Exercise Questions (Double Integrals in Polar Co-ordinates)

(5) Show that $\iint_R r^2 \sin\theta \, dr \, d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a\cos\theta$, above the initial line.

Solu: we want to evaluate $\iint_R r^2 \sin\theta \, dr \, d\theta$ where R is the region bounded by the semi-circle $r = 2a\cos\theta$.

Now $r = 2a\cos\theta$

Multiply by r on both sides

$\Rightarrow r^2 = 2ar\cos\theta$

if $x = r\cos\theta$, $y = r\sin\theta$ then $r^2 = x^2 + y^2$

$\Rightarrow x^2 + y^2 = 2ax$

$\Rightarrow (x^2 - 2ax + a^2) + y^2 = a^2$

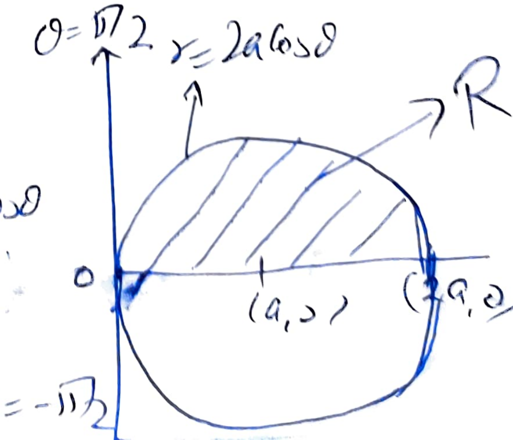
$\Rightarrow (x - a)^2 + y^2 = a^2$

which is an equ. of circle with centre $(a, 0)$ and radius a.

So, in region R

r lies between 0 and $2a\cos\theta$

and θ lies between 0 to $\pi/2$.



so, $\iint\limits_R r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$

$= \int_0^{\pi/2} \left(\left[\frac{r^3}{3}\right] \sin\theta \right)_0^{2a\cos\theta} d\theta$

$= \int_0^{\pi/2} \left[\frac{1}{3} \sin\theta \left(8a^3 \cos^3\theta - 0\right)\right] d\theta$

$= \frac{8a^3}{3} \left[\int_0^{\pi/2} \cos^3\theta . \sin\theta \, d\theta\right]$

$= \frac{8a^3}{-3} \int_0^{\pi/2} (\cos\theta)^3 (-\sin\theta) \, d\theta$

$= \frac{-8a^3}{3} \left[\frac{(\cos\theta)^4}{4}\right]_0^{\pi/2}$

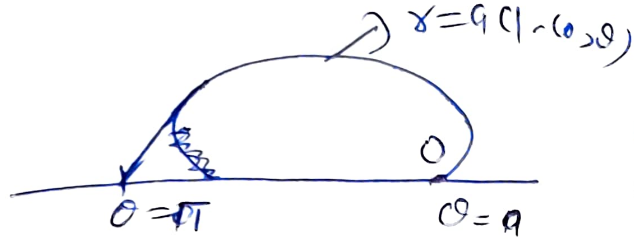
$= \frac{-8a^3}{4 . 3} \left[\left(\cos \frac{\pi}{2}\right)^4 - (\cos 0)^4\right]$

$= -\frac{2a^3}{3} [0 - 1] = \frac{2a^3}{3}.$

(7) Evaluate $\iint r \sin\theta \, dr \, d\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the intial line.

Solu: The region of integration R is covered by radial strips whose ends are $r=0$ and $r=a(1-\cos\theta)$, the strips starting from $\theta=0$ and ending at $\theta=\pi$

$$\iint_R r\sin\theta\, dr\, d\theta$$



$$= \int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \left(a^2(1-\cos\theta)^2 - 0^2\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta\,(1-\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 2\sin\theta/_2 \cdot \cos\theta/_2 \left(2\sin^2\theta/_2\right)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \sin^5\theta/_2\, \cos\theta/_2\, d\theta$$

Put $\sin\theta/_2 = t$, $\frac{1}{2}\cos\theta/_2\, d\theta = dt$

$$\Rightarrow \cos\theta/_2\, d\theta = 2\,dt$$

$$\Rightarrow \iint_R r\sin\theta\, dr\, d\theta = 4a^2 \int_0^1 t^5 (2)\,dt$$

$$= 8a^2 \int_0^1 t^5 \, dt = 8a^2 \left[\frac{t^6}{6} \right]_0^1 = \frac{4}{3} a^2 (1-0)$$

$$= \frac{4a^2}{3}.$$