1.9 HOMOGENEOUS LINEAR EQUATIONS (Cauchy-Euler Equations)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad \dots(i)$$

where a_i's are constants and X is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant co-efficients by the substitution $x = e^z$ or $z = \log x$

so that
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz} = Dy, \text{ where } D = \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or
$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = D^2 y - Dy = D(D-1)y$$

Similarly, $x^3 \dfrac{d^3 y}{dx^3} = D(D-1)(D-2)y$ and so on.

Substituting these values in equation (i), we get a linear differential equation with constant co-efficients, which can be solved by the methods already discussed.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $x^3 \dfrac{d^3 y}{dx^3} + 2x^2 \dfrac{d^2 y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $\quad x = e^z \quad$ *i.e.*, $\quad z = \log x$

so that

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y, \text{ where } D = \frac{d}{dz}$$

Substituting these values in the given equation, it reduces to

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

which is a linear equation with constant co-efficients.

Its A.E. is $D^3 - D^2 + 2 = 0$ or $(D+1)(D^2 - 2D + 2) = 0$

$$\therefore \quad D = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + e^z(c_2 \cos z + c_3 \sin z) = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$\text{P.I.} = 10\frac{1}{D^3 - D^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2}e^z + \frac{1}{D^3 - D^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + z \cdot \frac{1}{3D^2 - 2D}e^{-z}\right) = 10\left(\frac{1}{2}e^z + z \cdot \frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the C.S. is $y = \dfrac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \dfrac{2}{x}\log x.$

Example 2. *Solve* $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x.$

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.,* $z = \log x$ so that $x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D(D-1)y$, where $D = \dfrac{d}{dz}$.

Substituting these values in the given equation, it reduces to

$$[D(D-1) - D - 3]y = ze^{2z} \quad \text{or} \quad (D^2 - 2D - 3)y = ze^{2z}$$

which is a linear equation with constant co-efficients.

Its A.E. is $D^2 - 2D - 3 = 0$ or $(D-3)(D+1) = 0$

$$\therefore \quad D = 3, -1$$

$$\text{C.F.} = c_1e^{3z} + c_2e^{-z} = c_1x^3 + \frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3}(e^{2z} \cdot z)$$

$$= e^{2z}\frac{1}{(D+2)^2 - 2(D+2) - 3}z = e^{2z}\frac{1}{D^2 + 2D - 3}z$$

$$= e^{2z}\frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)}z = -\frac{1}{3}e^{2z}\left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1}z$$

$$= -\frac{1}{3}e^{2z}\left[1+\left(\frac{2D}{3}+\frac{D^2}{3}\right)+\ldots\ldots\right]z = -\frac{1}{3}e^{2z}\left(z+\frac{2}{3}\right) = -\frac{x^2}{3}\left(\log x+\frac{2}{3}\right)$$

Hence the C.S. is $\quad y = c_1x^3 + \dfrac{c_2}{x} - \dfrac{x^2}{3}\left(\log x + \dfrac{2}{3}\right)$.

Example 3. *Solve* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.*, $z = \log x$ so that $x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

where $D = \dfrac{d}{dz}$.

Substituting these values in the given equation, it reduces to

$$[D(D-1) + D + 1]y = z\sin z$$

or

$$(D^2 + 1)y = z\sin z$$

Its A.E. is $D^2 + 1 = 0$ so that $D = \pm i$

$$\text{C.F.} = c_1\cos z + c_2\sin z = c_1\cos(\log x) + c_2\sin(\log x)$$

$$\text{P.I.} = \frac{1}{D^2+1}z\sin z = \text{Imaginary part of }\frac{1}{D^2+1}ze^{iz}$$

$$= \text{I.P. of } e^{iz}\frac{1}{(D+i)^2+1}z = \text{I.P. of } e^{iz}\frac{1}{D^2+2iD}z$$

$$= \text{I.P. of } e^{iz}\frac{1}{2iD\left(1+\dfrac{D}{2i}\right)}z = \text{I.P of } e^{iz}\frac{1}{2iD\left(1-\dfrac{iD}{2}\right)}z$$

$$= \text{I.P. of }\frac{1}{2i}e^{iz}\frac{1}{D}\left(1-\frac{iD}{2}\right)^{-1}z = \text{I.P. of }\frac{1}{2i}e^{iz}\frac{1}{D}\left(1+\frac{iD}{2}+\ldots\ldots\right)z$$

$$= \text{I.P. of }\frac{1}{2i}e^{iz}\frac{1}{D}\left(z+\frac{i}{2}\right) = \text{I.P. of }\frac{1}{2i}e^{iz}\int\left(z+\frac{i}{2}\right)dz$$

$$= \text{I.P of } -\frac{i}{2}e^{iz}\left(\frac{z^2}{2}+\frac{i}{2}z\right) = \text{I.P. of } e^{iz}\left(-\frac{i}{4}z^2+\frac{z}{4}\right)$$

$$= \text{I.P. of }(\cos z + i\sin z)\left(-\frac{i}{4}z^2+\frac{z}{4}\right) = -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$$= -\frac{1}{4}(\log x)^2\cos(\log x) + \frac{1}{4}\log x\sin(\log x)$$

Hence the C.S. is

$$y = c_1\cos(\log x) + c_2\sin(\log x) - \tfrac{1}{4}(\log x)^2\cos(\log x) + \tfrac{1}{4}\log x\sin(\log x).$$

Example 4. *Solve:* $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$.

