CHAPTER

11

DIFFERENTIAL EQUATIONS OF FIRST ORDER

11.1 DEFINITION

An equation which involves differential co-efficient is called a differential equation.

For example,

1. $\frac{dy}{dx}=\frac{1+x^2}{1-y^2}$ **2.** $\frac{d^2y}{dx^2}=2\frac{dy}{dx}-8y=0$ **3.** $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=k\frac{d^2y}{dx^2}$

4. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$ **5.** $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial z}{\partial y}$

There are two types of differential equations :

(1) Ordinary Differential Equation

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

(2) Partial Differential Equation

A differential equation involving partial derivatives with respect to more than one independent variable is called a partial differential equation.

11.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

The *order* of a differential equation is the order of the highest differential co-efficient present in the equation. Consider

1. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E\sin wt.$ **2.** $\cos x\frac{d^2y}{dx^2}+\sin x\left(\frac{dy}{dx}\right)^2+8y=\tan x$

3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$

The order of the above equations is 2.

The degree of a differential equation is the degree of the highest derivative after removing the radical sign and fraction.

The *degree* of the equation (1) and (2) is 1. The degree of the equation (3) is 2.

11.3 FORMATION OF DIFFERENTIAL EQUATIONS

The differential equations can be formed by differentiating the ordinary equation and eliminating the arbitrary constants.

Example 1. *Form the differential equation by eliminating arbitrary constants, in the following cases and also write down the order of the differential equations obtained.*

(*a*) $y = A\,x + A^2$ (*b*) $y = A \cos x + B \sin x$ (*c*) $y^2 = Ax^2 + Bx + C$.

(*R.G.P.V. Bhopal, June 2008*)

Solution. (*a*) $y = Ax + A^2$ ... (1)

On differentiation $\frac{dy}{dx} = A$

Putting the value of A in (1), we get $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$ **Ans.**

On eliminating one constant A we get the differential equation of order 1.

(*b*) $y = A \cos x + B \sin x$

On differentiation $\frac{dy}{dx} = -A\sin x + B\cos x$

Again differentiating

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x \Rightarrow \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0 \qquad \textbf{Ans.}$$

This is differential equation of order 2 obtained by eliminating two constants A and B.

(*c*) $y^2 = Ax^2 + Bx + C$

On differentiation $2y\frac{dy}{dx} = 2Ax + B$

Again differentiating $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2A$

On differentiating again $y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0 \Rightarrow y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$ **Ans.**

This is the differential equation of order 3, obtained by eliminating three constants A, B, C.

Example 2. *Determine the differential equation whose set of independent solution is* $\{e^x, xe^x, x^2 e^x\}$ (*U.P., II Semester, Summer 2002*)

Solution. Here, we have

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x \qquad \text{... (1)}$$

Differentiating both sides, we get

$$\begin{aligned} y' &= c_1 e^x + c_2 x e^x + c_2 e^x + c_3 x^2 e^x + c_3 2x e^x \\ &= (c_1 e^x + c_2 x e^x + c_3 x^2 e^x) + c_2 e^x + c_3 2x e^x \\ &= y + c_2 e^x + c_3 2x e^x \qquad \text{[Using (1)]} \end{aligned}$$

$$\Rightarrow \quad y' - y = c_2 e^x + c_3 2x e^x \qquad \text{... (2)}$$

Again, differentiating both sides, we get

$$\Rightarrow \quad y'' - y' = c_2 e^x + 2c_3 e^x + 2x c_3 e^x$$

$$\Rightarrow \quad y'' - y' = (c_2 e^x + 2x c_3 e^x) + 2c_3 e^x$$

$$\Rightarrow \quad y'' - y' = y' - y + 2c_3 e^x \qquad \text{[Using (2)]}$$

$$\Rightarrow \quad y'' - 2y' + y = 2c_3 e^x \qquad \text{... (3)}$$

Finally, on differentiating both sides, we get

$$\Rightarrow \quad y''' - 2y'' + y' = 2c_3 e^x$$

$$\Rightarrow \quad y''' - 2y'' + y' = y'' - 2y' + y \qquad \text{[Using (3)]}$$

$$\Rightarrow \quad y''' - 2y'' - y'' + y' + 2y' - y = 0$$

$$\Rightarrow \quad y''' - 3y'' + 3y' - y = 0$$

$$\Rightarrow \quad (D-1)^3 y = 0 \qquad \textbf{Ans.}$$

Example 3. *By the elimination of the constants A and B obtain the differential equation of which* $xy = Ae^x + Be^{-x} + x^2$ *is the solution.* *(U.P.B. Pharma (C.O.) 2005)*

Solution. We have $xy = Ae^x + Be^{-x} + x^2$ .... (1)

On differentiating (1), we get $x\frac{dy}{dx} + y = Ae^x - Be^{-x} + 2x$ ... (2)

Again differentiating (2), we get $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = Ae^x + Be^{-x} + 2$ ... (3)

From (1), we have $Ae^x + Be^{-x} = xy - x^2$ ... (4)

Putting the value of $Ae^x + Be^{-x}$ From (4) in (3), we have

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2, \qquad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 2 - x^2 \qquad \textbf{Ans.}$$

EXERCISE 11.1

1. Write the order and the degree of the following differential equations.

(*i*) $\frac{d^2y}{dx^2} + a^2x = 0;$ (*ii*) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2};$ (*iii*) $x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^4 = 0.$

Ans. (*i*) 2,1 (*ii*) 2,2 (*iii*) 2,3

2. Give an example of each of the following type of differential equations.

(*i*) A linear-differential equation of second order and first degree **Ans.** *Q* 1 (*i*)

(*ii*) A non-linear differential equation of second order and second degree **Ans.** *Q* 1 (*ii*)

(*iii*) Second order and third degree. **Ans.** *Q* 1 (*iii*)

3. Obtain the differential equation of which $y^2 = 4a(x + a)$ is a solution.

Ans. $y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} - y^2 = 0$

4. Obtain the differential equation associated with the primitive $Ax^2 + By^2 = 1$.

Ans. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

5. Find the differential equation corresponding to

$y = a\,e^{3x} + be^x$. **Ans.** $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

6. By the elimination of constants A and B, find the differential equation of which

$y = e^x (A \cos x + B \sin x)$ is a solution. **Ans.** $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

7. Find the differential equation whose solution is $y = a \cos (x = 3)$. (*A.M.I.E., Summer 2000*)

Ans. $\frac{dy}{dx} = -\tan(x+3)$

8. Show that set of function $\left\{x, \frac{1}{x}\right\}$ forms a basis of the differential equation $x^2y'' + xy' - y = 0$.

