Exercise of Homogeneous Functions

(8) If $u = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Soln:- $u = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Here u is not a homogeneous function of x and y.

But $\cos^{-1}u = \frac{xy+yz+zx}{x^2+y^2+z^2} = \frac{x^2\left(\frac{y}{x}+\frac{y}{x}\cdot\frac{z}{x}+\frac{z}{x}\right)}{x^2\left(1+\frac{y^2}{x^2}+\frac{z^2}{x^2}\right)}$

$\Rightarrow \cos^{-1}u = x^0 f\left(\frac{y}{x}, \frac{z}{x}\right)$

$\Rightarrow \cos^{-1}u$ is homogeneous function of degree 0 in x, y, z.

So, by Euler's Theorem

$$x\frac{\partial}{\partial x}(\cos^{-1}u) + y\frac{\partial}{\partial y}(\cos^{-1}u) + z\frac{\partial}{\partial z}(\cos^{-1}u) = 0\cdot(\cos^{-1}u)$$

$$\Rightarrow x\left[\frac{-1}{\sqrt{1-u^2}}\right]\cdot\frac{\partial u}{\partial x} + y\cdot\left[\frac{-1}{\sqrt{1-u^2}}\right]\cdot\frac{\partial u}{\partial y} + z\left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{\partial u}{\partial z} = 0.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = 0.$$

(11) (ii) if $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$

show that $\frac{\partial u}{\partial x} = \frac{-y}{x}.\frac{\partial u}{\partial y}$.

Solu :- Given :- $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$

Here u is not homogeneous function of x and y.

But $\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{\sqrt{x}\left(1-\frac{\sqrt{y}}{\sqrt{x}}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$

$\Rightarrow \sin u = x^0 f(y/x)$

$\Rightarrow \sin u$ is homogenous fun. of degree 0 in x and y.

So, by Euler's Theorem

$$\cancel{x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0.(u)}$$

$$x.\frac{\partial}{\partial x}(\sin u) + y.\frac{\partial}{\partial y}(\sin u) = 0.(\sin u)$$

$$\Rightarrow x(\cos u)\cdot\frac{\partial u}{\partial x}+x(\cos u)\cdot\frac{\partial u}{\partial y}=0$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

$$\Rightarrow x\frac{\partial u}{\partial x}=-y\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{-y}{x}\cdot\frac{\partial u}{\partial y}.$$

(iii) Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$

where $\log u=\dfrac{x^3+y^3}{3x+4y}$

Solu :- $\log u=\dfrac{x^3+y^3}{3x+4y}=\dfrac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(3+\frac{4y}{x}\right)}$

$$\Rightarrow \log u=\frac{x^2\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[3+\frac{4y}{x}\right]}=x^2\cdot f(y/x)$$

$\Rightarrow \log u$ is homogenous function of degree 2 in x and y.

So, by Euler's theorem

$$x\frac{\partial}{\partial x}(\log u)+y\frac{\partial}{\partial y}(\log u)=2\log u$$

$$\Rightarrow x\cdot\frac{1}{u}\cdot\frac{\partial u}{\partial x}+y\cdot\frac{1}{u}\cdot\frac{\partial u}{\partial y}=2\log u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.\log u.$$

Hence Proved

(15) if $u = (x^2+y^2)^{1/3}$

Show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2.\frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$

Solⁿ :- Given $u = (x^2+y^2)^{1/3} = x^{2/3}\left(1+\frac{y^2}{x^2}\right)^{1/3}$

$\Rightarrow u = x^{2/3} f(y/x)$

$\Rightarrow u$ is homogenous fun. of x and y of degree $2/3$.

So, By Euler's Theorem

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{2}{3}u \quad — (1)$$

Take Partial derivative of (1) w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + 1.\frac{\partial u}{\partial x} + y.\frac{\partial^2 u}{\partial x\partial y} = \frac{2}{3}.\frac{\partial u}{\partial x} \quad — (2)$$

and take partial derivative of (1) w.r.t. y

$$x.\frac{\partial^2 u}{\partial y\partial x} + y.\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}.1 = \frac{2}{3}.\frac{\partial u}{\partial y} \quad — (3)$$

Multiply ② by x and ③ by y and then add

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\cdot\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}$$
$$+y^2\frac{\partial^2 u}{\partial y^2}+y\cdot\frac{\partial u}{\partial y}=\frac{2}{3}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

Since $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{2}{3}-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{-1}{3}\right)\left(2/3\cdot u\right)$$

(By Euler's thm)

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-2}{9}u.$$

16. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$
$$=2\cos 3u\cdot\sin u$$

Solⁿ:- Given $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Here u is not homogeneous function of x and y.

but $\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = x^2 f(y/x)$

$\Rightarrow \tan u$ is homogenous fun. of x and y of degree 2.

So, by Euler's theorem

$$x \cdot \frac{\partial}{\partial x}(\tan u) + y \cdot \frac{\partial}{\partial y}(\tan u) = 2(\tan u)$$

$$\Rightarrow x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \cdot \frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cdot \frac{\sin u}{\cos u} \cdot \cos^2 u = 2 \sin u \cos u$$

$$\Rightarrow x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u \quad \text{(1)}$$

Take partial derivative of (1) w.r.t. x and w.r.t y.

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot 1 + y \cdot \frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u \cdot \frac{\partial u}{\partial x} \quad \text{(2)}$$

and $$x \frac{\partial^2 u}{\partial y \partial x} + y \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \cdot 1 = 2\cos 2u \cdot \frac{\partial u}{\partial y} \quad \text{(3)}$$

Multiply (2) by x and (3) by y and then add.

$$x^2\frac{\partial^2 u}{\partial x^2} + x\cdot\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\left(x\cdot\frac{\partial u}{\partial x}\right)$$

$$yx\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2\cos 2u\left(y\frac{\partial u}{\partial y}\right)$$

By Adding

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= (2\cos 2u - 1)(\cancel{\tan u})(\sin 2u)$$ (By Euler's thm)

$$= 2\cos 2u\cdot\sin 2u - \sin 2u$$ (Using ①)

$$= \sin 4u - \sin 2u$$

$$= 2\cos 3u\cdot\sin u$$

Hence Proved