1

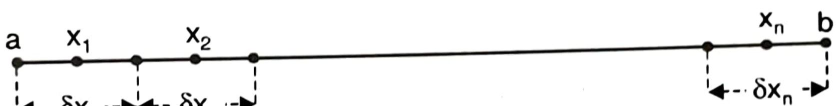
MULTIPLE INTEGRALS

1.1 DOUBLE INTEGRALS

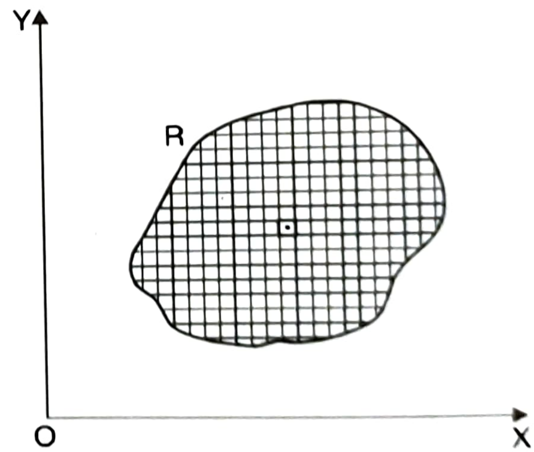
The definite integral $\int_a^b f(x)\,dx$ is defined as the limits of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \cdots + f(x_n)\delta x_n$$

when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, ..., \delta x_n$ tends to zero. Here $\delta x_1, \delta x_2, ..., \delta x_n$ are n sub-intervals into which the range $b - a$ has been divided and $x_1, x_2, ..., x_n$ are values of x lying respectively in the first, second, ..., nth sub-interval.



A double integral is its counterpart in two dimensions. Let a single-valued and bounded function $f(x, y)$ of two independent variables x, y be defined in a closed region R of the xy-plane. Divide the region R into sub-regions by drawing lines parallel to co-ordinate axes. Number the rectangles which lie entirely inside the region R, from 1 to n. Let (x_r, y_r) be any point inside the r^{th} rectangle whose area is δA_r.



Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \cdots + f(x_n, y_n)\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad ...(1)$$

Let the number of these sub-regions increase indefinitely, such that the largest linear dimension (*i.e.*, diagonal) of δA_r approaches zero. The limit of the sum (1), if it exists, irrespective of the mode of sub-division, is called the *double integral* of $f(x, y)$ over the region R and is denoted by $\iint_R f(x, y)\,dA$

In other words,
$$\lim_{\substack{n \to \infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r = \iint_R f(x, y)\, dA$$

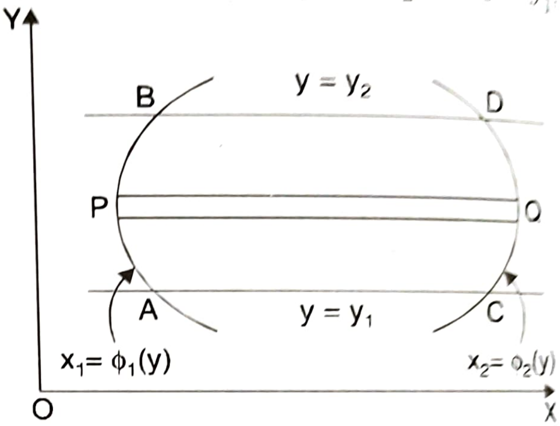
which is also expressed as
$$\iint_R f(x, y)\, dx dy \quad \text{or} \quad \iint_R f(x, y)\, dy\, dx$$

1.2 EVALUATION OF DOUBLE INTEGRALS

The methods of evaluating the double integrals depend upon the nature of the curves bounding the region R. Let the region R be bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$.

(i) When x_1, x_2 are functions of y and y_1, y_2 are constants. Let AB and CD be the curves $x_1 = \phi_1(y)$ and $x_2 = \phi_2(y)$.

