2

Linear Differential Equations of the First Order

2.1 DEFINITION

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

Thus, the standard form of a linear differential equation of the first order is

$\frac{dy}{dx} + Py = Q$, where P and Q are functions of x or constants (*i.e.*, independent of y).

2.2 TO SOLVE THE EQUATION $\frac{dy}{dx} + Py = Q$, WHERE P AND Q ARE FUNCTIONS OF x ONLY (Leibnitz's Equation)

The given equation is $\frac{dy}{dx} + Py = Q$

Multiplying throughout by $e^{\int P\,dx}$, we get

$$\frac{dy}{dx} \cdot e^{\int P\,dx} + Py \cdot e^{\int P\,dx} = Q \cdot e^{\int P\,dx} \qquad \ldots(1)$$

Now,
$$\frac{d}{dx}[ye^{\int P\,dx}] = \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot \frac{d}{dx}[e^{\int P\,dx}]$$

$$= \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot e^{\int P\,dx} \cdot \frac{d}{dx}[\int P\,dx]$$

$$\left[\because \frac{d}{dx}\left\{e^{f(x)}\right\} = e^{f(x)} \cdot \frac{d}{dx}\{f(x)\}\right]$$

$$= \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot e^{\int P\,dx} \cdot P = \frac{dy}{dx} \cdot e^{\int P\,dx} + Py \cdot e^{\int P\,dx}$$

$\therefore$ From (1), $\frac{d}{dx}[y \cdot e^{\int P\,dx}] = Q \cdot e^{\int P\,dx}$

Integrating both sides w.r.t. x, we have

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\, dx + c$$

which is the required solution of the given linear differential equation.

Note 1. The factor $e^{\int P\,dx}$, on multiplying by which the LHS of the differential equation becomes the differential co-efficient of some function of x and y, is called an integrating factor of the differential equation and is shortly written as I.F.

Note 2. The solution of the linear equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only, is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

Note 3. Sometimes a differential equation becomes linear if we take y as the independent variable and x as dependent variable. In that case, the equation can be put in the form $\frac{dx}{dy} + Px = Q$, **where P and Q are functions of y (and not of x) or constants.**

I.F. (in this case) = $e^{\int P\,dy}$, and the solution is

$$x(\text{I.F.}) = \int Q.\,(\text{I.F.})\,dy + c.$$

Note 4. While evaluating the I.F., it is very useful to remember that $e^{\log f(x)} = f(x)$.

Thus,
$$e^{\log x^2} = x^2.$$

Note 5. The co-efficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing throughout by it.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following :*

(i) $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ *(ii)* $\frac{dy}{dx} = y \tan x - 2 \sin x.$

Sol. (*i*) Given equation is $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Dividing throughout by $1 + x^2$, $\left(\text{to make the co-efficient of } \frac{dy}{dx} \text{ unity.}\right)$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{4x^2}{1+x^2} \qquad \ldots(i)$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here, $P = \frac{2x}{1+x^2}, \; Q = \frac{4x^2}{1+x^2}$

$\therefore$ $\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\,dx + c$$

or $$y(1 + x^2) = \int \frac{4x^2}{1+x^2}\,.\,(1 + x^2)\,dx + c$$

$$y(1 + x^2) = \int 4x^2 dx + c$$

or

$$y(1 + x^2) = \frac{4x^3}{3} + c.$$

or

(ii) Given equation is $\frac{dy}{dx} - (\tan x) \cdot y = -2 \sin x$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $\qquad P = -\tan x,\ Q = -2 \sin x$

$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{-\int \tan x\,dx} = e^{-(-\log \cos x)}$

$$= e^{\log \cos x} = \cos x$$

Hence the solution is

$$y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$y \cos x = \int -2 \sin x \cos x\,dx + c$$

or

$$= -\int \sin 2x\,dx + c = -\frac{-\cos 2x}{2} + c$$

or

$$y \cos x = \tfrac{1}{2} \cos 2x + c.$$

Example 2. *Solve the following:*

(i) $\sec x \frac{dy}{dx} = y + \sin x$ $\qquad$ (ii) $x \log x \frac{dy}{dx} + y = 2 \log x$

