2

PARTIAL DIFFERENTIATION

2.1 PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z with respect to x, treating y as constant, is called partial derivative of z w.r.t. x and is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

Thus,
$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative of z with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Thus,
$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

[**In general,** if z is a function of two or more independent variables, then the **partial derivative of z** w.r.t. **any one** of the independent variables is the **ordinary derivative of z** w.r.t. that variable, **treating all other variables as constant.**]

Geometrically. Let $z = f(x, y)$ be a function of two variables x and y. Then by Art. 10.1, it represents a surface S. If $y = k$, a constant, then $y = k$ represents a plane parallel to the zx-plane.

$\therefore$ $z = f(x, y)$ and $y = k$ represent a plane curve C which is the section of S by $y = k$.

$\frac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z).

Thus, $\frac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx-plane.

Similarly, $\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to yz-plane.

2.2 PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t. x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad f_{xx}; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad f_{xy}; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{yx}$$

In general, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad f_{xy} = f_{yx}.$$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$.

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots, x_n$, we get $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots, \frac{\partial z}{\partial x_n}$.

Note 2. (*i*) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

(*ii*) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(*iii*) If $z = \frac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(*iv*) If $z = f(u)$, where $u = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the first order partial derivatives of the following:*

(*i*) $u = \tan^{-1} \dfrac{x^2 + y^2}{x + y}$ (*ii*) $u = \cos^{-1} \left(\dfrac{x}{y}\right)$.

Sol. (*i*) $$u = \tan^{-1} \frac{x^2 + y^2}{x + y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\dfrac{x^2 + y^2}{x + y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{x + y}\right)$$

$$= \frac{(x + y)^2}{(x + y^2) + (x^2 + y^2)^2} \cdot \frac{(x + y)\dfrac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2)\dfrac{\partial}{\partial x}(x + y)}{(x + y)^2}$$

$$= \frac{(x + y) \cdot 2x - (x^2 + y^2) \cdot 1}{(x + y)^2 + (x^2 + y^2)^2}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{x^2 + 2xy - y^2}{(x + y)^2 + (x^2 + y^2)^2} \qquad ...(1)$$

[Since u remains the same if we interchange x and y, u is symmetrical w.r.t. x and y. Interchanging x and y in (1), we have]

Similarly, $$\frac{\partial u}{\partial y} = \frac{y^2 + 2xy - x^2}{(x + y)^2 + (x^2 + y^2)^2}$$

(*ii*) $$u = \cos^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1 - \left(\dfrac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2 - x^2}} \cdot \frac{1}{y} = \frac{-1}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1 - \left(\dfrac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2 - x^2}}\left(-\frac{x}{y^2}\right) = \frac{x}{y\sqrt{y^2 - x^2}}.$$

Example 2. *If* $z(x + y) = x^2 + y^2$, *show that* $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

Sol. $$z = \frac{x^2 + y^2}{x + y}$$ [z is symmetrical w.r.t. x and y]

$$\frac{\partial z}{\partial x} = \frac{(x + y)\dfrac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2)\dfrac{\partial}{\partial x}(x + y)}{(x + y)^2}$$

$$= \frac{(x + y) \cdot 2x - (x^2 + y^2) \cdot 1}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

Now $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2x^2 - 2y^2}{(x+y)^2}\right]^2 = \frac{4(x+y)^2(x-y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2}$$

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2 - 2xy + y^2)}{(x+y)^2} = \frac{4(x-y)^2}{(x+y)^2}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Example 3. *Prove that if* $f(x, y) = \frac{1}{\sqrt{y}} \cdot e^{-\frac{(x-a)^2}{4y}}$, *then* $f_{xy} = f_{yx}$.

