# 6

# Projections

## 6.1. INTRODUCTION

Projection transform a point in a coordinate system of dimensions n into another point in a coordinate system of dimensions less than *n*. We will consider the projection of objects from 3-D to 2-D only.

A categorical classification of different types of projection commonly taken into consideration is shown below.

or 172 difference

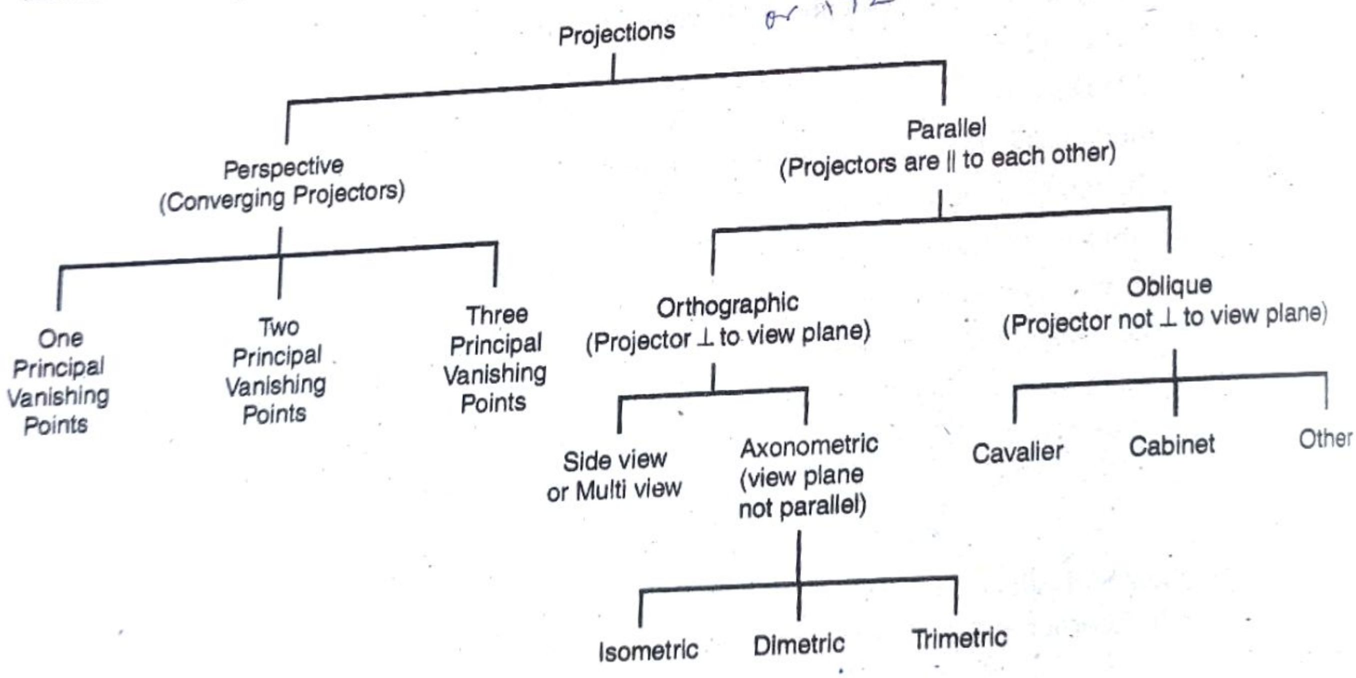


*Fig. 6.1. Classification of projection schemes.*

The projection of a 3-D object is defined by straight lines called projection rays or projectors emanating from the centre of projection, passing through each point of the object and intersecting a projection plane or view plane to form the projected view. In case of parallel projection, the distance between the view plane and the centre of projection is infinity. In perspective projection, this distance is finite and hence perspective foreshortening occurs. These two situations are depicted in Fig. 6.2. and 6.3. respectively.

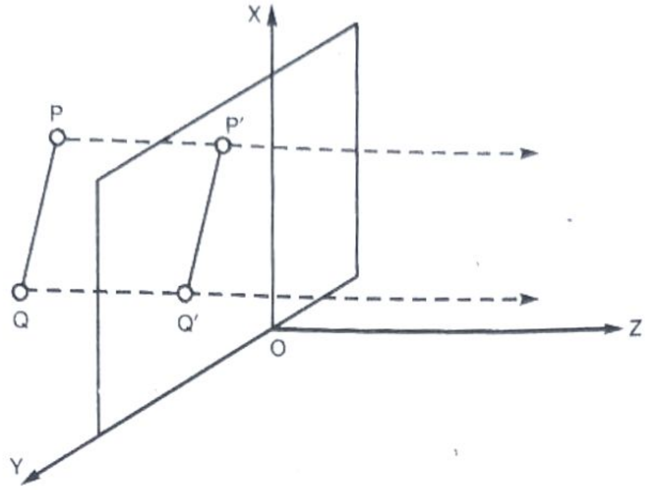


*Fig. 6.2. Parallel projection.*

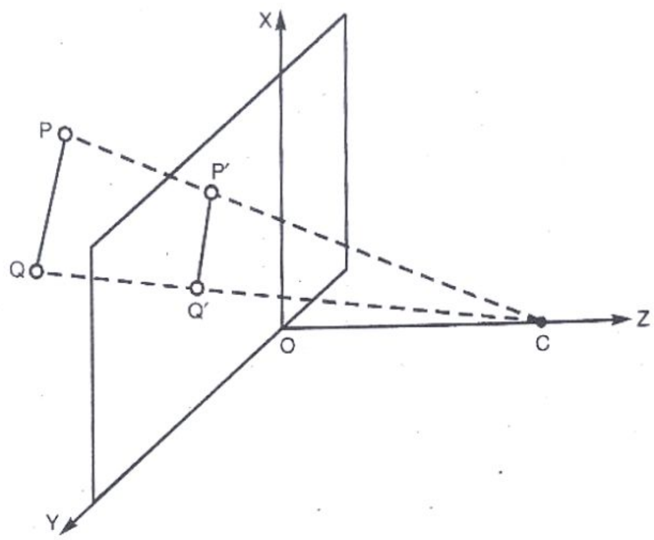


*Fig. 6.3. Perspective projection.*

## 6.2. PARALLEL PROJECTION

It is already mentioned that in parallel projection, the viewpoint is at an infinite distance from the object, and hence all the projectors are parallel to each other. Therefore, the object and its projected view are of the same size. Although infinite distance between the object and the viewer is impractical, we can derive a parallel transformation matrix assuming parallel projectors at the viewpoint.

