5. $$\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$x \quad y \quad z$

$$= \int_0^1\int_0^{1-x}\left[xy\left(\frac{z^2}{2}\right)_0^{1-x-y}\right]dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[xy\,\frac{(1-x-y)^2}{2}\right]dy\,dz$$

$$= \int_0^1\int_0^{1-x}\frac{x}{2}\left[y-2xy-2y^2+2xy^2+x^2y+y^3\right]dy\,dx$$

$$= \int_0^1 \frac{x}{2}\left[\frac{y^2}{2}-\frac{2xy^2}{2}-\frac{2y^3}{3}+\frac{2xy^3}{3}+\frac{x^2y^2}{2}+\frac{y^4}{4}\right]_0^{1-x}dx$$

$$= \int_0^1 \frac{x}{2}\left[\frac{6y^2-12xy^2-8y^3+8xy^3+6x^2y^2+3y^4}{12}\right]_0^{1-x}dx$$

$$= \int_0^1 \frac{x}{24}\Big[6(1-x)^2-12x(1-x)^2-8(1-x)^3+8x(1-x)^3 +6x^2(1-x)^2+3(1-x)^4\Big]dx$$

$$= \int_0^1 \frac{x}{24}\Big[6(1-2x+x^2)-12x(1-2x+x^2)-8(1-x^3-3x+3x^2) +8x(1-x^3-3x+3x^2)+6x^2(1-2x+x^2) +3(x^4-4x^3+6x^2-4x+1)\Big]dx$$

$$= \int_0^1 \frac{x}{24}\Big[6-12x+6x^2-12x+24x^2-12x^3-8+8x^3 +24x-24x^2+8x-8x^4-24x^2+24x^3 +6x^2-12x^3+6x^4+3x^4-12x^3+18x^2-12x+3\Big]dx$$

$$= \int_0^1 \frac{x}{24}\left[1-4x+6x^2-4x^3+x^4\right]dx$$

$$= \frac{1}{24}\int_0^1\left(x-4x^2+6x^3-4x^4+x^5\right)dx$$

$$= \frac{1}{24}\left[\frac{1}{2}-\frac{4x^3}{3}+\frac{6x^4}{4}-\frac{4x^5}{5}+\frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{24}\left[\frac{4}{2}-\frac{4}{3}-\frac{4}{5}+\frac{1}{6}\right] = \frac{1}{24}\left[\frac{60-40-24+5}{30}\right]$$

$$= \frac{1}{720}$$

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10. $$\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$
θ r z

$$=\int_0^{\pi/2}\int_0^{a\cos\theta}(rz)_0^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{a\cos\theta}\left[r\sqrt{a^2-r^2}\right]dr\,d\theta \quad -(1)$$

put $a^2-r^2=t$
$\Rightarrow -2r\,dr = dt$
$\Rightarrow r\,dr = -\frac{1}{2}dt$

$\Rightarrow$ from eq^n (1)
$$=\int_0^{\pi/2}\int_0^{a\cos\theta}\left(-\frac{1}{2}\sqrt{t}\right)dt\,d\theta = \int_0^{\pi/2}\int_0^{a\cos\theta} -\frac{1}{2}\times\frac{2}{3}t^{3/2}\,d\theta$$

$$=\int_0^{\pi/2} -\frac{1}{3}\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$=\int_0^{\pi/2} -\frac{1}{3}\left[(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right]d\theta$$

$$\Rightarrow\int_0^{\pi/2} -\frac{1}{3}\left[a^2(1-\cos^2\theta)\right]^{3/2}-\left[(a^2)^{3/2}\right]d\theta$$

$$\Rightarrow\int_0^{\pi/2} -\frac{1}{3}(a^2\sin^2\theta)^{3/2}-a^3\,d\theta$$

$$=\int_0^{\pi/2} -\frac{a^3}{3}(\sin^3\theta-1)\,d\theta$$

$$-\frac{a^3}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta = -\frac{a^3}{3}\int_0^{\pi/2}\sin^2\theta\sin\theta\,d\theta = -\frac{a^3}{3}\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,d\theta \quad -(2)$$

