Hence Proved

(16) (a) If $x^3+y^3-3axy=0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using partial derivative method.

Soln :- Given $f(x,y)=c$

i.e. $x^3+y^3-3axy=0$

Here $f_x=3x^2-3ay$, $f_y=3y^2-3ax$ — (1)

So, $$\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-(3x^2-3ay)}{3y^2-3ax}=\frac{-(x^2-ay)}{y^2-ax}$$

Now $$\frac{dy}{dx}=\frac{-f_x}{f_y}$$

So, $$\frac{d^2y}{dx^2}=-\left[\frac{f_y\frac{d}{dx}(f_x)-f_x\frac{d}{dx}(f_y)}{(f_y)^2}\right] \quad (2)$$

Now $$\frac{d}{dx}(f_x)=\frac{\partial f_x}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx}$$

$\Rightarrow \frac{d}{dx}(f_x) = f_{xx}(1) + f_{xy}\, y'$

and $\frac{d}{dx}(f_y) = \frac{\partial f_y}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f_y}{\partial y} \cdot \frac{dy}{dx}$

$= f_{xy}(1) + f_{yy}\, y'$

$\rightarrow \frac{d}{dx}(f_y) = f_{xy} + f_{yy} \cdot y'$

Put these values in (2)

So, $\frac{d^2y}{dx^2} = -\left[\frac{f_y(f_{xx} + f_{xy}\, y') - f_x(f_{xy} + f_{yy} \cdot y')}{(f_y)^2}\right]$

$= \frac{f_x(f_{xy} + f_{yy} \cdot y') - f_y(f_{xx} + f_{xy}\, y')}{(f_y)^2}$

Now $f_x = 3(x^2 - ay)$ $\qquad f_y = 3(y^2 - ax)$

$f_{yy} = 6y$

So, $f_{xx} = 6x$

$\cancel{f_{yx}}$

$f_{xy} = -3a = f_{yx}$

So, $\frac{d^2y}{dx^2} = \frac{3(x^2 - ay)\left[-3a + 6y\left(\frac{-(x^2 - ay)}{y^2 - ax}\right)\right] - 3(y^2 - ax)\left[6x - 3a\left(\frac{-(x^2 - ay)}{y^2 - ax}\right)\right]}{9(y^2 - ax)^2}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{9(x^2-ay)\left[\dfrac{-a(y^2-ax)-2y(x^2-ay)}{y^2-ax}\right] - 9(y^2-ax)\left[\dfrac{2x(y^2-ax)+a(x^2-ay)}{y^2-ax}\right]}{9(y^2-ax)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-9(x^2-ay)\left[\dfrac{ay^2-a^2x+2yx^2-2ay^2}{y^2-ax}\right] - 9(y^2-ax)\left[\dfrac{2xy^2-2ax^2+ax^2-ay^2}{y^2-ax}\right]}{9(y^2-ax)^2}$$

$$= \frac{-9\left[(x^2-ay)(-ay^2-a^2x+2yx^2)+(y^2-ax)(-ax^2-ay^2+2xy^2)\right]}{9(y^2-ax)^3}$$

$$= \frac{-\cancel{9}\left[\left(-ax^2y^2-\cancel{a^2x^3}+2x^4y-a^2y^3+a^3xy-2ax^2y^2 - ax^2y^2-ay^4+2xy^3+\cancel{a^2x^3}+a^2xy^2-2x^2ay^2\right)\right]}{\cancel{9}(y^2-ax)^3}$$

$$= \frac{-\left[-2ax^2y^2+2x^4y-a^2y^3+a^3xy-2ax^2y^2-ay^4+2xy^3 + xy^2a^2-2x^2ay^2\right]}{(y^2-ax)^3}$$

$$= \frac{\left[-6ax^2y^2+2yx^4-a^2y^3+a^3xy-ay^4+2xy^3+xy^2a^2\right]}{(y^2-ax)^3}$$

