(P) **Example**:- If $x = r\cos\theta$, $y = r\sin\theta$,

prove that

(i) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (ii) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$

(iii) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$

Solu :- (i) I.C $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$

Now $x = r\cos\theta$, $y = r\sin\theta$ — (1)

So, $\dfrac{\partial x}{\partial r} = \cos\theta$

Now to find $\dfrac{\partial r}{\partial x}$, we have to represent r as fun. of x and y

From (1) $r^2 = x^2 + y^2$

So, $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$

So, $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$

(ii) $\dfrac{1}{r} \cdot \dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$

Now $x = r\cos\theta$, $y = r\sin\theta$ — (1)

So, $\dfrac{\partial x}{\partial \theta} = -r\sin\theta$

And $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = \dfrac{1}{r}(-r\sin\theta) = -\sin\theta$ — (2)

Now to find $\dfrac{\partial \theta}{\partial x}$, we have to express θ as fun. of x and y

From (1) $\theta = \tan^{-1}\dfrac{y}{x}$

(28)

So, $\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{-y}{x^2}\right] = \frac{\cancel{x^2}}{x^2+y^2}\left[\frac{-y}{\cancel{x^2}}\right]$

$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$

Now $r\frac{\partial \theta}{\partial x} = r\left(\frac{-y}{x^2+y^2}\right) = r\left(\frac{-r\sin\theta}{r^2}\right)$

$\Rightarrow r\frac{\partial \theta}{\partial x} = -\sin\theta$ —— (3)

So, From (2) and (3)

$$\frac{1}{r}\cdot\frac{\partial x}{\partial \theta} = r\cdot\frac{\partial \theta}{\partial x}$$

(iii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$

Soln - Now $\theta = \tan^{-1}\frac{y}{x}$

$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$

$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$

$\frac{\partial^2 \theta}{\partial x^2} = \frac{-(-y)}{(x^2+y^2)^2}(2x) = \frac{2xy}{(x^2+y^2)^2}$ —— (1)

Now $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{1}{x}\right] = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right)$

$$\Rightarrow \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 \theta}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \quad \text{---(2)}$$

Adding (1) and (2)

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

Exercise Question No. (21)

(21) If $u = \log\sqrt{x^2+y^2+z^2}$

Prove that $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

sol:- $u = \log\sqrt{x^2+y^2+z^2} = \frac{1}{2}\log(x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2+z^2}(2x) = \frac{x}{x^2+y^2+z^2}$$

similarly $\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2+y^2+z^2}(2y) = \frac{y}{x^2+y^2+z^2}$

" $\frac{\partial u}{\partial z} = \frac{z}{x^2+y^2+z^2}$

Now $\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)(1) - x(2x)}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$

similarly $\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2+z^2)(1) - y(2y)}{(x^2+y^2+z^2)^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$

$$\frac{\partial^2 u}{\partial z^2} = \frac{(x^2+y^2+z^2)(1) - z(2z)}{(x^2+y^2+z^2)^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+\cancel{z^2})^2}$$

Now consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{y^2+z^2-\cancel{x^2}+x^2+\cancel{z^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}-\cancel{z^2}}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{(x^2+y^2+z^2)}$$

$$\Rightarrow (x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$$