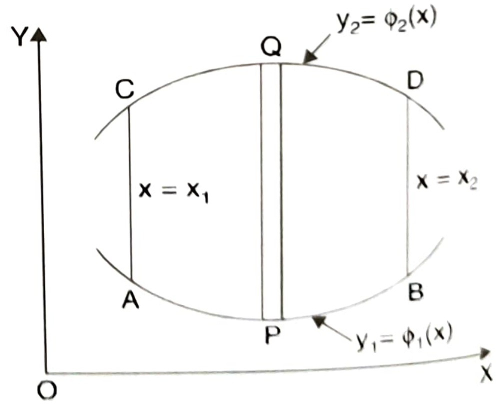
Take a horizontal strip PQ of width δy. Here the double integral is evaluated first w.r.t. x (treating y as a constant). The resulting expression which is a function of y is integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$. Thus,



$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1 = \phi_1(y)}^{x_2 = \phi_2(y)} f(x, y)\, dx}\; dy}$$

the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the horizontal strip PQ (keeping y constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

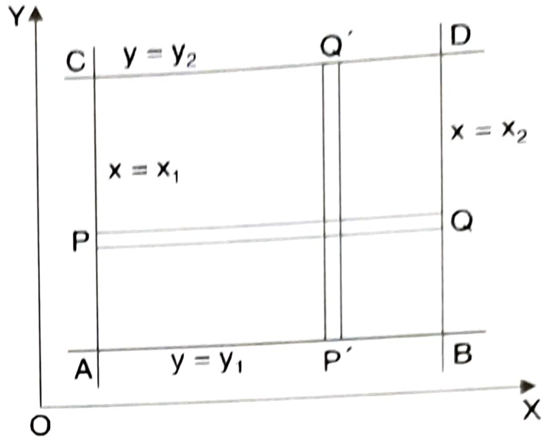
(ii) When y_1, y_2 are functions of x and x_1, x_2 are constants. Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx. Here the double integral is evaluated first w.r.t. y (treating x as constant). The resulting expression which is a function of x is integrated w.r.t. x between the limits $x = x_1$ and $x = x_2$. Thus,



$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1 = \phi_1(x)}^{y_2 = \phi_2(x)} f(x, y)\, dy}\; dx}$$

the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the vertical strip PQ (keeping x constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

***(iii)* When x_1, x_2, y_1, y_2 are constants.** Here the region of integration R is the rectangle ABDC. It is immaterial whether we integrate first along the horizontal strip PQ and then slide it from AB to CD; or we integrate first along the vertical strip P′Q′ and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.



$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy} = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\, dy}\, dx}$$

Note 1. From cases (*i*) and (*ii*) above, we observe that *integration is to be performed w.r.t. that variable having variable limits first and then w.r.t. the variable with constant limits.*

Note 2. If $f(x, y)$ has discontinuities within or on the boundary of the region of integration, then the change of the order of integration does not result into the same integrals.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that:* $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_3^4 \int_1^2 (xy + e^y)\, dx\, dy.$

Sol. $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_1^2 \left[\int_3^4 (xy + e^y)\, dy\right] dx$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right) dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3\right) dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3$$

$$\int_3^4 \int_1^2 (xy + e^y)\, dx\, dy = \int_3^4 \left[\int_1^2 (xy + e^y)\, dx\right] dy = \int_3^4 \left[\frac{yx^2}{2} + xe^y\right]_1^2 dy$$

$$= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y\right) dy = \int_3^4 \left(\frac{3y}{2} + e^y\right) dy$$

$$= \left[\frac{3y^2}{4} + e^y\right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

Hence the result.

Example 2. *Evaluate* $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$.

Sol. $$\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}\right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x\right]_0^1 dy = \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot \frac{\pi}{2}\, dy$$

$$= \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

Example 3. *Show that:* $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx.$

Sol. LHS $$= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\, dy = \int_0^1 dx \int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] dy$$

$$= \int_0^1 \left[2x \cdot \frac{(x+y)^{-2}}{-2} - \frac{(x+y)^{-1}}{-1}\right]_0^1 dx = \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{x+y}\right]_0^1 dx$$

$$= \int_0^1 \left[\frac{-x}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x}\right] dx = \int_0^1 \frac{-x+x+1}{(x+1)^2}\, dx = \int_0^1 \frac{1}{(x+1)^2}\, dx$$

$$= \left[-\frac{1}{x+1}\right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

RHS $$= \int_0^1 dy \int_0^1 \frac{(x+y)-2y}{(x+y)^3}\, dx = \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right] dx$$

$$= \int_0^1 \left[\frac{(x+y)^{-1}}{-1} - 2y \cdot \frac{(x+y)^{-2}}{-2}\right]_0^1 dy = \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{1}{y}\right] dy = \int_0^1 \frac{-1-y+y}{(1+y)^2}\, dy = -\int_0^1 \frac{1}{(1+y)^2}\, dy$$

$$= \left[\frac{1}{1+y}\right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$\therefore$ The two integrals are not equal.