Put $\cos\theta = t$
$-\sin\theta\,d\theta = dt$
$\Rightarrow \sin\theta\,d\theta = -dt$

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from eq^n (2)

$$= \frac{-a^3}{3}\int_0^{\pi/2} -(1-t^2)\,dt = \frac{-a^3}{3}\int_0^{\pi/2}(t^2-1)\,dt$$

$$= \frac{-a^3}{3}\left(\frac{t^3}{3}-t\right)_0^{\pi/2} = \frac{-a^3}{3}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)_0^{\pi/2} \quad -(3)$$

$$= \frac{-a^3}{3}\int 1\,d\theta = \frac{-a^3}{3}\theta \quad -(4)$$

from (3) & (4)

$$= \frac{-a^3}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta - \theta\right]_0^{\pi/2}$$

$$= \frac{-a^3}{3}\left[\left(\frac{\cos^3(\pi/2)}{3} - \cos(\pi/2) - \frac{\pi}{2}\right) - \left(\frac{\cos^3(0)}{3} - \cos 0\right)\right]$$

$$= \frac{-a^3}{3}\left[0 - 0 - \frac{\pi}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{-a^3}{3}\left[-\frac{\pi}{2} + \frac{2}{3}\right] = \frac{a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

(11) $$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$(x)\ (y)\ (z)$

put $x+y+z = t$

$1 \cdot dz = dt$

$$= \int_0^a \int_0^x \int_0^{x+y} e^t\,dt\,dy\,dx$$

$$= \int_0^a \int_0^x [e^t]_0^{x+y}\,dy\,dx = \int_0^a \int_0^x [e^{x+y+z}]_0^{x+y}\,dy\,dx$$

$$= \int_0^a \int_0^x \left(e^{x+y+x+y} - e^{x+y}\right) dy\,dx$$

$$= \int_0^a \int_0^x \left(e^{2x+2y} - e^{x+y}\right) dy\,dx$$

$$\int_0^a e^{2x}\int_0^x e^{2y}\,dy - e^x\int_0^x e^y\,dy\,dx \quad \text{— (1)}$$

solve : $\int e^{2y}\,dy \Rightarrow$ put $2y = t$

$2\,dy = dt \Rightarrow dy = \frac{1}{2}dt$

$= \int \frac{1}{2} e^t\,dt$

$= \frac{1}{2} e^t = \frac{1}{2} e^{2y}$

similarly, solve : $\int e^y\,dy = e^y$

from eq^n (1)

$$= \int_0^a e^{2x}\int_0^x e^{2y}\,dy - e^x\int_0^x e^y\,dy\,dx$$

$$= \int_0^a \left(e^{2x}\cdot\frac{e^{2y}}{2} - e^x\cdot e^y\right)_0^x dx$$

$$= \int_0^a \left(\frac{e^{2x+2y}}{2} - e^{x+y}\right)_0^x dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \left(\frac{e^{2x}}{2} + e^x\right)\right) dx$$

$$= \int_0^a \left(\frac{e^{4x} - 2e^{2x}}{2} - \frac{e^{2x} - 2e^x}{2}\right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x\right) dx$$

$$= \int_0^a \frac{e^{4x}}{2}\,dx - \int_0^a \frac{3}{2} e^{2x}\,dx + \int_0^a e^x\,dx \quad \text{— (2)}$$

solve : $\int e^{4x}\,dx$ put $4x = t$

$4\,dx = dt$

$dx = \frac{1}{4}dt$

$\Rightarrow \frac{1}{4}\int e^t\,dt$

$= \frac{1}{4} e^t = \frac{e^{4x}}{4}$

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similarly, $\int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$\int e^{x}\,dx = e^{x}$

from eq^n (2)