Sol. $$f(x, y) = \frac{1}{\sqrt{y}} \cdot e^{-\frac{(x-a)^2}{4y}} = y^{-\frac{1}{2}} e^{-\frac{(x-a)^2}{4y}}.$$

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= y^{-\frac{1}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \left[-\frac{2(x-a)}{4y}\right] = -\frac{1}{2} y^{-\frac{3}{2}} (x-a)\, e^{-\frac{(x-a)^2}{4y}}$$

$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{1}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial y}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-\frac{3}{2}} + y^{-\frac{1}{2}} \cdot \frac{(x-a)^2}{4y^2}\right]$$

$$= \frac{1}{4} y^{-\frac{3}{2}} e^{-\frac{(x-a)^2}{4y}} \quad [-2 + y^{-1}(x-a)^2]$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \left\{ e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right] \cdot [-2 + y^{-1}(x-a)^2] + e^{-\frac{(x-a)^2}{4y}} \cdot 2y^{-1}(x-a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \left\{ -\frac{2(x-a)}{4y} [-2 + y^{-1}(x-a)^2] + 2y^{-1}(x-a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{y}} \cdot \frac{x-a}{y} \left\{ -\frac{1}{2} [-2 + y^{-1}(x-a)^2] + 2 \right\}$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right]$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{2}(x-a)\left[-\frac{3}{2} y^{-\frac{5}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(x-a)^2}{4y^2} \right]$$

$$= -\frac{1}{4}(x-a) y^{-\frac{5}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \left[-3 + \frac{(x-a)^2}{2y} \right]$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right]$$

$\therefore \qquad f_{xy} = f_{yx}.$

Example 4. *If* $u = x^y$, *show that* $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.

$$u = x^y$$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = yx^{y-1} \log x + x^y \cdot \frac{1}{x} = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1} + yx^{y-1} \log x = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad \ldots(2)$$

From (1) and (2), $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Example 5. *If* $\theta = t^n e^{-\frac{r^2}{4t}}$, *find the value of n which will make* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

Sol. We have $\quad \theta = t^n e^{-\frac{r^2}{4t}}$

$$\frac{\partial \theta}{\partial r} = t^n \cdot e^{-\frac{r^2}{4t}} \cdot \left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 \cdot t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[3r^2 e^{-\frac{r^2}{4t}} + r^3 e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right] = -\frac{1}{2} t^{n-1} r^2 e^{-\frac{r^2}{4t}}\left[3 - \frac{r^2}{2t}\right]$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right)$$

Also, $$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}} \cdot \left(\frac{r^2}{4t^2}\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

Since, $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$ [Given]

$$\therefore \quad \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \quad \frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t} \quad \therefore \quad n = -\frac{3}{2}.$$

Example 6. *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that* $\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0$.

Sol. $$u = (1 - 2xy + y^2)^{-1/2} = V^{-1/2}, \quad \text{where } V = 1 - 2xy + y^2$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2} V^{-3/2} \cdot \frac{\partial V}{\partial x} = -\frac{1}{2} V^{-3/2}(-2y) = yV^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = y \cdot \frac{\partial}{\partial x}(V^{-3/2}) = y \cdot \left(-\frac{3}{2}\right) V^{-5/2} \cdot \frac{\partial V}{\partial x} = -\frac{3}{2} yV^{-5/2}(-2y)$$
$$= 3y^2 V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} = (1 - x^2) \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x}(1 - x^2)$$
$$= (1 - x^2) \cdot 3y^2 V^{-5/2} + yV^{-3/2}(-2x) = yV^{-3/2}[3yV^{-1}(1 - x^2) - 2x] \qquad \ldots(1)$$

Also, $$\frac{\partial u}{\partial y} = -\frac{1}{2}V^{-3/2}\frac{\partial V}{\partial y} = -\frac{1}{2}V^{-3/2}.(-2x+2y) = V^{-3/2}.(x-y)$$

$$\frac{\partial^2 u}{\partial y^2} = V^{-3/2}\cdot\frac{\partial}{\partial y}(x-y) + (x-y)\cdot\frac{\partial}{\partial y}(V^{-3/2})$$

$$= V^{-3/2}\cdot(-1) + (x-y)\cdot\left(-\frac{3}{2}V^{-5/3}\right)\cdot\frac{\partial V}{\partial y}$$

$$= -V^{-3/2} - \frac{3}{2}(x-y)V^{-5/2}\cdot(-2x+2y) = -V^{-3/2} + 3(x-y)^2V^{-5/2}$$

