(i) $\left(\frac{\partial u}{\partial x}\right)_y$ = The partial derivative of u w.r.t. x keeping y constant.

∴ We need a relation expressing u as a function of x and y.

From (1), $$\left(\frac{\partial u}{\partial x}\right)_y = l$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \text{The partial derivative of } x \text{ w.r.t. } u \text{ keeping } v \text{ constant.}$$

∴ We need a relation expressing x as a function of u and v.

Eliminating y between (1) and (2) by multiplying (1) by l, (2) by m and adding the products, we have

$$lu + mv = (l^2 + m^2)x \quad \text{or} \quad x = \frac{lu + mv}{l^2 + m^2}$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2}$$

Hence, $$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$$

(ii) $\left(\frac{\partial y}{\partial v}\right)_x$ = The partial derivative of y w.r.t. v keeping x constant.

∴ We need a relation expressing y as a function of v and x.

From (2), $$y = \frac{mx - v}{l} \qquad \therefore \quad \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{l}$$

Also $\left(\frac{\partial v}{\partial y}\right)_u$ = Partial derivative of v w.r.t. y keeping u constant

∴ We need a relation expressing v as a function of y and u.

Eliminating x between (1) and (2), we have $v = \dfrac{mu - (l^2 + m^2)y}{l}$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

Hence $$\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right)\left(-\frac{l^2 + m^2}{l}\right) = \frac{l^2 + m^2}{l^2}.$$

TEST YOUR KNOWLEDGE

1. Find the first order partial derivatives of the following functions:
   (i) $u = y^x$ (ii) $u = \log (x^2 + y^2)$
   (iii) $u = x^2 \sin \dfrac{y}{x}$ (iv) $u = \dfrac{x}{y} \tan^{-1}\left(\dfrac{y}{x}\right)$.
2. If $u = x^2 + y^2 + z^2$, prove that $xu_x + yu_y + zu_z = 2u$.
3. If $z = \log (x^2 + xy + y^2)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$.

4. If $u = x^2y + y^2z + z^2x$, prove that $u_x + u_y + u_z = (x + y + z)^2$.

5. If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

6. If $f(x, y) = x^3y - xy^3$, find $\left[\dfrac{1}{\frac{\partial f}{\partial x}} + \dfrac{1}{\frac{\partial f}{\partial y}}\right]_{\substack{x=1\\y=2}}$.

7. (*i*) If $u = \log(\tan x + \tan y)$, prove that $\sin 2x\dfrac{\partial u}{\partial x} + \sin 2y\dfrac{\partial u}{\partial y} = 2$.

(*ii*) If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x\dfrac{\partial u}{\partial x} + \sin 2y\dfrac{\partial u}{\partial y} + \sin 2z\dfrac{\partial u}{\partial z} = 2$.

(*iii*) Find first and second order derivatives from the relation $\log z = x + y + z$.

8. If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $f_x + f_y + f_z = 0$.

9. Verify that $\dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{\partial^2 u}{\partial y\partial x}$ for the following functions:

(*i*) $u = ax^2 + 2hxy + by^2$ (*ii*) $u = \tan^{-1}\left(\dfrac{x}{y}\right)$ (*iii*) $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$

(*iv*) $u = e^{ax}\sin by$ (*v*) $u = \log(x\sin y + y\sin x)$.

10. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$; where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$, $s = \dfrac{\partial^2 z}{\partial x\partial y}$.

11. If $u = \tan^{-1}\dfrac{xy}{\sqrt{1 + x^2 + y^2}}$, show that $\dfrac{\partial^2 u}{\partial x\partial y} = (1 + x^2 + y^2)^{-3/2}$.

12. If $u = e^{xyz}$, prove that $\dfrac{\partial^3 u}{\partial x\partial y\partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

13. If $u = \log(x^2 + y^2) + \tan^{-1}\dfrac{y}{x}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = \tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

15. Verify that $f_{xy} = f_{yx}$ when f is equal to

(*i*) $\sin^{-1}\left(\dfrac{y}{x}\right)$ (*ii*) $\log x\tan^{-1}(x^2 + y^2)$.

16. Find the value of n so that the equation $V = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

17. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$.

