$= \frac{m}{m^2+1}$

which is different for different values of m.

$\therefore \underset{(x,y)\to(0,0)}{Lt} \frac{x^3y}{x^6+y^2}$ doesn't exist.

Continuity of function of two variables

A function $f(x,y)$ is said to be continuous at the point (a,b) if

$\underset{(x,y)\to(a,b)}{Lt} f(x,y)$ exists and $= f(a,b)$

Thus $f(x,y)$ is said to be continuous at the point (a,b) if given $\varepsilon > 0$, $\exists$ a real no. $\delta > 0$ $\ni$

$|f(x,y) - f(a,b)| < \varepsilon$ for $|(x,y) - (a,b)| < \delta$

(1) Example:- Let $A = \{(x,y);\ 0 < x < 1,\ 0 < y < 1\}$ and $f : A \to R$ be defined by $f(x,y) = x+y$. Prove that f is continuous at each point of domain A.

Solution:- Let (α,β) be any pt. of A.

Now to prove:- $f(x,y)$ is continuous at (α,β).

We have to prove that $\underset{(x,y)\to(\alpha,\beta)}{Lt} f(x,y) = f(\alpha,\beta)$

Let $\varepsilon > 0$ be given

Consider $|f(x,y) - f(\alpha,\beta)| = |(x+y) - (\alpha+\beta)|$

$$= |(x-\alpha) + (y-\beta)|$$

$$\leq |x-\alpha| + |y-\beta|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $|x-\alpha| < \varepsilon/2$ and $|y-\beta| < \varepsilon/2$

So, Here $\delta = \varepsilon/2$

$\therefore$ for every $\varepsilon > 0$; $\exists$ $\delta = \varepsilon/2 > 0$ $\ni$

$|f(x,y) - f(\alpha,\beta)| < \varepsilon$ for $|x-\alpha| < \delta$ and $|y-\beta| < \delta$

where $\delta = \varepsilon/2$

Hence by definition of continuity, $f(x,y)$ is continuous at (α,β).

(2) Example :- Show that the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is continuous at $(0,0)$.

Solu :- Let $\varepsilon > 0$ be given

Consider $|f(x,y) - f(0,0)| = \left|\dfrac{xy(x^2-y^2)}{x^2+y^2} - 0\right|$

$$= |xy| \left|\frac{x^2-y^2}{x^2+y^2}\right|$$

$$\leq |x|.|y| \quad \text{since } |x^2-y^2| \leq |x^2+y^2|$$

$$< \sqrt{\varepsilon}.\sqrt{\varepsilon} = \varepsilon \quad \Rightarrow \left|\frac{x^2-y^2}{x^2+y^2}\right| \leq 1$$

whenever $|x| < \sqrt{\varepsilon}$ and $|y| < \sqrt{\varepsilon}$

So, Here $\delta = \sqrt{\varepsilon}$

So, for every $\varepsilon > 0$; $\exists$ $\delta = \sqrt{\varepsilon} > 0$ $\ni$

$|f(x,y) - f(0,0)| < \varepsilon$ for $|x| < \sqrt{\varepsilon}$ and $|y| < \sqrt{\varepsilon}$

So, by defi. of continuity, $f(x,y)$ is continuous at $(0,0)$.

(3) **Example**:- Show that $f(x,y)$ is discontinuous at the origin if

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4} ; & \text{if } (x,y) \neq (0,0) \\ 0 ; & \text{if } (x,y) = (0,0) \end{cases}$$

Solu:- We will show that

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ doesn't exist.}$$

Let $(x,y) \to (0,0)$ along the path $y = mx$

As $x \to 0$ and $y = mx \Rightarrow y \to 0$.

$$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^2y^2}{x^4+y^4}$$

$$= \underset{x\to 0}{Lt} \frac{x^2(m^2x^2)}{x^4+m^4x^4} = \underset{x\to 0}{Lt} \frac{x^4(m^2)}{x^4(1+m^4)}$$

$$= \underset{x\to 0}{Lt} \frac{m^2}{1+m^4} = \frac{m^2}{1+m^4}$$

which is different for different values of m.

So, $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$ doesn't exist.

$\Rightarrow$ $f(x,y)$ is discontinuous at $(0,0)$.