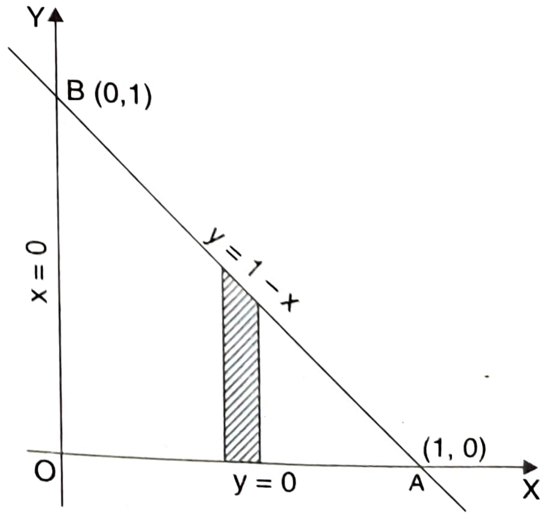
Example 4. *Evaluate* $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$.

Sol.
$$I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\, dy \right] dx$$
$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx$$
$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0]\, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$
$$= \frac{\pi}{4} \left[\log (x + \sqrt{1+x^2}) \right]_0^1 = \frac{\pi}{4} [\log (1+\sqrt{2}) - \log 1] = \frac{\pi}{4} \log (\sqrt{2}+1).$$

Example 5. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$.

Sol.
$$I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy.$$
$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$
$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$
$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1\, dy = \frac{\pi}{4} \int_0^a (a^2-y^2)\, dy$$
$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right] = \frac{\pi a^3}{6}.$$

Example 6. *Evaluate* $\iint e^{2x+3y}\, dx\, dy$ *over the triangle bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$.



Sol. The region R of integration is the triangle OAB. Here x varies from 0 to 1 and y varies from x-axis upto the line $x + y = 1$ *i.e.*, from 0 to $1 - x$.

∴ The region R can be expressed as
$$0 \le x \le 1,\ 0 \le y \le 1 - x$$

$$\therefore \iint_R e^{2x+3y}\, dx\, dy$$
$$= \int_0^1 \int_0^{1-x} e^{2x+3y}\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{3} e^{2x+3y}\right]_0^{1-x} dx$$

$$= \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x})\, dx$$

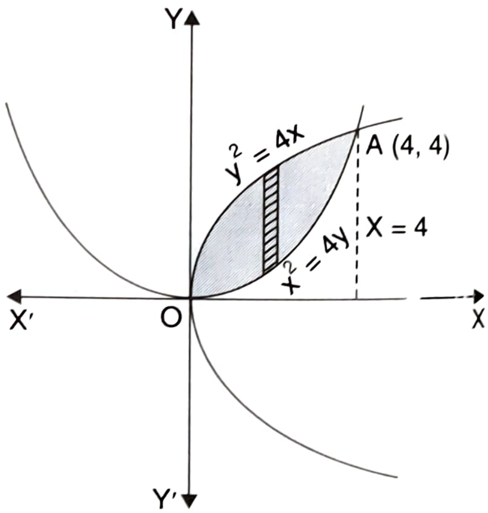
$$= \frac{1}{3}\left[-e^{3-x} - \frac{1}{2}e^{2x}\right]_0^1 = -\frac{1}{3}\left[\left(e^2 + \frac{1}{2}e^2\right) - \left(e^3 + \frac{1}{2}\right)\right]$$

$$= -\frac{1}{3}\left[-e^2(e-1) + \frac{1}{2}(e^2 - 1)\right]$$

$$= \frac{1}{6}(e-1)\,[2e^2 - (e+1)] = \frac{1}{6}(e-1)(2e^2 - e - 1)$$

$$= \frac{1}{6}(e-1)(e-1)(2e+1) = \frac{1}{6}(e-1)^2(2e+1).$$

Example 7. *Evaluate* $\iint_R y\, dx\, dy$, *where R is the region bounded by the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$.



