Travelling Salesman Problem-

You are given-

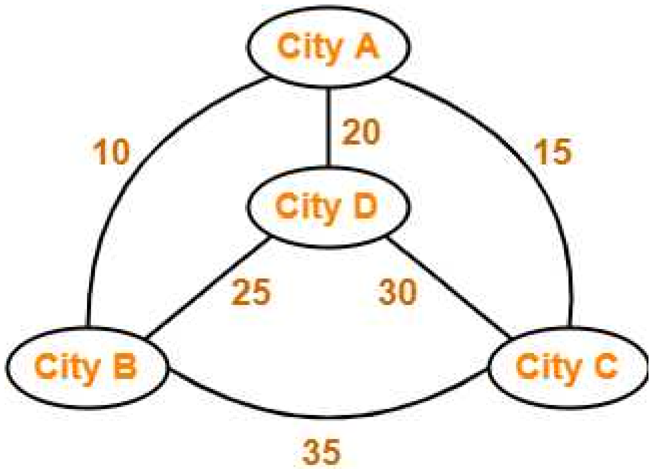
- A set of some cities
- Distance between every pair of cities

Travelling Salesman Problem states-

- A salesman has to visit every city exactly once.
- He has to come back to the city from where he starts his journey.
- What is the shortest possible route that the salesman must follow to complete his tour?

Example-

The following graph shows a set of cities and distance between every pair of cities-



Travelling Salesman Problem

If salesman starting city is A, then a TSP tour in the graph is-

A → B → D → C → A

Cost of the tour

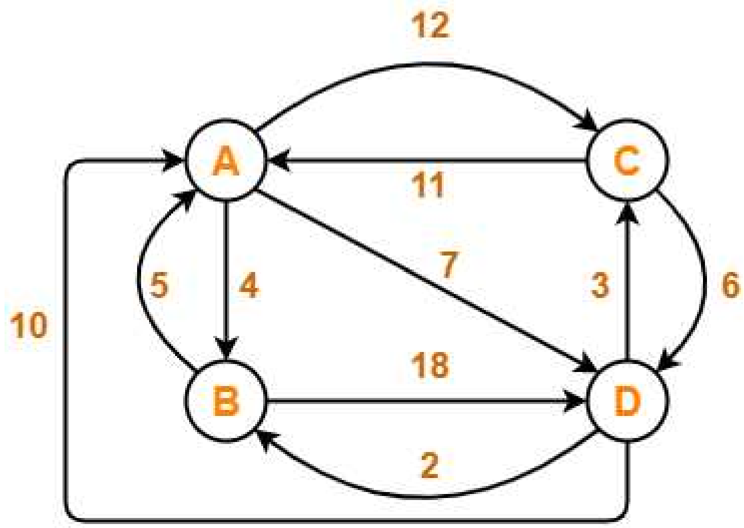
= 10 + 25 + 30 + 15

= **80 units**

PRACTICE PROBLEM BASED ON TRAVELLING SALESMAN PROBLEM USING BRANCH AND BOUND APPROACH-

Problem-

Solve Travelling Salesman Problem using Branch and Bound Algorithm in the following graph-



Solution-

Step-01:

Write the initial cost matrix and reduce it-

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} &
\begin{bmatrix}
\infty & 4 & 12 & 7 \\
5 & \infty & \infty & 18 \\
11 & \infty & \infty & 6 \\
10 & 2 & 3 & \infty
\end{bmatrix}
\end{array}
$$

Rules

- To reduce a matrix, perform the row reduction and column reduction of the matrix separately.
- A row or a column is said to be reduced if it contains at least one entry ‘0’ in it.

Row Reduction-

Consider the rows of above matrix one by one.

If the row already contains an entry ‘0’, then-

- There is no need to reduce that row.

If the row does not contains an entry ‘0’, then-

- Reduce that particular row.
- Select the least value element from that row.
- Subtract that element from each element of that row.
- This will create an entry ‘0’ in that row, thus reducing that row.

Following this, we have-

- Reduce the elements of row-1 by 4.
- Reduce the elements of row-2 by 5.
- Reduce the elements of row-3 by 6.
- Reduce the elements of row-4 by 2.

Performing this, we obtain the following row-reduced matrix-

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & 0 & 8 & 3 \\ 0 & \infty & \infty & 13 \\ 5 & \infty & \infty & 0 \\ 8 & 0 & 1 & \infty \end{bmatrix}
\end{array}
$$

Column Reduction-

Consider the columns of above row-reduced matrix one by one.

If the column already contains an entry ‘0’, then-

- There is no need to reduce that column.

If the column does not contains an entry ‘0’, then-

- Reduce that particular column.
- Select the least value element from that column.
- Subtract that element from each element of that column.
- This will create an entry ‘0’ in that column, thus reducing that column.

Following this, we have-

- There is no need to reduce column-1.
- There is no need to reduce column-2.
- Reduce the elements of column-3 by 1.
- There is no need to reduce column-4.

