1

EXACT DIFFERENTIAL EQUATIONS

1.1(A) RECAPITULATION DEFINITIONS

(*i*) A differential equation is an equation involving differentials or differential coefficients. Thus,

$$\frac{dy}{dx} = x^2 - 1 \qquad \ldots(1) \qquad\qquad \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \qquad \ldots(2)$$

$$(x + y^2 - 3y)\, dx = (x^2 + 3x + y)\, dy \qquad \ldots(3) \qquad\qquad y = x\frac{dy}{dx} + \frac{c}{\frac{dy}{dx}} \qquad \ldots(4)$$

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \qquad \ldots(5) \qquad\qquad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2} \qquad \ldots(6)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \ldots(7) \qquad\qquad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \ldots(8)$$

are all differential equations.

(*ii*) Differential equations which involve only one independent variable and the differential co-efficients with respect to it are called **ordinary differential equations.**

Thus equations (1) to (6) are all ordinary differential equations.

(*iii*) Differential equations which involve two or more independent variables and partial derivatives with respect to them are called **partial differential equations.**

Thus equations (7) and (8) are partial differential equations.

(*iv*) The **order** of a differential equation is the order of the highest order derivative occurring in the differential equation.

Thus equations (1), (3) and (4) are of first order; equations (2) and (6) are of the second order while equation (5) is of the third order.

(*v*) The **degree** of a differential equation is the degree of the highest order derivative which occurs in the differential equation provided the equation has been made free of the radical and fractional powers as far as the derivatives are concerned.

Thus, equations (1), (2), (3) and (5) are of the first degree.

Equation (4) is $$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$$

It is of the second degree.

Equation (6) is $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

It is of the second degree.

(*vi*) **Solution (or primitive) of a Differential Equation.** A solution (or integral) of a differential equation is a relation, free from derivatives, between the variables which satisfies the given equation.

Thus if $y = f(x)$ be the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

For example, $$y = c_1 \cos x + c_2 \sin x$$

is the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Since, $$\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x$$

$$\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

The **general (or complete) solution** of a differential equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation.

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

A **particular solution** of a differential equation is that which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

The solution of a differential equation of n^{th} order is its particular solution if it contains less than n arbitrary constants.

1.1(B) SOLUTION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

All differential equations of the first order and first degree cannot be solved. Only those among them which belong to (or can be reduced to) one of the following categories can be solved by the standard methods.

(i) Equations in which variables are separable.

(ii) Differential equation of the form $\frac{dy}{dx} = f(ax + by + c)$.

(iii) Homogeneous equations. (iv) Linear equations.

(v) Exact equations.

1.1(C) VARIABLES SEPARABLE FORM

If a differential equation of the first order and first degree can be put in the form where dx and all terms containing x are at one place, also dy and all terms containing y are at one place, then the variables are said to be separable.

Thus the general form of such an equation is $f(x)\,dx + \phi(y)\,dy = 0$

Integrating, we get $\int f(x)\,dx + \int \phi(y)\,dy = c$ which is the general solution, c being an arbitrary constant.

Note 1. Any equation of the form $f_1(x)\,\phi_2(y)\,dx + f_2(x)\,\phi_1(y)\,dy = 0$ can be expressed in the above form by dividing throughout by $f_2(x)\,\phi_2(y)$.

Thus, $$\frac{f_1(x)}{f_2(x)}dx + \frac{\phi_1(y)}{\phi_2(y)}dy = 0 \quad \text{or} \quad f(x)\,dx + \phi(y)\,dy = 0.$$

1.1(D) DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(ax + by + c)$

Differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c) \qquad \text{...(1)}$$

It can be reduced to a form in which the variables are separable by the substitution $ax + by + c = t$

so that $$a + b\frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$\therefore$ Equation (1) becomes $$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t) \quad \text{or} \quad \frac{dt}{dx} = a + bf(t).$$

$$\frac{dt}{a + bf(t)} = 2$$

After integrating both sides, t is to be replaced by its value.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.

Sol. The given equation can be written as $y(1 - ay) - (x + a)\frac{dy}{dx} = 0$

$$\frac{dx}{x + a} = \frac{dy}{y(1 - ay)} = 0$$

Integrating both sides, we have, $\int \frac{dx}{x+a} = \int \left(\frac{1}{y} + \frac{a}{1-ay}\right) dy + c$ [Partial Fractions]

$$\Rightarrow \quad \log(x+a) = \left[\log y + a \cdot \frac{\log(1-ay)}{-a}\right] + c$$

$$\Rightarrow \quad \log(x+a) - \log y + \log(1-ay) = \log C, \quad \text{where } c = \log C$$

$$\Rightarrow \quad \log \frac{(x+a)(1-ay)}{y} = \log C \Rightarrow (x+a)(1-ay) = Cy$$

which is the general solution of the given equation.

Note. Here c is replaced by log C to get a neat form of the solution.