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.*, $z = \log x$ so that $x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D(D-1)y,\quad$ where $D = \dfrac{d}{dz}$

Substituting these values in the given equation, it reduces to

$$[D(D-1)+4D+2]y = e^{e^z} \quad \text{or} \quad (D^2+3D+2)y = e^{e^z}$$

Its A.E. is $\quad D^2+3D+2=0 \quad$ or $\quad (D+1)(D+2)=0$

$\therefore \quad D = -1, -2$

$$\text{C.F.} = c_1e^{-z} + c_2e^{-2z} = c_1x^{-1} + c_2x^{-2}$$

$$\text{P.I.} = \frac{1}{D^2+3D+2} e^{e^z} = \frac{1}{(D+1)(D+2)} e^{e^z}$$

$$= \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z} = \frac{1}{D-(-1)} e^{e^z} - \frac{1}{D-(-2)} e^{e^z}$$

$$= e^{-z}\int e^{e^z} . e^z\, dz - e^{-2z}\int e^{e^z} . e^{2z} dz \quad \left[\because \frac{1}{D-a} X = e^{ax}\int X \cdot e^{-ax}\, dx\right]$$

$$= e^{-z}\int e^{e^z} . e^z\, dz - e^{-2z}\int e^{e^z} . e^z . e^z\, dz \qquad | \text{ Put } e^z = t$$

$$= e^{-z}\int e^t\, dt - e^{-2z}\int te^t\, dt$$

$$= e^{-z} . e^t - e^{-2z}(t-1)e^t \qquad | \text{ Integrating by parts}$$

$$= e^{-z} . e^{e^z} - e^{-2z}(e^z - 1)\, e^{e^z}$$

$$= (e^{-z} - e^{-z} + e^{-2z})\, e^{e^z} = e^{-2z} . e^{e^z}$$

$$= x^{-2}e^x$$

Hence the C.S. is $y = c_1x^{-1} + c_2x^{-2} + x^{-2}e^x$ or $y = (c_1x + c_2 + e^x)x^{-2}$.

1.10 LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

An equation of the form

$$(a+bx)^n \frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = X \qquad \ldots(i)$$

where a_i's are constants and X is a function of x, is called Legendre's linear equation.

Such equations can be reduced to linear differential equations with constant co-efficients, by the substitution $a + bx = e^z$ *i.e.*, $z = \log(a+bx)$ so that

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

or $$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\,Dy, \quad \text{where } D = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx} = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or $$(a+bx)^2\frac{d^2y}{dx^2} = b^2(D^2y - Dy) = b^2 D(D-1)y$$

Similarly, $(a + bx)^3 \dfrac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y$.

Substituting these values in equation (*i*), we get a linear differential equation with constant co-efficients, which can be solved by the methods already discussed.

Example 5. *Solve* $(3x + 2)^2 \dfrac{d^2y}{dx^2} + 3(3x + 2) \dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

Sol. Given equation is a Legendre's linear equation.

Put $\quad 3x + 2 = e^z \quad$ *i.e.*, $\quad z = \log(3x + 2) \quad$ so that $\quad (3x + 2)\dfrac{dy}{dx} = 3Dy$,

$$(3x + 2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1)y, \quad \text{where } D = \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2 D(D-1) + 3.3D - 36]y = 3\left(\frac{e^z - 2}{3}\right)^2 + 4\left(\frac{e^z - 2}{3}\right) + 1$$

or
$$9(D^2 - 4)y = \tfrac{1}{3}e^{2z} - \tfrac{1}{3} \quad \text{or} \quad (D^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$$

which is a linear equation with constant co-efficients.

Its A.E. is $\quad D^2 - 4 = 0 \quad \therefore \quad D = \pm 2$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-2z} = c_1(3x + 2)^2 + c_2(3x + 2)^{-2}$$

$$\text{P.I.} = \frac{1}{27} \cdot \frac{1}{D^2 - 4}(e^{2z} - 1) = \frac{1}{27}\left[\frac{1}{D^2 - 4}e^{2z} - \frac{1}{D^2 - 4}e^{0z}\right]$$

$$= \frac{1}{27}\left[z \cdot \frac{1}{2D}e^{2z} - \frac{1}{0 - 4}e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2}\int e^{2z}\,dz + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z} + 1) = \frac{1}{108}[(3x + 2)^2 \log(3x + 2) + 1]$$

Hence the C.S. is $y = c_1(3x + 2)^2 + c_2(3x + 2)^{-2} + \dfrac{1}{108}[(3x + 2)^2 \log(3x + 2) + 1]$.

