$\therefore$ Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}.$$

Example 9. *If* $z = \sqrt{x^2 + y^2}$ *and* $x^3 + y^3 + 3axy = 5a^2$, *find the value of* $\frac{dz}{dx}$, *when* $x = y = a$.

Sol. The given equations are of the form $z = f(x, y)$ and $\phi(x, y) = c$

$\therefore$ z is composite function of x.

$$\Rightarrow \quad \frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \qquad ...(1)$$

Now
$$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} . 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

Similarly,
$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

Also, differentiating $x^3 + y^3 + 3axy = 5a^2$ w.r.t. x, we have

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} + 3ay + 3ax \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad (y^2 + ax)\frac{dy}{dx} = -(x^2 + ay)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$$

$\therefore$ From (1),
$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{x^2 + ay}{y^2 + ax}\right)$$

$$\left[\frac{dz}{dx}\right]_{\substack{x = a \\ y = a}} = \frac{a}{\sqrt{a^2 + a^2}} + \frac{a}{\sqrt{a^2 + a^2}}\cdot\frac{a^2 + a^2}{a^2 + a^2} = 0.$$

Example 10. *If* $u = xe^y z$, *where* $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, *find* $\frac{du}{dx}$.

Sol. Here u is a function of x, y and z while y and z are functions of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= e^y z \cdot 1 + xe^y z \cdot \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + xe^y \cdot 2\sin x \cos x$$

$$= e^y \left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x \sin 2x \right].$$

Example 11. *Find* $\frac{du}{dx}$ *if* $u = \sin(x^2 + y^2)$, *where* $a^2x^2 + b^2y^2 = c^2$.

Sol. The given equations are the form $u = f(x, y)$ and $\phi(x, y) = k$

$\therefore$ u is a composite function of x.

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad ...(1)$$

Now $\dfrac{\partial u}{\partial x} = 2x \cos(x^2 + y^2),\ \dfrac{\partial u}{\partial y} = 2y \cos(x^2 + y^2)$

Also, differentiating $a^2x^2 + b^2y^2 = c$ w.r.t. x, we have

$$2a^2x + 2b^2y \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{a^2x}{b^2y}$$

$\therefore$ From (1), $\dfrac{du}{dx} = 2x \cos(x^2 + y^2) + 2y \cos(x^2 + y^2) \cdot \left[-\dfrac{a^2x}{b^2y}\right]$

$$= 2\left[x - \frac{a^2x}{b^2}\right] \cos(x^2 + y^2) = \frac{2(b^2 - a^2)x}{b^2} \cdot \cos(x^2 + y^2).$$

Example 12. *Find* $\dfrac{dy}{dx}$, *when*

(*i*) $x^y + y^x = c$ (*ii*) $(\cos x)^y = (\sin y)^x$.

Sol. (*i*) Let $f(x, y) = x^y + y^x$, then $f(x, y) = c$

[Using Cor. 4] $\therefore \dfrac{dy}{dx} = -\dfrac{f_x}{f_y} \Rightarrow \dfrac{dy}{dx} = -\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$.

(*ii*) Let $f(x, y) = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y(\cos x)^{y-1} \cdot (-\sin x) - (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\sin y)^{x-1} \cdot \cos y}$$

$$= \frac{y(\cos x)^{y-1} \sin x + (\cos x)^y \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\cos x)^y (\sin y)^{-1} \cos y}$$

$[\because \ (\sin y)^x = (\cos x)^y]$

$$= \frac{(\cos x)^y \left[y \cdot \dfrac{\sin x}{\cos x} + \log \sin y\right]}{(\cos x)^y [\log \cos x - x \cot y]} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}.$$

Example 13. *If* $f(x, y) = 0$, $\phi(y, z) = 0$, *show that* $\dfrac{\partial f}{\partial y} \cdot \dfrac{\partial \phi}{\partial z} \cdot \dfrac{dz}{dx} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial \phi}{\partial y}$.

Sol. $$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \quad \text{...(1)}$$

$f(x, y) = 0$ gives $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}$; $\phi(y, z) = 0$ gives $\dfrac{dz}{dy} = -\dfrac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial z}}$

$\therefore$ From (1), $\dfrac{dz}{dx} = \dfrac{\dfrac{\partial f}{\partial x} \cdot \dfrac{\partial \phi}{\partial y}}{\dfrac{\partial f}{\partial y} \cdot \dfrac{\partial \phi}{\partial z}} \Rightarrow \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial \phi}{\partial z} \cdot \dfrac{dz}{dx} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial \phi}{\partial y}$.

