5. CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function $f(x, y, z)$ is said to be continuous at the point (a, b, c) if

$$\underset{(x, y, z)\to(a, b, c)}{\text{Lt}} f(x, y, z) \text{ exists and } = f(a, b, c).$$

Thus $f(x, y, z)$ is said to be continuous at the point (a, b, c) if given $\varepsilon > 0$, there exists a
al number $\delta > 0$ such that $|f(x, y, z) - f(a, b, c)| < \varepsilon$ for $|(x, y, z) - (a, b, c)| < \delta$.

ILLUSTRATIVE EXAMPLES

Example 1. *Let $f : R^2 \to R$ be defined as $f(x, y) = x^2 + y^2$.*

Show that $\underset{(x, y)\to(0,0)}{Lt} f(x, y) = 0$

Sol. Let $\varepsilon > 0$ be given

$$|f(x, y) - 0| = |x^2 + y^2| = x^2 + y^2 < \varepsilon$$

henever $\sqrt{x^2 + y^2} < \sqrt{\varepsilon}$

e., whenever $|(x, y) - (0, 0)| < \delta$ where $\delta = \sqrt{\varepsilon}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta(= \sqrt{\varepsilon}) > 0$ such that

$$|f(x, y) - 0| < \varepsilon \text{ whenever } |(x, y) - (0, 0)| < \delta$$

Hence by definition of limit, $\underset{(x, y)\to(0, 0)}{\text{Lt}} f(x, y) = 0$

Example 2. *Let $f : R^3 \to R$ be defined by $f(x, y, z) = x^2 + y^2 + z^2$*

Show that $\underset{(x, y, z)\to(0,0,0)}{Lt} f(x, y, z) = 0.$

Sol. Let $\varepsilon > 0$ be given

$$|f(x, y, z) - 0| = |x^2 + y^2 + z^2|$$
$$= x^2 + y^2 + z^2 < \varepsilon \text{ whenever } \sqrt{x^2 + y^2 + z^2} < \sqrt{\varepsilon}$$

i.e., whenever $|(x, y, z) - (0, 0, 0)| < \delta$ where $\delta = \sqrt{\varepsilon}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta(= \sqrt{\varepsilon}) > 0$ such that $|f(x, y, z) - 0| < \varepsilon$
whenever $|(x, y, z) - (0, 0, 0)| < \delta$.

Hence by definition of limit, $\underset{(x, y, z)\to(0, 0, 0)}{\text{Lt}} f(x, y, z) = 0.$

Example 3. *Let $A = \{(x, y) : 0 < x < 1, 0 < y < 1, x, y \in R\}$. Let $f : A \to R$ defined by $f(x, y)$ $= x + y$. Show that*

$$\underset{\substack{(x, y)\to\left(0, \frac{1}{2}\right) \\ (x, y)\in A}}{Lt} f(x, y) = \frac{1}{2}.$$

Sol. Let $\varepsilon > 0$ be given.

$$\left|f(x, y) - \frac{1}{2}\right| = \left|x + y - \frac{1}{2}\right|$$

$$= \left| x + \left(y - \frac{1}{2}\right) \right| \le |x| + \left| y - \frac{1}{2} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $\quad |x| < \frac{\varepsilon}{2}$ and $\left| y - \frac{1}{2} \right| < \frac{\varepsilon}{2}$

i.e., whenever $|x - 0| < \delta$ and $\left| y - \frac{1}{2} \right| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta \left(= \frac{\varepsilon}{2} \right) > 0$ such that

$$\left| f(x, y) - \frac{1}{2} \right| < \varepsilon \text{ whenever } |x - 0| < \delta \text{ and } \left| y - \frac{1}{2} \right| < \delta$$

Hence by definition of limit, $\displaystyle \underset{(x,y)\to\left(0,\frac{1}{2}\right)}{\text{Lt}} f(x, y) = \frac{1}{2}$.

