Backtracking

BACKTRACKING

Principal

- Problems searching for a set of solutions or which require an optimal solution can be solved using the backtracking method .
- To apply the backtrack method, the solution must be expressible as an n-tuple(x_1,…,x_n), where the x_i are chosen from some finite set s_i
- The solution vector must satisfy the criterion function P(x_1 , ….. , x_n).

BACKTRACKING (Contd..)

- Suppose there are m n-tuples which are possible candidates for satisfying the function P.
- Then m= m_1, m_2…..m_n where m_i is size of set s_i 1<=i<=n.
- The brute force approach would be to form all of these n-tuples and evaluate each one with P, saving the optimum.

BACKTRACKING (Contd..)

- The backtracking algorithm has the ability to yield the same answer with far fewer than m-trials.
- In backtracking, the solution is built one component at a time.
- Modified criterion functions P_i $(x_1...x_n)$ called bounding functions are used to test whether the partial vector $(x_1,x_2,.....,x_i)$ can lead to an optimal solution.
- If $(x_1,...x_i)$ is not leading to a solution, $m_{i+1},....,m_n$ possible test vectors may be ignored.

BACKTRACKING – Constraints

EXPLICIT CONSTRAINTS are rules which restrict the values of x_i.

Examples $x_i \geq 0$ or $x_1 = 0$ or 1 or $l_i \leq x_i \leq u_i$.

IMPLICIT CONSTRAINTS describe the way in which the x_i must relate to each other .

Example : 8 queens problem.

BACKTRACKING: Solution Space

Tuples that satisfy the explicit constraints define a **solution space**.

The solution space can be organized into a tree.

Each node in the tree defines a **problem state**.

All paths from the root to other nodes define the **state-space** of the problem.

Solution states are those states leading to a tuple in the solution space.

Answer nodes are those solution states leading to an answer-tuple(i.e. tuples which satisfy implicit constraints).

BACKTRACKING -Terminology

LIVE NODE A node which has been generated and all of whose children are not yet been generated .

E-NODE (Node being expanded) - The live node whose children are currently being generated .

DEAD NODE - A node that is either not to be expanded further, or for which all of its children have been generated

DEPTH FIRST NODE GENERATION- In this, as soon as a new child C of the current E-node R is generated, C will become the new E-node.

BACKTRACKING -Terminology

BOUNDING FUNCTION - will be used to kill live nodes without generating all their children.

BACTRACKING-is depth – first node generation with bounding functions.

BRANCH-and-BOUND is a method in which E-node remains E-node until it is dead.

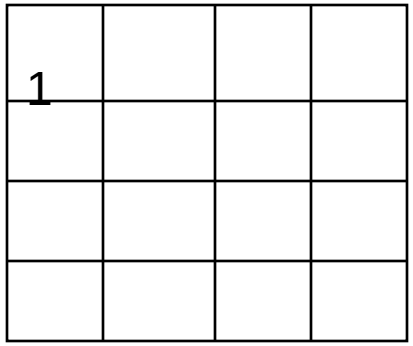
BACKTRACKING -Terminology

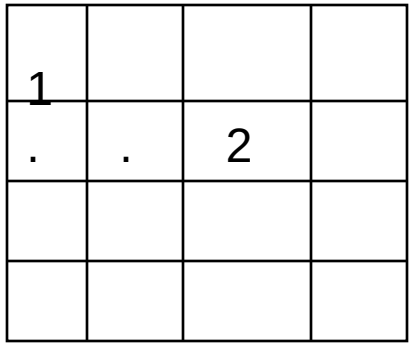
BREADTH-FIRST-SEARCH : Branch-and Bound with each new node placed in a queue .The front of the queen becomes the new E-node.

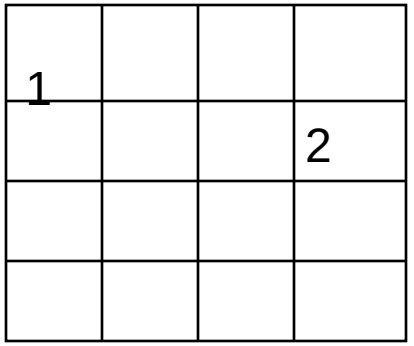
DEPTH-SEARCH (D-Search) : New nodes are placed in to a stack.The last node added is the first to be explored.

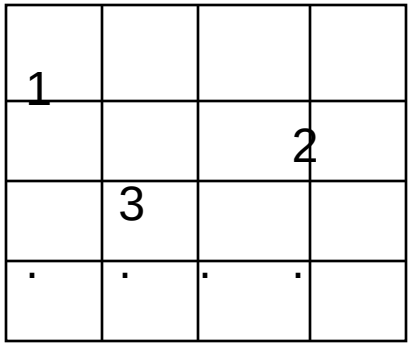
BACKTRACKING (4 Queens problem)

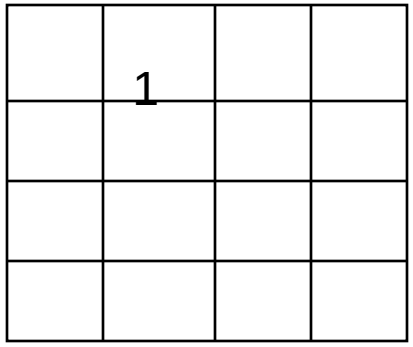
Example :

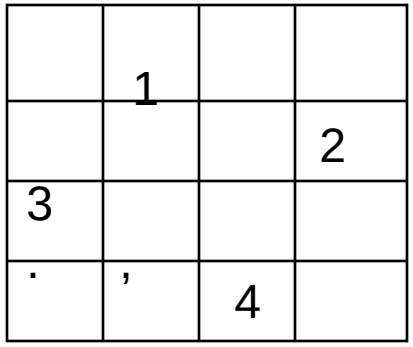












BACKTRACKING (Contd..)

- We start with root node as the only live node. The path is ( ); we generate a child node 2.
- The path is (1).This corresponds to placing queen 1 on column 1 .
- Node 2 becomes the E node. Node 3 is generated and immediately killed. (because x1=1,x2=2).
- As node 3 is killed, nodes 4,5,6,7 need not be generated.