$\therefore$ $$\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = y^2\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial y}(y^2)$$

$$= y^2[-V^{-3/2} + 3(x-y)^2V^{-5/2}] + V^{-3/2}(x-y)\cdot 2y$$
$$= y\,V^{-3/2}[-y + 3y(x-y)^2V^{-1} + 2(x-y)]$$
$$= y\,V^{-3/2}[3y(x-y)^2V^{-1} + (2x-3y)] \quad \ldots(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = yV^{-3/2}[3yV^{-1}(1-x^2) - 2x + 3y(x-y)^2V^{-1} + 2x - 3y]$$

$$= y\,V^{-3/2}[3yV^{-1}(1-x^2+x^2-2xy+y^2) - 3y]$$
$$= yV^{-2/3}[3yV^{-1}(1-2xy+y^2) - 3y]$$
$$= yV^{-3/2}[3y-3y] \qquad | \because \quad V = 1-2xy+y^2$$
$$= 0.$$

Example 7. *If $u = \log(x^3+y^3+z^3-3xyz)$, show that*

(i) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

(ii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y} = \frac{-9}{(x+y+z)^2}.$$

Sol. (i) $$u = \log(x^3+y^3+z^3-3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz};\quad \frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Adding, $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$$

$[\because\ x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)]$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}=-\frac{9}{(x+y+z)^2} \quad \ldots(1)$$

(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 u}{\partial z\partial y}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}$$

$$\left[\because \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 u}{\partial z\partial y}=\frac{\partial^2 u}{\partial y\partial z}, \frac{\partial^2 u}{\partial x\partial z}=\frac{\partial^2 u}{\partial z\partial x}\right]$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}$$

[from (1)]

Example 8. *If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x\partial y} = -(x \log ex)^{-1}$.*

Sol. $x^x y^y z^z = c$ defines z as a function of x and y.

Taking logs, $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y, we have

$$y \cdot \frac{1}{y} + 1 \cdot \log y + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial y} + 1 \cdot \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0 \quad \ldots(1)$$

Similarly,

$$\left.\begin{aligned} \frac{\partial z}{\partial y} &= -\frac{1+\log y}{1+\log z} \\ \frac{\partial z}{\partial x} &= -\frac{1+\log x}{1+\log z} \end{aligned}\right\} \quad \ldots(2)$$

Differentiating (1) partially w.r.t. x, we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y}+(1+\log z)\frac{\partial^2 z}{\partial x\partial y}=0 \quad \text{or} \quad \frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{z(1+\log z)}\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \qquad ...(3)$$

When $x = y = z$

From (2), $$\frac{\partial z}{\partial y}=-1,\ \frac{\partial z}{\partial x}=-1$$

From (3), $$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x(1+\log x)}(-1)(-1)$$

$$=-\frac{1}{x(\log e+\log x)}=-\frac{1}{x(\log ex)}=-(x\log ex)^{-1}.$$

Example 9. *If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that:*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

Sol. Given $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow$ $x^2 + y^2 = r^2(\cos^2\theta + \sin\theta)$

$\Rightarrow$ $r^2 = x^2 + y^2$ ...(1)

Differentiating partially w.r.t. x, we get $2r\frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}$

Similarly, $$\frac{\partial r}{\partial y}=\frac{y}{r}$$

Now $$u=f(r)$$

$\therefore$ $$\frac{\partial u}{\partial x}=f'(r)\cdot\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)$$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{r}f'(r)+x\cdot\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)f'(r)+\frac{x}{r}f''(r)\cdot\frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw)=vw\frac{\partial}{\partial x}(u)+uw\frac{\partial}{\partial x}(v)+uv\frac{\partial}{\partial x}(w)\right]$$

$$=\frac{1}{r}f'(r)-\frac{x}{r^2}\cdot\frac{x}{r}f'(r)+\frac{x}{r}\cdot f''(r)\cdot\frac{x}{r}=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$$