Example 2. *Solve $3e^x \tan y \, dx + (1+e^x)\sec^2 y \, dy = 0$, given $y = \frac{\pi}{4}$ when $x = 0$.*

Sol. The given equation can be written as $\frac{3e^x}{1+e^x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

Integrating, we have $\quad 3\log(1+e^x) + \log \tan y = \log c$

$$\Rightarrow \quad \log(1+e^x)^3 \tan y = \log c$$

$$\Rightarrow \quad (1+e^x)^3 \tan y = c \qquad ...(1)$$

which is the general solution of the given equation.

Since $y = \frac{\pi}{4}$ when $x = 0$, we have from (1)

$$(1+1)^3 \times 1 = c \Rightarrow c = 8$$

Therefore the required particular solution is $(1+e^x)^3 \tan y = 8$.

Example 3. *Solve $(x+y+1)^2 \frac{dy}{dx} = 1$.*

Sol. Putting $x + y + 1 = t$, we get $\quad 1 + \frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore the given equation becomes $t^2\left(\frac{dt}{dx} - 1\right) = 1$ or $\frac{dt}{dx} = \frac{1+t^2}{t^2}$

$$\Rightarrow \quad \frac{t^2}{1+t^2} dt = dx$$

Integrating, we have $\quad \int \left(1 - \frac{1}{1+t^2}\right) dt = \int dx + c = dx$ or $t - \tan^{-1} t = x + c$

or $\quad (x+y+1) - \tan^{-1}(x+y+1) = x + c$

or $\quad y = \tan^{-1}(x+y+1) + C$, where $C = c - 1$.

Example 4. *Solve $\frac{dy}{dx} = \sin(x+y)$.*

Sol. Put $x + y = t$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$\therefore$ Given equation changes to

$$\frac{dt}{dx} - 1 = \sin t$$

$$\frac{dt}{dx} = 1 + \sin t \quad \text{or} \quad \frac{dt}{1 + \sin t} = dx$$

Integrating both sides,

$$\int \frac{dt}{1 + \sin t} = \int dx + c$$

or $$\int \frac{1 - \sin t}{\cos^2 t} dt = x + c$$

or $$\int (\sec^2 t - \tan t \sec t) dt = x + c$$

or $$\tan t - \sec t = x + c$$

or $$\sin t - 1 = (x + c) \cos t$$

Substituting back the value of t,

$$\sin (x + y) - 1 = (x + c) \text{ os } (x + y).$$

1.1(E) HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$ ...(1)

is called a homogeneous differential equation if $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree in x and y.

If $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of degree r in x and y, then

$$f_1(x, y) = x^r \phi_1\left(\frac{y}{x}\right) \quad \text{and} \quad f_2(x, y) = x^r \phi_2\left(\frac{y}{x}\right)$$

Therefore, equation (1) reduces to $\frac{dy}{dx} = \frac{\phi_1\left(\frac{y}{x}\right)}{\phi_2\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$ ...(2)

Putting $\frac{y}{x} = v$ i.e., $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (2) becomes $$v + x\frac{dv}{dx} = F(v)$$

Separating the variables, $$\frac{dv}{F(v) - v} = \frac{dx}{x}$$

Integrating, we get the solution in terms of v and x. Replacing v by $\frac{y}{x}$, we get the required solution.

If $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$, put $x = vy$.

Example 5. *Solve:* $x\,dy - y\,dx = \sqrt{x^2 + y^2}\;dx.$

Sol. The given equation can be written as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = F\left(\frac{y}{x}\right) \qquad ...(1)$$

Putting $y = vx$, so that $\frac{dv}{dx} = v + x\frac{dv}{dx}$

Equation (1) becomes $v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$ or $x\frac{dv}{dx} = \sqrt{1 + v^2}$

Separating the variables, $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating both sides, $\log(v + \sqrt{1+v^2}) = \log x + \log c$

$$\left[\because \int \frac{1}{\sqrt{1+v^2}}dv = \cosh^{-1} v = \log(v + \sqrt{1 + v^2}\right]$$

or $\log(v + \sqrt{1+v^2}) = \log(cx)$ or $v + \sqrt{1+v^2} = cx$

or $\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$ $\left[\text{Since } v = \frac{y}{x}\right]$

or $y + \sqrt{x^2 + y^2} = cx^2$

which is the required solution.

Example 6. *Solve* $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

Sol. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

or $\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}} = F\left(\frac{x}{y}\right)$, which is homogeneous equation in $\frac{x}{y}$

$\therefore$ Put $\frac{x}{y} = v$ *i.e.,* $x = vy$ $\therefore$ $\frac{dx}{dy} = v + y\frac{dv}{dy}$

$v + y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v}$ or $y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} - v$

or $y\frac{dv}{dy} = \frac{-(e^v + v)}{1 + e^v}$ or $\frac{1 + e^v}{v + e^v}dv = -\frac{1}{y}dy$

Integrating both sides,

$$\log (v + e^{v}) = -\log y + \log c$$

$\therefore$ $$\log (v + e^{v})y = \log c \Rightarrow y(v + e^{v}) = c$$

$\therefore$ $$y\left(\frac{x}{y} + e^{x/y}\right) = c \quad \text{or} \quad x + ye^{x/y} = c.$$

1.1(F) EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\dfrac{a}{a'} \neq \dfrac{b}{b'}$

Putting $x = X + h$, $y = Y + k$ (h, k are constants)

so that $dx = dX, \quad dy = dY$

Equation (1) becomes $$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{a'(X + h) + b'(Y + k) + c'}$$

$$= \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad ...(2)$$

Choose h and k such that (2) becomes homogeneous.