Sol. (i) Given equation is $\sec x \cdot \frac{dy}{dx} - y = \sin x$

Dividing throughout by $\sec x$, to make the co-efficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} - (\cos x) \cdot y = \sin x \cos x$$

It is of the form $\qquad \frac{dy}{dx} + Py = Q$

Here, $\qquad P = -\cos x, \quad Q = \sin x \cos x$

$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\int -\cos x\,dx} = e^{-\sin x}$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

or

$$y \cdot e^{-\sin x} = \int \sin x \cos x \cdot e^{-\sin x}\,dx + c = \int te^{-t}\,dt + c, \text{ where } t = \sin x$$

$$= t \cdot \frac{e^{-t}}{-1} - \int 1 \cdot \frac{e^{-t}}{-1}\,dt + c = -te^{-t} - e^{-t} + c$$

or

$$= -e^{-t}(t + 1) + c = -e^{-\sin x}(\sin x + 1) + c$$

$$y = -(\sin x + 1) + c\,e^{\sin x}.$$

(*ii*) Given equation is $x \log x \dfrac{dy}{dx} + y = 2 \log x$

Dividing throughout by $x \log x$ to make the co-efficient of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{1}{x \log x} \cdot y = \frac{2}{x}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here, $P = \dfrac{1}{x \log x}$, $Q = \dfrac{2}{x}$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1/x}{\log x} dx} = e^{\log \log x} = \log x$$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$$

or $$y \log x = \int \frac{2}{x} \log x\, dx + c$$

or $$y \log x = 2 \int \frac{1}{x} \cdot \log x\, dx + c = 2 \cdot \frac{(\log x)^2}{2} + c$$

$\because \int [f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}, n \neq -1$

or $$y \log x = (\log x)^2 + c.$$

Example 3. *Solve:* $x(x-1) \dfrac{dy}{dx} - (x-2)y = x^3(2x-1)$.

Sol. Given equation is

$$x(x-1) \frac{dy}{dx} - (x-2)y = x^3(2x-1)$$

or $$\frac{dy}{dx} - \frac{x-2}{x(x-1)} y = \frac{x^2(2x-1)}{x-1}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here, $P = -\dfrac{x-2}{x(x-1)}$, $Q = \dfrac{x^2(2x-1)}{x-1}$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{x-2}{x(x-1)} dx} = e^{-\int \left(\frac{2}{x} - \frac{1}{x-1}\right) dx}$$

$$= e^{-[2 \log x - \log (x-1)]} = e^{-[\log x^2 - \log (x-1)]}$$

$$= e^{-\log \frac{x^2}{x-1}} = e^{\log \left(\frac{x^2}{x-1}\right)^{-1}} = \left(\frac{x^2}{x-1}\right)^{-1} = \frac{x-1}{x^2}$$

$\therefore$ The solution is

$$y \cdot \frac{x-1}{x^2} = \int \frac{x^2(2x-1)}{x-1} \cdot \frac{x-1}{x^2}\, dx + c = \int (2x-1)\, dx + c = x^2 - x + c$$

or

$$y(x-1) = x^2(x^2 - x + c).$$

Example 4. *Solve:* $x(1-x^2)\dfrac{dy}{dx} + (2x^2-1)y = x^3.$

Sol. Dividing by $x(1-x^2)$ to make the co-efficient of $\dfrac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}\, y = \frac{x^2}{1-x^2}$$

It is of the form $\dfrac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$

Here
$$\mathrm{P} = \frac{2x^2-1}{x(1-x^2)}, \quad \mathrm{Q} = \frac{x^2}{1-x^2}$$

Now
$$\mathrm{P} = \frac{2x^2-1}{x(1-x)(1+x)} = -\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

[Partial fractions]

$$\therefore \quad \int \mathrm{P}dx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$
$$= -\log[x(1-x)^{1/2}(1+x)^{1/2}]$$
$$= -\log[x\sqrt{1-x^2}] = \log(x\sqrt{1-x^2})^{-1}$$

$$\therefore \quad \text{I.F.} = e^{\int \mathrm{P}\,dx} = e^{\log(x\sqrt{1-x^2})^{-1}} = \frac{1}{x\sqrt{1-x^2}}$$