Sol. Solving $y^2 = 4x$ and $x^2 = 4y$, we have

$$\left(\frac{x^2}{4}\right)^2 = 4x \quad \text{or} \quad x(x^3 - 64) = 0$$

$\therefore$ $x = 0, 4$

When $x = 4,\ y = 4$

$\therefore$ Co-ordinates of A are (4, 4)

The region R can be expressed as

$$0 \le x \le 4,\ \frac{x^2}{4} \le y \le 2\sqrt{x}$$

$$\therefore \quad \iint_R y\, dx\, dy = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dy\, dx$$

$$= \int_0^4 \frac{1}{2}\left[y^2\right]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4 = \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}.$$

Example 8. *Evaluate* $\iint (x+y)^2\, dx\, dy$ *over the area bounded by the ellipse*

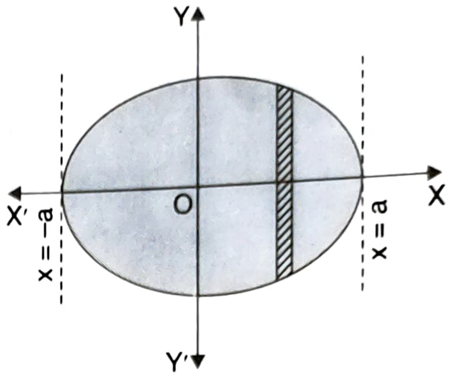
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol. For the ellipse

$$\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad \text{or} \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$\therefore$ The region of integration R can be expressed as

$$-a \le x \le a, \; -\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$



$$\therefore \quad \iint_R (x + y)^2 \, dx \, dy$$

$$= \iint_R (x^2 + y^2 + 2xy) \, dx \, dy$$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} (x^2 + y^2 + 2xy) \, dy \, dx$$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} (x^2 + y^2) \, dy \, dx + \int_{-a}^{a} \int_{-b/a\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} 2xy \, dy \, dx$$

$$= \int_{-a}^{a} \int_{0}^{b/a\sqrt{a^2 - x^2}} 2(x^2 + y^2) \, dy \, dx + 0$$

[Since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a} \left[2\left(x^2 y + \frac{y^3}{3} \right) \right]_0^{b/a\sqrt{a^2 - x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2 \cdot \frac{b}{a}\sqrt{a^2 - x^2} + \frac{1}{3} \cdot \frac{b^3}{a^3}(a^2 - x^2)^{3/2} \right] dx$$

$$= 4\int_{0}^{a} \left[\frac{b}{a} x^2 \sqrt{a^2 - x^2} + \frac{b^3}{3a^3}(a^2 - x^2)^{3/2} \right] dx$$

$$= 4\int_{0}^{\pi/2} \left(\frac{b}{a} \cdot a^2 \sin^2\theta \cdot a\cos\theta + \frac{b^3}{3a^3} \cdot a^3 \cos^3\theta \right) \times a\cos\theta \, d\theta$$

on putting $x = a\sin\theta$ and $dx = a\cos\theta \, d\theta$

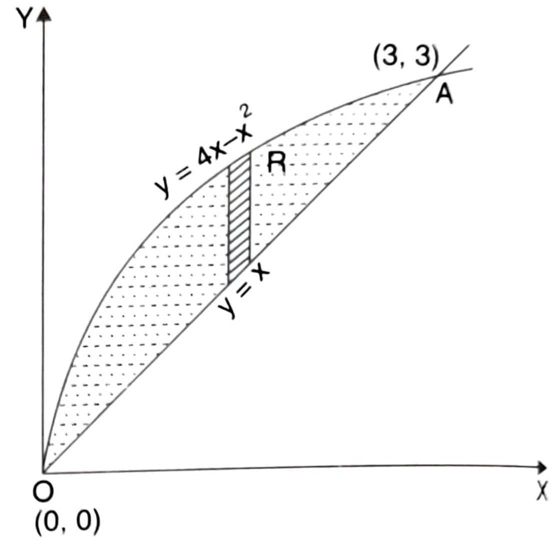
$$= 4\int_{0}^{\pi/2} \left(a^3 b \sin^2\theta \cos^2\theta + \frac{ab^3}{3}\cos^4\theta \right) d\theta$$

$$= 4\left[a^3 b \cdot \frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} + \frac{ab^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} \right] = \frac{\pi}{4}(a^3 b + ab^3) = \frac{\pi}{4} ab(a^2 + b^2).$$

Example 9. *Evaluate $\iint y \, dx \, dy$ over the part of the plane bounded by the line $y = x$ and the parabola $y = 4x - x^2$.*

Sol. The line $y = x$ meets the parabola $y = 4x - x^2$ in two distinct points O(0, 0) and A(3, 3).