This requires $ah + bk + c = 0$ and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - a'b} \quad \text{or} \quad h = \frac{bc' - b'c}{ab' - a'b}, k = \frac{ca' - c'a}{ab' - a'b}$$

Since $\dfrac{a}{a'} \neq \dfrac{b}{b'}$, $\therefore$ $ab' - a'b \neq 0$ so that h, k are finite.

$\therefore$ Equation (2) becomes $$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\dfrac{a}{a'} = \dfrac{b}{b'}$, $ab' - a'b = 0$ and the above method fails

Now, $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{m}$ (say) so that $a' = ma$, $b' = mb$

Equation (1) becomes $$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} = f(ax + by)$$

which can be solved by putting $ax + by = t$.

Example 7. *Solve:* $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0.$

Sol. The given equation can be written as $\dfrac{dy}{dx} = -\dfrac{3y - 7x + 7}{7y - 3x + 3}$ $\left[\text{Here } \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$ ...(1)

Putting $x = X + h,\ y = Y + k$ so that $dx = dX,\ dy = dY$ (h, k are constants)

Equation (1) becomes $$\frac{dY}{dX} = -\frac{3(Y+k) - 7(X+h) + 7}{7(Y+k) - 3(X+h) + 3}$$

$$= -\frac{3Y - 7X + (-7h + 3k + 7)}{7Y - 3X + (-3h + 7k + 3)} \quad ...(2)$$

Now, choosing, h, k such that $-7h + 3k + 7 = 0$ and $-3h + 7k + 3 = 0$

Solving these equations $h = 1,\ k = 0.$

With these values of h, k equation (2) reduces to $\dfrac{dY}{dX} = -\dfrac{3Y - 7X}{7Y - 3X}$ ...(3)

Putting $Y = vX$ so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

Equation (3) becomes $v + X\dfrac{dv}{dX} = -\dfrac{3vX - 7X}{7vX - 3X}$ or $X\dfrac{dv}{dX} = \dfrac{7 - 3v}{7v - 3} - v = \dfrac{7 - 7v^2}{7v - 3}$

Separating the variables $\dfrac{7v - 3}{1 - v^2}dv = 7\dfrac{dX}{X}$ or $\left(\dfrac{2}{1-v} - \dfrac{5}{1+v}\right)dv = 7\dfrac{dX}{X}$

Integrating $-2\log(1 - v) - 5\log(1 + v) = 7\log X + c$

or $7\log X + 2\log(1 - v) + 5\log(1 + v) = -c$

or $\log[X^7(1 - v)^2(1 + v)^5] = -c$ or $X^7\left(1 - \dfrac{Y}{X}\right)^2\left(1 + \dfrac{Y}{X}\right)^5 = e^{-c}$

or $(X - Y)^2(X + Y)^5 = C,$ where $C = e^{-c}$ ...(4)

Putting $X = x - h = x - 1,\ Y = y - k = y$

Equation (4) becomes $(x - y - 1)^2(x + y - 1)^5 = C$, which is the required solution.

Example 8. *Solve:* $(3y + 2x + 4)\,dx - (4x + 6y + 5)\,dy = 0.$

Sol. The given equation can be written as $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$ ...(1)

Here, $\dfrac{a}{a'} = \dfrac{b}{b'}$

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right)$

Equation (1) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

$$\frac{dt}{dx} = \frac{3t+12}{2t+5} + 2 = \frac{7t+22}{2t+5}$$

or

Separating the variables $\frac{2t+5}{7t+22} dt = dx$ or $\left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t+22}\right) dt = dx$

Integrating both sides $\frac{2}{7}t - \frac{9}{49} \log (7t+22) = x + c$

$\Rightarrow$

$$14t - 9 \log (7t+22) = 49x + 49c$$

Putting $t = 2x + 3y$, we have

$$14(2x+3y) - 9 \log (14x + 21y + 22) = 49x + 49c$$

or

$$21x - 42y + 9 \log (14x + 21y + 22) = -49c$$

or

$$7(x - 2y) + 3 \log (14x + 21y + 22) = C \qquad \left(\text{where } C = -\frac{49}{3}c\right)$$

which is the required solution.

1.2 EXACT DIFFERENTIAL EQUATIONS

A differential equation obtained from its primitive directly by differentiation, without any operation of multiplication, elimination or reduction etc. is said to be an exact differential equation.

Thus a differential equation of the form M $(x, y)\, dx$ + N $(x, y)\, dy = 0$ is an exact differential equation if it can be obtained directly by differentiating the equation $u(x, y) = c$, which is its primitive.

i.e., if $$du = \text{M}dx + \text{N}dy.$$

For example, the equation $xdx + ydy = 0$ is an exact differential equation, as it can be obtained from its primitive $x^2 + y^2 = c^2$ directly by differentiation.