Iterative Control Abstraction
(General Backtracking Method)

Procedure Backtrack(n)

{ K← 1

while **K > 0** do

if there remained an untried X(K) such that X(K) ∈ T (X (1) , …..X(k-1)) and B_k X (1) ,…,X(K)) = true then

  if (X (1),…, X(k)) is a path to an answer node then

print (X (1) ,…X (k))

  end if

K← K+1 // consider next set //

else K← K-1 // backtrack to previous set

   endif

repeat

end Backtrack

}

RECURSION Control Abstraction
(General Backtracking Method)

Procedure RBACKTRACK (k)

{ Global n , X(1:n)

for each X(k) such that

X(k) $\in$ T (X (1),..X(k-1)) and B_k (X(1)..,X(k-1), X(k))= true do

if (X (1) ,….,X(k)) is a path to an answer node

then print (X(1),…….,X(k)) end if

If (k<n)

CALL RBACKTRACK (k+1)

endif

repeat

End RBACKTRACK

}

EFFICIENCY OF BACKTRACKING ALGORITHM Depend on 4 Factors

- The time to generate the next X(k)

The no. of X(k) satisfying the explicit constraints

The time for bounding functions B_i

The no. of X(k) satisfying the B_i for all i

N queens problem
using Backtracking

The n-queens problem and solution

- In implementing the n – queens problem we imagine the chessboard as a two-dimensional array A (1 : n, 1 : n).
- The condition to test whether two queens, at positions (i, j) and (k, l) are on the same row or column is simply to check l = k or j = l
- The conditions to test whether two queens are on the same diagonal or not are to be found

The n-queens problem and solution contd..

Observe that

i) For the elements in the the upper left to lower Right diagonal, the row - column values are same or row- column = 0, e.g. 1-1=2-2=3-3=4-4=0

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

ii) For the elements in the upper right to the lower left diagonal, row + column value is the same e.g. 1+4=2+3=3+2=4+1=5

The n-queens problem and solution contd..

Thus two queens are placed at positions (i, j) and (k, l), then they are on the same diagonal only if

- $i - j = k - l$ or
- $i + j = k + l$

Two queens lie on the same diagonal if and only if

- $|j - l| = |i - k|$

The n-queens problem -Algorithm

Procedure PLACE (k)

```
{
    for i ← 1 to k-1 do
        if X(i) = X(k)  // two are in the same column
        or ABS(X(i) –X(k)) = ABS(i-k) // in the same diagonal
         then Return (false)
        end if
    repeat
    return (true)
    end
}
```

The n-queens problem -Algorithm contd..

Procedure N queens(n)

```
{ X(1)← 0 ; k←1 // k is the current row,//
    while k > 0 do
     X(k) ← X(k) + 1
While X(k) ≤ n and not PLACE(k)
    X(k) ← X(k) + 1
    repeat
if X(k) ≤ n then
     if k = n then print(X)
     else k ← k+1;
      X(k) = 0
    endif
 else  k← k-1
endif
repeat
end NQUEENS
```

Graph Coloring Problem
using Backtracking

GRAPH COLOURING PROBLEM

Let G be a graph and m be a positive integer .

The problem is to color the vertices of G using only m colors in such a way that no two adjacent nodes / vertices have the same color.

It is necessary to find the smallest integer m. m is referred to as the chromatic number of G.

GRAPH COLOURING PROBLEM (Contd..)

A map can be transformed into a graph by representing each region of map into a node and if two regions are adjacent, then the corresponding nodes are joined by an edge.

For many years it was known that 5 colors are required to color any map.

After a several hundred years, mathematicians with the help of a computer showed that 4 colours are sufficient.

Solving the Graph Colouring Problems

The graph is represented by its adjacency matrix Graph (1:n,1:n) where GRAPH (i,j) = true if <i,j> is an edge and Graph (i,j) = false otherwise.

The colours will be represented by the integers 1,2….m and the solution with n–tuple (X(1),….X(n)), where X(i) is the colour of node i.

Solving the Graph Colouring Problems (Contd..)

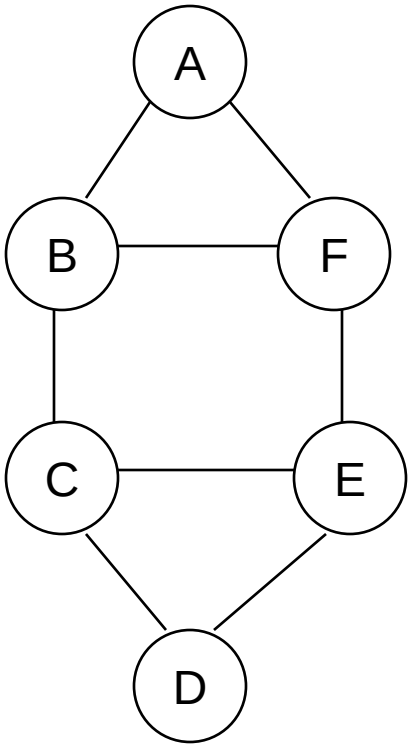
The solution can be represented as a state space tree.

Each node at level i has m children corresponding to m possible assignments to X(i) $1 \leq i \leq m$.

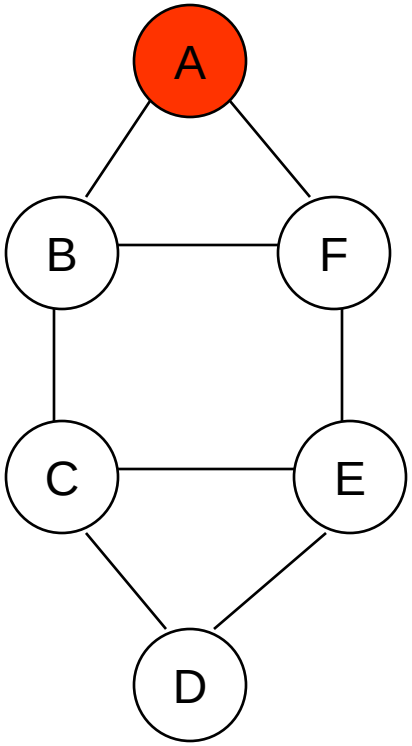
Nodes at level n+1, are leaf nodes. The tree has degree m with height n+1.

Graph Coloring

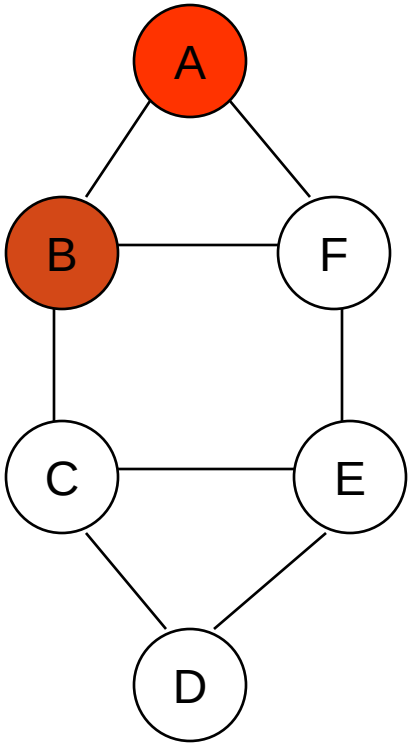
- As an example:
  - The vertices are enumerated in order A-F
  - The colors are given in order: R, G, B



