2

CHANGE OF VARIABLES

Quite often, the evaluation of a double or triple integral is greatly simplified by a suitable change of variables.

Let the variables x, y in the double integral $\iint_R f(x, y)\, dx\, dy$ be changed to u, v by means of the relations $x = \phi(u, v)$, $y = \psi(u, v)$, then the double integral is transformed into

$$\iint_{R'} f\{\phi(u, v), \psi(u, v)\} \mid J \mid du\, dv \text{ where } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$ is the Jacobian of

transformation from (x, y) to (u, v) co-ordinates and R′ is the region in the uv-plane which corresponds to the region R in the xy-plane.

(i) To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ).

Here we have $x = r \cos \theta$, $y = r \sin \theta$ so that $x^2 + y^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \quad \boxed{\iint_R f(x, y)\, dx\, dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$

i.e., replace x by $r \cos \theta$, y by $r \sin \theta$ and $dx\, dy$ by $r dr d\theta$.

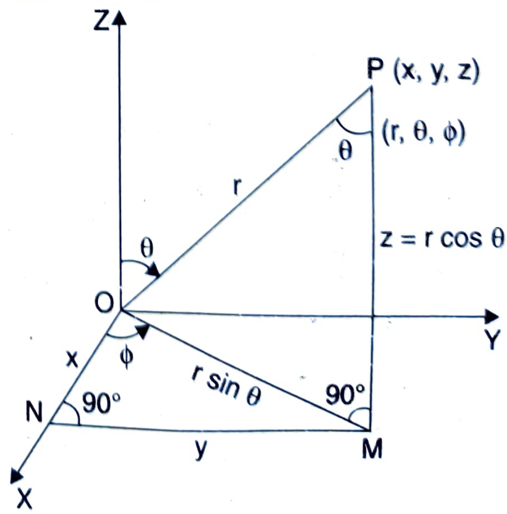
(ii) To change cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

Here, we have $x = r \sin \theta \cos \phi$

$y = r \sin \theta \sin \phi$

$z = r \cos \theta$

so that $x^2 + y^2 + z^2 = r^2$



$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\therefore \quad \iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\; r^2\sin\theta\, dr\, d\theta\, d\phi.$$

Note. Equation of sphere $x^2 + y^2 + z^2 = a^2$ in spherical polar coordinates is $r = a$.

(*i*) If the region of integration is the whole sphere, then

$$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$$

(*ii*) If the region of integration is the positive octant, then

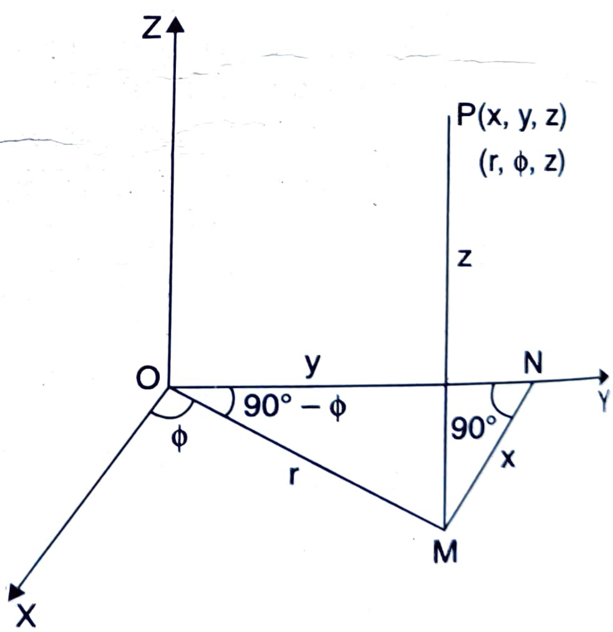
$$0 \le r \le a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{2}.$$

(*iii*) *To change cartesian co-ordinates (x, y, z) to cylindrical polar co-ordinates (r, φ, z).*

