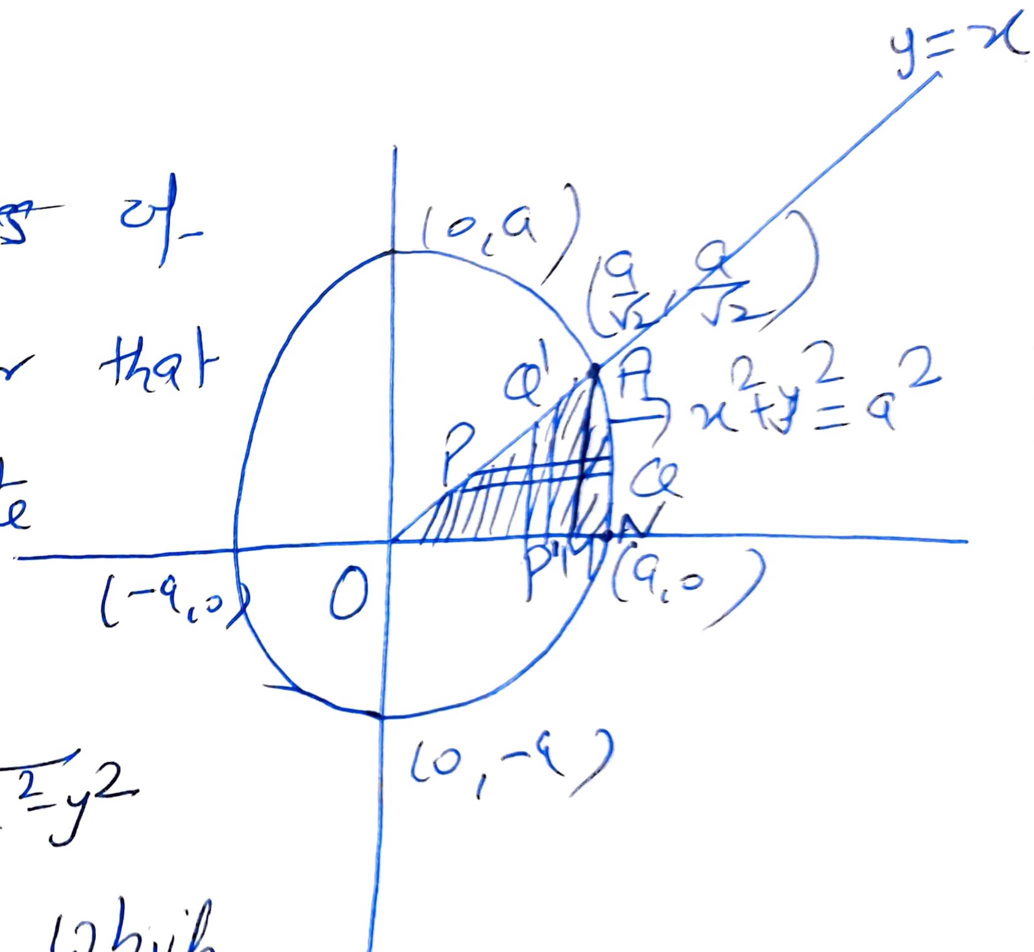
Exercise Questions of Change of order of Integration

(4) $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x \, dx \, dy$



Solu :- From the limits of integration, it is clear that we have to integrate first w.r.t. x which varies from y to $\sqrt{a^2-y^2}$ and then w.r.t. y which varies from 0 to $\frac{a}{\sqrt{2}}$.

Thus integration is first performed along the horizontal strip PQ which extends from a point P on $y = x$ to the point Q on the circle $x^2 + y^2 = a^2$ i.e. $x = \sqrt{a^2 - y^2}$. Then strip slides ~~from~~ in the shaded area.

For changing the order of integration, firstly ~~integration is performed along the vertical strip P'Q' which varies from~~ divide the region into two parts, OAM and AMP

(i) ~~in~~ OAM area, vertical strips varies from $y = 0$ to $y = x$. and it this area, ~~x~~ horizontal strip varies from $x = 0$ to $x = \frac{a}{\sqrt{2}}$.

(ii) In AMP area, vertical strip varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$ and horizontal strip varies from $x = \frac{a}{\sqrt{2}}$ to $x = a$.

Hence the given integral

$$\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x \, dx \, dy =$$

$$= \int_0^{a/\sqrt{2}} \int_0^{x} x \, dy \, dx + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} x \, dy \, dx$$

$$= \int_0^{a/\sqrt{2}} (xy)_0^{x} \, dx + \int_{a/\sqrt{2}}^{a} (xy)_0^{\sqrt{a^2-x^2}} \, dx$$

$$= \int_0^{a/\sqrt{2}} x^2 \, dx + \int_{a/\sqrt{2}}^{a} x(\sqrt{a^2-x^2}) \, dx$$

$$= \left[\frac{x^3}{3}\right]_0^{a/\sqrt{2}} + \left[\left(\frac{-1}{2}\right)\frac{(a^2-x^2)^{3/2}}{3/2}\right]_{a/\sqrt{2}}^{a}$$