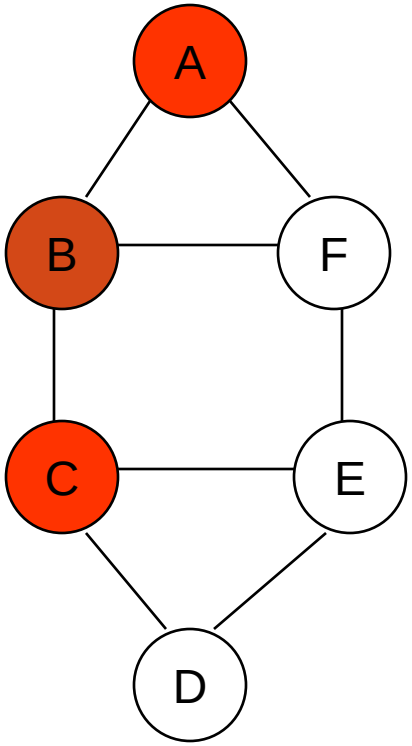
Graph Coloring



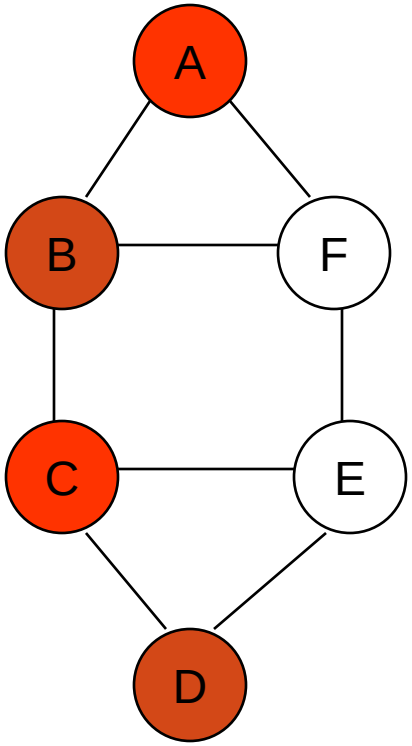
Graph Coloring



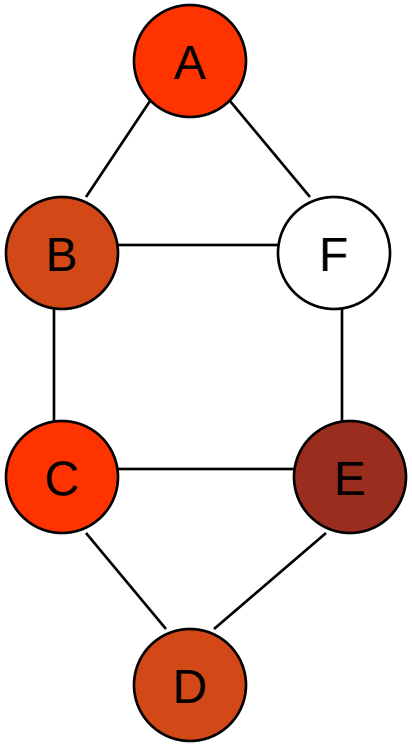
Graph Coloring



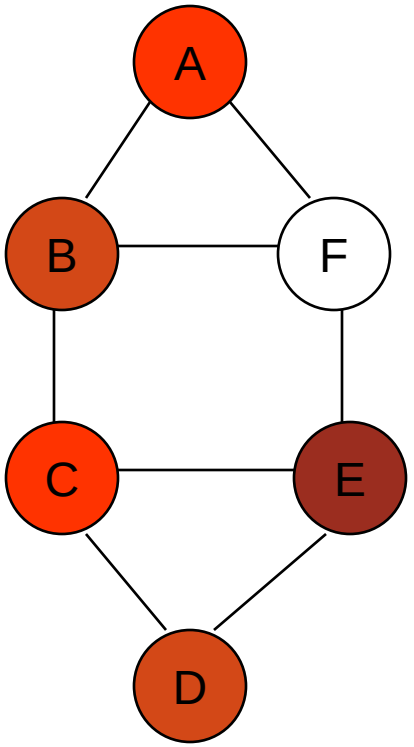
Graph Coloring



Graph Coloring

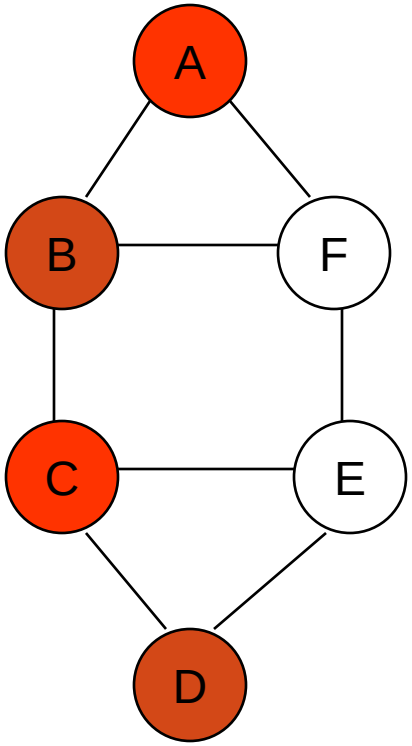


Graph Coloring

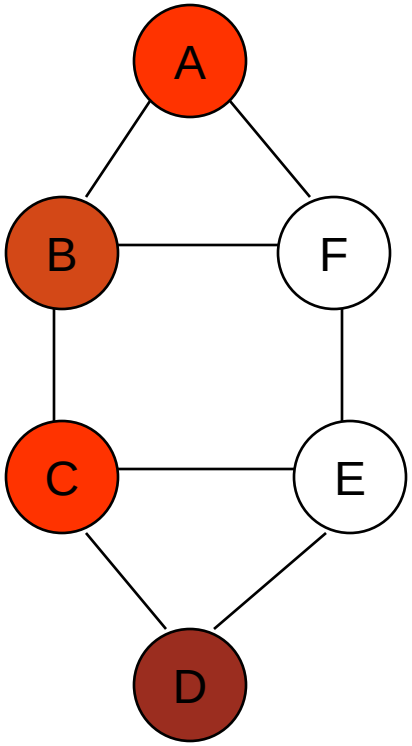


Stuck!

