Bezier Curve-

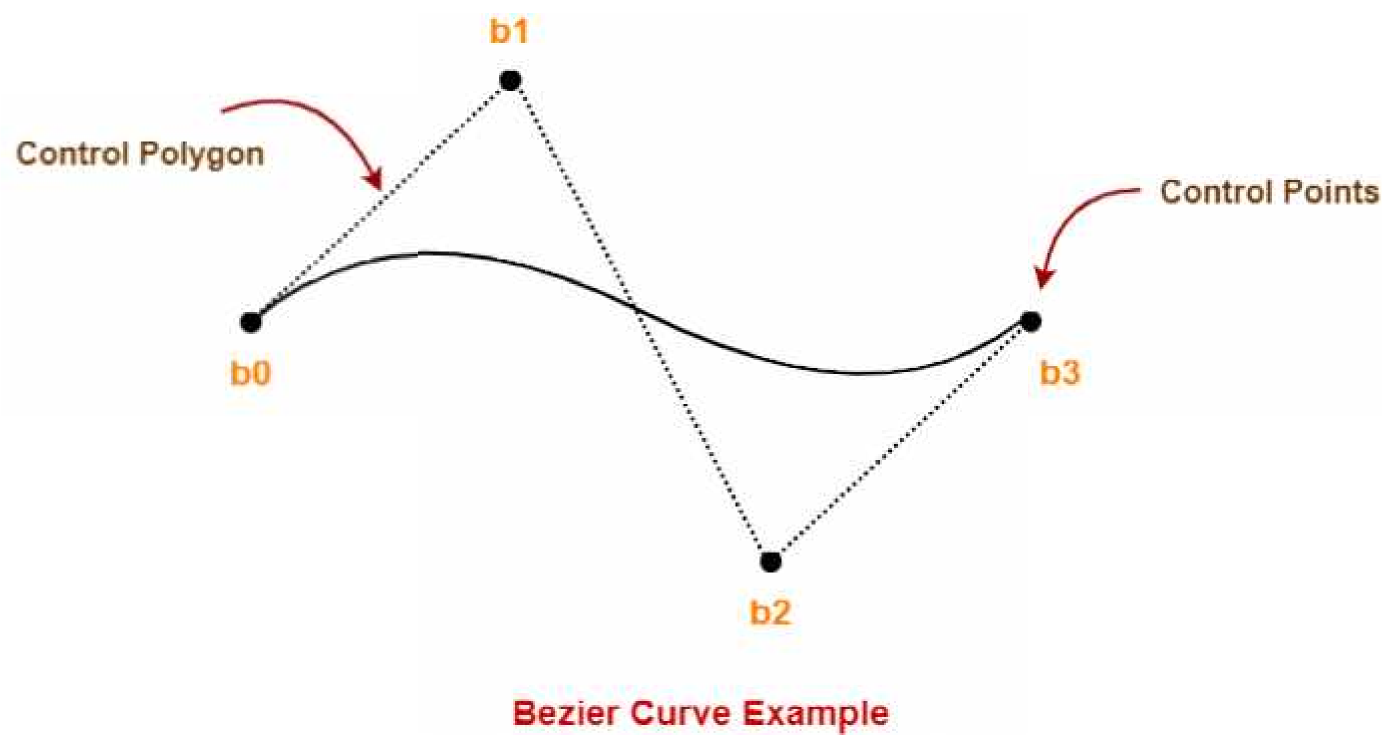
Bezier Curve may be defined as-

- Bezier Curve is parametric curve defined by a set of control points.
- Two points are ends of the curve.
- Other points determine the shape of the curve.

The concept of bezier curves was given by Pierre Bezier.

Bezier Curve Example-

The following curve is an example of a bezier curve-



Here,

- This bezier curve is defined by a set of control points b_0, b_1, b_2 and b_3.
- Points b_0 and b_3 are ends of the curve.
- Points b_1 and b_2 determine the shape of the curve.

-

Bezier Curve Properties-

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Few important properties of a bezier curve are-

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Property-01:

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Bezier curve is always contained within a polygon called as convex hull of its control points.