1.3 THEOREM

The necessary and sufficient condition for the differential equation Mdx + Ndy = 0 *to be exact is*

$$\frac{\partial \text{M}}{\partial y} = \frac{\partial \text{N}}{\partial x}.$$

The condition is necessary

The equation Mdx + Ndy = 0 will be exact, if $du = \text{M}dx + \text{N}dy$

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$\therefore$ $$\text{M}dx + \text{N}dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Equating co-efficients of dx and dy, we get

$$\text{M} = \frac{\partial u}{\partial x} \quad \text{and} \quad \text{N} = \frac{\partial u}{\partial y}$$

$\therefore \qquad \dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

But $\qquad \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

$\therefore \qquad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

which is the necessary condition of exactness.

The condition is sufficient.

Let $\qquad u = \displaystyle\int_{y \text{ constant}} M dx$

$\therefore \qquad \dfrac{\partial u}{\partial x} = M$ and $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial M}{\partial y}$

But $\qquad \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore \qquad \dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$

Integrating both sides w.r.t. x treating y as constant, we have $N = \dfrac{\partial u}{\partial y} + f(y)$

$$\therefore \qquad Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \qquad \left[\because\ M = \frac{\partial u}{\partial x}, N = \frac{\partial u}{\partial y} + f(y)\right]$$

$$= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + f(y)\, dy = du + f(y)\, dy = d[u + \textstyle\int f(y)\, dy]$$

which shows that $Mdx + Ndy$ is an exact differential and hence $Mdx + Ndy = 0$ is an exact differential equation.

Note. Since $\qquad Mdx + Ndy = d[u + \int f(y)\, dy]$

$\therefore \qquad Mdx + Ndy = 0 \Rightarrow d[u + \int f(y)\, dy] = 0$

Integrating, $\qquad u + \int f(y)\, dy = c$

But $\qquad u = \displaystyle\int_{y \text{ constant}} M dx$ and $f(y)$ = terms of N not containing x

Hence the solution of $Mdx + Ndy = 0$ is

$$\int_{y \text{ constant}} \mathbf{M}dx + \int (\text{terms of N not containing } x)\, dy = c.$$

Example 9. *Solve* $(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0.$

Sol. Here $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\therefore \quad \frac{\partial M}{\partial y} = 6x^2y - 6xy^2 = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y\ \text{constant}} M dx + \int (\text{terms of N not containing } x)\, dy = c$$

$$\int_{y\ \text{constant}} (5x^4 + 3x^2y^2 - 2xy^3)\, dx + \int -5y^4 dy = c$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = c.$$

Example 10. *Solve* $[\cos x \tan y + \cos (x + y)]\, dx + [\sin x \sec^2 y + \cos (x + y)]\, dy = 0.$

Sol. Here,

$$M = \cos x \tan y + \cos (x + y)$$

$$N = \sin x \sec^2 y + \cos (x + y)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin (x + y) = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y\ \text{constant}} M dx + \int (\text{terms of N not containing } x)\, dy = c$$

$$\int_{y\ \text{constant}} [\cos x \tan y + \cos (x + y)]\, dx = 0$$

$$\sin x \tan y + \sin (x + y) = c.$$

Example 11. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$

Sol. The given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y\ \text{constant}} M dx + \int (\text{terms of N not containing } x)\, dy = c$$

$$\int_{y\ \text{constant}} (y \cos x + \sin y + y)\, dx = c$$

$$y \sin x + (\sin y + y)\, x = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations (1 to 22):

1. $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)\,dy = 0$
2. $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)\,dy = 0$
3. $y(y^2 - 3x^2)dy + x(x^2 - 3y^2)\,dx = 0,\ y(0) = 1$
4. $(2x^3 - xy^2 - 2y + 3)\,dx - (x^2y + 2x)dy = 0$
5. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$
6. $\left[\dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}\right]dx + \left[\dfrac{1}{y} - \dfrac{x^2}{(x-y)^2}\right]dy = 0$
7. $xdy + ydx + \dfrac{xdy - ydx}{x^2 + y^2} = 0$
8. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$
9. $dx = \dfrac{y}{1 - x^2y^2}dx + \dfrac{x}{1 - x^2y^2}dy$
10. $2x\left(1 + \sqrt{x^2 - y}\right)dx = \sqrt{x^2 - y}\,dy$
11. $(y\cos x + 1)\,dx + \sin x\,dy = 0$
12. (*i*) $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)\,dy = 0$

