CHAPTER

16

APPLICATIONS TO DIFFERENTIAL EQUATIONS

16.1 INTRODUCTION

In this chapter, we shall study the application of differential equations to various physical problems.

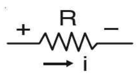
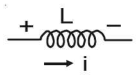
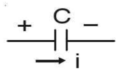
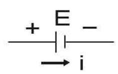
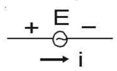
16.2 ELECTRICAL CIRCUIT

We will consider circuits made up of

(*i*) Voltage source which may be a battery or a generator.

(*ii*) Resistance, inductance and capacitance.

Table of Elements, Symbols and Units

	Element	*Symbol*		*Unit*
1.	Charge		q	coulomb
2.	Current		i	ampere
3.	Resistance,		R	ohm
4.	Inductance,		L	henry
5.	Capacitance,		C	farad
6.	Electromotive force or voltage (constant)		constant V	volt
7.	Variable voltage		variable V	volt

The formation of differential equation for an electric circuit depends upon the following laws.

(*i*) $i = \dfrac{dq}{dt}$, (*ii*) Voltage drop across resistance $R = Ri$

(*iii*) Voltage drop across inductance $L = L.\dfrac{di}{dt}$

(*iv*) Voltage drop across capacitance $C = \dfrac{q}{C}$

Kirchhoff's laws

I. Voltage law. The algebraic sum of the voltage drop around any closed circuit is equal to the resultant electromotive force in the circuit.

II. Current law. At a junction or node, current coming is equal to current going.

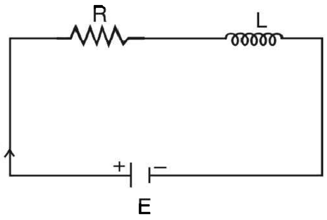
(*i*) **L - R series circuit.** Let *i* be the current flowing in the circuit containing resistance *R* and inductance L in series, with voltage source E, at any time *t*.

By voltage law $Ri + L\frac{di}{dt} = \text{E} \quad \Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ ...(1) (*M.U. II Semester, 2009*)

This is the linear differential equation

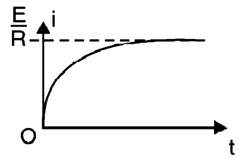
$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Its solution is $i.e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t}\, dt + A$



$\Rightarrow \qquad i.e^{\frac{R}{L}t} = \frac{E}{L} \times \frac{L}{R} e^{\frac{R}{L}t} + A$

$\Rightarrow \qquad i = \frac{E}{R} + Ae^{-\frac{Rt}{L}}$ ...(2)



At $t = 0$, $\qquad i = 0 \quad \Rightarrow \quad A = -\frac{E}{R}$

Thus, (2) becomes $i = \frac{E}{R}\left[1 - e^{-\frac{Rt}{L}}\right]$

(*ii*) **C-R series circuit.** Let *i* be current in the circuit containing resistance *R*, *L*, and capacitance *C* in series with voltage source *E*, at any time *t*.

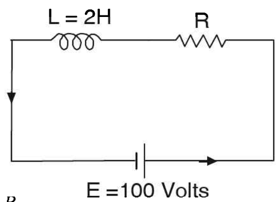
By voltage law

$$Ri + \frac{q}{C} = E \qquad \left[i = \frac{dq}{dt}\right]$$

$$\Rightarrow \qquad R\frac{dq}{dt} + \frac{q}{C} = E$$

Example 1. *An inductance of 2 henries and a resistance of 20 ohms are connected in series with an e.m.f. E volts. If the current is zero when t = 0, find the current at the end of 0.01 sec if E = 100 Volts.* (*U.P., II Semester, June 2008*)

Solution. Differential equation of the above circuit is as



1st case: $\qquad L\frac{di}{dt} + Ri = E \quad \Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

Its solution is $\quad ie^{\frac{Rt}{L}} = \frac{E}{L}\int e^{\frac{Rt}{L}} dt \quad \Rightarrow \quad ie^{\frac{Rt}{L}} = \frac{E}{L}\frac{L}{R}e^{\frac{R}{L}t} + A$

$\Rightarrow \qquad ie^{\frac{Rt}{L}} = \frac{E}{R}e^{\frac{R}{L}t} + A$ ...(1)

Putting $t = 0$, $i = 0$; in (1), we get $0 = \frac{E}{R} + A \quad \Rightarrow \quad A = -\frac{E}{R}$

Putting the value of A in (1), we get

$$i e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} - \frac{E}{R} \quad \Rightarrow \quad i = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L}t}$$

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] \qquad \ldots(2)$$

On putting the values of E, R and L in (2), we get

$$i = \frac{100}{20}\left[1 - e^{-\frac{20}{2}t}\right] = 5\,[1 - e^{-10t}]$$

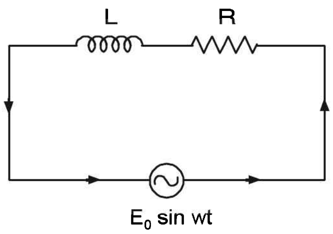
$$= 5\,[1 - e^{-10\times 0.01}] = 5\,[1 - e^{-0.1}] = 5\left[1 - \frac{1}{e^{0.1}}\right] \text{ at } [t = 0.01 \text{ sec}]$$