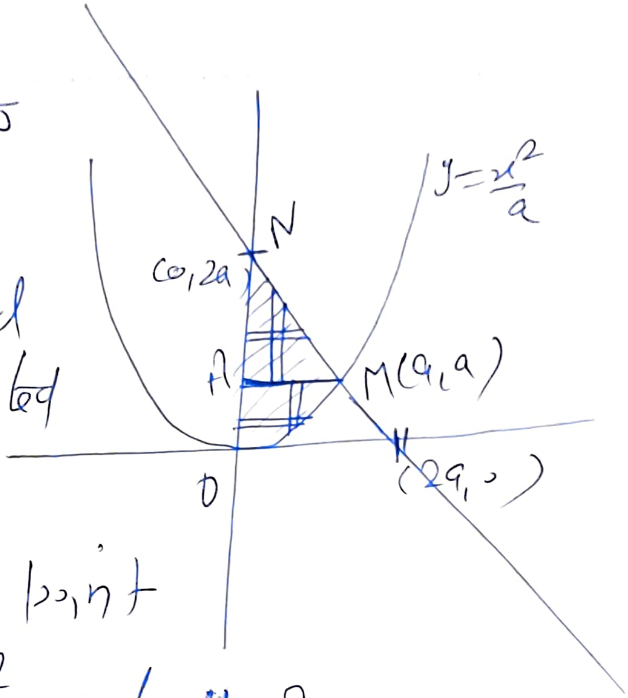
$$= \frac{1}{3}\left(\frac{a^3}{2\sqrt{2}}\right) - \frac{1}{3}\left[(a^2-a^2)^{3/2} - \left(a^2-\frac{a^2}{2}\right)^{3/2}\right]$$

$$= \frac{a^3}{6\sqrt{2}} - \frac{1}{3}\left[-\left(\frac{a^2}{2}\right)^{3/2}\right]$$

$$= \frac{a^3}{6\sqrt{2}} + \frac{1}{3}\frac{a^3}{2\sqrt{2}} = 2\left(\frac{a^3}{6\sqrt{2}}\right) = \frac{a^3\sqrt{2}}{6}$$

(6) $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$

Sol^n From the limits of integration, it is clear that shaded region is integrated area.



Now intersection point of curve $y = \frac{x^2}{a}$ and $y = 2a - x$ is

$\frac{x^2}{a} = 2a - x \Rightarrow x^2 = 2a^2 - ax$

$\Rightarrow x^2 + ax - 2a^2 = 0$

$\Rightarrow (x + 2a)(x - a) = 0$

$\Rightarrow x = a, -2a$

if $x = a$, $y = 2a - a = a \Rightarrow (a, a)$

if $x = -2a$, $y = 2a - (-2a) = 4a \Rightarrow (-2a, 4a)$

In shaded area, (a, a) is only intersection point.

For changing the order of integration, we divide the shaded area into two parts, ANM and AMO.

In ANM, x varies from $x=0$ to $x=2a-y$ and y varies from a to $2a$.

and In AMO, x varies from $x=0$ to $x=\sqrt{ay}$ and y varies from 0 to a.

Hence the given integral

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

$$= \int_0^a \left[y\frac{x^2}{2} \right]_0^{\sqrt{ay}} dy + \int_a^{2a} \left[y\frac{x^2}{2} \right]_0^{2a-y} dy$$

$$= \int_0^a \frac{y(ay)}{2}\,dy + \int_a^{2a} \frac{y}{2}(2a-y)^2\,dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \frac{1}{2}\int_a^{2a} y(4a^2+y^2-4ay)\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a + \frac{1}{2}\int_a^{2a}(4a^2y+y^3-4ay^2)\,dy$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right) + \frac{1}{2}\left[\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[2a^2(4a^2-a^2) + \frac{1}{4}(16a^4-a^4) - \frac{4a}{3}(8a^3-a^3)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[2a^2(3a^2) + \frac{1}{4}(15a^4) - \frac{4a}{3}(7a^3)\right]$$

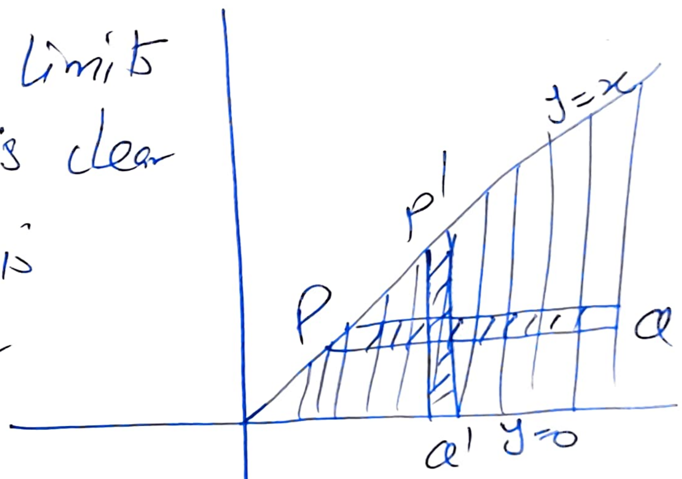
$$= \frac{a^4}{6} + \frac{1}{2}\left[6a^4 + \frac{15a^4}{4} - \frac{28a^4}{3}\right]$$

$$= \frac{a^4}{6} + 3a^4 + \frac{15}{8}a^4 - \frac{14}{3}a^4$$

$$= \frac{4a^4 + 72a^4 + 45a^4 - 112a^4}{24}$$

$$= \frac{(121-112)a^4}{24} = \frac{9a^4}{24} = \underline{\underline{\frac{3}{8}a^4}}$$

