13 Turing Machine

'Life needs positive extension which makes the system achieve higher transitions, understand the language of creation, and move towards excellence.'

Introduction

In the previous chapters, we discussed two of the major approaches to the modelling of computation viz. the automata approach and the grammatical approach. Under automata approach, we discussed finite automata and pushdown automata while under the grammatical approach, we discussed context-free languages.

Further, we made the following observations:

a. A finite automaton computational model is computationally equivalent to a regular language model.

b. A pushdown automaton model is computationally equivalent to a context-free language model.

c. A pushdown automaton model is more powerful, when compared to a finite automaton model in the sense that every language accepted by a finite automaton is also recognised by the pushdown automaton. However, there are languages, viz. the language $\{a^n b^n : n \in N\}$, which are recognised by pushdown automata but not by finite automata.

d. There are languages, including the language $\{a^n b^n c^n : n \in N\}$, which are not accepted even by pushdown automata.

This prompts us to introduce a more powerful model of automata approach, which recognises more languages than a pushdown automaton model, called Turing machine. It was first proposed by *Alan Turing* in 1936 and was designed to meet the following objectives:

a. They should be automata, i.e., their construction and function should be in the same general spirit as the other computational models.

b. They should be as simple as possible, to describe, to define formally and to reason about.

c. They should be as general as possible, in terms of the computations they can carry out.

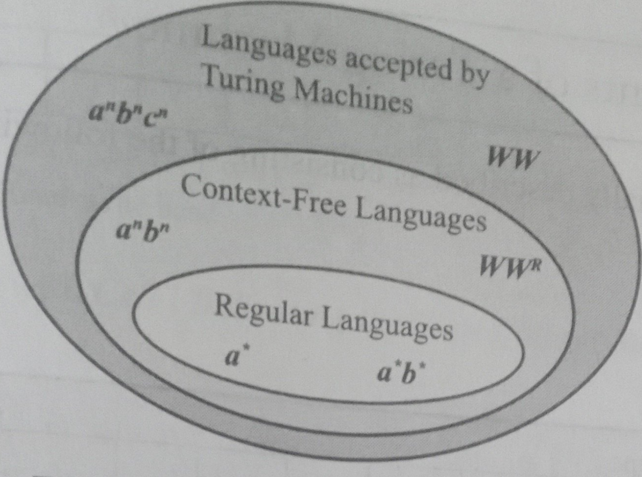

Figure 13. *The Language Hierarchy*

Definition:

There exists the following equivalent definitions for the concept of T.M:

a. Turing machines, are simple abstract computational devices intended to help investigate the extent and limitations of what can be computed.
b. A turing machine is a kind of state machine. At any time, the machine is in any one of the finite number of states. Instructions for a turing machine include the specification of conditions, under which the machine will make transitions from one state to other.

Alan Mathison Turing (1912–1954) was a British mathematician and cryptographer. He went to King's College, Cambridge in 1931 to study Mathematics. Turning graduated from Cambridge in Mathematics in 1934 and was a fellow at Kings for two years, during which period he wrote his now famous paper published in 1937–*On Computable Numbers, with an Application to the* Entscheidungs problem.

Turing is considered to be one of the fathers of modern computer science. He provided an influential formalisation of the concept of algorithm and computation–the Turing machine. He formulated the now widely accepted 'Turing' version of the Church-Turing thesis, that is any practical computing model has either the equivalent or a subset of the capabilities of a Turing machine. During World War II, he was the director of the Naval Enigma Hut at Bletchley Park for sometime and remained as the chief cryptanalyst of the Naval Enigma effort, throughout the war. After the war, he designed one of the earliest electronic programmable digital computers at the National Physical Laboratory and, shortly thereafter, actually built another early machine at the University of Manchester. He also, amongst many other things, made significant and characteristically provocative contributions to the discussion "Can machines think?".

13.1 Components of a Turing Machine

A Turing machine is usually described as consisting of the following three components.

- tape
- head
- control unit.

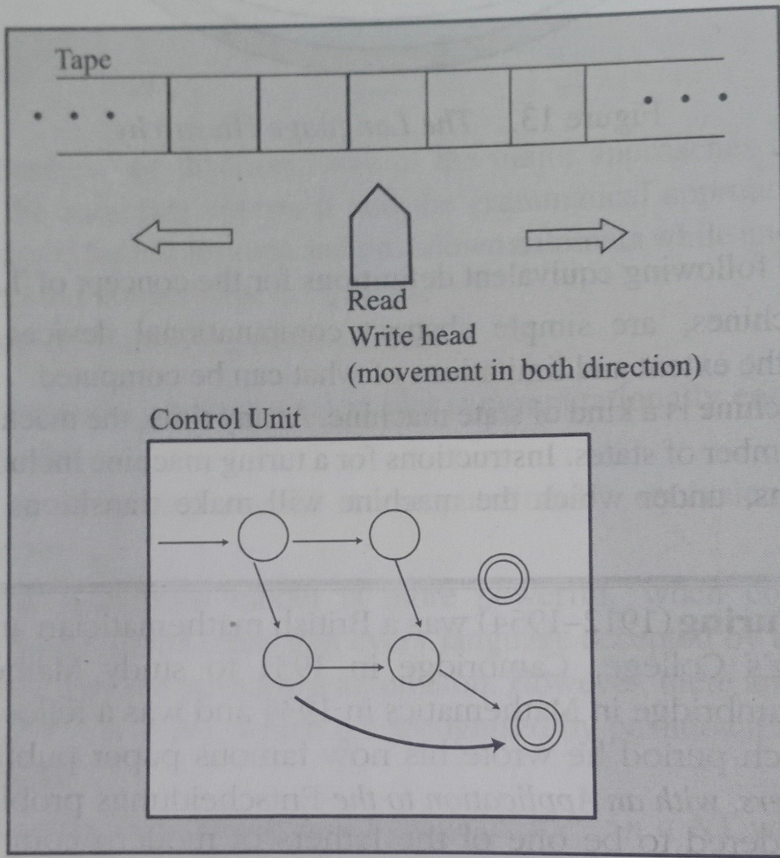

Figure 13.1. *Components of a Turing Machine*

a. TAPE

A tape is divided into a sequence of numbered cells or squares, one next to other. Each cell contains a symbol from some finite alphabet. The alphabet contains a blank symbol (B) and one or more other symbols. The set of symbols of the tape is denoted by Γ. The tape is assumed to be arbitrarily extensible to the left as well as to the right. This implies that the Turing machine is always supplied with as *much tape as it needs for its computation*. Cells that have not been written before are assumed to be filled with the blank symbol.

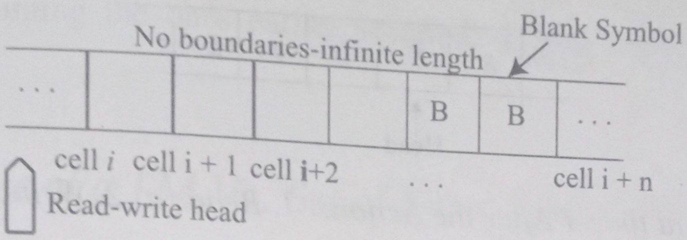

Figure 13.2. *The Tape with Cell i Indicating the Start Cell for the Current Computation*

b. **HEAD**

- A tape head, is always stationed at one of the tape cells and provides communication for the interaction between the tape and the control unit.
- In a single step, a tape head reads the contents of a cell on the tape (reads a symbol), replaces it with some other character (writes a symbol) and repositions itself to the next cell to the right or to the left of the one it has just read or does not move (moves left or right or does not move).
  This course of action is called the *move of a Turing machine.*
- At the beginning of the processing, the tape head always begins by reading the input in cell *i*. The head can never move left from the cell *i* and if it is given an order to do so, the machine crashes.

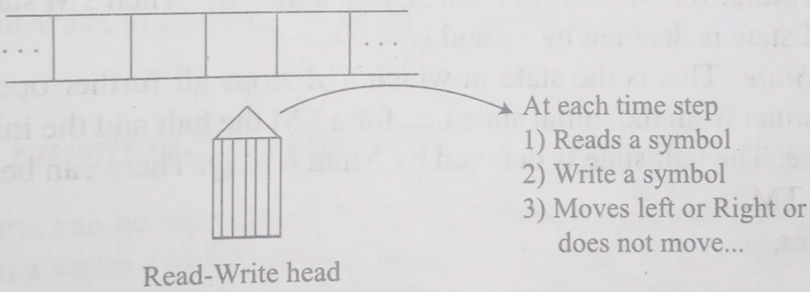

Figure 13.3. *Head*

EXAMPLE 13.1.1:

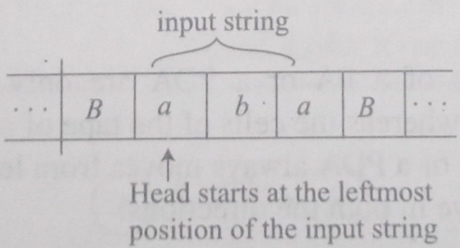

Figure 13.4. *Tape at Time-0*

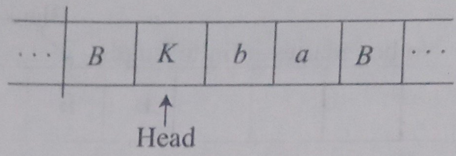

Figure 13.5. *Tape at time-1 After the Action: 1. Reads a 2. Writes K 3. Moves Right*

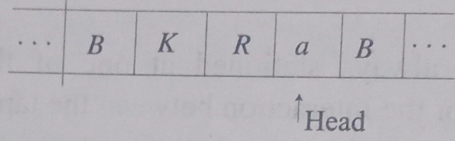

Figure 13.6. *Tape at Time-2 After the Action: 1. Reads b 2. Writes R 3. Moves Right*

c. **Control Unit**

The reading from the tape or writing into the tape is determined by the control unit. It contains a finite set of states Q. The states are categorised into three viz.,

(i) *The Initial State:* It is the state of control, just at the time when TM starts its operations. The initial state is denoted by q_0 and $q_0 \in Q$.
(ii) *The Halt State:* This is the state in which TM stops all further operations. The halt state is distinct from the initial state i.e., for a TM the halt and the initial states cannot be the same. The halt state is denoted by h and $h \sqsubset Q$. There can be one or more halt states in a TM.
(iii) *Other states.*

13.1.1 *Tape and Head of a FA/PDA Vs. Tape and Head of a TM*

The following are the differences in the roles of the tape and the tape head of a FA/PDA and the tape and head of a TM:

a. The cells of the tape of a FA or a PDA are only read/scanned but are never changed/written into, whereas the cells of the tape of a TM may be written also.
b. The tape head of a FA or a PDA always moves from left to right. However, the tape head of a TM can move in both the directions.

From the above two differences, it is clear that for a FA or a PDA, the information in the tape cells which is already scanned does not play any role in deciding the future moves of the automaton. On the other hand, in the case of a TM, the information contents of all the cells (including the ones earlier scanned) play a role in deciding the future moves.