Here we have $x = r\cos\phi$

$y = r\sin\phi$

$z = z$



$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\phi + \sin^2\phi) = r$$

$$\therefore \quad \iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_{V'} f(r\cos\phi, r\sin\phi, z)\, r\, dr\, d\phi\, dz.$$

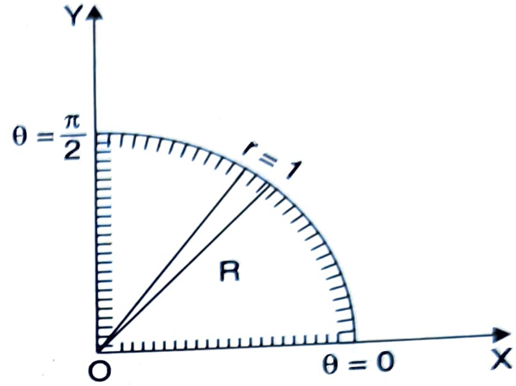
Note. For the cylinder $x^2 + y^2 = a^2$, $z = 0$, $z = h$, the limits of integration are

$$0 \le r \le a, \quad 0 \le \phi \le 2\pi, \; 0 \le z \le h$$

If the region of integration is a cylinder (or cone), change the problem to cylindrical polar coordinates.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\iint \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\, dxdy$ *over the positive quadrant of the circle* $x^2+y^2=1$.



Sol. Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$; $x^2+y^2=1$ transforms into $r = 1$. For the region of integration R, r varies from 0 to 1 and θ varies from 0 to $\dfrac{\pi}{2}$.

$$I = \iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta \qquad \mid\ dxdy \text{ is replaced by } r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\, dr\, d\theta$$

Now, $$\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\, dr = \int_0^1\left(\frac{r}{\sqrt{1-r^4}} - \frac{r^3}{\sqrt{1-r^4}}\right) dr$$

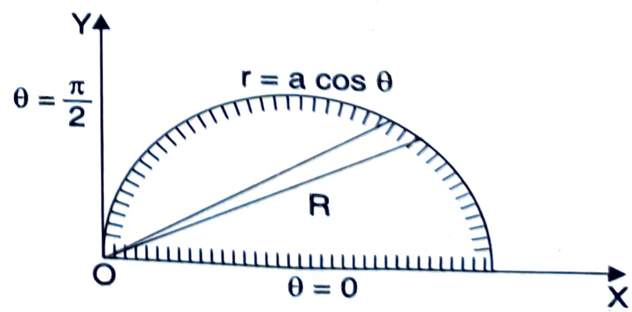
$$= \frac{1}{2}\int_0^1 \frac{2r}{\sqrt{1-r^4}}\, dr + \frac{1}{4}\int_0^1 -4r^3\,(1-r^4)^{-1/2}\, dr$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} + \frac{1}{4}\cdot\left[\frac{(1-r^4)^{1/2}}{1/2}\right]_0^1, \text{ where } t = r^2$$

$$= \frac{1}{2}\Big[\sin^{-1} t\Big]_0^1 + \frac{1}{2}\,(0-1) = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \qquad I = \int_0^{\pi/2}\left(\frac{\pi}{4}-\frac{1}{2}\right) d\theta = \left(\frac{\pi}{4}-\frac{1}{2}\right)\Big[\theta\Big]_0^{\pi/2} = \left(\frac{\pi}{4}-\frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi^2}{8} - \frac{\pi}{4}.$$

Example 2. *Evaluate* $\iint \sqrt{a^2-x^2-y^2}\; dx\, dy$ *over the semi-circle* $x^2+y^2 = ax$ *in the positive quadrant.*



Sol. Changing to polar co-ordinates, $x^2+y^2 = ax$ transforms into $r = a\cos\theta$. For the region of integration R, r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\dfrac{\pi}{2}$.

$$\therefore \quad \iint_R \sqrt{a^2-x^2-y^2}\; dx\, dy$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\,d\theta$$