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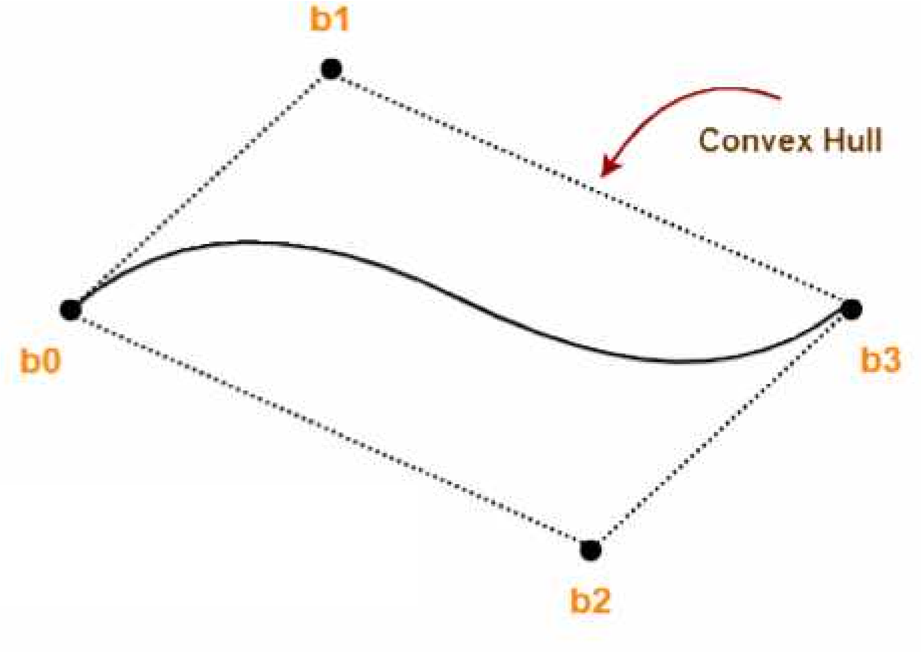

Bezier Curve With Convex Hull

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Property-02:

-

- Bezier curve generally follows the shape of its defining polygon.
- The first and last points of the curve are coincident with the first and last points of the defining polygon.

-

Property-03:

-

The degree of the polynomial defining the curve segment is one less than the total number of control points.

Degree = Number of Control Points – 1

Property-04:

The order of the polynomial defining the curve segment is equal to the total number of control points.

Order = Number of Control Points

Property-05:

- Bezier curve exhibits the variation diminishing property.
- It means the curve do not oscillate about any straight line more often than the defining polygon.

Bezier Curve Equation-

A bezier curve is parametrically represented by-

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t)$$

Bezier Curve Equation

Here,

- t is any parameter where 0 <= t <= 1

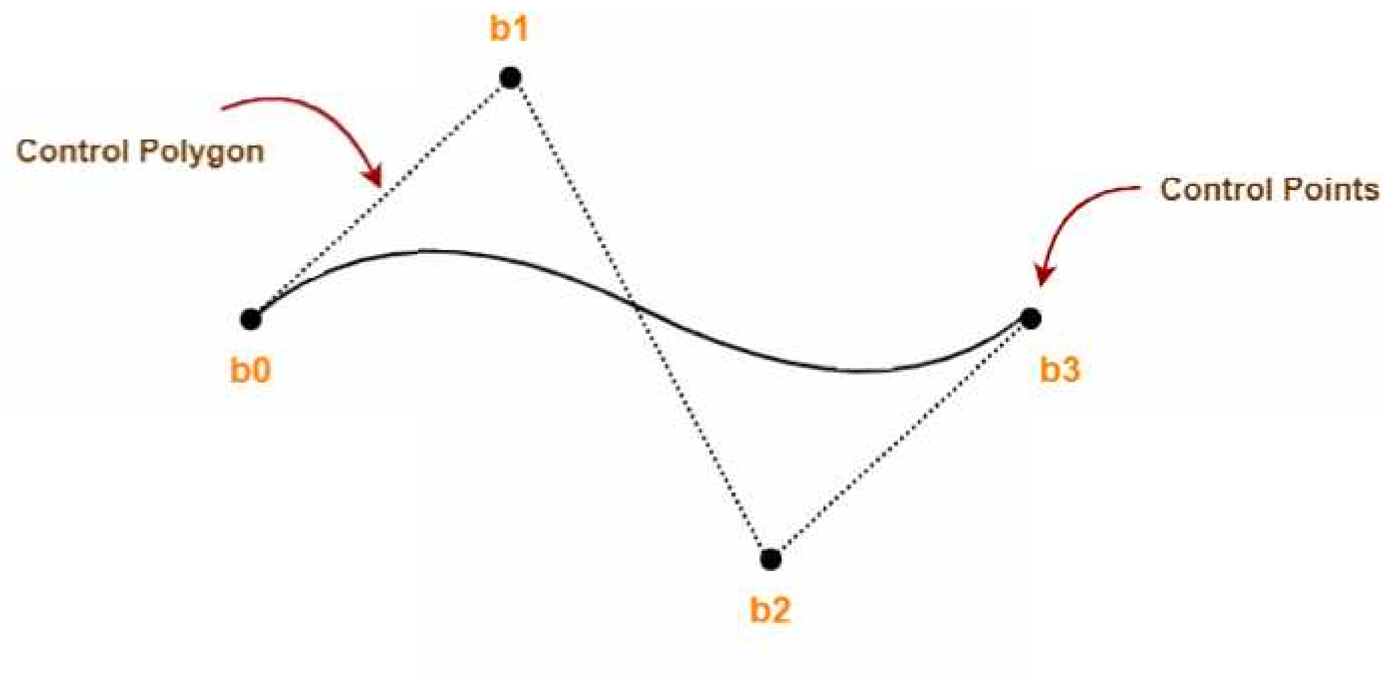
- $P(t)$ = Any point lying on the bezier curve
- B_i = i^{th} control point of the bezier curve
- n = degree of the curve
- $J_{n,i}(t)$ = Blending function = $C(n,i)t^i(1-t)^{n-i}$ where $C(n,i) = n! / i!(n-i)!$

