For purposes of evaluation, it can be expressed as the repeated integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\, dx\, dy\, dz \quad ...(1)$$

the order of integration depending upon the limits.

Let $x_1, x_2$ be function of $y, z$; $y_1, y_2$ be function of $z$ and $z_1, z_2$ be constants, *i.e.*,

Let $x_1 = f_1(y, z), x_2 = f_2(y, z), y_1 = \phi_1(z), y_2 = \phi_2(z)$ and $z_1 = a, z_2 = b$.

Then the integral (*i*) is evaluated as follows:

$$\boxed{\int_{z_1=a}^{z_2=b} \boxed{\int_{y_1=\phi_1(z)}^{y_2=\phi_2(z)} \boxed{\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\, dx} dy} dz}$$

First $f(x, y, z)$ is integrated w.r.t. $x$ (keeping $y$ and $z$ constant) between the limits $x_1$ and $x_2$. The resulting expression, which is a function of $y$ and $z$ is then integrated w.r.t. $y$ (keeping $z$ constant) between the limits $y_1$ and $y_2$. The resulting expression, which is a function of $z$ only is then integrated w.r.t. $z$ between the limits $z_1$ and $z_2$. The order of integration is from the innermost rectangle to the outermost rectangle.

**Limits involving two variables are kept innermost, then the limits involving one variable and finally the constant limits.**

If $x_1, x_2; y_1, y_2$ and $z_1, z_2$ are all constants, then the order of integration is immaterial, provided the limits are changed accordingly. Thus,

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x,y,z)\, dy\, dz\, dx$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dz\, dy\, dx, \text{ etc.}$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Evaluate* $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx.$

**Sol.**
$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\, dz\, dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} (\sin^{-1}1 - \sin^{-1}0)\, dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \int_0^1 \frac{\pi}{2}\Big[y\Big]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{4}[\sin^{-1}1] = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

**Example 2.** *Evaluate* $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$.

**Sol.**
$$I = \int_1^e \int_1^{\log y} \left[ \int_1^{e^x} \log z \, dz \right] dx \, dy$$

Since, $\int_1^{e^x} \log z \, dz = \int_1^{e^x} \log z \, . \, 1 \, dz$

Integrating by parts $= \left[ \log z \cdot z \right]_1^{e^x} - \int_1^{e^x} \frac{1}{z} \cdot z \, dz$

$$= e^x \log e^x - 0 - \left[ z \right]_1^{e^x} = xe^x - e^x + 1 = (x-1)\, e^x + 1$$

$\therefore$
$$I = \int_1^e \int_1^{\log y} [(x-1)\, e^x + 1] \, dx \, dy$$

Now, $\int_1^{\log y} [(x-1)\, e^x + 1] \, dx = \int_1^{\log y} (x-1)e^x \, dx + \left[ x \right]_1^{\log y}$

$$= \left[ (x-1)\, e^x \right]_1^{\log y} - \int_1^{\log y} 1 \cdot e^x \, dx + \log y - 1$$

$$= (\log y - 1)\, e^{\log y} - \left[ e^x \right]_1^{\log y} + \log y - 1$$

$$= y(\log y - 1) - (e^{\log y} - e) + \log y - 1 \qquad [\because \; e^{\log y} = y]$$

$$= y(\log y - 1) - y + e + \log y - 1 = (y+1) \log y - 2y + e - 1$$

$\therefore$
$$I = \int_1^e [\log y \cdot (y+1) - 2y + e - 1] \, dy$$

$$= \left[ \log y \cdot \left( \frac{y^2}{2} + y \right) \right]_1^e - \int_1^e \frac{1}{y} \left( \frac{y^2}{2} + y \right) dy - \left[ y^2 \right]_1^e + (e-1) \left[ y \right]_1^e$$

$$= \frac{e^2}{2} + e - \int_1^e \left( \frac{y}{2} + 1 \right) dy - (e^2 - 1) + (e-1)^2$$

$$= \frac{e^2}{2} + e - \left[ \frac{y^2}{4} + y \right]_1^e - 2e + 2 = \frac{e^2}{2} + e - \left[ \left( \frac{e^2}{4} + e \right) - \left( \frac{1}{4} + 1 \right) \right] - 2e + 2$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4} (e^2 - 8e + 13).$$

## TEST YOUR KNOWLEDGE

*Evaluate the following integrals (1–11):*

1. $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} \, dx \, dy \, dz.$
2. $\int_0^a \int_0^a \int_0^a (yz + zx + xy) \, dx \, dy \, dz.$
3. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) \, dy \, dx \, dz$.
4. $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz \, dz \, dy \, dx.$

5. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dz \, dy \, dx$.

6. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy$.

7. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$.

8. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} \, dz \, dy \, dx$.

9. $\int_0^{\pi/2} d\theta \int_0^{a \sin\theta} dr \int_0^{(a^2-r^2)/a} r dz$.

10. $\int_0^{\pi/2} \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r dz \, dr \, d\theta$.

11. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$

12. Evaluate $\iiint_V (x-2y+z) \, dz \, dy \, dx$, where V is the region determined by $0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le x+y$.

13. $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2+y^2+z^2) \, dx dy dz$

14. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z) \, dz dy dx$

15. $\int_0^2 \int_0^x \int_0^{x+y} e^x (y+2z) \, dz dy dx$

16. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4x-x^2}} dy dx dz$

## Answers

1. $(e-1)^3$
2. $\frac{3}{4}a^5$
3. $0$
4. $\frac{13}{9} - \frac{1}{6}\log 3$
5. $\frac{1}{720}$
6. $\frac{4}{35}$
7. $\frac{1}{48}$
8. $\frac{8}{3}\log 2 - \frac{19}{9}$
9. $\frac{5a^3\pi}{64}$
10. $\frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$
11. $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$
12. $\frac{8}{35}$.
13. $\frac{8}{3}abc(a^2+b^2+c^2)$
14. $\frac{a^4}{8}$
15. $19\left(\frac{e^2}{3}+1\right)$
16. $8\pi$.

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