    (*ii*) $\left[y\left(1 + \dfrac{1}{x}\right)\cos y\right]dx + (x + \log x)(\cos y - y\sin y)\,dy = 0$
13. $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$
14. $(1 + e^{x/y})\,dx + \left(1 - \dfrac{x}{y}\right)e^{x/y}dy = 0$
15. $e^y dx + (xe^y + 2y)\,dy = 0$
16. $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$
17. $(y^2e^{xy^2} + 4x^3)\,dx + (2xye^{xy^2} - 3y^2)\,dy = 0$
18. $(\sec x\tan x\tan y - e^x)dx + \sec x\sec^2 y\,dy = 0$
19. $(\sin x\cos y + e^{2x})\,dx + (\cos x\sin y + \tan y)\,dy = 0$
20. $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2)\,dy = 0$
21. $e^x(\cos y\,dx - \sin y\,dy) = 0,\ y(0) = 0$
22. $\left[\cos x\log(2y - 8) + \dfrac{1}{x}\right]dx + \dfrac{\sin x}{y - 4}dy = 0,\quad y(1) = \dfrac{9}{2}$
23. Find the value of λ for which the differential equation $(xy^2 + \lambda x^2y)dx + (x + y)x^2dy = 0$, is exact. Hence solve it.

Answers

1. $(x + y)\,(1 + 2xy) = c$
2. $x^3 + 3x^2y^2 + y^4 = c$
3. $x^4 - 6x^2y^2 + y^4 = 1$
4. $x^4 - x^2y^2 - 4xy + 6x = c$
5. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
6. $\dfrac{y^2}{y - x} + \log\dfrac{y}{x} = c$
7. $xy - \tan^{-1}\left(\dfrac{x}{y}\right) = c$
8. $x^2 + y^2 + 2a^2\tan^{-1}\left(\dfrac{x}{y}\right) = c$
9. $\log\dfrac{1 + xy}{1 - xy} - 2x = c$
10. $3x^2 + 2(x^2 - y)^{3/2} = c$
11. $y\sin x + x = c$
12. (*i*) $y(x + \log x) + x\cos y = c$ (*ii*) $y\cos y\,(x + \log x) = c$
13. $x^2y + xy - x\tan y + \tan y = c$
14. $x + ye^{x/y} = c$
15. $xe^y + y^2 = c$
16. $e^{xy} + y^2 = c$
17. $e^{xy^2} + x^4 - y^3 = c$
18. $\sec x\tan y - e^x = c$

19. $-\cos x \cos y + \frac{1}{2}e^{2x} + \log \sec y = c$

20. $y \sin x^2 - x^2y + x = c$

21. $e^x \cos y = 1$

22. $\sin x \log (2y - 8) + \log x = 0$

23. $\lambda = 3;\ \frac{1}{2}x^2y^2 + x^3y = c$.

7.4 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Differential equations which are not exact can sometimes be made exact after multiplying by a suitable factor (a function of x and/or y) called the integrating factor.

For example, consider the equation $y\,dx - x\,dy = 0$ ...(1)

Here, $M = y$ and $N = -x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ therefore the equation is not exact.}$$

(*i*) Multiplying the equation by $\frac{1}{y^2}$, it becomes $\frac{ydx - xdy}{y^2} = 0$ or $d\left(\frac{x}{y}\right) = 0$

which is exact.

(*ii*) Multiplying the equation by $\frac{1}{x^2}$, it becomes $\frac{ydx - xdy}{x^2} = 0$ or $d\left(\frac{y}{x}\right) = 0$

which is exact.

(*iii*) Multiplying the equation by $\frac{1}{xy}$, it becomes $\frac{dx}{x} - \frac{dy}{y} = 0$ or $d(\log x - \log y) = 0$

which is exact.

$\therefore$ $\frac{1}{y^2}, \frac{1}{x^2}$ and $\frac{1}{xy}$ are integrating factors of (1).

If a differential equation has one integrating factor, it has an infinite number of integrating factors.

(*a*) **I.F. found by inspection.** In a number of problems, a little analysis helps to find the integrating factor. The following differentials are useful in selecting a suitable integrating factor.

(*i*) $ydx + xdy = d(xy)$

(*ii*) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(*iii*) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(*iv*) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(*v*) $\frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$

(*vi*) $\frac{ydx + xdy}{xy} = d\,[\log (xy)]$

(*vii*) $\frac{xdx + ydy}{x^2 + y^2} = d\left[\frac{1}{2}\log (x^2 + y^2)\right]$

(*viii*) $\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $ydx - xdy + 3x^2y^2 e^{x^3} dx = 0$.

Sol. Since $3x^2e^{x^3} = d(e^{x^3})$, the term $3x^2y^2e^{x^3} dx$ should not involve y^2.

This suggests that $\frac{1}{y^2}$ may be an I.F.

Multiplying throughout by $\frac{1}{y^2}$, we have $\frac{ydx - xdy}{y^2} + 3x^2e^{x^3} dx = 0$

or $$d\left(\frac{x}{y}\right) + d\,(e^{x^3}) = 0, \text{ which is exact.}$$

Integrating, we get $\frac{x}{y} + e^{x^3} = c$, which is the required solution.

Example 2. *Solve* $xdy - ydx = x\sqrt{x^2 - y^2}\ dx$.

Sol. The given equation is $xdy - ydx = x^2\sqrt{1-\left(\frac{y}{x}\right)^2}\,dx$ or $\dfrac{\frac{xdy - ydx}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = dx$

or $$d\left(\sin^{-1}\frac{y}{x}\right) = dx, \text{ which is exact.}$$

Integrating, we get $\sin^{-1}\frac{y}{x} = x + c$ or $y = x \sin(x + c)$, which is the required solution.