Example 4. *Let $f(x, y) = x + y$. Show that $f(x, y)$ is continuous at $\left(\frac{1}{2}, \frac{1}{3}\right)$.*

Sol. $\quad f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

Let $\varepsilon > 0$ be given.

$$\left| f(x, y) - f\left(\frac{1}{2}, \frac{1}{3}\right) \right| = \left| (x + y) - \left(\frac{1}{2} + \frac{1}{3}\right) \right|$$

$$= \left| \left(x - \frac{1}{2}\right) + \left(y - \frac{1}{3}\right) \right| \le \left| x - \frac{1}{2} \right| + \left| y - \frac{1}{3} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $\quad \left| x - \frac{1}{2} \right| < \frac{\varepsilon}{2}$ and $\left| y - \frac{1}{3} \right| < \frac{\varepsilon}{2}$

i.e., whenever $\left| x - \frac{1}{2} \right| < \delta$ and $\left| y - \frac{1}{3} \right| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta \left(= \frac{\varepsilon}{2} \right) > 0$ such that

$$\left| f(x, y) - f\left(\frac{1}{2}, \frac{1}{3}\right) \right| < \varepsilon \text{ whenever } \left| x - \frac{1}{2} \right| < \delta \text{ and } \left| y - \frac{1}{3} \right| < \delta$$

Hence by definition of continuity, $f(x, y)$ is continuous at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Example 5. *Let $f : R^2 \to R$ be defined by*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Prove that $\underset{(x, y)\to(0,0)}{Lt} f(x, y)$ *does not exist.*

Sol. We know that if $\underset{(x, y)\to(a, b)}{\text{Lt}} f(x, y)$ exists, then this limit is independent of the path along which (x, y) approaches the point (a, b).

Here, let $(x, y) \to (0, 0)$ along the path $y = mx$ where m is any real number.

As $x \to 0$, from $y = mx$, we have $y \to 0$.

Now $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = \underset{(x,y)\to(0,0)}{\text{Lt}} \dfrac{xy}{x^2 + y^2}$ (Putting $y = mx$)

$$= \underset{x\to 0}{\text{Lt}} \frac{x \,.\, mx}{x^2 + m^2x^2} = \underset{x\to 0}{\text{Lt}} \frac{mx^2}{x^2\,(1 + m^2)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

which is different for different values of m.

$\therefore \underset{(x,y)\to(0,0)}{\text{Lt}} f(x)$ does not exist.

Example 6. *Prove that* $\underset{(x,y)\to(0,0)}{Lt} f(x, y)$ *does not exist, where*

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \; (x, y) \neq (0, 0).$$

Sol. We know that if $\underset{(x,y)\to(a,b)}{\text{Lt}} f(x, y)$ exists, then this limit is independent of the path along which (x, y) approaches the point (a, b).

Here, let $(x, y) \to (0, 0)$ along the path $y = mx$ where m is any real number.

As $x \to 0$, from $y = mx$, we have $y \to 0$

Now $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = \underset{(x,y)\to(0,0)}{\text{Lt}} \dfrac{x^2 - y^2}{x^2 + y^2}$ (Putting $y = mx$)

$$= \underset{x\to 0}{\text{Lt}} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \underset{x\to 0}{\text{Lt}} \frac{x^2\,(1 - m^2)}{x^2\,(1 + m^2)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1 - m^2}{1 + m^2} = \frac{1 - m^2}{1 + m^2}$$

which is different for different values of m.

$\therefore \underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y)$ does not exist.

Example 7. *Prove that* $\underset{(x,y)\to(0,0)}{Lt} f(x, y)$ *does not exist, where*

$$f(x, y) = \frac{xy^2}{x^2 + y^4}, \ (x, y) \neq (0, 0).$$

Sol. Let $(x, y) \to (0, 0)$ along the path $y = m\sqrt{x}$.

As $x \to 0$, from $y = m\sqrt{x}$, we have $y \to 0$.

$$\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = \underset{(x,y)\to(0,0)}{\text{Lt}} \frac{xy^2}{x^2 + y^4} \qquad \text{(Putting } y = m\sqrt{x}\text{)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x \cdot m^2 x}{x^2 + m^4 x^2} = \underset{x\to 0}{\text{Lt}} \frac{m^2 x^2}{x^2(1 + m^4)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{m^2}{1 + m^4} = \frac{m^2}{1 + m^4}$$

which is different for different values of m.

$\therefore \quad \underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y)$ does not exist.