$$=\frac{r^2-x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)=\frac{y^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r) \qquad |\ \text{Using (1)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2}=\frac{x^2}{r^3}f'(r)+\frac{y^2}{r^2}f''(r)$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{x^2+y^2}{r^3}f'(r)+\frac{x^2+y^2}{r^2}f''(r)$$

$$=\frac{r^2}{r^3}f'(r)+\frac{r^2}{r^2}f''(r)=f''(r)+\frac{1}{r}f'(r).$$

Example 10. *If* $x = r\cos\theta$, $y = r\sin\theta$, *prove that*

(i) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (ii) $\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial\theta} = r\dfrac{\partial\theta}{\partial x}$ (iii) $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0.$

Sol. (i) $\dfrac{\partial r}{\partial x}$ means $\left(\dfrac{\partial r}{\partial x}\right)_y$ = The partial derivative of r w.r.t. x, treating y as constant.

$\therefore$ We express r in terms of x and y.

Squaring and adding the given relations, $r^2 = x^2 + y^2$

Differentiating partially w.r.t. x, we get $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\dfrac{\partial x}{\partial r}$ means $\left(\dfrac{\partial x}{\partial r}\right)_\theta$ = The partial derivative of x w.r.t. r treating θ as constant.

$\therefore$ We express x in terms of r and θ.

Thus,
$$x = r\cos\theta \qquad \text{(given)}$$
$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \qquad \left(\because \cos\theta = \frac{x}{r}\right)$$
$$\therefore \quad \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

(ii) Expressing x in terms of r and θ, we have $x = r\cos\theta$
$$\Rightarrow \quad \frac{\partial x}{\partial\theta} = -r\sin\theta = -y \;\Rightarrow\; \frac{1}{r}\frac{\partial x}{\partial\theta} = -\frac{y}{r}$$

Expressing θ in terms of x and y, we have $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\dfrac{y}{x}$
$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-y}{r^2(\cos^2\theta+\sin^2\theta)} = -\frac{y}{r^2}$$
$$\Rightarrow \quad r\frac{\partial\theta}{\partial x} = -\frac{y}{r} \qquad \therefore \quad \frac{1}{r}\cdot\frac{\partial x}{\partial\theta} = r\frac{\partial\theta}{\partial x}.$$

(iii) Expressing θ in terms of x and y, we have $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\dfrac{y}{x}$
$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -y(x^2+y^2)^{-1}$$
$$\frac{\partial^2\theta}{\partial x^2} = y(x^2+y^2)^{-2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$
$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$
$$\frac{\partial^2\theta}{\partial x^2} = -x(x^2+y^2)^{-2}\cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$$
$$\therefore \quad \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Example 11. *If* $x = e^{r\cos\theta}\cos(r\sin\theta)$ *and* $y = e^{r\cos\theta}\sin(r\sin\theta)$, *prove that:*

$$\frac{\partial x}{\partial r} = \frac{1}{r}\cdot\frac{\partial y}{\partial \theta},\ \frac{\partial y}{\partial r} = -\frac{1}{r}\cdot\frac{\partial x}{\partial \theta}.$$

Hence deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = 0.$

Sol. $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore\quad \frac{\partial x}{\partial r} = e^{r\cos\theta}\cdot\cos\theta\cdot\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta)\cdot\sin\theta$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{...(1)}$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}\cdot(-r\sin\theta)\cdot\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta)\cdot r\cos\theta$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \text{...(2)}$$

Also, $y = e^{r\cos\theta}\sin(r\sin\theta)$

$$\therefore\quad \frac{\partial y}{\partial r} = e^{r\cos\theta}\cdot\cos\theta\cdot\sin(r\sin\theta) + e^{r\cos\theta}\cdot\cos(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \text{...(3)}$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta)\times r\cos\theta$$

$$= re^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \text{...(4)}$$

From (1) and (4), $$\frac{\partial x}{\partial r} = \frac{1}{r}\cdot\frac{\partial y}{\partial \theta} \qquad \text{...(5)}$$

