Lecture 1

A Transition to Higher Mathematics (MAT001 at Bayt-al-Hikmah)
Introduction to Set Theory

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1 Sets

Motivation
Life can be very chaotic sometimes. This is not exactly the most ideal circumstances under which a person can make the most of time, which is why organisation skills are necessary to make the most out of every moment. This emphasis on clarity, order and logic is even more profound in mathematics; thus, we ponder, is there a way to organise *mathematics itself*?

1.1. The Concept of Sets

When Georg Cantor (1845 - 1918) was doing research on Fourier series (which we will eventually cover in this course), the chaos of points at which trigonometric series failed to converge led him to seek a way of order. He began grouping these "exceptional" points into collections, trying to understand their structure; ultimately driving him to the concept of a **set** – an organised collection of distinct objects.

Now, we will proceed to formally introduce the notion of a mathematical set.

Definition 1.1.1. (Definition of a set).
A set S is a well-defined collection of objects; these are called elements of S. If an element x is in a set S, denote $x \in S$; else denote $x \notin S$. For any object y, it is **determined** whether $y \in S$ or $y \notin S$; there is no third state. Every element is **unique** in S.

Sets are usually denoted by capital letters (i.e., A, B, S), and their elements are listed inside curly brackets:

$$S = \{a, b, c, \dots\}$$

Where $a, b, c, \dots$ are distinct elements.

Example 1.1.2. The set $A = \{1, 3, 5\}$ is the set containing the numbers $1, 3,$ and 5. **Example 1.1.3.** The set containing the three biggest cities in the Gaza Strip can be denoted as $C = \{\text{Gaza City}, \text{Khan Yunis}, \text{Jabalia}\}$.
Example 1.1.4. The set containing all the positive numbers can be denoted as $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

Exercises.

Exercise 1.1.5. Which of the following are sets? Explain why/why not.

- The collection of all the red cars in the world.
- The collection of all beautiful paintings in the world.
- The set of all integers greater than 1.

Exercise 1.1.6. Write down the set containing the months of the year.
Exercise 1.1.7. What is incorrect about the 'set' $A = \{1, 1, 2, 2, 3, 3\}$?
Exercise 1.1.8. Determine whether the following statements are true or false.

- Australia $\in \{x \mid x \text{ is a country}\}$
- $\{1, 2\} \in \{1, 2, 3\}$
- $3, 5 \in \{1, 3, 5, \dots\}$

1.2. The Axioms of ZFC.

Early since its inception, there has been some inconsistencies associated with a naïve use of a set; therefore, an axiomatic approach was necessary to allow for a formalised treatment of sets mathematically. There has been a number of axiomisations of set theory; however, in this course we only introduce the standard axiomatic system for theory as it forms the widely accepted foundation for most of modern mathematics, known as the Zermelo-Fraenkel axioms plus the Axiom of Choice, abbreviated as ZFC set theory. The interested student may consult other texts for alternative axiomatic approaches, such as von Neumann-Bernays-Gödel (NBG) or Kripke-Platek (KP) set theory.

For the betterment of the student's understanding whilst adhering to the clear and formal logic of mathematics, we will proceed to state the formal form of the axiom before presenting a more informal one.

Axiom 1.2.1. (Extensionality).

$$\forall A\ \forall B\,((\forall x(x \in A \iff x \in B)) \Rightarrow A = B)$$

Informal. Two sets are equal if and only if they are comprised of exactly the same elements. If two sets A and B are equal, we denote $A = B$.

Axiom 1.2.2. (Empty set).

$$\exists S\ \forall x(x \notin S)$$

Informal. There exists a set with no elements; we call it the **empty set** and denote it by $\varnothing$.

Axiom 1.2.3. (Pairing).

$$\forall a\ \forall b\ \exists x\ \forall y\,(y \in x \iff (y = a\ \vee\ y = b))$$

Informal. For any two sets a and b, then there exists a set $S = \{a, b\}$.