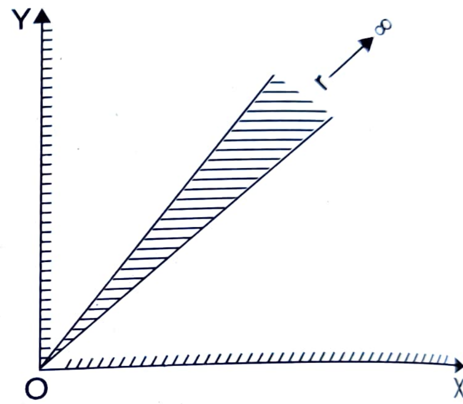
$$= \int_0^{\pi/2} -\frac{1}{2}\cdot\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta = -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

Example 3. *Change into polar co-ordinates and evaluate* $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$.

Hence show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.



Sol. For the region of integration in cartesian co-ordinates, y varies from 0 to ∞ and x also varies from 0 to ∞. Thus the region of integration is the plane XOY. Changing to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$; for the region of integration r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^\infty \frac{1}{2}\cdot e^{-r^2}\cdot 2r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^\infty \frac{1}{2}e^{-t}\,dt\,d\theta, \text{where } t = r^2$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-t}\right]_0^\infty d\theta = -\frac{1}{2}\int_0^{\pi/2}(0-1)\,d\theta = \frac{1}{2}\Big[\theta\Big]_0^{\pi/2} = \frac{\pi}{4}.$$

Now the above result can be written as

$$\int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy = \frac{\pi}{4}$$

or

$$\int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-x^2}\,dx = \frac{\pi}{4} \quad \text{or} \quad \left[\int_0^\infty e^{-x^2}\,dx\right]^2 = \frac{\pi}{4}$$

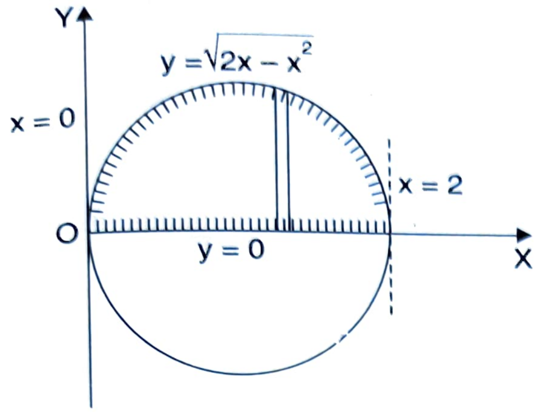
$$\therefore \quad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

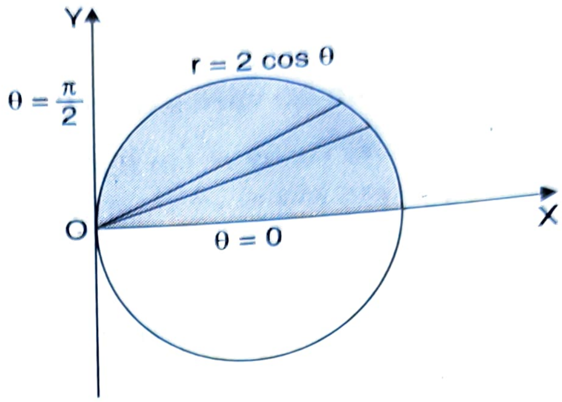
Example 4. *Evaluate* $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ *by changing to polar co-ordinates.*

Sol. In the given integral, y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2.

$$y = \sqrt{2x-x^2} \quad\Rightarrow\quad y^2 = 2x - x^2 \quad\Rightarrow\quad x^2+y^2 = 2x.$$

In polar co-ordinates, we have $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$.