The solution is

$$y \cdot \frac{1}{x\sqrt{1-x^2}} = \int \frac{x^2}{1-x^2} \cdot \frac{1}{x\sqrt{1-x^2}}\, dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}\, dx + c = -\frac{1}{2}\int (1-x^2)^{-3/2} \cdot (-2x)\, dx + c$$

$$= -\frac{1}{2} \cdot \frac{(1-x^2)^{-1/2}}{-\frac{1}{2}} + c$$

$$\Rightarrow \quad \frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c \quad \Rightarrow \quad y = x + cx\sqrt{1-x^2}$$

which is the required solution.

Example 5. *Solve:* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$

Sol. The given equation is

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

or
$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\text{I.F.} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{\int x^{-1/2}dx} = e^{2\sqrt{x}}$$

∴

∴ Hence the solution is

$$y \,.\, e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \,.\, e^{2\sqrt{x}}\, dx + c$$

or
$$y \,.\, e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + c$$

or
$$ye^{2\sqrt{x}} = 2\sqrt{x} + c \quad \text{or} \quad y = e^{-2\sqrt{x}}(2\sqrt{x} + c).$$

Equations of the Form $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only.

Example 6. *Solve the following:*

(*i*) $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$, (*ii*) $(2x - 10y^3)\frac{dy}{dx} + y = 0.$

Sol. (*i*) The given equation is

$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

or
$$(1 + y^2)\frac{dx}{dy} + x - e^{\tan^{-1} y} = 0$$

or
$$\frac{dx}{dy} + \frac{1}{1+y^2} \,.\, x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

It is of the form $\frac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + c = \int e^t \cdot e^t\, dt + c \quad \text{where } t = \tan^{-1} y$$

$$= \int e^{2t}\, dt + c = \tfrac{1}{2} e^{2t} + c$$

$$x \cdot e^{\tan^{-1} y} = \tfrac{1}{2} e^{2\tan^{-1} y} + c.$$

or

(*ii*) The given equation is $(2x - 10y^3) \dfrac{dy}{dx} + y = 0$

$$y \cdot \frac{dx}{dy} + 2x - 10y^3 = 0 \quad \text{or} \quad \frac{dx}{dy} + \frac{2}{y} \cdot x = 10y^2$$

or

It is of the form $\dfrac{dx}{dy} + \text{P}x = \text{Q}$

$$\text{I.F.} = e^{\int \text{P}\, dx} = e^{\int \frac{2}{y}\, dy} = e^{2 \log y} = e^{\log y^2} = y^2$$

$\therefore$ The solution is $xy^2 = \displaystyle\int 10y^2 \cdot y^2 dy + c = 10 \int y^4\, dy + c$

$$xy^2 = \frac{10y^5}{5} + c = 2y^5 + c.$$

or

TEST YOUR KNOWLEDGE

Solve *the following differential equations:*

1. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$
2. $\dfrac{dy}{dx} + y \sec x = \tan x$
3. $\dfrac{dy}{dx} + y \tan x = \sec x$
4. $(1 + x^2) \dfrac{dy}{dx} + 2xy = \cos x$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{x + 1}$
6. $(x + 1) \dfrac{dy}{dx} - ny = e^x (x + 1)^{n+1}$
7. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$
8. $(1 + x^2) \dfrac{dy}{dx} + y = \tan^{-1} x$
9. $\dfrac{dy}{dx} + \dfrac{2x}{x^2 + 1} \cdot y = \dfrac{1}{(x^2 + 1)^2}$ given that $y = 0$ when $x = 1$
10. $\dfrac{dy}{dx} + 2y \tan x = \sin x$ given that $y = 0$ when $x = \dfrac{\pi}{3}$
11. $x\dfrac{dy}{dx} + 2y = x^2 \log x$
12. $\dfrac{dy}{dx} + y \cos x = \sin 2x$
13. $\dfrac{dy}{dx} = x(x^2 - 2y)$
14. $\sin x \dfrac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$
15. $(1 - x^2) \dfrac{dy}{dx} + xy = ax$
16. $x(x - 1) \dfrac{dy}{dx} - y = x^2(x - 1)^2$