The region of integration R can be expressed as $0 \le x \le 3,\ x \le y \le 4x - x^2$.



$$\therefore \iint_R y\,dx\,dy = \int_0^3\int_x^{4x-x^2} y\,dy\,dx$$

$$= \int_0^3 \left(\frac{y^2}{2}\right)_x^{4x-x^2} dx$$

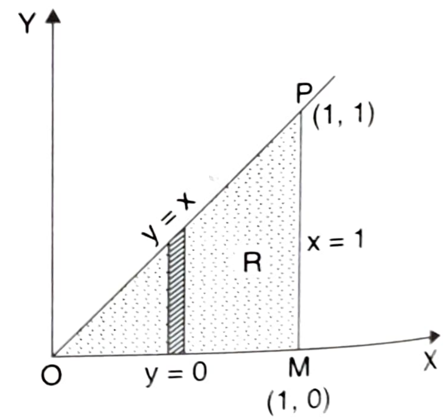
$$= \frac{1}{2}\int_0^3 [(4x-x^2)^2 - x^2]\,dx$$

$$= \frac{1}{2}\int_0^3 (15x^2 + x^4 - 8x^3)\,dx$$

$$= \frac{1}{2}\left(5x^3 + \frac{x^5}{5} - 2x^4\right)_0^3 = \frac{54}{5}.$$

Example 10. *When the region R of integration is the triangle given by $y = 0$, $y = x$ and $x = 1$, show that* $\iint_R \sqrt{4x^2 - y^2}\,dx\,dy = \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$.

Sol. The region of integration R can be expressed as $0 \le x \le 1,\ 0 \le y \le x$.



$$\therefore \iint_R \sqrt{4x^2 - y^2}\,dx\,dy = \int_0^1\int_0^x \sqrt{4x^2 - y^2}\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}y\sqrt{4x^2 - y^2} + \frac{1}{2}\cdot 4x^2 \sin^{-1}\frac{y}{2x}\right]_0^x dx$$

$$= \int_0^1 \frac{1}{2}\left(\sqrt{3}x^2 + 4x^2\cdot\frac{\pi}{6}\right)dx$$

$$= \frac{1}{2}\left(\sqrt{3}\,\frac{x^3}{3} + \frac{2\pi}{3}\cdot\frac{x^3}{3}\right)_0^1 = \frac{1}{2}\left(\sqrt{3}\cdot\frac{1}{3} + \frac{2\pi}{3}\cdot\frac{1}{3}\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right).$$

Example 11. *Evaluate* $\iint_S \sqrt{xy - y^2}\,dx\,dy$*, where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).*

Sol. Let OAB be the triangle whose vertices are (0, 0), (10, 1) and (1, 1) as shown in the figure.

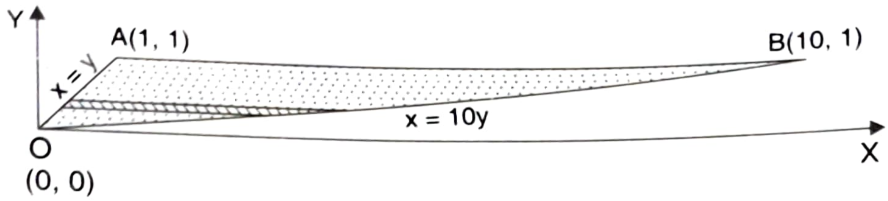
The equation of the line joining O(0, 0) and A(1, 1) is

$$y - 0 = \frac{1-0}{1-0}(x - 0) \quad \Rightarrow \quad y = x$$

The equation of the line joining O(0, 0) and B(10, 1) is

$$y - 0 = \frac{1-0}{10-0}(x - 0) \quad \Rightarrow \quad x = 10y$$

Hence the region of integration can be expressed as $y \le x \le 10y,\ 0 \le y \le 1$