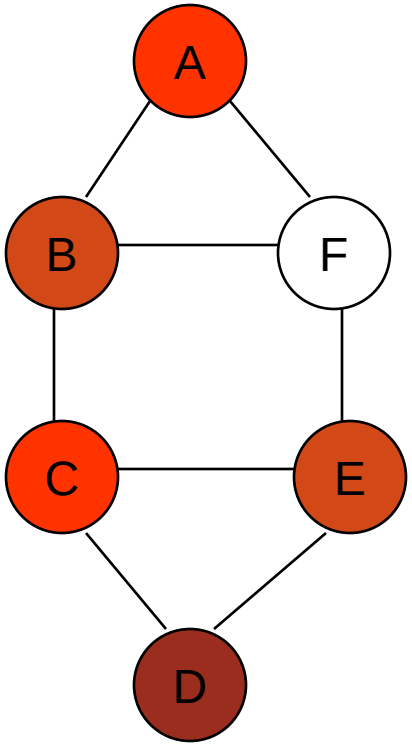
Graph Coloring



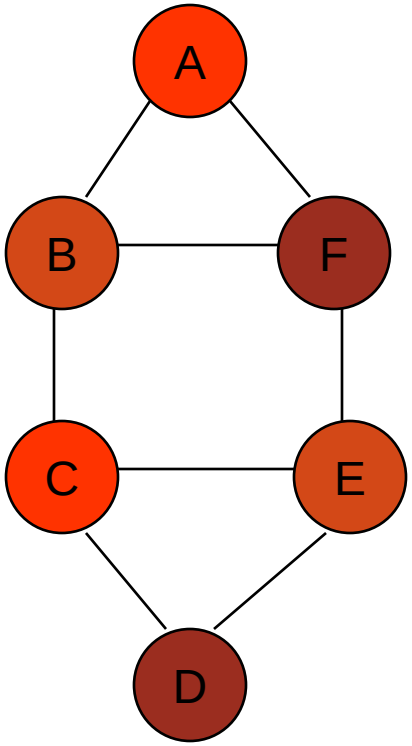
Graph Coloring



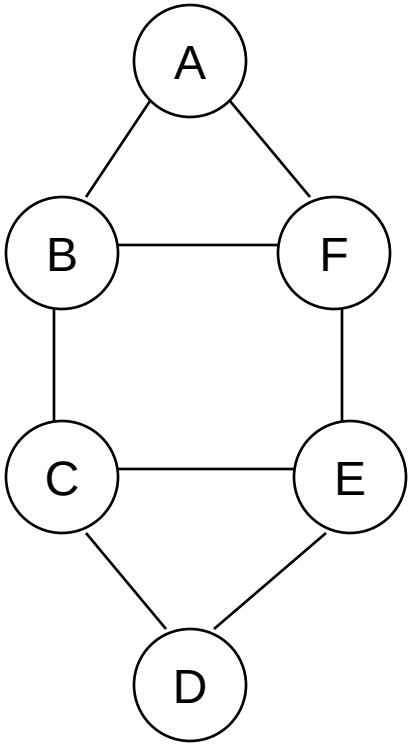
Graph Coloring



Graph Coloring



Graph Coloring



A

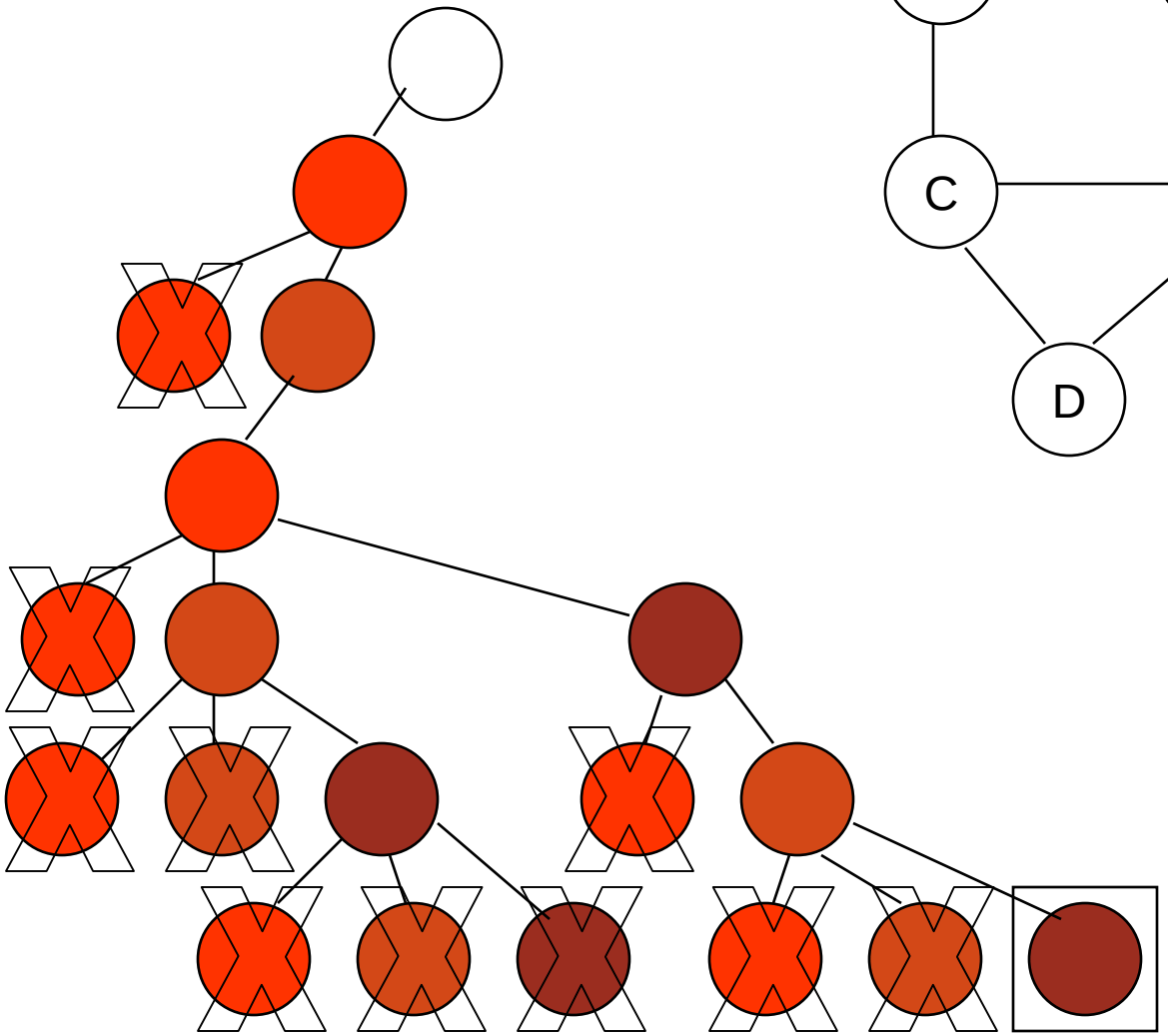
B

C

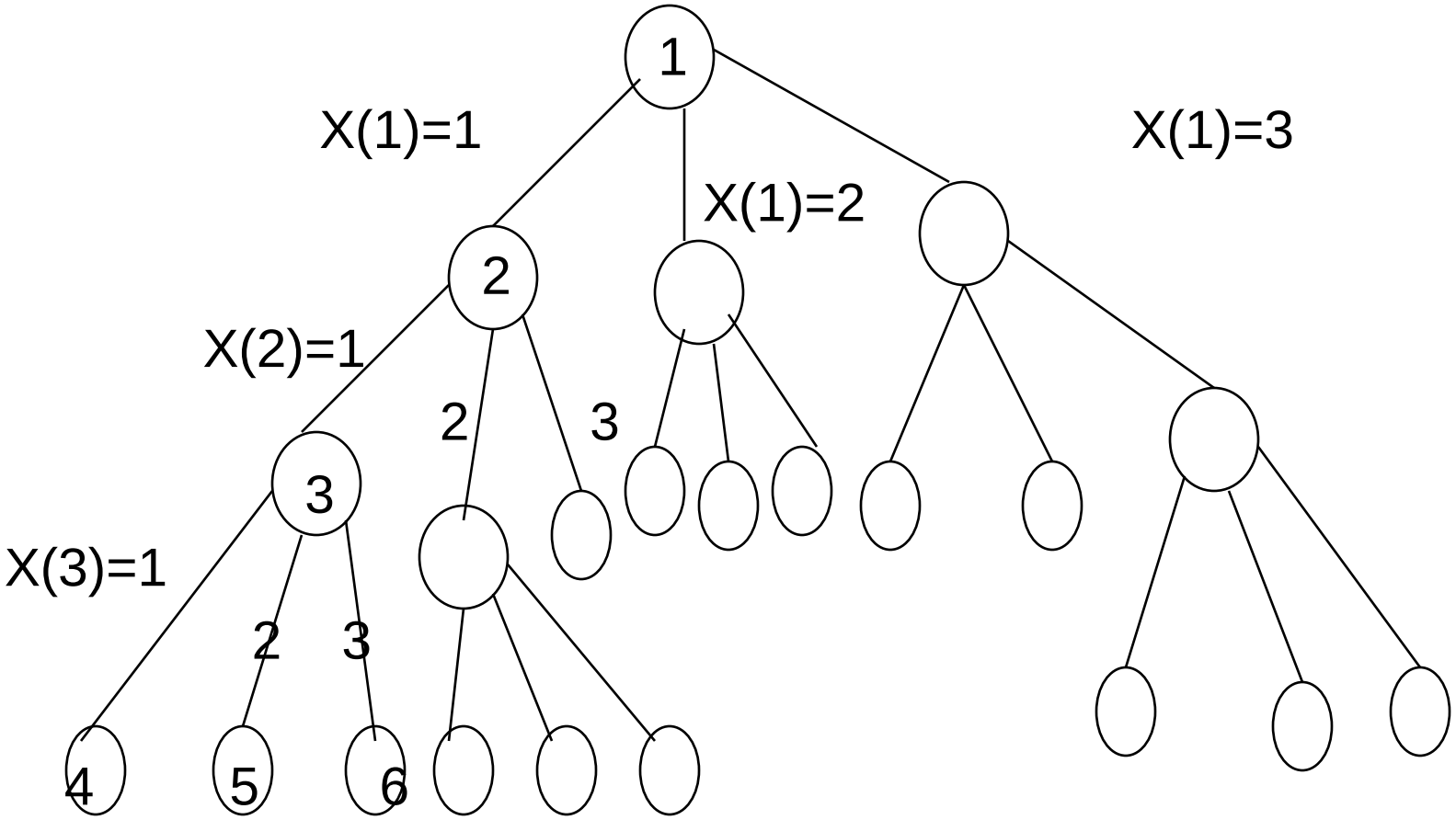
D

E

F



State space tree for m colouring problem with n = 3 and m = 3



Graph Colouring Problem- Algorithm

```
   Procedure MCOLORING (k)
{ global integer m,n,X(1:n);
     boolean GRAPH (1:n,1:n)
      integer  k
   loop
    Call NEXTVALUE(K)
    if X(k) = 0 then exit endif
    if k = n then Print X
       else call MCOLOURING(k+1)
    endif
  repeat
 end MCOLOURING
}
```

Graph Colouring Problem- Algorithm (Cont…)

Procedure NEXTVALUE(k)

```
{
Repeat
{
 X(k)←( X(k) +1 ) mod(m+1)
   if X(k)=0 then return endif
    for j← 1 to n do
     if GRAPH(k,j) and X(k)=X(j) then exit endif
     if j = n+1 then return endif
  } Until(False)
}
```

Sum of Subset Problem
using Backtracking

Sum-of-Subsets problem

- In this problem, we are given a vector of N values, called weights. The weights are usually given in ascending order of magnitude and are unique.
- **For example, W= (2, 4, 6, 8, 10) is a weight vector. We are also given a value M, for example 20.**
- The problem is to find all combinations of the weights that exactly add to M.
- **In this example, the weights that add to 20 are:**

  (2, 4, 6, 8); (2, 8, 10); and (4, 6, 10).
- Solutions to this problem are often expressed by an N-bit binary solution vector, X, where a 1 in position i indicates that Wi is part of the solution and a 0 indicates it is not.
- **In this manner the three solutions above could be expressed as: (1,1,1,1,0); (1,0,0,1,1); (0,1,1,0,1)**

Sum-of-Subsets problem

We are given ‘n’ positive numbers called weights and we have to find all combinations of these numbers whose sum is M. this is called sum of subsets problem.

If we consider backtracking procedure using fixed tuple strategy , the elements X(i) of the solution vector is either 1 or 0 depending on if the weight W(i) is included or not.