17. $y\,dx - x\,dy + \log x\,dx = 0$

18. $\dfrac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$

19. $\sin 2x \dfrac{dy}{dx} = y + \tan x$

20. $(x + 2y^3) \dfrac{dy}{dx} = y$

21. $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

22. $e^y\,dx + (1 + xe^y)\,dy = 0$

23. $\dfrac{dx}{dy} + 2x = 6e^y$

24. $2y' + 4y = x^2 - x$

25. $y' - 2y = \cos 3x$

26. $\dfrac{dy}{dx} + y \cot x = 2x + x^2 \cot x$

27. $y' + y = \dfrac{1 + x \log x}{x}$

28. $xy' - y = (x - 1)\,e^x$

Answers

1. $xy = \dfrac{1}{4} x^4 + c$

2. $y(\sec x + \tan x) = \sec x + \tan x - x + c$

3. $y = \sin x + c \cos x$

4. $y(1 + x^2) = \sin x + c$

5. $\dfrac{y}{x+1} = \log (x + 1) + c$

6. $y = (x + 1)^n (e^x + c)$

7. $y = \tan x - 1 + ce^{-\tan x}$

8. $y = \tan^{-1} x - 1 + ce^{-\tan x}$

9. $y(x^2 + 1) = \tan^{-1} x - \dfrac{\pi}{4}$

10. $y = \cos x - 2 \cos^2 x$

11. $x^2 y = \dfrac{x^4}{4} \log x - \dfrac{x^4}{16} + c$

12. $y = 2(\sin x - 1) + ce^{-\sin x}$

13. $y = \dfrac{1}{2} (x^2 - 1) + ce^{-x^2}$

14. $y \sin x = \dfrac{2}{3} \sin^3 x + c$

15. $y = a + c\sqrt{1 - x^2}$

16. $y = \left(1 - \dfrac{1}{x}\right)\left(\dfrac{x^3}{3} + c\right)$

17. $y + 1 + \log x = cx$

18. $y \sin^2 x = x^3 + c$

19. $y = \tan x + c\sqrt{\tan x}$

20. $x = y^3 + cy$

21. $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$

22. $xe^y + y = c$

23. $x = 2e^y + ce^{-2y}$

24. $y = \dfrac{1}{4} (x - 1)^2 + ce^{-2x}$

25. $y = \dfrac{1}{13} (3 \sin 3x - 2 \cos 3x) + ce^{2x}$

26. $y \sin x = x^2 \sin x + c$

27. $y = \log x + ce^{-x}$

28. $y = e^x + cx$

BERNOULLI'S EQUATION
(Equations Reducible to the Linear Form)

TO SOLVE THE EQUATION $\frac{dy}{dx} + Py = Qy^n$, WHERE P AND Q ARE FUNCTIONS OF x ONLY

The given equation is $\frac{dy}{dx} + Py = Qy^n$ ...(*i*)

Dividing both sides of (*i*) by y^n, **to make the RHS a function of x only.**

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad ...(ii)$$

Put $y^{1-n} = z$, then

$$(1-n) \,.\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{dz}{dx}$$

$\therefore$ (*ii*) becomes
$$\frac{1}{1-n} \cdot \frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + (1-n) \,.\, Pz = (1-n)\, Q.$$

hich is a linear equation in z and can be solved.

In the solution, putting $z = y^{1-n}$, we get the required solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Sol. The given equation is $2 \,.\, \frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$.

Dividing throughout by y^2

$$2y^{-2}\frac{dy}{dx} - \frac{1}{x} \,.\, y^{-1} = \frac{1}{x^2} \quad ...(i)$$

Put $y^{-1} = z$, then
$$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (*i*) becomes

$$-2\frac{dz}{dx} - \frac{1}{x}z = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} + \frac{1}{2x}z = -\frac{1}{2x^2}$$

ich is linear in z.

$$P = \frac{1}{2x}, \quad Q = -\frac{1}{2x^2}$$

$$\text{I.F.} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\log x} = e^{\log\sqrt{x}} = \sqrt{x}$$

$\therefore$ The solution is $z \cdot \sqrt{x} = \int -\frac{1}{2x^2}\sqrt{x}\,dx + c$

or $y^{-1}\sqrt{x} = -\frac{1}{2}\int x^{-3/2}\,dx + c$ or $\frac{\sqrt{x}}{y} = \frac{1}{\sqrt{x}} + c$

or $x = y(1 + c\sqrt{x})$.