13.1.2 *Halt State of a TM Vs. Set of Final States of a FA/PDA*

- A TM on entering the halt state stops making moves and whatever string is there on the tape will be taken as the output, irrespective of whether the position of head is at the end or in the middle of the string on the tape.
- If a FA/PDA enters a final state while scanning a symbol of the input tape, it can still go ahead with the repeated activities of moving to the right, scanning the symbol under the head and entering a new state etc. Further, the portion of a string from left to the symbol under the tape head is accepted, if the state is a final state, and is rejected if it is not.

13.2 Description of a Turing Machine

There are different ways to describe the task of a Turing machine:

13.2.1 *The Transition Diagram*

The turing machine can be represented using the transition diagram. For a directed graph, an *arc* going from a vertex (which corresponds to the state P) to the vertex that corresponds to the state q, and the also the *edge label*, can be represented in different forms as follows:

Form-1:

Read symbol (a) $\longrightarrow$ Write symbol (b), move Left(L) or move Right(R) or No move (N)

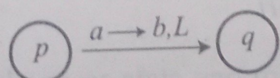

Figure 13.7. *Edge Label Format*

Form-2:

Read symbol (a)/Write symbol (b), move Left(L) or move Right (R) or do not move (N)

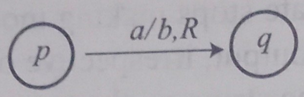

Figure 13.8. *Edge Label Format*

EXAMPLE 13.2.1: Consider the initial configuration of the tape, as shown in figure 13.9.

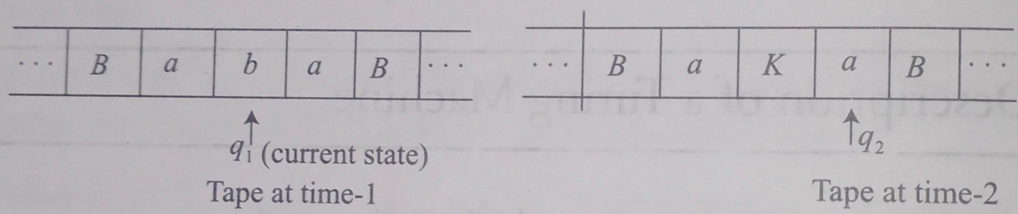

Figure 13.9.

The transition diagrams for this situation, by using the different edge label forms, are:

Form 1: **Form 2:**

Figure 13.10. *Transition Diagram Using the different Edge Label Forms*

13.2.2 5-Tuple Specification

The action performed by a TM, from one state to another state, can be specified by using the 5-tuple:

$$\boxed{\langle \text{State-1, Read Symbol, Write Symbol, } L/R/N, \text{ State-2} \rangle}$$

Form-2:

Read symbol (a)/Write symbol (b), move Left(L) or move Right (R) or do not move (N)

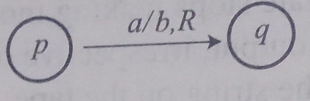

Figure 13.8. *Edge Label Format*

EXAMPLE 13.2.1: Consider the initial configuration of the tape, as shown in figure 13.9.

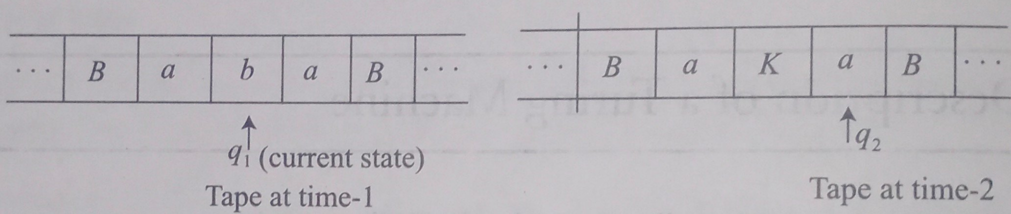

Figure 13.9.

The transition diagrams for this situation, by using the different edge label forms, are:

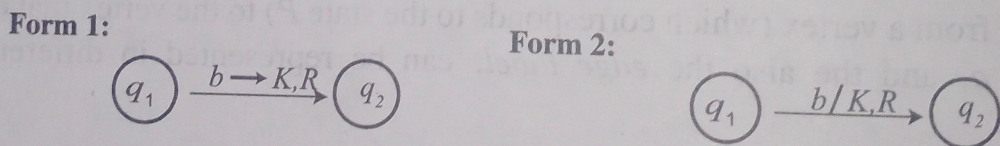

Figure 13.10. *Transition Diagram Using the different Edge Label Forms*

13.2.2 *5-Tuple Specification*

The action performed by a TM, from one state to another state, can be specified by using the 5-tuple:

$$\boxed{\langle \text{State-1, Read Symbol, Write Symbol, } L/R/N\text{, State-2}\rangle}$$

EXAMPLE: 5-tuple specification, for the action performed by the TM in figure 13.9, is:

$$< q_1, b, K, R, q_2 > .$$

13.2.3 *Transition Table*

The description of how a TM operates for a given set of symbols on the tape can be represented in a tabular format, called transition table or **action table**. In other words, the transition table describes the following for the given state and the symbol it currently reads:

a. write a symbol
b. move the head (left one step(L) or right one step(R) or no move (N))
c. assume the same or a new state, as prescribed.

The transition tables can be represented in different forms as shown:

Form-1:

Current State	Read Symbol	Write Symbol	Move Tape	Final State	5-tuples Specification

Table 13.1 *Action table form-1*

Form-2:

Current State	q_1			$\cdots$	q_n		
Tape Symbol	Write Symbol	Move Tape	Next State	$\cdots$	Write Symbol	Move Tape	Next State

Table 13.2 *Action table form-2*

Form 3:

Tape Symbol / States	a_1	a_2	a_3		a_n
	(q, b, d)				

where (q, b, d) = (new state, write symbol, move)

Table 13.3 Action table form 3

Example 13.2.2: Consider a TM, whose task is described in the transition diagram shown in figure 13.11.

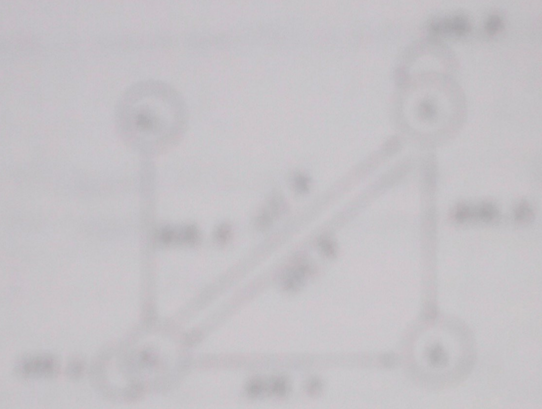

Figure 13.11 Transition Diagram

The transition table in different forms for the above TM is given below:

a.

Current State	Read Symbol	Write Symbol	Move Tape	Final State	5-tuple Specification
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table 13.4 Transition table in form 1

Form-3:

Tape Symbol / States	a_1	a_2	a_3	$\cdots$	a_n
	<action>				

where <action> = ⟨next state, write symbol, move⟩.

Table 13.3 *Action table form-3*

EXAMPLE 13.2.2: Consider a TM, whose task is described in the transition diagram shown in figure 13.11:

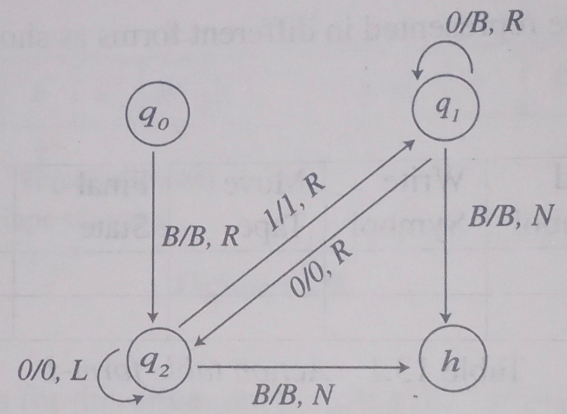

Figure 13.11. *Transition Diagram*

The transition table, in different forms for the above TM is given below:

a.

Current State	Read Symbol	Write Symbol	Move Tape	Final State	5-tuple Specification
q_0	B	B	R	q_2	(q_0, B, B, R, q_2)
q_1	0	0	R	q_2	$(q_1, 0, 0, R, q_2)$
q_1	1	B	R	q_1	$(q_1, 1, B, R, q_1)$
q_1	B	B	N	h	(q_1, B, B, N, h)
q_2	0	0	L	q_2	$(q_2, 0, 0, L, q_2)$
q_2	1	1	R	q_1	$(q_2, 1, 1, R, q_1)$

Table 13.4 *Transition table in form-1*

States \ Tape Symbols	0	1	B
q_0	–	–	$< q_2, B, R >$
q_1	$< q_2, 0, R >$	$< q_1, B, R >$	$< h, B, N >$
q_2	$< q_2, 0, L >$	$< q_1, 1, R >$	$< h, B, N >$

Table 13.5 *Transition table in form-3*

b.

Current States	q_0			q_1			q_2		
Tape Symbol	Write Symbol	Move	Next State	Write Symbol	Move	Next State	Write Symbol	Move	Next State
0	–	–	–	0	R	q_2	0	L	q_2
1	–	–	–	B	R	q_1	1	R	q_1
B	B	R	q_2	B	N	h	B	N	h

Table 13.6 *Transition table in form-2*

13.3 Observations on TM

a. **No $\in$-Transitions** are allowed in a TM.

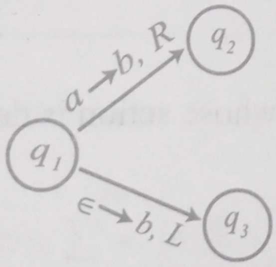

Figure 13.12. *Transition Diagram, $\in \to b, L$ is not Allowed*

b. **A Turing machine halts** if there are no possible transitions to follow.
c. In case the TM halts, we say that the word on the input tape is accepted by the TM.
d. In a TM, the halt states have no outgoing transitions.
e. **Infinite loop** in a TM: (Hanging in some states)

Because of the infinite loop:
(i) the final state cannot be reached.
(ii) the machine never halts.