18. If $V = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$.

19. If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that $V_{xx} + V_{yy} + V_{zz} = m(m + 1)r^{m-2}$.

20. If $u = \log(x^2 + y^2 + z^2)$, prove that $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$.

21. If $u = \log\sqrt{x^2 + y^2 + z^2}$, prove that $(x^2 + y^2 + z^2)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 1$.

22. If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, find the value of $\dfrac{\partial^2 u}{\partial x \partial y}$.

23. If $x^2 + y^2 + z^2 = \dfrac{1}{u^2}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

24. If $u = \sqrt{x^2 + y^2 + z^2}$, show that

(i) $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2 = 1$ (ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$.

25. If $u = e^{x-at}\cos(x - at)$, show that $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

26. If $v = \dfrac{1}{\sqrt{t}} e^{\frac{-x^2}{4a^2 t}}$, prove that $\dfrac{\partial v}{\partial t} = a^2\dfrac{\partial^2 v}{\partial x^2}$.

27. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = y^2 u^3$.

28. If $u = e^x(x\cos y - y\sin y)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

29. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\dfrac{\partial^2 r}{\partial x^2}\cdot\dfrac{\partial^2 r}{\partial y^2} = \left(\dfrac{\partial^2 r}{\partial x \partial y}\right)^2$ (ii) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$

(iii) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$.

30. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2} = \left(\dfrac{\partial v}{\partial y}\right)_x \cdot \left(\dfrac{\partial y}{\partial v}\right)_u$.

31. Show that $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$, where $z = xf(x + y) + yg(x + y)$.

32. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$, prove that $\dfrac{\partial}{\partial y}\left(u\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(u\dfrac{\partial v}{\partial y}\right)$.

[**Hint.** Given $u = f(z)$, $v = \phi(z)$, where $z = ax^2 + 2hxy + by^2$.

We have to prove that

$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial x \partial y} \quad \text{or} \quad \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} \qquad \left(\because \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}\right)$$

or $$\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} : \frac{\partial v}{\partial y}\Bigg]$$

33. If $u = \log (x^3 + y^3 - x^2y - xy^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = -4(x+y)^{-2}$.

[**Hint.** $u = \log \{x^2(x-y) - y^2(x-y)\} = \log (x-y)(x^2-y^2) = \log (x-y)^2(x+y)$
$= 2 \log (x-y) + \log (x+y)$]

34. (*a*) If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

(*b*) If $V = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $V_{xx} + V_{yy} + V_{zz} = f''(r) + \dfrac{2}{r} f'(r)$.

35. If $z = f(x + ay) + \phi(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

36. Find p and q, if $x = \sqrt{a}\ (\sin u + \cos v)$, $y = \sqrt{a}\ (\cos u - \sin v)$, $z = 1 + \sin (u - v)$

where p and q mean $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ respectively.

$\left[\textbf{Hint. } x^2 + y^2 = 2az, \quad \therefore \quad z = \dfrac{x^2 + y^2}{2a}\right]$

37. The equation $\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation.

Show that if $u = Ae^{-gx} \sin (nt - gx)$, where A, g, n are positive constants then $g = \sqrt{\dfrac{n}{2\mu}}$.

Answers

1. (*i*) $y^x \log y,\ xy^{x-1}$ (*ii*) $\dfrac{2x}{x^2 + y^2}, \dfrac{2y}{x^2 + y^2}$

(*iii*) $2x \sin \dfrac{y}{x} - y \cos \dfrac{y}{x},\ x \cos \dfrac{y}{x}$ (*iv*) $\dfrac{-x}{x^2 + y^2} + \dfrac{1}{y} \tan^{-1} \dfrac{y}{x}, \dfrac{x^2}{y(x^2 + y^2)} - \dfrac{x}{y^2} \tan^{-1} \dfrac{y}{x}$

6. $-\dfrac{13}{22}$ 7. (*iii*) $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = \dfrac{z}{1-z}; \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{z}{(1-z)^3}$

16. 2, –3 22. $\dfrac{x^2 - y^2}{x^2 + y^2}$ 36. $p = \dfrac{x}{a}, q = \dfrac{y}{a}$.

2.3 HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\dfrac{y}{x}\right)$ or $y^n \phi\left(\dfrac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$