Example 2. *Solve the following :*

(i) $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ (ii) $\frac{dy}{dx} + \frac{x}{1-x^2}\,y = x\sqrt{y}$.

Sol. (i) The given equation is $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing throughout by e^y

$$e^{-y}\frac{dy}{dx} + e^{-y}\frac{dy}{dx} + \frac{1}{x} = \frac{1}{x^2} \quad \ldots(i)$$

Put $e^{-y} = z$, then $-e^{-y}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (i) becomes

$$-\frac{dz}{dx} + z \cdot \frac{1}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x} \cdot z = -\frac{1}{x^2}$$

which is linear in z. $P = -\frac{1}{x},\ Q = -\frac{1}{x^2}$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

$\therefore$ The solution is $z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x}\,dx + c$

or $e^{-y} \cdot \frac{1}{x} = -\int \frac{1}{x^3}\,dx + c$ or $e^{-y} \cdot \frac{1}{x} = \frac{1}{2x^2} + c$

or $2x = e^y + 2cx^2e^y$.

(ii) The given equation is $\frac{dy}{dx} + \frac{x}{1-x^2}\,y = x\sqrt{y}$

Dividing throughout by $\sqrt{y}$,

$$y^{1/2} \cdot \frac{dy}{dx} + \frac{x}{1-x^2}\,y^{1/2} = x \quad \ldots(i)$$

Put $y^{1/2} = z$; then $\frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (i) becomes

$$2 \cdot \frac{dz}{dx} + \frac{x}{1-x^2} \cdot z = x \quad \text{or} \quad \frac{dz}{dx} + \frac{x}{2(1-x^2)} \cdot z = \frac{x}{2}$$

which is linear in z.

$$P = \frac{x}{2(1-x^2)}, \; Q = \frac{x}{2}$$

$\therefore$
$$\text{I.F.} = e^{\int \frac{x}{2(1-x^2)} dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2} dx} \qquad \text{| Note}$$

$$= e^{-\frac{1}{4}\log(1-x^2)} = e^{\log(1-x^2)^{-1/4}} = (1-x^2)^{-1/4}$$

$\therefore$ The solution is
$$z \,.\, (1-x^2)^{-1/4} = \int \frac{x}{2}(1-x^2)^{-1/4}\, dx + c$$

or
$$\sqrt{y} \,.\, (1-x^2)^{-1/4} = -\frac{1}{4}\int -2x(1-x^2)^{-1/4}\, dx + c$$

or
$$\sqrt{y} \,.\, (1-x^2)^{-1/4} = -\frac{1}{4} \cdot \frac{(1-x^2)^{3/4}}{\frac{3}{4}} + c$$

or
$$\sqrt{y} = -\tfrac{1}{3}(1-x^2) + c(1-x^2)^{1/4}.$$

Example 3. *Solve:* $(x^2y^3 + xy)\, dy = dx.$

Sol. The given equation is $(x^2y^3 + xy)dy = dx$

or
$$\frac{dx}{dy} = x^2y^3 + xy$$

or
$$\frac{dx}{dy} - xy = x^2y^3 \qquad \Big|\ \text{Form } \frac{dx}{dy} + Px = Qx^n$$

Dividing throughout by x^2

$$x^{-2}\frac{dx}{dy} - x^{-1}\, y = y^3 \qquad ...(i)$$

Put $x^{-1} = z$, then $\quad -x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$

$\therefore$ (*i*) becomes

$$-\frac{dz}{dy} - zy = y^3 \quad \text{or} \quad \frac{dz}{dy} + y.z = -y^3$$

which is linear in z.