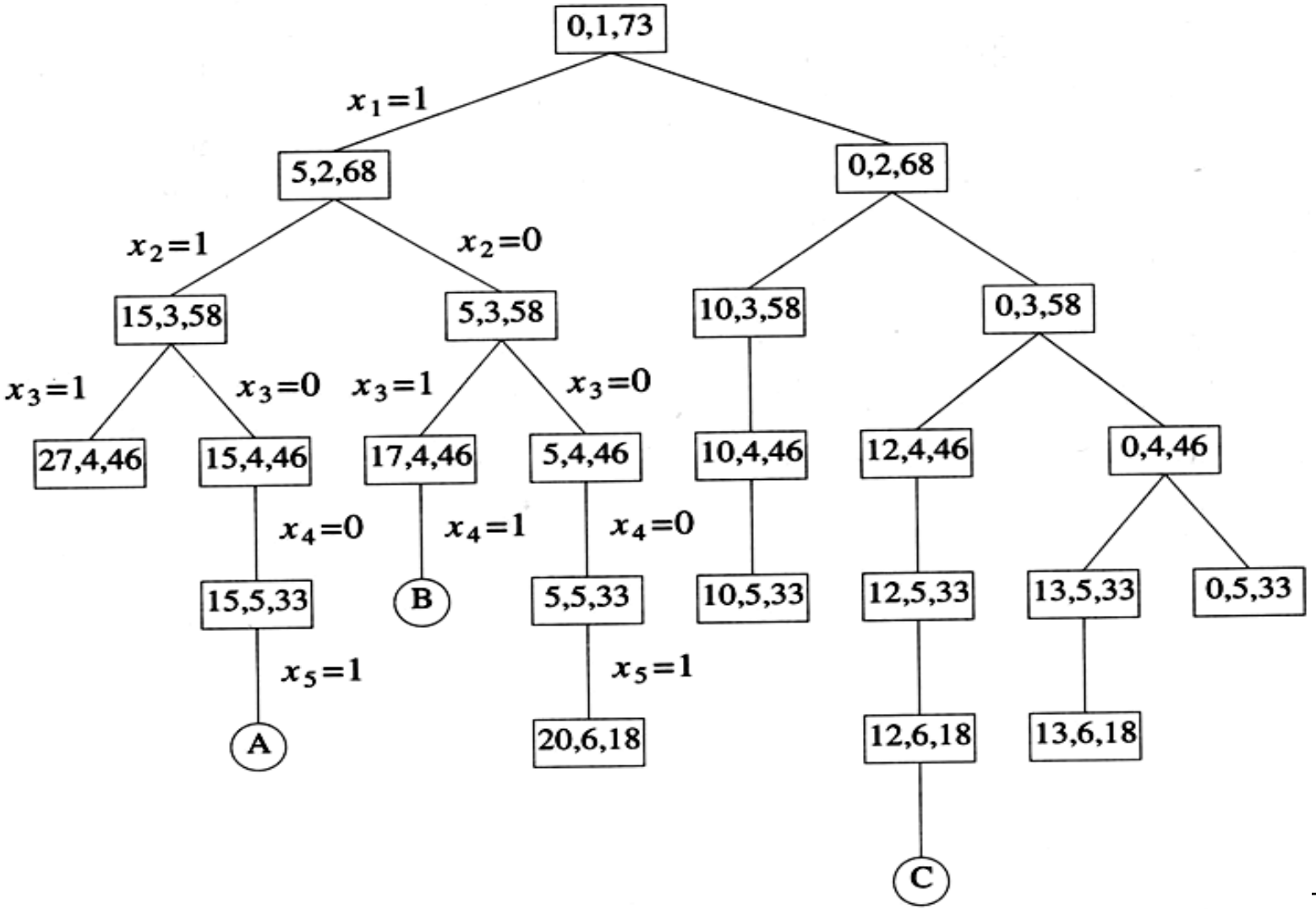
If the state space tree of the solution, for a node at level I, the left child corresponds to X(i)=1 and right to X(i)=0.

Sum of Subsets Algorithm

```
void SumOfSub(float s, int k, float r)
 {
   // Generate left child.
  x[k] = 1;
   if (s+w[k] == m)
   {   for (int j=1; j<=k; j++)
          Print (x[j] )
      }
   else if (s+w[k]+w[k+1] <= m)
        SumOfSub(s+w[k], k+1, r-w[k]);
   // Generate right child and evaluate
   if ((s+r-w[k] >= m) && (s+w[k+1] <= m)) {
        x[k] = 0;
        SumOfSub(s, k+1, r-w[k]);
   }
}
```

Sum of Subsets State Space Tree

- Example n=6, w[1:6]={5,10,12,13,15,18}, m=30



Branch and Bound

Branch and Bound Principal

The term branch-and-bound refers to all state space search methods in which all children of the £-node are generated before any other live node can become the £-node.

We have already seen two graph search strategies, BFS and D-search, in which the exploration of a new node cannot begin until the node currently being explored is fully explored.

Both of these generalize to branch-and-bound strategies.

In branch-and-bound terminology, a BFS-like state space search will be called FIFO (First In First Out) search as the list of live nodes is a first-in-first-out list (or queue).

A D-search-like state space search will be called LIFO (Last In First Out) search as the list of live nodes is a last-in-first-out list (or stack).

Control Abstraction for Branch and Bound(LC Method)

```
line   procedure LC (T, ĉ)
       //search T for an answer node//
 0     if T is an answer node then output T; return; endif
 1     E ← T   //E-node//
 2     initialize the list of live nodes to be empty
 3       loop
 4         for each child X of E do
 5           if X is an answer node then output the path from X to T
 6                                                    return
 7           endif
 8           call ADD(X)   //X is a new live node//
 9           PARENT(X) ← E   //pointer for path to root//
10         repeat
11         if there are no more live nodes then print ('no answer node')
12                                                    stop
13         endif
14         call LEAST(E)
15       repeat
16     end LC
```

LC Method Control Abstarction Explanation

- The search for an answer node can often be speeded by using an "intelligent" ranking function, c(.), for live nodes.
- The next £-node is selected on the basis of this ranking function.
- Let T be a state space tree and c( ) a cost function for the nodes in T. If X is a node in T then c(X) is the minimum cost of any answer node in the subtree with root X. Thus, c(T) is the cost of a minimum cost answer node

LC Method Control Abstarction Explanation

- The algorithm uses two subalgorithms LEAST(X) and ADD(X) to respectively delete and add a live node from or to the list of live nodes.
- LEAST{X) finds a live node with least c( ). This node is deleted from the list of live nodes and returned in variable X.
- ADD(X) adds the new live node X to the list of live nodes.
- Procedure LC outputs the path from the answer node it finds to the root node T.