Cubic Bezier Curve-

- Cubic bezier curve is a bezier curve with degree 3.
- The total number of control points in a cubic bezier curve is 4.

Example-

The following curve is an example of a cubic bezier curve-



Cubic Bezier Curve

Here,

- This curve is defined by 4 control points b_0, b_1, b_2 and b_3.
- The degree of this curve is 3.
- So, it is a cubic bezier curve.

-

Cubic Bezier Curve Equation-

-

The parametric equation of a bezier curve is-

-

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t)$$

Bezier Curve Equation

-

Substituting n = 3 for a cubic bezier curve, we get-

-

$$P(t) = \sum_{i=0}^{3} B_i J_{3,i}(t)$$

-

Expanding the above equation, we get-

$P(t) = B_0J_{3,0}(t) + B_1J_{3,1}(t) + B_2J_{3,2}(t) + B_3J_{3,3}(t)$ ……….(1)

-

Now,

-

$$J_{3,0}(t) = \frac{3!}{0!\,(3-0)!} t^{0} (1-t)^{3-0}$$

$$J_{3,0}(t) = (1-t)^{3} \qquad \text{.............}(2)$$

$$J_{3,1}(t) = \frac{3!}{1!\,(3-1)!} t^{1} (1-t)^{3-1}$$

$$J_{3,1}(t) = 3t(1-t)^{2} \qquad \text{.............}(3)$$

$$J_{3,2}(t) = \frac{3!}{2!\,(3-2)!} t^{2} (1-t)^{3-2}$$

$$J_{3,2}(t) = 3t^{2}(1-t)^{2} \qquad \text{.............}(4)$$

$$J_{3,3}(t) = \frac{3!}{3!\,(3-3)!} t^{3} (1-t)^{3-3}$$

$$J_{3,3}(t) = t^{3} \qquad \text{.............}(5)$$

-

Using (2), (3), (4) and (5) in (1), we get-

-

$$P(t) = B_0(1-t)^3 + B_1 3t(1-t)^2 + B_2 3t^2(1-t) + B_3 t^3$$

-

This is the required parametric equation for a cubic bezier curve.

-

Applications of Bezier Curves-

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Bezier curves have their applications in the following fields-

-

1. Computer Graphics-

-

- Bezier curves are widely used in computer graphics to model smooth curves.
- The curve is completely contained in the convex hull of its control points.
- So, the points can be graphically displayed & used to manipulate the curve intuitively.

-

2. Animation-

-

- Bezier curves are used to outline movement in animation applications such as Adobe Flash and synfig.
- Users outline the wanted path in bezier curves.
- The application creates the needed frames for the object to move along the path.
- For 3D animation, bezier curves are often used to define 3D paths as well as 2D curves.

-

3. Fonts-

-

- True type fonts use composite bezier curves composed of quadratic bezier curves.
- Modern imaging systems like postscript, asymptote etc use composite bezier curves composed of cubic bezier curves for drawing curved shapes.

-

PRACTICE PROBLEMS BASED ON BEZIER CURVE IN COMPUTER GRAPHICS-

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Problem-01:

-

Given a bezier curve with 4 control points-

$B_0[1\ 0]$, $B_1[3\ 3]$, $B_2[6\ 3]$, $B_3[8\ 1]$

Determine any 5 points lying on the curve. Also, draw a rough sketch of the curve.

-

Solution-

-

We have-

- The given curve is defined by 4 control points.
- So, the given curve is a cubic bezier curve.