Performing this, we obtain the following column-reduced matrix-

$$
\begin{array}{c c}
 & \begin{matrix} A & B & C & D \end{matrix} \\
\begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} \infty & 0 & 7 & 3 \\ 0 & \infty & \infty & 13 \\ 5 & \infty & \infty & 0 \\ 8 & 0 & 0 & \infty \end{bmatrix}
\end{array}
$$

Finally, the initial distance matrix is completely reduced.

Now, we calculate the cost of node-1 by adding all the reduction elements.

Cost(1)

= Sum of all reduction elements

= 4 + 5 + 6 + 2 + 1

= 18

Step-02:

- We consider all other vertices one by one.
- We select the best vertex where we can land upon to minimize the tour cost.

Choosing To Go To Vertex-B: Node-2 (Path A → B)

- From the reduced matrix of step-01, M[A,B] = 0
- Set row-A and column-B to ∞
- Set M[B,A] = ∞

Now, resulting cost matrix is-

$$
\begin{array}{cc}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 13 \\ 5 & \infty & \infty & 0 \\ 8 & \infty & 0 & \infty \end{bmatrix}
\end{array}
$$

Now,

- We reduce this matrix.
- Then, we find out the cost of node-02.

Row Reduction-

- We can not reduce row-1 as all its elements are ∞.
- Reduce all the elements of row-2 by 13.
- There is no need to reduce row-3.
- There is no need to reduce row-4.

Performing this, we obtain the following row-reduced matrix-

$$
\begin{array}{cc}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 \\ 5 & \infty & \infty & 0 \\ 8 & \infty & 0 & \infty \end{bmatrix}
\end{array}
$$

Column Reduction-

- Reduce the elements of column-1 by 5.
- We can not reduce column-2 as all its elements are ∞.
- There is no need to reduce column-3.
- There is no need to reduce column-4.

Performing this, we obtain the following column-reduced matrix-

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \left[\begin{array}{cccc} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 \\ 0 & \infty & \infty & 0 \\ 3 & \infty & 0 & \infty \end{array}\right]
\end{array}
$$

Finally, the matrix is completely reduced.

Now, we calculate the cost of node-2.

Cost(2)

= Cost(1) + Sum of reduction elements + M[A,B]

= 18 + (13 + 5) + 0

= 36

Choosing To Go To Vertex-C: Node-3 (Path A → C)

- From the reduced matrix of step-01, M[A,C] = 7
- Set row-A and column-C to ∞
- Set M[C,A] = ∞

Now, resulting cost matrix is-

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \left[\begin{array}{cccc} \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 13 \\ \infty & \infty & \infty & 0 \\ 8 & 0 & \infty & \infty \end{array}\right]
\end{array}
$$

Now,

- We reduce this matrix.
- Then, we find out the cost of node-03.

Row Reduction-

- We can not reduce row-1 as all its elements are ∞.
- There is no need to reduce row-2.
- There is no need to reduce row-3.
- There is no need to reduce row-4.

Thus, the matrix is already row-reduced.

Column Reduction-

- There is no need to reduce column-1.
- There is no need to reduce column-2.
- We can not reduce column-3 as all its elements are ∞.
- There is no need to reduce column-4.

Thus, the matrix is already column reduced.

Finally, the matrix is completely reduced.

Now, we calculate the cost of node-3.

Cost(3)

= Cost(1) + Sum of reduction elements + M[A,C]

= 18 + 0 + 7

= 25

Choosing To Go To Vertex-D: Node-4 (Path A → D)

- From the reduced matrix of step-01, M[A,D] = 3
- Set row-A and column-D to ∞
- Set M[D,A] = ∞

Now, resulting cost matrix is-

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty \\ 5 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \end{bmatrix}
\end{array}
$$

Now,

- We reduce this matrix.
- Then, we find out the cost of node-04.

Row Reduction-

- We can not reduce row-1 as all its elements are ∞.
- There is no need to reduce row-2.
- Reduce all the elements of row-3 by 5.
- There is no need to reduce row-4.

Performing this, we obtain the following row-reduced matrix-

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \end{bmatrix} \end{array}$$

Column Reduction-

- There is no need to reduce column-1.
- There is no need to reduce column-2.
- There is no need to reduce column-3.
- We can not reduce column-4 as all its elements are ∞.

Thus, the matrix is already column-reduced.

Finally, the matrix is completely reduced.

Now, we calculate the cost of node-4.

Cost(4)

= Cost(1) + Sum of reduction elements + M[A,D]

= 18 + 5 + 3

= 26

Thus, we have-

- Cost(2) = 36 (for Path A → B)
- Cost(3) = 25 (for Path A → C)
- Cost(4) = 26 (for Path A → D)

We choose the node with the lowest cost.

Since cost for node-3 is lowest, so we prefer to visit node-3.

Thus, we choose node-3 i.e. path **A → C**.