$= 0.475$ Approx. **Ans.**

Example 2. *Solve the equation* $L\frac{di}{dt} + Ri = E_0 \sin wt$

where L, R and E_0 *are constants and discuss the case when t increases indefinitely.*

Solution. $L\frac{di}{dt} + Ri = E_0 \sin wt$



$$\Rightarrow \qquad \frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin wt$$

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Solution is

$$i.e^{\frac{R}{L}t} = \frac{E_0}{L} \int e^{\frac{R}{L}t} \sin wt\, dt + A$$

$$\left[\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right)\right]$$

$$\Rightarrow \qquad i.e^{\frac{R}{L}t} = \frac{E_0}{L} \frac{e^{\frac{R}{L}t}}{\sqrt{\frac{R^2}{L^2} + w^2}} \sin\left(wt - \tan^{-1}\frac{Lw}{R}\right) + A$$

$$i = \frac{E_0}{\sqrt{R^2 + L^2 w^2}} \sin\left(wt - \tan^{-1}\frac{Lw}{R}\right) + Ae^{-\frac{R}{L}t}$$

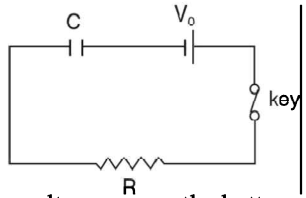
As t increases indefinitely, then $Ae^{-\frac{Rt}{L}}$ tends to zero.

so $$i = \frac{E_0}{\sqrt{R^2 + L^2 w^2}} \sin\left(wt - \tan^{-1}\frac{Lw}{R}\right)$$ **Ans.**

Example 3. *A condenser of capacity C farads with* V_0 *is discharged through a resistance R ohms. Show that if q coulomb is the charge on the condenser, i ampere the current and V the voltage at time t.*

$$q = CV,\ V = Ri \text{ and } i = \frac{dq}{dt}, \text{ hence show that } V = V_0 e^{-\frac{t}{RC}}$$

Solution.



Voltage across $R = Ri$

Voltage drop across capacitance $= \dfrac{q}{C}$

$\therefore$ The equation of discharge of condenser can be written, when after release of key the condenser gets discharged and at that time voltage across the battery gets zero so that $V_0 = 0$

The differential equation of the above circuit is

$$Ri + \frac{q}{C} = 0 \quad \Rightarrow \quad R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad \left(\text{as } i = \frac{dq}{dt}\right)$$

$$\Rightarrow \quad \frac{dq}{dt} + \frac{q}{RC} = 0 \quad \Rightarrow \quad \frac{dq}{dt} = -\frac{q}{RC} \Rightarrow \frac{dq}{q} = -\frac{1}{RC}dt$$

Integrating both sides, we get

$$\int \frac{dq}{q} = -\frac{1}{RC}\int dt \quad \Rightarrow \quad \log q = -\frac{1}{RC}t + A \qquad ...(1)$$

But at $t = 0$, the charge at the condenser is q_0 such that

$$\log q_0 = -\frac{1}{RC}(0) + A \quad \Rightarrow \quad A = \log q_0 \qquad ...(2)$$

Putting the value of A from (2) in (1), we have

$$\log q = -\frac{1}{RC}t + \log q_0 \quad \Rightarrow \quad \log q - \log q_0 = -\frac{1}{RC}t$$

$$\Rightarrow \quad \log\frac{q}{q_0} = -\frac{1}{RC}t \quad \Rightarrow \quad \frac{q}{q_0} = e^{-\frac{t}{RC}}$$

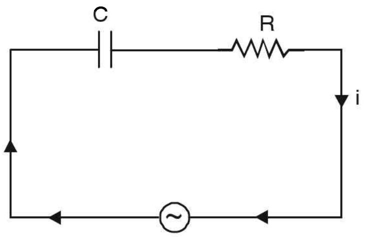
$$\Rightarrow \quad q = q_0\, e^{-\frac{t}{RC}} \qquad ...(3)$$

Dividing both side of (3) by C, we get

$$\frac{q}{C} = \frac{q_0}{C}e^{-\frac{t}{RC}} \quad \Rightarrow \quad V = V_0 e^{-\frac{t}{RC}} \qquad \left[\text{as } \frac{q}{C} = V\right]$$ **Proved.**

Example 4. *The equations of electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of capacity C, in series, is* $E = Ri + \int \frac{i}{C}dt$. *Find the current i at any time t, when* $E = E_0 \sin wt$. (*U.P. II Semester, Summer 2006*)