Let the $X$-$Y$ plane at $z = 0$ is the view plane and the direction of projection is specified by the direction vector $[x_p, y_p, z_p] = xp\mathbf{i} + yp\mathbf{j} + zp\mathbf{k}$. Let $(x, y, z)$ be a point on the object and $(x', y', z')$ be its projected view. The parametric equations of the line passing through $(x, y, z)$ are as follows :

$$x(u) = x + xpu$$
$$y(u) = y + ypu \qquad \text{...(6.1)}$$
$$z(u) = z + zpu$$

This line intersects the $X$-$Y$ plane at $z = 0$. Therefore,

$$u = -\frac{z}{z_p} \qquad \text{...(6.2)}$$

and

$$x' = x - \left(\frac{x_p}{z_p}\right) z \qquad \text{...(6.3)}$$

$$y' = y - \left(\frac{y_p}{z_p}\right) z$$

In a homogeneous coordinate representation, we have

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{x_p}{z_p} & 0 \\ 0 & 1 & -\frac{y_p}{z_p} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \text{...(6.4)}$$

Since $z' = 0$ on the plane of projection, we can rewrite the above transformation matrix as

$$T_{\text{parallel}} = \begin{bmatrix} 1 & 0 & -\frac{x_p}{z_p} & 0 \\ 0 & 1 & -\frac{y_p}{z_p} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{...(6.5)}$$

**Example 6.1**

*Derive the equation of parallel projection onto the xy-plane in the direction of projection $V = aI + bJ + cK$.*

**Solution.**

From Fig. 6.4 we see that the vectors $V$ and $\overline{PP'}$ have the same direction. This means that $\overline{PP'} = kV$. Comparing components, we see that

$$x' - x = ka \qquad y' - y = kb \qquad z' - z = kc$$

So $\quad k = -\frac{z}{c} \qquad x' = x - \frac{a}{c}z \qquad$ and $\quad y' = y - \frac{b}{c}z$

In $3 \times 3$ matrix form, this is

$$
\mathrm{Par}_V = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ & \left[\begin{array}{ccc} 1 & 0 & -\dfrac{a}{c} \\ 0 & 1 & -\dfrac{b}{c} \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} x \\ y \\ z \end{array} \end{array}
$$

and so

$$P' = \mathrm{Par}_V \cdot P$$

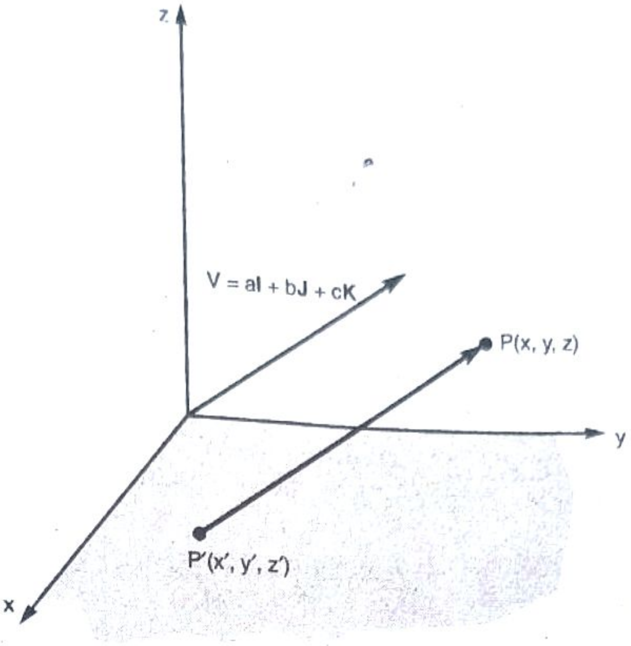


*Fig. 6.4.*

**Example 6.2**

*Derive the general equation of parallel projection onto a given view plane in the direction of a given projector V (see Fig. 6.5).*

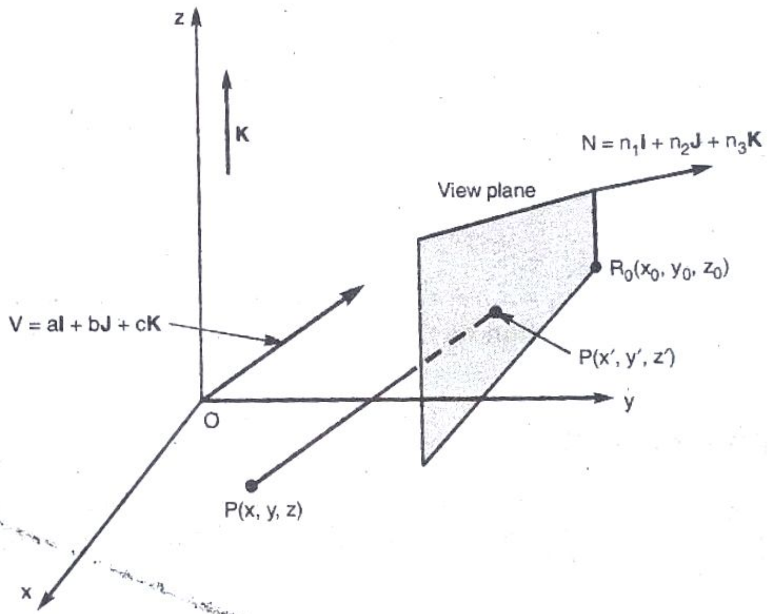


*Fig. 6.5.*

**Solution.**

We reduce the problem to parallel projection onto the $xy$-plane in the direction of the projector $V = aI + bJ + cK$ by means of these steps :

1. Translate the view reference point $R_0$ of the view plane to the origin using the translation matrix $T - R_0$.
2. Perform an alignment transformation $A_N$ so that the view normal vector $N$ of the view plane points in the direction $K$ of the normal to the $xy$-plane. The direction of projection vector $V$ is transformed to a new vector $V' = A_N V$.
3. Project onto the xy-plane using $Par_{V'}$.
4. Perform the inverse of steps 2 and 1. So finally $Par_{V.N.R_0} = T_{-R_0}^{-1} . A_N^{-1} . Par_{V'} . A_N . T_{-R_0}$. From what we learned in Chapter-6, we know that

$$T_{-R_0} = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\lambda = \sqrt{n_2^2 + n_3^2}$ and $\lambda \neq 0$, that

$$A_N = \begin{bmatrix} \frac{\lambda}{|\mathbf{N}|} & \frac{-n_1 n_2}{\lambda |\mathbf{N}|} & \frac{-n_1 n_3}{\lambda |\mathbf{N}|} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{-n_2}{\lambda} & 0 \\ \frac{n_1}{|\mathbf{N}|} & \frac{n_2}{|\mathbf{N}|} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, after multiplying, we find

$$Par_{\mathbf{V.N.}R_0} = \begin{bmatrix} d_1 - an_1 & -an_2 & -an_3 & ad_0 \\ -bn_1 & d_1 - bn_2 & -bn_3 & bd_0 \\ -cn_1 & -cn_2 & d_1 - cn_3 & cd_0 \\ 0 & 0 & 0 & d_1 \end{bmatrix}$$

Here $d_0 = n_1 x_0 + n_2 y_0 + n_3 z_0$ and $d_1 = n_1 a + n_2 b + n_3 c$. An alternative and much easier method to derive this matrix is by finding the intersection of the projector through $P$ with the equation of the view plane.

### 6.2.1. Orthographic Projection

In orthographic projection, the direction of projection is perpendicular to the plane of projection and center of projection lies at infinity. Therefore, the transformation matrices, with the principal axes as the direction of projection, become

$$T_{ortho_x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ortho_z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ...(6.7)$$

$$T_{ortho_y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ...(6.8)$$

Different kinds of orthographic projections are shown in Fig. 6.6.