$$P = y, \; Q = -y^3$$

$$\text{I.F.} = e^{\int y\, dy} = e^{y^2/2}$$

$\therefore$ The solution is
$$z \,.\, e^{1/2y^2} = \int -y^3 \,.\, e^{1/2y^2}\, dy + c$$

or
$$x^{-1} \,.\, e^{1/2y^2} = -\int y^2 \,.\, y \,.\, e^{1/2y^2}\, dy + c$$

$$= -\int 2t\, e^t\, dt + c, \quad \text{where } t = \tfrac{1}{2}y^2$$

or
$$x^{-1} \,.\, e^{1/2y^2} = -2e^t(t-1) + c$$

or
$$x^{-1} \,.\, e^{1/2y^2} = -2e^{1/2y^2}\left(\tfrac{1}{2}y^2 - 1\right) + c \quad \text{or} \quad x^{-1} = -y^2 + 2 + ce^{-1/2y^2}.$$

Example 4. *Solve the following:*

(*i*) $(x+1)\dfrac{dy}{dx} + 1 = 2e^{-y}$ (*ii*) $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$

Sol. (*i*) The given equation is $(x+1)\dfrac{dy}{dx}+1=2e^{-y}$

or $$\frac{dy}{dx}+\frac{1}{x+1}=\frac{2e^{-y}}{x+1}$$

or $$e^y\cdot\frac{dy}{dx}+\frac{1}{x+1}\cdot e^y=\frac{2}{x+1} \qquad ...(i)$$

Put $e^y=z$, then $$e^y\cdot\frac{dy}{dx}=\frac{dz}{dx}$$

$\therefore$ From (*i*), $$\frac{dz}{dx}+\frac{1}{x+1}\cdot z=\frac{2}{x+1}$$

which is linear in z. $$P=\frac{1}{x+1},\ Q=\frac{2}{x+1}$$

$$\text{I.F.}=e^{\int\frac{1}{x+1}dx}=e^{\log(x+1)}=x+1$$

$\therefore$ The solution is $$z(x+1)=\int\frac{2}{x+1}\cdot(x+1)\,dx+c$$

or $$e^y\cdot(x+1)=2x+c.$$

(*ii*) The given equation is

$$\frac{dy}{dx}=e^{x-y}(e^x-e^y) \quad \text{or} \quad \frac{dy}{dx}=e^{2x}\cdot e^{-y}-e^x$$

or $$\frac{dy}{dx}+e^x=e^{2x}\cdot e^{-y} \quad \text{or} \quad e^y\cdot\frac{dy}{dx}+e^x\cdot e^y=e^{2x} \qquad ...(i)$$

Put $e^y=z$, then $$e^y\cdot\frac{dy}{dx}=\frac{dz}{dx}$$

$\therefore$ (*i*) becomes $$\frac{dz}{dx}+e^x\cdot z=e^{2x}$$

which is linear in z. $$P=e^x,\ Q=e^{2x}$$

$$\text{I.F.}=e^{\int e^x\,dx}=e^{e^x}$$

$\therefore$ The solution is $$z\cdot e^{e^x}=\int e^{2x}\cdot e^{e^x}\,dx+c$$

or $$e^y\cdot e^{e^x}=\int e^x\cdot e^x\cdot e^{e^x}\,dx+c$$
$$=\int t\,e^t\,dt+c, \quad \text{where } t=e^x$$
$$=e^t(t-1)+c$$

or $$e^y\cdot e^{e^x}=e^{e^x}(e^x-1)+c \quad \text{or} \quad e^y=e^x-1+c\,e^{-e^x}.$$

Example 5. *Solve:* $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0.$

Sol. The given equation is

$$\frac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$$

or $$\frac{1}{1+y^2}\cdot\frac{dy}{dx}+2x\tan^{-1}y-x^3=0$$

or $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} + 2x \tan^{-1} y = x^3 \qquad ...(i)$$

Put $\tan^{-1} y = z$, then $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ From (i), $$\frac{dz}{dx} + 2xz = x^3$$

which is linear in z.