-

The parametric equation for a cubic bezier curve is-

-

$$\mathbf{P(t) = B_0(1-t)^3 + B_13t(1-t)^2 + B_23t^2(1-t) + B_3t^3}$$

-

Substituting the control points B_0, B_1, B_2 and B_3, we get-

$P(t) = [1\ 0](1-t)^3 + [3\ 3]3t(1-t)^2 + [6\ 3]3t^2(1-t) + [8\ 1]t^3$ ……..(1)

-

Now,

To get 5 points lying on the curve, assume any 5 values of t lying in the range 0 <= t <= 1.

Let 5 values of t are 0, 0.2, 0.5, 0.7, 1

-

For t = 0:

-

Substituting t=0 in (1), we get-

$P(0) = [1\ 0](1-0)^3 + [3\ 3]3(0)(1-t)^2 + [6\ 3]3(0)^2(1-0) + [8\ 1](0)^3$

$P(0) = [1\ 0] + 0 + 0 + 0$

$P(0) = [1\ 0]$

-

For t = 0.2:

-

Substituting t=0.2 in (1), we get-

$P(0.2) = [1\ 0](1-0.2)^3 + [3\ 3]3(0.2)(1-0.2)^2 + [6\ 3]3(0.2)^2(1-0.2) + [8\ 1](0.2)^3$

$P(0.2) = [1\ 0](0.8)^3 + [3\ 3]3(0.2)(0.8)^2 + [6\ 3]3(0.2)^2(0.8) + [8\ 1](0.2)^3$

$P(0.2) = [1\ 0] \times 0.512 + [3\ 3] \times 3 \times 0.2 \times 0.64 + [6\ 3] \times 3 \times 0.04 \times 0.8 + [8\ 1] \times 0.008$

$P(0.2) = [1\ 0] \times 0.512 + [3\ 3] \times 0.384 + [6\ 3] \times 0.096 + [8\ 1] \times 0.008$

$P(0.2) = [0.512\ 0] + [1.152\ 1.152] + [0.576\ 0.288] + [0.064\ 0.008]$

$P(0.2) = [2.304\ 1.448]$

-

For t = 0.5:

-

Substituting t=0.5 in (1), we get-

$P(0.5) = [1\ 0](1-0.5)^3 + [3\ 3]3(0.5)(1-0.5)^2 + [6\ 3]3(0.5)^2(1-0.5) + [8\ 1](0.5)^3$

$P(0.5) = [1\ 0](0.5)^3 + [3\ 3]3(0.5)(0.5)^2 + [6\ 3]3(0.5)^2(0.5) + [8\ 1](0.5)^3$

$P(0.5) = [1\ 0] \times 0.125 + [3\ 3] \times 3 \times 0.5 \times 0.25 + [6\ 3] \times 3 \times 0.25 \times 0.5 + [8\ 1] \times 0.125$

$P(0.5) = [1\ 0] \times 0.125 + [3\ 3] \times 0.375 + [6\ 3] \times 0.375 + [8\ 1] \times 0.125$

$P(0.5) = [0.125\ 0] + [1.125\ 1.125] + [2.25\ 1.125] + [1\ 0.125]$

$P(0.5) = [4.5\ 2.375]$

-

For t = 0.7:

-

Substituting t=0.7 in (1), we get-

$P(t) = [1\ 0](1-t)^3 + [3\ 3]3t(1-t)^2 + [6\ 3]3t^2(1-t) + [8\ 1]t^3$

$P(0.7) = [1\ 0](1-0.7)^3 + [3\ 3]3(0.7)(1-0.7)^2 + [6\ 3]3(0.7)^2(1-0.7) + [8\ 1](0.7)^3$

$P(0.7) = [1\ 0](0.3)^3 + [3\ 3]3(0.7)(0.3)^2 + [6\ 3]3(0.7)^2(0.3) + [8\ 1](0.7)^3$

$P(0.7) = [1\ 0] \times 0.027 + [3\ 3] \times 3 \times 0.7 \times 0.09 + [6\ 3] \times 3 \times 0.49 \times 0.3 + [8\ 1] \times 0.343$

P(0.7) = [1 0] x 0.027 + [3 3] x 0.189 + [6 3] x 0.441 + [8 1] x 0.343

P(0.7) = [0.027 0] + [0.567 0.567] + [2.646 1.323] + [2.744 0.343]

P(0.7) = [5.984 2.233]

-

For t = 1:

-

Substituting t=1 in (1), we get-

P(1) = [1 0]$(1-1)^3$ + [3 3]3(1)$(1-1)^2$ + [6 3]3$(1)^2$(1-1) + [8 1]$(1)^3$

P(1) = [1 0] x 0 + [3 3] x 3 x 1 x 0 + [6 3] x 3 x 1 x 0 + [8 1] x 1

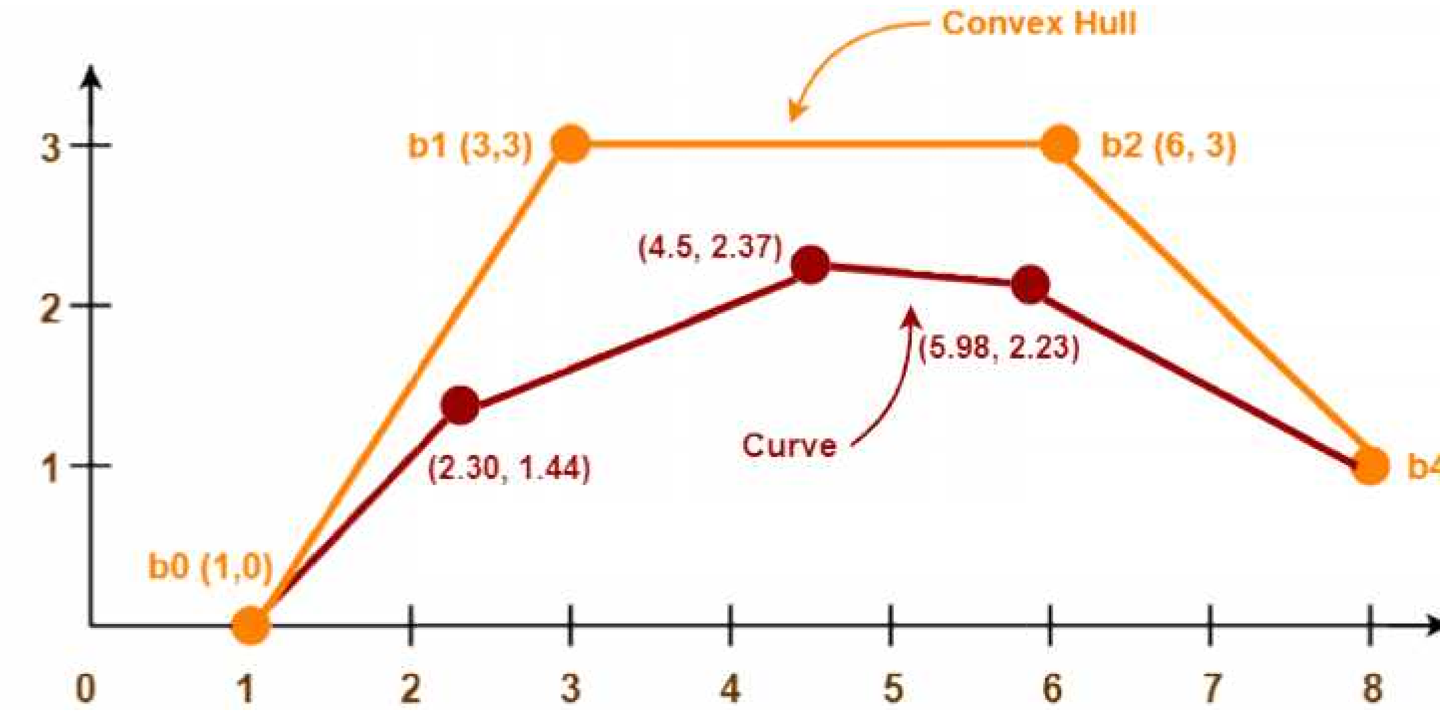
P(1) = 0 + 0 + 0 + [8 1]

P(1) = [8 1]

-

Following is the required rough sketch of the curve-

-



-

To gain better understanding about Bezier Curves in Computer Graphics,

Types of Curves

A curve is an infinitely large set of points. Each point has two neighbors except endpoints. Curves can be broadly classified into three categories − **explicit, implicit,** and **parametric curves**.

Implicit Curves

Implicit curve representations define the set of points on a curve by employing a procedure that can test to see if a point in on the curve. Usually, an implicit curve is defined by an implicit function of the form −

fx,yx,y = 0

It can represent multivalued curves multipleyvaluesforanxvaluemultipleyvaluesforanxvalue. A common example is the circle, whose implicit representation is

$$\mathbf{x^2 + y^2 - R^2 = 0}$$

Explicit Curves

A mathematical function y = fxx can be plotted as a curve. Such a function is the explicit representation of the curve. The explicit representation is not general, since it cannot represent vertical lines and is also single-valued. For each value of x, only a single value of y is normally computed by the function.