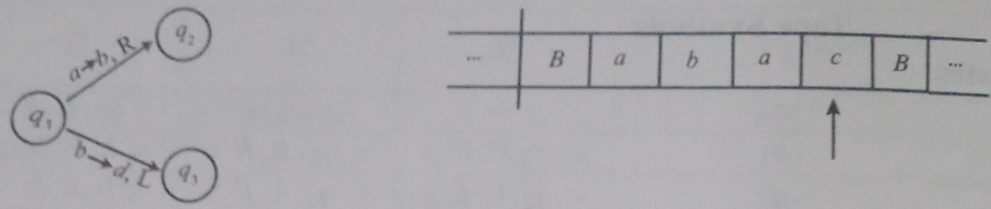

Figure 13.13. *A TM which Halts, Since no Possible Transition*

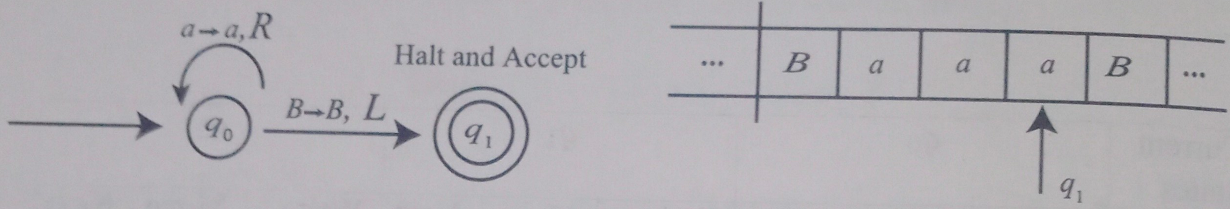

Figure 13.14. *A TM that accepts the word $w = aaa$*

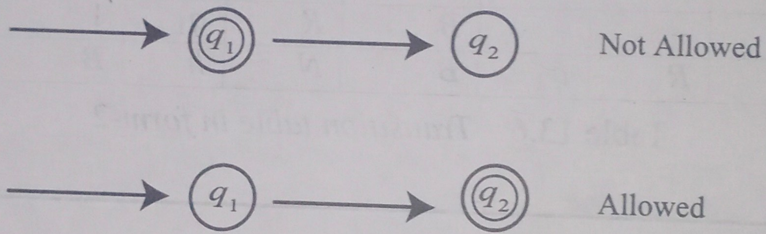

Figure 13.15. *Transition Diagram Showing No Outgoing Transition for Halt State*

(iii) the input is not accepted.

EXAMPLE 13.3.1: Consider a TM, whose action is described in figure 13.16:

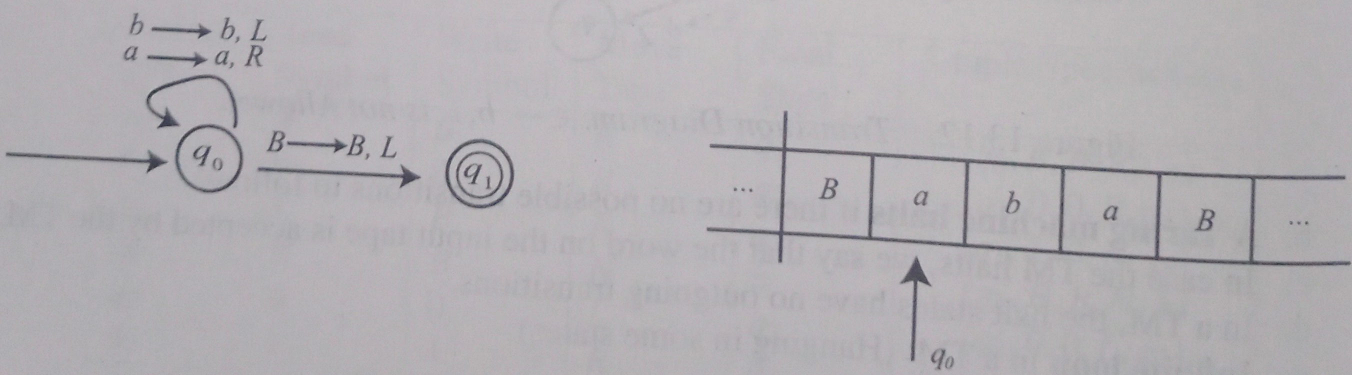

Figure 13.16. *A TM at time 0*

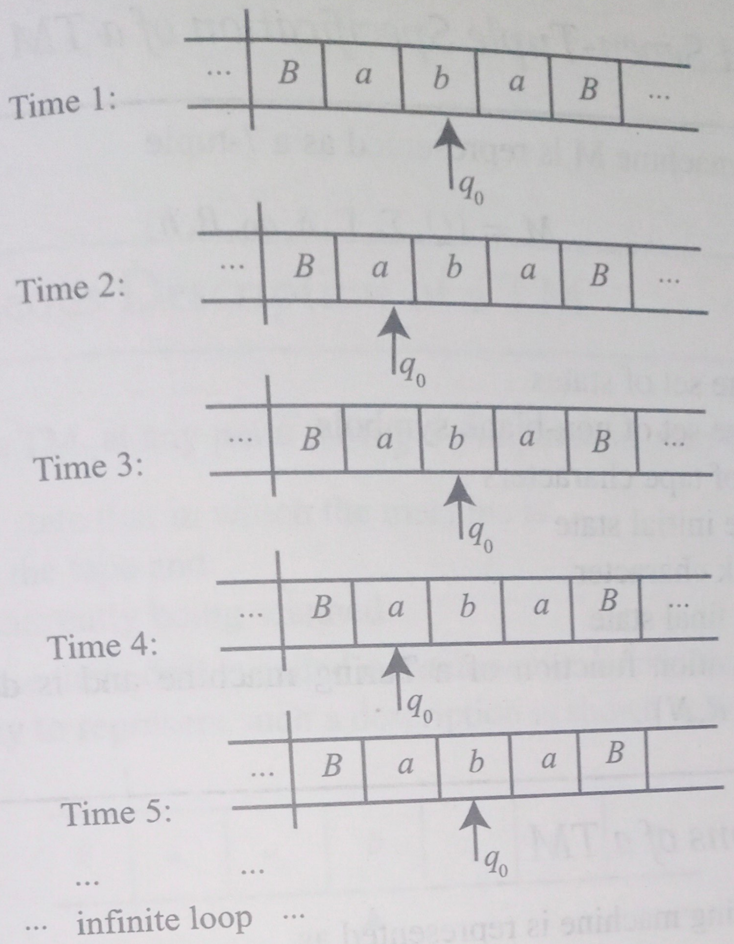

Figure 13.17. *Infinite Loop*

13.4 Elements of TM

A TM has the following seven characteristics:

a. A finite set Q of states, $q_0, q_1, \cdots, q_n$.
b. A finite input alphabet of letters, $\sum = \{a, b, \ldots\}$.
c. A finite alphabet Γ, of tape characters. The tape alphabet does not contain blank B, although a TM can write B onto its tape which is called *erasing*.
d. A transition function δ, which tells how the machine goes from one step to the next i.e., δ describes the following to be performed for a given character scanned at the current state:
   - what character to be written on the tape and,
   - tape head movements (Left, Right, No move) or (L, R, N).
e. An initial state q_0.
f. A special symbol B indicating blank character. $\sum$ does not include B.
g. A set of halt states 'h'.

13.4.1 *Ordered Seven-Tuple Specification of a TM*

Formally a turing machine M is represented as a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$$

where

a. Q is the finite set of states
b. Σ is the finite set of non-blank symbols
c. Γ is the set of tape characters
d. $q_0 \in Q$ is the initial state
e. B is the blank character
f. $h \sqsubseteq Q$ is the final state
g. δ is the transition function of a Turing machine and is defined as $Q \times \Gamma$ to $Q \times \Gamma \times \{L, R, N\}$.

13.4.2 *Transitions of a TM*

The transition of a turing machine is represented as

$$\boxed{\delta(q_i, a_k) = (q_j, a_1, x)}$$

for $q_i \in Q$, $(a_k, a_1) \in \Gamma$ and x is any one of the values 'L', 'R' and 'N'.

The meaning of $\delta(q_i, a_k) = (q_j, a_1, k)$ is that, if q_i is the current state of the TM and a_k is the cell currently under the head, then TM writes a_1 in the cell currently under the head, enters the state q_j and the head moves to the adjacent cell to the right, if the value of x is R. Otherwise, the head moves to the adjacent cell to the left, if the value of x is L and continues scanning the same cell, if the value of x is N.

EXAMPLE 13.4.1: For the TM in figure 13.11, the transition functions are:

a. $\delta(q_0, B) = (q_2, B, R)$.
This means that for q_0 as the current state and B as the cell currently under the head, the TM writes B in the cell currently under the head, enters the state q_2 and the head moves to the adjacent cell at right. Similarly, the other transitions of TM are:
b. $\delta(q_1, 0) = (q_2, 0, R)$
c. $\delta(q_1, 1) = (q_1, B, R)$
d. $\delta(q_1, B) = (h, B, N)$
e. $\delta(q_2, 0) = (q_2, 0, L)$

13.5 Instantaneous Description of a TM

The complete state of a TM, at any point during a computation, may be described by:

a. the name of the state that in which the machine is
b. the symbols on the tape and
c. the cell that is currently being scanned.

A description of these three facts is called instantaneous description(ID) or configuration of a TM. A simple way to represent such a description is shown below:

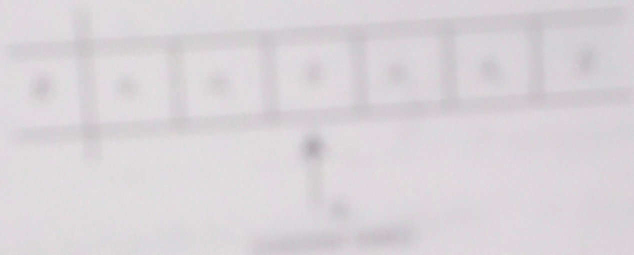

Figure 13.18 Instantaneous Description [illegible]

Formal Definition: An ID of a TM is a string [illegible]

Example 13.8.9: Consider a TM whose action is described in the transition table shown in table 13.7.

Input / States	0	1	B
q_0	[illegible]	[illegible]	[illegible]
q_1	[illegible]	[illegible]	[illegible]
q_2	[illegible]	[illegible]	[illegible]

Table 13.7 Transition table

f. $\delta(q_2, 1) = (q_1, 1, R)$
g. $\delta(q_2, B) = (h, B, N)$.

13.5 Instantaneous Description of a TM

The complete state of a TM, at any point during a computation, may be described by.

a. the name of the state that in which the machine is
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A description of these three data is called *instantaneous description*(ID) or *configuration* of a TM. A simple way to represent such a description is shown below:

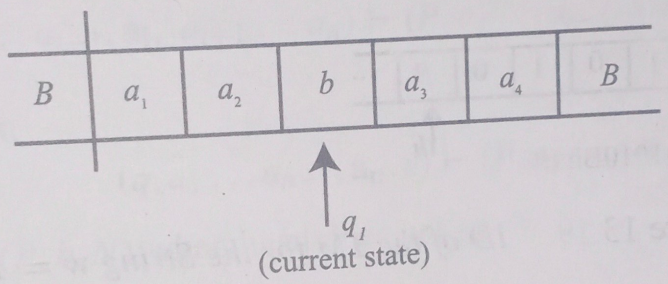


Figure 13.18. *Instantaneous Description:* $a_1\ a_2\ q_1\ ba_3\ a_4$

Formal Definition: An ID of a TM is a string xqy, where q is the current state, xy is the string made form the tape symbols Γ. The head points to the first character of the substring y. The initial ID is denoted by qxy, where q is the start state and the head points to the first symbol from left–x. The final ID is denoted by $xyqB$, where $q \in h$ is the final state and the head points to the blank character denoted by B.