Step-03:

We explore the vertices B and D from node-3.

We now start from the cost matrix at node-3 which is-

$$
\begin{array}{c}
\\ A \\ B \\ C \\ D
\end{array}
\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array} \\
\begin{bmatrix}
\infty & \infty & \infty & \infty \\
0 & \infty & \infty & 13 \\
\infty & \infty & \infty & 0 \\
8 & 0 & \infty & \infty
\end{bmatrix}
\end{array}
$$

Cost(3) = 25

Choosing To Go To Vertex-B: Node-5 (Path A → C → B)

- From the reduced matrix of step-02, M[C,B] = ∞
- Set row-C and column-B to ∞
- Set M[B,A] = ∞

Now, resulting cost matrix is-

$$\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
A & \infty & \infty & \infty & \infty \\
B & \infty & \infty & \infty & 13 \\
C & \infty & \infty & \infty & \infty \\
D & 8 & \infty & \infty & \infty
\end{array}$$

Now,

- We reduce this matrix.
- Then, we find out the cost of node-5.

Row Reduction-

- We can not reduce row-1 as all its elements are ∞.
- Reduce all the elements of row-2 by 13.
- We can not reduce row-3 as all its elements are ∞.
- Reduce all the elements of row-4 by 8.

Performing this, we obtain the following row-reduced matrix-

$$\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
A & \infty & \infty & \infty & \infty \\
B & \infty & \infty & \infty & 0 \\
C & \infty & \infty & \infty & \infty \\
D & 0 & \infty & \infty & \infty
\end{array}$$

Column Reduction-

- There is no need to reduce column-1.
- We can not reduce column-2 as all its elements are ∞.
- We can not reduce column-3 as all its elements are ∞.
- There is no need to reduce column-4.

Thus, the matrix is already column reduced.

Finally, the matrix is completely reduced.

Now, we calculate the cost of node-5.

Cost(5)

= cost(3) + Sum of reduction elements + M[C,B]

= 25 + (13 + 8) + ∞

= ∞

Choosing To Go To Vertex-D: Node-6 (Path A → C → D)

- From the reduced matrix of step-02, M[C,D] = ∞
- Set row-C and column-D to ∞
- Set M[D,A] = ∞

Now, resulting cost matrix is-

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \end{bmatrix} \end{array}$$

Now,

- We reduce this matrix.
- Then, we find out the cost of node-6.

Row Reduction-

- We can not reduce row-1 as all its elements are ∞.
- There is no need to reduce row-2.
- We can not reduce row-3 as all its elements are ∞.
- We can not reduce row-4 as all its elements are ∞.

Thus, the matrix is already row reduced.

Column Reduction-

- There is no need to reduce column-1.
- We can not reduce column-2 as all its elements are ∞.
- We can not reduce column-3 as all its elements are ∞.
- We can not reduce column-4 as all its elements are ∞.

Thus, the matrix is already column reduced.

Finally, the matrix is completely reduced.

Now, we calculate the cost of node-6.

Cost(6)

= cost(3) + Sum of reduction elements + M[C,D]

= 25 + 0 + 0

= 25

Thus, we have-

- Cost(5) = ∞ (for Path A → C → B)
- Cost(6) = 25 (for Path A → C → D)

We choose the node with the lowest cost.

Since cost for node-6 is lowest, so we prefer to visit node-6.

Thus, we choose node-6 i.e. path **C → D**.

Step-04:

We explore vertex B from node-6.

We start with the cost matrix at node-6 which is-

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} &
\left[\begin{array}{cccc}
\infty & \infty & \infty & \infty \\
0 & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty \\
\infty & 0 & \infty & \infty
\end{array}\right]
\end{array}
$$

Cost(6) = 25

Choosing To Go To Vertex-B: Node-7 (Path A → C → D → B)

- From the reduced matrix of step-03, M[D,B] = 0
- Set row-D and column-B to ∞
- Set M[B,A] = ∞

Now, resulting cost matrix is-

$$
\begin{array}{c|c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}
\end{array}
$$

Now,

- We reduce this matrix.
- Then, we find out the cost of node-7.

Row Reduction-

- We can not reduce row-1 as all its elements are ∞.
- We can not reduce row-2 as all its elements are ∞.
- We can not reduce row-3 as all its elements are ∞.
- We can not reduce row-4 as all its elements are ∞.

Column Reduction-

- We can not reduce column-1 as all its elements are ∞.
- We can not reduce column-2 as all its elements are ∞.
- We can not reduce column-3 as all its elements are ∞.
- We can not reduce column-4 as all its elements are ∞.

Thus, the matrix is already column reduced.

Finally, the matrix is completely reduced.

All the entries have become ∞.

Now, we calculate the cost of node-7.

Cost(7)

= cost(6) + Sum of reduction elements + M[D,B]

= 25 + 0 + 0

= 25

Thus,

- Optimal path is: **A → C → D → B → A**
- Cost of Optimal path = **25 units**