Example 14. *If* $\phi(x, y, z) = 0$, *show that* $\left(\dfrac{\partial y}{\partial z}\right)_x \left(\dfrac{\partial z}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial y}\right)_z = -1.$

Sol. The given relation defines y as a function of x and z. Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\dfrac{\partial \phi}{\partial z}}{\dfrac{\partial \phi}{\partial y}}$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\dfrac{\partial \phi}{\partial x}}{\dfrac{\partial \phi}{\partial z}}$$

Similarly, $$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial x}}$$

Multiplying, we get the desired result.

Example 15. *Prove that* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$

where $x = u \cos \alpha - v \sin \alpha$, $y = u \sin \alpha + v \cos \alpha$

Or

By changing the independent variables u *and* v *to* x *and* y *by means of the relations* $x = u \cos \alpha - v \sin \alpha$, $y = u \sin \alpha + v \cos \alpha$, *show that* $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$ *transforms into* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$.

Sol. Here z is a composite function of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \cos \alpha \frac{\partial z}{\partial x} + \sin \alpha \frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial u}(z) = \left(\cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}\right) z \Rightarrow \frac{\partial}{\partial u} \equiv \cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y} \quad \ldots(1)$$

Also $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = -\sin \alpha \frac{\partial z}{\partial x} + \cos \alpha \frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial v}(z) = \left(-\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y}\right) z$$

$\Rightarrow$ $$\frac{\partial}{\partial v} \equiv -\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y} \quad \ldots(2)$$

Now we shall make use of the equivalence of operators as given by (1) and (2).

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial y\partial x} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2} \quad \ldots(3)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} - \cos\alpha\sin\alpha\frac{\partial^2 z}{\partial y\partial x} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2} \quad \ldots(4)$$

Adding (3) and (4), $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$.

Example 16. *Transform the equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *into polar co-ordinates.*

Sol. The relations connecting cartesian co-ordinates, (x, y) with polar co-ordinates (r, θ) are

$$x = r\cos\theta,\ y = r\sin\theta$$

Squaring and adding, $r^2 = x^2 + y^2$

Dividing, $\tan\theta = \dfrac{y}{x}$

$$r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Here u is a composite function of x and y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial\theta}$$

Or

$$\frac{\partial}{\partial x}(u) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\right)u$$

$$\Rightarrow \qquad \frac{\partial}{\partial x} \equiv \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta} \qquad ...(1)$$

Also,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial \theta}$$

or
$$\frac{\partial}{\partial y}(u) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)u$$

$$\Rightarrow \qquad \frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta} \qquad ...(2)$$

Now we shall make use of the equivalence of cartesian and polar operators as given by (1) and (2).

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\frac{\partial u}{\partial \theta}\left(-\frac{1}{r^2}\right) - \frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial \theta}\right]$$

$$- \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial \theta\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\cdot\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial r} - \frac{2\cos\theta\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} \qquad ...(3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r^2}\cdot\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial \theta}\right]$$

$$+ \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{\partial^2 u}{\partial \theta\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r^2}\cdot\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r}\cdot\frac{\partial u}{\partial r} + \frac{2\cos\theta\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} \qquad ...(4)$$

Adding (3) and (4), $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\cdot\dfrac{\partial^2 u}{\partial \theta^2}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ transforms into } \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0.$$

1. (a) Find $\frac{du}{dt}$ when $u = x$
   (b) If $u = x^2 + y^2 + z^2$ and
   verify the result by
2. If $u = \sin\frac{x}{y}$, $x = e^t$, $y = t$
3. If $u = x^3 + y^3$, where $x =$
4. (a) At a given instant increasing at the r area is increasing
   (b) At a given instant tively and they ar the rate at which
   (c) If $z = 2xy - 3x^2y$ and $y = 2$ cm, at w nor decreasing?
5. (a) If $z = u^2 + v^2$, $u =$
   (b) If $z = \log(u^2 + v)$,
   (c) If $u = \tan^{-1}\left(\frac{y}{x}\right)$
6. If $u = f(r, s)$, $r = x +$
7. If $z = e^{ax+by} f(ax - b$
8. (i) If $x = u + v$, $y =$
   (ii) If $u = f(r, s)$, $r =$
9. If $u = x\log(xy)$, wh
10. (a) If $u = f(r, s, t)$
    (b) If $x = u + v + w$
    (c) If $u = f(2x - 3y$
    (d) If $u = f(x^2 + 2$
11. If $z = x^2y$ and x^2