0/1 knapsack problem using Branch and Bound

The 0/1 knapsack problem

- Positive integer $P_1, P_2, \ldots, P_n$ (profit)
  $W_1, W_2, \ldots, W_n$ (weight)
  M (capacity)

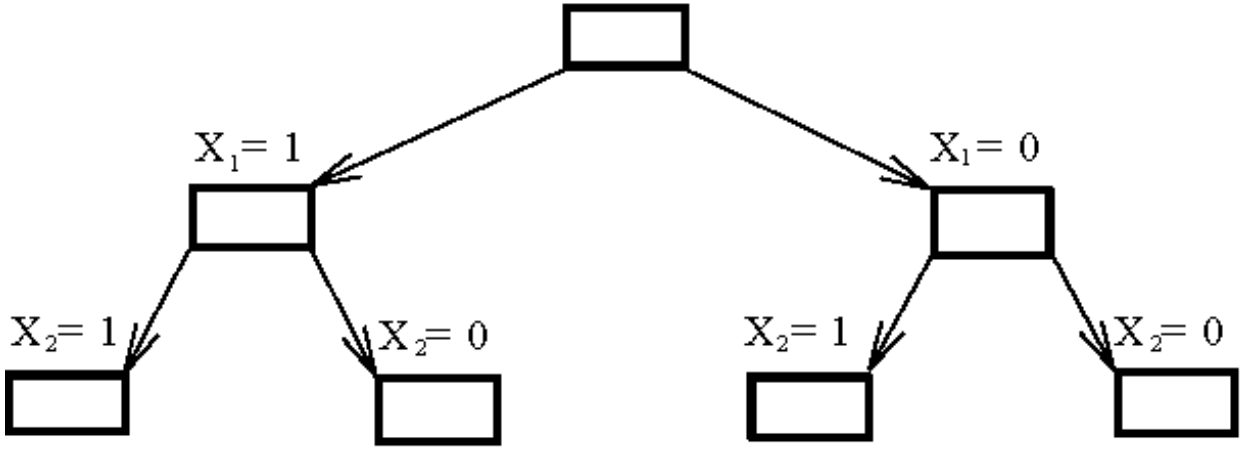
$$\text{maximize} \quad \sum_{i=1}^{n} P_i X_i$$

$$\text{subject to} \quad \sum_{i=1}^{n} W_i X_i \le M \quad X_i = 0 \text{ or } 1,\ i = 1, \ldots, n.$$

The problem is modified:

$$\text{minimize} \quad -\sum_{i=1}^{n} P_i X_i$$

The 0/1 knapsack problem



The Branching Mechanism in the Branch-and-Bound Strategy to Solve 0/1 Knapsack Problem.

How to find the upper bound?

- Ans: by quickly finding a feasible solution in a greedy manner: starting from the smallest available i, scanning towards the largest i's until M is exceeded. The upper bound can be calculated.

How to find the ranking Function

- Ans: by relaxing our restriction from $X_i = 0$ or 1 to $0 \le X_i \le 1$ (knapsack problem)

  Let $-\sum_{i=1}^{n} P_i X_i$ be an optimal solution for 0/1 knapsack problem and $-\sum_{i=1}^{n} P_i X_i'$ be an optimal solution for fractional knapsack problem. Let

  $Y = -\sum_{i=1}^{n} P_i X_i$, $Y' = -\sum_{i=1}^{n} P_i X_i'$.

  $\Rightarrow Y' \le Y$

How to expand the tree?

- By the best-first search scheme
- That is, by expanding the node with the best lower bound. If two nodes have the same lower bounds, expand the node with the lower upper bound.

0/1 Knapsack algorithm using BB

```
procedure UBOUND (p, w, k, M)
    //p, w, k and M have the same meaning as in Algorithm 7.11//
    //W(i) and P(i) are respectively the weight and profit of the ith object//
      global W(1:n), P(1:n); integer i, k, n
      b ← p; c ← w
      for i ← k + 1 to n do
        if c + W(i) ≤ M then c ← c + W(i); b ← b − P(i) endif
      repeat
      return (b)
  end UBOUND
```

Algorithm 8.5 Function $u(\cdot)$ for knapsack problem

0/1 Knapsack Example using LCBB (Least Cost)

- Example (LCBB)
- Consider the knapsack instance:
- n = 4;
- (pi, p2,
- p3, p4) = (10, 10, 12, 18);
- (wi. w2, w3, w4) = (2, 4, 6, 9) and
- M = 15.