Obtain a particular solution when $y(1) = 1, y(1) = 2$. **Ans.** $y = \frac{3x}{2} - \frac{1}{2x}$

11.4 SOLUTION OF A DIFFERENTIAL EQUATION

In the example 1(*b*), $y = A\cos x + B\sin x$, on eliminating A and B we get the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

$y = A\cos x + B\sin x$ is called the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

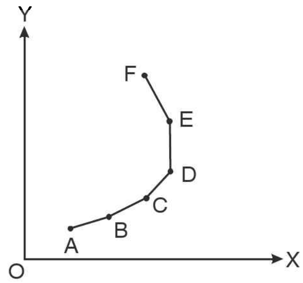
The order of the differential equation $\frac{d^2y}{dx^2} + y = 0$ is two and the solution $y = A\cos x + B\sin x$ contains two arbitrary constants. The number of arbitrary constants in the solution is equal to the order of the differential equation.

An equation containing dependent variable (y) and independent variable (x) and free from derivative, which satisfies the differential equation, is called the solution (primative) of the differential equation.

11.5 GEOMETRICAL MEANING OF THE DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Let $$f\left(x,\ y,\ \frac{dy}{dx}\right) = 0 \qquad \text{... (1)}$$

be a differential equation of the first order and first degree.



It is known that a direction of a curve at a particular point is given by the tangent line at that point and the slope of the tangent is $\frac{dy}{dx}$ at that point. Let A (x_0, y_0) be any initial point. From (1), we can find $\frac{dy}{dx}$ at A (x_0, y_0).

With the help of $\frac{dy}{dx}$ at A (x_0, y_0) draw the tangent at the point A. On the tangent line take a neighbouring point B (x_1, y_1).

Find $\frac{dy}{dx}$ at the point B (x_1, y_1) from equation (1) and draw the tangent at B with the help of $\frac{dy}{dx}$ at (x_1, y_1).

Take a neighbouring point C (x_2, y_2) on this tangent and in this way draw another tangent at the point C. Similarly draw, some more tangents by taking the neighbouring points on them. They form a smooth curve *i.e.* $y = f_1(x)$ which is the solution (1).

Again we take another starting point A' (x_0', y_0'). We can draw another curve starting from A'. In this way we can draw a number of curves.

The given differential equation represents a family of curves.

11.6 DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

We will discuss the standard methods of solving the differential equations of the following types:

(*i*) Equations solvable by separation of the variables. (*ii*) Homogeneous equations.
(*iii*) Linear equations of the first order. (*iv*) Exact differential equations.

11.7 VARIABLES SEPARABLE

If a differential equation can be written in the form

$$f(y)\,dy = \phi(x)\,dx$$

We say that variables are separable, y on left hand side and x on right hand side.

We get the solution by integrating both sides.

Working Rule:

Step 1. Separate the variables as $f(y)\,dy = \phi(x)\,dx$

Step 2. Integrate both sides as $\int f(y)\,dy = \int \phi(x)\,dx$

Step 3. Add an arbitrary constant C on R.H.S.

Example 4. *Solve* : $\dfrac{dy}{dx} = \dfrac{x(2\log x+1)}{\sin y + y\cos y}$ *(U.P., II, 2008, U.P.B. Pharm (C.O.) 2005)*

Solution. We have, $\dfrac{dy}{dx} = \dfrac{x(2\log x+1)}{\sin y + y\cos y}$

Separating the variables, we get

$$(\sin y + y\cos y)\,dy = \{x(2\log x + 1)\}\,dx$$

Integrating both the sides, we get $\int (\sin y + y\cos y)dy = \int \{x(2\log x+1)\}\,dx + C$

$$-\cos y + y\sin y - \int (1)\cdot \sin y\,dy = 2\int \log x\cdot x\,dx + \int x\,dx + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = 2\left[\log x\cdot\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x\cdot\frac{x^2}{2} - \int x\,dx + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x\cdot\frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C$$

Ans.

Example 5. *Solve the differential equation.*

$$x^4\frac{dy}{dx} + x^3 y = -\sec(xy).$$

(A.M.I.E.T.E., Winter 2003)

Solution. $x^4\dfrac{dy}{dx} + x^3 y = -\sec(xy) \quad \Rightarrow \quad x^3\left(x\dfrac{dy}{dx} + y\right) = -\sec xy$

Put $v = xy$, $\dfrac{dv}{dx} = x\dfrac{dy}{dx} + y \quad \Rightarrow \quad x^3\dfrac{dv}{dx} = -\sec v$

$\Rightarrow \quad \dfrac{dv}{\sec v} = -\dfrac{dx}{x^3} \quad \Rightarrow \quad \int \cos v\,dv = -\int \dfrac{dx}{x^3} + c$

$\Rightarrow \quad \sin v = \dfrac{1}{2x^2} + c \quad \Rightarrow \quad \sin xy = \dfrac{1}{2x^2} + c$ **Ans.**

Example 6. *Solve* : $\cos(x+y)dy = dx$

Solution. $\cos(x+y)\,dy = dx \quad \Rightarrow \quad \dfrac{dy}{dx} = \sec(x+y)$

On putting $x + y = z$

So that $$1+\frac{dy}{dx}=\frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{dz}{dx}-1$$

$$\frac{dz}{dx}-1=\sec z \quad \Rightarrow \quad \frac{dz}{dx}=1+\sec z$$

Separating the variables, we get

$$\frac{dz}{1+\sec z}=dx$$

On integrating,

$$\int\frac{\cos z}{\cos z+1}dz=\int dx \quad \Rightarrow \quad \int\left[1-\frac{1}{\cos z+1}\right]dz=x+C$$

$$\int\left(1-\frac{1}{2\cos^2\frac{z}{2}-1+1}\right)dz=x+C$$

$$\int\left(1-\frac{1}{2}\sec^2\frac{z}{2}\right)dz=x+C \quad \Rightarrow \quad z-\tan\frac{z}{2}=x+C$$

$$x+y-\tan\frac{x+y}{2}=x+C$$

$$y-\tan\frac{x+y}{2}=C$$ **Ans.**

Example 7. *Solve the equation.*

$(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y\, dy = 0$ (*U.P. II Semester, Summer 2005*)

Solution. We have

$$(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y\, dy = 0$$

Re-arranging (1), we get $\frac{x\,dx}{y\,dy}=\frac{3x^2+2y^2-8}{2x^2+3y^2-7}$

Applying componendo and dividendo rule, we get

$$\frac{x\,dx+y\,dy}{x\,dx-y\,dy}=\frac{5x^2+5y^2-15}{x^2-y^2-1} \quad \Rightarrow \quad \frac{x\,dx+y\,dy}{x^2+y^2-3}=5\left(\frac{x\,dx-y\,dy}{x^2-y^2-1}\right)$$

Multiplying by 2 both the sides, we get

$$\Rightarrow \quad \left(\frac{2x\,dx+2y\,dy}{x^2+y^2-3}\right)=5\left(\frac{2x\,dx-2y\,dy}{x^2-y^2-1}\right)$$

Integrating both sides, we get

$$\log(x^2+y^2-3)=5\log(x^2-y^2-1)+\log C$$

$$\Rightarrow \quad x^2+y^2-3=C(x^2-y^2-1)^5$$ **Ans.**

where C is arbitrary constant of integration.