Solution. We have,



E_0 sin wt

$$Ri + \int \frac{i}{C}dt = E_0 \sin wt$$

Differentiating both the sides, we get

$$\frac{R\,di}{dt} + \frac{i}{C} = E_0 w \cos wt$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{i}{RC} = \frac{E_0}{R} w \cos wt$$

$$I.F. = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

Its solution is $i.\,(I.F.) = \int \frac{E_0 w}{R}\cos wt\,(I.F.)\,dt \Rightarrow i.e^{\frac{t}{RC}} = \frac{E_0 w}{R}\int \cos wt.\, e^{\frac{t}{RC}}dt + A$

$$\left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]\right]$$

$$\Rightarrow \qquad i.e^{\frac{t}{RC}} = \frac{E_0 w}{R}\frac{e^{\frac{t}{RC}}}{\frac{1}{R^2C^2}+w^2}\left[\frac{1}{RC}\cos wt + w\sin wt\right] + A$$

$$= \frac{E_0 w}{R}\frac{R^2C^2\,e^{\frac{t}{RC}}}{1+w^2R^2C^2}\left[\frac{1}{RC}\cos wt + w\sin wt\right] + A$$

$$i = E_0 w.\frac{RC^2}{1+w^2R^2C^2}\left[\frac{1}{RC}\cos wt + w\sin wt\right] + Ae^{-\frac{t}{RC}}$$

$$i = E_0 w.\frac{C}{1+w^2R^2C^2}[\cos wt + w\,RC\sin wt] + Ae^{-\frac{t}{RC}} \qquad \textbf{Ans.}$$

EXERCISE 16.1

1. A coil having a resistance of 15 ohms and an inductance of 10 henries is connected to 90 volts supply. Determine the value of current after 2 seconds. ($e^{-3} = 0.05$) **Ans.** 5.985 amp.
2. A resistance of 70 ohms, an inductance of 0.80 henry are connected in series with a battery of 10 volts. Determine the expression for current as a function of time after $t = 0$.

   Ans. $i = \frac{1}{7}\left(1 - e^{-\frac{175}{2}t}\right)$
3. A circuit consists of resistance R ohms and a condenser of C farads connected to a constant e.m.f. E; if $\frac{q}{C}$ is the voltage of the condenser at time t after closing the circuit Show that $\frac{q}{C} = E - Ri$ and hence show that the voltage at time t is $E\left(1 - e^{-\frac{t}{CR}}\right)$.
4. Show that the current $i = \frac{q}{CR}e^{-\frac{t}{R.C.}}$ during the discharge of a condenser of charge Q coulomb through a resistance R ohms.
5. A condenser of capacity C farads with voltage v_0 is discharged through a resistance R ohms. Show that if q coulomb is the charge on the condenser, i ampere the current and v the voltage at time t.

   $q = Cv$, $v = Ri$ and $i = -\frac{dq}{dt}$, hence show that $v = v_0\, e^{-\frac{1}{Rc}}$.
6. Solve $L\frac{di}{dt} + Ri = E\cos wt$ **Ans.** $i = \frac{E}{L^2w^2+R^2}\left(R\cos wt + Lw\sin wt - \mathrm{R}e^{-\frac{Rt}{L}}\right)$
7. A circuit consists of a resistance R ohms and an inductance of L henry connected to a generator of $E\cos(wt + \alpha)$ voltage. Find the current in the circuit. ($i = 0$, when $t = 0$).

   Ans. $i = \frac{E}{\sqrt{R^2+L^2w^2}}\cos\left[wt + \alpha - \tan^{-1}\frac{Lw}{R}\right] - \frac{E}{\sqrt{R^2+L^2w^2}}.e^{-\frac{R}{L}t}\cos\left[\alpha - \tan^{-1}\frac{Lw}{R}\right]$

16.3 SECOND ORDER DIFFERENTIAL EQUATION

We have already discussed $R - L$ and $R - L - C$ electric circuits. Here we want to do circuit problems involving second order differential equations.

Example 5. *The damped LCR circuit is governed by the equation*

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\left(\frac{1}{C}\right)q=0$$

where L, R, C are positive constants. Find the conditions under which the circuit is overdamped, underdamped and critically damped. Find also the critical resistance.

(*U.P. II Semester, Summer 2005*)

Solution. Given equation is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\left(\frac{1}{C}\right)q=0 \quad\Rightarrow\quad \frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\left(\frac{1}{LC}\right)q=0 \qquad ...(1)$$

Let $\frac{R}{L}=2p$ and $\frac{1}{LC}=w^2$

Thus equation (1) becomes

$$\frac{d^2q}{dt^2}+2p\frac{dq}{dt}+w^2q=0$$

Its auxiliary equation is

$$m^2+2pm+w^2=0$$

$$\Rightarrow \qquad m=-p\pm\sqrt{p^2-w^2}$$

Case 1. When $p > w$, roots are real and distinct solution of equation (1) is

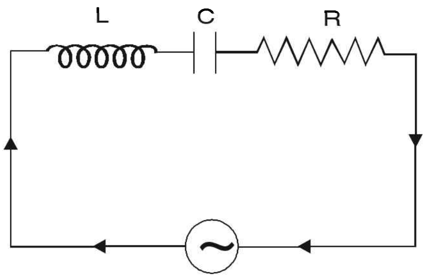
$$q=Ae^{(-p+\sqrt{p^2-w^2})t}+Be^{(-p-\sqrt{p^2-w^2})t}$$

In this case q is always positive, this is a condition of over damping.