Axiom 1.2.4. (Union).

$$\forall A\ \exists U\ \forall x\,(x \in U \iff \exists B(x \in B\ \wedge B\ \in A))$$

Informal. There exists a set U that contains all elements of the elements of A, which are themselves sets. From henceforth we will denote U as $\bigcup A$.

Axiom 1.2.5. (Power set).

$$\forall A\ \exists S\ \forall x\,(x \in S \iff \forall y(y \in S \Rightarrow y \in x))$$

Informal. For any set A, there exists a set S such that all its elements contain only elements of A. We call it the **power set** of A and denote it as $\mathcal{P}(A)$.

Remark. We can now define the notion x is a **subset** of y, denoted as $x \subseteq y$ as $\forall z(z \in x \Rightarrow z \in y)$. Thus Axiom 1.2.5. can be simplified as

$$\forall A\ \exists S\ \forall x\,(x \in S \iff x \subseteq A)$$

Axiom 1.2.6. (Infinity).

$$\exists I\ \left(\varnothing \in I\ \wedge\ \forall x(x \in I \Rightarrow \bigcup\{x, \{x\}\} \in I\right)$$

Informal. There exists a set containing infintely many elements.

Axiom 1.2.7. (Separation Schema).

$$\forall A\ \exists B\ \forall x\,(x \in B \iff (x \in A\ \wedge\ \varphi(x))$$

Informal. For every property given by the formula $\varphi(x)$ with parameters and for every set A, there exists a subset of A containing exactly the elements satisfying φ.

Remark. We can now describe sets more conveniently using the separation schema. Let A be an existing set, and a property be given by a formula with parameters $\lambda(x)$. Therefore, we denote the set of every element x of A that satisfies $\lambda(x)$ to be:

$$S = \{x \in A \mid \lambda(x)\}$$

Axiom 1.2.8. (Replacement Schema).

$$\forall A\,(\forall x \in A\ \exists! y \phi(x, y) \Rightarrow \exists B\ \forall y\,(y \in B \iff \exists x \in A \phi(x, y)))$$

Informal. For every **definable function** $\phi(x, y)$ that maps each x to a unique y, and with domain A, the image set exists.

Axiom 1.2.9. (Foundation/Regularity).

$$\forall A\,(A \neq \varnothing \Rightarrow \exists x \in A\ \forall y \in x(y \notin A))$$

Informal. For every non-empty set A, there is an element $x \in A$ that is disjoint from A.

Axiom 1.2.10. (Choice).

$$\forall A \Big((\forall x \in A\,(x \neq \varnothing)) \Rightarrow \exists f : A \to \bigcup A\ \forall x \in A(f(x) \in x)\Big)$$

Informal. If every element in a set A is non-empty, then there exists a choice function defined on A such that all $f(x) \in x$ for all $x \in A$.

If the student is unfamiliar with the conventional mathematical logical operators ($\forall, \exists, \iff, \dots$), please review the prerequisite material on basic mathematical logic.

These axioms may seem redundant; however, as we move through the course later on, we will soon see that these are very useful in maintaining the order and logic of mathematics. As an immediate example to satisfy one's curiosity, consider this set proposed by the mathematician Bertrand Russell in 1901:

Example 1.2.11. (Russell's Paradox). Consider the set $R = \{x \mid x \notin x\}$. If $R \in R$, then it must follow that $R \notin R$, which is absurd. But if $R \notin R$ that means $R \in R$ by definition; this is a clear logical contradiction.

How do we go about resolving this paradox? This is where the axioms of ZFC are useful. We see that this problem arises due to *unrestricted comprehension*, i.e. the ability to form a set based on any given property. Instead, ZFC requires *restricted comprehension* by Axiom 1.2.7, since one can only define a subset of an existing set using a property. Therefore, the set $R = \{x \mid x \notin x\}$ is not a valid set; rather, the set $S = \{x \in A \mid x \notin x\}$ for some already existing set A would be a valid version. This is not problematic as if $S \in A$ then we have a contradiction (why?); but if $S \notin A$ everything is valid. Therefore, we conclude that $S \notin A$ and hence this paradox is resolved.

