26/8/2020

Unit-I Multivariate Differential Calculus ①

Limit:-

Before discussing limit of a function of two variables, firstly we will discuss about function of two variables.

Function of two variables:-

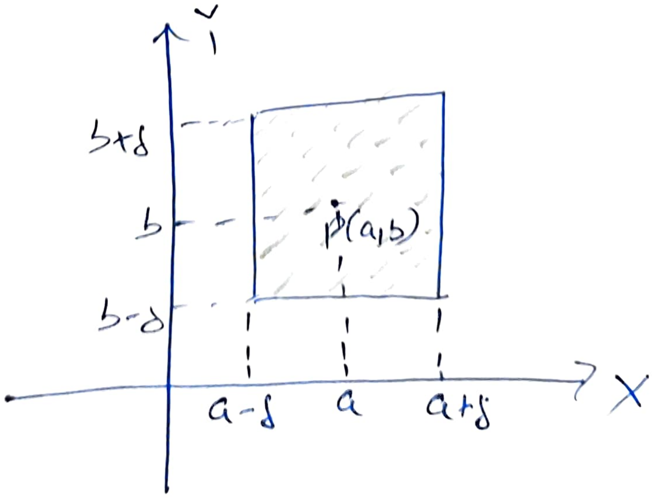
Let $z = x^2 + y^2$ where $0 \le x \le 1, \ \& \ y \le 1$

Here x and y are independent variables and z is function of x and y.

Neighbourhood of a point (a,b):-

Let δ be any positive number. Then δ-nbh. of point (a,b) is defined as $|(x,y) - (a,b)| < \delta$

$$\Rightarrow \{(x,y);\ a-\delta < x < a+\delta,\ b-\delta < y < b+\delta\}$$



Limit of a function of two variables

$\lim f(x,y) = l$ as $(x,y) \to (a,b)$, if for given $\varepsilon > 0$, $\exists$ a +ve no. $\delta \ni$

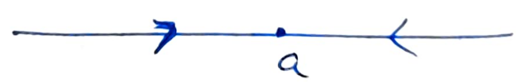
$$|f(x,y) - l| < \varepsilon \quad \forall \quad |(x,y) - (a,b)| < \delta$$

$$\Rightarrow |f(x,y) - l| < \varepsilon \;\; \forall \; \{(x,y) : a-\delta < x < a+\delta,\; b-\delta < y < b+\delta\}$$

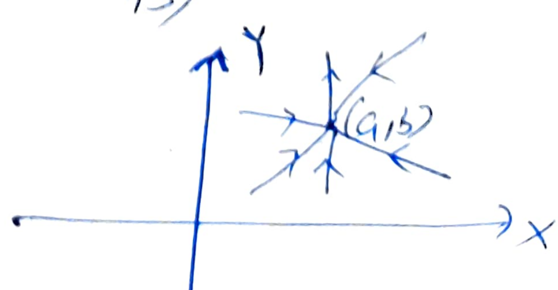
Then we write it as $\underset{(x,y)\to(a,b)}{Lt} f(x,y) = l$

① **Note** :- $\underset{(x,y)\to(a,b)}{Lt} f(x,y)$ if it exists, is unique.

② **Note** :- We know that if f is a function of single variable x, then $\underset{x\to a}{Lt} f(x)$ exists iff $\underset{x\to a^+}{Lt} f(x) = \underset{x\to a^-}{Lt} f(x)$



Similarly, if f is a function of two variables x and y, then $\underset{(x,y)\to(a,b)}{Lt} f(x,y)$ exists if this limit is independent of the path along which we approach the point (a,b)



(1) **Example**:- Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x,y) = x^2+y^2$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

solution:- Let $\varepsilon > 0$ be given

Now $|f(x,y)-0| < \varepsilon \Rightarrow |x^2+y^2-0| < \varepsilon$

$\Rightarrow |x^2+y^2-0| < \varepsilon$ whenever $|(x,y)-(0,0)| < \delta$

$\Rightarrow |x^2+y^2| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$\Rightarrow |x^2+y^2| < \varepsilon \Rightarrow \sqrt{x^2+y^2} < \sqrt{\varepsilon}$