$\therefore$ For the region of integration, r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral, replacing x by $r\cos\theta$, y by $r\sin\theta$, $dy\,dx$ by $r\,dr\,d\theta$, we have

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta}\frac{r\cos\theta\cdot r\,dr\,d\theta}{r} = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2\cdot\frac{2}{3} = \frac{4}{3}.$$

Example 5. *Evaluate $\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$ through the volume of the cylinder $x^2+y^2=a^2$ intercepted by the planes $z=0$ and $z=h$.*

Sol. Changing to cylindrical co-ordinates by changing x to $r\cos\phi$, y to $r\sin\theta$ and replacing $dx\,dy\,dz$ by $r\,dr\,d\phi\,dz$

$$I = \int_0^h\int_0^{2\pi}\int_0^a z(r^2+z^2)\,r\,dr\,d\phi\,dz = \int_0^h\int_0^{2\pi}\int_0^a (zr^3+z^3r)\,dr\,d\phi\,dz$$

$$= \int_0^h\int_0^{2\pi}\left[z\cdot\frac{r^4}{4}+z^3\cdot\frac{r^2}{2}\right]_0^a d\phi\,dz = \int_0^h\int_0^{2\pi}\left(\frac{a^4}{4}z+\frac{a^2}{2}z^3\right)d\phi\,dz$$

$$= \int_0^h\left(\frac{a^4}{4}z+\frac{a^2}{2}z^3\right)\Big[\phi\Big]_0^{2\pi} dz = \int_0^h 2\pi\left(\frac{a^4}{4}z+\frac{a^2}{2}z^3\right)dz$$

$$= 2\pi\left[\frac{a^4z^2}{8}+\frac{a^2z^4}{8}\right]_0^h = \frac{\pi}{4}(a^4h^2+a^2h^4) = \frac{\pi}{4}a^2h^2(a^2+h^2).$$

Example 6. *Evaluate* $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$, *by changing to spherical polar co-ordinates.*

Sol. Here the region of integration is bounded by

$z = 0,\quad z = \sqrt{1-x^2-y^2}\quad$ (*i.e.*, $x^2+y^2+z^2=1$)

$y = 0,\quad y = \sqrt{1-x^2}\quad$ (*i.e.*, $x^2+y^2=1$)

$x = 0,\quad x = 1$

which is the volume of the sphere $x^2+y^2+z^2=1$ in the positive octant.

Changing to spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $x^2 + y^2 + z^2 = r^2$.

For the volume of sphere $x^2 + y^2 + z^2 = 1$ in the positive octant, r varies from 0 to 1, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

Replacing $dz\,dy\,dx$ by $r^2 \sin\theta\, dr\, d\theta\, d\phi$, we have

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2 \sin\theta\, dr\, d\theta\, d\phi}{\sqrt{1-r^2}}$$

Now, $\displaystyle\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr$ Putting $r = \sin t$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{\cos t}\cos t\,dt = \int_0^{\frac{\pi}{2}} \sin^2 t\,dt = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

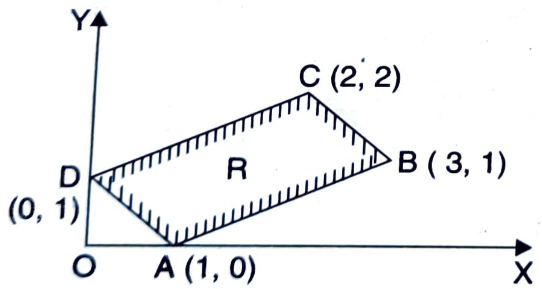
$$\therefore \quad I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\pi}{4}\sin\theta\, d\theta\, d\phi = \int_0^{\frac{\pi}{2}} \frac{\pi}{4}\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

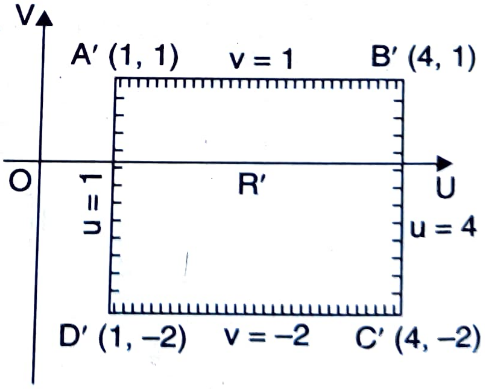
Note. For the whole volume of the sphere $x^2 + y^2 + z^2 = a^2$.

$$0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

Example 7. *Evaluate* $\iint_R (x+y)^2\, dx\, dy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1), using the transformation* $u = x + y$ *and* $v = x - 2y$.

