Differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + 1\cdot\frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\,\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{2}\sec^2 u - 1\right)\frac{\partial u}{\partial y} \qquad \ldots(3) \qquad \left[\because \frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}\right]$$

Multiplying (2) by x, (3) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{2}\sec^2 u - 1\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{1}{2\cos^2 u} - 1\right)\cdot\frac{1}{2}\tan u \qquad \text{[Using (1)]}$$

$$= -\frac{2\cos^2 u - 1}{2\cos^2 u}\cdot\frac{1}{2}\frac{\sin u}{\cos u} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$

$$[\because \; 2\cos^2 u - 1 = \cos 2u]$$

Example 7. *If* $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, *show that* $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0.$

Sol. Let $\qquad u = xf\left(\frac{y}{x}\right)$ *and* $v = g\left(\frac{y}{x}\right) = x^0 g\left(\frac{y}{x}\right)$

so that $\qquad z = u + v \qquad \ldots(1)$

Since u is a homogeneous function of degree $n = 1$ in x, y

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 0 \qquad \ldots(2)$$

Since v is a homogeneous function of degree $n = 0$ in x, y

$$\therefore \quad x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0 \qquad \ldots(3)$$

Adding (2) and (3), we have

$$x^2\frac{\partial^2}{\partial x^2}(u+v) + 2xy\frac{\partial^2}{\partial x\partial y}(u+v) + y^2\frac{\partial^2}{\partial y^2}(u+v) = 0$$

or $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0. \qquad \text{[Using (1)]}$$

TEST YOUR KNOWLEDGE

1. Verify Euler's theorem for the functions:

(i) $f(x, y) = ax^2 + 2hxy + by^2$

(ii) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

(iii) $f(x, y) = \dfrac{x^2(x^2 - y^2)^3}{(x^2 + y^2)^3}$

(iv) $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$

(v) $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$.

Homogeneous Function.

① (22) verify Euler's Theorem for function:-

$$u = \log\left(\frac{x^2+y^2}{xy}\right)$$

Solution- Given $u = \log\left(\frac{x^2+y^2}{xy}\right)$

$$\Rightarrow e^u = \frac{x^2+y^2}{xy}$$

Say $V = e^u$ $\Rightarrow V = \frac{x^2+y^2}{xy} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^2(y/x)}$

$$\Rightarrow V = \frac{x^0\left(1+\frac{y^2}{x^2}\right)}{y/x} = x^0\phi(y/x)$$

$\Rightarrow$ V is homogeneous function of degree 0 in x, y.

We have to verify that $x.\frac{\partial V}{\partial x} + y.\frac{\partial V}{\partial y} = 0.V = 0$

$\because V = e^u \Rightarrow \frac{\partial V}{\partial x} = e^u \frac{\partial u}{\partial x}$

Now $u = \log\left(\frac{x^2+y^2}{xy}\right) = \log(x^2+y^2) - \log(xy)$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{y}{xy} = \frac{2x^2y - x^2y - y^3}{xy(x^2+y^2)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{x^2y - y^3}{xy(x^2+y^2)} = \frac{x^2-y^2}{x(x^2+y^2)}$$

Also $\frac{\partial v}{\partial y} = e^{u} \cdot \frac{\partial u}{\partial y}$

$u = \log(x^2+y^2) - \log(xy)$

$\Rightarrow \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{x}{xy} = \frac{2xy^2 - x^3 - xy^2}{xy(x^2+y^2)}$

$\Rightarrow \frac{\partial u}{\partial y} = \frac{xy^2 - x^3}{xy(x^2+y^2)} = \frac{y^2-x^2}{y(x^2+y^2)}$

Now consider $x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y}$

$= x\left(e^{u} \cdot \frac{\partial u}{\partial x}\right) + y\left(e^{u} \cdot \frac{\partial u}{\partial y}\right)$

$= e^{u}\left[x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}\right]$

$= e^{u}\left[x\left(\frac{x^2-y^2}{x(x^2+y^2)}\right) + y\left(\frac{y^2-x^2}{y(x^2+y^2)}\right)\right]$

$= e^{u}\left[\frac{x^2-y^2}{x^2+y^2} + \frac{y^2-x^2}{x^2+y^2}\right]$

$= e^{u}\left[\frac{\cancel{x^2} - \cancel{y^2} + \cancel{y^2} - \cancel{x^2}}{x^2+y^2}\right] = 0$

Hence Proved

(6) If $f(x,y) = \sqrt{y^2-x^2} \cdot \sin^{-1}\frac{x}{y} + \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

Prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - f(x,y) = 0$

Sol^n^:- $f(x,y) = \sqrt{y^2-x^2}\,\sin^{-1}\frac{x}{y} + \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

$$= y\sqrt{1-\frac{x^2}{y^2}}\,\sin^{-1}\frac{x}{y} + \frac{y^2\left(\frac{x^2}{y^2}-1\right)}{y\sqrt{\frac{x^2}{y^2}+1}}$$

$$= y\sqrt{1-\frac{x^2}{y^2}}\,\sin^{-1}\frac{x}{y} + \frac{y\left(\frac{x^2}{y^2}-1\right)}{\sqrt{\frac{x^2}{y^2}+1}}$$

$$= y\left[\sqrt{1-\frac{x^2}{y^2}}\,.\sin^{-1}\frac{x}{y} + \frac{\left(\frac{x^2}{y^2}-1\right)}{\sqrt{\frac{x^2}{y^2}+1}}\right]$$

$$= y^1\phi\left(\frac{x}{y}\right)$$

$\Rightarrow$ f is homogeneous fun. of degree 1 in x and y.

So, by Euler's theorem; $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1.f(x,y)$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - f(x,y) = 0.$$