Thus if $p > w$

$$\frac{R}{2L}>\frac{1}{\sqrt{LC}}$$

$$R>2\sqrt{\frac{L}{C}}$$



Case 2. When $p < w$, roots are imaginary

$$q=e^{-pt}\left(A\cos\sqrt{w^2-p^2}+B\sin\sqrt{w^2-p^2}\right)$$

period of oscillation decreases and this condition is of under damping.

Case 3. When $p = w$, roots are equal $q=(A+Bt)e^{-pt}$,

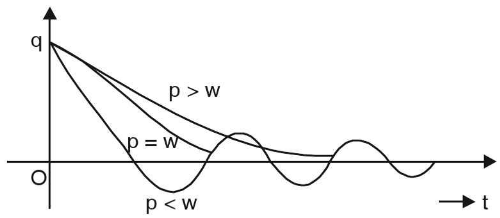
This is a condition of critically damped.

Critical resistance is given by

$$p=w$$

$$\Rightarrow \qquad \frac{R}{2L}=\frac{1}{\sqrt{LC}}$$

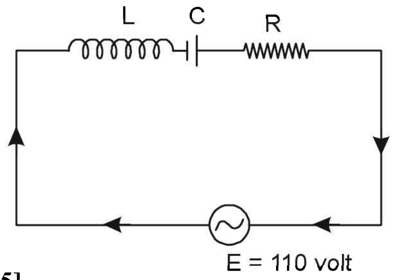
$$R=2\sqrt{\frac{L}{C}} \quad \textbf{Ans.}$$



Example 6. *A circuit consists of resistance of 5ohms, inductance of 0.05 Henrys and capacitance of 4×10^{-4} farads. If q (0) = 0, i (0) = 0 find q (t) and i (t), when an emf of 110 volts is applied.* (*M.D.U., 2010*)

Solution.

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\left(\frac{1}{C}\right)Q=110 \qquad \text{... (1)}$$



$$\Rightarrow \qquad \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{1}{LC}Q=\frac{110}{L}$$

Let $\frac{R}{L}=2p$ and $\frac{1}{LC}=w^2$

Thus equation is

$$\frac{d^2Q}{dt^2}+2p\frac{dQ}{dt}+w^2Q=\frac{110}{0.05} \qquad [L = 0.05]$$

$$\Rightarrow \qquad \frac{d^2Q}{dt^2}+2p\frac{dQ}{dt}+w^2Q=2200$$

Its auxiliary equation is

$m^2 + 2\,p\,m + w^2 = 0$

$$m=-p\pm\sqrt{p^2-w^2} \qquad \text{... (2)}$$

Here, we have

$R = 5$ ohms, $L = 0.05$ Henrys, $C = 4 \times 10^{-4}$ farads

$$\therefore \qquad 2p=\frac{R}{L}=\frac{5}{0.05}=100 \qquad \Rightarrow \qquad p=50$$

$$w^2=\frac{1}{LC}=\frac{1}{0.05\times 4\times 10^{-4}}=50000$$

Putting the values of p and w in (2), we get

$$m=-50\pm\sqrt{(50)^2-50000}=-50\pm\sqrt{2500-50000}$$

$$\Rightarrow \qquad m=-50\pm\sqrt{-47500}=-50\pm 50\sqrt{-19}=-50\pm 50\sqrt{19}\,i$$

$$\text{C.F.}=e^{-50t}\left(A\cos 50\sqrt{19}\,t+B\sin 50\sqrt{19}\,t\right) \qquad \text{... (3)}$$

$$\text{P.I.}=\frac{1}{D^2+2PD+w^2}2200$$

$$=\frac{1}{D^2+100\,D+50000}2200 \qquad [D=0]$$

$$=\frac{2200}{50000}=\frac{22}{500}=\frac{11}{250}$$

Complete solution = C.F. + P.I.

$$Q=e^{-50t}\left[A\cos 50\sqrt{19}\,t+B\sin 50\sqrt{19}\,t\right]+\frac{11}{250} \qquad \text{... (4)}$$

On putting $Q = 0$, $t = 0$ in (4), we get

$$Q = A + \frac{11}{250} \qquad \Rightarrow \qquad A = -\frac{11}{250}$$

On differentiating (4), we get

$$i = \frac{dQ}{dt} = -50\, e^{-50t}\, [A \cos 50\sqrt{19}\, t + B \sin 50\sqrt{19}\, t] +$$
$$e^{-50t}\, [-50\sqrt{19}\, A \sin 50\sqrt{19}\, t + 50\sqrt{19}\, B \cos 50\sqrt{19}\, t] \qquad \ldots (5)$$

On putting $i = 0$, $t = 0$ in (5), we get

$$0 = -50\, A + 50\sqrt{19}\, B \qquad \ldots (6)$$

On putting $A = -\frac{11}{250}$ in (6), we get

$$0 = -50\left(-\frac{11}{250}\right) + 50\sqrt{19}\, B \qquad \Rightarrow \qquad B = -\frac{11}{5 \times 50\sqrt{19}}$$

$$B = -\frac{11}{250\sqrt{19}}$$

On putting the values of A and B in (4), we get

$$Q = e^{-50t}\left[-\frac{11}{250}\cos 50\sqrt{19}\, t - \frac{11}{250\sqrt{19}}\sin 50\sqrt{19}\, t\right] + \frac{11}{250}$$

