33. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = -4(x+y)^{-2}$.

[**Hint.** $u = \log\{x^2(x-y) - y^2(x-y)\} = \log(x-y)(x^2-y^2) = \log(x-y)^2(x+y)$
$= 2\log(x-y) + \log(x+y)$]

34. (*a*) If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

(*b*) If $V = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $V_{xx} + V_{yy} + V_{zz} = f''(r) + \dfrac{2}{r} f'(r)$.

35. If $z = f(x + ay) + \phi(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

36. Find p and q, if $x = \sqrt{a}(\sin u + \cos v)$, $y = \sqrt{a}(\cos u - \sin v)$, $z = 1 + \sin(u - v)$

where p and q mean $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ respectively.

$\left[\textbf{Hint. } x^2 + y^2 = 2az, \quad \therefore \quad z = \dfrac{x^2 + y^2}{2a}\right]$

37. The equation $\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation.

Show that if $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants then $g = \sqrt{\dfrac{n}{2\mu}}$.

Answers

1. (*i*) $y^x \log y,\ xy^{x-1}$ (*ii*) $\dfrac{2x}{x^2 + y^2}, \dfrac{2y}{x^2 + y^2}$

(*iii*) $2x \sin \dfrac{y}{x} - y \cos \dfrac{y}{x},\ x \cos \dfrac{y}{x}$ (*iv*) $\dfrac{-x}{x^2 + y^2} + \dfrac{1}{y}\tan^{-1}\dfrac{y}{x}, \dfrac{x^2}{y(x^2 + y^2)} - \dfrac{x}{y^2}\tan^{-1}\dfrac{y}{x}$

6. $-\dfrac{13}{22}$ **7.** (*iii*) $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = \dfrac{z}{1-z}$; $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{z}{(1-z)^3}$

16. 2, –3 **22.** $\dfrac{x^2 - y^2}{x^2 + y^2}$ **36.** $p = \dfrac{x}{a}, q = \dfrac{y}{a}$.

4.3 HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\dfrac{y}{x}\right)$ or $y^n \phi\left(\dfrac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$

For example, if $f(x, y) = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, then

(i) $$f(x, y) = \frac{x\left(1 + \dfrac{y}{x}\right)}{\sqrt{x}\left(1 + \sqrt{\dfrac{y}{x}}\right)} = x^{1/2}\,\phi\left(\frac{y}{x}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\dfrac{1}{2}$ in x and y.

(ii) $$f(x, y) = \frac{y\left(\dfrac{x}{y} + 1\right)}{\sqrt{y}\left(\sqrt{\dfrac{x}{y}} + 1\right)} = y^{1/2}\,\phi\left(\frac{x}{y}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\dfrac{1}{2}$ in x and y.

(iii) $$f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x + y)}{\sqrt{t}\,(\sqrt{x} + \sqrt{y})} = t^{1/2} f(x, y)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\dfrac{1}{2}$ in x and y.

Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree (or order) n in the variables x, y, z if

$$f(x, y, z) = x^n\,\phi\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad y^n\phi\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or} \quad z^n\,\phi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Alternative test is $f(tx, ty, tz) = t^n f(x, y, z)$.

2.4 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If u is a homogeneous function of degree n in x and y, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

Since u is a homogeneous function of degree n in x and y, it can be expressed as

$$u = x^n f\left(\frac{y}{x}\right)$$

$\therefore$ $$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$\Rightarrow$ $$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} yf'\left(\frac{y}{x}\right) \qquad \ldots(1)$$

Also $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$\Rightarrow$ $$y\frac{\partial u}{\partial y} = x^{n-1} yf'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Adding (1) and (2), we get $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nx^n f\left(\dfrac{y}{x}\right)=nu.$

Note. Euler's theorem can be extended to a homogeneous function of any number of variables.

Thus, if u is a homogeneous function of degree n in x, y and z, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=nu.$

2.5 IF u IS A HOMOGENEOUS FUNCTION OF DEGREE n IN x AND y, THEN $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=n(n-1)u$

Since u is a homogeneous function of degree n in x and y

$\therefore$ By Euler's Theorem, we have $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu$ ...(1)

Differentiating (1) partially w.r.t. x, we have

$$1\cdot\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\cdot\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad ...(2)$$

Differentiating (1) partially, w.r.t. y, we have

$$x\frac{\partial^2 u}{\partial y\partial x}+1\cdot\frac{\partial u}{\partial y}+y\cdot\frac{\partial^2 u}{\partial y^2}=n\cdot\frac{\partial u}{\partial y}$$

But $$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$$

$$\therefore \quad x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y} \quad ...(3)$$

Multiplying (2) by x, (3) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu=n\cdot nu \quad \text{[Using (1)]}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n^2u-nu=n(n-1)u.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Euler's theorem for the functions:*

(i) $u=(x^{1/2}+y^{1/2})(x^n+y^n)$ (ii) $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}.$

