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FUNCTIONS OF TWO OR MORE VARIABLES (LIMIT AND CONTINUITY)

1.1 INTRODUCTION

Multivariable functions or functions of more than one independent variable play a very important role in the study of statistics, fluid dynamics, electricity etc. In higher studies, they occur more frequently than functions of a single variable. Their derivatives and integrals have a vast variety of applications in engineering.

1.2 FUNCTION OF TWO VARIABLES

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called a function of two variables x and y and this is denoted by $z = f(x, y)$.

z is called the dependent variable while x and y are called independent variables.

For example, the area of a triangle is determined when its base and altitude are known. Thus, area of a triangle is a function of two variables, base and altitude.

In a similar way, a function of more than two variables can be defined.

Domain of a function of two variables is a subset of $R^2 = R \times R = \{(x, y) : x, y \in R\}$ and range is a subset of R. Thus a function f of two variables is denoted as

$$f : S \to R \quad \text{where } S \subset R^2.$$

Similarly, a function f of three variables is denoted as

$$f : S \to R \quad \text{where } S \subset R^3.$$

Geometrically, Let $z = f(x, y)$ be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy-plane. Then to each point (x, y) of this area corresponds a value of z given by the relation $z = f(x, y)$. Representing all these values (x, y, z) by points in space, we get a surface.

Hence the function $z = f(x, y)$ represents a surface.

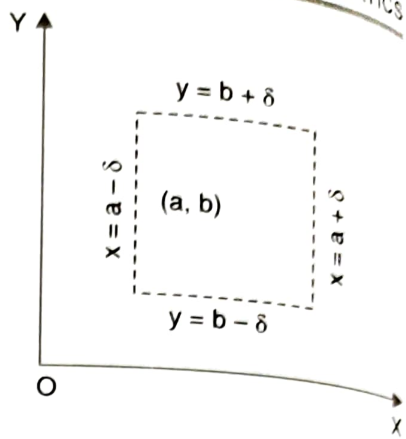
1.3 NEIGHBOURHOOD OF A POINT (a, b)

Every point (a, b) in R^2 has two types of neighbourhoods:

***(i)* Square Neighbourhood**

The interior of the square with centre at (a, b), sides parallel to the coordinate axes and each side $= 2\delta$ is called a square neighbourhood of the point (a, b). For every positive value of δ, we get a square neighbourhood of (a, b).

Thus a square neighbourhood of (a, b) is
$\{(x, y) : a - \delta < x < a + \delta, b - \delta < y < b + \delta\}$
$= \{(x, y) : |x - a| < \delta, |y - b| < \delta\}$

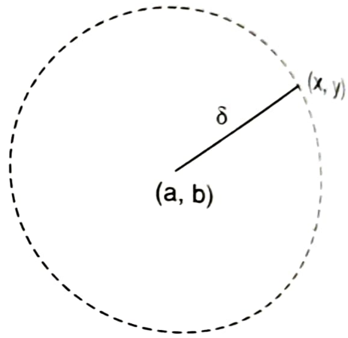


Similarly a neighbourhood of (a, b, c) in the form of a cube is

$$\{(x, y, z) : a - \delta < x < a + \delta, b - \delta < y < b + \delta, c - \delta < z < c + \delta\}$$
$$= \{(x, y, z) : |x - a| < \delta, |y - b| < \delta, |z - c| < \delta\}$$

(ii) **Circular Neighbourhood**

The interior of the circle with centre at (a, b) and radius δ is called a circular neighbourhood of the point (a, b). For every positive value of δ, we get a circular neighbouhood of (a, b).



Thus a circular neighbourhood of (a, b) is $\{(x, y) : |(x, y) - (a, b)| < \delta\}$ where $|(x, y) - (a, b)|$ stands for the distance between the points (x, y) and (a, b)

i.e.,
$$|(x, y) - (a, b)| = \sqrt{(x - a)^2 + (y - b)^2}$$

Similarly, a spherical neighbourhood of (a, b, c) is

$$\{(x, y, z) : |(x, y, z) - (a, b, c)| < \delta\}$$

where $|(x, y, z) - (a, b, c)| = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

1.4 LIMIT OF A FUNCTION OF TWO VARIABLES

A function $f(x, y)$ is said approach to a limit l as the point (x, y) approaches the point (a, b) if corresponding to any pre-assigned positive number ε, however small, we can find a positive number δ (depending on ε) such that

Def. 1. $|f(x, y) - l| < \varepsilon$

for all points (x, y) *other than* (a, b) for which $|x - a| < \delta$ and $|y - b| < \delta$

i.e. for all points (x, y) for which $0 < |x - a| < \delta$ and $0 < |y - b| < \delta$

This definition of limit is based on square neighbourhood of a point.