Sol. The vertices A(1, 0), B(3, 1), C(2, 2), D(0, 1) of the parallelogram ABCD in the xy-plane become A′(1, 1), B′(4, 1), C′(4, –2), D′(1, –2) in the uv-plane under the given transformation.





The region R in the xy-plane becomes the region R′ in the uv-plane which is a square bounded by the line $u = 1$, $u = 4$ and $v = -2$, $v = 1$.

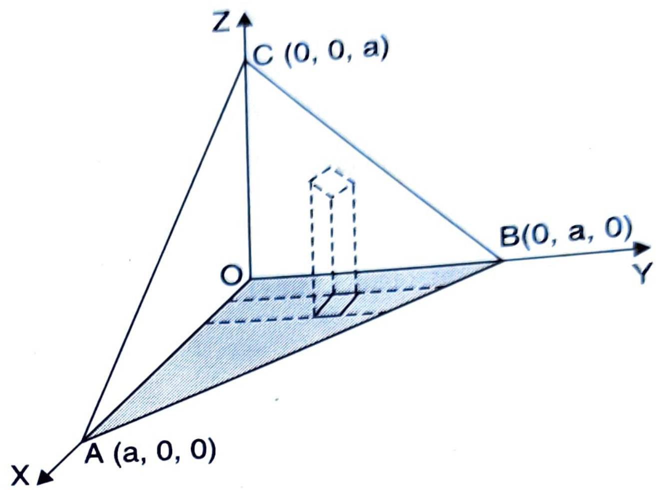
Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \Rightarrow J = \frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{3}$$

$$\therefore \quad \iint_R (x+y)^2\, dx\, dy = \iint_{R'} u^2\, |J|\, du\, dv = \int_{-2}^{1}\int_1^4 u^2 \cdot \frac{1}{3}\, du\, dv$$

$$= \int_{-2}^{1} \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7\, dv = 7\Big[v\Big]_{-2}^{1} = 7 \times 3 = 21.$$

Example 8. *Evaluate* $\iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$, *where R denotes the region bounded by* $x = 0,\ y = 0,\ z = 0$ *and* $x + y + z = a,\ (a > 0)$.

Sol. The plane $x + y + z = a,\ (a > 0)$ meets the coordinate axes in A $(a, 0, 0)$, B$(0, a, 0)$ and C$(0, 0, a)$. On the face ABC, $z = a - x - y$. The projection of plane ABC on the xy-plane is the triangle OAB bounded by the lines OB $(x = 0)$, OA $(y = 0)$ and AB $(x + y = a)$.



$\therefore$ R $= \{(x, y, z)\colon 0 \le x \le a,\ 0 \le y \le a - x,\ 0 \le z \le a - x - y\}$

$$I = \iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= \int_0^a \left[\int_0^{a-x} \left[\int_0^{a-x-y} (x^2 + y^2 + z^2)\, dz \right] dy \right] dx$$

$$= \int_0^a \left[\int_0^{a-x} \left[(x^2 + y^2)\, z + \frac{z^3}{3} \right]_0^{a-x-y} dy \right] dx$$

$$= \int_0^a \left[\int_0^{a-x} \left[(x^2 + y^2)(a - x - y) + \frac{1}{3}(a - x - y)^3 \right] dy \right] dx$$

$$= \int_0^a \left[\int_0^{a-x} \left[(a - x)x^2 + (a - x)y^2 - x^2 y - y^3 + \frac{1}{3}(a - x - y)^3 \right] dy \right] dx$$

$$= \int_0^a \left[(a - x)x^2 y + (a - x) \cdot \frac{y^3}{3} - x^2 \cdot \frac{y^2}{2} - \frac{y^4}{4} + \frac{1}{3} \cdot \frac{(a - x - y)^4}{-4} \right]_0^{a-x} dx$$

