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LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

1.1 DEFINITIONS

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the n^{th} order is of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_{n-1}\frac{dy}{dx} + P_n y = X$, where $P_1, P_2, \ldots\ldots, P_{n-1}, P_n$ and X are functions of x only.

A **linear differential equation with constant co-efficients** is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_{n-1}\frac{dy}{dx} + a_n y = X \quad \ldots(1)$$

where $a_1, a_2, \ldots\ldots, a_{n-1}, a_n$ are constants and X is either a constant or a function of x only.

1.2 THE OPERATOR D

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y.

Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots, \frac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2, \ldots\ldots, \frac{d^n}{dx^n} = D^n$

Thus, the symbol D is a **differential operator** or simply an **operator.**

Written in symbolic form, equation (1) becomes

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1}D + a_n)y = X$$

or $$f(D)y = X$$

where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1}D + a_n$

i.e., $f(D)$ is a polynomial in D.

The operator D can be treated as an algebraic quantity.

Thus
$$D(u + v) = Du + Dv$$
$$D(\lambda u) = \lambda\, Du$$
$$D^p D^q\, u = D^{p+q}\, u$$
$$D^p D^q u = D^q D^p u$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

1.3 THEOREMS

Theorem 1. If $y = y_1, y = y_2, \ldots, y = y_n$ *are n linearly independent solutions of the differential equation*

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)y = 0 \qquad \ldots(i)$$

then $u = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ *is also its solution, where* $c_1, c_2, \ldots, c_n$ *are arbitrary constants.*

Proof. Since $y = y_1, y = y_2, \ldots, y = y_n$ are solution of equation (*i*).

$$\therefore \quad \left.\begin{array}{l} D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \cdots + a_n y_1 = 0 \\ D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \cdots + a_n y_2 = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \cdots + a_n y_n = 0 \end{array}\right\} \qquad \ldots(ii)$$

Now
$$\begin{aligned} & D^n u + a_1 D^{n-1} u + a_2 D^{n-2} u + \cdots + a_n u \\ &= D^n (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &\quad + a_1 D^{n-1}(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &\quad + a_2 D^{n-2}(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &\quad + \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\quad + a_n (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) \\ &= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \cdots + a_n y_1) \\ &\quad + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_n + \cdots + a_n y_2) \\ &\quad + \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\quad + c_n (D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_2 + \cdots + a_n y_n) \\ &= c_1(0) + c_2(0) + \cdots + c_n(0) \qquad [\because \text{ of } (ii)] \\ &= 0 \end{aligned}$$

which shows that $u = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$ is also the solution of equation (*i*).

Since this solution contains n arbitrary constants, it is the general or complete solution of equation (*i*).

Theorem 2. *If* $y = u$ *is the complete solution of the equation* $f(D)y = 0$ *and* $y = v$ *is a particular solution (containing no arbitrary constants) of the equation* $f(D)y = X$, *then the complete solution of the equation* $f(D)y = X$ *is* $y = u + v$.

Proof. Since $y = u$ is the complete solution of the equation $f(D)y = 0$ ...(*i*)

$$\therefore \qquad f(D)u = 0 \qquad \ldots(ii)$$

Also $y = v$ is a particular solution of the equation $f(D)y = X$ ...(*iii*)

$$\therefore \qquad f(D)v = X \qquad \ldots(iv)$$

Adding (*ii*) and (*iv*), we have $f(D)(u+v) = X$

Thus $y = u + v$ satisfies the equation (*iii*), hence it is the **complete solution (C.S.)** because it contains n arbitrary constants.

The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation (*iii*).

$\therefore$ The complete solution of equation (*iii*), is **y = C.F. + P.I.**

Thus in order to solve the equation (iii), we first find the C.F. i.e., the C.S. of equation (i) and then the P.I. i.e., a particular solution of equation (iii).

1.4 AUXILIARY EQUATION (A.E.)

Consider the differential equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_n)y = 0$ ...(*i*)

Let $y = e^{mx}$ be a solution of (*i*), then $Dy = me^{mx}$, $D^2y = m^2e^{mx}$,......., $D^{n-2}y = m^{n-2}e^{mx}$

$$D^{n-1}y = m^{n-1}e^{mx},\ D^n y = m^n e^{mx}$$

Substituting the values of y, Dy, D^2y, ..., D^ny in (*i*), we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n)\, e^{mx} = 0$$

or $$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \ldots(ii)$$

Thus $y = e^{mx}$ will be a solution of equation (*i*) if m satisfies equation (*ii*).

Equation (*ii*) is called the auxiliary equation for the differential equation (*i*).

Replacing m by D in (*ii*), we get $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_n = 0$ ...(*iii*)

Equation (*ii*) gives the same values of m *as* equation (*iii*) gives of D. In practice, we take equation (*iii*) as the auxiliary equation which is obtained by equating to zero the symbolic co-efficient of y in equation (*i*).

Definition. The equation obtained by equating to zero the symbolic co-efficient of y is called the **auxiliary equation**, briefly written as **A.E.**

1.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_n)y = 0$ ...(*i*)

where all the a_i's are constant.

Its auxiliary equation is $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_n = 0$ ...(*ii*)

Let $D = m_1, m_2, m_3, \ldots\ldots, m_n$ be the roots of the A.E. The solution of equation (*i*) depends upon the nature of roots of the A.E. The following cases arise:

Case I. *If all the roots of the A.E. are real and distinct*, then equation (*ii*) is equivalent to

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n) = 0 \qquad \ldots(iii)$$

Equation (*iii*) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0,\ (D - m_2)y = 0, \ldots\ldots, (D - m_n)y = 0$$

Now, consider the equation $(D - m_1)y = 0$, *i.e.*, $\dfrac{dy}{dx} - m_1 y = 0$

It is a linear equation and I.F. $= e^{\int -m_1 dx} = e^{-m_1 x}$

$\therefore$ its solution is $\quad y \,.\, e^{-m_1 x} = \int 0 \,.\, e^{-m_1 x}\, dx + c_1 \quad$ or $\quad y = c_1 e^{m_1 x}$

Similarly, the solution of $(D - m_2)y = 0$ is $y = c_2 e^{m_2 x}$

....

the solution of $(D - m_n)y = 0$ is $y = c_n e^{m_n x}$

Hence the complete solution of equation (*i*) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x} \qquad \ldots(iv)$$

Case II. *If two roots of the A.E. are equal,* let $m_1 = m_2$.

The solution obtained in equation (*iv*) becomes

$$y = (c_1 + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= c e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

It contains $(n - 1)$ arbitrary constants and is, therefore, not the complete solution of equation (*i*).