$$= \left(\frac{e^{4x}}{8} - \frac{3}{4}e^{2x} + e^{x}\right)_0^a$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$= \frac{e^{4a} - 6e^{2a} + 8e^{a}}{8} - \left(\frac{1-6+8}{8}\right)$$

$$= \frac{e^{4a} - 6e^{2a} + 8e^{a}}{8} - \frac{3}{8}$$

$$= \left(\frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}\right)$$

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8. $\int_{x=0}^{\log 2}\int_{y=0}^{x}\int_{z=0}^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

put $x+y+z=t$

$dz = dt$

$\Rightarrow \int e^t\,dt = e^t = e^{x+y+z}$

$\Rightarrow \int_0^{\log 2}\int_0^{x}\left(\int_0^{x+\log y} e^{x+y+z}\,dz\right)dy\,dx$

$\Rightarrow \int_0^{\log 2}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+\log y} dy\,dx$

$\Rightarrow \int_0^{\log 2}\int_0^{x}\left(e^{x+y+x+\log y} - e^{x+y}\right)dy\,dx$

$= \int_0^{\log 2}\int_0^{x}\left(e^{2x+y}\cdot e^{\log y} - e^{x+y}\right)dy\,dx$

$= \int_0^{\log 2}\int_0^{x}\left(y e^{2x+y} - e^{x+y}\right)dy\,dx$

$= \int_0^{\log 2} e^{2x}\int_0^{x} y e^{y}\,dy - e^{x}\int_0^{x} e^{y}\,dy \quad -(1)$

solve : $\int y e^y\,dy$ using by parts

$\int uv\,dx = u\left(\int v\,dx\right) - \int\left[\left(\frac{du}{dx}\right)\cdot\int v\,dx\right]dx$

$u = y,\ v = e^y$

$\Rightarrow y e^y - \int 1\cdot e^y\,dy = y e^y - e^y$

from eq^n (1)

$= \int_0^{\log 2}\left[e^{2x}\left(y e^y - e^y\right) - e^{x}\cdot e^{y}\right]_0^{x} dx$

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$$\int_0^{\log 2} \left(xe^{3x} - e^{3x} - \cancel{e^{2x}} - 0 + \cancel{e^{2x}} + e^{x}\right)dx$$

$$\int_0^{\log 2} e^{3x}(x-1) + e^{x}\,dx \quad —(2)$$

solve : $\int (x-1)e^{3x}\,dx$

using by parts : $\int uv\,dx = u\int v\,dx - \int\left[\left(\frac{du}{dx}\right)\cdot\int v\,dx\right]dx$

$u = x-1$, $v = e^{3x}$

$$\Rightarrow \int (x-1)e^{3x}\,dx = (x-1)\frac{e^{3x}}{3} - \int 1\cdot\frac{e^{3x}}{3}\,dx$$

$$= \frac{(x-1)e^{3x}}{3} - \frac{e^{3x}}{9}$$

so, from eq^n (2)

$$\int_0^{\log 2} e^{3x}(x-1) + e^{x}\,dx = \left[\frac{(x-1)e^{3x}}{3} - \frac{e^{3x}}{9} + e^{x}\right]_0^{\log 2}$$

$$\cancel{= \left[\frac{(\log 2-1)e^{3\log 2}}{3} - \frac{e^{\log 2}}{9} + e^{\log 2} - 0 + \frac{1}{9} + 1\right]}$$

$$= \left[\frac{(\log 2-1)e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + e^{\log 2}\right] - \left[\frac{-1}{3} - \frac{1}{9} + 1\right]$$

$$= \left(\frac{8\log 2}{3} - \frac{8}{3} - \frac{8}{9} + 2\right) - \left(\frac{-3-1+9}{9}\right)$$

$$= \frac{8\log 2}{3} + \left(\frac{-24-8+18}{9}\right) - \frac{5}{9}$$

$$= \frac{8\log 2}{3} - \frac{14}{9} - \frac{5}{9} = \frac{8\log 2}{3} - \frac{19}{9}$$