On putting the values of A and B in (5), we get

$$i = -50\, e^{-50t}\left[\left(-\frac{11}{250}\right)\cos 50\sqrt{19}\, t - \frac{11}{250\sqrt{19}}\sin 50\sqrt{19}\, t\right] +$$
$$e^{-50t}\left[-50\sqrt{19}\,.\left(\frac{-11}{250}\right)\sin 50\sqrt{19}\, t + 50\sqrt{19}\left(\frac{-11}{250\sqrt{19}}\right)\cos 50\sqrt{19}\, t\right]$$
$$= e^{-50t}\left[\left(\frac{11}{5} - \frac{11}{5}\right)\cos 50\sqrt{19}\, t + \left(\frac{11}{5\sqrt{19}} + \frac{11\sqrt{19}}{5}\right)\sin 50\sqrt{19}\, t\right]$$

$$\Rightarrow \qquad i = e^{-50t}\,\frac{11 + 11 \times 19}{5 \times 19}\sin 50\sqrt{19}\, t$$

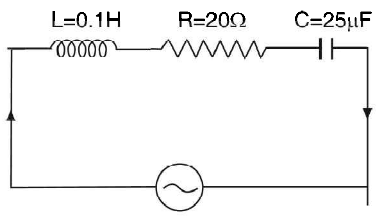
$$\Rightarrow \qquad i = e^{-50t}\,\frac{44}{\sqrt{19}}\sin 50\sqrt{19}\, t = \frac{44}{\sqrt{19}}\, e^{-50t}\sin 50\sqrt{19}\, t$$ **Ans.**

Example 7. *An electric circuit consists of an inductance 0.1 henry, a resistance of 20 ohms and a condenser of capacitance 25 microfarads. Find the charge q and the current i at time t, given the initial conditions* $q = 0.05$ *coulombs,* $i = 0$ *when* $t = 0$

Solution. The differential equation of the above given circuit can be written as

$$L\frac{di}{dt} + Ri + \frac{q}{C} = 0 \qquad \Rightarrow \qquad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad \left[i = \frac{dq}{dt}\right]$$

$$\Rightarrow \qquad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$



First, we will solve the equation and then put the values of R, L and C. For convenience we put

$$\frac{R}{L} = 2b,\ b = \frac{R}{2L} = \frac{20}{2 \times 0.1} = 100$$

Let $\frac{1}{LC} = k^2$

$$\Rightarrow \qquad k = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.1 \times 25 \times 10^{-6}}} = \sqrt{\frac{10^7}{25}} = 632.5 \geq 100$$

Our equation reduces to

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + k^2 q = 0$$

A.E. is $m^2 + 2b\, m + k^2 = 0$

So that
$$m = \frac{-2b \pm \sqrt{4b^2 - 4k^2}}{2} = -b \pm \sqrt{b^2 - k^2} = -b \pm j\sqrt{k^2 - b^2}$$

C.F. is $q = e^{-bt}[A \cos\sqrt{k^2 - b^2}\, t + B \sin\sqrt{k^2 - b^2}\, t]$ ...(1)

On putting $q = 0.05$ and $t = 0$ in (1), we get $0.05 = A$

On differentiating (1), we get

$$\frac{dq}{dt} = -be^{-bt}[A \cos\sqrt{k^2 - b^2}\, t + B \sin\sqrt{k^2 - b^2}\, t]$$
$$+ e^{-bt}[-A\sqrt{k^2 - b^2} \sin\sqrt{k^2 - b^2}\, t + B\sqrt{k^2 - b^2} \cos\sqrt{k^2 - b^2}\, t] \quad ...(2)$$

On putting $\frac{dq}{dt} = 0$ and $t = 0$ in (2), we get

$$0 = -bA + B\sqrt{k^2 - b^2} \quad \Rightarrow \quad B = \frac{bA}{\sqrt{k^2 - b^2}} = \frac{0.05\, b}{\sqrt{k^2 - b^2}}$$

Substituting the values of A and B in (1), we have

$$q = e^{-bt}[0.05 \cos\sqrt{k^2 - b^2}\, t + \frac{0.05\, b}{\sqrt{k^2 - b^2}} \sin\sqrt{k^2 - b^2}\, t] \quad ...(3)$$

Now,
$$\sqrt{k^2 - b^2} = \sqrt{\frac{10^7}{25} - (100)^2} = \sqrt{400000 - 10000} = \sqrt{390000} = 624.5$$

On putting these values in (3), we have

$$q = e^{-100t}[0.05 \cos 624.5t + \frac{0.05 \times 100}{624.5} \sin 624.5t]$$

$$\Rightarrow \qquad q = e^{-100t}[0.05 \cos 624.5t + 0.008 \sin 624.5t] \quad ...(4)\ \textbf{Ans.}$$

