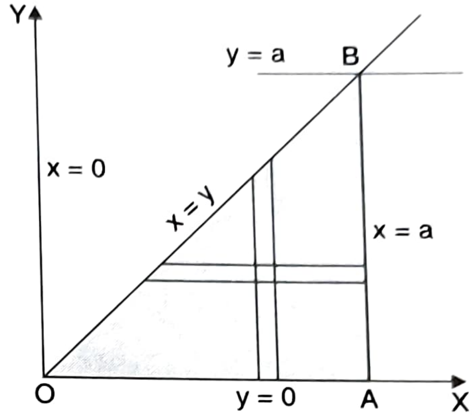
ILLUSTRATIVE EXAMPLES

Example 1. *Change the order of integration in* $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2 + y^2}$ *and hence evaluate the same.*

Sol. From the limits of integration, it is clear that the region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Thus the region of integration is the ΔOAB and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become: y varies from 0 to x and x varies from 0 to a.



$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2 + y^2} = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2 + y^2}$$

$$= \int_0^a x \cdot \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}\cdot\Big[x\Big]_0^a = \frac{\pi a}{4}.$$

Example 2. *Evaluate the following integral by changing the order of integration*

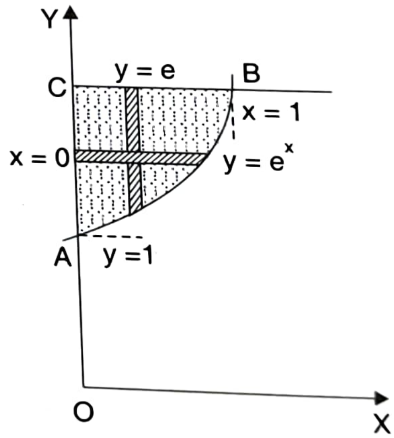
$$\int_0^1 \int_{e^x}^e \frac{dy\,dx}{\log y}.$$

Sol. The given limits show that the region of integration is bounded by the curves

$$y = e^x,\ y = e,\ x = 0,\ x = 1$$

It is the area shaded in the diagram. It can also be considered as bounded by

$$x = 0,\ x = \log y,\ y = 1,\ y = e$$



Hence, $$\int_0^1 \int_{e^x}^e \frac{dy\,dx}{\log y} = \int_1^e \int_0^{\log y} \frac{dx\,dy}{\log y}$$

$$= \int_1^e \left(\frac{x}{\log y}\right)_0^{\log y} dy$$

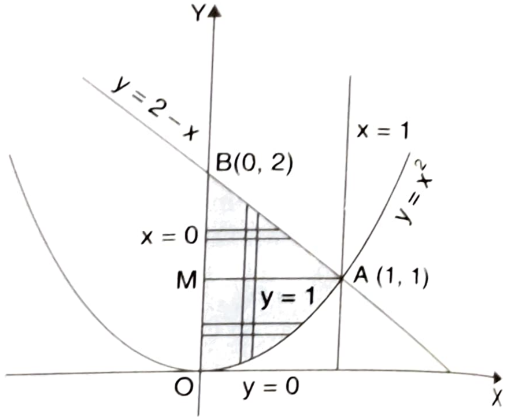
$$= \int_1^e 1.\,dy = \Big(y\Big)_1^e = e - 1.$$

Example 3. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ *and hence evaluate the same.*

Sol. From the limits of integration, it is clear that we have to integrate first with respect to y which varies from $y = x^2$ to $y = 2 - x$ and then with respect to x which varies from $x = 0$ to $x = 1$. The region of integration (shown shaded) is divided into vertical strips. For changing the order of integration, we divide the region of integration into horizontal strips.

Solving $y = x^2$ and $y = 2 - x$, the co-ordinates of A are (1, 1). Draw AM $\perp$ OY. The region of integration is divided into two parts, OAM and MAB.