Example 3. *Solve:* $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$.

Sol. The given equation is $xdx + ydy - a^2d\left(\tan^{-1}\frac{y}{x}\right) = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} - a^2 \tan^{-1}\frac{y}{x} = c$

or $$x^2 + y^2 - 2a^2 \tan^{-1}\frac{y}{x} = C, \quad \text{where } C = 2c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $xdy - ydx = (x^2 + y^2)\, dx$
2. $xdy - ydx = (x^2 + y^2)\,(dx + dy)$
3. $y(2xy + e^x)\, dx = e^x\, dy$
4. $(y \log y - 2xy)\, dx + (x + y)\, dy = 0$
5. $xdy - ydx = xy^2\, dx$
6. $xdy = (x^2y^2 - y)\, dx$
7. $(x + y)^2\left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)$
8. $xdy - ydx = (4x^2 + y^2)dy$
9. $(y + y^2 \cos x)\, dx - (x - y^3)\, dy = 0.$

Answers

1. $\tan^{-1}\dfrac{y}{x} = x + c$

2. $\tan^{-1}\dfrac{y}{x} = x + y + c$

3. $x^2 + \dfrac{e^x}{y} = c$

4. $x \log y - x^2 + y = c$

5. $\dfrac{x}{y} + \dfrac{x^2}{2} = c$

6. $-\dfrac{1}{xy} = x + c$

7. $\log(xy) = -\dfrac{1}{x+y} + c$

8. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{y}{2x}\right) = y + c$

9. $\dfrac{x}{y} + \sin x + \dfrac{y^2}{2} = c.$

Hints

1. $\dfrac{xdy - ydx}{x^2+y^2} = dx \Rightarrow d\left(\tan^{-1}\dfrac{y}{x}\right) = dx$

3. I.F. $= \dfrac{1}{y^2}$ and $\dfrac{ye^x dx - e^x dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

4. I.F. $= \dfrac{1}{y}$ and $\dfrac{x}{y}dy + \log y dx = d(x \log y)$

6. $\dfrac{xdy + ydx}{x^2y^2} = dx \Rightarrow \dfrac{d(xy)}{(xy)^2} = dx$

7. $\dfrac{xdy + ydx}{xy} = \dfrac{dx + dy}{(x+y)^2} \Rightarrow \dfrac{d(xy)}{xy} = \dfrac{d(x+y)}{(x+y)^2}$

8. $\dfrac{xdy - ydx}{4x^2 + y^2} = dy \Rightarrow \dfrac{(xdy - ydx)/x^2}{4 + (y/x)^2} = dy \Rightarrow \dfrac{d\left(\dfrac{y}{x}\right)}{4 + \left(\dfrac{y}{x}\right)^2} = dy.$

(b) **I.F. for a Homogeneous Equation.** If $Mdx + Ndy = 0$ is a homogeneous equation in x and y, then $\dfrac{1}{Mx + Ny}$ is an I.F. provided $Mx + Ny \neq 0$.

Note. If $Mx + Ny$ consists of only one term, use the above method of I.F. otherwise, proceed by putting $y = vx$.

Example. *Solve:* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

Sol. The given equation is homogeneous in x and y with

Now,
$$M = x^2y - 2xy^2 \quad \text{and} \quad N = -x^3 + 3x^2y$$

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$$

$\therefore$
$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}.$$

Multiplying throughout by $\dfrac{1}{x^2y^2}$, the given equation becomes $\left(\dfrac{1}{y} - \dfrac{2}{x}\right)dx - \left(\dfrac{x}{y^2} - \dfrac{3}{y}\right)dy = 0$, which is exact. The solution is

$$\int_{y \text{ constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(xy - 2y^2)\, dx - (x^2 - 3xy)\, dy = 0$
2. $x^2y\, dx - (x^3 + y^3)\, dy = 0$
3. $(3xy^2 - y^3)\, dx - (2x^2y - xy^2)\, dy = 0$
4. $(x^2 - 3xy + 2y^2)\, dx + x(3x - 2y)dy = 0.$

Answers

1. $\dfrac{x}{y} - 2\log x + 3\log y = c$
2. $\log y - \dfrac{x^3}{3y^3} = c$
3. $3\log x - 2\log y + \dfrac{y}{x} = c$
4. $x^2 \log x + 3xy - y^2 = cx^2$

(c) I.F. for an equation of the form $f_1(xy)\, ydx + f_2(xy)\, xdy = 0$.

If $Mdx + Ndy = 0$ is of the form $f_1(xy)\, ydx + f_2(xy)\, xdy = 0$, then $\dfrac{1}{Mx - Ny}$ is an I.F. provided

$$Mx - Ny \neq 0.$$

Example. *Solve:* $y\,(xy + 2x^2y^2)\, dx + x(xy - x^2y^2)\, dy = 0.$

Sol. The given equation is of the form $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$.