The part of the complete solution corresponding to the repeated root is the complete solution of

$$(D - m_1)(D - m_1)y = 0$$

Putting $(D - m_1)y = v$, it becomes $(D - m_1)v = 0 \quad$ *i.e.*, $\quad \dfrac{dv}{dx} - m_1 v = 0$

As in case I, its solution is $v = c_1 e^{m_1 x}$

$\therefore$

$$(D - m_1)y = c_1 e^{m_1 x} \quad \text{or} \quad \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

which is a linear equation and I.F. $= e^{-m_1 x}$

$\therefore$ its solution is $\quad y \,.\, e^{-m_1 x} = \int c_1 e^{m_1 x} \,.\, e^{-m_1 x}\, dx + c_2 = c_1 x + c_2$

or

$$y = (c_1 x + c_2)\, e^{m_1 x}$$

Thus, the complete solution of equation (*i*) is

$$y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

If, however, three roots of the A.E. are equal, say $m_1 = m_2 = m_3$, then proceeding as above, the solution becomes

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}$$

Case III. *If two roots of the A.E. are imaginary,* let

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta$$

The solution obtained in equation (*iv*) becomes

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)]$$

$$+ c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx} \quad [\because \text{ By Euler's Theorem, } e^{i\theta} = \cos\theta + i \sin\theta]$$

$$= e^{\alpha x}[(c_1 + c_2) \cos \beta x + i (c_1 - c_2) \sin \beta x] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$$

$$= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$$

[Taking $c_1 + c_2 = C_1$, $i(c_1 - c_2) = C_2$]

Case IV. *If two pairs of imaginary roots be equal,* let

$$m_1 = m_2 = \alpha + i\beta \quad \text{and} \quad m_3 = m_4 = \alpha - i\beta$$

Then by case II, the complete solution is

$$y = e^{\alpha x} [(c_1x + c_2) \cos \beta x + (c_3x + c_4) \sin \beta x] + c_5e^{m_5x} + \ldots\ldots + c_ne^{m_nx}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0.$

Sol. Given equation in symbolic form is $(D^3 - 7D - 6)y = 0$

Its A.E. is $D^3 - 7D - 6 = 0$ or $(D + 1)(D + 2)(D - 3) = 0$

whence $D = -1, -2, 3$

Hence the C.S. is $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x}$.

Example 2. *Solve:* $(D^3 - 4D^2 + 4D)y = 0.$

Sol. The A.E. is $D^3 - 4D^2 + 4D = 0$ or $D(D^2 - 4D + 4) = 0$

or $D(D - 2)^2 = 0$

whence $D = 0, 2, 2$

Hence, the C.S is $y = c_1e^{0x} + (c_2x + c_3)e^{2x}$ or $y = c_1 + (c_2x + c_3)e^{2x}$.

Example 3. *Solve:* $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0.$

Sol. Given equation in symbolic form is $(D^4 + 13D^2 + 36)y = 0$

Its A.E. is $D^4 + 13D^2 + 36 = 0$

or $(D^2 + 4)(D^2 + 9) = 0$ $\therefore$ $D = \pm 2i, \pm 3i$

Hence the C.S. is $y = e^{0x} (c_1 \cos 2x + c_2 \sin 2x) + e^{0x} (c_3 \cos 3x + c_4 \sin 3x)$

or $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x$.

Example 4. *Solve:* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Sol. Given equation in symbolic form is $(D^4 + 4)x = 0$, where $D = \dfrac{d}{dt}$

Its A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$

or $(D^2 + 2)^2 - (2D)^2 = 0$ or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

whence $D = \dfrac{-2 \pm \sqrt{-4}}{2}$ and $\dfrac{2 \pm \sqrt{-4}}{2}$ *i.e.,* $D = -1 \pm i$ and $1 \pm i$

Hence the C.S. is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{t}(c_3 \cos t + c_4 \sin t)$.

Example 5. *Solve:* $y'' - 2y' + 10y = 0$, *given* $y(0) = 4$, $y'(0) = 1$.

Sol. Given equation in symbolic form is

$$(D^2 - 2D + 10)y = 0$$

Its A.E. is $\quad D^2 - 2D + 10 = 0$

$$\Rightarrow \quad D = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

The C.S. is $\quad y = e^x (c_1 \cos 3x + c_2 \sin 3x)$ ...(1

Now $y(0) = 4 \Rightarrow y = 4$, when $x = 0$

$\therefore \quad 4 = c_1$

Equation (1) becomes $\quad y = e^x (4 \cos 3x + c_2 \sin 3x)$ ...(2

so that $\quad y' = e^x (4 \cos 3x + c_2 \sin 3x) + e^x(-12 \sin 3x + 3c_2 \cos 3x$

Since $y'(0) = 1$ *i.e.,* $y' = 1$, when $x = 0$

$\therefore \quad 1 = 4 + 3c_2 \Rightarrow c_2 = -1$

Equation (2) becomes $y = e^x (4 \cos 3x - \sin 3x)$, which is the required particul solution.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = 0$.
2. $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$.
3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$.
4. $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 0$.
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$.
6. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0$.
7. $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 4y = 0$.
8. $\dfrac{d^4y}{dx^4} + 6\dfrac{d^2y}{dx^2} + 9y = 0$.
9. $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.
10. $\dfrac{d^3y}{dx^3} + y = 0$.
11. $\dfrac{d^2y}{dx^2} + y = 0$, given that $y(0) = 2$ and $y\left(\dfrac{\pi}{2}\right) = -2$.
12. $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$, given that, when $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 0$.
13. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 29y = 0$, given that, when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 15$.
14. If $\dfrac{d^4x}{dt^4} = m^4x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$.
15. Solve the differential equation: $9y''' + 3y'' - 5y' + y = 0$.

16. Solve the differential equation $\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}+12\dfrac{dy}{dx}+8y=0$ under the conditions $y(0)=0$, $y'(0)=0$ and $y''(0)=2$.

17. Solve the differential equation $\dfrac{d^2i}{dt^2}+\dfrac{R}{L}\dfrac{di}{dt}+\dfrac{i}{LC}=0$, where $R^2C=4L$ and R, C, L are constants.

Answers

1. $y=c_1e^{5x}+c_2e^{-x}$ **2.** $y=c_1e^{-ax}+c_2e^{-bx}$

3. $y=c_1e^{(2+\sqrt{3})x}+c_2e^{(2-\sqrt{3})x}$ **4.** $x=(c_1+c_2t)e^{-4t}$

5. $y=(c_1+c_2x+c_3x^2)e^x$ **6.** $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$

7. $y=c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-2x}$ **8.** $y=(c_1+c_2x)\cos\sqrt{3}\,x+(c_3+c_4x)\sin\sqrt{3}\,x$

9. $y=(c_1+c_2x+c_3x^2)\cos x+(c_4+c_5x+c_6x^2)\sin x$

$$+e^{-\frac{1}{2}x}\left[(c_7+c_8x)\cos\frac{\sqrt{3}}{2}x+(c_9+c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$$

10. $y=c_1e^{-x}+e^{x/2}\left(c_2\cos\dfrac{\sqrt{3}x}{2}+c_3\sin\dfrac{\sqrt{3}x}{2}\right)$ **11.** $y=2(\cos x-\sin x)$

12. $x=0$ **13.** $y=3e^{-2x}\sin 5x$ **15.** $y=c_1e^{-x}+(c_2+c_3x)\,e^{\frac{1}{3}x}$

16. $y=x^2e^{-2x}$ **17.** $i=(c_1+c_2t)\,e^{-\frac{Rt}{2L}}$.