On differentiating (4), we have

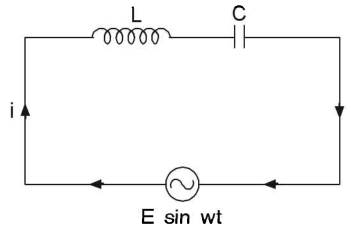
$$\frac{dq}{dt} = -100\, e^{-100t}[0.05 \cos 624.5t + 0.008 \sin 624.5t]$$
$$+ e^{-100t}[-0.05 \times 624.5 \sin 624.5t + 0.008 \times 624.5 \cos 624.5t]$$

$$\Rightarrow \qquad i = e^{-100t}[(-5 + 4.996) \cos 624.5t - (0.8 + 31.225) \sin 624.5t]$$
$$= e^{-100t}[-0.004 \cos 624.5t - 32.025 \sin 624.5t]$$
$$= -32\, e^{-100t} \sin 624.5t. \text{ approximately} \qquad \textbf{Ans.}$$

Example 8. *An alternating e.m.f. E sin wt is applied to an inductance L and capacitance C in series. Show that the current in the circuit is* $\frac{Ew}{(n^2-w^2)L}$ *(cos wt – cos nt), where* $n^2 = \frac{1}{LC}$.

(*U.P. II Semester, June 2010, 2009*)

Solution. The differential equation for the above circuit is



$$L\frac{d^2q}{dt^2}+\frac{q}{C} = E \sin wt$$

$$\Rightarrow \quad \frac{d^2q}{dt^2}+\frac{q}{LC} = \frac{E}{L}\sin wt$$

$$\Rightarrow \quad \left(D^2+\frac{1}{LC}\right)q = \frac{E}{L}\sin wt$$

A.E. is $m^2+\frac{1}{LC} = 0 \Rightarrow m^2+n^2 = 0 \Rightarrow m = \pm i\, n$ $\qquad \left(\because \frac{1}{LC} = n^2\right)$

$$\text{C.F.} = A\cos nt + B\sin nt$$

$$\text{P.I.} = \frac{1}{D^2+n^2}\frac{E}{L}\sin wt$$

$$\Rightarrow \quad \text{P.I.} = \frac{1}{-w^2+n^2}\frac{E}{L}\sin wt$$

Complete solution is $\quad q = A\cos n\,t + B\sin n\,t + \frac{E}{(n^2-w^2)\,L}\sin wt \qquad$ ...(1)

On putting $q = 0$, $t = 0$ in (1), we get

$$0 = A$$

On putting the value of A in (1), we get

$$q = B\sin nt + \frac{E}{(n^2-w^2)L}\sin wt \qquad \text{...(2)}$$

On differentiating (2) w.r.t., 't', we get

$$\frac{dq}{dt} = B\,n\cos nt + \frac{Ew}{(n^2-w^2)\,L}\cos wt$$

$$\Rightarrow \quad i = B\,n\cos nt + \frac{Ew}{(n^2-w^2)\,L}\cos wt \qquad \text{...(3)}$$

On putting $i = 0$, $t = 0$ in (3), we get

$$0 = Bn + \frac{Ew}{(n^2-w^2)L} \quad \Rightarrow \quad B = -\frac{Ew}{n(n^2-w^2)L}$$

Putting the value of B in (3), we get

$$i = -\frac{Ewn}{n(n^2-w^2)L}\cos nt + \frac{Ew}{(n^2-w^2)L}\cos wt$$

$$\Rightarrow \quad i = \frac{Ew}{(n^2-w^2)L}(\cos wt - \cos nt)$$ **Proved.**

Example 9. *For an electric circuit with circuit constants, L, R, C the charge q on a plate condenser is given by*

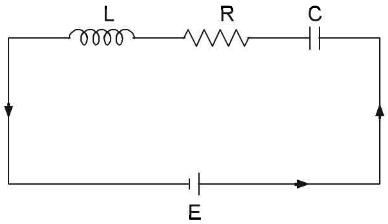
$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = E \text{ and the current by } i = \frac{dq}{dt}$$

Let L = 1 henry, $\quad C = 10^{-4}$ *farad,* $\quad$ *R = 100 ohms,* $\quad$ *E = 100 volts,*

Suppose that no charge present and no current is flowing at time t = 0, when the e.m.f. is applied. Determine q and i at any time t.