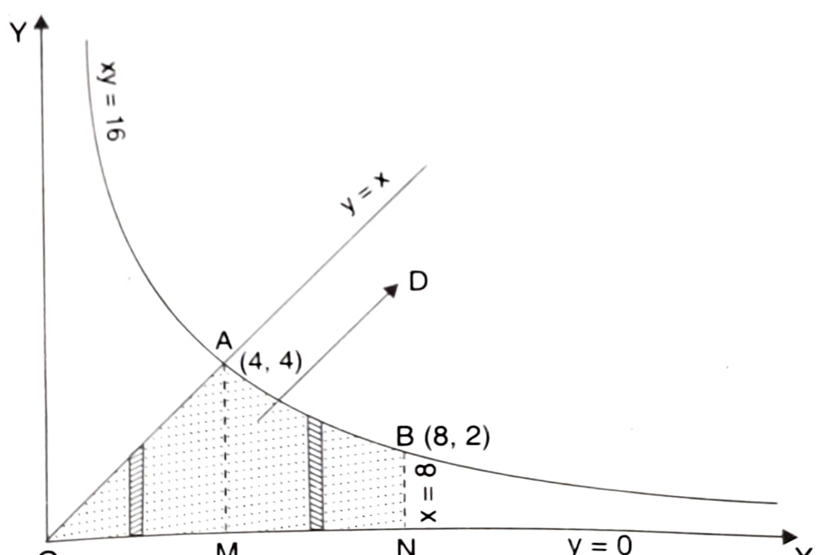
$$\therefore \quad \iint_S \sqrt{xy - y^2}\, dx\, dy = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\, dx\, dy$$

$$= \int_0^1 \left[\frac{2}{3} \frac{(xy - y^2)^{3/2}}{y} \right]_y^{10y} dy = \int_0^1 \frac{2}{3y} (9y^2)^{3/2}\, dy$$

$$= 18 \int_0^1 y^2\, dy = 18 \left(\frac{y^3}{3} \right)_0^1 = 18 \left(\frac{1}{3} \right) = 6.$$

Example 12. *Let D be the region in the first quadrant bounded by the curves $xy = 16$, $x = y$, $y = 0$ and $x = 8$. Sketch the region of integration of the following integral $\iint_D x^2\, dx\, dy$ and evaluate it by expressing it as an appropriate repeated integral.*

Sol. The straight line $x = y$ intersects the hyperbola $xy = 16$ at A(4, 4). The line $x = 8$ intersects the hyperbola $xy = 16$ at B(8, 2). The shaded portion in the following figure is the region of integration.



To evaluate the given integral, we divide the area OABNO into two parts by AM as shown in the figure.

$$\text{Then, } \iint_D x^2\, dx\, dy = \int_{x=0}^{x=4} \int_{y=0}^{y=x} x^2\, dy\, dx + \int_{x=4}^{x=8} \int_{y=0}^{y=\frac{16}{x}} x^2\, dy\, dx$$

$$= \int_0^4 x^2\, dx \int_0^x dy + \int_4^8 x^2\, dx \int_0^{\frac{16}{x}} dy = \int_0^4 x^2 (y)_0^x\, dx + \int_4^8 x^2 \left(y \right)_0^{16/x} dx$$

$$= \int_0^4 x^3\, dx + \int_4^8 16x\, dx = \left(\frac{x^4}{4}\right)_0^4 + (8x^2)_4^8$$

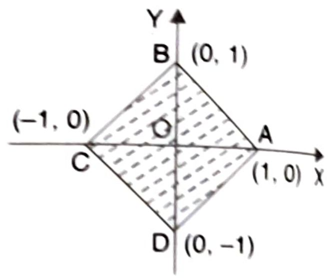
$$= 64 + 8\,(64 - 16) = 64 + 384 = 448.$$

Example 13. *Sketch the region of integration and evaluate* $\iint_R (y - 2x^2)\, dx\, dy$ *where R is the region inside the square* $|\,x\,| + |\,y\,| = 1$.