EXERCISE 11.2

Solve the following differential equations :

1. $\frac{dx}{x}=\tan y\cdot dy$ **Ans.** $x\cos y = C$
2. $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **Ans.** $\sin^{-1} y = \sin^{-1} x + C$
3. $y(1+x^2)^{1/2}dy+x\sqrt{1+y^2}\,dx=0$ **Ans.** $\sqrt{1+y^2}+\sqrt{1+x^2}=C$
4. $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$ **Ans.** $\tan x\tan y = C$

5. $(1 + x^2)\, dy - x\, y\, dx = 0$ **Ans.** $y^2 = C\,(1 + x^2)$

6. $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$ **Ans.** $(e^y + 1) \sin x = C$

7. $3\, e^x \tan y\, dx + (1 - e^x)\, sec^2\, y\, dy = 0$ **Ans.** $(1 - e^x)^3 = C \tan y$

8. $(e^y + 2) \sin x\, dx - e^y \cos x\, dy = 0$ **Ans.** $(e^y + 2) \cos x = C$

9. $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ **Ans.** $e^y = e^x + \dfrac{x^3}{3} + C$

10. $\dfrac{dy}{dx} = 1 + \tan(y - x)$ [Put $y - x = z$] **Ans.** $\sin(y - x) = e^{x+c}$

11. $(4x + y)^2 \dfrac{dx}{dy} = 1$ **Ans.** $\tan^{-1} \dfrac{4x + y}{2} = 2x + C$

12. $\dfrac{dy}{dx} = (4x + y + 1)^2$ [**Hint.** Put $4x + y + 1 = z$] **Ans.** $\tan^{-1} \dfrac{4x + y + 1}{2} = 2x + C$

11.8 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$

is called a homogeneous equation if each term of $f(x, y)$ and $\phi(x, y)$ is of the same degree *i.e.*,

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$

In such case we put $\boxed{y = vx. \text{ and } \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}}$

The reduced equation involves v and x only. This new differential equation can be solved by *variables separable* method.

Working Rule

Step 1. Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ **Step 2.** Separate the variables.

Step 3. Integrate both the sides. **Step 4.** Put $v = \dfrac{y}{x}$ and simplify.

Example 8. *Solve the following differential equation*

$(2xy + x^2)\, y = 3y^2 + 2xy$ (*A.M.I.E.T.E. Dec. 2006*)

Solution. We have, $(2xy + x^2) \dfrac{dy}{dx} = 3y^2 + 2xy \Rightarrow \dfrac{dy}{dx} = \dfrac{3y^2 + 2xy}{2xy + x^2}$

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

On substituting, the given equation becomes $v + x\dfrac{dv}{dx} = \dfrac{3v^2x^2 + 2vx^2}{2vx^2 + x^2} = \dfrac{3v^2 + 2v}{2v + 1}$

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{3v^2 + 2v - 2v^2 - v}{2v + 1} \qquad \Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{v^2 + v}{2v + 1} \Rightarrow \left(\dfrac{2v + 1}{v^2 + v}\right) dv = \dfrac{dx}{x}$

$\Rightarrow \quad \displaystyle\int \left(\frac{2v + 1}{v^2 + v}\right) dv = \int \frac{dx}{x} \qquad \Rightarrow \quad \log(v^2 + v) \log x + \log c$

$\Rightarrow \quad v^2 + v = cx \qquad \Rightarrow \quad \dfrac{y^2}{x^2} + \dfrac{y}{x} = cx$

$\Rightarrow \quad y^2 + xy = cx^3$

Example 9. *Solve the equation* :

$$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x}$$

Solution. $$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x} \qquad \ldots (1)$$

Put $y = vx$ in (1) so that $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$$v+x\frac{dv}{dx}=v+x\sin v$$

$$\Rightarrow \quad x\frac{dv}{dx}=x\sin v \quad \Rightarrow \quad \frac{dv}{dx}=\sin v$$

Separating the variable, we get

$$\Rightarrow \quad \frac{dv}{\sin v}=dx \quad \Rightarrow \quad \int \operatorname{cosec} v\,dv=\int dx+C$$

$$\log\tan\frac{v}{2}=x+C \quad \Rightarrow \quad \log\tan\frac{y}{2x}=x+C$$ **Ans.**

EXERCISE 11.3

Solve the following differential equations:

1. $(y^2 - xy)\,dx + x^2 dy = 0$ **Ans.** $\dfrac{x}{y}=\log x+C$

2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$ (*AMIETE, June 2009*) **Ans.** $x^2 + y^2 = ax$

3. $x(y-x)\dfrac{dy}{dx}=y(y+x).$ **Ans.** $\dfrac{y}{x}-\log xy=a$

4. $x\,(x - y)\,dy + y^2\,dx = 0$ (*U.P. B. Pharm (C.O.) 2005*) **Ans.** $y = x \log C y$

5. $\dfrac{dy}{dx}+\dfrac{x-2y}{2x-y}=0$ **Ans.** $y - x = C\,(x + y)^3$

6. $\dfrac{dy}{dx}=\tan\dfrac{y}{x}+\dfrac{y}{x}$ **Ans.** $\sin\dfrac{y}{x}=Cx$

7. $\dfrac{dy}{dx}=\dfrac{3xy+y^2}{3x^2}$ **Ans.** $3x + y \log x + Cy = 0$

8. $\dfrac{dy}{dx}=\dfrac{x^2-2y^2}{2xy}$ **Ans.** $4y^2 - x^2 = \dfrac{C}{x^2}$

9. $(x^2 + y^2)\,dy = xy\,dx$ **Ans.** $-\dfrac{x^2}{2y^2}+\log y=C$

10. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $\dfrac{-x^3}{3y^3}+\log y=C$

11. $(y^2 + 2xy)\,dx + (2x^2 + 3xy)\,dy = 0$ (*AMIETE, Summer 2004*) **Ans.** $xy^2\,(x + y) = C$

12. $(2xy^2 - x^3)\,dy + (y^3 - 2yx^2)\,dx = 0$ **Ans.** $y^2\,(y^2 - x^2) = Cx^{-2}$

13. $(x^3 - 3xy^2)\,dx + (y^3 - 3x^2y)\,dy = 0,\ y(0) = 1$ **Ans.** $x^4 - 6x^2y^2 + y^4 = 1$