$$P = 2x, \ Q = x^3$$

$$\text{I.F.} = e^{\int 2x\, dx} = e^{x^2}$$

$\therefore$ The solution is $$z \,.\, e^{x^2} = \int x^3 \,.\, e^{x^2}\, dx + c$$

or $$\tan^{-1} y \,.\, e^{x^2} = \frac{1}{2} \int 2x \,.\, x^2 e^{x^2}\, dx + c$$

$$= \frac{1}{2} \int t\, e^t\, dt + c, \quad \text{where } t = x^2$$

$$= \frac{1}{2} e^t (t - 1) + c$$

or $$\tan^{-1} y \,.\, e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

or $$\tan^{-1} y = \frac{1}{2} (x^2 - 1) + ce^{-x^2}.$$

Example 6. *Solve the following differential equations:*

(i) $(x^3y^2 + xy)\, dx = dy$ *(ii)* $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$

Sol. *(i)* The given equation is $(x^3y^2 + xy)\, dx = dy$

or $$\frac{dy}{dx} = x^3y^2 + xy \quad \text{or} \quad \frac{dy}{dx} - xy = x^3y^2$$

Dividing both sides by y^2, $$y^{-2} \frac{dy}{dx} - xy^{-1} = x^3 \qquad ...(i)$$

Put $y^{-1} = z$, then $$-y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (i) becomes $$\frac{dz}{dx} - xz = x^3 \quad \text{or} \quad \frac{dz}{dx} + xz = -x^3$$

which is linear in z.

$$P = x, \quad Q = -x^3$$

$$\text{I.F.} = e^{\int x\, dx} = e^{\frac{x^2}{2}}$$

$\therefore$ The solution is $$z \,.\, e^{\frac{x^2}{2}} = \int -x^3 \,.\, e^{\frac{x^2}{2}}\, dx + c = -\int x^2 \,.\, xe^{\frac{x^2}{2}}\, dx + c$$

$$= -\int 2te^t\, dt + c, \quad \text{where } t = \frac{x^2}{2}$$

$$= -\int 2te^t dt + c = -2e^t (t - 1) + c$$

$$y^{-1} \,.\, e^{\frac{x^2}{2}} = -2e^{\frac{x^2}{2}} \left(\frac{x^2}{2} - 1\right) + c$$

or $$y^{-1} = -x^2 + 2 + ce^{-\frac{x^2}{2}}.$$

(ii) The given equation is $$\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$$

Dividing both sides by y $(\log y)^2$, we get

$$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} + \frac{1}{\log y} \cdot \frac{1}{x} = \frac{1}{x^2} \quad ...(i)$$

Put $\frac{1}{\log y} = (\log y)^{-1} = z$, then $-(\log y)^{-2} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$

or $$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} = -\frac{dz}{dx}$$

$\therefore$ From (i), $$-\frac{dz}{dx} + z \cdot \frac{1}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x} z = -\frac{1}{x^2}$$

which is linear in z. $$P = -\frac{1}{x}, \; Q = -\frac{1}{x^2}.$$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is $$z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x} dx + c$$

or $$z \cdot \frac{1}{x} = -\int x^{-3} dx + c = -\frac{x^{-2}}{-2} + c$$

or $$\frac{1}{\log y} \cdot \frac{1}{x} = \frac{1}{2x^2} + c \quad \text{or} \quad \frac{1}{\log y} = \frac{1}{2x} + cx.$$

Example 7. *Show how to solve an equation of the form*

$$f'(y) \frac{dy}{dx} + Pf(y) = Q \quad \textit{where P, Q are functions of x only.}$$

Sol. (a) The given equation is

$$f'(y) \frac{dy}{dx} + Pf(y) = Q \quad ...(i)$$

where P, Q are functions of x only.

Put $f(y) = z$, then $$f'(y) \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (i) becomes $$\frac{dz}{dx} + Pz = Q$$

which is linear in z and can be solved.

$$\text{I.F.} = e^{\int P\, dx} \text{ and the solution is}$$
$$z(\text{I.F.}) = \int Q \,.\, (\text{I.F.})\, dx + c$$
or $$f(y) \,.\, (\text{I.F.}) = \int Q \,.\, (\text{I.F.})\, dx + c$$

Example 8. *Solve the following differential equations:*

(i) $(x+1)\dfrac{dy}{dx}+1=e^{x-y}$ (ii) $\dfrac{dy}{dx}=y\tan x-y^2\sec x.$

Sol. (i) The given equation is

$$(x+1)\frac{dy}{dx}+1=\frac{e^x}{e^y} \quad \text{or} \quad e^y\frac{dy}{dx}+\frac{e^y}{x+1}=\frac{e^x}{x+1} \qquad ...(i)$$