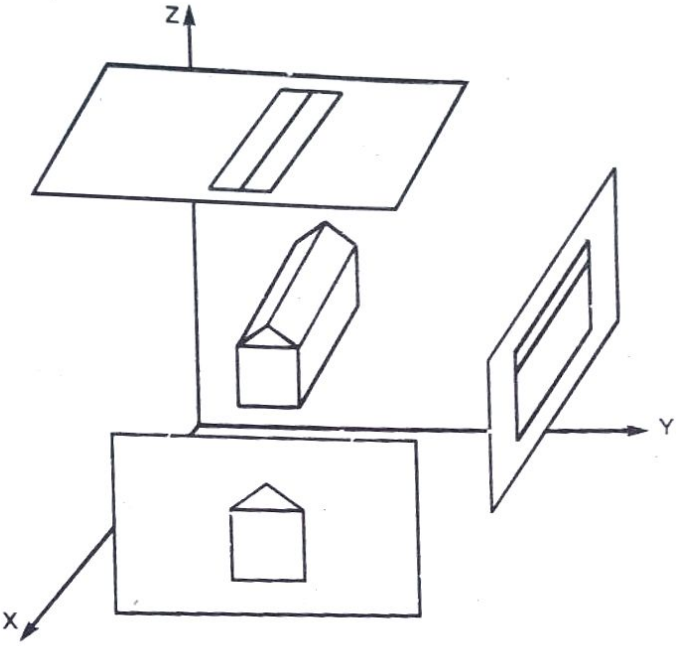


*Fig. 6.6. Orhtographic projection (side views).*

#### 6.2.1.1. Side View or Multi View

On 53 (Ignore)

In this case view planes is parallel to the object faces or side views which are projected. It creates multiple views or different sides of an object from 3D to multiple 2D images of same object.

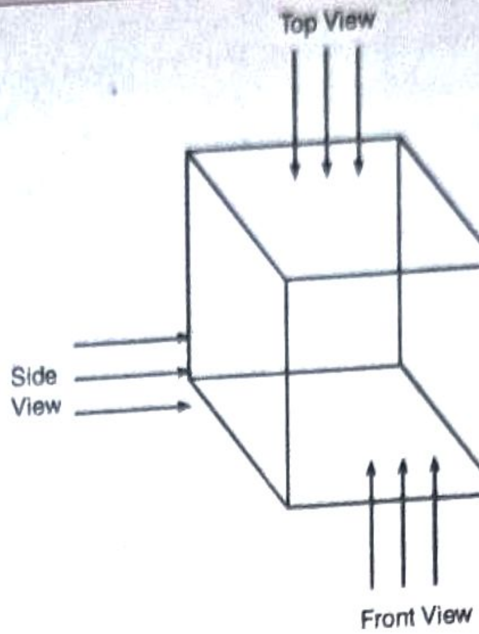


*Fig. 6.7.*

### 6.2.2. Axonometric Projection

In orthographic projection, only one of the surfaces of a 3-D object is visible and hence, in general, it is unable to show the shape of the object. But in axonometric projection, the object is rotated prior to projection so that at least three surfaces of the object become visible.

Here, all the object lines that are not parallel to the view plane are foreshortened. But relative lengths of such parallel lines remain constant, *i.e.*, a number of parallel lines are equally foreshortened.

#### *Trimetric Projection*

A trimetric projection is obtained by rotating the object arbitrarily about the coordinate axes. Here, in general, the foreshortening factors along the three principal axes are different.

Let us apply a trimetric projection over the unit vectors along the three principal axes. Let $T$ be the combined transformation matrix (combining all rotation followed by projections) and $U$ be the unit vector matrix, then the transformation can be expressed as follows :

$$T \times U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_x^* & x_y^* & x_z^* & 0 \\ y_x^* & y_y^* & y_z^* & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad ...(6.9)$$

Now, foreshortening factors along the three axes are as follows :

$$f_x = \sqrt{x_x^{*2} + y_x^{*2}}$$

$$f_y = \sqrt{x_y^{*2} + y_y^{*2}} \quad ...(6.10)$$

$$f_z = \sqrt{x_z^{*2} + y_z^{*2}}$$

All different

If rotation are performed along all the axes and then projection is taken on the X-Y plane, then the combined transformation matrix is formed as

$$T = T_{\text{orhto}_z} \times T_{\text{rot}_z} \times T_{\text{rot}_y} \times T_{\text{rot}_x}$$

Note that rotation about all the axes is not a must.

**Example 6.3**

*Consider a unit cube in the standard position with its vertices at A(0, 0, 0), B(1, 0, 0), C(1, 1, 0), D(0, 1, 0), E(0, 1, 1), F(0, 0, 1), G(1, 0, 1) and H(1, 1, 1) respectively as shown in Fig. 6.7. Perform a trimetric projection of the cube when the view direction is along the negative Z-axis. Consider the rotation angles of θ = 30° and φ = 60° about Y-and X-axes respectively.*

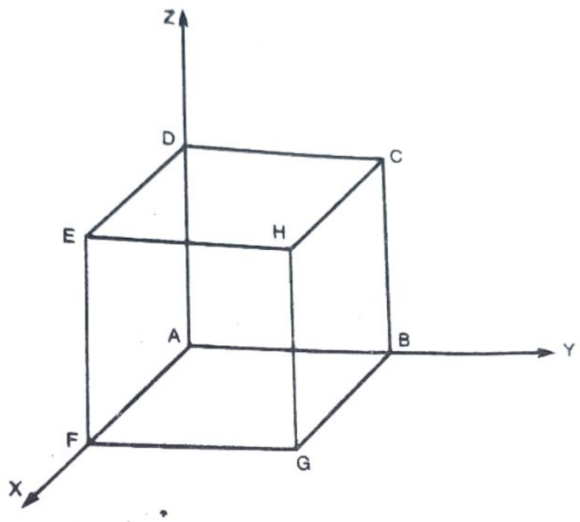


*Fig. 6.7. Unit cube in standard position.*

**Solution.** $T_{\text{trimetric}} = T_{\text{ortho}_z} \times T_{\text{rot}_x} \times T_{\text{rot}_y}$

θ → x    φ → Y

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi & 0 \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & -\cos\phi\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The vertex matrix for the cube is

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, the transformed vertex matrix is

$$P' = \text{Ttrimetric} \times P$$

or

$$P' = \begin{bmatrix} 0 & 0.5000 & 0.5000 & 0 & 0.8660 & 0.8660 & 1.3660 & 1.3660 \\ 0 & 0.4330 & 1.2990 & 0.8660 & 0.6160 & -0.2500 & 0.1830 & 1.0490 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, we can compute the foreshortening factors along different axes following Eq. (6.10).

### Dimetric Projection

A dimetric projection is a special case of trimetric projection where two of the three foreshortening factors are equal. The third is arbitrary.