EXAMPLE 13.5.1: Consider a TM, whose action is described in the transition table shown in table 13.7:

States \ Input	0	1	B
q_0	–	–	(q_2, B, R)
q_1	$(q_2, 0, R)$	(q_1, B, R)	(h, B, N)
q_2	$(q_2, 0, L)$	$(q_1, 1, R)$	(h, B, N)

Table 13.7 *Transition table*

The action of the TM with the string $w = 1010$ is shown in figure 13.19:

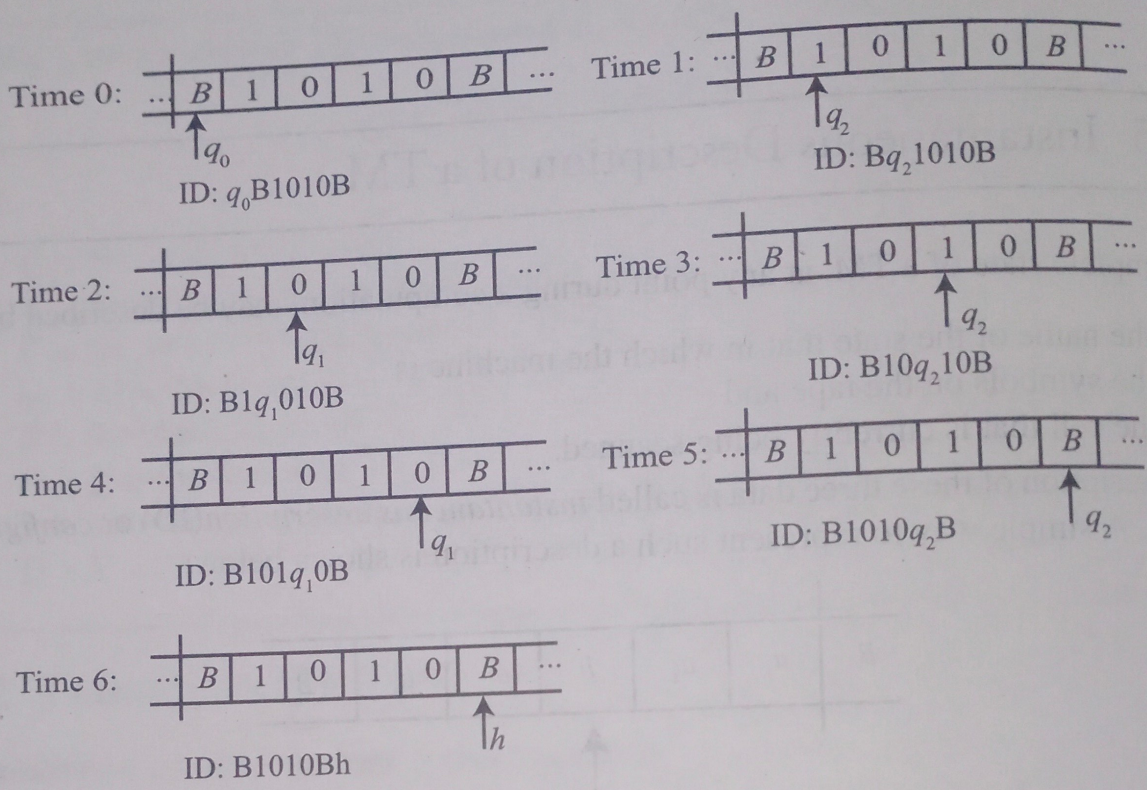


Figure 13.19. *ID of the TM for the String* $w = 1010$

13.6 Moves of a TM

As discussed in the previous sections, there are three possible different types of moves, viz.,

- Move to the left
- Move to the right and
- No move.

In this section, we give the formal definition to the moves of a TM.

Formally, let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ be a TM. Let the ID of M be

$$(q, a_1, a_2, \ldots, a_i-1, a_i, a_i+1 \ldots a_n).$$

Consider the following transitions:

a. $\delta(q, a_i) = \delta(P, b, L)$, for moving to the left.

The action of the TM with the string $w = 1010$ is shown in figure 13.19:

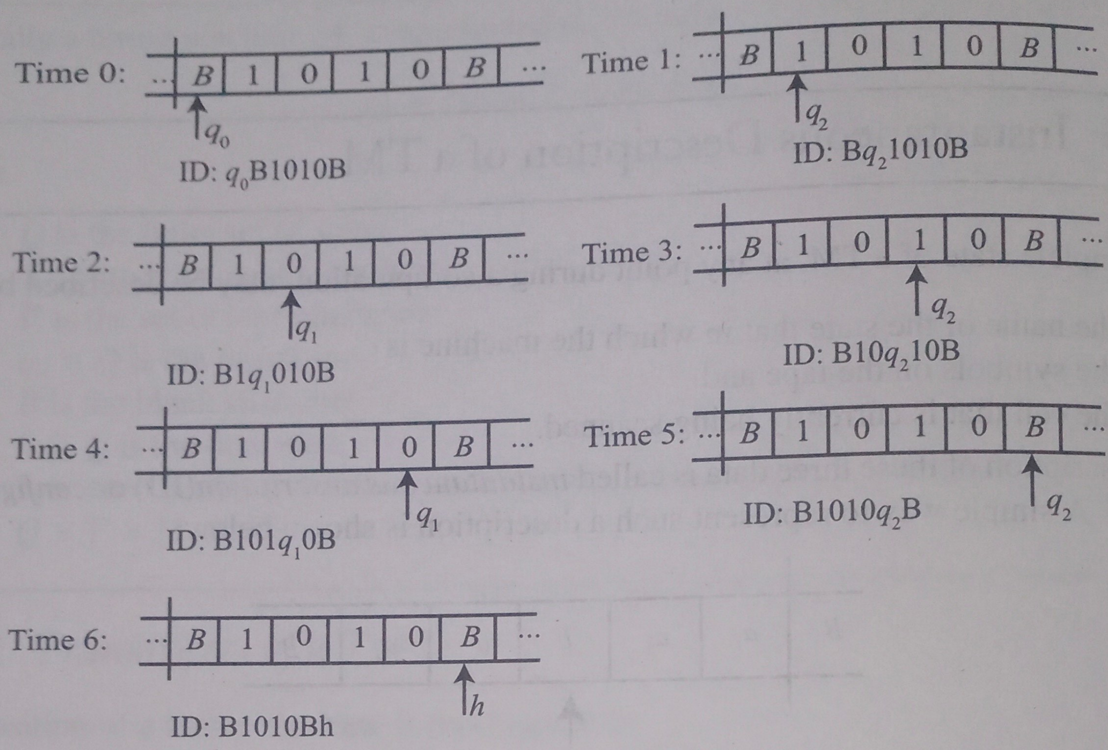


Figure 13.19. *ID of the TM for the String $w = 1010$*

13.6 Moves of a TM

As discussed in the previous sections, there are three possible different types of moves, viz.,

- Move to the left
- Move to the right and
- No move.

In this section, we give the formal definition to the moves of a TM.

Formally, let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ be a TM. Let the ID of M be

$$(q, a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1} \ldots a_n).$$

Consider the following transitions:

a. $\delta(q, a_i) = \delta(P, b, L)$, for moving to the left.

Case i: If $i > 1$, then the move of TM in going from the ID $(q, a_1a_2, \ldots, a_{i-1}, \mathbf{a_i}, a_{i+1} \ldots a_n)$ to ID $(P, a_1 \ldots a_{i-2}, \mathbf{a_{i-1}}, a_ia_{i+1}..a_n)$ is denoted by:

$$(q, a_1a_2 \ldots a_{i-1}\mathbf{a_i},\ a_{i+1} \ldots a_n) \vdash (P, a_1 \ldots a_{i-2}, \mathbf{a_{i-1}}, b, a_{i+1} \ldots a_n).$$

Case ii: If $i = 1$, the TM crashes, as it is already scanning the leftmost symbol at cell i and attempts to move to the left, which is not possible. Hence, move is not defined.

Case iii: If $i = n$ and B is the blank symbol, then

$$(q, a_1a_2 \ldots a_{n-1}, \mathbf{a_n}, e) \vdash (q, a_1a_2 \ldots a_{n-2}, a_{n-1}, B, e).$$

b. $\delta(q, a_i) = \delta(P, b, R)$, for moving to the right.

Case i: If $i < n$, then the move of TM is .

$$(q, a_1 \ldots a_{i-1}, \mathbf{a_i},\ a_{i+1} \ldots a_n) \vdash (P, a_1 \ldots a_{i-1}, b\ \mathbf{a_{i+1}}, a_{i+2} \ldots a_n).$$

Case ii: If $i = n$ then

$$(q, a_1 \ldots a_{n-1}, \mathbf{a_n}, e) \vdash (P, a_1 \ldots, B, e).$$

$\delta(q, a_i) = (P, b, N)$ when head does not move.

Then, the move is denoted as

$$(q, a_1 \ldots a_{i-1}, \mathbf{a_i}, a_{i+1} \ldots a_n) \vdash (P, a_1 \ldots a_{i-1}, \mathbf{b}, a_{i+1} \ldots a_n).$$

Note-1: e marks the end of the string.

EXAMPLE 13.6.1: Consider a TM, whose action is described in the transition table shown in table 13.8.

States \ Input	0	1	B
q_0	–	–	(q_2, B, R)
q_1	$(q_2, 0, R)$	(q_1, B, R)	(h, B, N)
q_2	$(q_2, 1, L)$	$(q_1, 1, R)$	(q_2, B, R)

Table 13.8 *Transition table*

The action of TM for the string $w = 0101$ is as follows:

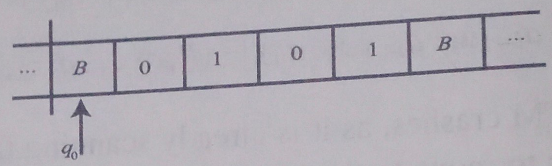


Figure 13.20. *Tape with Symbols 0101*

Consider the following transitions:

a. $\delta(q_0, B) = (q_2, B, R)$, this is represented by the move as

$$q_0B0101B \vdash Bq_20101B$$

b. $\delta(q_2, 0) = (q_2, 1, L)$

$$Bq_20101B \vdash q_2B1101B$$

c. $\delta(q_2, B) = (q_2, B, R)$

$$q_2B1101B \vdash Bq_21101B$$

d. $\delta(q_2, 1) = (q_1, 1, R)$

$$Bq_21101B \vdash B1q_1101B$$

e. $\delta(q_1, 1) = (q_1, B, R)$

$$B1q_1101B \vdash B1Bq_101B$$

f. $\delta(q_1, 0) = (q_2, 0, R)$

$$B1Bq_101B \vdash B1B0q_21B$$

g. $\delta(q_2, 1) = (q_1, 1, R)$

$$B1B0q_21B \vdash B1B01q_1B$$

h. $\delta(q_1, B) = (h, B, N)$

$$B1B01q_1B \vdash B1B01Bh$$

Thus the **move** is:

$$q_0B0101B \vdash Bq_20101B \vdash q_2B1101B \vdash Bq_21101B \vdash B1q_1101B \vdash B1Bq_101B \vdash B1B0q_21B \vdash B1B01q_1B \vdash B1B01Bh$$

The **equivalent notation** is:

$$q_0B0101B \overset{*}{\vdash} B1B01Bh.$$

13.7 String Classes in TM

Every Turing machine TM, over the alphabet Σ, divides the set of input string w into three classes:

a. **Accept (TM)** is the set of all strings $w \in \Sigma^*$, such that, if the tape initially contains w and the TM is then run, then TM ends in a HALT state.

b. **Loop (TM)** is the set of all strings $w \in \Sigma^*$, such that, if the tape initially contains w and the TM is then run, then the TM loops forever (infinite loop).

c. **Reject (TM)** is the set of all strings $w \in \Sigma^*$ such that any of the following three cases arise:

Case (i): There may be a state and a symbol under the tape head, for which δ does not have a value.