14. $2xy^2\,dy - (x^3 + 2y^3)\,dx = 0$ **Ans.** $2y^3 = 3x^3 \log x + 3x^3 + C$

15. $x\sin\dfrac{y}{x}\,dy=\left(y\sin\dfrac{y}{x}-x\right)dx$ **Ans.** $\cos\dfrac{y}{x}=\log x+C$

16. $\left\{x\cos\dfrac{y}{x}+y\sin\dfrac{y}{x}\right\}y-\left\{y\sin\dfrac{y}{x}-x\cos\dfrac{y}{x}\right\}x\dfrac{dy}{dx}=0$ **Ans.** $xy\cos\dfrac{y}{x}=a$

17. $\dfrac{dy}{dx}=\dfrac{y+\sqrt{x^2+y^2}}{x}$ **Ans.** $y+\sqrt{x^2+y^2}=Cx^2$

18. $x\frac{dy}{dx}=y(\log y-\log x+1)$ (*AMIETE, Summer 2004*) **Ans.** $\log\frac{y}{x}=Cx$

19. $xy\log\frac{x}{y}dx+\left(y^2-x^2\log\frac{x}{y}\right)dy=0$ given that $y(1)=0$

Ans. $\frac{x^2}{2y^2}\log\frac{x}{y}-\frac{x^2}{4y^2}+\log y=1-\frac{3}{4e^2}$

20. $(1+e^{\frac{x}{y}})\,dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$ (*AMIETE, June 2009*) **Ans.** $e^{\frac{x}{y}}+\frac{x}{y}=e^{-y}+C$

11.9 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

Case I. $\frac{a}{A}+\frac{b}{B}$

The equations of the form

$$\boxed{\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}}$$

can be reduced to the homogeneous form by the substitution if $\frac{a}{A}+\frac{b}{B}$

$$x=X+h,\quad y=Y+k \qquad (h,k \text{ being constants})$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dY}{dX}$$

The given differential equation reduces to

$$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C}=\frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C}$$

Choose h, k so that
$$ah+bk+c=0$$
$$Ah+Kk+C=0$$

Then the given equation becomes homogeneous $\frac{dY}{dX}=\frac{aX+bY}{AX+BY}$

Case II. If $\frac{a}{A}=\frac{b}{B}$ then the value of h, k will not be finite.

$$\frac{a}{A}=\frac{b}{B}=\frac{1}{m} \qquad \text{(say)}$$

$$A=am,\ B=bm$$

The given equation becomes $\frac{dy}{dx}=\frac{ax+by+c}{m(ax+by)+c}$

Now put $ax+by=z$ and apply the method of variables separable.

Example 10. *Solve* : $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Solution. Put $x=X+h,\qquad y=Y+k.$

The given equation reduces to

$$\therefore \qquad \frac{dY}{dX}=\frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3} \qquad \left(\frac{1}{2}\neq\frac{2}{1}\right)$$

$$=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \qquad \text{.... (1)}$$

Now choose h and k so that $h + 2k - 3 = 0$, $2h + k - 3 = 0$
Solving these equations we get $h = k = 1$

$$\therefore \qquad \frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \qquad \dots (2)$$

Put $Y = vX$, so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

The equation (2) is transformed as

$$v + X\frac{dv}{dX} = \frac{X + 2vX}{2X + vX} = \frac{1 + 2v}{2 + v}$$

$$X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v} \qquad \Rightarrow \qquad \left(\frac{2 + v}{1 - v^2}\right)dv = \frac{dX}{X}$$

$$\Rightarrow \qquad \frac{1}{2}\frac{1}{(1 + v)}dv + \frac{3}{2}\frac{1}{1 - v}dv = \frac{dX}{X} \qquad \text{(Partial fractions)}$$

On integrating, we have

$$\frac{1}{2}\log(1 + v) - \frac{3}{2}\log(1 - v) = \log X + \log C$$

$$\Rightarrow \qquad \log\frac{1 + v}{(1 - v)^3} = \log C^2X^2 \qquad \Rightarrow \qquad \frac{1 + v}{(1 - v)^3} = C^2X^2$$

$$\frac{1 + \frac{Y}{X}}{\left(1 - \frac{Y}{X}\right)^3} = C^2X^2 \qquad \Rightarrow \qquad \frac{X + Y}{(X - Y)^3} = C^2 \quad \text{or} \quad X + Y = C^2(X - Y)^3$$

Put $X = x - 1$ and $Y = y - 1$ $\Rightarrow$ $x + y - 2 = a(x - y)^3$ **Ans.**

Example 11. *Solve* : $(x + 2y)(dx - dy) = dx + dy$

Solution. $(x + 2y)(dx - dy) = dx + dy \Rightarrow (x + 2y - 1)\,dx - (x + 2y + 1)\,dy = 0$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1} \qquad \dots(1)$$

Hence $\dfrac{a}{A} = \dfrac{b}{B}$ *i.e.*, $\left(\dfrac{1}{1} = \dfrac{2}{2}\right)$ (Case of failure)

Now put $x + 2y = z$ so that $1 + 2\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Equation (1) becomes

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{2} = \frac{z - 1}{z + 1} \qquad \Rightarrow \qquad \frac{dz}{dx} = 2\frac{(z - 1)}{z + 1} + 1 = \frac{3z - 1}{z + 1}$$

$$\Rightarrow \qquad \frac{z + 1}{3z - 1}dz = dx \qquad \Rightarrow \qquad \left(\frac{1}{3} + \frac{4}{3}\frac{1}{3z - 1}\right)dz = dx$$

On integrating, $\dfrac{z}{3} + \dfrac{4}{9}\log(3z - 1) = x + C$

$$3z + 4\log(3z - 1) = 9x + 9C$$

$$\Rightarrow \qquad 3(x + 2y) + 4\log(3x + 6y - 1) = 9x + 9C$$

$$3x - 3y + a = 2\log(3x + 6y - 1)$$ **Ans.**

EXERCISE 11.4

Solve the following differential equations :

1. $\dfrac{dy}{dx}=\dfrac{2x+9y-20}{6x+2y-10}$ **Ans.** $(2x-y)^2=C\,(x+2y-5)$
2. $\dfrac{dy}{dx}=\dfrac{y-x+1}{y+x+5}$ **Ans.** $\log[(y+3)^2+(x+2)^2]+2\tan^{-1}\dfrac{y+3}{x+2}=a$
3. $\dfrac{dy}{dx}=\dfrac{x-y-2}{x+y+6}$ **Ans.** $(y+4)^2+2\,(x+2)\,(y+4)-(x+2)^2=a^2$
4. $\dfrac{dy}{dx}=\dfrac{y+x-2}{y-x-4}$ (*AMIETE, Dec. 2009*) **Ans.** $-(y-3)^2+2(x+1)\,(y-3)+(x+1)^2=a$
5. $\dfrac{dy}{dx}=\dfrac{2x-5y+3}{2x+4y-6}$ **Ans.** $(x-4y+3)\,(2x+y-3)=a$
6. $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$ **Ans.** $2\,(2x+y)+\log\,(2x+y-1)=3x+C$
7. $(x-y-2)\,dx-(2x-2y-3)\,dy=0$ **Ans.** $\log\,(x-y-1)=x-2y+C$ (*U.P. B. Pharm (C.O.) 2005*)
8. $(6x-4y+1)\,dy-(3x-2y+1)\,dx=0$ (*A.M.I.E.T. E., Dec. 2006*) **Ans.** $4x-8y-\log\,(12x-xy+1)=c$
9. $\dfrac{dy}{dx}=-\dfrac{3y-2x+7}{7y-3x+3}$ (*A.M.I.E.T.E., Summer 2004*) **Ans.** $(x+y-1)^5\,(x-y-1)^2=1$
10. $\dfrac{dy}{dx}=\dfrac{2y-x-4}{y-3x+3}$ (*AMIETE, Dec. 2010*)