From (2) and (3), $$\frac{\partial y}{\partial r} = -\frac{1}{r}\cdot\frac{\partial x}{\partial \theta} \qquad \text{...(6)}$$

From (5), $$\frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial \theta}$$

From (6), $$\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$$

$$\therefore\quad \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta\partial r} = -r\frac{\partial^2 y}{\partial r\partial \theta}$$

$$\therefore\quad \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial x}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial \theta} + \frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial \theta} = 0.$$

Example 12. *If* $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Sol. Given
$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \qquad ...(1)$$

$$x^2(a^2+u)^{-1}+y^2(b^2+u)^{-1}+z^2(c^2+u)^{-1}=1$$

Differentiating partially w.r.t. x, we have

$$2x(a^2+u)^{-1}-x^2(a^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-y^2(b^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-z^2(c^2+u)^{-2}\cdot\frac{\partial u}{\partial x}=0$$

$$\frac{2x}{a^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}$$

$$\frac{2x}{a^2+u}=V\frac{\partial u}{\partial x} \text{ where } V=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x}{V(a^2+u)}$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{2y}{V(b^2+u)} \quad\text{and}\quad \frac{\partial u}{\partial z}=\frac{2z}{V(c^2+u)}$$

$$\therefore\quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{V^2}(V)=\frac{4}{V} \qquad ...(2)$$

Now,
$$2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=2\left[\frac{2x^2}{V(a^2+u)}+\frac{2y^2}{V(b^2+u)}+\frac{2z^2}{V(c^2+u)}\right]$$

$$=\frac{4}{V}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

$$=\frac{4}{V}(1) \qquad \text{[Using (1)]}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 \qquad \text{[Using (2)]}$$

Example 13. *If* $u=lx+my$, $v=mx-ly$, *show that:*

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v=\frac{l^2}{l^2+m^2},\quad \left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u=\frac{l^2+m^2}{l^2}.$$

Sol. Given
$$u=lx+my \qquad ...(1)$$
$$v=mx-ly \qquad ...(2)$$

(i) $\left(\dfrac{\partial u}{\partial x}\right)_y$ = The partial derivative of u w.r.t. x keeping y constant.

$\therefore$ We need a relation expressing u as a function of x and y.

From (1), $$\left(\frac{\partial u}{\partial x}\right)_y = l$$

$\left(\dfrac{\partial x}{\partial u}\right)_v$ = The partial derivative of x w.r.t. u keeping v constant.

$\therefore$ We need a relation expressing x as a function of u and v.

Eliminating y between (1) and (2) by multiplying (1) by l, (2) by m and adding the products, we have

$$lu + mv = (l^2 + m^2)x \quad \text{or} \quad x = \frac{lu + mv}{l^2 + m^2}$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2}$$

Hence, $$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$$

(ii) $\left(\dfrac{\partial y}{\partial v}\right)_x$ = The partial derivative of y w.r.t. v keeping x constant.

$\therefore$ We need a relation expressing y as a function of v and x.

From (2), $$y = \frac{mx - v}{l} \qquad \therefore \quad \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{l}$$

Also $\left(\dfrac{\partial v}{\partial y}\right)_u$ = Partial derivative of v w.r.t. y keeping u constant

$\therefore$ We need a relation expressing v as a function of y and u.

Eliminating x between (1) and (2), we have $v = \dfrac{mu - (l^2 + m^2)y}{l}$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

Hence $$\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right)\left(-\frac{l^2 + m^2}{l}\right) = \frac{l^2 + m^2}{l^2}.$$

TEST YOUR KNOWLEDGE

1. Find the first order partial derivatives of the following functions:
   (i) $u = y^x$ (ii) $u = \log(x^2 + y^2)$
   (iii) $u = x^2 \sin \dfrac{y}{x}$ (iv) $u = \dfrac{x}{y} \tan^{-1}\left(\dfrac{y}{x}\right)$.
2. If $u = x^2 + y^2 + z^2$, prove that $xu_x + yu_y + zu_z = 2u$.
3. If $z = \log(x^2 + xy + y^2)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$.