Introduction to Stochastic Differential Equations: Quiz II

Pr. Jean-François Jabir

Fall 2017

First name: .... **Last name:** ....

Course question.
• Formulate the complete definition of a weak solution to a one-dimensional Stochastic Differential Equation of the form:

$$(*)\quad \begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)dW_t,\ 0 \leq t \leq T \\ X_0 = \xi \sim \mu_0, \end{cases}$$

for T a finite time horizon, μ_0 a given probability measure on $\mathbb{R}$ such that $\int |x|\mu_0(dx) < \infty$, $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ two given Borel functions. **Bonus**: Formulate the differences between a weak solution and a strong solution to $(*)$.

Exercise 1. Let σ, b be two given real numbers and let $(W_t;\, t \geq 0)$ be a standard Brownian motion. Define the process:

$$X_t = bt + \sigma W_t,\ t \geq 0.$$

Apply the Itô formula to the following cases:

$$(a)\ \ (X_t)^3;\ \ (b)\ \ \sin(X_t);\ \ (c)\ \ \frac{1}{1+(X_t)^2}.$$

Exercise 2. Consider the SDE

$$(**)\ dX_t = -X_t\,dt + dW_t,\ X_0 = 1.$$

1. Justify that $(**)$ admits a unique strong solution $(X_t;\, t \geq 0)$.
2. Show that $Y_t = e^t X_t$ is a square integrable martingale given by
$$Y_t = 1 + \int_0^t e^s\,dW_s.$$
3. Deduce from the preceding question the expectation and variance of $(X_t;\, t \geq 0)$ and show that $\lim_{t\to\infty} \mathbb{E}[X_t] = 0$.