Ans. $X^2-5XY+Y^2=c\left[\dfrac{2Y+(-5+\sqrt{21})\,X}{2Y-(5+\sqrt{21})\,X}\right]^{\frac{1}{\sqrt{21}}},\quad X=x-2,\ Y=y-3$

11.10 LINEAR DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\boxed{\frac{dy}{dx}+Py=Q} \qquad \text{... (1)}$$

is called a linear differential equation, where P and Q, are functions of x (but not of y) or constants.

In such case, multiply both sides of (1) by $e^{\int P dx}$

$$e^{\int P dx}\left(\frac{dy}{dx}+Py\right)=Q\,e^{\int P dx} \qquad \text{... (2)}$$

The left hand side of (2) is

$$\frac{d}{dx}\left[y.e^{\int P dx}\right]$$

(2) becomes $$\frac{d}{dx}\left[y.e^{\int P\,dx}\right]=Q.e^{\int P\,dx}$$

Integrating both sides, we get

$$y.e^{\int P\,dx}=\int Q.e^{\int P\,dx}\,dx+C$$

This is the required solution.

Note. $e^{\int P\,dx}$ is called the integrating factor.

Solution is $$\boxed{y\times[I.F.]=\int Q\,[I.F.]\,dx+C}$$

Working Rule

Step 1. Convert the given equation to the standard form of linear differential equation

i.e. $$\frac{dy}{dx} + Py = Q$$

Step 2. Find the integrating factor i.e. I.F. = $e^{\int P dx}$

Step 3. Then the solution is $y(I.F.) = \int Q\ (I.F.) dx + C$

Example 12. *Solve*: $$(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$$ (*A.M.I.E.T.E., Summer 2002*)

Solution. $$\frac{dy}{dx} - \frac{y}{x+1} = e^x (x+1)$$

$$\text{Integrating factor} = e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

The solution is $$y.\frac{1}{x+1} = \int e^x .(x+1).\frac{1}{x+1} dx = \int e^x dx$$

$$\frac{y}{x+1} = e^x + C$$ **Ans.**

Example 13. *Solve a differential equation*

$$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x.$$ (*Nagpur University, Summer 2008*)

Solution. We have $(x^3 - x)\dfrac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x} y = \frac{x^5 - 2x^3 + x}{x^3 - x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x} y = x^2 - 1$$

$$\text{I.F.} = e^{\int -\frac{3x^2-1}{x^3-x} dx} = e^{-\log(x^3 - x)} = e^{\log(x^3-x)^{-1}} = \frac{1}{x^3 - x}$$

Its solution is

$$y(\text{I.F.}) = \int Q(I.F.)\, dx + C \quad \Rightarrow \quad y\left(\frac{1}{x^3 - x}\right) = \int \frac{x^2 - 1}{x^3 - x} dx + C$$

$$\Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{x^2 - 1}{x(x^2 - 1)} dx + C \quad \Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{1}{x} dx + C$$

$$\Rightarrow \quad \frac{y}{x^3 - x} = \log x + C \quad \Rightarrow \quad y = (x^3 - x) \log x + (x^3 - x) C$$ **Ans.**

Example 14. *Solve* $\sin x \dfrac{dy}{dx} + 2y = \tan^3\left(\dfrac{x}{2}\right)$ (*Nagpur University, Summer 2004*)

Solution. Given equation : $\sin x \dfrac{dy}{dx} + 2y = \tan^3 \dfrac{x}{2} \Rightarrow \dfrac{dy}{dx} + \dfrac{2}{\sin x} y = \dfrac{\tan^3 \frac{x}{2}}{\sin x}$

This is linear form of $\dfrac{dy}{dx} + Py = Q$

$$\therefore \quad P = \frac{2}{\sin x} \quad \text{and} \quad Q = \frac{\tan^3 \frac{x}{2}}{\sin x}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{2}{\sin x} dx} = e^{2\int \text{cosec}\, x\, dx} = e^{2 \log \tan \frac{x}{2}} = \tan^2 \frac{x}{2}$$

$\therefore$ Solution is $\quad y.(\text{I.F.}) = \int \text{I.F.}(Q\,dx) + C$

$$y\tan^2\frac{x}{2} = \int \tan^2\frac{x}{2}\cdot\frac{\tan^3\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}} + C = \frac{1}{2}\int\frac{\tan^4\frac{x}{2}}{\cos^2\frac{x}{2}}dx + C$$

$$= \frac{1}{2}\int \tan^4\frac{x}{2}.\sec^2\frac{x}{2}\,dx + C \qquad \ldots (1)$$

Putting $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$ on R.H.S. (1), we get

$$y.\tan^2\frac{x}{2} = \frac{1}{2}\int t^4(2dt) + C \quad \Rightarrow \quad y\tan^2\frac{x}{2} = \frac{t^5}{5} + C$$