Let rotations be done about $Y$-and $X$-axes, in that order, angles $\phi$ and $\theta$, and the projection is taken on the $X$-$Y$ plane. Then

$$T_{\text{dimetric}} = T_{\text{ortho}_z} \times T_{\text{rot}_x} \times T_{\text{rot}_y}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi & 0 \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\phi\sin\phi & \cos\theta & -\cos\phi\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ...(6.11)$$

Applying $T_{\text{dimetric}}$ over the unit vectors matrix $U$, we get

$$U^* = T_{\text{dimetric}} \times U$$

$$= T_{\text{dimetric}} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad ...(6.12)$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & -\cos\phi\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$f_x^2 = x_x^{*2} + y_x^{*2} = \cos^2\phi + \sin^2\phi\sin^2\theta \quad ...(6.13)$$

$$f_y^2 = x_y^{*2} + y_y^{*2} = \cos^2\theta \quad ...(6.14)$$

$$f_z^2 = x_z^{*2} + y_z^{*2} = \sin^2\phi + \cos^2\phi\sin^2\theta \quad ...(6.15)$$

Equating eqn. (6.13) and (6.14), we get

$$\sin^2\phi = \frac{\sin^2\theta}{1 - \sin^2\theta} \quad ...(6.16)$$

Using $\sin^2\phi$ from eqn. (6.16) in eqn. (6.15), we get

$$\sin^2\theta = \frac{f_z^2}{2} \quad ...(6.17)$$

and thus

$$\theta = \sin^{-1}\left(\pm\frac{f_z}{\sqrt{2}}\right) \quad ...(6.18)$$

for scale

Using $\theta$ from eqn. (6.18) in eqn. (6.15), we get

$$\phi = \sin^{-1}\left(\pm\frac{f_z}{\sqrt{2 - f_z^2}}\right) \quad ...(6.19)$$

### Isometric Projection

The isometric projection of any object is pictorial view in distorted shaped with equal shortened dimension. This means the square, rectangle and circular faces are seen as rhombus, paralleogram and ellipse.

In an isometric projection, all the three foreshortening factors are equal. Equating eqn. (6.13) and (6.14), we get,

$$\sin^2\phi = \frac{\sin^2\theta}{1-\sin^2\theta} \qquad \text{[Eqn. 6.16]}$$

Again, equating eqn. (6.14) and (6.15), we get

$$\sin^2\phi = \frac{1-2\sin^2\theta}{1-\sin^2\theta} \qquad ...(6.20)$$

From eqn. (6.16) and (6.20),

$$\sin^2\theta = \frac{1}{3}$$

*i.e.,* $$\sin\theta = \pm\sqrt{\frac{1}{3}}$$

*i.e.,* $$\theta = \pm 35.26°$$

And then, $$\sin^2\phi = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

*i.e.,* $$\phi = \pm 45°$$

All Same

The foreshortening factor is, therefore,

$$f = \sqrt{\cos^2\theta} = \sqrt{\frac{2}{3}} = 0.8165$$

### 6.2.3. Oblique Projection

For othographic and axonometric projections the projectors are perpendicular to the plane of projection. But in the case of oblique projections, projectors intersect the plane of projection at an oblique angle. The general 3-D shape of the object can be illustrated using oblique projections. However, only faces of the object that are parallel to the projection plane can be viewed in their true shape and sizes. Other faces are foreshortened differently. An oblique projection is formed by parallel projection from a center of projection at infinity that intersects the plane of projection at an oblique angle, providing a general 3D shape of object. Oblique means slanting or receding axis at an angle. There are 3 axis vertical, horizontal and oblique.

**Example 6.4**

*Find the general form of an oblique projection on to the xy-plane. (Fig. 6.8)*

**Solution.** Let $\phi$ is the angle between the projection of any line perpendicular to $XY$ plane and $X$-axis.

Let $\alpha$ is angle between vector $V$ or $PP'$ (projector) and $P'O$.

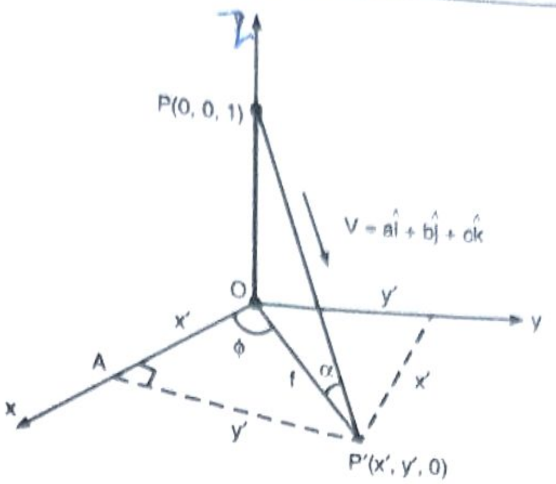


*Fig. 6.8*

Also assume if $f$ is the for-shortening factor *i.e.*, $P'O$ then

In $\Delta POP'$

$$\frac{1}{f} = \tan\alpha$$

$\Rightarrow$ $$f = \frac{1}{\tan\alpha}$$

if $$V = ai + bj + ck$$

then

$$x' - 0 = a \Rightarrow x' = a$$

$$y' - 0 = b \Rightarrow y' = b$$

$$0 - 1 = c \Rightarrow c = -1$$

In $\Delta P'AO$ $$\frac{x'}{f} = \cos\phi \quad \text{so} \quad x' = f\cos\phi = a$$

$$y' = f\sin\phi = b$$

$$\frac{y'}{f} = \sin\phi$$

Thus, the matrix for oblique projection is

$\Rightarrow$ $$\begin{pmatrix} 1 & 0 & f\cos\phi & 0 \\ 0 & 1 & f\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Althrough, by varying the angle of projection, various types of oblique projections may takes place, only two of them, viz. cavalier and cabinet are of particular interest.

### Cavalier Projection

Here, the angle between the projectors and the plane of projections is 45°, and there will be no foreshortening of lines perpendicualr to the plane of projection.

Let us consider the situation depicted in Fig. 6.9. Here, the projection of a unit vector $OP$ along the Z-axis on the X-Y plane is shown. The length of the projected view $OP' = f$. In general, let us consider the projected view $OP' = f$, which makes an angle $\alpha$ with the +ve X-axis. Then we have

$$x' = f\cos\alpha \quad \text{and} \quad y' = f\sin\alpha \qquad \ldots(6.2)$$

The general transformation matrix for parallel projection, therefore, comes out to be

$$T_{\text{parallel}} = \begin{bmatrix} 1 & 0 & -\dfrac{x_p}{z_p} & 0 \\ 0 & 1 & -\dfrac{y_p}{z_p} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots(6.2)$$

Here, $x_p = f\cos\alpha$, $y_p = f\sin\alpha$ and $z_p = 1$. Therefore, we can rewrite the matrix as

$$T_{\text{parallel}} = \begin{bmatrix} 1 & 0 & -f\cos\alpha & 0 \\ 0 & 1 & -f\sin\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots(6.2)$$

For the cavalier projection $f = 1$. In Fig. 6.9, the angle $\angle PP' = \beta$ and $\beta = \cot^{-1}(f) = 45°$. For different values of $\alpha$, we get different types of cavalier projections.