Case (ii): If the head is reading the leftmost cell (cell i), containing the symbol x, the state of TM is say q then $\delta(q, x)$ suggests a move to the left of the current cell. However, as there is no cell to the left as of the leftmost cell, no move is possible.

Case (iii): If TM enters an infinite loop or if a TM rejects a given string w because of above two cases, we say that the TM crashes (terminates unsuccessfully).

13.8 Language Accepted by a TM

The language accepted by a TM is the set of accepted strings $w \in \Sigma^*$.

Formally, Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ be a TM. The language accepted by M denoted by $L(M)$ is defined as

$L(M) = \{w | w \in \Sigma^*$ and if $w = a_1, \dots a_n$, then

$$(q_0, e, a_1, a_2, \cdots a_n) \overset{*}{\vdash} (h, b_1, \dots b_{j-1}, b_j \dots b_n)$$

for some $b_1, b_2 \dots b_n \in \Gamma^*\}$

or

$$L(M) = \{w : q_0 w \overset{*}{\vdash} x_1 \, h \, x_2\}.$$

a. **Turing Acceptable Language**
A language L over some alphabet is said to be Turing Acceptable language, if there exists a TM, M such that $L = L(M)$.

b. **Turing Decidable Language**
A language L over Σ i.e., $L \sqsubseteq \Sigma^*$ is said to be Turing decidable, if both the languages L and its complement $\Sigma^* - L$ are Turing acceptable.

c. **Recursively enumerable language**
A language L is recursively enumerable, if it is accepted by a TM.

We discuss in detail, the TM languages, in chapter 14-section 14.7.

13.9 Role of TM's

The TMs are designed to play atleast the following three roles:

a. Accepting devices for languages (similar to the role played by FAs and PDAs).

b. Computer of functions
In this role, a TM represents a particular function (say the SUM function which gives as output, the sum of two positive integers given as input). Here the initial input represents an argument of the function and the (final) string on the tape (when the TM enters the Halt State) is treated as the value obtained by the application of the function to the argument represented by the initial string.

c. An enumerator of strings of a language, that outputs the strings of a language (one at a time) in some systematic order i.e. as a list.

13.10 Design of TM's

The basic strategy for designing a TM is given below:

a. The objective of scanning a symbol by the tape head is to know about the future status. The machine must remember the symbols scanned previously, by going to the next unique state.

b. The number of states must be minimised. This can be achieved by changing the states:
   - only when there is a change in the written symbol or
   - when there is a change in the movement of the tape head.

13.10.1 TM as Accepting Devices for Languages

This concept is illustrated through the following examples.

EXAMPLE 13.10.1: Design a TM that erases all non-blank symbols on the tape, over the alphabets $\{a, b\}$.

Design Strategy: The TM in state q_0 must perform the following operations:

- On input symbol a, replace a by B, move the tape head towards right and stay at q_0.
- On input symbol b, replace b by B, move the tape head towards right and stay at q_0.
- On input symbol B, replace B by B, change the state to h and do not move tape head.

Thus the TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ is,

$$M = (\{q_0, h\}, \{a, b\}, \{a, b, B\}, \delta, q_0, B, h)$$

where δ is given by:

Transition diagram:

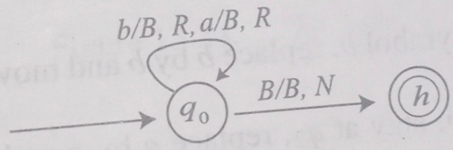


Figure 13.21. *Transition Diagram for M to Erase All Non-Blank Symbols*

Transition table:

Tape Symbol / States	a	b	B
q_0	$< q_0, B, R >$	$< q_0, B, R >$	$< h, B, N >$
h	–	–	Accept

Table 13.9 *Transition table for M*

TM action for the string $w = abab$

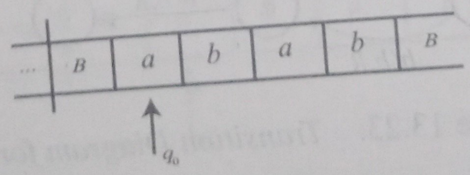


Figure 13.22. *The Tape*

ID is: $q_0ababB \vdash Bq_0babB$
$\vdash BBq_0abB$
$\vdash BBBq_0bB$
$\vdash BBBBq_0B$
$\vdash BBBBBh$

Since the final state h is reached, the string $abab$ is accepted.

EXAMPLE 13.10.2: Design a TM that accepts the language of all strings, over the alphabet $\Sigma = \{a, b\}$, whose second letter is b.

Design Strategy:

Step-1: In state q_0

- On input symbol a, change to state q_1, replace a by a and move the tape head towards right.
- On input symbol b, change to state q_1, replace b by b and move the tape head towards right.

Step-2: In state q_1 on input symbol b, replace b by b and move the tape head towards right.

Step-3: In state q_2

- On input symbol a, stay at q_2, replace a by a and move the tape head towards right.
- On input symbol b, stay at q_2, replace b by b and move the tape head towards right.
- On input symbol B, change to state h and do not move the tape head.

Thus, the TM, $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ is

$$M = (\{q_0, q_1, q_2, h\}, \{a, b\}, \{a, b, B\}, \delta, q_0, B, h),$$

where δ is given by:

Transition diagram:

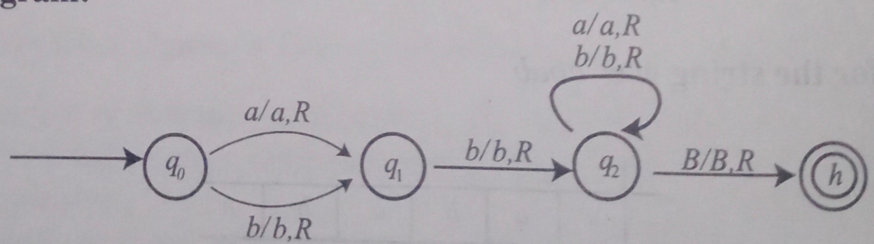


Figure 13.23. *Transition Diagram for TM*

Transition table:

States \ Tape Symbol	a	b	B
q_0	$< q_1, a, R >$	$< q_1, b, R >$	–
q_1	–	$< q_2, b, R >$	–
q_2	$< q_2, a, R >$	$< q_2, b, R >$	$< h, B, N >$
h	–	–	Accept

Table 13.10 *Transition table for TM*

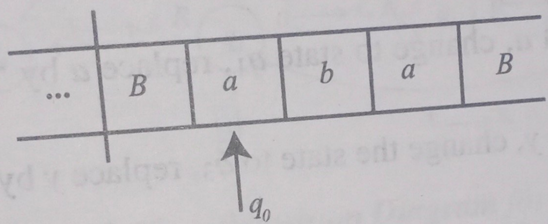


Figure 13.24. *The Tape*

a. **TM action for the string** $w = aba$

ID is: $q_0abaB \vdash aq_1baB$

$\vdash abq_2aB$

$\vdash abaq_2B$

$\vdash abaBh$

Since the final state h is reached, the string aba is accepted.

b. **TM action for the string** $w = aaa$

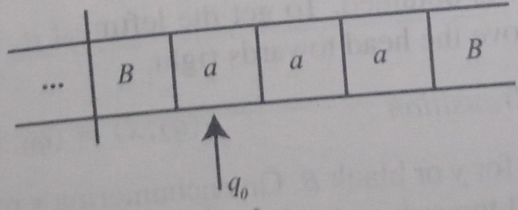


Figure 13.25. *The Tape*

$$\text{ID is: } \quad q_0aaaB \vdash aq_1aaB$$

Transition is not defined for (q_1, a). The machine halts and the string $w = aaa$ is rejected. In other words, we say that the TM crashes on giving string w as input.

EXAMPLE 13.10.3: Design a TM which accepts all strings of the form $a^n b^n$ for $n \geqslant 1$.

Design strategy:

Let q_0 be the start state and the tape head point to the first symbol of the string to be scanned.

Step-1: In state q_0,

- on input symbol a, change to state q_1, replace a by 'x' and move the tape head towards right.
- if you encounter y, change the state to q_3, replace y by y and move the tape head towards right.

Step-2: In state q_1, search for the leftmost b and replace it by y. Now, move the head to point to the leftmost b. When the head is moved towards b, the symbol encountered may be a or y. Irrespective of what symbol is encountered, replace a by a, y by y, remain in state q_1 and move the read towards right.

Transitions are :
$$\delta(q_1, a) = (q_1, a, R)$$
$$\delta(q_1, y) = (q_1, y, R)$$

Step-3: In state q_2, search for the rightmost x to get leftmost a. During this process, the symbols encountered may be y's and a's. Replace y by y, a by a, remain in state q_2 and move the head towards left.

Transitions are :
$$\delta(q_2, y) = (q_2, y, L)$$
$$\delta(q_2, a) = (q_2, a, L)$$

Once rightmost x is obtained. To get the leftmost a, replace x by x, change the state to q_0 and move the head towards right.

Transition :
$$\delta(q_2, x) = (q_0, x, R)$$

Step-4: In state q_3, search for y or blank B. On encountering y replace it by y, remain in q_3 and move the head towards right. On encountering B, change to state q_4, replace B by B and move towards left.