Solution. The differential equation is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$



$$\Rightarrow \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}$$

Putting $\frac{R}{L} = 2b$ and $\frac{1}{LC} = k^2$ in (1), we have

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + k^2q = \frac{E}{L} \qquad ...(1)$$

This equation is exactly identical, we have

$$\Rightarrow \quad q = \frac{E}{k^2L} + e^{-bt}[A\cos\sqrt{k^2-b^2}\,t + B\sin\sqrt{k^2-b^2}\,t] \qquad ...(2)$$

On putting $q = 0$ and $t = 0$ in (1), we get

$$0 = \frac{E}{k^2L} + A \quad \Rightarrow \quad A = -\frac{E}{k^2L}$$

Differentiating (2), we have

$$\frac{dq}{dt} = -be^{-bt}[A\cos\sqrt{k^2-b^2}\,t + B\sin\sqrt{k^2-b^2}\,t]$$

$$+ e^{-bt}[-A\sqrt{k^2-b^2}\sin\sqrt{k^2-b^2}\,t + B\sqrt{k^2-b^2}\cos\sqrt{k^2-b^2}\,t] \qquad ...(3)$$

On putting $\frac{dq}{dt} = 0$ and $t = 0$ in (3), we have

$$0 = -bA + B\sqrt{k^2-b^2} \quad \Rightarrow \quad B = \frac{bA}{\sqrt{k^2-b^2}} = \frac{-\frac{bE}{k^2L}}{\sqrt{k^2-b^2}} = -\frac{bE}{k^2L\sqrt{k^2-b^2}}$$

Substituting the values of A and B in (2), we have

$$q = \frac{E}{k^2L} + e^{-bt}\left[-\frac{E}{k^2L}\cos\sqrt{k^2-b^2}\,t - \frac{bE}{k^2L\sqrt{k^2-b^2}}\sin\sqrt{k^2-b^2}\,t\right]$$

$$= \frac{E}{k^2L}\left[1 - e^{-bt}\left(\cos\sqrt{k^2-b^2}\,t + \frac{b}{\sqrt{k^2-b^2}}\sin\sqrt{k^2-b^2}\,t\right)\right] \qquad ...(4)$$

Now, $$\frac{E}{k^2L} = \frac{E}{\frac{1}{LC}\cdot L} = EC = 100 \times 10^{-4} = \frac{1}{100}$$

$$b = \frac{R}{2L} = \frac{100}{2\times 1} = 50$$

$$\sqrt{k^2-b^2} = \sqrt{\frac{1}{LC} - (50)^2} = \sqrt{\frac{1}{10^{-4}} - (50)^2} = \sqrt{10000 - 2500} = \sqrt{7500} = 50\sqrt{3}$$

On putting these values in (4), we get

$$q = \frac{1}{100}\left[1 - e^{-50t}(\cos 50\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin 50\sqrt{3}\,t)\right]$$ **Ans.**

Example 10. *The voltage V and the current i at a distance x from the sending end of the transmission line satisfy the equations.*

$$-\frac{dV}{dx} = Ri, \quad -\frac{di}{dx} = GV$$

where R and G are constants. If $V = V_0$ *at the sending end* $(x = 0)$ *and* $V = 0$ *at receiving end* $(x = 1)$.

Show that $$V = V_0\left\{\frac{\sinh n(l - x)}{\sinh nl}\right\}, \text{ when } n^2 = RG$$

Solution. We have, $$-\frac{dV}{dx} = Ri \qquad ...(1)$$

$$-\frac{di}{dx} = GV \qquad ...(2)$$

When $x = 0$, $V = V_0$; When $x = l$, $V = 0$

Putting the value of i from (1) in (2), we get

$$-\frac{d}{dx}\left(-\frac{dV}{dx}\frac{1}{R}\right) = GV \quad \Rightarrow \quad \frac{d^2V}{dx^2} = RGV$$

$$\Rightarrow \quad \frac{d^2V}{dx^2} - (RG)V = 0 \quad \Rightarrow \quad (D^2 - RG)V = 0 \qquad (RG = n^2)$$

A.E. is $\quad m^2 - n^2 = 0, \quad m = \pm n$

$$\therefore \quad V = A\,e^{nx} + B\,e^{-nx} \qquad ...(3)$$

Now, we have to find out the values of A and B with the help of given conditions.

On putting $x = 0$ and $V = V_0$ in (3), we get

$$V_0 = A + B \qquad ...(4)$$

On putting $x = 1$ and $V = 0$ in (3), we get

$$0 = A\,e^{nl} + B\,e^{-nl} \qquad ...(5)$$

On solving (4) and (5), we have

$$A = \frac{V_0}{1 - e^{2nl}}, \quad B = \frac{-V_0\,e^{2nl}}{1 - e^{2nl}}$$

Substituting the values of A and B in (3), we have

$$V = \frac{V_0\,e^{nx}}{1 - e^{2nl}} - \frac{V_0\,e^{2nl}\,e^{-nx}}{1 - e^{2nl}} = \frac{V_0\,[e^{nx} - e^{2nl - nx}]}{1 - e^{2nl}}$$

$$= \frac{V_0\,[e^{(nl - nx)} - e^{-(nl - nx)}]}{e^{nl} - e^{-nl}} = V_0\left\{\frac{\sinh n(l - x)}{\sinh nl}\right\}$$ **Proved.**

EXERCISE 16.2

1. For an electric circuit with circuit constants L, R, C the charge q on the plate of the condensor is given by :

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

Find q at any time t.