Sol. (i) $u=(x^{1/2}+y^{1/2})(x^n+y^n)$ ...(1)

$$=x^{1/2}\left(1+\frac{y^{1/2}}{x^{1/2}}\right)x^n\left(1+\frac{y^n}{x^n}\right)$$

$$= x^{n+1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]\left[1+\left(\frac{y}{x}\right)^{n}\right] = x^{n+1/2} f\left(\frac{y}{x}\right) \quad [\text{OR } f(tx, ty) = t^{n+1/2} f(x, y)]$$

$\Rightarrow$ u is a homogeneous function of degree $\left(n+\frac{1}{2}\right)$ in x and y

$\therefore$ By Euler's theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)u$ ...(2)

From (1), $\frac{\partial u}{\partial x} = \frac{1}{2}x^{-1/2}(x^n + y^n) + nx^{n-1}(x^{1/2} + y^{1/2})$

$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{1/2}(x^n + y^n) + nx^n(x^{1/2} + y^{1/2})$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}y^{-1/2}(x^n + y^n) + ny^{n-1}(x^{1/2} + y^{1/2})$$

$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{1/2}(x^n + y^n) + ny^n(x^{1/2} + y^{1/2})$$

$\therefore$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(x^{1/2} + y^{1/2})(x^n + y^n) + n(x^n + y^n)(x^{1/2} + y^{1/2})$$

$$= \frac{1}{2}u + nu = \left(n+\frac{1}{2}\right)u$$

which is the same as (2). Hence the verification.

(*ii*)
$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} \quad ...(1)$$

$$= \operatorname{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x} = x^0 f\left(\frac{y}{x}\right) \quad [\text{OR } f(tx, ty) = f(x, y) = t^0 f(x, y)]$$

$\Rightarrow$ u is a homogeneous function of degree 0 in x and y.

$\therefore$ By Euler's theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \times u = 0$ ...(2)

From (1), $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{\sqrt{x^2+y^2}}$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$\therefore$ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ which is the same as (2). Hence the verification.

Example 2. *If* $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$, *prove that* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u$.

Sol. Here u is not a homogeneous function but

$$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\frac{y}{x}\right]} = x^2 f\left(\frac{y}{x}\right)$$

is a homogeneous function of degree 2 in x and y.

∴ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u \quad \text{or} \quad x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

or

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \sin u}{\cos u} \cdot \cos^2 u = 2 \sin u \cos u = \sin 2u.$$

Example 3. *If* $u = \sin^{-1}\left(\dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}\right)$, *show that* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} + 3 \tan u = 0$.

Sol. Here u is not a homogeneous function.

$$\sin u = f(x, y, z) = \frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$$

$$f(tx, ty, tz) = \frac{t\,(x + 2y + 3z)}{t^4 \sqrt{x^8 + y^8 + z^8}} = t^{-3} f(x, y, z)$$

⇒ $\sin u$ is a homogeneous function of degree – 3 in x, y, z.

∴ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) + z \frac{\partial}{\partial z} (\sin u) = -3 \sin u$$

or

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} + 3 \sin u = 0$$

or

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0.$$

Example 4. *If* $u = \log \dfrac{x^4 + y^4}{x + y}$, *show that* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 3$.

Sol. Here u is not a homogeneous function

$$u = \log \frac{x^4 + y^4}{x + y} \Rightarrow u = \log_e \left(\frac{x^4 + y^4}{x + y}\right) \Rightarrow e^u = \frac{x^4 + y^4}{x + y}$$

which is a homogeneous function of degree 3 in x, y.

$\therefore$ By Euler's theorem, we have $x \dfrac{\partial}{\partial x}(e^u) + y \dfrac{\partial}{\partial y}(e^u) = 3 \times e^u$

or $$xe^u \frac{\partial u}{\partial x} + ye^u \frac{\partial u}{\partial y} = 3e^u \quad \text{or} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3.$$

Example 5. *If* $u = \dfrac{x^2 y}{x + y}$, *show that* $x \dfrac{\partial^2 u}{\partial x^2} + y \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial u}{\partial x}$.

Sol. Here $u = \dfrac{x^2 y}{x + y}$ is a homogeneous function of degree 2 in x and y.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \quad \ldots(1)$$

Differentiating (1) partially w.r.t. x,

$$x \frac{\partial^2 u}{\partial x^2} + 1 \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x} \qquad \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right]$$

Example 6. *If* $u = \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, *prove that:*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4 \cos^3 u}.$$

Sol. $u = \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$ is not a homogeneous function but

$\sin u = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree $\dfrac{1}{2}$ in x and y.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = \frac{1}{2} \sin u$$

or $$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u \quad \text{or} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u \quad \ldots(1)$$

Differentiating (1) partially w.r.t. x,

$$x \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2} \sec^2 u \frac{\partial u}{\partial x}$$

or $$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{1}{2} \sec^2 u - 1\right) \frac{\partial u}{\partial x} \quad \ldots(2)$$