Turing Machine

Then, the TM, $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ is

$$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{a, b\}, \{a, b, x, y, B\}, \delta, q_0, B, \{q_4\})$$

where δ is given by:

Transition diagram:

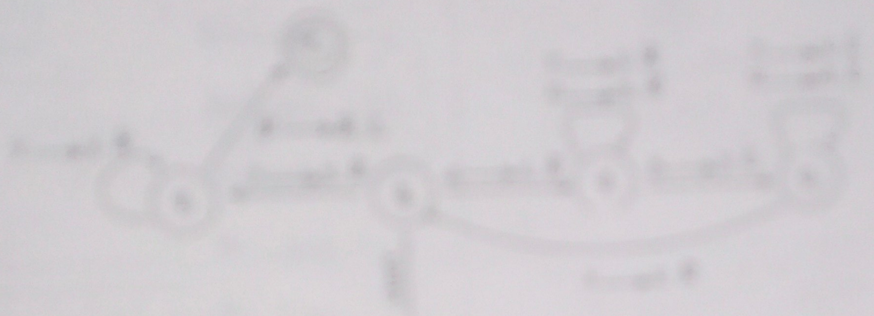


Figure 13.26 Transition Diagram for TM

Transition table:

Tape Symbol / States	a	b	x	y	B
q_0	(q_1, x, R)	–	–	(q_3, y, R)	–
q_1	(q_1, a, R)	(q_2, y, L)	–	(q_1, y, R)	–
q_2	(q_2, a, L)	–	(q_0, x, R)	(q_2, y, L)	–
q_3	–	–	–	(q_3, y, R)	(q_4, B, L)
q_4	–	–	–	–	–

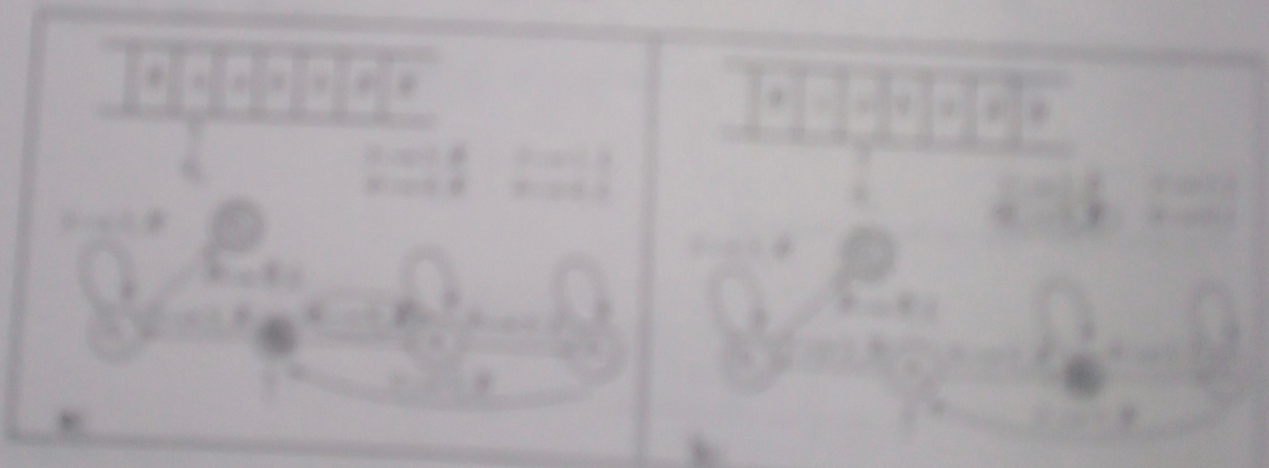
Table 13.13 Transition table for TM

TM action for the string $x = aabb$

Figure 13.27 The Tape

ID is [illegible]

Since the final state [illegible] is reached, the string [illegible] is accepted. The figure [illegible] below shows the action of the TM for the string [illegible].



ID is: $q_0aabbB \vdash xq_1abbB$

$\vdash xaq_1bbB$

$\vdash xq_2aybB$

$\vdash q_2xaybB$

$\vdash xq_0aybB$

$\vdash xxq_1ybB$

$\vdash xxyq_1bB$

$\vdash xxq_2yyB$

$\vdash xq_2xyyB$

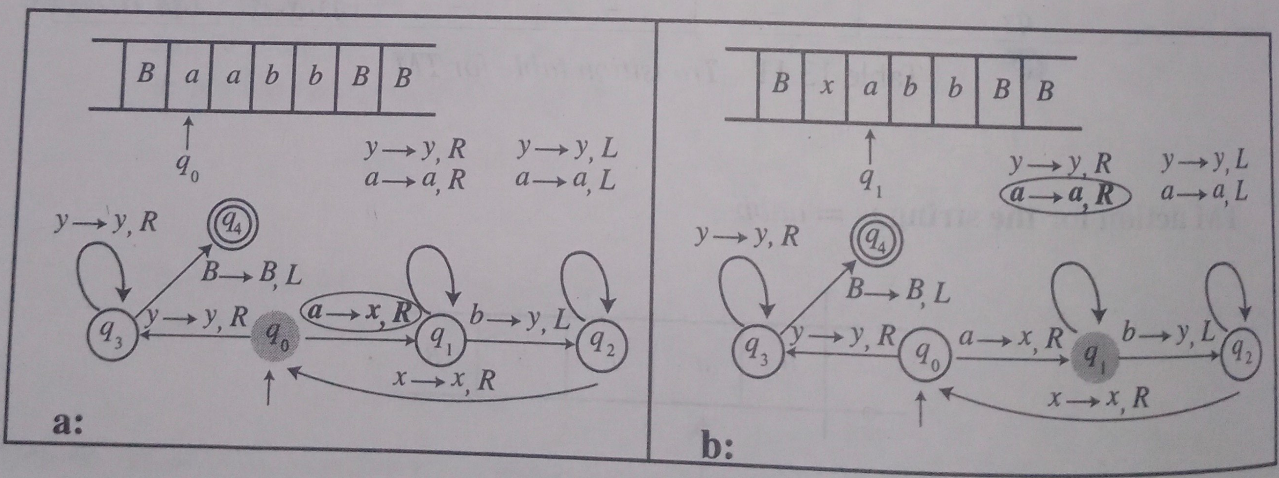
$\vdash xxq_0yyB$

$\vdash xxyq_3yB$

$\vdash xxyyq_3B$

$\vdash xxyyq_4$

Since the final state q_4 is reached, the string *aabb* is accepted. The figure 13.28(a-n) below shows the action of the TM for the string *aabb*.