Putting $e^y = z$ so that $e^y\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ (i) becomes $\dfrac{dz}{dx}+\dfrac{z}{x+1}=\dfrac{e^x}{x+1}$

which is linear in z with $P=\dfrac{1}{x+1}$, $Q=\dfrac{e^x}{x+1}$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int\frac{1}{x+1}dx}=e^{\log(x+1)}=x+1$$

$\therefore$ The solution is $z(x+1)=\displaystyle\int\frac{e^x}{x+1}\cdot(x+1)\,dx+c$ or $e^y(x+1)=e^x+c.$

(ii) The given equation is

$$\frac{dy}{dx}-y\tan x=-y^2\sec x$$

or

$$-\frac{1}{y^2}\cdot\frac{dy}{dx}+\frac{1}{y}\tan x=\sec x \qquad ...(1)$$

Putting $\dfrac{1}{y}=z$ so that $-\dfrac{1}{y^2}\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ Equation (1) becomes $\dfrac{dz}{dx}+z\tan x=\sec x$

which is linear in z with $P=\tan x$, $Q=\sec x$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int\tan x\,dx}=e^{\log\sec x}=\sec x$$

$\therefore$ The solution is $z\cdot\sec x=\displaystyle\int\sec x\cdot\sec x\,dx+c$

$$\frac{1}{y}\sec x=\tan x+c \quad \text{or} \quad \frac{1}{y}=\sin x+c\cos x.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\dfrac{dy}{dx}+\dfrac{y}{x}=y^2$
2. $y'+y=y^2$
3. $\dfrac{dy}{dx}=x^3y^3-xy$
4. $3\dfrac{dy}{dx}+\dfrac{2}{x+1}y=\dfrac{x^3}{y^2}$

5. $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$

6. $x\frac{dy}{dx} + y = x^3y^4$

7. $\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$

8. $(y\log x - 1)y\,dx = x\,dy$

9. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

10. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$

11. $(x - y^2)\,dx + 2xy\,dy = 0$

12. $\cos x\,dy = y(\sin x - y)\,dx$

13. $xy - \frac{dy}{dx} = y^3e^{-x^2}$

14. $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$

15. $e^y\left(\frac{dy}{dx} + 1\right) = e^x$

16. $(xy - 2x\log x)\,dy = 2y\,dx$

17. $\frac{dy}{dx} + \frac{y}{x} = y^2\log x$

18. $y\,(2xy + e^x)\,dx = e^xy$

19. $x\frac{dy}{dx} + y = y^2x^3\cos x$

20. $\sin y\frac{dy}{dx} = \cos y\,(1 - x\cos y)$

Answers

1. $\frac{1}{xy} + \log x = c$

2. $y = \frac{1}{1 + ce^x}$

3. $y^{-2} = x^2 + 1 + ce^{x^2}$

4. $y^2(x+1)^2 = \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + c$

5. $\frac{1}{y^5} = \frac{5}{2}x^3 + cx^5$

6. $\frac{1}{y^3} = -3x^3\log x + cx^3$

7. $\frac{1}{y}\sec^2 x = -\frac{\tan^3 x}{3} + c$

8. $\frac{1}{y} = \log x + 1 + cx$

9. $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

10. $\sin y = (1+x)(e^x + c)$

11. $y^2 = x\,(c - \log x)$

12. $\frac{1}{y} = \sin x + c\cos x$

13. $y^{-2}\,.\,e^{x^2} = 2x + c$

14. $3y^2 = 2x^2e^{\frac{1}{x^3}} + cx^2$

15. $e^{x+y} = \frac{1}{2}e^{2x} + c$

16. $y\log x = \frac{y^2}{4} + c$

17. $\frac{1}{y} = -\frac{1}{2}(\log x)^2 + cx$

18. $e^x = y(c - x^2)$

19. $\frac{1}{xy} = -x\sin x - \cos x + c$

20. $\sec y = x + 1 + ce^x$

□□□