$$= \int_0^a \left[x^2 (a - x)^2 + \frac{1}{3}(a - x)^4 - \frac{1}{2} x^2 (a - x)^2 - \frac{1}{4}(a - x)^4 - \frac{1}{12}(0) + \frac{1}{12}(a - x)^4 \right] dx$$

$$= \int_0^a \left[\frac{1}{2} x^2 (a - x)^2 + \frac{1}{6}(a - x)^4 \right] dx = \int_0^a \left[\frac{1}{2} a^2 x^2 - a x^3 + \frac{1}{2} x^4 + \frac{1}{6}(a - x)^4 \right] dx$$

$$= \left[\frac{1}{2} a^2 \cdot \frac{x^3}{3} - a \cdot \frac{x^4}{4} + \frac{1}{2} \cdot \frac{x^5}{5} + \frac{1}{6} \cdot \frac{(a - x)^5}{-5} \right]_0^a$$

$$= \frac{a^5}{6} - \frac{a^5}{4} + \frac{a^5}{10} - \frac{1}{30}(0) + \frac{1}{30} \cdot a^5$$

$$= \left(\frac{10 - 15 + 6 + 2}{60} \right) a^5 = \frac{3}{60} a^5 = \frac{a^5}{20}$$

TEST YOUR KNOWLEDGE

1. Evaluate $\iint \sin \pi (x^2 + y^2)\, dx\, dy$ over the region bounded by the circle $x^2 + y^2 = 1$ by changing to polar co-ordinates.
2. Evaluate $\iint (a^2 - x^2 - y^2)\, dx\, dy$ over the semi-circle $x^2 + y^2 = ax$ in the positive quadrant by changing to polar co-ordinates.
3. Evaluate $\iint (x^2 + y^2)^{7/2}\, dx\, dy$ over the circle $x^2 + y^2 = 1$.
4. Evaluate $\iint xy\,(x^2 + y^2)^{3/2}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.
5. Evaluate the following by changing into polar co-ordinates:

(i) $\displaystyle\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2}$ (ii) $\displaystyle\int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} dy\, dx$

(iii) $\displaystyle\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy$ (iv) $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2)\, dx\, dy$

(v) $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - y^2}} y^2 \sqrt{x^2 + y^2}\, dx\, dy$ (vi) $\displaystyle\int_0^a \int_y^a \frac{x^2\, dx\, dy}{\sqrt{x^2 + y^2}}$.

(vii) $\displaystyle\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2 - y^2}} \log (x^2 + y^2)\, dx\, dy\,,\ (a > 0)$

(viii) $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \log_e (x^2 + y^2 + 1)\, dx\, dy$

6. Evaluate $\displaystyle\iint_D e^{-(x^2 + y^2)}\, dy\, dx$, where D is the region bounded by $x^2 + y^2 = a^2$.
7. Evaluate $\displaystyle\iint xy\,(x^2 + y^2)^{n/2}\, dx\, dy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$.
8. Evaluate $\displaystyle\iint \sqrt{\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\, dx\, dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[**Hint.** Put $x = aX$, $y = bY$ so that $\displaystyle I = \iint \sqrt{\frac{1 - X^2 - Y^2}{1 + X^2 + Y^2}}\, ab\, dX dY$]

9. Transform the following to cartesian form and hence evaluate $\displaystyle\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta$.

10. Evaluate the integral $\iint_R \sqrt{x^2+y^2}\, dxdy$ by changing to polar coordinates where R is the region in the xy-plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$.

$\left[\textbf{Hint.}\ I = \int_0^{2\pi}\int_2^3 r(r\, drd\theta)\right]$

11. Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

12. Evaluate $\iiint \dfrac{dx\, dy\, dz}{\sqrt{a^2-x^2-y^2-z^2}}$,

(i) Over the positive octant of the sphere $x^2+y^2+z^2=a^2$.
(ii) Throughout the volume of the sphere $x^2+y^2+z^2=a^2$.