1.6 THE INVERSE OPERATOR $\dfrac{1}{f(D)}$

Definition. $\dfrac{1}{f(D)}X$ is that function of x, *free from arbitrary constants,* which when operated upon by $f(D)$ gives X.

Thus $f(D)\left\{\dfrac{1}{f(D)}X\right\}=X$

$\therefore$ $f(D)$ and $\dfrac{1}{f(D)}$ are inverse operators.

Theorem 1. *$\dfrac{1}{f(D)}X$ is the particular integral of $f(D)y=X$.*

Proof. The given equation is $f(D)y=X$ ...(1)

Putting $y=\dfrac{1}{f(D)}X$ in (1), we have $f(D)\left\{\dfrac{1}{f(D)}X\right\}=X$ or $X=X$

which is true.

$\therefore$ $y=\dfrac{1}{f(D)}X$ is a solution of (1).

Since it contains no arbitrary constants, it is the particular integral of $f(D)\,y=X$.

Theorem 2. $\frac{1}{D} X = \int X \, dx.$

Proof. Let $\frac{1}{D} X = y$

Operating both sides by D, we have $D\left(\frac{1}{D} X\right) = Dy$ or $X = \frac{dy}{dx}$

Integrating both sides w.r.t. x

$$y = \int X \, dx,$$

no arbitrary constant being added since $y = \frac{1}{D} X$ contains no arbitrary constant.

$$\therefore \quad \frac{1}{D} X = \int X \, dx.$$

Theorem 3. $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx.$

Proof. Let $\frac{1}{D-a} X = y$

Operating on both sides by $(D - a)$, $(D - a)\left(\frac{1}{D-a} X\right) = (D - a)y$

or $$X = \frac{dy}{dx} - ay \quad i.e., \quad \frac{dy}{dx} - ay = X$$

which is a linear equation and I.F. $= e^{\int -a dx} = e^{-ax}$

$\therefore$ Its solution is $ye^{-ax} = \int X e^{-ax} \, dx$, no constant being added or $y = e^{ax} \int X e^{-ax} \, dx$

Hence, $$\frac{1}{D-a} X = e^{ax} \int e^{-ax} X \, dx.$$

1.7 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the differential equation, $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n)y = X$
It can be written as $f(D)y = X$

$$\therefore \quad \text{P.I.} = \frac{1}{f(D)} X$$

Case I. *When* $X = e^{ax}$

Since,
$$D e^{ax} = a e^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^{n-1} e^{ax} = a^{n-1} e^{ax}$$
$$D^n e^{ax} = a^n e^{ax}$$

$\therefore \quad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots\cdots + a_{n-1}D + a_n)\, e^{ax}$
$$= (a^n + a_1 a^{n-1} + a_2 a^{n-2} + \cdots\cdots + a_{n-1} a + a_n)\, e^{ax}$$

or
$$f(D)\, e^{ax} = f(a)\, e^{ax}$$

Operating on both sides by $\dfrac{1}{f(D)}$.

$$\frac{1}{f(D)}(f(D)e^{ax}) = \frac{1}{f(D)}(f(a)e^{ax}) \quad \text{or} \quad e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

Dividing both sides by $f(a)$, $\dfrac{1}{f(a)}e^{ax} = \dfrac{1}{f(D)}e^{ax}$, provided $f(a) \neq 0$

Hence,
$$\boldsymbol{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}}\text{, provided } \boldsymbol{f(a) \neq 0.}$$

Case of failure. If $f(a) = 0$, the above method fails.

Since $f(a) = 0$, $D = a$ is a root of A.E. $f(D) = 0$

$\therefore$ $D - a$ is a factor of $f(D)$.

Let
$$f(D) = (D - a)\,\phi(D), \text{ where } \phi(a) \neq 0 \quad \ldots(i)$$

Then
$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)\,\phi(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(a)}e^{ax}$$
$$= \frac{1}{\phi(a)}\cdot\frac{1}{D-a}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{ax}\,.\,e^{-ax}dx \quad \text{[By Theorem 3]}$$
$$= \frac{1}{\phi(a)}e^{ax}\int 1\,dx = x\,.\,\frac{1}{\phi(a)}e^{ax} \quad \ldots(ii)$$

Differentiating both sides of (*i*) w.r.t. D, we have $f'(D) = (D - a)\,\phi'(D) + \phi(D)$

$\Rightarrow$
$$f'(a) = \phi(a)$$

$\therefore$ From (*ii*), we have $\boldsymbol{\dfrac{1}{f(D)}e^{ax} = x\,.\,\dfrac{1}{f'(a)}e^{ax}}$, **provided $\boldsymbol{f'(a) \neq 0}$**

If $f'(a) = 0$, then $\dfrac{1}{f(D)}e^{ax} = x^2\,.\,\dfrac{1}{f''(a)}e^{ax}$, provided $f''(a) \neq 0$

and so on.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of $(4D^2 + 4D - 3)\, y = e^{2x}$.*

Sol.
$$\text{P.I.} = \frac{1}{4D^2 + 4D - 3}e^{2x} = \frac{1}{4(2)^2 + 4(2) - 3}e^{2x} \quad \text{(replacing D by 2)}$$
$$= \frac{1}{21}e^{2x}.$$

Example 2. *Find the P.I. of $(D^2 + 3D + 2)y = 5$.*

Sol.
$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}(5e^{0x}) \quad [\because \; e^{0x} = 1]$$

$$= 5 \cdot \frac{1}{0+0+2} e^{0x} \qquad \text{(replacing D by 0)}$$

$$= \frac{5}{2}.$$

Example 3. *Find the P.I. of* $(D^3 - 3D^2 + 4)y = e^{2x}$.

Sol.
$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4} e^{2x}.$$

Here the denom. vanishes, when D is replaced by 2. It is a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \quad \text{P.I.} = x \cdot \frac{1}{3D^2 - 6D} e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \quad \text{P.I.} = x^2 \cdot \frac{1}{6D - 6} e^{2x} = x^2 \cdot \frac{1}{6(2) - 6} e^{2x} = \frac{x^2}{6} e^{2x}.$$

Case II. *When* $X = \sin(ax + b)$ *or* $\cos(ax + b)$

$$D \sin(ax + b) = a \cos(ax + b)$$
$$\mathbf{D^2 \sin(ax + b) = (-a^2) \sin(ax + b)}$$
$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$
$$D^4 \sin(ax + b) = a^4 \sin(ax + b)$$

or
$$\mathbf{(D^2)^2 \sin(ax + b) = (-a^2)^2 \sin(ax + b)}$$

....