For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. For the region MAB, x varies from 0 to $2 - y$ and y varies from 1 to 2.



$$\therefore \quad \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \cdot \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

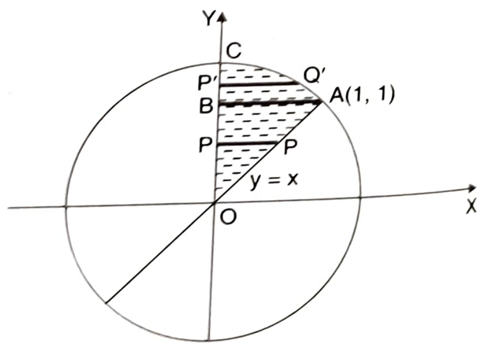
$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2} \cdot \left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\,dy = \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right] = \frac{3}{8}.$$

Example 4. *Evaluate by changing the order of integration of* $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2 + y^2}}$.

Sol. From limits of integration it is clear that we have to first integrate w.r.t. y and then w.r.t. x. y varies from $y = x$ to $y = \sqrt{2 - x^2}$ *i.e.*, $y^2 = 2 - x^2$ or $x^2 + y^2 = 2$ *i.e.*, a circle with centre at (0, 0) and radius $= \sqrt{2}$ and x varies 0 to 1. This integration is firstly performed along vertical strips and then along horizontal strips. For change of the order of integration we have to first perform integration along horizontal strips and then along vertical strips. The region of integration is shown shaded in the figure.



For horizontal strips the whole region is divided into two portion OAB and ACB. Let the horizontal strip in the portion OAB be PQ and that of in ACB be P′Q′.

The curve and the line intersect at A(1, 1).

For OAB: x varies from 0 to x and y varies for 0 to 1.

For ACB: x varies from 0 to $\sqrt{2 - y^2}$ and y varies for 1 to $\sqrt{2}$

$$\therefore \quad \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_0^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\int_0^1 \int_0^y (x^2+y^2)^{-1/2}\,(2x)\,dx\,dy + \frac{1}{2}\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} (2x)\,(x^2+y^2)^{-1/2}\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 \left.\frac{(x^2+y^2)^{1/2}}{1/2}\right|_0^y dy + \frac{1}{2}\int_1^{\sqrt{2}} \left.\frac{(x^2+y^2)^{1/2}}{1/2}\,.\right|_0^{\sqrt{2-y^2}} dy$$

$$\left|\ \text{by using } \int [f(x)]\,f'(x)dx = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1\right.$$

$$= \int_0^1 \left(\sqrt{2}y - y\right)dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right)dy$$

$$= \left.\left(\frac{\sqrt{2}y^2}{2} - \frac{y^2}{2}\right)\right|_0^1 + \left.\left(\sqrt{2}y - \frac{y^2}{2}\right)\right|_1^{\sqrt{2}} = \frac{\sqrt{2}-1}{2} + 2 - 1 - \sqrt{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left[\sqrt{2} - 1 + 4 - 2 - 2\sqrt{2} + 1\right] = \frac{1}{2}\left[2 - \sqrt{2}\right] = 1 - \frac{1}{\sqrt{2}}.$$

Example 5. *Change the order of integration in the following integral and evaluate:*

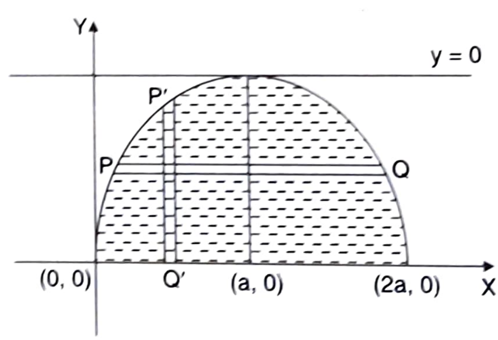
$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$$

Sol. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$

From the limits of integration it is clear that we have to first integrate w.r.t. x and then w.r.t. y

x varies from $a - \sqrt{a^2-y^2}$ to $a + \sqrt{a^2-y^2}$

i.e., $x = a \pm \sqrt{a^2-y^2}$; $(x-a)^2 = a^2 - y^2$ or $x^2 + y^2 - 2ax = 0$ *i.e.*, inside the circle with centre at $(a, 0)$ and radius a and y varies from 0 to a.