$$y\tan^2\frac{x}{2} = \frac{\tan^5\frac{x}{2}}{5} + C$$ **Ans.**

EXERCISE 11.5

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$ **Ans.** $xy = \frac{x^5}{5} - \frac{3x^2}{2} + C$
2. $(2y - 3x)\,dx + x\,dy = 0$ **Ans.** $y\,x^2 = x^3 + C$
3. $\frac{dy}{dx} + y\cot x = \cos x$ **Ans.** $y\sin x = \frac{\sin^2 x}{2} + C$
4. $\frac{dy}{dx} + y\sec x = \tan x$ **Ans.** $y = \frac{C - x}{\sec x + \tan x} + 1$
5. $\cos^2 x\frac{dy}{dx} + y = \tan x$ **Ans.** $y = \tan x - 1 + Ce^{-\tan x}$
6. $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$ **Ans.** $2y = (x + a)^5 + 2C\,(x + a)^3$
7. $x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$ **Ans.** $x\,y = \sin x + C\cos x$
8. $x\log x\frac{dy}{dx} + y = 2\log x$ **Ans.** $y\log x = (\log x)^2 + C$
9. $x\frac{dy}{dx} + 2y = x^2\log x$ **Ans.** $yx^2 = \frac{x^4}{4}\log x - \frac{x^4}{16} + C$
10. $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0$ **Ans.** $r\sin^2\theta = \frac{-\sin^4\theta}{2} + C$
11. $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$ **Ans.** $y = \sin x - 1 + Ce^{-\sin x}$
12. $(1-x^2)\frac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$ **Ans.** $y = \sqrt{1-x^2} + C(1-x^2)$
13. $\sec x\frac{dy}{dx} = y + \sin x$ (*A.M.I.E.T.E., Dec 2005*) **Ans.** $y = -\sin x - 1 + ce^{\sin x}$
14. $y' + y\tan x = \cos x, y(0) = 0$ (*A.M.I.E.T.E., June 2006*) **Ans.** $y = x\cos x$
15. Solve $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$ (*AMIETE, Dec. 2009*) **Ans.** $x = -\tan^{-1} y - 1 + ce^{\tan^{-1} y}$
16. Find the value of α so that e^2 is an integrating factor of differential equation $x(1 - y)\,dx - dy = 0$. (*A.M.I.E.T.E., Summer 2005*) **Ans.** $\alpha = \frac{1}{2}$

17. Slove the differential equation $\cot 3x \dfrac{dy}{dx} - 3y = \cos 3x + \sin 3x,\ 0 < x < \dfrac{\pi}{2}$.

(*AMIETE, Dec. 2009*) **Ans.** $y \cos 3x = \dfrac{1}{12}\left[6x - \sin 6x - \cos 6x\right]$

18. The value of α so that $e^{\alpha y^2}$ is an integrating factor of the differential equation

$(e^{\frac{-y^2}{2}} - xy)\, dy - dx = 0$ is (*A.M.I.E.T.E. Dec., 2005*)

(*a*) -1 (*b*) 1 (*c*) $\dfrac{1}{2}$ (*d*) $-\dfrac{1}{2}$ **Ans.** (c)

19. The solution of the differential equation $(y + x)^2 \dfrac{dy}{dx} = a^2$ is given by

(*a*) $y + x = a \tan\left(\dfrac{y-c}{a}\right)$ (*b*) $y - x = \tan\left(\dfrac{y-c}{a}\right)$

(*c*) $y - x = a \tan(y - c)$ (*d*) $a(y - x) = \tan\left(y - \dfrac{c}{a}\right)$ **Ans.** (*a*)

(*AMIETE, June 2010*)

11.11 EQUATIONS REDUCIBLE TO THE LINEAR FORM (BERNOULLI EQUATION)

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad \ldots(1)$$

where **P** and **Q** are constants or functions of x can be reduced to the linear form on dividing by y^n and substituting $\dfrac{1}{y^{n-1}} = z$

On dividing bothsides of (1) by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}} P = Q \qquad \ldots(2)$$

Put $\dfrac{1}{y^{n-1}} = z$, so that $\dfrac{(1-n)}{y^n}\dfrac{dy}{dx} = \dfrac{dz}{dx} \Rightarrow \dfrac{1}{y^n}\dfrac{dy}{dx} = \dfrac{dz}{1-n}$

$\therefore$ (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$ or $\dfrac{dz}{dx} + P(1-n)z = Q(1-n)$

which is a linear equation and can be solved easily by the previous method discussed in article 11.10 on page 217.

Example 15. *Solve* $x^2 dy + y(x + y)\, dx = 0$ (*U.P. II Semester Summer 2006*)

Solution. We have, $x^2\, dy + y\,(x + y)\, dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

Put $-\dfrac{1}{y} = z$ so that $\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

The given equation reduces to a linear differential equation in z.

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

$$\textbf{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log 1/x} = \frac{1}{x}.$$

Hence the solution is

$$z.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\,dx + C \qquad \Rightarrow \qquad \frac{z}{x} = \int -x^{-3}\,dx + C$$

$$\Rightarrow \qquad -\frac{1}{xy} = -\frac{x^{-2}}{-2} + C \qquad \Rightarrow \qquad \frac{1}{xy} = -\frac{1}{2x^2} - C \qquad \textbf{Ans.}$$

Example 16. *Solve*: $x\dfrac{dy}{dx} + y\log y = xy\ e^x$ (*A.M.I.E., Summer 2000*)

Solution. $x\dfrac{dy}{dx} + y\log y = xye^x$

Dividing by xy, we get

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x \qquad ...(1)$$

Put $\log y = z$, so that $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Equation (1) becomes, $\dfrac{dz}{dx} + \dfrac{z}{x} = e^x$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Solution is $\qquad zx = \displaystyle\int xe^x\,dx + C$

$$zx = xe^x - e^x + C$$

$$\Rightarrow \qquad x\log y = xe^x - e^x + C \qquad \textbf{Ans.}$$

Example 17. *Solve*: $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y.$ (*Nagpur University, Summer 2000*)

Solution. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$

$$\Rightarrow \qquad \cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x \qquad ...(1)$$

Put $\sin y = z$, so that $\cos y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

(1) becomes $\qquad \dfrac{dz}{dx} - \dfrac{z}{1+x} = (1+x)e^x$

$$I.F. = e^{-\int \frac{1}{1+x}dx} = e^{-\log(1+x)} = e^{\log \frac{1}{1+x}} = \frac{1}{1+x}$$

Solution is $\qquad z.\dfrac{1}{1+x} = \displaystyle\int (1+x)e^x.\frac{1}{1+x}\,dx + C = \int e^x dx + C$

$$\frac{\sin y}{1+x} = e^x + C \qquad \textbf{Ans.}$$

Example 18. *Solve*: $\tan y\dfrac{dy}{dx} + \tan x = \cos y\cos^2 x$ (*Nagpur University, Summer 2000*)

Solution. $\tan y\dfrac{dy}{dx} + \tan x = \cos y\cos^2 x$

$$\sec y\tan y\frac{dy}{dx} + \sec y\ \tan x = \cos^2 x$$

Writing $z = \sec y$, so that $\dfrac{dz}{dx} = \sec y \tan y \dfrac{dy}{dx}$

The equation becomes $\dfrac{dz}{dx} + z\tan x = \cos^2 x$

$$\text{I.F.} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution of the equation is

$$z\sec x = \int \cos^2 x \sec x\,dx + C$$

$$\sec y \sec x = \int \cos x\,dx + C = \sin x + C$$

$$\sec y = (\sin x + C)\cos x \qquad \textbf{Ans.}$$

Example 19. *Solve differential equation*

$$\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$$

(*Nagpur University, Summer 2000*)