Sol. The given region R is $|\,x\,| + |\,y\,| = 1$. It intersects x-axis (put $y = 0$) at $|\,x\,| = 1$ *i.e.*, $x = \pm 1$ *i.e.*, at $(-1, 0)$ and $(1, 0)$

Similarly it intersects y-axis (put $x = 0$) at $(0, -1)$ and $(0, 1)$

Region R is shown in the adjoining figure. (Shaded portion). It is a square with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$



Limits of integration for the region OAB are

$$0 \le x \le 1 - y \quad \text{and} \quad 0 \le y \le 1$$

$$\therefore \iint_R (y - 2x^2)\, dxdy = 4\int_0^1 \int_0^{1-y} (y - 2x^2)\, dxdy$$

$$= 4\int_0^1 \left(yx - \frac{2x^3}{3}\right)\Bigg|_0^{1-y} dy = 4\int_0^1 \left[y\,(1-y) - \frac{2}{3}(1-y)^3\right] dy$$

$$= 4\left\{\frac{y^2}{2} - \frac{y^3}{3} - \frac{2}{3}\,\frac{(1-y)^4}{4\,(-1)}\Bigg|_0^1\right\}$$

$$= 4\left\{\frac{1}{2} - \frac{1}{3} + \frac{1}{6}(-1)\right\} = 4\,\frac{3 - 2 - 1}{6} = 0.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1–10):

1. $\int_0^3 \int_0^1 (x^2 + 3y^2)\, dy\, dx.$
2. $\int_0^3 \int_1^2 xy\,(1 + x + y)\, dy\, dx.$
3. $\int_1^a \int_1^b \frac{dy\, dx}{xy}.$
4. $\int_1^2 \int_0^x \frac{dy\, dx}{x^2 + y^2}.$
5. $\int_0^1 dx \int_0^x e^{y/x}\, dy.$
6. $\int_0^1 \int_{x^2}^x (x^2 + 3y + 2)\, dy\, dx.$
7. $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx.$
8. $\int_0^1 \int_y^{y^2+1} x^2 y\, dx\, dy.$
9. $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\, dy\, dx.$
10. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$
11. Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the region in the positive quadrant for which $x + y \le 1$.
12. Evaluate $\iint x^2 y^2\, dx\, dy$ over the circle $x^2 + y^2 = 1$.

13. Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

14. Compute the value of $\iint_R y\,dx\,dy$, where R is the region in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

15. Evaluate $\iint xy\,(x + y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.

16. Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Evaluate the following integrals (17–20):

17. $\int_0^1 \int_0^1 \frac{dxdy}{(1+x^2)(1+y^2)}$

18. $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)}\, x\,dxdy$.

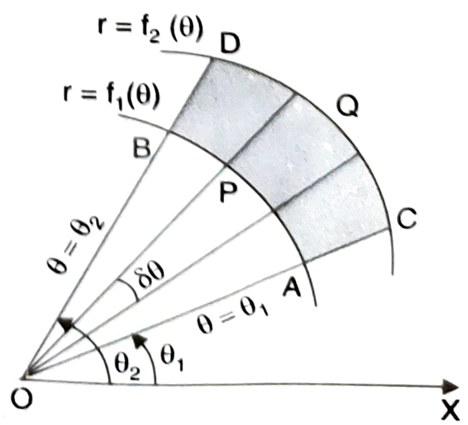
19. $\int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x\,dydx}{y^2+x^2+a^2}$.

20. $\int_0^1 \int_0^{\sqrt{1-x^2}} 4xye^{x^2}\,dy\,dx$.

Answers

1. 12	2. $30\frac{3}{4}$	3. $\log a \log b$	4. $\frac{\pi}{4}\log 2$
5. $\frac{1}{2}(e-1)$	6. $\frac{7}{12}$	7. $\frac{1}{2}$	8. $\frac{67}{120}$
9. $\frac{3\pi a^4}{4}$	10. $\frac{3}{35}$	11. $\frac{1}{6}$	12. $\frac{\pi}{24}$
13. $\frac{a^4}{8}$	14. $\frac{ab^2}{3}$	15. $\frac{3}{56}$	16. $\frac{a^4}{3}$
17. $\frac{\pi^2}{16}$	18. $\frac{\pi}{4}$	19. $\frac{\pi a}{4}$	20. $e-2$

1.3 EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES



To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the straight lines $\theta=\theta_1, \theta=\theta_2$ and the curves $r=r_1, r=r_2$, we first integrate w.r.t. r between the limits $r=r_1$ and $r=r_2$ (treating θ as a constant). The resulting expression is then integrated w.r.t. θ between the limits $\theta=\theta_1$ and $\theta=\theta_2$.

Geometrically, AB and CD are the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta=\theta_2$ so that ACDB is the region of integration. PQ is a