## 1.8 METHOD OF VARIATION OF PARAMETERS TO FIND P.I.

Consider the linear equation of **second order with constant co-efficients**

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X \qquad ...(1)$$

Let its C.F. be $y = c_1y_1 + c_2y_2$ so that $y_1$ and $y_2$ satisfy the equation

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \qquad ...(2)$$

Now, replacing $c_1$, $c_2$ (regarded as parameters) by unknown functions $u(x)$ and $v(x)$ let us assume that the P.I. of (1) is $y = uy_1 + vy_2$ ...(3)

Differentiating (3) w.r.t. $x$, we have $y' = uy_1' + vy_2' + u'y_1 + v'y_2 = uy_1' + vy_2'$ ...(4)

assuming that $u$, $v$ satisfy the equation $u'y_1 + v'y_2 = 0$ ...(5)

Differentiating (4) w.r.t. $x$, we have $y'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$

Substituting the values of $y$, $y'$ and $y''$ in (1), we get

$$(uy_1'' + u'y_1' + vy_2'' + v'y_2') + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_2) = X$$

or
$$u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = X$$

or
$$u'y_1' + v'y_2' = X \qquad ...(6)$$

since $y_1$ and $y_2$ satisfy (2).

Solving (5) and (6), we get $$u' = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{y_2X}{W}$$

and
$$v' = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \frac{y_1X}{W}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of $y_1, y_2$.

Integrating, $$u = -\int \frac{y_2 X}{W}\, dx, \quad v = \int \frac{y_1 X}{W}\, dx$$

Substituting in (3), the P.I. is known. Thus P.I. $= -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$.

**Note 1.** As the solution is obtained by varying the arbitrary constants $c_1, c_2$ of the C.F., the method is known as *variation of parameters.*

**Note 2.** Method of variation of parameters is to be used if instructed to do so.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Apply the method of variation of parameters to solve*

$$\frac{d^2y}{dx^2} + 4y = 4 \sec^2 2x.$$

**Sol.** Given equation in symbolic form is $(D^2 + 4)y = 4 \sec^2 2x$

Its A.E. is $D^2 + 4 = 0$ so that $D = \pm 2i$

$\therefore$ C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$

Here, $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = 4 \sec^2 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -\cos 2x \int \frac{\sin 2x \,.\, 4 \sec^2 2x}{2}\, dx + \sin 2x \int \frac{\cos 2x \,.\, 4 \sec^2 2x}{2}\, dx$$

$$= -2 \cos 2x \int \sec 2x \tan 2x\, dx + 2 \sin 2x \int \sec 2x\, dx$$

$$= -2 \cos 2x \,.\, \frac{\sec 2x}{2} + 2 \sin 2x \,.\, \frac{1}{2} \log(\sec 2x + \tan 2x)$$

$$= -1 + \sin 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$.

**Example 2.** *Solve by the method of variation of parameters:*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}.$$

**Sol.** Given equation in symbolic form is

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

Its A.E. is $(D - 3)^2 = 0 \quad \Rightarrow \quad D = 3, 3$

$\therefore$ C.F. is $y = (c_1 + c_2 x)e^{3x}$

Here, $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and $X = \dfrac{e^{3x}}{x^2}$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -e^{3x} \int \frac{xe^{3x} \cdot \dfrac{e^{3x}}{x^2}}{e^{6x}}\, dx + xe^{3x} \int \frac{e^{3x} \cdot \dfrac{e^{3x}}{x^2}}{e^{6x}}\, dx$$

$$= -e^{3x} \int \frac{1}{x}\, dx + xe^{3x} \int \frac{1}{x^2}\, dx$$

$$= -e^{3x} \log x + xe^{3x}\left(-\frac{1}{x}\right) = -(1 + \log x)\, e^{3x}$$

Hence, C.S. is $\quad y = (c_1 + c_2 x)\, e^{3x} - (1 + \log x)\, e^{3x}$

or $\quad y = [(c_1 - 1) + c_2 x - \log x]\, e^{3x}$

or $\quad y = [(C_1 + c_2 x - \log x)\, e^{3x}$, $\quad$ where $C_1 = c_1 - 1$.

