Eliminating x between (1) and (2), we have $v = \dfrac{mu - (l^2 + m^2)y}{l}$

$$\therefore \qquad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

Hence
$$\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right)\left(-\frac{l^2 + m^2}{l}\right) = \frac{l^2 + m^2}{l^2}.$$

TEST YOUR KNOWLEDGE

1. Find the first order partial derivatives of the following functions:

   (i) $u = y^x$ (ii) $u = \log (x^2 + y^2)$

   (iii) $u = x^2 \sin \dfrac{y}{x}$ (iv) $u = \dfrac{x}{y} \tan^{-1}\left(\dfrac{y}{x}\right)$.

2. If $u = x^2 + y^2 + z^2$, prove that $xu_x + yu_y + zu_z = 2u$.

3. If $z = \log (x^2 + xy + y^2)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$.

4. If $u = x^2y + y^2z + z^2x$, prove that $u_x + u_y + u_z = (x + y + z)^2$.

5. If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

6. If $f(x, y) = x^3y - xy^3$, find $\left[\dfrac{1}{\frac{\partial f}{\partial x}} + \dfrac{1}{\frac{\partial f}{\partial y}}\right]_{\substack{x=1\\y=2}}$.

7. (i) If $u = \log(\tan x + \tan y)$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} = 2$.

(ii) If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2$.

(iii) Find first and second order derivatives from the relation $\log z = x + y + z$.

8. If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $f_x + f_y + f_z = 0$.

9. Verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ for the following functions:

(i) $u = ax^2 + 2hxy + by^2$ (ii) $u = \tan^{-1}\left(\dfrac{x}{y}\right)$ (iii) $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$

(iv) $u = e^{ax} \sin by$ (v) $u = \log(x \sin y + y \sin x)$.

10. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$; where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$.

11. If $u = \tan^{-1} \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$, show that $\dfrac{\partial^2 u}{\partial x \partial y} = (1 + x^2 + y^2)^{-3/2}$.

12. If $u = e^{xyz}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

13. If $u = \log(x^2 + y^2) + \tan^{-1} \dfrac{y}{x}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = \tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

15. Verify that $f_{xy} = f_{yx}$ when f is equal to

(i) $\sin^{-1}\left(\dfrac{y}{x}\right)$ (ii) $\log x \tan^{-1}(x^2 + y^2)$.

16. Find the value of n so that the equation $V = r^n (3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0.$$

17. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.

18. If $V = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$.

19. If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that $V_{xx} + V_{yy} + V_{zz} = m(m + 1) r^{m-2}$.