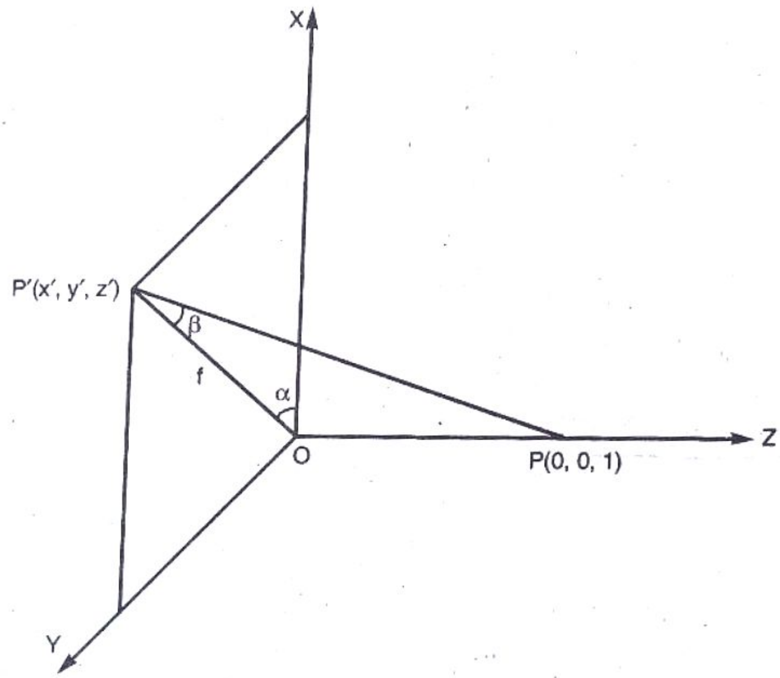


*Fig. 6.9. Cavalier Projection.*

## Cabinet Projection

For the cabinet projection, the foreshortening factor is $f = \dfrac{1}{2}$. Thus, in this case, $\beta = \cot^{-1}\left(\dfrac{1}{2}\right) = 63.435°$. That is, here the projectors make an angle of 63.435° with the view plane. As in the case of cavalier projections, here also for different values of $\alpha$, we get different kinds of cabinet projections. Generally projecting with $\alpha = 30°$ and $45°$ are considered for thorough observation.

**Example 6.5**

*Find the transformation for (a) cavalier with $\theta = 45°$ and (b) cabinet projections with $\theta = 30°$. (c) Draw the projection of the unit cube for each transformation.*

**Solution.**

(*a*) A cavalier projection is an oblique projection where there is no forehortening of lines perpendicular to they *xy*-plane. We then see that $f = 1$. With $\theta = 45°$, we have

$$Par_{V_1} = \begin{pmatrix} 1 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(*b*) A cabinet projection is an oblique projection with $f = \frac{1}{2}$. With $\theta = 30°$, we have

$$Par_{V_2} = \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To construct the projections, we represent the vertices of the unit cube by a matrix whose columns are homogeneous coordinates of the vertices.

$$V = (ABCDEFGH) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(*c*) To draw the cavalier projection, we find the image coordinates by applying the transformation matrix $Par_{V_1}$ to the coordinate matrix $V$:

$$Par_{V_1}.V = \begin{bmatrix} 0 & 1 & 1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 1 & 1+\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The image coordinates are then

$$A' = (0, 0, 0) \qquad E' = \left(\frac{\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}, 0\right)$$

$$B' = (1, 0, 0) \qquad F' = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$C' = (1, 1, 0) \qquad G' = \left(1+\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$D' = (0, 1, 0) \qquad H' = \left(1+\frac{\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}, 0\right)$$

Refer to Fig. 6.10.

To draw the cabinet projection :

$$Par_{V_2}.V = \begin{bmatrix} 0 & 1 & 1 & 0 & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 1+\frac{\sqrt{3}}{4} & 1+\frac{\sqrt{3}}{4} \\ 0 & 0 & 1 & 1 & 1\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

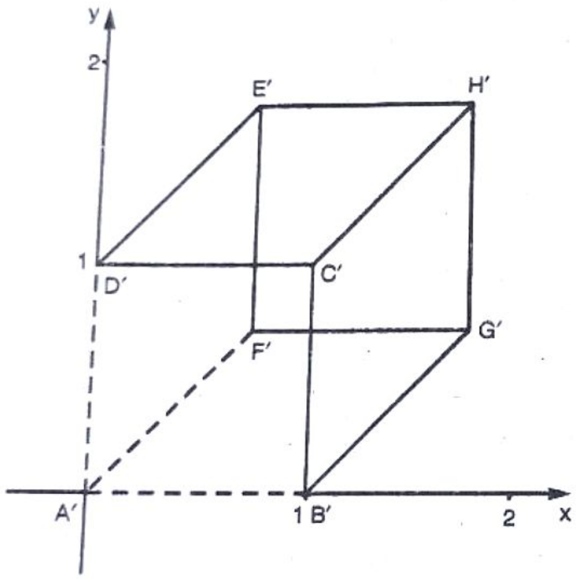


Fig. 6.10

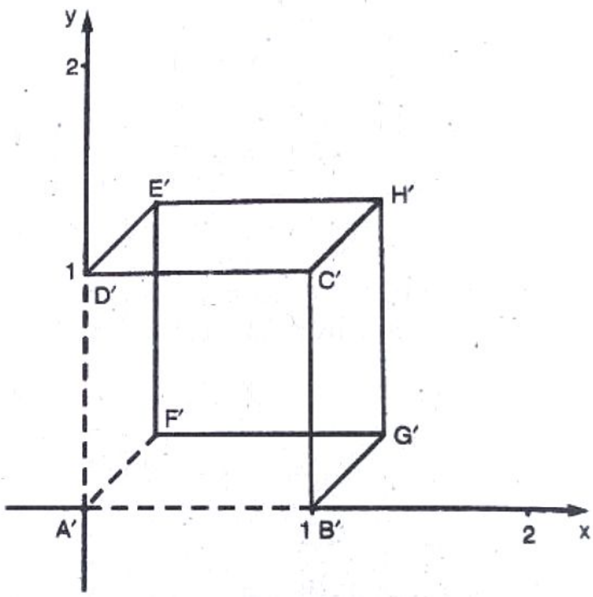


Fig. 6.11

The image coordinates are then (see Fig. 6.11)

$$A' = (0, 0, 0) \qquad E' = \left(\frac{\sqrt{3}}{4}, 1\frac{1}{4}, 0\right)$$

$$B' = (1, 0, 0) \qquad F' = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, 0\right)$$

$$C' = (1, 1, 0) \qquad G' = \left(1+\frac{\sqrt{3}}{4}, \frac{1}{4}, 0\right)$$

$$D' = (0, 1, 0) \qquad H' = \left(1+\frac{\sqrt{3}}{4}, 1\frac{1}{4}, 0\right)$$

## 6.3. PERSPECTIVE PROJECTION

In the perspective projection, as far as an object is from the viewer, the smaller it appears on the view plane. Here, lines of projection meet at the centre of projection and the projected image is viewed by keeping an eye at that position. In this case generalization techniques used by artists for drawing 3D objects are used Eye of artist is placed at the center of projection and the canvas becomes view plane. Image point is determined by projector that goes from an object point to the center of projection. In perspective projection projectors are considered to be converging in the eye of the viewer.

### *General Description of Perspective Projection*

In perspective projection center of projection and view plane is needed. Viewplane is determined by view reference point *i.e.* $R_0$ and view plane normal.