Parametric Curves

Curves having parametric form are called parametric curves. The explicit and implicit curve representations can be used only when the function is known. In practice the parametric curves are used. A two-dimensional parametric curve has the following form −

Ptt = ftt, gtt or Ptt = xtt, ytt

The functions f and g become the x,yx,y coordinates of any point on the curve, and the points are obtained when the parameter t is varied over a certain interval [a, b], normally [0, 1].

Bezier Curves

Bezier curve is discovered by the French engineer **Pierre Bézier**. These curves can be generated under the control of other points. Approximate tangents by using control points are used to generate curve. The Bezier curve can be represented mathematically as −

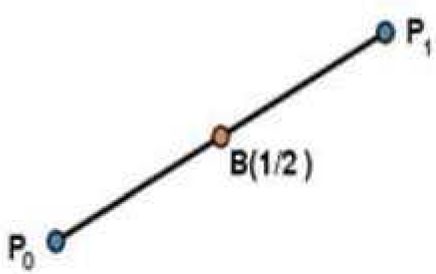
$$\sum_{k=0}^{n} P_i B_i^n(t)$$∑k=0nPiBin(t)

Where $p_i$pi is the set of points and $B_i^n(t)$Bin(t) represents the Bernstein polynomials which are given by −

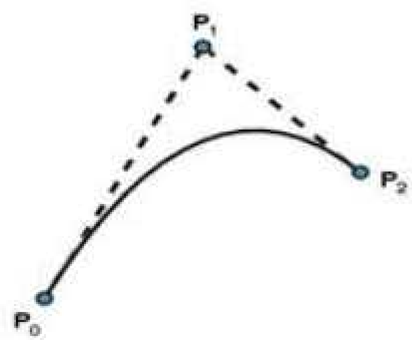
$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$$Bin(t)=(ni)(1−t)n−iti

Where **n** is the polynomial degree, **i** is the index, and **t** is the variable.

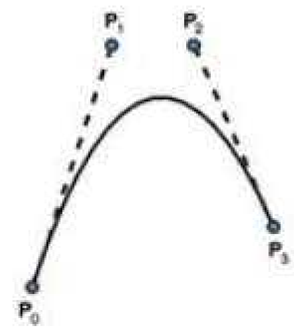
The simplest Bézier curve is the straight line from the point P_0P0 to P_1P1. A quadratic Bezier curve is determined by three control points. A cubic Bezier curve is determined by four control points.



Simple Bezier Curve



Quadratic Bazier Curve



Cubic Bazier Curve

Properties of Bezier Curves

Bezier curves have the following properties −

- They generally follow the shape of the control polygon, which consists of the segments joining the control points.
- They always pass through the first and last control points.
- They are contained in the convex hull of their defining control points.
- The degree of the polynomial defining the curve segment is one less that the number of defining polygon point. Therefore, for 4 control points, the degree of the polynomial is 3, i.e. cubic polynomial.
- A Bezier curve generally follows the shape of the defining polygon.
- The direction of the tangent vector at the end points is same as that of the vector determined by first and last segments.
- The convex hull property for a Bezier curve ensures that the polynomial smoothly follows the control points.

- No straight line intersects a Bezier curve more times than it intersects its control polygon.
- They are invariant under an affine transformation.
- Bezier curves exhibit global control means moving a control point alters the shape of the whole curve.
- A given Bezier curve can be subdivided at a point t=t0 into two Bezier segments which join together at the point corresponding to the parameter value t=t0.

B-Spline Curves

The Bezier-curve produced by the Bernstein basis function has limited flexibility.

- First, the number of specified polygon vertices fixes the order of the resulting polynomial which defines the curve.
- The second limiting characteristic is that the value of the blending function is nonzero for all parameter values over the entire curve.

The B-spline basis contains the Bernstein basis as the special case. The B-spline basis is non-global.