Solution. We have, $\dfrac{dy}{dx} + \dfrac{1}{x}\tan y = \dfrac{1}{x^2}\tan y\sin y \quad \Rightarrow \quad \dfrac{1}{\tan y \sin y}\dfrac{dy}{dx} + \dfrac{1}{x}\dfrac{1}{\sin y} = \dfrac{1}{x^2}$

$$\Rightarrow \quad \cot y \operatorname{cosec} y \frac{dy}{dx} + \frac{1}{x}\operatorname{cosec} y = \frac{1}{x^2} \qquad \ldots (1)$$

Putting $\operatorname{cosec} y = z$, and $-\operatorname{cosec} y \cot y \dfrac{dy}{dx} = \dfrac{dz}{dx}$ in (1), we get

$$-\frac{dz}{dx} + \frac{1}{x}z = \frac{1}{x^2}$$

$$\frac{dz}{dx} - \frac{1}{x}z = -\frac{1}{x^2}$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Its solution is $z\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$

$$z \cdot \frac{1}{x} = \int\left(\frac{-1}{x^2}\right)\frac{1}{x}dx + C \;= -\int \frac{1}{x^3}dx + C = -\frac{x^{-2}}{-2} + C$$

$$z = \frac{1}{2x^2} + Cx \quad \Rightarrow \quad \operatorname{cosec} y = \frac{1}{2x^2} + Cx \qquad \textbf{Ans.}$$

Example 20. $x\left[\dfrac{dx}{dy} + y\right] = 1 - y$ (*Nagpur University, Summer 2004*)

Solution. $x\left(\dfrac{dy}{dx} + y\right) = (1 - y)$

$$\Rightarrow \quad \frac{dy}{dx} + y = \frac{1}{x} - \frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}$$

which is in linear form of $\dfrac{dy}{dx} + Py = Q$.

$$\therefore \quad P = \left(1 + \frac{1}{x}\right), \qquad Q = \frac{1}{x}$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x + \log x} = e^x . e^{\log x} = e^x . x = xe^x$$

Its solution is

$$y(\text{I.F.}) = \int \text{I.F.}(Q\,dx) + C$$

$$y(x.e^x) = \int (x.e^x) \times \frac{1}{x}\,dx + C \quad \Rightarrow \quad y(x.e^x) = \int e^x\,dx + C$$

$$y\,(x\,.\,e^x) = e^x + C$$

$\therefore$ $$y = \frac{1}{x} + \frac{C}{x}e^{-x}$$ **Ans.**

Example 21. *Solve the differential equation.*

$$y \log y\,dx + (x - \log y)\,dy = 0$$ (*Uttarakhand II Semester, June 2007*)

Solution. We have,

$$y \log y\,dx + (x - \log y)\,dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-x + \log y}{y \log y} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{-x}{y \log y} + \frac{\log y}{y \log y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log(\log y)} = \log y$$

Its solution is $$x.\log y = \int \frac{1}{y}(\log y)\,dy$$

$$x.\log y = \frac{(\log y)^2}{2} + C$$ **Ans.**

Example 22. $$y e^y\,dx = (y^3 + 2x e^y)\,dy$$ (*Nagpur University, Winter 2003*)

Solution. $$y e^y\,dx = (y^3 + 2x e^y)\,dy \quad \Rightarrow \quad \frac{dx}{dy} - \frac{2x}{y} = \frac{y^2}{e^y}$$

which is linear in x

$\therefore$ $$P = \frac{-2}{y} \quad \text{and} \quad Q = \frac{y^2}{e^y}$$

$$\text{I.F.} = e^{\int P\,dy} = e^{\int \frac{-2}{y} dy} = e^{-2 \log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Its solution is $$x(\text{I.F.}) = \int Q.(\text{I.F.}\,dy) + C \quad \Rightarrow \quad x.\frac{1}{y^2} = \int \frac{y^2}{e^y} \times \frac{1}{y^2}\,dy + C$$

$$\frac{x}{y^2} = \int e^{-y}.dy + C \quad \Rightarrow \quad \frac{x}{y^2} = -e^{-y} + C$$

$\therefore$ $$\frac{x}{y^2} + e^{-y} = C$$ **Ans.**

Example 23. *Solve* $\dfrac{dy}{dx} = \dfrac{y}{2y \log y + y - x}$ (*Nagpur University, Summer 2003*)

Solution. $$\frac{dx}{dy} = \frac{2y \log y + y - x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y}x = 1 + 2 \log y$$

Which is of the form $\frac{dx}{dy}+Px=Q$

Here $\quad P=\frac{1}{y}$ and $Q = 1 + 2\log y$

$$\text{I.F.}=e^{\int P\,dy}=e^{\int \frac{1}{y}dy}=e^{\log y}=y$$

Its solution is

$$x.(I.F.)=\int Q(I.F.)\,dy+C \quad\Rightarrow\quad xy=\int (1+2\log y)y\,dy+C$$

$$\Rightarrow \quad xy=\int (y+2y\log y)\,dy+C \;=\frac{y^2}{2}+2\left[\log y.\frac{y^2}{2}-\int \frac{1}{y}.\frac{y^2}{2}dy\right]+C$$

$$=\frac{y^2}{2}+2\left[\frac{y^2}{2}\log y-\frac{1}{4}y^2\right]+C \;= y^2\log y + C$$

$$\Rightarrow \quad x = y\log y + \frac{C}{y}$$ **Ans.**

Example 24. *Solve* : $\frac{dy}{dx}=\frac{y+1}{(y+2)\,e^y-x}$ (*Nagpur University, Winter 2004*)

Solution. $\frac{dx}{dy}=\frac{(y+2)\,e^y}{y+1}-\frac{x}{y+1}$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{x}{y+1}=\frac{(y+2)\,e^y}{y+1}$$

which is linear in x

Here $\quad P=\frac{1}{y+1}$ and $Q=\frac{y+2}{y+1}e^y$

$$\text{I.F.}=e^{\int P\,dy}=e^{\int \frac{1}{y+1}dy}=e^{\log(y+1)}=y+1$$

Its solution is

$$x(\text{I.F.})=\int Q(\text{I.F.}\,dy)+C$$

$$\Rightarrow \quad x(y+1)=\int \frac{y+2}{y+1}e^y(y+1)\,dy+C$$

$$\Rightarrow \quad x(y+1)=\int (y+2)e^y\,dy+C \;=(y+2)e^y-\int \frac{d}{dy}(y+2).e^y\,dy+C$$

$$\Rightarrow \quad x\,(y+1)=(y+2)e^y-\int e^y\,dy+C$$

$$\Rightarrow \quad x\,(y+1) = (y+2)\,e^y - e^y + C$$

$$\Rightarrow \quad x\,(y+1) = (y+1)\,e^y + C \quad\Rightarrow\quad x=e^y+\frac{C}{y+1}$$ **Ans.**

Example 25. *Solve*: $(1 + y^2)\,dx = (\tan^{-1}y - x)\,dy$.