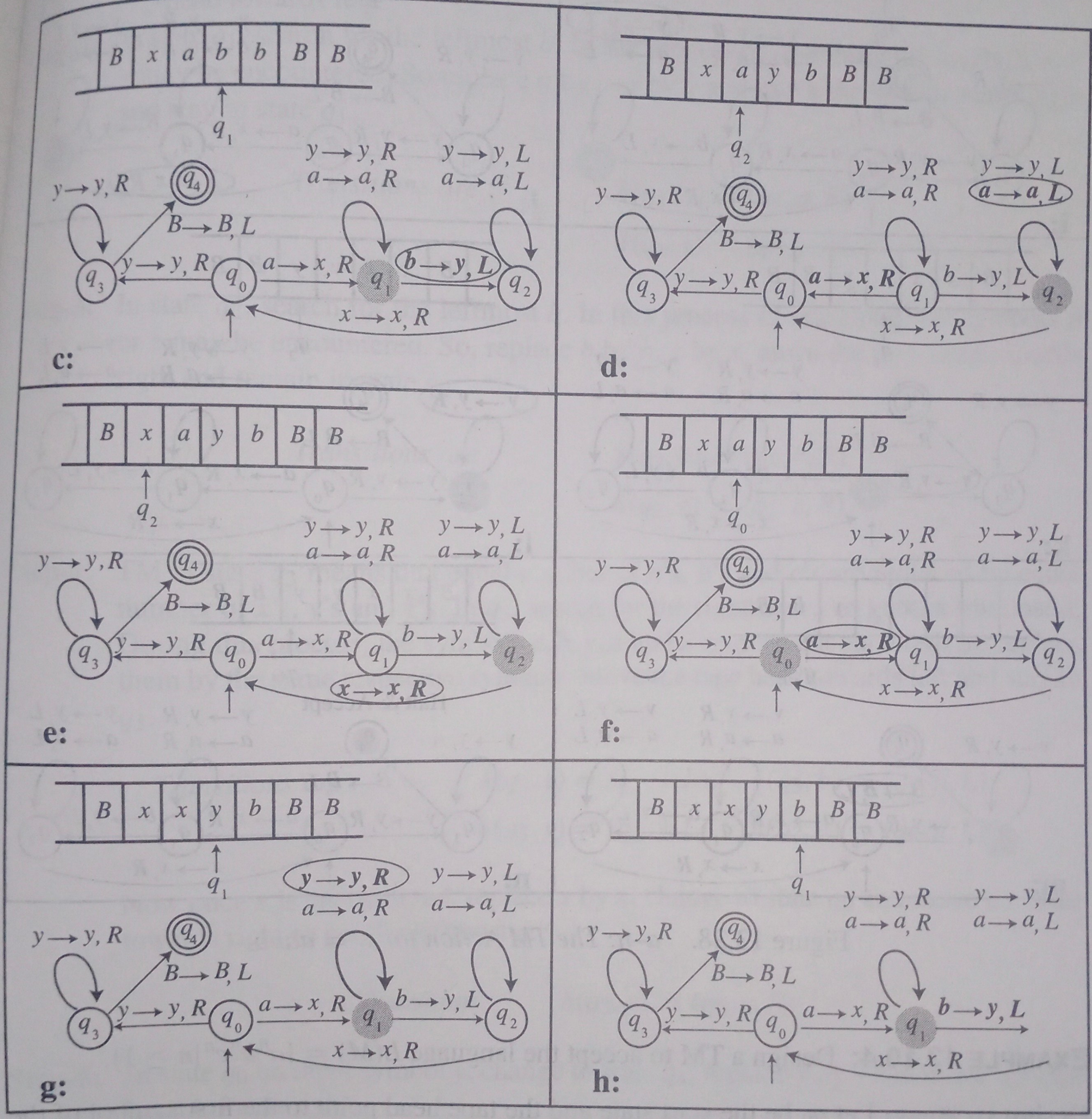


Figure 13.28. *a–n: The TM action for w = aabb.*

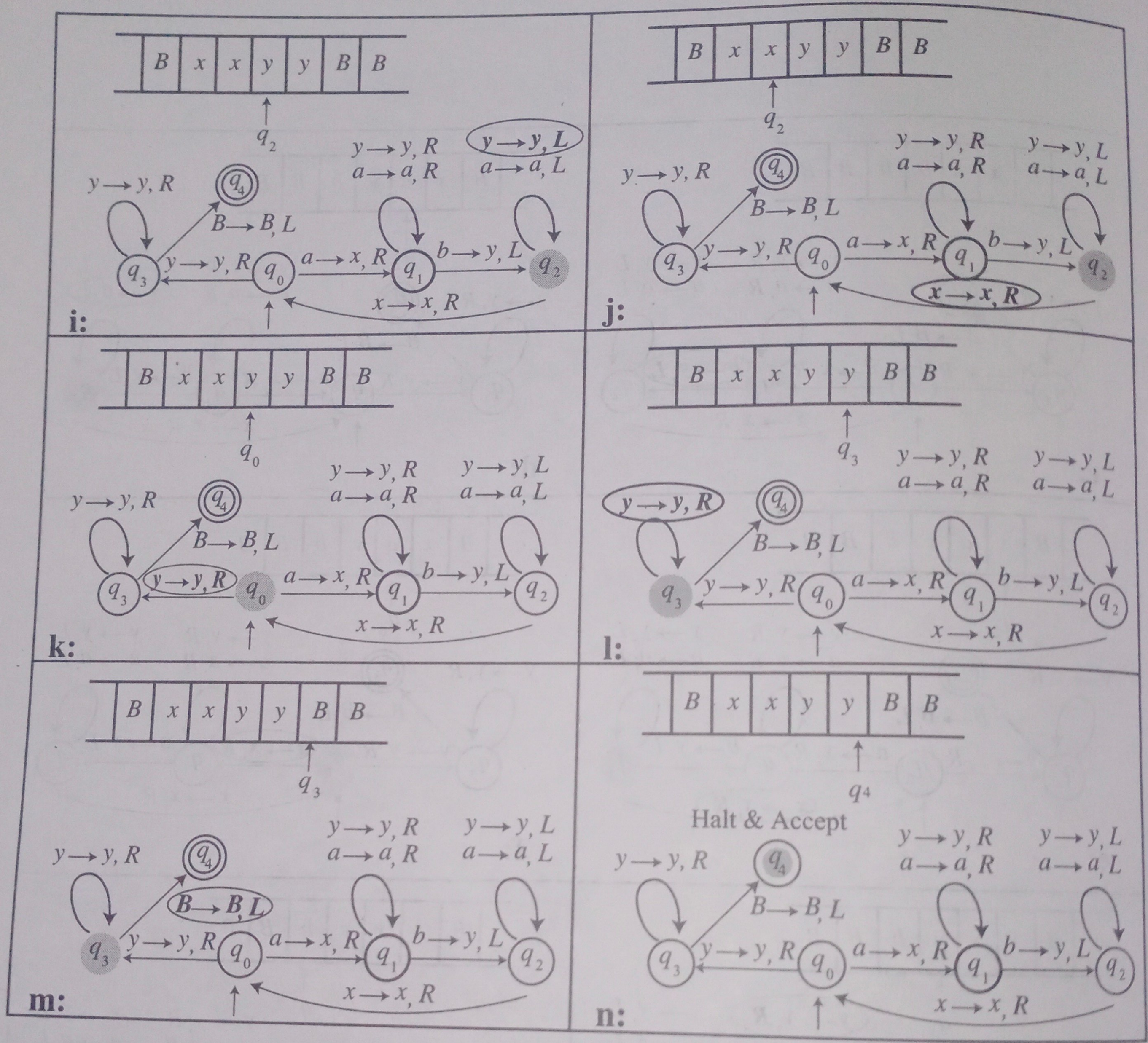


Figure 13.28. *a–n: The TM Action for $w = aabb$*

EXAMPLE 13.10.4: Design a TM to accept the language $L(M) = \{a^n b^n c^n | n \geq 1\}$.

Design strategy: Let q_0 be the start state and the tape head point to the first symbol of the string to be scanned.

Step-1: ■ In state q_0 on input symbol a, change to state q_1, replace a by x and move the tape head towards right.

- In state q_1 on input symbol b, change to state q_2, replace b by y and move the tape head towards right.
- In state q_2 on input symbol c, change to q_3, replace c by z and move the tape head towards left.

-2: In state q_1, search for the leftmost b. In the process of searching the symbols a or y may be encountered. So replace a by a, y by y and move the head towards right and stay in state q_1

Transitions are : $\delta(q_1, a) = (q_1, a, R)$

$\delta(q_1, y) = (q_1, y, R)$.

p-3: In state q_2, search for the leftmost b. In this process of searching, the symbols b or z may be encountered. So, replace b by b, z by z, move the tape head towards right and remain in state q_2.

Transitions are : $\delta(q_2, b) = (q_2, b, R)$

$\delta(q_2, z) = (q_2, z, R)$.

ep-4: TM in state q_3 means that equal number of a's, b's and c's are replaced by equal number of x's, y's and z's. In q_3, search for the rightmost x to get the leftmost a. During this process, the symbols z, b, y, a and x may be encountered. So replace them by the same respective symbols, move the tape head towards left and stay in q_3.

Transitions are : $\delta(q_3, z) = (q_3, z, L)$, $\delta(q_3, b) = (q_3, b, L)$

$\delta(q_3, y) = (q_3, y, L)$, $\delta(q_3, a) = (q_3, a, L)$

Now once x is encountered, replace x by x, change to state q_0 and move the head towards right to get the leftmost a.

Transition : $\delta(q_3, x) = (q_0, x, R)$

Step-5: In state q_0 on input symbol y, change to state q_4, replace y by y and move the tape head towards right.

Step-6: In state q_4 on input symbol y, stay in q_4, replace y by y and move the head towards right.

Transition : $\delta(q_4, y) = (q_4, y, R)$.

If z is encountered in q_4, change to q_5, replace z by z and move the head towards right.

$$\textit{Transition}: \qquad \delta(q_4, z) = (q_5, z, R).$$

(TM in q_4, with input z means that there are no b's and no c's).

Step-7: In state $q5$ on input symbol z, stay in q_5, replace z by z and move the head towards right, continue to be in q_5 for the input symbol z, so that there are only z's and no more c's. However, if B is encountered once, change to state q_6, replace B by B and move the head towards right.

$$\textit{Transitions}: \qquad \begin{aligned} \delta(q_5, z) &= (q_5, z, R) \\ \delta(q_5, B) &= (q_6, B, R) \end{aligned}$$

Thus $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ i.e.,

$$M = \big(\{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}, \{a, b, c\}, \{a, b, c, x, y, z\}, \delta, q_0, B, q_6\big),$$

where δ is given by:

Transition Diagram:

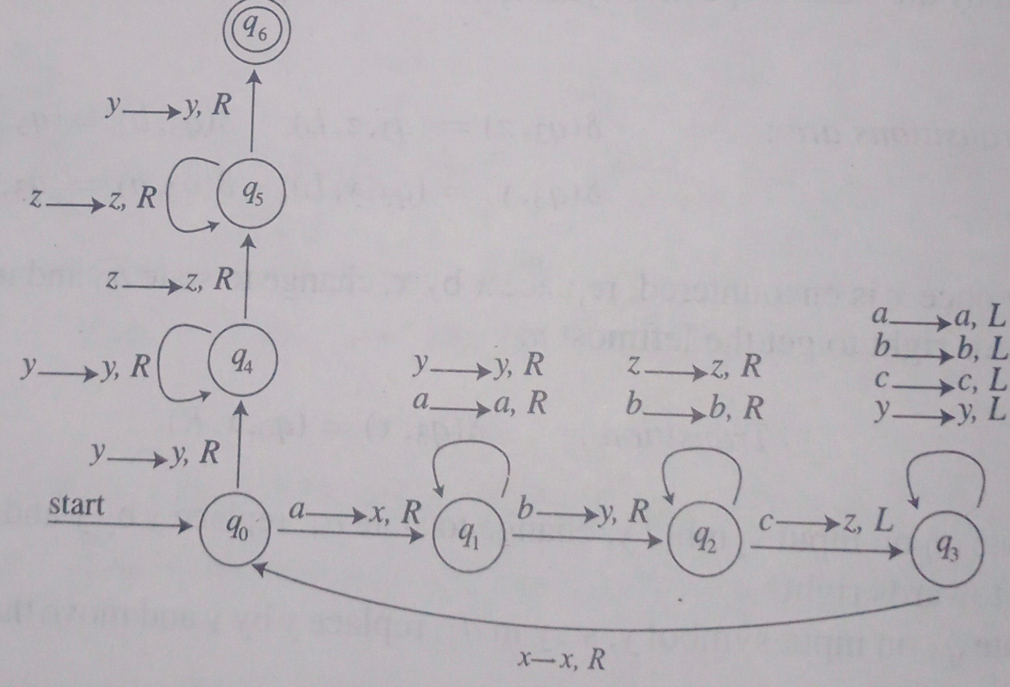


Figure 13.29. *Transition Diagram for M*

Transition table:

States \ Tape Symbols	a	b	c	x	y	z	B
q_0	(q_1, x, R)	–	–	–	(q_4, y, R)	–	–
q_1	(q_1, a, R)	(q_2, y, R)	–	–	(q_1, y, R)	–	–
q_2	–	(q_2, b, R)	(q_3, z, L)	–	–	(q_2, z, R)	–
q_3	(q_3, a, L)	(q_3, b, L)	–	(q_0, x, R)	(q_3, y, L)	(q_3, z, L)	–
q_4	–	–	–	–	(q_4, y, R)	(q_5, z, R)	–
q_5	–	–	–	–	–	(q_5, z, R)	(q_6, B, R)

Table 13.12 *Transition table for M.*

TM action for the string $w = abc$

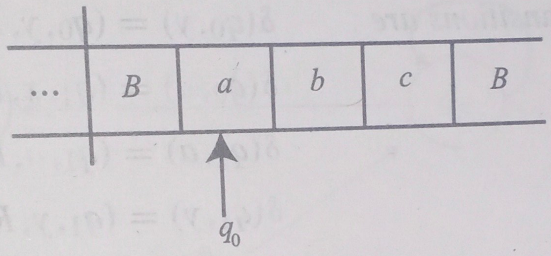


Figure 13.30. *The Tape*

ID is: $q_0abcB \vdash xq_1bcB$

$\vdash xyq_2cB$

$\vdash xq_3yzB$

$\vdash q_3xyzB$

$\vdash xq_0yzB$

$\vdash xyq_4zB$

$\vdash xyzq_5B$

$\vdash xyzq_6$.

Since the final state q_6 is reached, the string $w = abc$ is accepted.

EXAMPLE 13.10.5: Design a TM that recognises the language L of all strings, over $\{a, b\}$, with number of a's equal to the number of b's.

Design Strategy: Let q_0 be the initial state and the tape head points to the first symbol of the string to be scanned, which can either be a or b. The following cases are considered, based on the next input symbol to be scanned.

Case-1: Next input symbol to be scanned is B.

Change the state from q_0 to h, replace B by B and move the head towards right.

Transition : $\delta(q_0, B) = (h, B, R)$.

Case-2: Next input symbol to be scanned is a.

In state q_0 on input symbol a, skip all subsequent symbols till the symbol b is encountered. Then, come back to the next leftmost symbol and repeat any of the three cases based on the next symbol to be scanned.

Transitions are :

$$\begin{aligned}
\delta(q_0, y) &= (q_0, y, R)\\
\delta(q_0, a) &= (q_1, x, R)\\
\delta(q_1, a) &= (q_1, a, R)\\
\delta(q_1, y) &= (q_1, y, R)\\
\delta(q_1, b) &= (q_2, y, L)\\
\delta(q_2, y) &= (q_2, y, L)\\
\delta(q_2, a) &= (q_2, a, L)\\
\delta(q_2, x) &= (q_0, x, R).
\end{aligned}$$

Case-3: Next input symbol to be scanned is b.