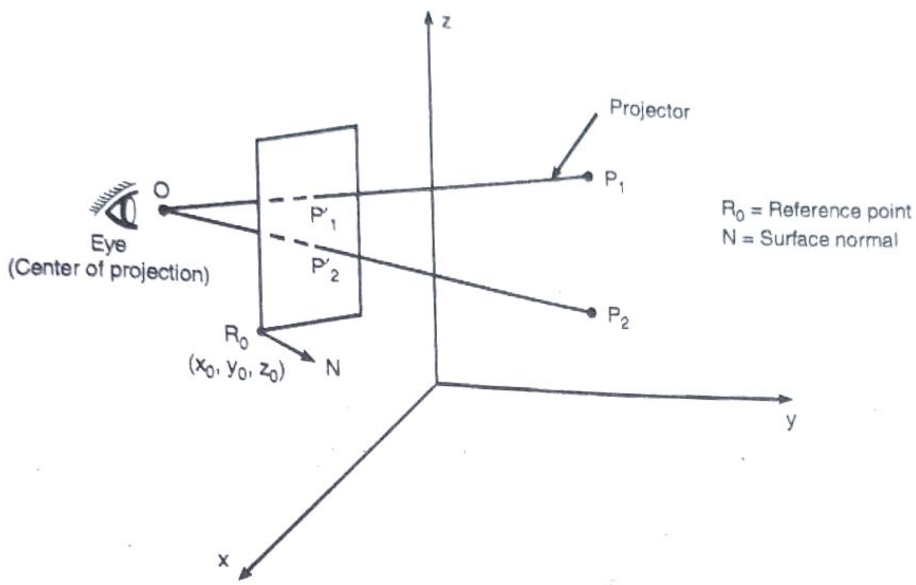


*Fig. 6.12. General perspective projection*

In the above Fig. 6.12, if $P_1$ and $P_2$ are two endpoints of an object which is viewed, then $P'_1$ and $P'_2$ will be the points on view plane. If $P$ has co-ordinate of $(x, y, z)$ then $P'$ will have co-ordinate of $(x', y', z')$ *i.e.*, of image.

### *Perspective Anamolies*

With perspective projection actual image can be distorted in shape due to some anamolies of perspective projection which are as follows:

(1) **Perspective Forshortening:** This means farther an object is from the center of projection smaller it appears.

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(2) **Vanishing Points:** Projection of lines that are not parallel to viewplane appear to meet at some point on the view plane. For example, railway track appears to meet at some point.

(3) **View Confusion:** Objects behind the center of projection are projected upside down and backward on to the view plane.

### 6.3.1. Standard Perspective Projection

Here, the view plane is considered to be the $X$-$Y$ plane at $z = 0$ and the viewing direction is along the negative $Z$-axis. If the centre of projection is $(x_c, y_c, z_c)$ and $(x, y, z)$ is a point on the object, the projection beam can be expressed mathematically as

$$\begin{aligned} x(u) &= x_c + (x - x_c)\,u \\ y(u) &= y_c + (y - y_c)\,u \\ z(u) &= z_c + (z - z_c)\,u \end{aligned} \qquad \text{...(6.24)}$$

The projected point $(x', y', 0)$ can be found out using $z(u) = 0$ in eqn. (6.24). Therefore,

$$u = \frac{z_c}{z - z_c} \qquad \text{...(6.25)}$$

and we get

$$x' = x_c - z_c \frac{x - x_c}{z - z_c} \qquad \text{...(6.26)}$$

$$y' = y_c - z_c \frac{y - y_c}{z - z_c} \qquad \text{...(6.27)}$$

Simplifying eqn. (6.26) and (6.27),

$$x' = \frac{x_c z - x z_c}{z - z_c} \qquad \text{...(6.28)}$$

$$y' = \frac{y_c z - y z_c}{z - z_c} \qquad \text{...(6.29)}$$

Hence, the transformation can be written as

$$T_{\text{perspective}} = \begin{bmatrix} -z_c & 0 & x_c & 0 \\ 0 & -z_c & y_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -z_c \end{bmatrix} \qquad \text{...(6.30)}$$

and

$$\begin{bmatrix} wx' \\ wy' \\ wz' \\ w \end{bmatrix} = \begin{bmatrix} -z_c & 0 & x_c & 0 \\ 0 & -z_c & y_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -z_c \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \text{...(6.31)}$$

where $w = z - z_c$.

Now, if we consider that the viewer is viewing the projected object keeping an eye at the centre of projection which is situated at a distance $d$ from the origin and the view plane is the $X$-$Y$ plane at $z = 0$ (as shown in Fig. 6.13) and then we have $z_c = -d$.

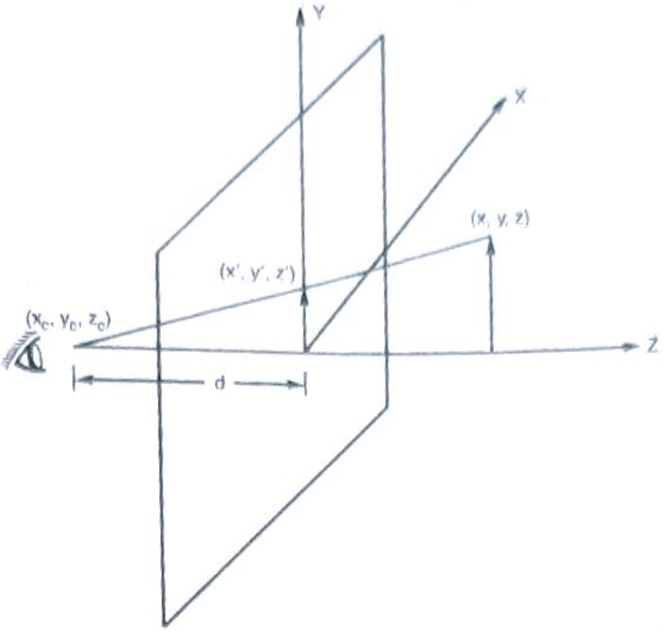


*Fig. 6.13. Perspective projection on the X-Y plane at z = 0.*

Therefore, we can write

$$T_{\text{perspective}} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \end{bmatrix} \quad \text{...(6.32)}$$

or

$$T_{\text{perspective}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \quad \text{...(6.33)}$$

## *Another Derivation of Perspective Projection*

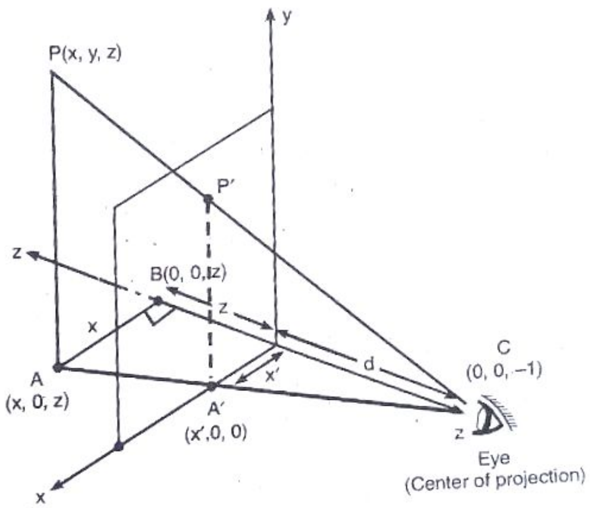


*Fig. 6.14*

In the Fig. 6.14, if $P$ is the original point object which has co-ordinates of $(x, y, z)$ and $P'$ with co-ordinates $(x', y', z')$ is viewed in view plane of $x$-$y$. In case center of project is $C$ which is on negative $z$ axis at destance of $d$. If $A$ is the projection of $P$ on $x$-$z$ plane and we draw $AB \perp z$-**axis.**

Then using similar triangles $\Delta ABC$ and $\Delta A'OC$

$$\frac{x'}{x} = \frac{d}{z+d} \Rightarrow x' = \frac{d.x}{z+d}$$

Similarly,

$$y' = \frac{d.y}{z+d} \text{ and } z' = 0$$

By using homogeneous co-ordinate, we can represent it in 4 × 4 matrix.