This integration is first perfomed along horizontal strips PQ. For changing the order of integration divide the region into vertical strips P′Q′ where y varies from 0 to $\sqrt{2ax - x^2}$ and x varies from 0 to $2a$. The region of integration is shown shaded in the figure.

$$\therefore \quad \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dxdy = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\,dx = \int_0^{2a} y \Big|_0^{\sqrt{2ax-x^2}} dx$$

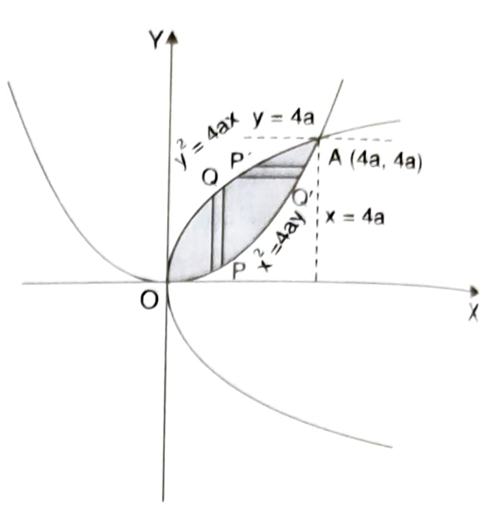
$$= \int_0^{2a} \sqrt{2ax - x^2}\,dx = \int_0^{2a} \sqrt{a^2 - (x-a)^2}\,dx$$

$$= \left.\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}\right|_0^{2a} = \frac{a^2}{2}\frac{\pi}{2} - \frac{a^2}{2}\left(-\frac{\pi}{2}\right) = \frac{a^2\pi}{2}.$$

Example 6. *Change the order of integration in the following integral and evaluate*

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx.$$

Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies from $y = \dfrac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and then w.r.t. x which varies from $x = 0$ to $x = 4a$. Thus integration is first performed along the vertical strip PQ which extends from a point P on the parabola $y = \dfrac{x^2}{4a}$ (*i.e.*, $x^2 = 4ay$) to the point Q on the parabola $y = 2\sqrt{ax}$ (*i.e.*, $y^2 = 4ax$). Then the strip slides from O to A $(4a, 4a)$, the point of intersection of the two parabolas.



For changing the order of integration, we divide the region of integration OPAQO into horizontal strips P′Q′ which extend from P′ on the parabola $y^2 = 4ax$ *i.e.*, $x = \dfrac{y^2}{4a}$ to Q′ on the parabola $x^2 = 4ay$ *i.e.*, $x = 2\sqrt{ay}$. Then this strip slides from O to A$(4a, 4a)$, *i.e.*, varies from 0 to $4a$.

$$\therefore \quad \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\, dy$$

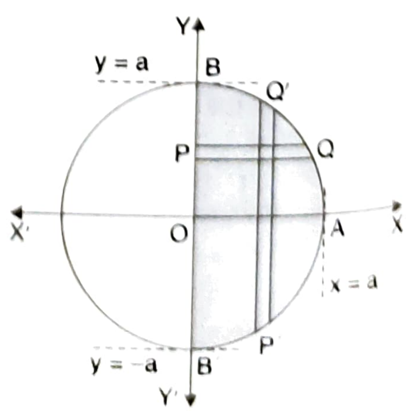
$$= \int_0^{4a}\Big[x\Big]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left[2\sqrt{a}\cdot\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a}\cdot(4a)^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{4}{3}\sqrt{a}\cdot 8a^{3/2} - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 7. *Change the order of integration in the integral* $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy.$

Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. x which varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and then w.r.t. y which varies from $y = -a$ to $y = a$. Thus integration is first performed along the horizontal strip PQ which extends from a point P on $x = 0$ (*i.e.*, y-axis) to the point Q on the circle $x = \sqrt{a^2 - y^2}$ (*i.e.*, $x^2 = a^2 - y^2$ or $x^2 + y^2 = a^2$). Then the strip slides from B′ to B.