Here, $\quad M = xy^2 + 2x^2y^3$ and $N = x^2y - x^3y^2$

Now, $\quad Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying throughout by $\dfrac{1}{3x^3y^3}$, the given equation becomes

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right) dy = 0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \int -\frac{1}{3y}\, dy = c$

or
$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or
$$-\frac{1}{xy} + 2\log x - \log y = C, \quad \text{where } C = 3c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(1 + xy)\, ydx + (1 - xy)xdy = 0.$
2. $(x^2y^2 + xy + 1)\, ydx + (x^2y^2 - xy + 1)xdy = 0.$
3. $y(2xy + 1)dx + x(1 + 2xy - x^3y^3)dy = 0.$
4. $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)\, dy = 0.$

5. $(y - xy^2)dx - (x + x^2y)\,dy = 0.$

6. $(xy \sin xy + \cos xy)\,ydx + (xy \sin xy - \cos xy)\,xdy = 0.$

Answers

1. $-\dfrac{1}{xy} + \log\left(\dfrac{x}{y}\right) = c$

2. $xy + \log\left(\dfrac{x}{y}\right) - \dfrac{1}{xy} = c$

3. $\dfrac{1}{x^2y^2} + \dfrac{1}{3x^3y^3} + \log y = c$

4. $-\dfrac{1}{xy} + 2\log x - \log y = c$

5. $\log\left(\dfrac{x}{y}\right) - xy = c$

6. $y \cos xy = cx.$

(*d*) **For the equation Mdx + Ndy = 0**

(*i*) If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, a function of x only, then $e^{\int f(x)dx}$ is an I.F.

(*ii*) If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$, a function of y only, then $e^{\int g(y)dy}$ is an I.F.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $(xy^2 - e^{\frac{1}{x^3}})\,dx - x^2y\,dy = 0.$

Sol. Here, $\quad M = xy^2 - e^{\frac{1}{x^3}}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x},$$ which is a function of x only.

$\therefore$

$$\text{I.F.} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$$

Multiplying throughout by $\dfrac{1}{x^4}$, we have $\left(\dfrac{y^2}{x^3} - \dfrac{1}{x^4}e^{\frac{1}{x^3}}\right)dx - \dfrac{y}{x^2}\,dy = 0$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx = c$

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int -\frac{3}{x^4}e^{\frac{1}{x^3}}\,dx = c \quad \text{or} \quad -\frac{y^2}{2x^2} + \frac{1}{3}\int e^t\,dt = c, \quad \text{where } t = \frac{1}{x^3}$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^t = c \quad \text{or} \quad -\frac{3y^2}{x^2} + 2e^{\frac{1}{x^3}} = C, \quad \text{where } C = 6c.$$

Example 2. *Solve:* $(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)\, dy = 0.$

Sol. Here, $M = xy^3 + y$ and $N = 2x^2y^2 + 2x + 2y^4$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3 + y} = \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y}$$

which is a function of y only.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Multiplying throughout by y, we have $(xy^4 + y^2)\, dx + 2\, (x^2y^3 + xy + y^5)\, dy = 0$

which is exact. The solution is $\displaystyle\int_{y \text{ constant}} (xy^4 + y^2)\, dx + \int 2y^5\, dy = c$

or $$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(x^2 + y^2 + x)\, dx + xydy = 0$
2. $(x^2 + y^2 + 1)\, dx - 2xydy = 0.$
3. $(x^2 + y^2 + 2x)dx + 2ydy = 0.$
4. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\, dy = 0.$
5. $\left(y + \dfrac{y^3}{3} + \dfrac{x^2}{2}\right)dx + \dfrac{1}{4}(x + xy^2)dy = 0.$
6. $(x \sec^2 y - x^2 \cos y)dy = (\tan y - 3x^4)dx.$
7. $(xye^{x/y} + y^2)\, dx - x^2e^{x/y}\, dy = 0.$
8. $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0.$
9. $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0.$
10. $y(2x^2y + e^x)dx = (e^x + y^3)dy.$
11. $ydx - xdy + \log x\, dx = 0.$
12. $(2x \log x - xy)\, dy + 2ydx = 0.$
13. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)\, dy = 0.$
14. $y \log y\, dx + (x - \log y)\, dy = 0.$
15. $(xy^3 + y)dx + 2\, (x^2y^2 + x + y^4)dy = 0.$

Answers

1. $3x^4 + 6x^2y^2 + 4x^3 = c$
2. $x - \dfrac{y^2}{x} - \dfrac{1}{x} = c$
3. $e^x(x^2 + y^2) = c$
4. $\left(y + \dfrac{2}{y^2}\right)x + y^2 = c$
5. $3x^2y + x^4y^3 + x^6 = c$
6. $\dfrac{\tan y}{x} + x^3 - \sin y = c$
7. $e^{x/y} + \log x = c$
8. $x^2y(x - ay) = c$
9. $e^x + m\left(\dfrac{y}{x}\right)^2 = c$
10. $\dfrac{2x^3}{3} + \dfrac{e^x}{y} - \dfrac{y^2}{2} = c$
11. $1 + y + \log x = cx$
12. $2y \log x - \dfrac{1}{2}y^2 = c$
13. $x^3y^2 + \dfrac{x^2}{y} = c$
14. $x \log y - \dfrac{1}{2}(\log y)^2 = c$
15. $\dfrac{x^2y^4}{2} + xy^2 + \dfrac{y^6}{3} = c.$

(e) **I.F. for an equation of the form**

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

where a, b, c, d, m, n, p, q are all constant is $x^h y^k$, where h, k are so chosen that after multiplication by $x^h y^k$ the equation becomes exact.