Discuss the case when R is negligible and show that q is oscillatory. Calculate its period and frequency.

$$\textbf{Ans. } q = e^{-\frac{R}{2L}t}\left[A\cos\frac{\sqrt{4CL - R^2C^2}}{2LC}t + B\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right]$$

$$q = A\cos\frac{1}{\sqrt{LC}}t + B\sin\frac{1}{\sqrt{LC}}t, \text{Period} = 2\pi\sqrt{LC}, \text{frequency} = \frac{1}{2\pi\sqrt{LC}}$$

2. A condenser of capacity C is discharged through an iductance L and a resistance R in series and the charge q at any time t is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

If L = 10 milli henry, R = 200 ohms, $C = 0.1\ \mu\ F$ and also when $t = 0$, charge q = 0.01 coulomb and current $\frac{dq}{dt} = 0$. Find the value of q at any time t and find the frequency of the circuit if the discharge is oscillatory. **Ans.** $q = e^{-10000\,t}\,[\,0.01\cos 3\times10^4\,t + 0.33\times10^{-2}\sin 3\times10^4\,t]$

$$\text{frequency} = \frac{3\times10^4}{2\pi}$$

3. A condenser of capacity C is discharged through L and a resistance R in series and the charge q at any time t is given by the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

If L = 0.5 henry, R = 300 ohms, $C = 2\times10^{-6}$ farad and also when $t = 0$, charge q = 0.01 and current $\frac{dq}{dt} = 0$, find the value of q in terms of t. **Ans.** $q = e^{-200\,t}\,[0.01\cos 100\sqrt{91}\,t + 0.0031\sin 100\sqrt{91}\,t]$

4. A 10^{-3} farad capacitor is connected in series with 0.05 henry inductor and 10 ohms resistor. Initially, the current in the circuit is zero and the charge on the capacitor is also zero. If the e.m.f. is 50 sin 200 t. Find the charge t seconds after the circuit is closed.

Ans. $q = 0.125\,e^{-100\,t}\,[\cos 100\,t + \sin 100\,t] - 25\cos 200\,t + 0.125\sin 200\,t$

5. For an electric circuit with circuit constants L, R, C, the charge q on a plate on the condenser is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin wt$$

and the current $i = \frac{dq}{dt}$. The circuit is tuned to resonance so that $w^2 = \frac{1}{LC}$

If $R^2 = \frac{4L}{C}$ and $q = i = 0$ at $t = 0$, show that

$$q = \frac{E}{Rw}\left[-\cos wt + e^{-\frac{Rt}{2L}}\left(\cos pt + \frac{R}{2LP}\sin pt\right)\right] \qquad \text{where } p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

6. If the charge on one of the coatings of a layden jar be q when a force $E\cos pt$ acts in the circuit connecting the coatings and the circuit contains inductance L. Resistance R and capacitance C satisfying the differential equation:

$$\left(LD^2 + RD + \frac{1}{C}\right)q = E\cos pt$$

$D = \frac{d}{dt}$, find an expression for the charge given

$q = \frac{dq}{dt} = 0$ when $t = 0$.

7. An e.m.f. $E\sin pt$ is applied at $t = 0$ to a circuit containing a condenser C and inductance L in series. The current x satisfies the equation

$$L\frac{dx}{dt}+\frac{1}{C}\int x\,dt = E\sin pt$$

If $p^2 = \frac{1}{LC}$, and initially the current x and the charge q are zero, show that the current in the circuit at time t is given by

$$x=\frac{E}{2L}t\sin pt,\text{ where } x=-\frac{dq}{dt}.$$

8. An L-C-R circuit has $R = 180$ ohm, $C = \frac{1}{280}$ farad, $L = 20$ henries and an applied voltage $E(t) = 10\sin t$. Assuming that no charge is present but an initial current of i ampere is flowing at $t = 0$ when the voltage is first applied, find q and $i=\frac{dq}{dt}$ at any time t. q is given by the differential equation

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E(t)$$

Ans. $q=\frac{11}{50}e^{-2t}-\frac{101}{500}e^{-7t}+\frac{1}{1000}[26\sin t-18\cos t]$

16.4 MECHANICAL ENGINEERING PROBLEMS

Rectilinear Motion

When a body moves in a straight line the motion is called rectilinear motion. If x be the distance of the body at any time t from starting point then we have its velocity v given by

$$\text{velocity} = v = \frac{dx}{dt} \qquad \boxed{v = \frac{dx}{dt}}$$

If the acceleration of the body be 'a' then

$$\text{Acceleration} = a = \frac{dv}{dt}=\frac{d^2x}{dt^2}$$

Since

$$a = \frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dx}$$

$$\boxed{a = \frac{d^2x}{dt^2}}$$

$$\boxed{a = v\frac{dv}{dx}}$$

If mass of a body is m and the body is moving with acceleration a by a force F acting on it, then

$$\boxed{F = ma}$$

$$\boxed{F = m\frac{d^2x}{dt^2}}$$

$$\boxed{F = mv\frac{dv}{dx}}$$

Example 11. *A moving body is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.*

Solution. By Newton's second law of motion, the equation of motion of the body is

$$v\frac{dv}{dx} = -cx-bv^2 \quad\Rightarrow\quad v\frac{dv}{dx}+bv^2=-cx \qquad ...(1)$$