$\Rightarrow \delta = \sqrt{\varepsilon}$

So, for every $\varepsilon > 0$, $\exists$ $\delta = \sqrt{\varepsilon} > 0$ $\ni$

$|f(x,y)-0| < \varepsilon$ for $|(x,y)-(0,0)| < \delta$

Hence by definition of limit, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

(2) **Example**:- Let $f(x,y) = x+y$, Show that ~~f(x,y)~~

~~is continuous~~ ~~&~~ $\lim_{(x,y)\to(\frac{1}{2},\frac{1}{3})} f(x,y) = 5/6$

Solu:- Let $\varepsilon > 0$, be given

Now $\left| f(x,y) - \underbrace{f\left(\frac{1}{2},\frac{1}{3}\right)}_{=5/6} \right| = \left| (x+y) - \left(\frac{1}{2}+\frac{1}{3}\right) \right|$

$$= \left|\left(x-\frac{1}{2}\right)+\left(y-\frac{1}{3}\right)\right| \leq \left|x-\frac{1}{2}\right| + \left|y-\frac{1}{3}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

whenever $\left|x-\frac{1}{2}\right| < \frac{\varepsilon}{2} = \delta$ and $\left|y-\frac{1}{3}\right| < \varepsilon/2 = \delta$

So, for every $\varepsilon > 0$, $\exists$ $\delta = \varepsilon/2 > 0$ $\ni$

~~$|f(x,y) - f(\frac{1}{2},$~~

$\left|f(x,y) - \frac{5}{6}\right| < \varepsilon$ whenever $\left|x-\frac{1}{2}\right| < \delta$ and $\left|y-\frac{1}{3}\right| < \delta$

So, by definition of limit, $\underset{(x,y)\to(\frac{1}{2},\frac{1}{3})}{Lt} f(x,y) = 5/6$

(3) **Example**:- Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

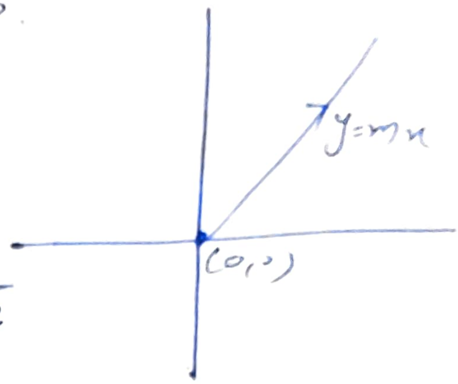
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

Prove that $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$ doesn't exist.

Solution:- We know that if $\underset{(x,y)\to(a,b)}{Lt} f(x,y)$ exists, then this limit is independent of the path along which $(x,y) \to (a,b)$.

Let $(x,y) \to (0,0)$ along the line $y = mx$ where m is any real no.

As $x \to 0$, as $y = mx \Rightarrow y \to 0$



Now $\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{(x,y)\to(0,0)}{Lt} \frac{xy}{x^2+y^2}$

$$= \underset{(x,y)\to(0,0)}{Lt} \frac{x(mx)}{x^2+m^2x^2}$$

$$= \underset{x\to 0}{Lt} \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \underset{x\to 0}{Lt} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

which is different for different values of m.

$\therefore \underset{(x,y)\to(0,0)}{Lt} f(x,y)$ doesn't exist.

④ Example :- Prove that $\underset{(x,y)\to(0,0)}{Lt} \frac{x^3y}{x^6+y^2}$ doesn't exist.

Solution :- Let $(x,y) \to (0,0)$ along the path $x^3 = my$

As $y \to 0$ from $x^3 = my \Rightarrow x \to 0$

$\therefore \underset{(x,y)\to(0,0)}{Lt} \frac{x^3y}{x^6+y^2} = \underset{y\to 0}{Lt} \frac{(my)y}{m^2y^2+y^2} = \underset{y\to 0}{Lt} \frac{m}{m^2+1}$

$$= \frac{m}{m^2+1}$$

which is different for different values of m.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2} \text{ doesn't exist.}$$