Exercise Question No. (12)

(12) If $u = e^{xyz}$

Prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

Soln :- $u = e^{xyz}$

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot [xy]$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}[x] + [xy]\, e^{xyz}[xz]$$

$$= e^{xyz}[x + x^2yz]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}[1 + 2xyz]$$

$$+ e^{xyz}[yz][x + x^2yz]$$

$$= e^{xyz}(1 + 2xyz + xyz + x^2y^2z^2)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

(19) if $V = r^m$ where $r^2 = x^2 + y^2 + z^2$

Show that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$

Soln :- $V = r^m$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial x} = m r^{m-1} \cdot \frac{\partial r}{\partial x} \quad \text{—(1)}$$

Given $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow \cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{—(2)}$$

$$\Rightarrow \frac{\partial V}{\partial x} = m r^{m-1} \cdot \frac{x}{r} = m x \cdot r^{m-2} \quad \left[\text{From (1) and (2)}\right]$$

$$\frac{\partial^2 V}{\partial x^2} = m\left[x \cdot (m-2) \cdot r^{m-3} \cdot \frac{\partial r}{\partial x} + r^{m-2}(1)\right]$$

$$= m\left[x(m-2) \cdot r^{m-3} \frac{x}{r} + r^{m-2}\right]$$

$$= m\left[x^2(m-2) r^{m-4} + r^{m-2}\right]$$

$$= m r^{m-2}\left[x^2(m-2) r^{-2} + 1\right]$$

$$= m r^{m-2}\left[\frac{x^2}{r^2}(m-2) + 1\right]$$

~~∂V~~ Similarly $\frac{\partial^2 V}{\partial y^2} = m \cdot r^{m-2}\left[\frac{y^2}{r^2}(m-2) + 1\right]$

$$\frac{\partial^2 V}{\partial z^2} = m r^{m-2}\left[\frac{z^2}{r^2}(m-2) + 1\right]$$

Consider $V_{xx} + V_{yy} + V_{zz}$

$$= m r^{m-2}\left[\frac{x^2+y^2+z^2}{r^2}(m-2) + 3\right]$$

$$= m r^{m-2}\left[\frac{r^2}{r^2}(m-2)+3\right]$$

$[\because r^2 = x^2+y^2+z^2]$

$$= m r^{m-2}\left[m+1\right] = m(m+1)r^{m-2}$$

34 (a) If $u = f(r)$ where $r^2 = x^2+y^2$

Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Solu:- $u = f(r)$ and $r^2 = x^2+y^2$

$$\frac{\partial u}{\partial x} = \cancel{\frac{\partial u}{\partial r}}\, f'(r)\cdot\frac{\partial r}{\partial x} \cancel{= 2x}$$

$$r^2 = x^2+y^2 \Rightarrow 2r\cdot\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\Rightarrow \frac{\partial u}{\partial x} = f'(r)\cdot\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\cdot\frac{\partial r}{\partial x}\left(\frac{x}{r}\right) + f'(r)\left(\frac{-x}{r^2}\right)\cdot\frac{\partial r}{\partial x} + \frac{f'(r)}{r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\left(\frac{x}{r^2}\right)\left(\frac{x}{r}\right) + \frac{f'(r)}{r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r) + \frac{f'(r)}{r}$$

similarly $\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f''(r) - \frac{y^2}{r^3} f'(r) + \frac{f'(r)}{r}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2}{r^2} f''(r) \frac{-(x^2+y^2)}{r^3} f'(r) + \frac{2 f'(r)}{r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{r^2}{r^2} f''(r) - \frac{r^2}{r^3} f'(r) + \frac{2}{r} f'(r)$$

$$= f''(r) - \frac{1}{r} f'(r) + \frac{2}{r} f'(r)$$

$$= f''(r) + \frac{1}{r} f'(r)$$

27) If $u = (1-2xy+y^2)^{-1/2}$

Prove that $x \cdot \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = y^2 u^3$

Soln:- Given $u = (1-2xy+y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = \left(\frac{-1}{2}\right)(1-2xy+y^2)^{-3/2}(-2y)$$

$$= \frac{y}{(1-2xy+y^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = \left(\frac{-1}{2}\right)(1-2xy+y^2)^{-3/2}(-2x+2y)$$

$$= \frac{x-y}{(1-2xy+y^2)^{3/2}}$$

Consider $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$

$$= x\left(\frac{y}{(1-2xy+y^2)^{3/2}}\right) - y\left(\frac{x-y}{(1-2xy+y^2)^{3/2}}\right)$$

$$= \frac{\cancel{xy}-\cancel{xy}+y^2}{(1-2xy+y^2)^{3/2}} = \frac{y^2}{(1-2xy+y^2)^{3/2}}$$

$$= y^2u^3 \quad \left[\because u=(1-2xy+y^2)^{-1/2}\right]$$

8) Example :- If $x^x y^y z^z = c$

Show that at $x=y=z$; $\frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$

Soln :- say z is a fun. of x and y.

Now Given - $x^x y^y z^z = c$

Apply log on both sides

$$x\log x + y\log y + z\log z = \log c$$

$\Rightarrow$ Firstly diff. partially w.r.t. x.

$$x\cdot\left(\frac{1}{x}\right) + \log x(1) + 0 + \left(z\cdot\frac{1}{z}+\log z\cdot 1\right)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)(1+\log z) = -1-\log x$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}$$

similarly $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{1+\log z}$

[$\because$ z is symmetric fun. of x and y]

Now $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-(1+\log y)}{1+\log z}\right)$

$= \frac{-(1+\log y)(-1)}{(1+\log z)^2}\left(\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right)$

$= \frac{1+\log y}{(1+\log z)^2}\cdot\frac{1}{z}\left(\frac{-(1+\log x)}{1+\log z}\right)$

$= \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}$

at $x=y=z$;

$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log x)^2}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)}$

$= \frac{-1}{x(\log e+\log x)} = -(x\log ex)^{-1}$

(10) Example:- If $x = r\cos\theta$, $y = r\sin\theta$;
prove that

(i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ (ii) $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

(iii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$