This is one of the many examples where ZFC is able to resolve logical paradoxes associated with unrestricted comprehension. Though this is a single example, one can already appreciate the importance of a comprehensive axiomatic approach, as ZFC offers, in providing a versitale but unambiguous logical framework for set theory.

Example 1.2.12. List all the subsets of the set $\{1, 2, 3\}$.
Solution.
The sets $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}$, and $\{1, 2, 3\}$ are the desired subsets. □

Example 1.2.13. For any three unique sets a, b and c, prove the existence of the set $\{a, b, c\}$ using Axiom 1.2.3 and Axiom 1.2.4.
Proof.
By Axiom 1.2.3 the set $x = \{a, b\}$ and the set $y = \{b, c\}$ exists; furthermore, the set $Z = \{x, y\}$ also exists by the Axiom. Now, by Axiom 1.2.4 one also knows that the set $\bigcup Z = \{a, b, c\}$ exists; we are done. □

Example 1.2.14. (Disproof of the existence of an absolute universal set). Prove that the set U such that

$$\forall x(x \in U)$$

does not exist.
Proof.
Assume for the sake of contradiction that such a set U does exist. Therefore, we use Axiom 1.2.7 to define the Russell set $R = \{x \in U \mid x \notin x\}$. Thus, by definition $R \in U$. However, to prevent a contradiction occurring (by Example 1.2.11), $R \notin U$. Hence contradiction; no universal set U exists. $\square$

Exercises.

Exercise 1.2.15. Use Axiom 1.2.7 to define a set $B \subseteq A = \{1, 2, 3, 4, 5\}$ containing all the even elements.

Exercise 1.2.16. Use Axiom 1.2.3 and Axiom 1.2.7 to justify the existence of the set $\{1, 2, 3\}$ from $\{1, 2\}$ and $\{1, 3\}$.

Exercise 1.2.17. (Existence of singletons). Use Axiom 1.2.3 to prove the existence of the set $\{x\}$ where x is a set. (This type of set is called a **singleton**).

Exercise 1.2.18. Prove that the empty set $\varnothing$ is unique.

Problem 1.2.19. Prove that no set is a member of itself.

Exercise 1.2.20. Prove that the empty set is the subset of every set.

Problem 1.2.21. (Cantor's Theorem). Prove that there is no function f such that $f : A \to \mathcal{P}(A)$ is a surjection. (This is equivalent to $|A| < |\mathcal{P}(A)|$).

From henceforth, unless stated otherwise, we will assume that any given set exists and satisfies the axioms of ZFC.

1.3. Set Operations

Addition, multiplication, division, etc. are all operations associated with elementary arithmetic. Do similar operations exist for sets and set manipulation? It turns out there are indeed. We will now proceed to define the four most common of these operations.

Definition 1.3.1. (Union of sets). For any two sets A, B, denote $\bigcup\{A, B\}$ as $A \cup B$. Call it *the union of A and B.*

From Axiom 1.2.4, we see that $A \cup B$ is the set containing all the elements of A and B. More formally,

$$\forall x \in A \cup B(x \in A \vee x \in B)$$

Before we introduce the intersection operation, we will first introduce the intersection set.

Theorem 1.3.2. (Existence of the intersection set). Show that for any set S, then the set $\bigcap A := \{x \in C \mid \forall B \in A, x \in B\}$ for some $C \in A$ exists.
Proof.
We differentiate two separate cases: $A \neq \varnothing$ and $A = \varnothing$. For the latter, define $\bigcap A = \varnothing$. Now we will consider the former. As $A \neq \varnothing$, by Axiom 1.2.4 and 1.2.3, some $C \in A$ exists. Therefore, by Axiom 1.2.7, the set $I = \{x \in C \mid \forall B \in A, x \in B\}$ must exist and we are done. $\square$

This leads us to the intersection operation:

Definition 1.3.3. (Intersection of sets). For any two sets A, B, denote $\bigcap\{A, B\} = A \cap B$. Call it the *intersection of set A and B*

From Theorem 1.3.2 it is clear that $A \cap B$ is the set containing all the elements that are both in A and in B. Formally,

$$\forall x \in A \cap B(x \in A \wedge x \in B)$$

Definition 1.3.4. (Difference of sets). For any two sets $A, B \subseteq C$, call $A \backslash B$ *the difference of the two sets A and B* (in this order), and define it as the set $A \backslash B := \{x \in C \mid x \notin B \wedge x \in A\}$.