**Example 3.** *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2} - y = e^{-x} \sin (e^{-x}) + \cos (e^{-x}).$$

**Sol.** Given equation in symbolic form is

$$(D^2 - 1)y = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$$

Its A.E. is $\quad D^2 - 1 = 0 \quad \Rightarrow \quad D = \pm 1$

$\therefore$ C.F. is $\quad y = c_1 e^{x} + c_2 e^{-x}$

Here, $y_1 = e^{x}$, $y_2 = e^{-x}$ and $X = e^{-x} \sin (e^{-x}) + \cos (e^{-x})$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -e^{x} \int \frac{e^{-x}[e^{-x} \sin (e^{-x}) + \cos (e^{-x})]}{-2}\, dx + e^{-x} \int \frac{e^{x}[e^{-x} \sin (e^{-x}) + \cos (e^{-x})]}{-2}\, dx$$

$$= \frac{1}{2} e^{x} \int e^{-x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx - \frac{1}{2} e^{-x} \int e^{x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx \qquad \ldots(1)$$

Now, $\displaystyle\int e^{-x} [e^{-x} \sin (e^{-x}) + \cos (e^{-x})]\, dx$

$$= -\int (t \sin t + \cos t)\, dt, \quad \text{where } t = e^{-x}$$

$$= -\left[t(-\cos t) - \int 1 \,.\, (-\cos t)\, dt + \sin t\right]$$

$$= -(-t \cos t + 2 \sin t) = e^{-x} \cos (e^{-x}) - 2 \sin (e^{-x})$$

Also, $\int e^x [\cos(e^{-x}) + e^{-x} \sin(e^{-x})]\, dx$ | Form $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

$= e^x \cos(e^{-x})$

$\therefore$ From (1), we have

$$\text{P.I.} = \frac{1}{2} e^x [e^{-x} \cos(e^{-x}) - 2 \sin(e^{-x})] - \frac{1}{2} e^{-x} . e^x \cos(e^{-x})$$

$$= \frac{1}{2} \cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2} \cos(e^{-x}) = - e^x \sin(e^{-x})$$

Hence, C.S. is $y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x})$.

## TEST YOUR KNOWLEDGE

*Solve by the method of variation of parameters:*

1. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x.$

2. (i) $\dfrac{d^2y}{dx^2} + 16y = 32 \sec 2x$ (ii) $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$

(iii) $y'' + y = \sec^2 x$ (iv) $y'' + 3y' + 2y = \sin(e^x)$

3. $\dfrac{d^2y}{dx^2} + y = \tan x.$

4. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x.$

5. (i) $\dfrac{d^2y}{dx^2} + y = x \sin x.$ (ii) $(D^2 + 1)y = \operatorname{cosec} x \cot x$

6. (i) $y'' - 2y' + 2y = e^x \tan x.$ (ii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x.$

7. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = \dfrac{1}{x^3} e^{-3x}.$

8. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = \dfrac{12e^{4x}}{x^4}.$

9. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x} \sec^2 x.$

10. $y'' - 2y' + y = e^x \log x.$

11. $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}.$

12. $\dfrac{d^2y}{dx^2} + y = \dfrac{1}{1+\sin x}.$

### Answers

1. $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x$

2. (i) $y = c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x - 4 \sin 4x \log(\sec 2x + \tan 2x)$

(ii) $y = c_1 \cos ax + c_2 \sin ax + \dfrac{1}{a^2} \cos ax \log(\cos ax) + \dfrac{1}{a} x \sin ax$

(iii) $y = c_1 \cos x + c_2 \sin x - 1 + \sin x \log(\sec x + \tan x)$

(iv) $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \sin(e^x)$

3. $y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$

4. $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$

5. (i) $y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x$

(ii) $y = c_1 \cos x + c_2 \sin x + \cos x \log \sin x - x \sin x$

6. (i) $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$

(ii) $y_1 = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$

7. $y = \left(c_1 + c_2 x + \frac{1}{2x}\right) e^{-3x}$

8. $y = \left(c_1 + c_2 x + \frac{2}{x^2}\right) e^{4x}$

9. $y = (c_1 + c_2 x - \log \cos x) e^{2x}$

10. $y = (c_1 + c_2 x) e^x + \frac{1}{4} x^2 e^x (2 \log x - 3)$

11. $y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + (e^x - e^{-x}) \log (1 + e^x)$

12. $y = c_1 \cos x + c_2 \sin x + \sin x \log (1 + \sin x) - x \cos x - 1.$