Def. 2. $|f(x, y) - l| < \varepsilon$

for all points (x, y) *other than* (a, b) for which $|(x, y) - (a, b)| < \delta$

i.e. for all points (x, y) for which $0 < |(x, y) - (a, b)| < \delta$ or $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$

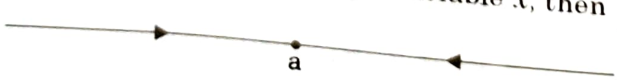
This definition of limit is based on circular neighbourhood of a point.

Note 1. A function $f(x, y)$ tends to a limit l as the point (x, y) tends to point (a, b) is symbolically written as

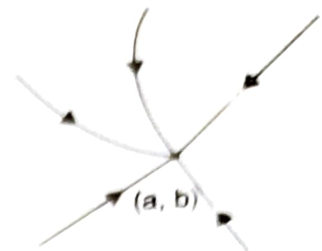
$$\underset{(x, y) \to (a, b)}{\text{Lt}} f(x, y) = l$$

Note 2. $\underset{(x, y) \to (a, b)}{\text{Lt}} f(x, y)$ if it exists, is unique

Note 3. We know that if f is a function of single variable x, then $\underset{x \to a}{\text{Lt}} f(x)$ exists iff



$\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a'}{\text{Lt}} f(x)$, *i.e.*, the limit is independent of the path along which x approaches a.



Similarly, if f is a function of two variables x and y, then $\underset{(x, y) \to (a, b)}{\text{Lt}} f(x, y)$ exists iff this limit is independent of the path along which (x, y) approaches (a, b).

Note 4. If a function $f(x, y)$ approaches two different numbers l_1 and l_2 as (x, y) approaches (a, b) along two different paths, then $\lim\limits_{(x, y) \to (a, b)} f(x, y)$ does not exist. This may be taken as a test for non-existence of limit.

Note 5. If $\lim\limits_{(x, y) \to (a, b)} f(x, y) = l$ and $\lim\limits_{(x, y) \to (a, b)} g(x, y) = m$, then

(*i*) $\lim\limits_{(x, y) \to (a, b)} [f(x, y) + g(x, y)] = l + m$

(*ii*) $\lim\limits_{(x, y) \to (a, b)} [f(x, y) - g(x, y)] = l - m$

(*iii*) $\lim\limits_{(x, y) \to (a, b)} f(x, y) \cdot g(x, y) = lm$

(*iv*) $\lim\limits_{(x, y) \to (a, b)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{l}{m}$, provided $m \neq 0$

(*v*) $\lim\limits_{(x, y) \to (a, b)} kf(x, y) = kl$

(*vi*) $\lim\limits_{(x, y) \to (a, b)} [f(x, y)]^{p/q} = l^{p/q}$ where p, q are integers.

Note 6. In functions of a single variable, $\lim\limits_{x \to a} f(x) = f(a)$, obtained by replacing x by a, provided $f(a) \in \mathbb{R}$. Similarly, in functions of two or more variables

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b) \text{ provided } f(a, b) \in \mathbb{R}$$

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c) \text{ provided } f(a, b, c) \in \mathbb{R}.$$

1.5 CONTINUITY OF A FUNCTION OF TWO VARIABLES

A function $f(x, y)$ is said to be continuous at the point (a, b) if

$$\underset{(x, y) \to (a, b)}{\text{Lt}} f(x, y) = f(a, b)$$

i.e. if given $\varepsilon > 0$, there exists a positive real number δ (depending on ε) such that

$$|f(x, y) - f(a, b)| < \varepsilon \text{ for } |(x, y) - (a, b)| < \delta$$

Thus, a function $f(x, y)$ is continuous at the point (a, b) if

(*i*) f is defined at (a, b)

(*ii*) $\lim\limits_{(x, y) \to (a, b)} f(x, y)$ exists

(*iii*) $\lim\limits_{(x, y) \to (a, b)} f(x, y) = f(a, b)$

Also f is continuous if f is continuous at every point of its domain.

1.6 CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function $f(x, y, z)$ is said to be continuous at the point (a, b, c) if

$$\underset{(x, y, z) \to (a, b, c)}{\text{Lt}} f(x, y, z) = f(a, b, c).$$

i.e. if given $\varepsilon > 0$, there exists a positive real number δ (depending on ε) such that

$$|f(x, y, z) - f(a, b, c)| < \varepsilon \text{ for } |(x, y, z) - (a, b, c)| < \delta.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits:*

(*i*) $\underset{(x, y) \to (1, 2)}{Lt} \dfrac{x^2 + xy - y^2}{x^3y^2 - 2xy^3 + y^4}$ (*ii*) $\underset{(x, y) \to (0, 0)}{Lt} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

(*iii*) $\underset{(x, y, z) \to (3, 2, 1)}{Lt} \left(\dfrac{1}{x} + \dfrac{2}{y} + \dfrac{3}{z}\right)$

