}_t\}$ is progressively measurable if for each $t \geq 0$, that the map

$$(s, \omega) \to X_s(\omega)$$

defined on $[0, t] \times \Omega$ into $\mathbb{R}$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ measurable. Here $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ is the smallest σ-algebra generated by sets of the form $A \times B$ where A is a Borel subset of $[0, t]$ and $B \in \mathcal{F}_t$. Thus $\{X_t\}$ with respect to $\{\mathcal{F}_t\}$ is progressive if for all reals a and for any $t \geq 0$,

$$\{(s, \omega) \in [0, t] \times \Omega \,|\, X_s(\omega) \leq a\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t$$

However if $\{X_t\}$ is adapted to $\{\mathcal{F}_t\}$, and is right continuous, then $\{X_t\}$ with respect to $\{\mathcal{F}_t\}$ is progressive, and so X_{T} is $\mathcal{F}_{\mathrm{T}}$ measurable.

These results are discussed correctly in Elliott (1982), pages 14-15. Some authors assume right continuous filtrations for this result, but this is not required.

We will assume these results for the time being and return to a careful study of these results as necessary.

This remark also indicates why some processes are assumed to be progressive. As we can assume martingales are càdlàg, we can assume that martingales with a continuous time parameter are progressive.

This lecture made use of some good presentations in R. F. Bass, “Probabilistic Techniques in Analysis” Springer 1995.