(20) If $x = e^{r}\cos\theta$, $y = e^{r}\sin\theta$

Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right)$

Soln :- $x = e^{r}\cos\theta$, $y = e^{r}\sin\theta$

Now $\cancel{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}}$

Now u is composite fun. of r and θ.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cdot(e^{r}\cos\theta) + \frac{\partial u}{\partial y}\cdot(e^{r}\sin\theta)$$

$$= \cancel{e^{r}\left[\frac{\partial}{\partial x}\right.}$$

$$= e^{r}\left[\cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}\right]$$

So, $\frac{\partial}{\partial r}(u) = e^{r}\left[\cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}\right]u$

$$\Rightarrow \frac{\partial}{\partial r} = e^{r}\left(\cos\theta\cdot\frac{\partial}{\partial x} + \sin\theta\cdot\frac{\partial}{\partial y}\right)$$

Now $\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} = \left(e^r \cos\theta \frac{\partial}{\partial x} + e^r \sin\theta \frac{\partial}{\partial y}\right)\left(e^r \cos\theta \frac{\partial u}{\partial x} + e^r \sin\theta \frac{\partial u}{\partial y}\right)$$

$$= e^{2r}.\cos^2\theta \frac{\partial^2 u}{\partial x^2} + e^{2r}\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y} + e^{2r}.\sin\theta\cos\theta \frac{\partial^2 u}{\partial y \partial x}$$

$$+ e^{2r}.\sin^2\theta . \frac{\partial^2 u}{\partial y^2}$$

$$= e^{2r}\cos^2\theta \frac{\partial^2 u}{\partial x^2} + \sin 2\theta . e^{2r} . \frac{\partial^2 u}{\partial x \partial y} + e^{2r}\sin^2\theta . \frac{\partial^2 u}{\partial y^2}$$

$$\left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right]$$

similarly $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} . \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial \theta}$

$$\Rightarrow \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-e^r \sin\theta) + \frac{\partial u}{\partial y}(e^r \cos\theta)$$

$$= -e^r \sin\theta \frac{\partial u}{\partial x} + e^r \cos\theta . \frac{\partial u}{\partial y}$$

$$= \left(-e^r \sin\theta \frac{\partial}{\partial x} + e^r \cos\theta \frac{\partial}{\partial y}\right) u$$

Now $\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)$

$$= \left(-e^r \sin\theta . \frac{\partial}{\partial x} + e^r \cos\theta \frac{\partial}{\partial y}\right).$$

$$\left(-e^r \sin\theta \frac{\partial u}{\partial x} + e^r \cos\theta . \frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial u}{\partial \theta^2} = e^{2r}\sin^2\theta \frac{\partial^2 u}{\partial x^2} - \sin\theta\cos\theta\, e^{2r} \frac{\partial^2 u}{\partial x \partial y}$$

$$- \sin\theta\cos\theta\, e^{2r} \frac{\partial^2 u}{\partial y \partial x} + e^{2r}\cos^2\theta \cdot \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial \theta^2} = e^{2r} \cdot \sin^2\theta \frac{\partial^2 u}{\partial x^2} - 2\sin\theta\cos\theta\, e^{2r} \cdot \frac{\partial^2 u}{\partial x \partial y}$$

$$+ e^{2r} \cdot \cos^2\theta \cdot \frac{\partial^2 u}{\partial y^2} \quad \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial \theta^2} = e^{2r} \sin^2\theta \frac{\partial^2 u}{\partial x^2} - \sin 2\theta \cdot e^{2r} \cdot \frac{\partial^2 u}{\partial x \partial y}$$

$$+ e^{2r} \cdot \cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$\cancel{\Rightarrow \frac{\partial^2 u}{\partial \theta^2} =}$$

Now $$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial r^2} = e^{2r}(\sin^2\theta + \cos^2\theta) \cdot \frac{\partial^2 u}{\partial x^2}$$

$$+ e^{2r}(\sin^2\theta + \cos^2\theta) \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial r^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) e^{2r}$$

$$\Rightarrow e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$