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} d.x \\ d.y \\ 0 \\ z+d \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

**Example 6.7**

*Consider a unit cube (Fig. 6.7) in standard position wih its vertices at A(0, 0, 0), B(1, 0, 0), C(1, 1, 0), D(0, 1, 0). E(0, 1, 1), F(0, 0, 1), G(1, 0, 1) and H(1, 1, 1) respectively. Perform a perspective projection of the cube when the view direction is along the negative Z-axis.*

**Solution.**

Let the $X$-$Y$ plane be the plane of projection and centre of projection is at $(0, 0, -d)$ on the $Z$-axis. Here, for each of the vertices $(x, y, z)$ of the cube, we have a projected view $(x', y', z')$ such that

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

*i.e.*,

$$V' = T_{\text{perspective}} \times V$$

Now, in our example, if we consider $d = 1$, then

$$V' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times V$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Therefore, the projected vertices (see Fig. 6.15) are $A'(0, 0, 0)$, $B'(1, 0, 0)$, $C'(1, 1, 0)$, $D'(0, 1, 0)$, $E'\left(0, \frac{1}{2}, 0\right)$, $F'(0, 0, 0)$, $G'\left(\frac{1}{2}, 0, 0\right)$ and $H'\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Again, if we consider $d = 5$, then

$$V' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 1 \end{bmatrix} \times V$$

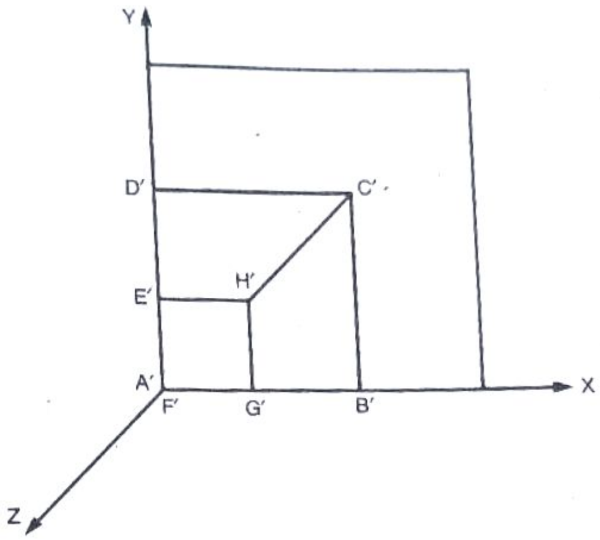


*Fig. 6.15. Perspective projection of the cube when d = 1.*

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} \end{bmatrix}$$

Therefore, the projected vertices (see Fig. 6.16) are $A'(0, 0, 0)$, $B'(1, 0, 0)$, $C'(1, 1, 0)$, $D'(0, 1, 0)$, $E'\left(0, \frac{5}{6}, 0\right)$, $F'(0, 0, 0)$, $G'\left(\frac{5}{6}, 0, 0\right)$ and $H'\left(\frac{5}{6}, \frac{5}{6}, 0\right)$.

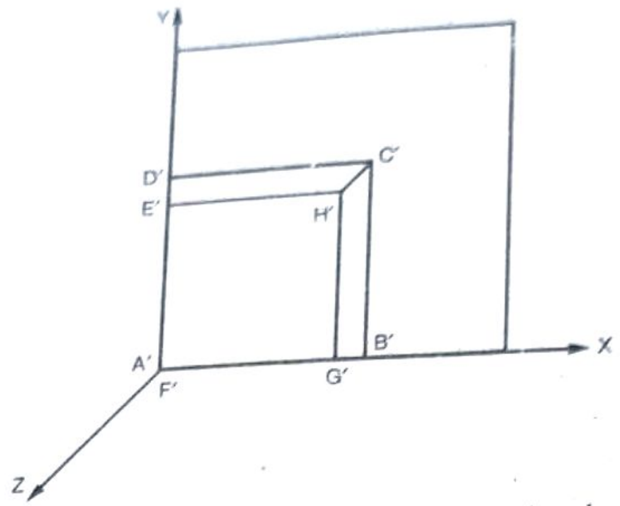


*Fig. 6.16. Perspective projection of the cube when d = 5.*

### 6.3.2. Vanishing Point

For standard perspective projection, the view plane is considered to be the *X-Y* plane, the centre of projection is on the *Z*-axis at $z = d$ and the viewing direction is along the negative *Z*-axis. In perspective projection, parallel lines in the object which are not parallel to the view plane, when extended, meet at a point on the view plane. This point is called a **vanishing point**. Depending on the number of vanishing points generated, when a cubical object is projected, we can classify perspective projection into three different classes of interest. These are as follows :

1. **One-point Perspective :** This occurs when one of the cube faces is parallel to the view plane.
2. **Two-point Perspective :** This occurs when none of the cube faces is parallel to the view plane, but a set of parallel edges is parallel to the view plane.
3. **Three-point Perspective :** This occurs when none of the cube edges is parallel to the view plane.

Figure 6.17 to 6.19 illustrate the formation of vanishing points in three different situations.