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13. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

z y x

$=\int_{-c}^{c}\int_{-b}^{b}\left(\frac{x^3}{3}+xy^2+xz^2\right)_{-a}^{a}dy\,dz$

$=\int_{-c}^{c}\int_{-b}^{b}\left(\frac{a^3}{3}+\frac{a^3}{3}\right)+\left(ay^2+ay^2\right)+\left(az^2+az^2\right)dy\,dz$

$=\int_{-c}^{c}\int_{-b}^{b}\left(\frac{2a^3}{3}+2ay^2+2az^2\right)dy\,dz$

$=\int_{-c}^{c}\left(\frac{2a^3y}{3}+\frac{2ay^3}{3}+2ayz^2\right)_{-b}^{b}dz$

$=\int_{-c}^{c}\left[\left(\frac{2a^3b}{3}+\frac{2a^3b}{3}\right)+\left(\frac{2ab^3+2ab^3}{3}\right)+\left(2abz^2+2abz^2\right)\right]dz$

$=\int_{-c}^{c}\left(\frac{4a^3b}{3}+\frac{4ab^3}{3}+4abz^2\right)dz$

$=\left[\frac{4a^3bz}{3}+\frac{4ab^3z}{3}+\frac{4abz^3}{3}\right]_{-c}^{c}$

$=\left(\frac{4a^3bc+4a^3bc}{3}\right)+\left(\frac{4ab^3c+4ab^3c}{3}\right)+\left(\frac{4abc^3+4abc^3}{3}\right)$

$=\frac{8a^3bc}{3}+\frac{8ab^3c}{3}+\frac{8abc^3}{3}$

$=\frac{8}{3}abc\left(a^2+b^2+c^2\right)$

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15. $$\int_0^2\int_0^x\int_0^{x+y} e^x(y+2z)\,dz\,dy\,dx$$

solve : $\int e^x(y+2z)\,dz$

$$= ye^x\int 1\,dz + 2e^x\int z\,dz$$

$$\cancel{= \int_0^{x+y}\left(e^x yz + e^x z^2\right)_0^{x+y}} = e^x yz + e^x z^2$$

$$\cancel{= e^x\left[yz+z^2\right]_0^{x+y}}$$

$$\cancel{= e^x\left[y(x+y)+x^2+y^2+2xy\right]}$$

$$= \int_0^2\int_0^x\int_0^{x+y} e^x(y+2z)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^x e^x\left(yz+z^2\right)_0^{x+y} dy\,dx$$

$$= \int_0^2\int_0^x e^x\left(y(x+y)+x^2+y^2+2xy\right) dy\,dx$$

$$= \int_0^2\int_0^x e^x\left[x^2+2y^2+3xy\right] dy\,dx$$

$$= \int_0^2\left(x^2e^x\int_0^x 1\,dy + 2e^x\int_0^x y^2\,dy + 3xe^x\int_0^x y\,dy\right)$$

$$= \int_0^2\left(x^2e^x y + \frac{2}{3}e^x y^3 + \frac{3}{2}xe^x y^2\right)_0^x dx$$

$$= \int_0^2\left(x^3e^x + \frac{2}{3}e^x x^3 + \frac{3}{2}x^3e^x\right) dx$$

$$= \int_0^2 \frac{19}{6}x^3e^x\,dx$$

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$$= \frac{19}{6}\int_0^2 x^3 e^x dx \quad — (1)$$

solve : $\int x^3 e^x dx$

using by parts ; $u = x^3$, $v = e^x$

$= x^3e^x - \int 3x^2 e^x dx$

Now solve : $\int 3x^2 e^x dx$

~~again~~ using by parts ; $u = x^2$, $V = e^x$

$\Rightarrow 3\int x^2 e^x dx = 3\left(x^2 e^x - \int 2x \cdot e^x dx\right)$

Now solve : $\int 2x e^x dx$

$\Rightarrow 2\int x e^x dx$; using by parts : $u = x$, $V = e^x$

$\Rightarrow 2\left(xe^x - \int e^x dx\right)$

solve : $\int e^x dx = e^x$

from eq^n (1)