....

In general, $(D^2)^n \sin(ax + b) = (-a^2)^n \sin(ax + b)$

$$\therefore \quad f(D^2) \sin(ax + b) = f(-a^2) \sin(ax + b)$$

Operating on both sides by $\frac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)} (f(D^2) \sin(ax + b)) = \frac{1}{f(D^2)} [f(-a^2) \sin(ax + b)]$$

or
$$\sin(ax + b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax + b).$$

Dividing both sides by $f(-a^2)$,

$$\frac{1}{f(-a^2)} \sin(ax + b) = \frac{1}{f(D^2)} \sin(ax + b), \qquad \text{provided } f(-a^2) \neq 0.$$

Hence,
$$\mathbf{\frac{1}{f(D^2)} \sin(ax + b) = \frac{1}{f(-a^2)} \sin(ax + b),} \qquad \textbf{provided } \mathbf{f(-a^2) \neq 0}$$

Similarly,
$$\mathbf{\frac{1}{f(D^2)} \cos(ax + b) = \frac{1}{f(-a^2)} \cos(ax + b),} \qquad \textbf{provided } \mathbf{f(-a^2) \neq 0}$$

Case of Failure. If $f(-a^2) = 0$, the above method fails.

Since $\cos(ax + b) + i \sin(ax + b) = e^{i(ax + b)}$ | Euler's Theorem

$$\therefore \quad \frac{1}{f(D^2)} [\cos(ax + b) + i \sin(ax + b)] = \frac{1}{f(D^2)} e^{i(ax + b)}$$

[If we replace D by ia, $f(D^2) = f(-a^2) = 0$, so that it is a case of failure]

$$= x \cdot \frac{1}{f'(D^2)} e^{i(ax+b)} = x \cdot \frac{1}{f'(D^2)} [\cos(ax + b) + i \sin(ax + b)]$$

Equating real parts

$$\frac{1}{f(D^2)} \cos(ax + b) = x \cdot \frac{1}{f'(D^2)} \cos(ax + b), \qquad \text{provided } f'(-a^2) \neq 0$$

Equating imaginary parts

$$\frac{1}{f(D^2)} \sin(ax + b) = x \cdot \frac{1}{f'(D^2)} \sin(ax + b), \qquad \text{provided } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$, then

$$\frac{1}{f(D^2)} \sin(ax + b) = x^2 \cdot \frac{1}{f''(D^2)} \sin(ax + b), \qquad \text{provided } f''(-a^2) \neq 0$$

$$\frac{1}{f(D^2)} \cos(ax + b) = x^2 \cdot \frac{1}{f''(D^2)} \cos(ax + b), \qquad \text{provided } f''(-a^2) \neq 0$$

and so on.

Example 4. *Find the P.I. of $(D^3 + 1)y = \sin(2x + 3)$.*

Sol. $$\text{P.I.} = \frac{1}{D^3 + 1} \sin(2x + 3) = \frac{1}{D(-2^2) + 1} \sin(2x + 3)$$

[Putting $D^2 = -2^2$]

$$= \frac{1}{1 - 4D} \sin(2x + 3)$$

Multiplying and dividing by $(1 + 4D)$

$$= \frac{1 + 4D}{(1 - 4D)(1 + 4D)} \sin(2x + 3) = \frac{1 + 4D}{1 - 16D^2} \sin(2x + 3)$$

$$= \frac{1 + 4D}{1 - 16(-2^2)} \sin(2x + 3) \qquad \text{[Putting } D^2 = -2^2]$$

$$= \frac{1}{65} [\sin(2x + 3) + 4D \sin(2x + 3)]$$

$$= \frac{1}{65} [\sin(2x + 3) + 8 \cos(2x + 3)] \qquad \left[\because D = \frac{d}{dx}\right]$$

Example 5. *Find the P.I. of $(D^2 + 4)y = \cos 2x$.*

Sol. $$\text{P.I.} = \frac{1}{D^2 + 4} \cos 2x$$

Here the denominator vanishes when D is replaced by $-2^2 = -4$. It is a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \qquad \text{P.I.} = x \cdot \frac{1}{2D} \cos 2x = \frac{x}{2} \int \cos 2x \, dx \qquad \left[\because \ \frac{1}{D} f(x) = \int f(x) dx\right]$$

$$= \frac{x}{4} \sin 2x.$$

Case III. *When* $X = x^m$, m *being a positive integer.*

Here, $\qquad \text{P.I.} = \dfrac{1}{f(D)} x^m$

Take out the lowest degree term from $f(D)$ to make the first term unity (so that Binomial Theorem for a negative index is applicable).

The remaining factor will be of the form $1 + \phi(D)$ or $1 - \phi(D)$

Take this factor in the numerator. It takes the form $[1 + \phi(D)]^{-1}$ or $[1 - \phi(D)]^{-1}$

Expand it in ascending powers of D as far as the term containing D^m, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.

Operate on x^m term by term.

Example 6. *Find the P.I. of* $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

Sol. $$\text{P.I.} = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \frac{1}{4\left(1 + \dfrac{5D}{4} + \dfrac{D^2}{4}\right)}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} + \frac{21D^2}{16} \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4} D(x^2 + 7x + 9) + \frac{21}{16} D^2(x^2 + 7x + 9)\right]$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}(2x + 7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Case IV. *When* $X = e^{ax} V$, *where* V *is a function of* x.

Let u be a function of x, then by successive differentiation, we have

$$D(e^{ax} u) = e^{ax} Du + a e^{ax} u = e^{ax}(D + a)u$$

$$D^2(e^{ax} u) = D[e^{ax}(D + a) u] = e^{ax}(D^2 + aD) u + ae^{ax}(D + a)u$$

$$= e^{ax}(D^2 + 2aD + a^2) u = e^{ax}(D + a)^2 u$$

Similarly, $\quad D^3 (e^{ax} u) = e^{ax} (D + a)^3 u$

In general, $\quad D^n (e^{ax} u) = e^{ax} (D + a)^n u$

$\therefore \quad f(D) (e^{ax} u) = e^{ax} f (D + a) u$

Operating on both sides by $\dfrac{1}{f(D)}$,

$$\frac{1}{f(D)} [f(D) (e^{ax} u)] = \frac{1}{f(D)} [e^{ax} f(D + a)u]$$

$$\Rightarrow \quad e^{ax} u = \frac{1}{f(D)} [e^{ax} f(D + a)u] \qquad ...(i)$$

Now let $\quad f(D + a) u = V, \quad i.e., \quad u = \dfrac{1}{f(D+a)} V$

$\therefore$ From (*i*), we have $\quad e^{ax} \dfrac{1}{f(D+a)} V = \dfrac{1}{f(D)} (e^{ax} V)$

or
$$\mathbf{\frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V.}$$

Thus e^{ax} which is on the right of $\dfrac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + *a*.