Example 8. *Prove that* $\underset{(x,y)\to(0,0)}{Lt} \dfrac{x^2 y}{x^4 + y^2}$ *does not exist.*

Sol. Let $(x, y) \to (0, 0)$ along the path $x = m\sqrt{y}$

As $y \to 0$, from $x = m\sqrt{y}$, we have $x \to 0$.

$$\therefore \quad \underset{(x,y)\to(0,0)}{\text{Lt}} \frac{x^2 y}{x^4 + y^2} = \underset{y\to 0}{\text{Lt}} \frac{m^2 y \cdot y}{m^4 y^2 + y^2} = \underset{y\to 0}{\text{Lt}} \frac{m^2 y^2}{y^2 (m^4 + 1)}$$

$$= \underset{y\to 0}{\text{Lt}} \frac{m^2}{m^4 + 1} = \frac{m^2}{m^4 + 1}$$

which is different for different values of m.

$\therefore \quad \underset{(x,y)\to(0,0)}{\text{Lt}} \dfrac{x^2 y}{x^4 + y^2}$ does not exist.

Example 9. *Let* $f(x, y) = y \sin \dfrac{1}{x} + x \sin \dfrac{1}{y}$, *where* $x \neq 0, y \neq 0$. *Prove that* $f(x, y) \to 0$ *as* $(x, y) \to (0, 0)$.

Sol. Let $\varepsilon > 0$ be given.

$$|f(x, y) - 0| = \left| y \sin \frac{1}{x} + x \sin \frac{1}{y} \right| \leq \left| y \sin \frac{1}{x} \right| + \left| x \sin \frac{1}{y} \right|$$

$$= |y| \left| \sin \frac{1}{x} \right| + |x| \left| \sin \frac{1}{y} \right|$$

$$\leq |y| + |x| \qquad \left[\because \left|\sin\frac{1}{x}\right| \leq 1 \text{ and } \left|\sin\frac{1}{y}\right| \leq 1\right]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$

i.e., whenever $|x| < \delta$ and $|y| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\left(=\frac{\varepsilon}{2}\right) > 0$ such that $|f(x, y) - 0| < \varepsilon$ whenever $|x - 0| < \delta$ and $|y - 0| < \delta$. Hence by definition of limit,

$$\underset{(x,y)\to(0,0)}{\text{Lt}} f(x, y) = 0.$$

Example 10. *Let $A = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ and $f : A \to R$ be defined by $f(x, y) = x + y$. Prove that f is continuous at every point of the domain A.*

Sol. Let (α, β) be any point of A.

Let us prove that $f(x, y)$ is continuous at (α, β)

i.e.,
$$\underset{(x,y)\to(\alpha,\beta)}{\text{Lt}} f(x, y) = f(\alpha, \beta)$$

Let $\varepsilon > 0$ be given

$$|f(x, y) - f(\alpha, \beta)| = |(x + y) - (\alpha + \beta)| = |(x - \alpha) + (y - \beta)|$$
$$\leq |x - \alpha| + |y - \beta|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|x - \alpha| < \frac{\varepsilon}{2}$ and $|y - \beta| < \frac{\varepsilon}{2}$

i.e., whenever $|x - \alpha| < \delta$ and $|y - \beta| < \delta$ where $\delta = \frac{\varepsilon}{2}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\left(=\frac{\varepsilon}{2}\right) > 0$ such that

$$|f(x, y) - f(\alpha, \beta)| < \varepsilon \text{ whenever } |x - \alpha| < \delta \text{ and } |y - \beta| < \delta.$$

Hence by definition of continuity, $f(x, y)$ is continuous at (α, β). Since (α, β) is any point of A, therefore, f is continuous at every point of A.

Example 11. *Show that the function $f : R^2 \to R$ defined by*

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right), & (x, y) \neq (0, 0) \\ 0, & \text{Otherwise} \end{cases}$$

is continuous at (0, 0).