(11) $\int_0^\infty \int_0^x x e^{-x^2/y}\, dy\, dx$



Sol: From the given limits of integration, it is clear that shaded area is required area for integration.

Now For changing order of integration, horizontal strip PQ varies from $x=y$ to $x=\infty$ and vertical strip P'Q' varies from $y=0$ to $y=\infty$.

Hence the given integral

$$\int_0^\infty \int_0^x x e^{-x^2/y}\, dy\, dx = \int_0^\infty \int_y^\infty x e^{-x^2/y}\, dx\, dy$$

$$= \int_0^\infty \int_y^\infty \frac{-2xy}{-2y} e^{-x^2/y}\, dx\, dy$$

$$= \int_0^\infty \int_y^\infty \frac{-y}{2}\left(\frac{-2x}{y} e^{-x^2/y}\right) dx\, dy$$

$$= \int_0^\infty \left(e^{-x^2/y}\right)_y^\infty \left(\frac{-y}{2}\right) dy$$

$$= \int_0^\infty \left(e^{-\infty} - e^{-y^2/y}\right)\left(\frac{-y}{2}\right) dy$$

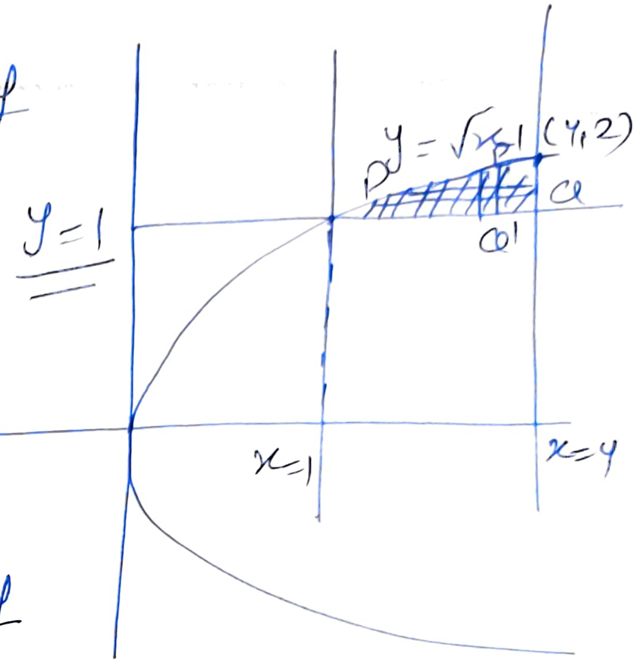
$$= \int_0^\infty (0 - e^{-y})\left(\frac{-y}{2}\right) dy$$

$$= \frac{1}{2} \int_0^\infty y e^{-y}\, dy$$

$$= \frac{1}{2}\left[\left(\frac{y e^{-y}}{-1}\right)_0^\infty - \int_0^\infty (1) \frac{e^{-y}}{-1}\, dy\right]$$

$$= \frac{1}{2}\left[(0+0) - (e^{-\infty} - e^{-0})\right] = \frac{1}{2}$$

(14) $\int_1^4 \int_1^{\sqrt{x}} (x+y^2)\,dy\,dx$



Solu:- From given limits of integration, it is clear that shaded area is required integrated area.

For changing the order of integration, horizontal strip PQ varies from $x=y^2$ to $x=4$ and vertical strip P'Q' varies from $y=1$ to $y=2$

Hence the given integral is

$$\int_1^4 \int_1^{\sqrt{x}} (x+y^2)\,dy\,dx = \int_1^2 \int_{y^2}^4 (x+y^2)\,dx\,dy$$

$$= \int_1^2 \left(\frac{x^2}{2} + xy^2\right)_{y^2}^4 dy$$

$$= \int_1^2 \left(\frac{16}{2} + 4y^2 - \frac{y^4}{2} - y^4\right) dy$$

$$= \int_1^2 \left(8 + 4y^2 - \frac{3}{2}y^4\right) dy$$

$$= \left[8y + \frac{4y^3}{3} - \frac{3}{2}\left(\frac{y^5}{5}\right)\right]_1^2$$

$$= 8(2-1) + \frac{4}{3}(8-1) - \frac{3}{10}(32-1)$$

$$= 8 + \frac{28}{3} - \frac{93}{10}$$

$$= \frac{240 + 280 - 279}{30} = \frac{241}{30}$$