Sol. (*i*) $\underset{(x, y) \to (1, 2)}{\lim} \dfrac{x^2 + xy - y^2}{x^3y^2 - 2xy^3 + y^4} = \dfrac{(1)^2 + (1)(2) - (2)^2}{(1)^3(2)^2 - 2(1)(2)^3 + (2)^4} = \dfrac{-1}{4}$

(*ii*) $\underset{(x, y) \to (0, 0)}{\lim} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ | Form $\dfrac{0}{0}$

$$= \underset{(x, y) \to (0, 0)}{\text{Lt}} \frac{x(x - y)}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

$$= \underset{(x, y) \to (0, 0)}{\text{Lt}} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} = \underset{(x, y) \to (0, 0)}{\text{Lt}} x(\sqrt{x} + \sqrt{y})$$

$$= 0(\sqrt{0} + \sqrt{0}) = 0$$

(*iii*) $\underset{(x, y, z) \to (3, 2, 1)}{\text{Lt}} \left(\dfrac{1}{x} + \dfrac{2}{y} + \dfrac{3}{z}\right) = \dfrac{1}{3} + \dfrac{2}{2} + \dfrac{3}{1} = \dfrac{13}{3}.$

Example 2. *Show that the function f given by*

$$f(x, y) = \begin{cases} xy^2 + x^2y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{cases}$$

is not continuous at (1, 2).

Sol. $\underset{(x, y) \to (1, 2)}{\text{Lt}} f(x, y) = \underset{(x, y) \to (1, 2)}{\text{Lt}} (xy^2 + x^2y)$

$$= (1)(2)^2 + (1)^2(2) = 6$$

But $f(1, 2) = 0$ (given)

$\Rightarrow$ $\underset{(x, y) \to (1, 2)}{\text{Lt}} f(x, y) \neq f(1, 2)$

$\Rightarrow$ f is not continuous at (1, 2).

TEST YOUR KNOWLEDGE

1. Evaluate the following limits:

(i) $\underset{(x,y)\to(1,2)}{\text{Lt}} \left(\dfrac{2x^2 y}{x^2 + y^2 + 1} \right)$

(ii) $\underset{(x,y)\to(1,-2)}{\text{Lt}} \left(\dfrac{1}{x} - \dfrac{1}{y} \right)^3$

(iii) $\underset{(x,y)\to(4,4)}{\text{Lt}} (\sin^2 x + \cos^2 y)$

(iv) $\underset{\substack{(x,y)\to(1,1) \\ x \neq 1}}{\text{Lt}} \dfrac{xy - y - 3x + 3}{x - 1}$

(*v*) $\underset{(x,y)\to(2,1)}{\text{Lt}} \sin\sqrt{|xy|-2}$

(*vi*) $\underset{(x,y,z)\to\left(\frac{\pi}{2},\frac{1}{3},\frac{3}{2}\right)}{\text{Lt}} \tan^{-1}(xyz)$

(*vii*) $\underset{(x,y,z)\to(1,-1,0)}{\text{Lt}} \dfrac{e^{x+y}}{y^2+\cos\sqrt{xz}}$

2. Show that $f(x, y)$ is discontinuous at (2, 3) where

$$f(x,y)=\begin{cases}3xy, & \text{if } (x,y)\neq(2,3)\\ 6, & \text{if } (x,y)=(2,3)\end{cases}$$

Can f be suitably redefined to make it continuous at (2, 3)?

3. Prove that the function $f : A \to R$, $(A \subset R^2)$, defined by $f(x,y)=\begin{cases}1, & \text{if } y\neq 0\\ 0, & \text{if } y=0\end{cases}$ is not continuous at (a, b) if $b = 0$.

4. Show that the following functions are continuous at the origin:

(*i*) $f(x, y) = x^2 + y^2$

(*ii*) $f(x, y, z) = x^2 + y^2 + z^2$

(*iii*) $f(x,y)=\begin{cases}x\sin\dfrac{1}{y}, & y\neq 0\\ 0, & y=0\end{cases}$

(*iv*) $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(*v*) $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ [**Hint.** Put $x =$ cos θ and $y = r \sin\theta$]

5. Show that the following functions are discontinuous at the origin:

(*i*) $f(x,y)=\begin{cases}\dfrac{2x^2y^2}{x^4+y^4}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(*ii*) $f(x,y)=\begin{cases}\dfrac{x-y}{x+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(*iii*) $f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^6}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(*iv*) $f(x,y)=\begin{cases}\dfrac{x^4-y^2}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(*v*) $f(x,y)=\begin{cases}\dfrac{y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

6. Using ε-δ approach, show that $\underset{(x,y)\to(2,1)}{\text{Lt}} (3x + 4y) = 10.$

7. Show that the function $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Answers

1. (*i*) $\dfrac{2}{3}$ (*ii*) $\dfrac{27}{8}$ (*iii*) 1 (*iv*) −2 (*v*) 0

(*vi*) 1 (*vii*) $\dfrac{1}{2}$

3. Yes, $f(x, y) = 18$ when $(x, y) = (2, 3)$.

□□□