Example 7. *Find the P.I. of $(D^2 - 4D + 3)y = e^x \cos 2x$.*

Sol.
$$P.I. = \frac{1}{D^2 - 4D + 3} e^x \cos 2x = e^x \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x$$

$$= e^x \frac{1}{D^2 - 2D} \cos 2x = e^x \frac{1}{-2^2 - 2D} \cos 2x \qquad [\text{Putting } D^2 = -2^2]$$

$$= -\frac{1}{2} e^x \frac{1}{2+D} \cos 2x = -\frac{1}{2} e^x \frac{2-D}{(2+D)(2-D)} \cos 2x$$

$$= -\frac{1}{2} e^x \frac{2-D}{4-D^2} \cos 2x = -\frac{1}{2} e^x \frac{2-D}{4-(-2^2)} \cos 2x$$

$$= -\frac{1}{16} e^x (2 \cos 2x - D \cos 2x) = -\frac{1}{16} e^x (2 \cos 2x + 2 \sin 2x)$$

$$= -\frac{1}{8} e^x (\cos 2x + \sin 2x).$$

Case V. *When X is any other function of x.*

Resolve $f(D)$ into linear factors.

Let $\quad f(D) = (D - m_1)(D - m_2) \cdots\cdots (D - m_n)$

Then $$P.I. = \frac{1}{f(D)} X = \frac{1}{(D - m_1)(D - m_2) \cdots\cdots (D - m_n)} X$$

$$= \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots\cdots + \frac{A_n}{D - m_2} \right) X \qquad \text{(Partial Fractions)}$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \ldots\ldots + A_n \frac{1}{D - m_n} X$$

$$= A_1 e^{m_1 x} \int X e^{-m_1 x}\, dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx + \ldots\ldots + A_n e^{m_n x} \int X e^{-m_n x} dx$$

$$\left[\because \quad \frac{1}{D - m} X = e^{mx} \int X e^{-mx} dx\right.$$

Remark. We know that $e^{i\theta} = \cos\theta + i \sin\theta$ (Euler's Theorem)

$\therefore$ $x^n \sin ax$ = Imaginary part of $x^n (\cos ax + i \sin ax)$

$= \text{I.P. of } x^n e^{iax}$

and $x^n \cos ax$ = Real part of $x^n (\cos ax + i \sin ax)$

$= \text{R.P. of } x^n e^{iax}$.

Example 8. *Solve* $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Sol. A.E. is $D^3 - 6D^2 + 11D - 6 = 0$ or $(D - 1)(D - 2)(D - 3) = 0$

whence $D = 1, 2, 3$

$\therefore$ $\text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$$\text{P.I.} = \frac{1}{D^3 - 6D^2 + 11D - 3} (e^{-2x} + e^{-3x})$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} e^{-3x}$$

$$= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x} = -\frac{1}{120} (2e^{-2x} + e^{-3x})$$

Hence the C.S. is $y = \text{C.F.} + \text{P.I.}$

i.e.,

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{120} (2e^{-2x} + e^{-3x}).$$

Example 9. *Solve* $(D - 2)^2 y = 8(e^{2x} + \sin 2x + x^2)$.

Sol. A.E. is $(D - 2)^2 = 0$ whence $D = 2, 2$

$\therefore$ $\text{C.F.} = (c_1 + c_2 x)e^{2x}$

$$\text{P.I.} = \frac{1}{(D - 2)^2} [8(e^{2x} + \sin 2x + x^2)]$$

$$= 8\left[\frac{1}{(D - 2)^2} e^{2x} + \frac{1}{(D - 2)^2} \sin 2x + \frac{1}{(D - 2)^2} x^2\right]$$

Now, $\dfrac{1}{(D - 2)^2} e^{2x} = x \cdot \dfrac{1}{2(D - 2)} e^{2x}$ | Case of failure

$= x^2 \cdot \dfrac{1}{2} e^{2x}$ | Case of failure

$= \dfrac{x^2}{2} e^{2x}$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-2^2-4D+4}\sin 2x \quad [\text{Putting } D^2 = -2^2]$$

$$= -\frac{1}{4D}\sin 2x = -\frac{1}{4}\int \sin 2x\, dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{(2-D)^2}x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}\left[1-2\left(-\frac{D}{2}\right)+\frac{(-2)(-3)}{2}\left(\frac{D}{2}\right)^2 \ldots\ldots\right]x^2$$

$$= \frac{1}{4}\left[1+D+\frac{3}{4}D^2+\ldots\ldots\right]x^2$$

$$= \frac{1}{4}\left[x^2+D(x^2)+\frac{3}{4}D^2(x^2)\right]$$

$$\therefore \quad \text{P.I.} = 8\left[\frac{x^2}{2}e^{2x}+\frac{1}{8}\cos 2x+\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)\right]$$

$$= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence the C.S. is $y = (c_1 + c_2 x)\, e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$

Example 10. *Solve:* $(D + 2)(D - 1)^2\, y = e^{-2x} + 2 \sinh x.$

Sol. A.E. is $(D + 2)(D - 1)^2 = 0$ so that $D = -2, 1, 1$

$$\therefore \quad \text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)\, e^{x}$$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2}(e^{-2x}+e^{x}-e^{-x}) \quad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now $$\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right]$$

$$= \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x} \quad | \text{ Case of failure}$$

$$= \frac{1}{9}x\cdot\frac{1}{1}e^{-2x} = \frac{x}{9}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^{x}\right]$$

$$= \frac{1}{3}\cdot\frac{1}{(D-1)^2}e^{x} \quad | \text{ Case of failure}$$

$$= \frac{1}{3} \cdot x \frac{1}{2(D-1)} e^x \qquad | \text{ Case of failure}$$

$$= \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^x = \frac{1}{6} x^2 e^x$$

$$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$$\therefore \qquad \text{P.I.} = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$$

Hence the C.S. is $\qquad y = c_1 e^{-2x} + (c_2 + c_3 x) e^x + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}.$

Example 11. *Solve* $\dfrac{d^2y}{dx^2} - 4y = x \sinh x.$

Sol. Given equation in symbolic form is $(D^2 - 4)y = x \sinh x$

A.E. is $D^2 - 4 = 0$ so that $D = \pm 2$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4} x \sinh x = \frac{1}{D^2-4} x \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} e^x \cdot x - \frac{1}{D^2-4} e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2-4} x - e^{-x} \frac{1}{(D-1)^2-4} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D-3} x - e^{-x} \frac{1}{D^2-2D-3} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-3\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)} x - e^{-x} \frac{1}{-3\left(1+\frac{2D}{3}-\frac{D^2}{3}\right)} x\right]$$

$$= -\frac{1}{6}\left[e^x \left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1} x - e^{-x}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1} x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}\ldots\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}\ldots\ldots\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{x}{3} \sinh x - \frac{2}{9} \cosh x$$

Hence the C.S. is $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{x}{3} \sinh x - \dfrac{2}{9} \cosh x.$

Example 12. *Solve* $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.