Sol. Let $\varepsilon > 0$ be given.

$$|f(x, y) - f(0, 0)| = \left| xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right) - 0 \right|$$

$$= |xy| \left| \frac{x^2 - y^2}{x^2 + y^2} \right|$$

$$\leq |xy| \qquad \left[\because |x^2 - y^2| \leq |x^2 + y^2| \therefore \left|\frac{x^2 - y^2}{x^2 + y^2}\right| \leq 1\right]$$

or $\qquad [f(x, y) - f(0, 0)] \leq |x|\,|y|$

$$< \sqrt{\varepsilon} \times \sqrt{\varepsilon} = \varepsilon$$

whenever $\quad |x| < \sqrt{\varepsilon}$ and $|y| < \sqrt{\varepsilon}$

i.e., $\quad$ whenever $|x| < \delta$ and $|y| < \delta$ where $\delta = \sqrt{\varepsilon}$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\ (= \sqrt{\varepsilon}) > 0$ such that $|f(x, y) - f(0, 0)| < \varepsilon$ whenever $|x - 0| < \delta$ and $|y - 0| < \delta$.

Hence by definition of continuity, $f(x, y)$ is continuous at $(0, 0)$.

Example 12. *Let $f(x, y) = \sqrt{|xy|}$. Show that $f(x, y)$ is continuous at the origin.*

Sol. Let $\varepsilon > 0$ be given

$$|f(x, y) - f(0, 0)| = |\sqrt{|xy|} - 0| = \sqrt{|xy|} = \sqrt{|x||y|} = \sqrt{|x|} \cdot \sqrt{|y|} < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$$

whenever $\qquad \sqrt{|x|} < \sqrt{\varepsilon}$ and $\sqrt{|y|} < \sqrt{\varepsilon}$

i.e., $\quad$ whenever $|x| < \varepsilon$ and $|y| < \varepsilon$

i.e., $\quad$ whenever $|x - 0| < \delta$ and $|y - 0| < \delta$ where $\delta = \varepsilon$

$\therefore$ For every $\varepsilon > 0$, there exists $\delta\ (= \varepsilon) > 0$ such that

$$|f(x, y) - f(0, 0)| < \varepsilon \text{ whenever } |x - 0| < \delta \text{ and } |y - 0| < \delta.$$

Hence by definition of continuity, $f(x, y)$ is continuous at $(0, 0)$ the origin.

Example 13. *Let* $\phi(y, z) = \dfrac{yz}{\sqrt{y^2 + z^2}},\ (y, z) \neq (0, 0)$

$$= 0, \text{ when } (y, z) = (0, 0).$$

Show that $\phi(y, z)$ is continuous at $(0, 0)$.

Sol. Let $\varepsilon > 0$ be given.

$$|\phi(y, z) - \phi(0, 0)| = \left| \frac{yz}{\sqrt{y^2 + z^2}} - 0 \right| = \left| \frac{yz}{\sqrt{y^2 + z^2}} \right|$$

(Put $y = r\cos\theta$ and $z = r\sin\theta$ so that $y^2 + z^2 = r^2$)

$$= \left| \frac{r\cos\theta \cdot r\sin\theta}{r} \right| = \left| \frac{1}{2} r \sin 2\theta \right| = \frac{1}{2} r\,|\sin 2\theta| \leq \frac{1}{2} r < r$$

$$|\phi(y, z) - \phi(0, 0)| < \sqrt{y^2 + z^2} < \varepsilon$$

or

whenever $$\sqrt{y^2 + z^2} < \varepsilon$$

i.e., whenever $|(y, z) - (0, 0)| < \delta$ where $\delta = \varepsilon$

$\therefore$ For ever y $\varepsilon > 0$, there exists $\delta\ (= \varepsilon) > 0$ such that

$|\phi(y, z) - \phi(0, 0)| < \varepsilon$ whenever $|(y, z) - (0, 0)| < \delta$.

Hence by definition of continuity, $\phi\ (y, z)$ is continuous at (0, 0).

EXERCISE 9

1. Let $f(x, y) = x^2 + y^2$. Show that $f(x, y)$ is continuous at the origin.
2. Let $f(x, y, z) = x^2 + y^2 + z^2$. Show that $f(x, y, z)$ is continuous at the origin.
3. Let $f : R^2 \to R$ be defined as

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that f is not continuous at (0, 0).

4. Show that $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$

is not continuous at (0, 0).

5. Prove that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist, where

$$f(x, y) = \frac{2xy}{x^2 + y^2},\ (x, y) \neq (0, 0)$$

6. Show that the following functions are discontinuous at the origin.

(*i*) $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^4 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$

(*ii*) $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0 \quad , & \text{if } (x, y) = (0, 0) \end{cases}$