TEST YOUR KNOWLEDGE

Solve:

1. (*i*) $x^2 y'' + 4xy' + 2y = 0$. (*ii*) $x^2 \dfrac{d^2y}{dx^2} + 9x \dfrac{dy}{dx} + 25y = 50$.

2. $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$.

3. $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 20y = (x + 1)^2$.

4. $x^2 \dfrac{d^3y}{dx^3} - 4x \dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$. **[Hint.** Multiply throughout by x]

5. (*i*) $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1$. (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = x^2$.

   (*iii*) $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$.

6. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

7. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$

8. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x \log x.$

9. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3 \log x.$

10. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2 \log x.$

11. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin (\log x).$

12. $x^3 \dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 65 \cos (\log x).$

13. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \sin (\log x).$

14. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin (\log x) + 1}{x}.$

15. (i) $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}.\dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}.$

(ii) $x^2y'' - 4xy' + 8y = 4x^3 + 2 \sin (\log x)$

16. (i) $(1 + x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4 \cos \log (1 + x).$

(ii) $(1 + x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2 \sin [\log (1 + x)]$

(iii) $(1 + x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin [2 \log (1 + x)]$

17. $(x + 1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x + 3)(2x + 4)$

18. $(1 + 2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1 + 2x)^2$

19. $(2x + 3)^2 \dfrac{d^2y}{dx^2} - 2(2x+3)\dfrac{dy}{dx} - 12y = 6x.$

Answers

1. (i) $y = c_1x^{-1} + c_2x^{-2}$

(ii) $y = x^{-4} [c_1 \cos (3 \log x) + c_2 \sin (3 \log x)] + 2$

2. $y = c_1x^2 + \dfrac{c_2}{x} + \dfrac{1}{3}\left(x^2 - \dfrac{1}{x}\right) \log x$

3. $y = c_1x^{-5} + c_2x^{-4} - \dfrac{x^2}{14} - \dfrac{x}{9} - \dfrac{1}{20}$

4. $y = c_1 + c_2x^3 + c_3x^4 + \dfrac{2}{3} x$

5. (i) $y = (c_1 + c_2 \log x)x + c_2x^{-1} + \dfrac{1}{4x} \log x$

(ii) $y = c_1x^2 + c_2x^3 - x^2 \log x$

(iii) $y = \dfrac{1}{x}(c_1 + c_2 \log x) + \dfrac{1}{x} \log \dfrac{x}{1-x}$

6. $u = \dfrac{kr}{8}(a^2 - r^2)$

7. $y = (c_1 + c_2 \log x)x + \log x + 2$

8. $y = x[c_1 \cos (\log x) + c_2 \sin (\log x)] + x \log x$

9. $y = c_1x^3 + c_2x^{-4} + \dfrac{x^3}{98} \log x (7 \log x - 2)$

10. $y = c_1x^{-1} + c_2x^4 - \dfrac{x^2}{6} - \dfrac{1}{2} \log x + \dfrac{3}{8}$

11. $y = x^2[c_1 \cos (\log x) + c_2 \sin (\log x)] + \dfrac{1}{8} [\sin (\log x) + \cos (\log x)]$

12. $y = c_1x^{-2} + x[c_2 \cos (\sqrt{3} \log x) + c_3 \sin (\sqrt{3} \log x)] + 8 \cos (\log x) - \sin (\log x)$

13. $y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \dfrac{1}{2} x^2 \log x \cos(\log x)$

14. $y = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}}$

$$+ \frac{1}{61x}\left[\log x\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{61}\{27\sin(\log x) + 191\cos(\log x)\}\right] + \frac{1}{6x}(1 + \log x)$$

15. (i) $y = c_1 + c_2 \log x + 2(\log x)^3$

(ii) $y = x^{5/2}\left[c_1 \cos\left(\dfrac{\sqrt{7}}{2}\log x\right) + c_2 \sin\left(\dfrac{\sqrt{7}}{2}\log x\right) + 2x^3 + \dfrac{5}{37}\cos(\log x) + \dfrac{7}{37}\sin(\log x)\right]$

16. (i) $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)]$

(ii) $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$

(iii) $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \dfrac{1}{3}\sin[2\log(1+x)]$

17. $y = c_1 + c_2 \log(x+1) + [\log(x+1)]^2 + x^2 + 8x$

18. $y = (1+2x)^2[c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2]$

19. $y = c_1(2x+3)^{-1} + c_2(2x+3)^3 - \dfrac{3}{16}(2x+3) + \dfrac{3}{4}$.