$= \frac{19}{6}\left[x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right]_0^2$

$= \frac{19}{6}\left[8e^2 - 12e^2 + 12e^2 - 6e^2 + 6\right]$

$= \frac{19}{6}\left(2e^2 + 6\right) = \frac{19}{3}\left(e^2 + 3\right)$ or $19\left(\frac{e^2}{3} + 1\right)$

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12. $$\int_0^1\int_0^{x^2}\int_0^{x+y}(x-2y+z)\,dz\,dy\,dx$$

x y z

$$= \int_0^1\int_0^{x^2}\left(xz-2yz+\frac{z^2}{2}\right)_0^{x+y} dy\,dx$$

$$= \int_0^1\int_0^{x^2}\left[x(x+y)-2y(x+y)+\frac{(x+y)^2}{2}\right] dy\,dx$$

$$= \int_0^1\int_0^{x^2}\left(\frac{2x^2-2xy-4y^2+x^2+y^2+2xy}{2}\right) dy\,dx$$

$$= \int_0^1\int_0^{x^2}\frac{3}{2}(x^2-y^2)\,dy\,dx$$

$$= \int_0^1\frac{3}{2}\left[x^2y-\frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^1\frac{3}{6}\left(3x^2y-y^3\right)_0^{x^2} dx$$

$$= \frac{3}{6}\int_0^1\left(3x^4-x^6\right)dx$$

$$= \frac{3}{6}\left[\frac{3x^5}{5}-\frac{x^7}{7}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{3}{5}-\frac{1}{7}\right) = \frac{21-5}{2\times35} = \frac{16}{2\times35} = \frac{8}{35}$$

14. $$\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left(xz+yz+\frac{z^2}{2}\right)_0^{a-x-y} dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left[x(a-x-y)+y(a-x-y)+\frac{(a-x-y)^2}{2}\right] dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left(ax-x^2-xy+ay-xy-y^2+\frac{(a-x-y)^2}{2}\right) dy\,dx$$

$$= \int_0^a \int_0^{a-x} \frac{1}{2}\left(\begin{array}{l} 2ax+2ay-2x^2-2y^2-4xy+a^2-2ax \\ -2ay+2xy+x^2+y^2 \end{array}\right) dy\,dx$$

$$= \int_0^a \int_0^{a-x} \frac{(a^2-x^2-y^2-2xy)}{2}\,dy$$

$$= \int_0^a \frac{1}{2}\left[a^2y-x^2y-\frac{y^3}{3}-\frac{2xy^2}{2}\right]_0^{a-x}$$

$$= \int_0^a \frac{1}{2}\left[\begin{array}{l} a^3-a^2x-ax^2+x^3-\frac{(a^3-x^3-3a^2x+3ax^2)}{3} \\ -x(a^2-2ax+x^2) \end{array}\right]$$

$$= \int_0^a \frac{1}{2}\left[\frac{1}{3}\left(\begin{array}{l} 3a^3-3a^2x-3ax^2+3x^3-a^3+x^3+3a^2x-3ax^2 \\ -3xa^2+6ax^2-3x^2 \end{array}\right)\right]$$

$$= \frac{1}{6}\int_0^a \left(2a^3-3a^2x+4x^3-3x^2\right) dx$$

$$= \frac{1}{6}\left[2a^3x-\frac{3}{2}a^2x^2+x^4-\frac{3}{4}x^4\right]_0^a$$

$$= \frac{1}{6}\left(2a^4-\frac{3}{2}a^4+a^4-\frac{3}{4}a^4\right)$$

$$= \frac{1}{6}\left(\frac{8a^4-6a^4+4a^4-3a^4}{4}\right) = \frac{3a^4}{6\times 4} = \frac{a^4}{8}$$