Example. *Solve* $(2x^2y^2 + y)\,dx + (3x - x^3y)\,dy = 0.$

Sol. The equation can be written as $2(x^2y^2\,dx - x^3y dy) + (y\,dx + 3x dy) = 0$

or
$$x^2y\,(2y dx - x dy) + x^0y^0\,(y\,dx + 3x dy) = 0$$

which is of the form mentioned above. Therefore, it has an I.F. of the form x^hy^k.

Multiplying the given equation by x^hy^k, we have

$$(2x^{h+2}y^{k+2} + x^h\,y^{k+1})\,dx + (3x^{h+1}y^k - x^{h+3}y^{k+1})\,dy = 0$$

For this equation to be exact, we must have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

i.e.,
$$2(k+2)\,x^{h+2}y^{k+1} + (k+1)\,x^hy^k = 3(h+1)\,x^hy^k - (h+3)\,x^{h+2}y^{k+1}$$

which holds when $\quad 2(k+2) = -(h+3) \quad$ and $\quad k + 1 = 3(h+1)$

i.e., when $\quad h + 2k + 7 = 0 \quad$ and $\quad 3h - k + 2 = 0$

Solving these equation, we have $h = -\dfrac{11}{7}, k = -\dfrac{19}{7}$

$$\therefore \qquad \text{I.F.} = x^{-\frac{11}{7}}y^{-\frac{19}{7}}$$

Multiplying the given equation by $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}} + x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx + \left(3x^{-\frac{4}{7}}y^{-\frac{19}{7}} - x^{\frac{10}{7}}y^{-\frac{12}{7}}\right)dy = 0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(2x^{\frac{3}{7}}y^{-\frac{5}{7}} + x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx = c$

or
$$\frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}} = c \quad \text{or} \quad 4x^{\frac{10}{7}}y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}}y^{-\frac{12}{7}} = C, \text{ where } C = \frac{20}{7}c.$$

Note. The values of h and k can also be determined from the relations

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \text{and} \quad \frac{c+h+1}{p} = \frac{d+k+1}{q}.$$

Comparing the given equation

$$x^2y(2y\,dx - x\,dy) + x^0y^0(y\,dx + 3x\,dy) = 0$$

with

$$x^ay^b\,(my\,dx + nx\,dy) + x^cy^d\,(py\,dx + qx\,dy) = 0$$

we have

$$a = 2,\ b = 1,\ c = 0,\ d = 0$$
$$m = 2,\ n = -1,\ p = 1,\ q = 3$$

$$\therefore \quad \frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \Rightarrow \quad \frac{2+h+1}{2} = \frac{1+k+1}{-1}$$

or
$$3 + h = -4 - 2k \quad \text{or} \quad h + 2k + 7 = 0 \qquad \ldots(1)$$

Also, $\dfrac{c+h+1}{p} = \dfrac{d+k+1}{q} \Rightarrow \dfrac{0+h+1}{1} = \dfrac{0+k+1}{3}$

or $3h - k + 2 = 0$ ...(2)

Solving (1) and (2), we have $h = -\dfrac{11}{7}, k = -\dfrac{19}{7}$.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(x^2y + y^4)dx + (2x^3 + 4xy^3)dy = 0$
2. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$
3. $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$
4. $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$
5. $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$
6. $x(3ydx + 2xdy) + 8y^4(ydx + 3xdy) = 0$
7. $(2y^2 - 4x^2y)\, dx + (4xy + 3x^3)\, dy = 0$.

Answers

1. $7x^{11/2}y^{11} + 11x^{7/2}y^{1/4} = c$ (I.F. $= x^{5/2}y^{10}$)
2. $6\sqrt{xy} - \left(\dfrac{y}{x}\right)^{3/2} = c$ (I.F. $= x^{-5/2}y^{-1/2}$)
3. $5x^{-36/13}\, y^{24/13} - 12x^{-10/13}\, y^{-15/13} = c$ (I.F. $= x^{-49/13}y^{-28/13}$)
4. $x^2y^4 - y^2x^4 = c$ (I.F. $= xy$)
5. $x^2y^3\,(1 + 2xy) = c$ (I.F. $= xy^2$)
6. $x^3y^2 + 4x^2y^6 = c$ (I.F. $= xy$)
7. $5x^{-\frac{2}{11}}y^{-\frac{4}{11}} + x^{\frac{20}{11}}y^{-\frac{5}{11}} = c$ (I.F. $= x^{-13/11}y^{-26/11}$)

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