Exercise Questions

(2) If $u = \sin\left(\frac{x}{y}\right)$; $x = e^t$, $y = t^2$, find $\frac{du}{dt}$ as a function of t. Verify your result by direct substitution.

Sol :- Now $u = f(x, y) = \sin\left(\frac{x}{y}\right)$

where $x = e^t$, $y = t^2$

$\Rightarrow u$ is composite fun. of t.

Now $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = \cos\left(\frac{x}{y}\right) \frac{1}{y} \cdot (e^t) + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \cdot 2t$$

$$= \frac{e^t}{t^2} \cos\left(\frac{e^t}{t^2}\right) - \frac{e^t}{t^4}(2t) \cos\left(\frac{e^t}{t^2}\right)$$

$$= \frac{e^t}{t^2} \cos\left(\frac{e^t}{t^2}\right)\left[1 - \frac{2}{t}\right]$$

Verification of result by direct substitution

$u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$

$$\Rightarrow u = \sin\left(\frac{e^t}{t^2}\right)$$

Now $\frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right)\left[e^t\left(\frac{-2}{t^3}\right) + \frac{1}{t^2}\cdot e^t\right]$

$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)\left[1 - \frac{2}{t}\right]$$

Hence Proved

(4) (a) Let $z = xy$, which represents the ~~Given~~ area of rectangle with x and y as sides.

Now Given at a given instant, $x = 4m$, $y = 3m$

and $\frac{dx}{dt} = 1.5 m/sec.$ and $\frac{dy}{dt} = 0.5 m/sec.$

Now $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$$= (y)(1.5) + (x)(0.5)$$

$$= 3(1.5) + 4(0.5)$$

$$= 4.5 + 2.0 = 6.5$$

So, ~~at~~ The rate at which the area is increasing at that instant is $6.5 m^2/sec.$

(7) If $z = e^{ax+by} f(ax-by)$

Prove that $b \cdot \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$

Solu:- $z = e^{ax+by} f(ax-by)$

$$\frac{\partial z}{\partial x} = e^{ax+by}[f'(ax-by)(a)] + f(ax-by). e^{ax+by}(a)$$

$$\Rightarrow b\frac{\partial z}{\partial x} = abe^{ax+by}[f'(ax-by) + f(ax-by)]$$

Now $\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + e^{ax+by}(b) \cdot f(ax-by)$

$$\Rightarrow a\frac{\partial z}{\partial y} = abe^{ax+by}[f(ax-by) - f'(ax-by)]$$

Consider $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by}(2)f(ax-by)$

$$= 2ab[e^{ax+by} \cdot f(ax-by)]$$

$$= 2abz$$

Hence Proved

(10)(d) If $u = f(x^2+2yz, y^2+2zx)$

Prove that $(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$

Solu:- Let $u = f(v, w)$

where $v = x^2 + 2yz$, $w = y^2 + 2zx$

Now $u = f(v, w)$ where v and w are functions of x, z and y only.

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x}$

Now $\frac{\partial v}{\partial x} = 2x$, $\frac{\partial w}{\partial x} = 2z$

So, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}(2x) + \frac{\partial u}{\partial w}(2z)$

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y}$

$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot (2z) + \frac{\partial u}{\partial w}(2y)$

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z}$

$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \cdot (2y) + \frac{\partial u}{\partial w}(2x)$

Now consider $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$

$= (y^2 - zx)\left[(2x)\left(\frac{\partial u}{\partial v}\right) + 2z \cdot \frac{\partial u}{\partial w}\right]$

$+ (x^2 - yz)\left[2z\frac{\partial u}{\partial v} + 2y\frac{\partial u}{\partial w}\right]$

$+ (z^2 - xy)\left[\frac{\partial u}{\partial v} \cdot 2y + \frac{\partial u}{\partial w}(2x)\right]$

$$= \frac{\partial u}{\partial v}\left[(y^2-zx)2x + (x^2-yz)2z + (z^2-xy)(2y)\right]$$

$$+ \frac{\partial u}{\partial w}\left[(y^2-zx)(2z) + (x^2-yz)(2y) + (z^2-xy)(2x)\right]$$

$$= \frac{\partial u}{\partial v}\left[\cancel{2xy^2} - \cancel{2zx^2} + \cancel{2x^2z} - \cancel{2yz^2} + \cancel{2yz^2} - \cancel{2xy^2}\right]$$

$$+ \frac{\partial u}{\partial w}\left[\cancel{2zy^2} - \cancel{2z^2x} + \cancel{2x^2y} - \cancel{2y^2z} + \cancel{2xz^2} - \cancel{2x^2y}\right]$$

$$= \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}(0) = 0.$$

(11) If $z = x^2 y$ and $x^2 + xy + y^2 = 1$

Show that $\frac{dz}{dx} = 2xy - \frac{x^2(2x+y)}{x+2y}$

Solu:- $z = x^2y$ and $x^2+xy+y^2=1$

Now $\frac{\partial z}{\partial x} = 2xy$, $\frac{\partial z}{\partial y} = x^2$

Given $f(x,y) = c$

i.e. $x^2+xy+y^2=1$

Taking diff w.r.t. x.

$$2x + x\frac{dy}{dx} + y\cdot 1 + 2y\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x+2y) = \frac{-2x-y}{\cancel{x+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$$

Here $\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$

$$= (2xy)(1) + (x^2)\left(\frac{-(2x+y)}{x+2y}\right)$$

$$= 2xy - \frac{x^2(2x+y)}{x+2y}$$

Hence Proved