$$\ldots \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u = 0 \qquad \ldots(2)$$

Since v is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0 \qquad \ldots(3)$$

Adding (2) and (3), we have

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v) + y^2 \frac{\partial^2}{\partial y^2}(u+v) = 0$$

or

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0. \qquad \text{[Using (1)]}$$

TEST YOUR KNOWLEDGE

1. Verify Euler's theorem for the functions:

(i) $f(x, y) = ax^2 + 2hxy + by^2$

(ii) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

(iii) $f(x, y) = \dfrac{x^2 (x^2 - y^2)^3}{(x^2 + y^2)^3}$

(iv) $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$

(v) $u = \log \left(\dfrac{x^2 + y^2}{xy} \right)$.

2. (i) If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ (ii) If $u = xf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

(iii) If $z = xyf\left(\frac{x}{y}\right)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

3. If $V = \frac{x^3y^3}{x^3 + y^3}$, show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 3V$.

4. If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

5. If $f(x, y) = \sqrt{x^2 - y^2}\,\sin^{-1}\frac{y}{x}$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$.

6. If $f(x, y) = \sqrt{y^2 - x^2}\,\sin^{-1}\frac{x}{y} + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - f(x, y) = 0$.

7. If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f(x, y) = 0$.

8. If $u = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

9. If $u = \cos^{-1}\frac{x + y}{\sqrt{x} + \sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$.

10. If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

11. (i) If $u = \sin^{-1}\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\cdot\frac{\partial u}{\partial y}$.

(ii) If $\sin u = \frac{x^2y^2}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

(iii) Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ where $\log u = \frac{x^3 + y^3}{3x + 4y}$.

12. (i) If $u = \log\left(\frac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3$.

(ii) Show that $xu_x + yu_y + zu_z = 2\tan u$, where $u = \sin^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$.

(iii) If $u = \frac{x}{y + z} + \frac{y}{z + x} + \frac{z}{x + y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

13. If $u = \frac{x^2y^2}{x + y}$, show that

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = 2\frac{\partial u}{\partial x}$ (ii) $x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial y}$.

14. Given $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$.

15. If $u = (x^2 + y^2)^{1/3}$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$.

16. If $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = 2 \cos 3u \sin u$.

17. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$, show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

18. If $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, then evaluate $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2}$.

19. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144} (13 + \tan^2 u)$.

Answers

4. 0 **18.** $2u$.

2.6 COMPOSITE FUNCTIONS

(*i*) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$

then u is called a composite function of (the **single variable**) t and we can find $\dfrac{du}{dt}$.

(*ii*) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$

then z is called a composite function of (**two variables**) u and v so that we can find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

2.7 DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$,