The proof of existence for such a set is left to the interested student as an exercise in the exercises section.

Definition 1.3.5. (Complement of a set in U). Let $A \subseteq U$. Then, define $\complement_U A := \{x \in U \mid x \notin A\}$ and call it the *complementary set of A in U*. It is also often called the *the complement of A in U*.

Exercises.

Exercise 1.3.6. Prove that for any two sets $A, B \subseteq C$ where A, B, C are all well-defined, the difference $A \backslash B$ of the sets A and B exists.

Problem 1.3.7. Prove that $U \backslash A$ is equivalent to $\complement_U A$ where $A \subseteq U$.

1.4. Properties of the Set Operations

Now we have defined our basic operations between sets, the question of their properties naturally arouses our curiosity. What properties do they satisfy? What laws govern such operations? How can we visualise them? We are going to address these questions in this section.

To begin, we first introduce a very important property of a set:

Definition 1.4.1. (The Cardinality of a Set). If a set S has an n number of elements where $n \in \mathbb{Z}^+ \cup \{0\}$ ($\mathbb{Z}^+$ is the set of all positive integers), then define the cardinality of S as n and denote $|S| = n$.

This may not appear as logical and as founded upon our ZFC axioms as we may want to be; however we will eventually redefine the notion of cardinality using more logical, structured and precise mathematical language based upon our ZFC axiomatic approach after having learned *ordinals* in our next lecture on set theory. The interested student may do some research on ordinals online.

Theorem 1.4.2. (The Properties of Set Operations). Let $A, B, C \subseteq U$. Then, the following equalities hold:

- **Commutativity:**
$$A \cup B = B \cup A,\ A \cap B = B \cap A$$
- **Associativity:**
$$(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$$
- **Distributivity:**
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\ A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$$
- **Identities:**
$$A \cap \varnothing = \varnothing,\ A \cup \varnothing = A$$
- **Domination Laws:**
$$A \cup U = U,\ A \cap U = A$$
- **Idempotent Laws:**
$$A \cup A = A,\ A \cap A = A$$
- **Complement Laws:**
$$A \cup \complement_U A = U,\ A \cap \complement_U A = \varnothing$$
- **Double Complement Law:**
$$\complement_U(\complement_U A) = A$$
- **De Morgan's Laws:**
$$\complement_U(A \cup B) = \complement_U A \cap \complement_U B,\ \complement_U(A \cap B) = \complement_U A \cup \complement_U B$$

Example 1.4.3. Prove the distributive property from Theorem 1.4.2.
Proof.
By definition,
$$x \in A \cap (B \cup C) \Rightarrow x \in A \wedge x \in (B \cup C)$$
$$\Rightarrow x \in A \wedge (x \in B \vee x \in C)$$
$$\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$$
$$\Rightarrow x \in (A \cup B) \cap (A \cup C)$$

Similarly, we can prove the converse meaning that $x \in A \cap (B \cup C) \iff x \in (A \cup B) \cap (A \cup C)$, i.e. $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$ by Axiom 1.2.1. The remainder of the proof can be easily derived using a similar approach which the interested student can finish off. $\square$

We leave the proofs of the rest of the properties as exercises for the student.

Exercises

Exercise 1.4.4. Prove the commutative property from Theorem 1.4.2.

Exercise 1.4.5. Prove the associative property from Theorem 1.4.2.

Exercise 1.4.6. Prove the identities from Theorem 1.4.2.

Exercise 1.4.7. Prove the domination law from Theorem 1.4.2.

Exercise 1.4.8. Prove the idempotent law from Theorem 1.4.2.

Exercise 1.4.9. Prove the complement laws from Theorem 1.4.2.