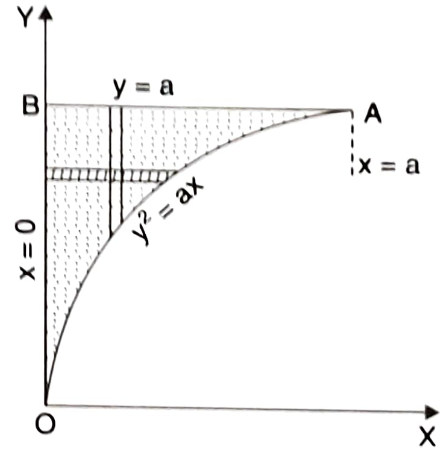


For changing the order of integration, we divide the region of integration B′AQBPB′ into vertical strips P′Q′ which extend from P′ on the circle $y = -\sqrt{a^2 - x^2}$ to Q′ on the circle $y = \sqrt{a^2 - x^2}$. $[x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$; for points in the 4^{th} quadrant. $y = -\sqrt{a^2 - x^2}$ and for points in the first quadrant, $y = \sqrt{a^2 - x^2}]$. Then this strip slides from y-axis ($x = 0$) to A, where $x = a$.

$$\therefore \quad \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x, y)\, dx\, dy = \int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy\, dx.$$

Example 8. *Change the order of integration and hence evaluate*

$$\int_0^a\int_{\sqrt{ax}}^{a} \frac{y^2\, dy\, dx}{\sqrt{y^4 - a^2x^2}}.$$



Sol. The given limits show that the area of integration lies between $y^2 = ax$, $y = a$, $x = 0$ and $x = a$.

We can consider it as lying between $y = 0$, $y = a$, $x = 0$ and $x = y^2/a$ by changing the order of integration. Hence the given integral,

$$\int_{x=0}^{a}\int_{y=\sqrt{ax}}^{a} \frac{y^2\, dy\, dx}{\sqrt{y^4 - a^2x^2}} = \int_{y=0}^{a}\int_{x=0}^{\frac{y^2}{a}} \frac{y^2\, dx\, dy}{\sqrt{y^4 - a^2x^2}}$$

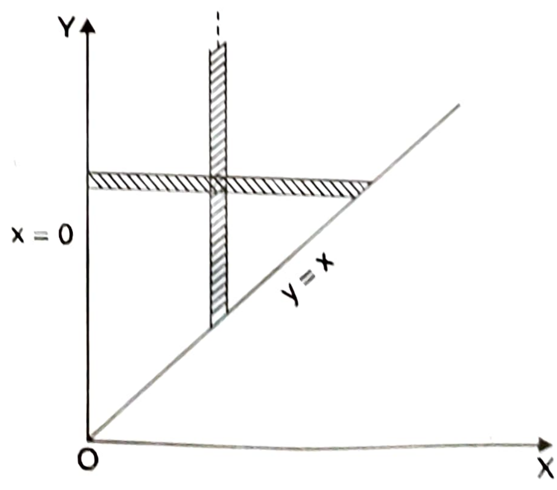
$$= \frac{1}{a}\int_0^a\int_0^{\frac{y^2}{a}} \frac{y^2\, dx\, dy}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}} = \frac{1}{a}\int_0^a y^2\left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a y^2\, [\sin^{-1}(1) - \sin^{-1}(0)]\, dy$$

$$= \frac{\pi}{2a}\int_0^a y^2\, dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a = \frac{\pi}{6a}(a^3) = \frac{\pi a^2}{6}.$$

Example 9. *Evaluate the following integral by changing the order of integration:*

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$$



Sol. The given limits show that the area of integration lies between $y = x$, $y = \infty$, $x = 0$ and $x = \infty$.

We can consider it as lying between $x = 0$, $x = y$, $y = 0$ and $y = \infty$ by changing the order of integration.

