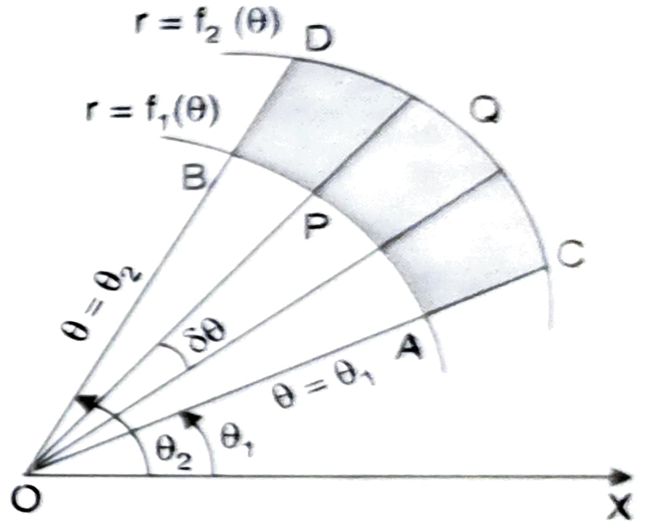
1.3 EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$ over the region bounded by the straight lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$, $r = r_2$, we first integrate w.r.t. r between the limits $r = r_1$ and $r = r_2$ (treating θ as a constant). The resulting expression is then integrated w.r.t. θ between the limits $\theta = \theta_1$ and $\theta = \theta_2$.



Geometrically, AB and CD are the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$ so that ACDB is the region of integration. PQ is a wedge of angular thickness $\delta\theta$. Then $\int_{r=r_1}^{r=r_2} f(r,\theta)\,dr$ indicates that the integration is performed along PQ (*i.e.*, r varies, θ is constant) and the integration w.r.t. θ means rotation of this strip PQ from AC to BD. dr indicates that the integration is performed along PQ (*i.e.*, r varies, θ is constant) and the integration w.r.t. θ means rotation of this strip PQ from AC to BD.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta.$

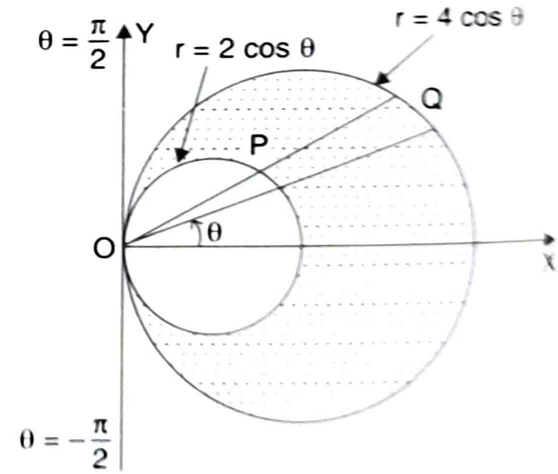
Sol.
$$I = \int_0^{\pi/2}\left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta = -\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{a^3}{18}(3\pi-4).$$

Example 2. *Evaluate* $\iint r^3\,dr\,d\theta$, *over the area bounded between the circles* $r = 2\cos\theta$ *and* $r = 4\cos\theta$.

Sol. The region of integration R is shown shaded. Here r varies from $2\cos\theta$ to $4\cos\theta$ while θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.



$$\therefore \iint_R r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1}{4}(256\cos^4\theta - 16\cos^4\theta)\,d\theta$$

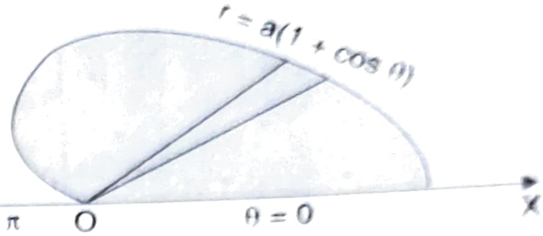
$$= 60\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$= 120\int_0^{\pi/2}\cos^4\theta\,d\theta \quad \text{[Since } \cos^4\theta \text{ is an even function of } \theta\text{]}$$

$$= 120\times\frac{3\times1}{4\times2}\cdot\frac{\pi}{2} = \frac{45}{2}\pi.$$

Example 3. *Evaluate* $\iint r \sin \theta \, dr \, d\theta$ *over the area of the cardioid* $r = a(1 + \cos \theta)$ *above the initial line.*

Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a(1 + \cos \theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$.



$$\therefore \iint_R r \sin \theta \, dr \, d\theta = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \sin \theta \, dr \, d\theta$$

$$= \int_0^{\pi} \sin \theta \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2} \int_0^{\pi} \sin \theta \, . \, a^2 (1 + \cos \theta)^2 \, d\theta$$

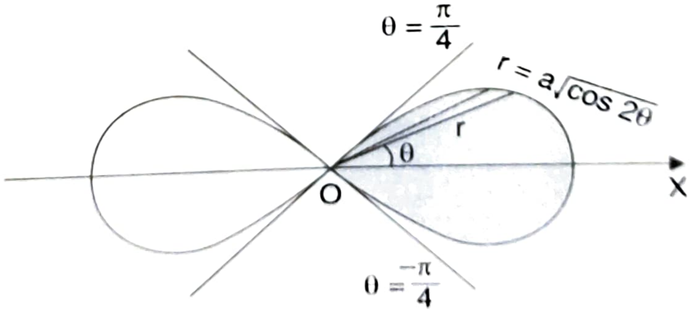
$$= \frac{a^2}{2} \int_0^{\pi} 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} . \left(2 \cos^2 \frac{\theta}{2} \right)^2 d\theta = 4a^2 \int_0^{\pi} \sin \frac{\theta}{2} \cos^5 \frac{\theta}{2} \, d\theta$$

Putting $\cos \frac{\theta}{2} = t$ so that $\sin \frac{\theta}{2} \, d\theta = -2dt$

$$\iint_R r \sin \theta \, dr \, d\theta = 4a^2 \int_1^0 t^5 (-2dt) = 8a^2 \int_0^1 t^5 dt = 8a^2 \left[\frac{t^6}{6} \right]_0^1 = 8a^2 \left(\frac{1}{6} \right) = \frac{4a^2}{3}$$

Example 4. *Evaluate* $\iint \frac{r \, dr \, d\theta}{\sqrt{a^2 + r^2}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a \sqrt{\cos 2\theta}$, the strips starting from $\theta = -\frac{\pi}{4}$ and ending at $\theta = \frac{\pi}{4}$.



$$\therefore \iint_R \frac{r \, dr \, d\theta}{\sqrt{a^2 + r^2}} = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2} (a^2 + r^2)^{-1/2} \cdot 2r \, dr \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} \cdot \frac{(a^2 + r^2)^{1/2}}{1/2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4} [(a^2 + a^2 \cos 2\theta)^{1/2} - a] \, d\theta$$

$$= a \int_{-\pi/4}^{\pi/4} [(1 + \cos 2\theta)^{1/2} - 1] \, d\theta = a \int_{-\pi/4}^{\pi/4} [(2 \cos^2 \theta)^{1/2} - 1] \, d\theta$$

$$= a\int_{-\pi/4}^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta$$

$$= 2a\int_{0}^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta = 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left(1 - \frac{\pi}{4}\right).$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1–4):

1. $\int_0^{\pi}\int_0^{a\sin\theta} r\, dr\, d\theta$.
2. $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\, dr\, d\theta$.
3. $\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\, dr\, d\theta$.
4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, dr\, d\theta$.
5. Show that $\iint_R r^2 \sin\theta\, dr\, d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a\cos\theta$, above the initial line.
6. Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.
7. Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line.
8. Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2a\cos\theta$, $r = 2b\cos\theta$, $(b < a)$.

Answers

1. $\frac{\pi a^2}{4}$
2. $\frac{a^2}{6}$
3. $a^2\left(1 + \frac{\pi}{8}\right)$
4. $\frac{5}{8}\pi a^3$
6. $\frac{45\pi}{2}$
7. $\frac{4a^2}{3}$
8. $\frac{3\pi}{2}(a^4 - b^4)$.

1.4 CHANGE OF ORDER OF INTEGRATION

In a double integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\int_c^d\int_a^b f(x, y)\, dx\, dy = \int_a^b\int_c^d f(x, y)\, dy\, dx.$$

But if the limits of integration are variable, a change in the order of integration necessitates change in the limits of integration. A rough sketch of the region of integration helps in fixing the new limits of integration.