Exercise 1.4.10. Prove the double complement law from Theorem 1.4.2.

Problem 1.4.11. Prove De Morgan's Laws from Theorem 1.4.2.

Exercise 1.4.12. Prove that if $A \subseteq B$ then $|A| \leq |B|$ using an informal and intuitive proof.

Problem 1.4.13. Prove that if $A \subseteq B$ and $B \subseteq A$ for any two sets A, B, then $A = B$.

1.5. Conclusion.

At the end of this lecture, you should be able to:

- Understand the notion of a mathematical set and the motivation behind it
- Understand and use the ZFC axioms in solving problems and its purpose
- Understand and use set operations in solving problems

Now, we want to apply our newfound knowledge to form and develop a more nuanced understanding of mathematics with more solid foundations formed upon set theory. We conclude with two open ended questions as the student's homework:

Question 1.5.1. Suggest possible real-world applications of set theory in a paper for submission.

Question 1.5.2. Is it possible to base the natural numbers upon the foundations of set theory? Propose viable and mathematically sound approaches in a paper for submission.

1.6. Solutions to Problems.

Problem 1.2.19.
Proof.
We proceed by contradiction; assume there exists such a set a; thus $a \in \{a\}$. We immediately notice that $a \neq \varnothing$ by definition. Now, by Axiom 1.2.9, a and $\{a\}$ must be disjoint; this contradicts the fact that $a \in \{a\}$ because that means it is not disjoint (i.e. they share common elements). Contradiction; therefore no set is a member of itself.

Problem 1.2.21.
Proof.
Proceed with contradiction. Let a function $f : A \to \mathcal{P}(A)$ be a surjection. Using Axiom 1.2.7, define the set $D = \{x \in A \mid x \notin f(x)\}$. Therefore, $D \subseteq \mathcal{P}(A)$; hence it has a pre-image under f, say:

$$f(d) = D \qquad \text{For some } d \in A.$$

We identify two cases: $d \in D$ or $d \notin D$. If $d \in D$, then $d \notin f(d) = D$; contradiction. On the other hand, if $d \notin D$ then by definition $d \in f(d) = D$, which is absurd. Therefore, having derived a contradiction in both cases, we conclude that there must not exist such a function f. $\square$

Problem 1.3.7.
Proof.
By Definition 1.3.4 we see that $U \backslash A = \{x \in U \mid x \notin A \land x \in U\} = \{x \in U \mid x \notin A\}$, which is equivalent to $\complement_U A$ by Definition 1.3.5. $\square$

Problem 1.4.11.

Proof.

Let $x \in \complement_U(A \cup B)$. Then by definition,

$$x \in \complement_U(A \cup B) \Rightarrow x \in U \wedge x \notin A \cup B$$

$$\Rightarrow x \in U \wedge (x \notin A \wedge x \notin B)$$
$$\Rightarrow (x \in U \wedge x \notin A) \wedge (x \in U \wedge x \notin B)$$
$$\Rightarrow x \in \complement_U A \cap \complement_U B$$

Conversely, let $x \in \complement_U A \cap \complement_U B$. Then,

$$x \in \complement_U A \wedge x \in \complement_U B \Rightarrow (x \in U \wedge x \notin A) \wedge (x \in U \wedge x \notin B)$$

$$\Rightarrow x \in U \wedge (x \notin A \wedge x \notin B)$$
$$\Rightarrow x \in U \wedge x \notin A \cup B$$
$$\Rightarrow x \in \complement_U(A \cup B)$$

Therefore,

$$\complement_U(A \cup B) = \complement_U A \cap \complement_U B$$

by Axiom 1.2.1. Using similar logic, one can prove that $\complement_U(A \cap B) = \complement_U A \cup \complement_U B$. □

Problem 1.4.13.

Proof.

By definition, we see that $x \in A \Rightarrow x \in B$. Similarly, observe that $x \in B \Rightarrow x \in A$. Therefore, $x \in B \iff x \in A$ which by Axiom 1.2.1 means that $A = B$ as desired. □