13. Evaluate $\iiint \dfrac{dx\, dy\, dz}{(x+y+z+1)^3}$ over the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$.

14. Evaluate $\iiint z(x^2+y^2)\, dx\, dy\, dz$ over the volume of the cylinder $x^2+y^2=1$ intercepted by the planes $z=2$ and $z=3$.

15. Evaluate the following integrals through the volume of the sphere $x^2+y^2+z^2=1$, by changing into spherical polar co-ordinates:

(i) $\iiint z^2\, dx\, dy\, dz$ (ii) $\iiint (x^2+y^2+z^2)^m\, dx\, dy\, dz$. $(m>0)$

16. By using the transformation $x+y=u$, $y=uv$, show that $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}} dy\, dx = \frac{1}{2}(e-1)$.

[**Hint.** Here x varies from 0 to 1 and y varies from 0 to $1-x$. The region D of integration is the triangle OAB in xy-plane. Under the given transformation

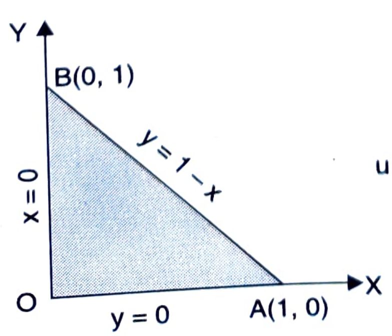
$$x = u - uv = u(1-v),\ y = uv$$

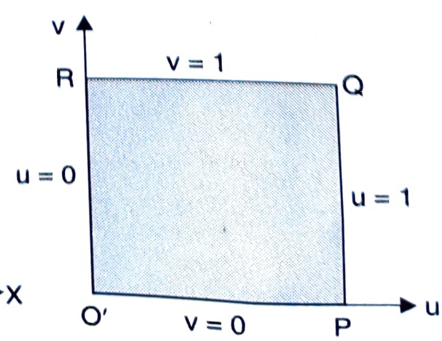
Now,
$$x=0 \Rightarrow u=0 \text{ or } v=1$$
$$y=0 \Rightarrow u=0 \text{ or } v=0$$
$$x+y=1 \Rightarrow u=1$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = u \text{ and } dxdy = |J|\, dudv = u\, dudv$$

The region D transforms into the region
$$D' = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$$

which is square OPQR is uv-plane.





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17. Evaluate $\iiint x^2yz\,dxdydz$ throughout the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$.

[**Hint.** Put $x = au, y = bv, z = cw$, $I = a^2bc \int_0^1 \int_0^{1-u} \int_0^{1-u-v} u^2vw\,abc\,du\,dv\,dw$]

Answers

1. 2

2. $\frac{5\pi a^4}{64}$

3. $\frac{2\pi}{9}$

4. $\frac{1}{14}$

5. (i) $\frac{\pi a}{4}$ (ii) πa^2 (iii) $8\left(\frac{\pi}{2}-\frac{5}{3}\right)a^2$ (iv) $\frac{\pi a^4}{8}$
(v) $\frac{\pi a^5}{20}$ (vi) $\frac{a^3}{3}\log(\sqrt{2}+1)$ (vii) $\frac{\pi a^2}{4}\left(\log a-\frac{1}{2}\right)$ (viii) $\frac{\pi}{2}(2\log 2-1)$

6. $\pi(1-e^{-a^2})$

7. $\frac{2^{n+3}}{n+4}$

8. $\frac{\pi ab}{8}(\pi-2)$

9. 0

10. $\frac{38\pi}{3}$

11. $\frac{1}{8}$

12. (i) $\frac{\pi^2a^2}{8}$ (ii) π^2a^2

13. $\frac{1}{2}\log 2-\frac{5}{16}$

14. $\frac{5\pi}{4}$

15. (i) $\frac{4\pi}{15}$ (ii) $\frac{4\pi}{2m+3}$

17. $\frac{a^3b^2c^2}{2520}$

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