(9) $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$

$= \int_0^{2a} \left[x^2y + \frac{y^3}{3}\right]_0^{\sqrt{2ax-x^2}} dx$

$= \int_0^{2a} \left[x^2(2ax-x^2)^{1/2} + \frac{(2ax-x^2)^{3/2}}{3}\right] dx$

$= \int_0^{2a} x^2(2ax-x^2)^{1/2}\,dx + \int_0^{2a} \frac{(2ax-x^2)^{3/2}}{3}\,dx$

$= I_1 + I_2$ (say)

$I_1 = \int_0^{2a} x^2(2ax-x^2)^{1/2}\,dx$

$= \int_0^1 4a^2t^2\left(2a(2at)-(2at)^2\right)^{1/2}(2a)\,dt$ | Put $x = 2at$, if $x=0,\ t=0$; if $x=2a,\ t=1$; $dx = 2a\,dt$

$= \int_0^1 4a^2t^2\left[4a^2t - 4a^2t^2\right]^{1/2} 2a\,dt$

$= \int_0^1 4a^2(2a)(4a^2)^{1/2}\, t^2 .\, t^{1/2}(1-t)^{1/2}\,dt$

$= \cancel{32a^5}\ 16a^4 \int_0^1 t^{5/2}(1-t)^{1/2}\,dt$

we know that

Beta Fun. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = B(m,n)$

$$\Rightarrow I_1 = 16a^4\, B\left(\frac{7}{2}, \frac{3}{2}\right)$$

$$= 16a^4 \frac{\Gamma\frac{7}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{7}{2}+\frac{3}{2}\right)} = \frac{16a^4 \cdot \Gamma\frac{7}{2} \cdot \Gamma\frac{3}{2}}{\Gamma 5}$$

$$= \frac{16 \cdot a^4\ \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}\ \frac{1}{2}\cdot\Gamma\frac{1}{2}}{4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{\cancel{16}\cdot a^4\ \ 5\cdot\cancel{3}\cdot 1\cdot(\sqrt{\pi})^2}{\cancel{16}\cdot 4\cdot\cancel{3}\cdot 2\cdot 1}$$

$$= \frac{5}{8} a^4 \cdot \pi$$

$$I_2 = \frac{1}{3}\int_0^{2a} (2ax - x^2)^{3/2}\, dx$$

$$= \frac{1}{3}\int_0^1 \left(2a(2at) - 4a^2t^2\right)^{3/2} (2a\,dt)$$

Put $x = 2at$

$dx = 2a\,dt$

if $x=0$, $t=0$

if $x=2a$, $t=1$

$$= \frac{1}{3}\int_0^1 (4a^2)^{3/2} (t)^{3/2} (1-t)^{3/2} \cdot 2a\, dt$$

$$= \frac{1}{3}\, 16a^4 \int_0^1 t^{3/2} (1-t)^{3/2}\, dt$$

$$\Rightarrow I_2 = \frac{16}{3}a^4 \frac{\Gamma\frac{5}{2}\,\Gamma\frac{5}{2}}{\Gamma\frac{10}{2}} = \frac{16}{3}a^4 \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\right)^2}{\Gamma 5}$$

$$\Rightarrow I_2 = \frac{1\cancel{6}}{\cancel{3}}a^4 \cdot \frac{\cancel{9}}{\cancel{16}} \frac{(\sqrt{\pi})^2}{\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1}$$

$$\Rightarrow I_2 = \frac{\pi a^4}{2(4)} = \frac{\pi a^4}{8}$$

So, $I_1 + I_2 = \frac{5\pi a^4}{8} + \frac{\pi a^4}{8} = \frac{6}{8}\pi a^4 = \frac{3}{4}\pi a^4$