A B-spline curve is defined as a linear combination of control points Pi and B-spline basis function $N_{i,}$Ni, k tt given by

$C(t)=\sum_{i=0}^{n} P_i N_{i,k}(t)$,C(t)=∑i=0nPiNi,k(t), $n \geq k-1$,n≥k−1, tϵ$[t_{k-1},t_{n+1}]$tϵ[tk−1,tn+1]

Where,

- {$p_i$pi: i=0, 1, 2….n} are the control points
- k is the order of the polynomial segments of the B-spline curve. Order k means that the curve is made up of piecewise polynomial segments of degree k - 1,
- the $N_{i,k}(t)$Ni,k(t) are the "normalized B-spline blending functions". They are described by the order k and by a non-decreasing sequence of real numbers normally called the "knot sequence".

$$t_i:i=0,...n+K \quad ti:i=0,...n+K$$

The N_i, k functions are described as follows −

$$N_{i,1}(t)=\{1,0,\text{if}\, u\epsilon[t_i,t_{i+1})\,\text{Otherwise} \quad Ni,1(t)=\{1,ifuϵ[ti,ti+1)0,Otherwise$$

and if k > 1,

$$N_{i,k}(t)=t-t_i t_{i+k-1} N_{i,k-1}(t)+t_{i+k}-t t_{i+k}-t_{i+1} N_{i+1,k-1}(t) \quad Ni,k(t)=t-titi+k-1Ni,k-1(t)+ti+k-tti+k-ti+1Ni+1,k-1(t)$$

and

$$t\epsilon[t_{k-1},t_{n+1}) \quad t\epsilon[tk-1,tn+1)$$

Properties of B-spline Curve

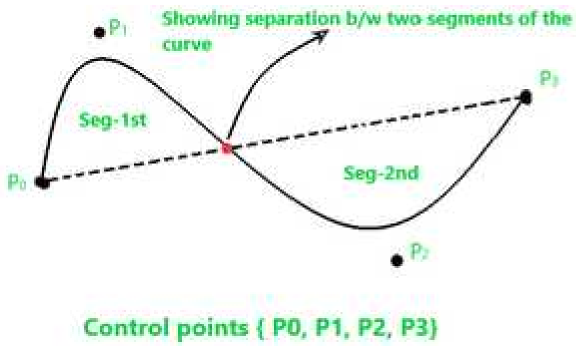
B-spline curves have the following properties −

- The sum of the B-spline basis functions for any parameter value is 1.
- Each basis function is positive or zero for all parameter values.
- Each basis function has precisely one maximum value, except for k=1.
- The maximum order of the curve is equal to the number of vertices of defining polygon.
- The degree of B-spline polynomial is independent on the number of vertices of defining polygon.
- B-spline allows the local control over the curve surface because each vertex affects the shape of a curve only over a range of parameter values where its associated basis function is nonzero.
- The curve exhibits the variation diminishing property.
- The curve generally follows the shape of defining polygon.
- Any affine transformation can be applied to the curve by applying it to the vertices of defining polygon.
- The curve line within the convex hull of its defining polygon.

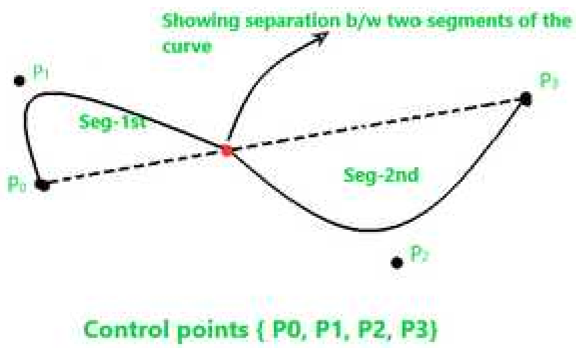
Concept of **B-spline** curve came to resolve the disadvantages having by **Bezier curve,** as we all know that both curves are parametric in nature. In Bezier curve we face a problem, when we change any of the control point respective location the whole curve shape gets change. But here in B-spline curve, the only a specific segment of the curve-shape gets changes or affected by the changing of the corresponding location of the control points.

In the **B-spline curve**, the control points impart local control over the curve-shape rather than the global control like **Bezier-curve.**

B-spline curve shape before changing the position of control point P_1 –



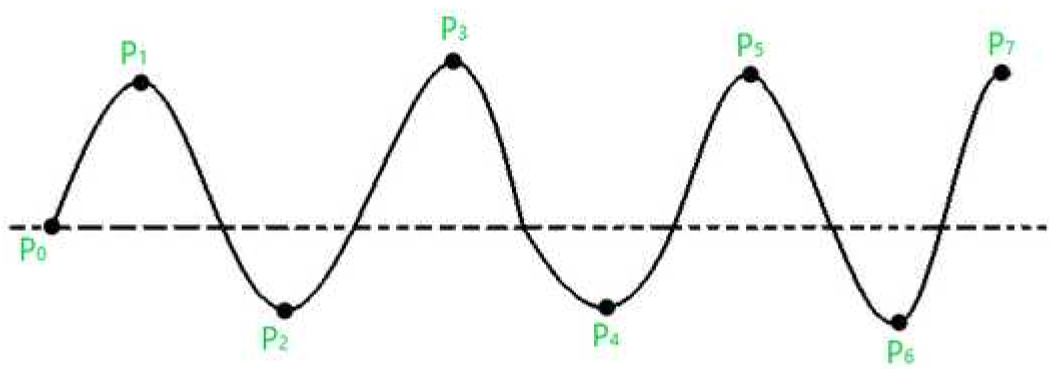
B-spline curve shape after changing the position of control point P_1 –



You can see in the above figure that only the **segment-1st** shape as we have only changed the control point P_1, and the shape of segment-2nd remains intact.