Sol. Given equation in symbolic form is $(D^4 - 1)y = \cos x \cosh x$

A.E. is $D^4 - 1 = 0$ or $(D^2 - 1)(D^2 + 1) = 0$ so that $D = \pm 1, \pm i$

$\therefore$ $\quad$ C.F. $= c_1e^x + c_2e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$

$= c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$

$$\text{P.I.} = \frac{1}{D^4-1}\cos x \cosh x = \frac{1}{D^4-1}\cos x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4-1}e^x \cos x + \frac{1}{D^4-1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^4-1}\cos x + e^{-x}\frac{1}{(D-1)^4-1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4+4D^3+6D^2+4D}\cos x + e^{-x}\frac{1}{D^4-4D^3+6D^2-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1^2)^2+4D(-1^2)+6(-1^2)+4D}\cos x + e^{-x}\frac{1}{(-1^2)^2-4D(-1^2)+6(-1^2)-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{-5}\cos x + e^{-x}\frac{1}{-5}\cos x\right] = -\frac{1}{5}\left(\frac{e^x+e^{-x}}{2}\right)\cos x = -\frac{1}{5}\cosh x \cos x$$

Hence the C.S. is $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$.

Example 13. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$.

Sol. Given equation in symbolic form is $(D^2 - 2D + 1)y = xe^x \sin x$

A.E. is $D^2 - 2D + 1 = 0$ or $(D - 1)^2 = 0$ so that $D = 1, 1$

$\therefore$ $\quad$ C.F. $= (c_1 + c_2x)e^x$

$$\text{P.I} = \frac{1}{(D-1)^2}e^x . x \sin x = e^x . \frac{1}{(D+1-1)^2}x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D}\int x \sin x \, dx \quad \text{Integrating by parts}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1(-\cos x)dx\right] = e^x \frac{1}{D}(-x \cos x + \sin x)$$

$$= e^x \int(-x\cos x + \sin x)\,dx = e^x\left[-\left\{x \sin x - \int 1.\sin x\,dx\right\} - \cos x\right]$$

$$= e^x[-x \sin x - \cos x - \cos x] = -e^x(x \sin x + 2\cos x)$$

Hence the C.S. is $y = (c_1 + c_2x)e^x - e^x(x \sin x + 2 \cos x)$.

Example 14. *Solve* $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$.

Sol. Given equation in symbolic form is

$$(D^2 - 4)y = \cosh(2x-1) + 3^x$$

A.E. is $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$\therefore$ C.F. $= c_1e^{2x} + c_2e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4}[\cosh(2x-1) + 3^x]$$

$$= \frac{1}{D^2-4}\left[\frac{e^{2x-1} + e^{-(2x-1)}}{2} + e^{\log 3^x}\right] \left[\because \cosh t = \frac{e^t + e^{-t}}{2} \text{ and } u = e^{\log u}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^{2x-1} + \frac{1}{D^2-4}e^{-(2x-1)}\right] + \frac{1}{D^2-4}e^{x\log 3}$$

$$= \frac{1}{2}\left[x \cdot \frac{1}{2D}e^{2x-1} + x \cdot \frac{1}{2D}e^{-(2x-1)}\right] + \frac{1}{(\log 3)^2 - 4}e^{x\log 3}$$

$$= \frac{1}{2}\left[x \cdot \frac{1}{2D}e^{2x-1} + x \cdot \frac{1}{2D}e^{-(2x-1)}\right] + \frac{1}{(\log 3)^2 - 4}e^{x\log 3}$$

$$= \frac{x}{4}\left[\int e^{2x-1}\,dx + \int e^{-(2x-1)}dx\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\left[\frac{e^{2x-1}}{2} + \frac{e^{-(2x-1)}}{-2}\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\left[\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} + \dfrac{x}{4}\sinh(2x-1) + \dfrac{3^x}{(\log 3)^2 - 4}$.

Example 15. *Solve* $(D^2 + 1)y = x^2 \sin 2x$.

Sol. A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2+1}x^2 \sin 2x = \text{I.P. of } \frac{1}{D^2+1}x^2e^{2ix}$$

$$= \text{I.P. of } e^{2ix}\frac{1}{(D+2i)^2+1}x^2 = \text{I.P. of } e^{2ix}\frac{1}{D^2+4iD-3}x^2$$

$$= \text{I.P. of } e^{2ix}\frac{1}{-3\left(1 - \frac{4}{3}iD - \frac{D^2}{3}\right)}x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3}\left[1-\left(\frac{4iD+D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{I.P. of } -\frac{1}{3} e^{2ix}\left[1+\left(\frac{4iD+D^2}{3}\right)+\left(\frac{4iD+D^2}{3}\right)^2+\ldots\ldots\right] x^2$$

$$= \text{I.P. of } -\frac{1}{3} e^{2ix}\left[1+\frac{4iD}{3}+\left(\frac{1}{3}-\frac{16}{9}\right)D^2+\ldots\ldots\right] x^2$$

$$= \text{I.P. of } -\frac{1}{3} e^{2ix}\left[x^2+\frac{4i}{3}(2x)-\frac{13}{9}(2)\right]$$

$$= \text{I.P. of } -\frac{1}{3}(\cos 2x + i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+\left(\frac{8x}{3}\right)i\right]$$

$$= -\frac{1}{3}\left[\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$$

$$= -\frac{1}{27}\,[24x\cos 2x+(9x^2-26)\sin 2x]$$

Hence the C.S. is $y = c_1 \cos x + c_2 \sin x - \dfrac{1}{27}\,[24x \cos 2x + (9x^2 - 26) \sin 2x]$.

Example 16. *Solve* $(D^4 + 2D^2 + 1)y = x^2 \cos x$.