Hence the given integral,

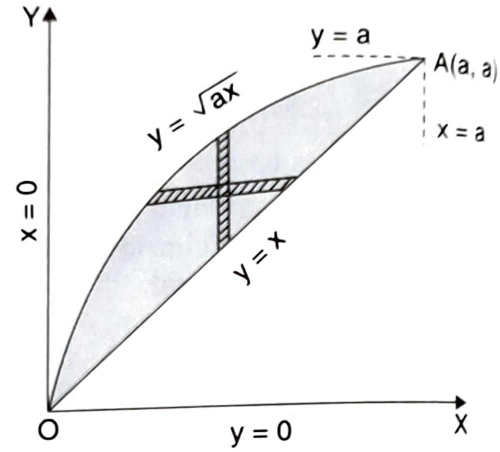
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\left(x\right)_0^y dy = \int_0^\infty e^{-y} dy$$

$$= \left(-e^{-y}\right)_0^\infty = 1 - 0 = 1.$$

Example 10. *By changing the order of integration, evaluate*

$$\int_0^a \int_{y^2/a}^y \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy.$$



Sol. The given limits show that the area of integration (shown shaded) lies between $x = \dfrac{y^2}{a}$, $x = y$, $y = 0$, $y = a$. We can consider it as lying between

$$y = x,\ y = \sqrt{ax},\ x = 0,\ x = a.$$

$$\therefore \int_0^a \int_{\frac{y^2}{a}}^y \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy = \int_0^a \int_x^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dy\, dx$$

$$= \int_0^a \int_x^{\sqrt{ax}} -\frac{1}{2}\cdot\frac{1}{a-x}\cdot(ax-y^2)^{-1/2}\cdot(-2y)\, dy dx$$

$$= \int_0^a \frac{-1}{2(a-x)}\cdot\left[\frac{(ax-y^2)^{1/2}}{\frac{1}{2}}\right]_x^{\sqrt{ax}} dx = \int_0^a \frac{-1}{a-x}\left[0-(ax-x^2)^{1/2}\right] dx$$

$$= \int_0^a \frac{(ax-x^2)^{1/2}}{a-x}\, dx = \int_0^a \frac{\left[x(a-x)\right]^{1/2}}{a-x}\, dx$$

$$= \int_0^a \left(\frac{x}{a-x}\right)^{1/2} dx \qquad \text{Put } x = a\sin^2\theta,\ dx = 2a\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a\sin^2\theta}{a\cos^2\theta}\right)^{1/2}\cdot 2a\sin\theta\cos\theta\, d\theta = 2a\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = 2a\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a}{2}.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals by changing the order of integration:

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$

2. $\int_0^1 \int_{4y}^{4} e^{x^2}\, dx\, dy$

3. $\int_0^4 \int_y^4 \frac{x\, dx\, dy}{x^2 + y^2}$

4. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} x\, dx\, dy$

5. $\int_0^1 \int_{y^2}^{2-y} xy\, dx\, dy$

6. $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$

7. $\int_0^b \int_0^{a/b\sqrt{b^2 - y^2}} xy\, dx\, dy$

8. $\int_0^2 \int_{\frac{x^2}{4}}^{3-x} xy\, dy\, dx$

9. $\int_0^2 \int_{\sqrt{2y}}^{2} \frac{x^2\, dx\, dy}{\sqrt{x^4 - 4y^2}}$

10. $\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}}\, dx\, dy$

11. $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\, dy\, dx$

12. $\int_0^3 \int_1^{\sqrt{4-y}} (x + y)\, dx\, dy$

13. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$

14. $\int_1^4 \int_1^{\sqrt{x}} (x + y^2)\, dy\, dx$.

Answers

1. $\frac{\pi}{16}$
2. $\frac{1}{8}(e^{16} - 1)$
3. π
4. $\frac{a^3\sqrt{2}}{6}$
5. $\frac{3}{8}$
6. $\frac{3a^4}{8}$
7. $\frac{1}{8}a^2b^2$
8. $\frac{8}{3}$
9. $\frac{2\pi}{3}$
10. $\frac{1}{2}$
11. $\frac{1}{2}$
12. $\frac{241}{60}$
13. $\frac{a^3}{28} + \frac{a}{20}$
14. $\frac{241}{30}$.

1.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ which is continuous at every point of a finite region V of three dimensional space. Divide the region V into n sub-regions of respective volumes δV_1, δV_2, ..., δV_n. Let (x_r, y_r, z_r) be an arbitrary point in the r^{th} sub-region. Consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r .$$

The limit of this sum as $n \to \infty$ and $\delta V_r \to 0$, if it exists, is called the *triple integral* of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\, dV$.