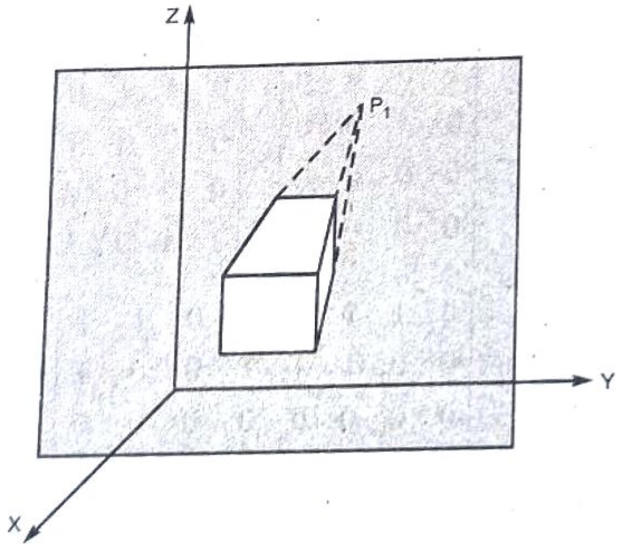


*Fig. 6.17. One principal vanishing point.*

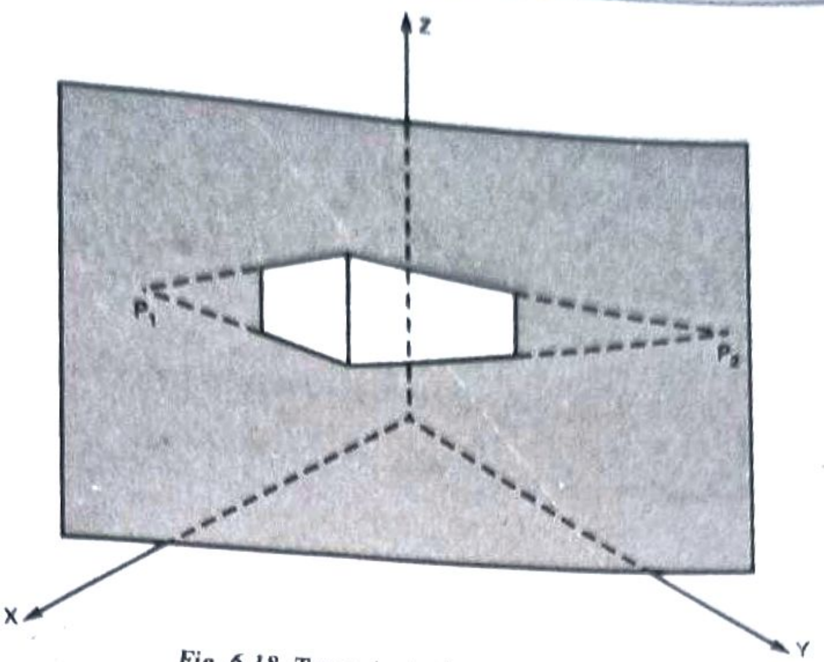


*Fig. 6.18. Two principal vanishing points.*

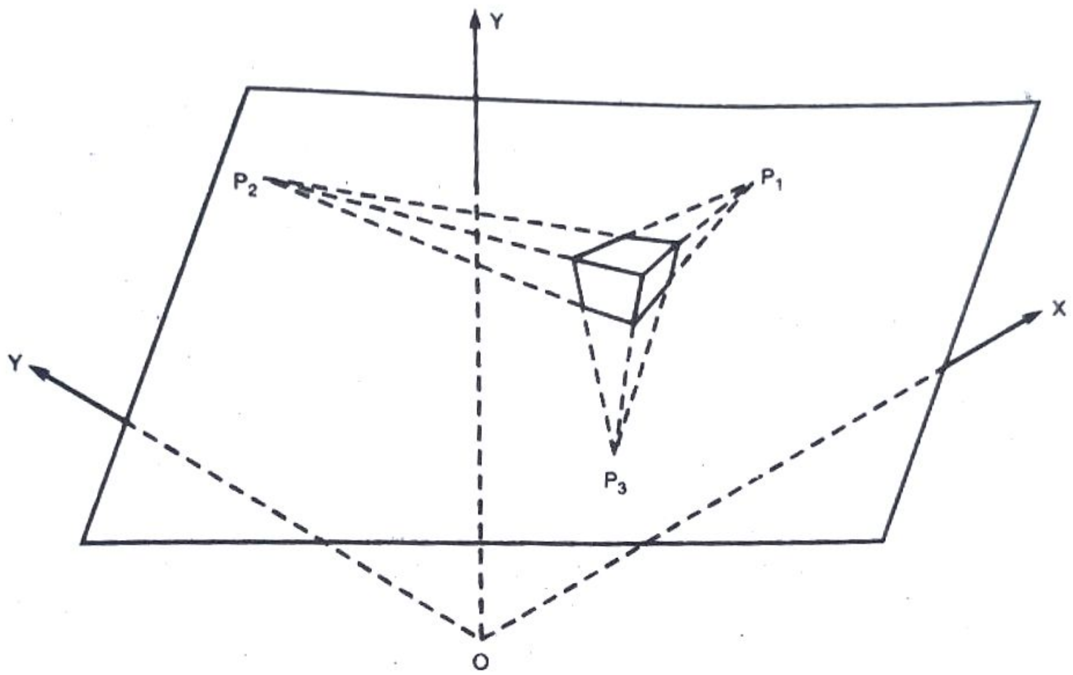


*Fig. 6.19. Three principal vanishing points.*

## Determining Vanishing Points

As already mentioned, vanishing points are those points where a set of parallel lines meets (not parallel to the view plane) on the view plane. Let us consider a perspective projection with the following features :

- $C(x_c, y_c, z_c)$ = centre of projection
- $X$-$Y$ plane = view plane or plane of projection
- $P(x, y, z)$ = a point on the object to be projected
- $P'(x', y', z')$ = projected view of $P$

For the perspective projection, we have

$$x' = \frac{x}{1 + \frac{z}{z_c}} \quad \text{and} \quad y' = \frac{y}{1 + \frac{z}{z_c}}$$

Now let us consider a line from $P_1$ to $P_2$ on the object, whose projected view is $P'_1P'_2$. Let a point be on the line $P_1P_2$ and $P'(x', y', z')$ be the projected view of $P$.

The line $P_1P_2$ can be parametrically expressed as

$$P(t) = P_1 + (P_2 - P_1)\,t\,; \qquad 0 \le t$$
$$= P_1 + Dt\,; \qquad \ldots(6.3$$

where $D = (dx, dy, dz)$ is the direction vector of the line $P_1P_2$

For each point $P(x, y, z)$ on the line $P_1P_2$, we can write

$$x'(t) = \frac{z_c x(t)}{z_c + z(t)} \quad \text{and} \quad y'(t) = \frac{z_c y(t)}{z_c + z(t)} \qquad \ldots(6.36$$

or

$$x'(t) = \frac{z_c(x_1 + d_x t)}{z_c + z_1 + d_z t} \quad \text{and} \quad y'(t) = \frac{z_c(y_1 + d_y t)}{z_c + z_1 + d_z t} \qquad \ldots(6.3$$

Now let us consider extension of the $P_1P_2$. As the parameter $t \to \infty$, in eqn. (6.37), the terms containing will become dominant. Then we can simplify eqn. (6.37) as

$$x'(t) = \frac{z_c d_x}{d_z} \quad \text{and} \quad y'(t) = \frac{z_c d_y}{d_z} \qquad \ldots(6.3$$

Therefore, vanishing points depend only on the direction vector of the parallel lines concerned. If the line is parallel to the view plane, then $dz = 0$ and vanishing points are formed only at infinity.

### 6.3.3. General Perspective Projection

In general, for perspective projection, the view plane orientation and the position of the centre of projection may be arbitrary. In such a situation, we have to deal with the process of deriving a perspective transformation matrix differently. Let us consider the derivation process in two steps. In the first step, we shall try to develop the transformation matrix for any arbitrary view plane with the origin as the centre of projection. In the second step, we shall develop the general transformation by combining it with other transformations.

#### *Centre of Projection at Origin*

Let us consider the following for a general perspective projection (Refer to Fig. 6.20) :

- $O(0, 0, 0)$ = centre of projection
- $N = (n_1 i + n_2 j + n_3 k)$ = normal to the plane of projection