B-spline Curve :

As we see above that the B-splines curves are independent of the number of control points and made up of joining the several segments smoothly, where each segment shape is decided by some specific control points that come in that region of segment. Consider a curve given below –



Attributes of this curve are –

- We have "n+1" control points in the above, so, n+1=8, so n=7.
- Let's assume that the order of this curve is 'k', so the curve that we get will be of a polynomial degree of "k-1". Conventionally it's said that the value of 'k' must be in the range: $2 \le k \le n+1$. So, let us assume k=4, so the curve degree will be k-1 = 3.
- The total number of segments for this curve will be calculated through the following formula –
  Total no. of seg = n – k + 2 = 7 – 4 + 2 = 5.

Segments	Control points	Parameter
S_0	P_0,P_1,P_2,P_3	$0 \le t \le 2$
S_1	P_1,P_2,P_3,P_4	$2 \le t \le 3$
S_2	P_2,P_3,P_4,P_5	$3 \le t \le 4$
S_3	P_3,P_4,P_5,P_6	$4 \le t \le 5$
S_4	P_4,P_5,P_6,P_7	$5 \le t \le 6$

Knots in B-spline Curve :
The point between two segments of a curve that joins each other such points are known as knots in **B-spline curve.** In the case of the cubic polynomial degree curve, the knots are **"n+4".** But in other common cases, we have **"n+k+1"** knots. So, for the above curve, the total knots vectors will be –

```
Total knots = n+k+1 = 7 + 4 + 1 = 12
```

These knot vectors could be of three types –

- Uniform (periodic)
- Open-Uniform
- Non-Uniform

B-spline Curve Equation : The equation of the spline-curve is as follows –

Where **P_i, k, t** correspondingly represents the control points, degree, parameter of the curve.

And following are some conditions for **x_i** are as follows –

Some cases of Basis function :

Properties of B-spline Curve :

- Each basis function has 0 or +ve value for all parameters.
- Each basis function has one maximum value except for k=1.
- The degree of B-spline curve polynomial does not depend on the number of control points which makes it more reliable to use than Bezier curve.
- B-spline curve provides the local control through control points over each segment of the curve.
- The sum of basis functions for a given parameter is one.

Difference between Spline, B-Spline and Bezier Curves

- Last Updated : 16 Jun, 2020

1. Spline :
A spline curve is a mathematical representation for which it is easy to build an interface that will allow a user to design and control the shape of complex curves and surfaces.
2. B-Spline :
B-Spline is a basis function that contains a set of control points. The B-Spline curves are specified by Bernstein basis function that has limited flexibility.

3. Bezier :
These curves are specified with boundary conditions, with a characterizing matrix or with

blending function. A Bezier curve section can be filled by any number of control points. The number of control points to be approximated and their relative position determine the degree of Bezier polynomial.

Difference between Spline, B-Spline and Bezier Curves :

Spline	B-Spline	Bezier
A spline curve can be specified by giving a specified set of coordinate positions, called control points which indicate the general shape of the curve.	The B-Spline curves are specified by Bernstein basis function that has limited flexibiity.	The Bezier curves can be specified with boundary conditions, with a characterizing matrix or with blending function.
It follows the general shape of the curve.	These curves are a result of the use of open uniform basis function.	The curve generally follows the shape of a defining polygon.
Typical CAD application for spline include the design of automobile bodies, aircraft and spacecraft surfaces and ship hulls.	These curves can be used to construct blending curves.	These are found in painting and drawing packages as well as in CAD applications.
It possess a high degree of smoothness at the places where the polynomial pieces connect.	The B-Spline allows the order of the basis function and hence the degree of the resulting curve is	The degree of the polynomial defining the curve segment is one less than the number of defining

	independent of number of vertices.	polygon point.
A spline curve is a mathematical representation for which it is easy to build an interface that will allow a user to design and control the shape of complex curves and surfaces.	In B-Spline, there is local control over the curve surface and the shape of the curve is affected by every	