Sol. A.E. is $\quad (D^2 + 1)^2 = 0 \;\Rightarrow\; D = \pm i, \pm i$

$\therefore \quad$ C.F. $= (c_1 x + c_2) \cos x + (c_3 x + c_4) \sin x$

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 (\cos x + i \sin x)$$

$$= \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix} = \text{R.P. of } e^{ix}\frac{1}{[(D+i)^2+1]^2} x^2$$

$$= \text{R.P. of } e^{ix}\frac{1}{(D^2+2iD)^2} x^2 = \text{R.P. of } e^{ix}\frac{1}{\left[2iD\left(1+\frac{D}{2i}\right)\right]^2} x^2$$

$$= \text{R.P. of } e^{ix}\frac{1}{-4D^2\left(1-\frac{iD}{2}\right)^2} x^2 = \text{R.P. of } \frac{e^{ix}}{-4}\cdot\frac{1}{D^2}\left(1-\frac{iD}{2}\right)^{-2} x^2$$

$$= \text{R.P. of } -\frac{1}{4} e^{ix}\cdot\frac{1}{D^2}\left[1+2\left(\frac{iD}{2}\right)+3\left(\frac{iD}{2}\right)^2+\ldots\ldots\right] x^2$$

$$= \text{R.P. of } -\frac{1}{4} e^{ix}\cdot\frac{1}{D^2}\left[x^2+i(2x)-\frac{3}{4}(2)\right]$$

$$= \text{R.P. of } -\frac{1}{4} e^{ix}\cdot\frac{1}{D}\left[\frac{x^3}{3}+ix^2-\frac{3}{2}x\right]$$

$$= \text{R.P. of } -\frac{1}{4} e^{ix} \left[\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3x^2}{4} \right]$$

$$= \text{R.P. of } -\frac{1}{48} (\cos x + i \sin x) \left[(x^4 - 9x^2) + (4x^3)i \right]$$

$$= -\frac{1}{48} \left[(x^4 - 9x^2) \cos x - 4x^3 \sin x \right]$$

Hence the C.S. is $y = (c_1 x + c_2) \cos x + (c_3 x + c_4) \sin x - \frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x]$.

Example 17. *Solve* $\frac{d^2y}{dx^2} + y = \text{cosec } x$.

Sol. Given equation in symbolic form is $(D^2 + 1)y = \text{cosec } x$

A.E. is $\quad D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore \quad$ C.F. $= c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2 + 1} \text{cosec } x = \frac{1}{(D + i)(D - i)} \text{cosec } x$$

$$= \frac{1}{2i} \left(\frac{1}{D - i} - \frac{1}{D + i} \right) \text{cosec } x \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2i} \left(\frac{1}{D - i} \text{cosec } x - \frac{1}{D + i} \text{cosec } x \right)$$

Now $\quad \frac{1}{D - i} \text{cosec } x = e^{ix} \int \text{cosec } x \, e^{-ix} \, dx \qquad \left[\because \frac{1}{D - a} X = e^{ax} \int X e^{-ax} \, dx \right]$

$$= e^{ix} \int \text{cosec } x (\cos x - i \sin x) \, dx = e^{ix} \int (\cot x - i) \, dx$$

$$= e^{ix} (\log \sin x - ix)$$

Changing i to $-i$, we have $\frac{1}{D + i} \text{cosec } x = e^{-ix} (\log \sin x + ix)$

$\therefore$
$$\text{P.I.} = \frac{1}{2i} \left[e^{ix} (\log \sin x - ix) - e^{-ix} (\log \sin x + ix) \right]$$

$$= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i} \right) - x \left(\frac{e^{ix} + e^{-ix}}{2} \right)$$

$$= \log \sin x \, . \, \sin x - x \cos x$$

Hence the C.S. is $\quad y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x$.

Example 18. *Solve* $\frac{d^2y}{dx^2} + a^2 y = \tan ax$.

Sol. Given equation in symbolic form is $(D^2 + a^2)y = \tan ax$

A.E. is $\quad D^2 + a^2 = 0 \Rightarrow D = \pm ia$

$\therefore \quad$ C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \tan ax = \frac{1}{(D + ia)(D - ia)} \tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \tan ax \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} \tan ax - \frac{1}{D + ia} \tan ax\right]$$

Now $\frac{1}{D - ia} \tan ax = e^{iax} \int \tan ax \,.\, e^{-iax}\, dx$

$$= e^{iax} \int \tan ax (\cos ax - i \sin ax)\, dx = e^{iax} \int \left(\sin ax - i \frac{\sin^2 ax}{\cos ax}\right) dx$$

$$= e^{iax} \int \left(\sin ax - i \frac{1 - \cos^2 ax}{\cos ax}\right) dx = e^{iax} \int [\sin ax - i(\sec ax - \cos ax)]\, dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a} \log(\sec ax + \tan ax) + i\frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a} e^{iax} [(\cos ax - i \sin ax) + i \log(\sec ax + \tan ax)]$$

$$= -\frac{1}{a} e^{iax} [e^{-iax} + i \log(\sec ax + \tan ax)] = -\frac{1}{a} [1 + ie^{iax} \log(\sec ax + \tan ax)]$$

Changing i to $-i$, we have $\frac{1}{D + ia} \tan ax = -\frac{1}{a} [1 - ie^{-iax} \log(\sec ax + \tan ax)]$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[-\frac{1}{a}\left\{1 + ie^{iax} \log(\sec ax + \tan ax) + \frac{1}{a}\{1 - ie^{-iax} \log(\sec ax + \tan ax)\right\}\right]$$

$$= -\frac{1}{a^2} \log(\sec ax + \tan ax)\left(\frac{e^{iax} + e^{-iax}}{2}\right) = -\frac{1}{a^2} \log(\sec ax + \tan ax) \,.\, \cos ax$$

Hence the C.S. is $y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log(\sec ax + \tan ax)$.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\frac{d^3y}{dx^3} + y = 3 + 5e^x$.
2. $\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2$.
3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2 \cosh x$.
4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = \sin 3x$.
5. (*i*) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$. (*ii*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos 2x$
6. (*i*) $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \frac{x}{2}$ (*ii*) $(D^3 + 1)y = 2 \cos^2 x$

(iii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$

(iv) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$

(v) $(D^3 - D)\,z = 2y + 1 + 4\cos y + 2e^y$, where $D \equiv \frac{d}{dy}$

(vi) $(D^2 + D + 1)\,y = (1 + \sin x)^2$

7. $(D^2 - 4D + 3)y = \sin 3x \cos 2x.$

8. $(D^2 - 3D + 2)y = 6e^{-3x} + \sin 2x.$

9. $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x.$

10. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = e^{2x} + \sin 2x.$

11. $\frac{d^2y}{dx^2} - 4y = x^2 + 2x.$

12. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2.$

13. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4.$

14. $\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3.$

15. $(D^2 - 3D + 2)y = 2e^x \cos \frac{x}{2}.$

16. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x.$

17. $\frac{d^4y}{dx^4} - y = e^x \cos x.$

18. (i) $(D^2 - 2D)y = e^x \sin x.$

(ii) $y'' - 2y' + 2y = x + e^x \cos x$

19. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x.$

20. (i) $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x \cos 2x.$

(ii) $(D^2 + 4D + 3)y = e^{-x} \sin x + x\,e^{3x}.$

21. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x.$

22. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x.$

23. $(D - 1)^2(D + 1)^2y = \sin^2 \frac{x}{2} + e^x + x.$

24. $\frac{d^2y}{dx^2} + 4y = x \sin x.$

25. $(D^2 - 1)y = x^2 \sin x.$

26. $\frac{d^2y}{dx^2} - 9y = x \cos 2x.$

27. $(D^2 - 1)y = x \sin x + (1 + x^2)e^x.$

28. $(D^2 - 1)y = x \sin 3x + \cos x.$

29. $\frac{d^2y}{dx^2} + a^2y = \sec ax.$

30. $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x.$

31. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}.$

32. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37 \sin 3x = 0$ and find the value of y when $x = \frac{\pi}{2}$ being given tha

