(4) If $u = \frac{y}{z} + \frac{x}{y} + \frac{z}{x}$ (Exercise Question (16) (5))

then $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0$

Soln :- $u = \frac{y}{z} + \frac{x}{y} + \frac{z}{x}$

$$\frac{\partial u}{\partial x} = 0 + \frac{1}{y} - \frac{z}{x^2} = \frac{1}{y} - \frac{z}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{z} - \frac{x}{y^2} + 0 = \frac{1}{z} - \frac{x}{y^2}$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + 0 + \frac{1}{x} = \frac{1}{x} - \frac{y}{z^2}$$

then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= x\left(\frac{1}{y} - \frac{z}{x^2}\right) + y\left(\frac{1}{z} - \frac{x}{y^2}\right) + z\left(\frac{1}{x} - \frac{y}{z^2}\right)$$

$$= \cancel{\frac{x}{y}} - \cancel{\frac{z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{x}{y}} + \cancel{\frac{z}{x}} - \cancel{\frac{y}{z}} = 0.$$

Note :- $\frac{\partial^2 z}{\partial x^2} = r$ (Notation), $\frac{\partial z}{\partial x} = p$

if $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s$ (Notation), $\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial y^2} = t$ (Notation)

Exercise Question (7)

(7) (i) if $u = \log(\tan x + \tan y)$

Prove that $\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} = 2$

Sol :- $u = \log(\tan x + \tan y)$

Now $\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y} (\sec^2 x)$

$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y} \sec^2 y$

Now consider $\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y}$

$= 2\sin x \cdot \cos x \left[\frac{1}{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}\right] \cdot \frac{1}{\cos^2 x}$

$+ 2\sin y \cdot \cos y \left[\frac{1}{\frac{\sin y}{\cos y} + \frac{\sin x}{\cos x}}\right] \cdot \frac{1}{\cos^2 y}$

$= \frac{2\sin x}{\cancel{\cos x}} \left[\frac{\cancel{\cos x} \cdot \cos y}{\sin x \cos y + \cos x \sin y}\right]$

$+ \frac{2\sin y}{\cancel{\cos y}} \left[\frac{\cancel{\cos y} \cdot \cancel{\cos y} \cos x}{\sin x \cos y + \cos x \sin y}\right]$

$= \frac{2(\sin x \cos y + \cos x \cdot \sin y)}{\sin x \cos y + \cos x \sin y} = 2$

similarly, we can do (ii)

(iii) Find first and second order derivatives from the relation $\log z = x+y+z$

Soln: Given, $\log z = x+y+z$

$$\frac{1}{z}\cdot\frac{\partial z}{\partial x} = 1+\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x}\left(\frac{1}{z}-1\right)=1$$

$$\Rightarrow \frac{\partial z}{\partial x}\left(\frac{1-z}{z}\right)=1 \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{1-z}.$$

Now $$\frac{1}{z}\cdot\frac{\partial z}{\partial y} = 1+\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y}\left(\frac{1}{z}-1\right)=1$$

$$\Rightarrow \frac{\partial z}{\partial y}\left(\frac{1-z}{z}\right)=1 \Rightarrow \frac{\partial z}{\partial y} = \frac{z}{1-z}.$$

Now $$\frac{\partial z}{\partial x} = \frac{z}{1-z}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \left[\frac{(1-z)(1)-z(-1)}{(1-z)^2}\right]\cdot\frac{\partial z}{\partial x}$$

$$= \frac{1-\cancel{z}+\cancel{z}}{(1-z)^2}\cdot\frac{z}{1-z}$$

$$= \frac{z}{(1-z)^3}$$

~~Given~~ $\frac{\partial z}{\partial x} = \frac{z}{1-z}$

So, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \left[\frac{(1-z)(1) - z(-1)}{(1-z)^2}\right] \cdot \frac{\partial z}{\partial y}$

$= \left[\frac{1 - \cancel{z} + \cancel{z}}{(1-z)^2}\right] \frac{z}{1-z} = \frac{z}{(1-z)^3}$

$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = \frac{z}{(1-z)^3}$

similarly $\frac{\partial^2 z}{\partial x \partial y} = \frac{z}{(1-z)^3} = \frac{\partial^2 z}{\partial y^2}$

Exercise Question No. 8

(8) If $f(x,y,z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Prove that $f_x + f_y + f_z = 0$.

Solu :- Given $f(x,y,z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} x^2 - y^2 & y^2 - z^2 & z^2 \\ x - y & y - z & z \\ 0 & 0 & 1 \end{vmatrix}$

Applying the operation
$C_1 \to C_1 - C_2$
$C_2 \to C_2 - C_3$

$$= (x-y)(y-z)\begin{vmatrix} x+y & y+z & z^2 \\ 1\ \cancel{y+z} & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)\left[x+\cancel{y}-\cancel{y}-z\right]$$ By expanding last row

$$= (x-y)(y-z)(x-z)$$

Now $f(x,y,z) = (x-y)(y-z)(x-z)$

$$\Rightarrow f_x = (y-z)\left[(x-y)(1) + (x-z)(1)\right]$$

$$= (y-z)\left[x-y\right] + (y-z)(x-z)$$

$$f_y = (x-z)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$= (x-z)(x-y) - (x-z)(y-z)$$

$$f_z = (x-y)\left[(y-z)(-1) + (x-z)(-1)\right]$$

$$= -(x-y)(y-z) - (x-y)(x-z)$$

Now consider $f_x + f_y + f_z$

$$= (x-y)(y-z) + (y-z)(x-z) + (x-z)(x-y)$$
$$- (x-z)(y-z) - (x-y)(y-z) - (x-y)(x-z)$$

$$= 0$$