Partial Differentiation

Let $z = f(x,y)$ be a function of two independent variables.

If y is kept constant and x is allowed to vary only. Then z becomes a function of x only.

The derivative of z with respect to x treating y as constant, is called partial derivative of z w.r.t. x and is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or z_x.

Thus $\frac{\partial z}{\partial x} = \underset{h \to 0}{Lt} \frac{f(x+h,y) - f(x,y)}{h}$

Similarly, the derivative of z w.r.t. y, treating x as a constant, is called partial derivative of z w.r.t. y and is denoted by $\frac{\partial z}{\partial y}$ or z_y or f_y or $\frac{\partial f}{\partial y}$

Thus $\frac{\partial z}{\partial y} = \underset{k \to 0}{Lt} \frac{f(x,y+k) - f(x,y)}{k}$

Here $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called first order partial derivatives of z.

Partial Derivatives of Higher Order

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \text{ or } f_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \text{ or } f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \text{ or } f_{xy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \text{ or } f_{yx}$$

In general; $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

i.e. $f_{xy} = f_{yx}$ (In general, but not always)

Rules :-

(1) If $z = u + v$ where $u = f(x,y)$, $v = \phi(x,y)$

then $\dfrac{\partial z}{\partial x} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x}$, $\dfrac{\partial z}{\partial y} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y}$

(2) If $z = uv$ where $u = f(x,y)$, $v = \phi(x,y)$

Then $\dfrac{\partial z}{\partial x} = u \cdot \dfrac{\partial v}{\partial x} + v \dfrac{\partial u}{\partial x}$

(3) If $z = \dfrac{u}{v}$, where $u = f(x,y)$, $v = \phi(x,y)$

then $\frac{\partial z}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

and $\frac{\partial z}{\partial y} = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$

(4) if $z = f(u)$ where $u = \phi(x,y)$

then $\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$; $\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$

Practice Questions:-

(1) Find the partial derivative of the following.

(i) $u = \tan^{-1}\frac{(x^2+y^2)}{x+y}$

Sol:- Now we know that $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

Here $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ $\left[\because \frac{d}{dx}\left[\tan^{-1}(f(x))\right] = \frac{1 \cdot \frac{d}{dx}(f(x))}{1+(f(x))^2}\right]$

So, $\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} \cdot \frac{\partial}{\partial x}\left[\frac{x^2+y^2}{x+y}\right]$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\cancel{(x+y)^2}}{(x+y)^2+(x^2+y^2)^2}\left[\frac{(x+y)(2x)-(x^2+y^2)(1)}{\cancel{(x+y)^2}}\right]$

$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2+(x^2+y^2)} = \frac{x^2-y^2+2xy}{(x+y)^2+(x^2+y^2)}$

Since u is symmetrical function of x and y.

So, if $\frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}$

similarly $\frac{\partial u}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}$

② Example :- If $u = x^y$,

show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Soln :- If $u = x^y$

$$\frac{\partial u}{\partial y} = x^y . \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y . \log x)$$

$$= y x^{y-1} . \log x + x^y . \frac{1}{x}$$

$$= y x^{y-1} . \log x + x^{y-1}$$

$$= x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x^{y-1}(y \log x + 1)\right] \quad \text{—— 1}$$

$= x^{y-1}\left(y\cdot\frac{1}{x}\right) + (y\log x+1).(y-1)x^{y-2}$ (14)

$= yx^{y-2} + (y-1)x^{y-2} + y(y-1).\log x.\,x^{y-2}$

$= (2y-1)x^{y-2} + y(y-1)\log x\,.x^{y-2}$

Now $\frac{\partial u}{\partial x} = yx^{y-1}$

$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial}{\partial y}(yx^{y-1})$

$= x^{y-1} + yx^{y-1}\log x$

$= x^{y-1}(1+y\log x)$

Now $\frac{\partial^3 u}{\partial x\partial y\partial x} = \frac{\partial}{\partial x}\left[x^{y-1}(1+y\log x)\right]$ — (2)

which is equal to (1) equ.

So, from (1) and (2)

$$\frac{\partial^3 u}{\partial x^2\partial y} = \frac{\partial^3 u}{\partial x\partial y\partial x}$$

(3) If $\theta = t^n.e^{\frac{-r^2}{4t}}$, find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

Solu.:- $\theta = t^n.e^{\frac{-r^2}{4t}}$

$\frac{\partial\theta}{\partial r} = t^n.\,e^{-\frac{r^2}{4t}}\left(\frac{-2r}{4t}\right) = -\frac{1}{2}rt^{n-1}.e^{\frac{-r^2}{4t}}$

$$r^2\frac{\partial v}{\partial r} = \frac{-1}{2} r^3 t^{n-1} e^{\frac{-r^2}{4t}}$$ (15)

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = \frac{-1}{2}t^{n-1}\left[\frac{\partial}{\partial r}\left(r^3 \cdot e^{\frac{-r^2}{4t}}\right)\right]$$

$$= \frac{-1}{2}t^{n-1}\left[3r^2 \cdot e^{\frac{-r^2}{4t}} + r^3 \cdot e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)\right]$$

$$= \frac{-1}{2}t^{n-1}\left[e^{\frac{-r^2}{4t}}\right]\left[3r^2 - \frac{r^4}{2t}\right]$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = \frac{1}{2}t^{n-1} \cdot e^{\frac{-r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) \quad (1)$$

$$\frac{\partial v}{\partial t} = nt^{n-1} \cdot e^{\frac{-r^2}{4t}} + t^n \cdot e^{\frac{-r^2}{4t}}\left(\frac{r^2}{4t^2}\right)$$

$$\Rightarrow \frac{\partial v}{\partial t} = t^{n-1} \cdot e^{\frac{-r^2}{4t}}\left(n + \frac{r^2}{4t}\right) \quad (2)$$

By given $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = \frac{\partial v}{\partial t}$

So, from (1) and (2)

$$\frac{1}{2}t^{n-1} e^{\frac{-r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) = t^{n-1} \cdot e^{\frac{-r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{r^2}{2t} - 3\right) = \left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \cancel{\frac{r^2}{4t}} - \frac{3}{2} = n + \cancel{\frac{r^2}{4t}}$$

$$\Rightarrow n = -3/2$$