$y = 3,\ \frac{dy}{dx} = 0$ when $x = 0.$

Answers

1. $y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2}e^x$

2. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}$

3. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$

4. $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{26}(3 \cos 3x - 2 \sin 3x)$

5. (*i*) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \dfrac{1}{15}(2\cos 2x - \sin 2x)$

(*ii*) $y = c_1 + c_2 e^{-x} + \dfrac{1}{10}(\sin 2x - 2\cos 2x)$

6. (*i*) $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \dfrac{\sqrt{3}}{2}x + c_3 \sin \dfrac{\sqrt{3}}{2}x\right) + \dfrac{1}{730}(\sin 3x + 27\cos 3x) - \dfrac{1}{2} - \dfrac{1}{4}(\cos x - \sin x)$

(*ii*) $y = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos \dfrac{\sqrt{3}}{2}x + c_3 \sin \dfrac{\sqrt{3}}{2}x\right) + 1 + \dfrac{1}{65}(\cos 2x - 8\sin 2x)$

(*iii*) $y = (c_1 + c_2 x)e^{-x} + \dfrac{1}{9}e^{2x} - \dfrac{1}{2} + \dfrac{1}{50}(3\cos 2x - 4\sin 2x)$

(*iv*) $y = c_1 + (c_2 + c_3 x)e^{-x} - \dfrac{x^2}{2}e^{-x} + \dfrac{1}{50}(3\cos 2x - 4\sin 2x)$

(*v*) $z = c_1 + c_2 e^{y} + c_3 e^{-y} - y^2 - y - 2\sin y + ye^{y}$

(*vi*) $y = e^{-\frac{x}{2}}\left(c_1 \cos \dfrac{\sqrt{3}}{2}x + c_2 \sin \dfrac{\sqrt{3}}{2}x\right) + \dfrac{3}{2} - 2\cos x - \dfrac{1}{13}\sin 2x + \dfrac{3}{26}\cos 2x$

7. $y = c_1 e^{x} + c_2 e^{3x} + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$

8. $y = c_1 e^{x} + c_2 e^{2x} + \dfrac{3}{10}e^{-3x} + \dfrac{1}{20}(3\cos 2x - \sin 2x)$

9. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{5}e^{x} - \dfrac{x}{4}\cos 2x$

10. $y = c_1 + e^{x}(c_2 \cos \sqrt{3}\,x + c_3 \sin \sqrt{3}\,x) + \dfrac{1}{8}(e^{2x} + \sin 2x)$

11. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4}\left(x^2 + 2x + \dfrac{1}{2}\right)$

12. $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \dfrac{1}{18}\left(x^3 - \dfrac{x^2}{2} + \dfrac{25}{6}x\right)$

13. $y = c_1 + c_2 e^{-x} + \dfrac{x^3}{3} + 4x$

14. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{5}e^{2x} + \dfrac{1}{5}\cosh 2x + x^3 - 6x$

15. $y = c_1 e^{x} + c_2 e^{2x} - \dfrac{8}{5}e^{x}\left(2\sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right)$

16. $y = c_1 e^{x} + c_2 e^{2x} + \dfrac{1}{4}e^{3x}(2x - 3) + \dfrac{1}{20}(3\cos 2x - \sin 2x)$

17. $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \dfrac{1}{5}e^{x}\cos x$

18. (*i*) $y = c_1 + c_2 e^{2x} - \dfrac{1}{2}e^{x}\sin x$ (*ii*) $y = e^{x}(c_1 \cos x + c_2 \sin x) + \dfrac{1}{2}(x + 1 + xe^{x}\sin x)$

19. $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \dfrac{1}{2}e^{-2x}(3x\sin 2x + \cos 4x)$

20. (*i*) $y = c_1 \cos \sqrt{2}\,x + c_2 \sin \sqrt{2}\,x + \dfrac{e^{3x}}{11}\left(x^2 - \dfrac{12}{11}x + \dfrac{50}{121}\right) + \dfrac{e^{x}}{17}(4\sin 2x - \cos 2x)$

(*ii*) $y = c_1 e^{-x} + c_2 e^{-3x} - \dfrac{1}{5}e^{-x}(\sin x + 2\cos x) + \dfrac{1}{24}e^{3x}\left(x - \dfrac{5}{12}\right)$

21. $y = c_1 + (c_2 + c_3 x)e^{-x} + \dfrac{1}{108}e^{2x}(6x^2 - 14x + 11) + \dfrac{1}{2}x + \dfrac{1}{100}(3\sin 2x + 4\cos 2x)$

22. $y = (c_1 + c_2 x)e^{2x} - e^{2x}[4x \cos 2x + (2x^2 - 3) \sin 2x]$

23. $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)e^{-x} + \dfrac{1}{2} - \dfrac{1}{8} \cos x + \dfrac{x^2}{8} e^{x} + x$

24. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{9} (3x \sin x - 2 \cos x)$

25. $y = c_1 e^{x} + c_2 e^{-x} - x \cos x - \dfrac{1}{2} (x^2 - 1) \sin x$

26. $y = c_1 e^{3x} + c_2 e^{-3x} - \dfrac{1}{169} (13x \cos 2x - 4 \sin 2x)$

27. $y = c_1 e^{x} + c_2 e^{-x} - \dfrac{1}{2} (x \sin x + \cos x) + \dfrac{1}{12} x e^{x} (2x^2 - 3x + 9)$

28. $y = c_1 e^{x} + c_2 e^{-x} - \dfrac{1}{10} \left(x \sin 3x + \dfrac{3}{5} \cos 3x \right) - \dfrac{1}{2} \cos x$

29. $y = c_1 \cos ax + c_2 \sin ax + \dfrac{1}{a} \left(x \sin ax + \cos ax \dfrac{\log \cos ax}{a} \right)$

30. $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log (\sec 2x + \tan 2x)$

31. $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^{x}}$

32. $y = e^{-x} (c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x \,;\, y = 1.$