In state q_0 on input symbol b, skip all subsequent symbols till the symbol a is encountered. Then, come back to the next leftmost symbol and repeat any of the three cases based on the next symbol to be scanned.

Transitions are :

$$\begin{aligned}
\delta(q_0, y) &= (q_0, y, R)\\
\delta(q_0, b) &= (q_3, x, R)\\
\delta(q_3, b) &= (q_3, b, R)\\
\delta(q_3, y) &= (q_3, y, R)\\
\delta(q_3, a) &= (q_4, y, L)
\end{aligned}$$

[illegible]

[illegible]

and [illegible]

[illegible] TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ is

[illegible]

where δ is given by:

Transition diagram:

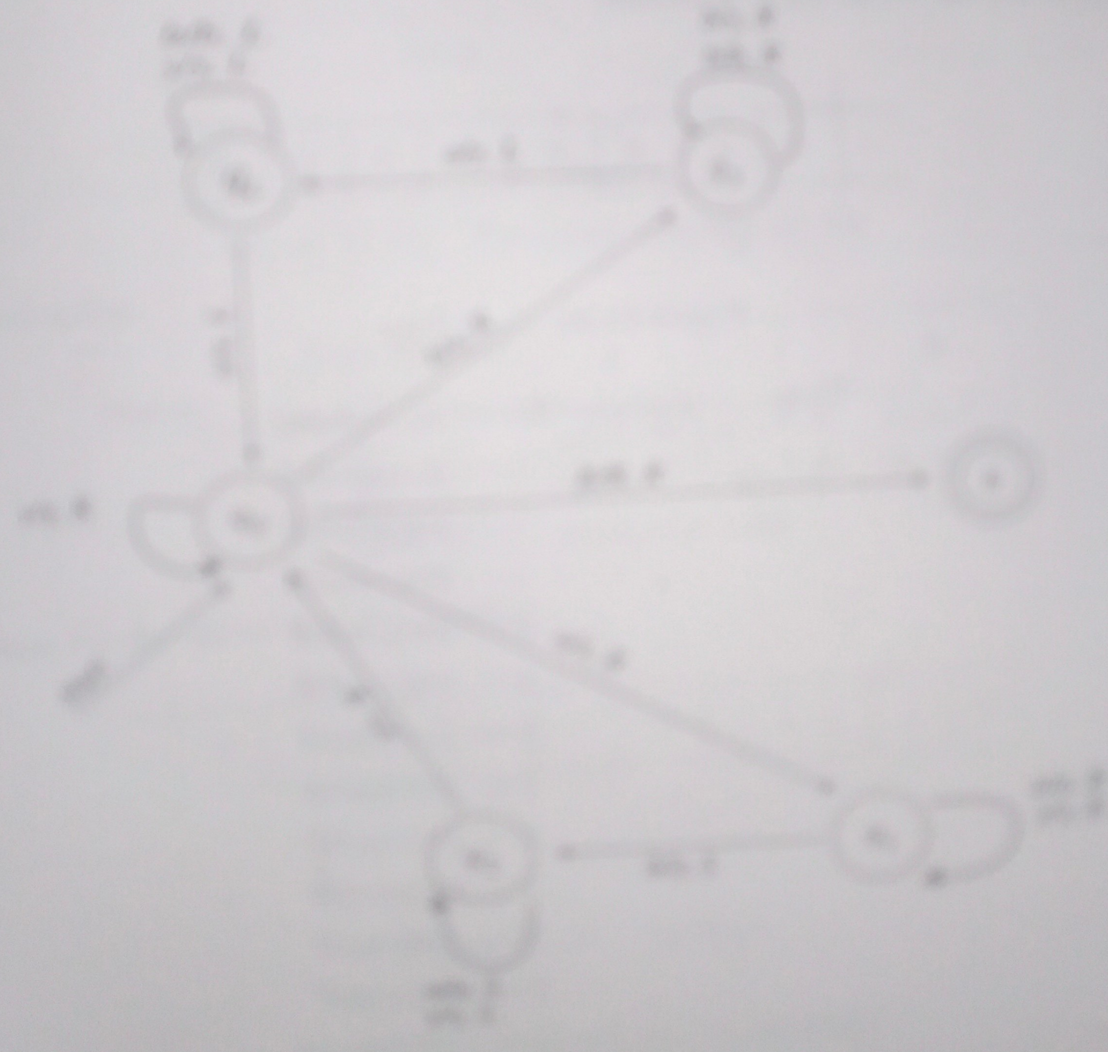


Figure [illegible] Transition diagram for M

$$\delta(q_4, y) = (q_4, y, L)$$
$$\delta(q_4, b) = (q_4, b, L)$$
and $$\delta(q_4, x) = (q_0, x, R).$$

Thus, the TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, h)$ is

$$M = (\{q_0, q_1, q_2, q_3, q_4, h\}, \{a, b\}\{a, b, x, y\}, \delta, q_0, B, h)$$

where δ is given by:

Transition diagram:

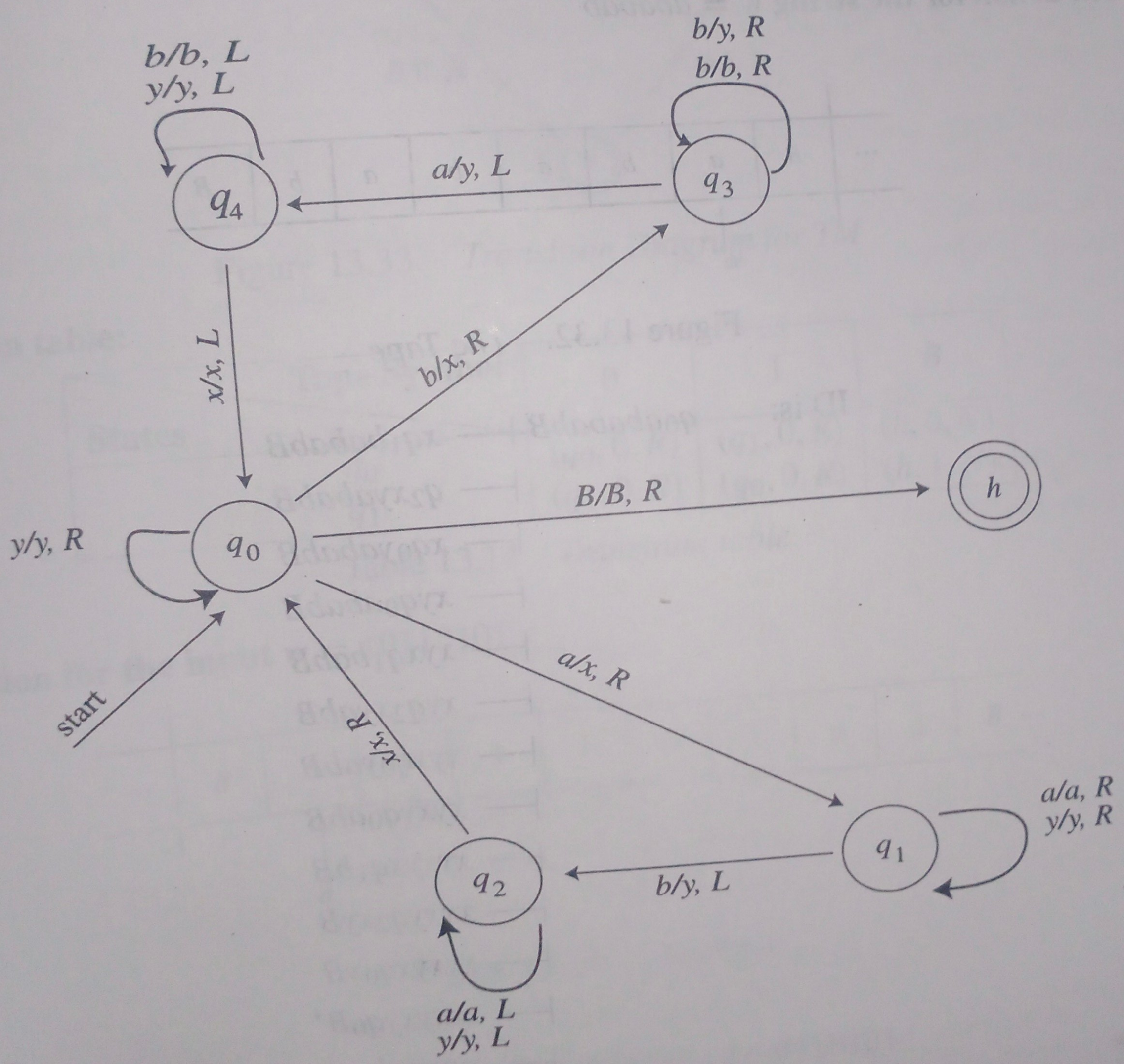


Figure 13.31. *Transition Diagram for M*

Transition table:

Tape Symbol / States	[illegible]	[illegible]	[illegible]	[illegible]	B
[illegible]	[illegible]	[illegible]	–	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	–	[illegible]	–
[illegible]	[illegible]	–	[illegible]	[illegible]	–
[illegible]	[illegible]	[illegible]	–	[illegible]	–
[illegible]	–	[illegible]	[illegible]	[illegible]	–

Table 13.13 Transition table

TM action for the string x = [illegible]

Figure 13.32 The Tape

ID is: [illegible]

[illegible]

EXAMPLE 13.10.8: Design a TM "parity counter" that outputs 0 or 1, depending on whether the number of 1's in the input sequence is even or odd respectively.

Transition table:

States \ Tape Symbol	a	b	x	y	B
q_0	(q_1, x, R)	(q_3, x, R)	–	(q_0, y, R)	(h, B, R)
q_1	(q_1, a, R)	(q_2, y, L)	–	(q_1, y, R)	–
q_2	(q_2, a, L)	–	(q_0, x, R)	(q_2, y, L)	–
q_3	(q_4, y, L)	(q_3, b, R)	–	(q_3, y, R)	–
q_4	–	(q_4, b, L)	(q_0, x, R)	(q_4, y, L)	–

Table 13.13 *Transition table.*

TM action for the string $w = ababab$

…	B	a	b	a	b	a	b	B
		$\uparrow q_0$						

Figure 13.32. *The Tape*

ID is: $q_0abababB \vdash xq_1bababB$

$\vdash q_2xyababB$

$\vdash xq_0yababB$

$\vdash xyq_0ababB$

$\vdash xyxq_1babB$

$\vdash xyq_2xyabB$

$\vdash xyxq_0yabB$

$\vdash xyxyq_0abB$

$\vdash xyxyxq_1bB$

$\vdash xyxyq_2xyB$

$\vdash xyxyxq_0yB$

$\vdash xyxyxyq_0B$

$\vdash xyxyxyh.$

EXAMPLE 13.10.6: Design a TM 'parity counter' that outputs 0 or 1, depending on wheth the number of 1's in the input sequence is even or odd respectively.