0/1 Knapsack State Space tree of Example using LCBB

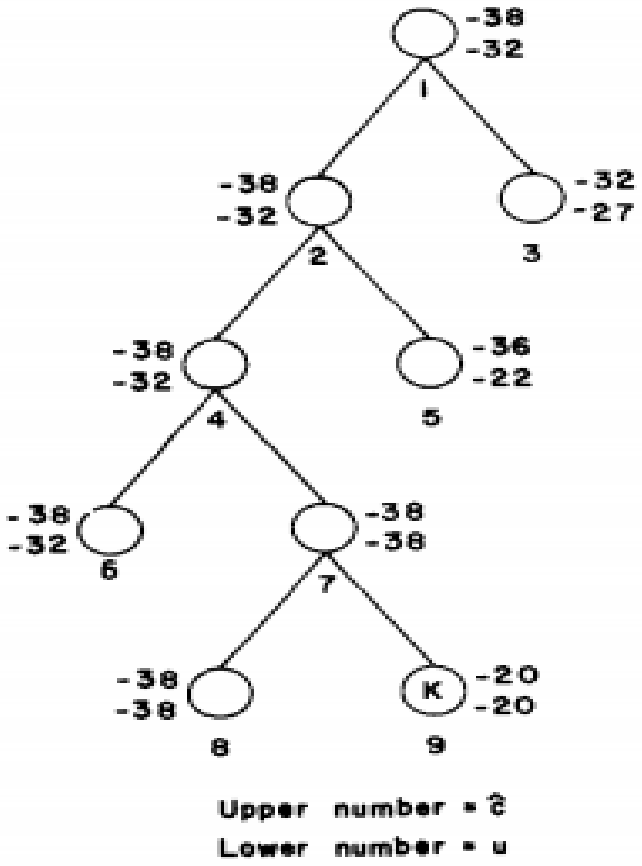


Figure 8.9 LC Branch-and-bound tree for Example 8.2

0/1 Knapsack State Space tree of Example using FIFO BB

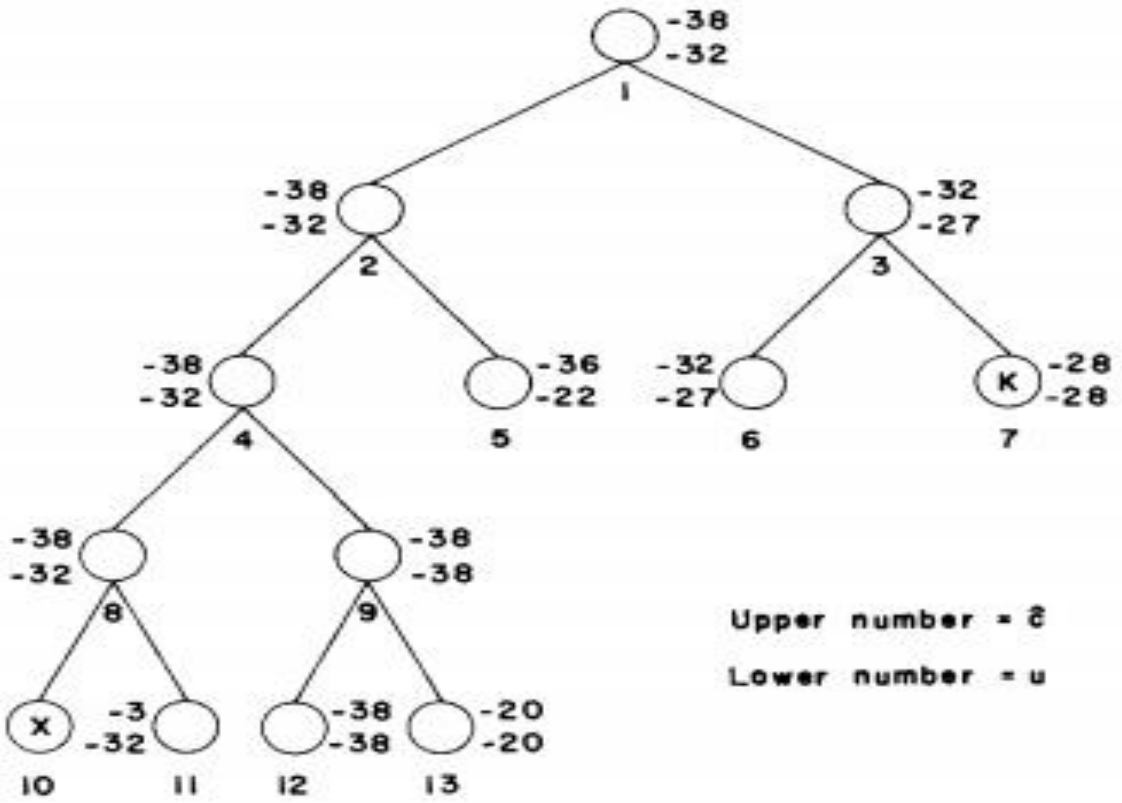


Figure 8.10 FIFO branch-and-bound tree for Example 8.3

Traveling Salesman problem (TSP) using Branch and Bound

The traveling salesperson problem

- Given a graph, the TSP Optimization problem is to find a tour, starting from any vertex, visiting every other vertex and returning to the starting vertex, with minimal cost.

The basic idea

- There is a way to split the solution space (branch)
- There is a way to predict a lower bound for a class of solutions. There is also a way to find a upper bound of an optimal solution. If the lower bound of a solution exceeds the upper bound, this solution cannot be optimal and thus we should terminate the branching associated with this solution.

Example- TSP

- Example with Cost Matrix(a) and its Reduced Cost Matrix (b)
- Reduced matrix means every row and column of matrix should contain at least one Zero and all other entries should be non negative.

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

(a) Cost Matrix

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

(b) Reduced Cost Matrix

L = 25

Reduced Matrix for node 2,3…10 of State Space tree using LC Method

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix}$$

a) path 1,2; node 2

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{bmatrix}$$

b) path 1,3; node 3

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$

c) path 1,4;node 4

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}$$

d) path 1,5; node 5

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

e) path 1,4,2; node 6

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix}$$

f) path 1,4,3; node 7

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$

g) path 1,4,5; node 8

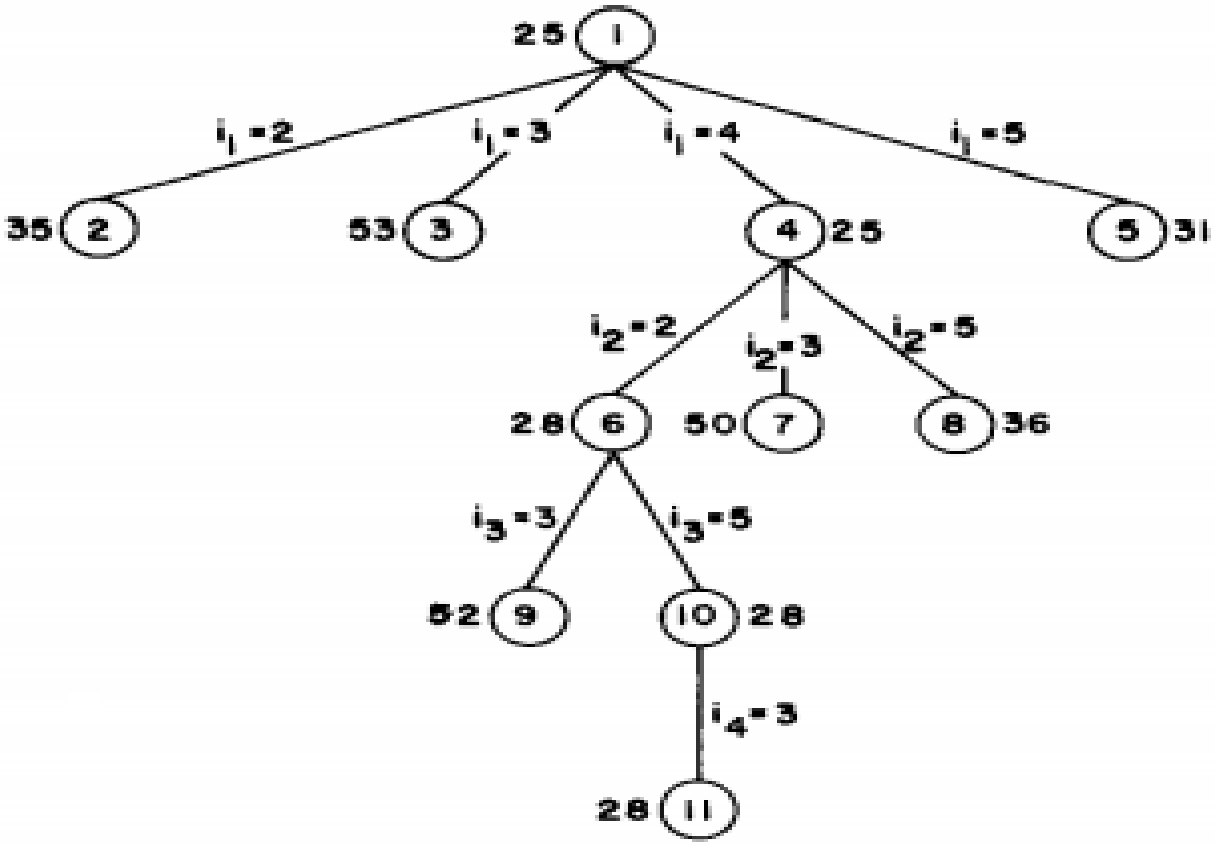
$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \end{bmatrix}$$

h) path 1,4,2,3; node 9

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix}$$

i) path 1,4,2,5; node 10

State Space tree of Example using LC Method



Numbers outside the node are $\hat{c}$ values

State space tree generated by procedure LCBB.