(*AMIETE, June 2010, 2004, R.G.P.V., Bhopal, April 2010, June 2008, U.P. (B. Pharm) 2005*)

Solution. $(1 + y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\frac{dx}{dy}=\frac{\tan^{-1}y-x}{1+y^2} \quad\Rightarrow\quad \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

This is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Its solution is $\quad x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \dfrac{\tan^{-1} y}{1+y^2} dy + C$

Put $\quad \tan^{-1} y = t$ on R.H.S., so that $\dfrac{1}{1+y^2} dy = dt$

$$x.e^{\tan^{-1} y} = \int e^t . t\, dt + C = t.e^t - e^t + C = e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) + C$$

$$x = \left(\tan^{-1} y - 1\right) + Ce^{-\tan^{-1} y}$$ **Ans.**

Example 26. *Solve* : $r \sin\theta - \dfrac{dr}{d\theta}\cos\theta = r^2$ (*Nagpur University, Summer 2005*)

Solution. The given equation can be written as $-\dfrac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$ ... (1)

Dividing (1) by $r^2 \cos\theta,$ we get $-r^{-2}\dfrac{dr}{d\theta} + r^{-1}\tan\theta = \sec\theta$ ... (2)

Putting $\quad r^{-1} = v$ so that $-r^{-2}\dfrac{dr}{d\theta} = \dfrac{dv}{d\theta}$ in (2), we get

$$\frac{dv}{d\theta} + v\tan\theta = \sec\theta$$

$$\text{I.F.} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

Solution is $\quad v\sec\theta = \int \sec\theta,\ \sec\theta + C \quad \Rightarrow \quad v\sec\theta = \int \sec^2\theta\, d\theta + C$

$$\frac{\sec\theta}{r} = \tan\theta + C \quad \Rightarrow \quad r^{-1} = (\sin\theta + C\cos\theta)$$

$\therefore$ $$r = \frac{1}{\sin\theta + C\cos\theta}$$ **Ans.**

EXERCISE 11.6

Solve the following differential equations:

1. $\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y} = 2xe^{-x}$ **Ans.** $e^x + x^2 y + Cy = 0$

2. $3\dfrac{dy}{dx} + 3\dfrac{y}{x} = 2x^4 y^4$ **Ans.** $\dfrac{1}{y^3} = x^5 + Cx^3$

3. $\dfrac{dy}{dx} = y\tan x - y^2 \sec x$ **Ans.** $\sec x = (\tan x + C)\, y$

4. $\dfrac{dy}{dx} = 2y\tan x + y^2\tan^2 x$, if $y = 1$ at $x = 0$ **Ans.** $\dfrac{1}{y}\sec^2 x = -\dfrac{\tan^3 x}{3} + 1$

5. $\dfrac{dy}{dx} + \tan x \tan y = \cos x \sec y$ **Ans.** $\sin y \sec x = x + C$

6. $dy + y\tan x \,.\, dx = y^2 \sec x \,.\, dx$ **Ans.** $y\,(x + C) + \cos x = 0$

7. $(x^2 y^2 + xy)\, y\, dx + (x^2 y^2 - 1)\, x\, dy = 0$ **Ans.** $x\, y = \log C\, y$

8. $(x^2 + y^2 + x)\, dx + xy\, dy = 0$ **Ans.** $x^2 y^2 = -\dfrac{x^4}{2} - \dfrac{2x^3}{3} + C$

9. $\dfrac{dy}{dx}+y=3e^x y^3$ **Ans.** $\dfrac{1}{y^2}=6e^x+Ce^{2x}$

10. $(x-y^2)\,dx+2\,x\,y\,dy=0$ **Ans.** $\dfrac{y^2}{x}+\log x=C$

11. $e^y\left(\dfrac{dy}{dx}+1\right)=e^x$ **Ans.** $e^{x+y}=\dfrac{e^{2x}}{2}+C$

12. $x^2y-x^3\dfrac{dy}{dx}=y^4\cos x$ **Ans.** $x^3=y^3(3\sin x-C)$

13. $3\dfrac{dy}{dx}+\dfrac{2}{x+1}.y=\dfrac{x^2}{y^2}$ **Ans.** $y^3(x+1)^2=\dfrac{x^5}{5}+\dfrac{x^4}{2}+\dfrac{x^3}{3}+C$

14. $\cos x\dfrac{dy}{dx}+4y\sin x=4\sqrt{y}\sec x$ **Ans.** $\sqrt{y}\sec^2 x=2\left[\tan x+\dfrac{\tan^3 x}{3}\right]+C$

15. $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$ **Ans.** $\tan y=\dfrac{1}{2}(x^2-1)+Ce^{-x^2}$

16. $\dfrac{1}{1+y^2}\dfrac{dy}{dx}+2x\tan^{-1}y=x^3$ **Ans.** $\tan^{-1}y=\dfrac{1}{2}(x^2-1)+Ce^{-x^2}$

17. $e^{-y}\sec^2 y\,dy=dx+x\,dy$ **Ans.** $x\,e^y=\tan y+C$

18. $(x+y+1)\dfrac{dy}{dx}=1$ **Ans.** $x+y+2=C\,e^y$

19. $\dfrac{dy}{dx}=\dfrac{y^3}{e^{2x}+y^2}$ **Ans.** $e^{-2x}y^2+2\log y+C=0$

20. $dx-xy\,(1+xy^2)\,dy=0$ **Ans.** $-\dfrac{1}{x}=y^2-2+Ce^{-y^2/2}$

21. $\dfrac{dy}{dx}+\dfrac{y}{x}\log y=\dfrac{y}{x^2}(\log y)^2$ (*A.M.I.E.T.E., Summer 2004, 2003, Winter 2003, 2001*)

Ans. $\dfrac{1}{x\log y}=\dfrac{1}{2x^2}+C$

22. $3\dfrac{dy}{dx}+xy=xy^{-2}$ (*A.M.I.E.T.E., June 2009*) **Ans.** $y^3=1+Ce^{-x^2/2}$

27. $x\dfrac{dy}{dx}+y=x^3y^6$ (*AMIETE, June 2010*) **Ans.** $\dfrac{1}{y^5x^5}=\dfrac{5}{2x^2}+C$

23. General solution of linear differential equation of first order $\dfrac{dx}{dy}+Px=Q$ (where P and Q are constants or functions of y) is

(*a*) $ye^{\int P.dx}=\int Q\,e^{\int P.dx}\,dx+c$ (*b*) $xe^{\int P.dy}=\int Q\,e^{\int P.dy}\,dy+c$

(*c*) $y=\int Q\,e^{\int P.dx}\,dx+c$ (*d*) $x=\int Q\,e^{\int P.dy}\,dy+c$ (*AMIETE, June, 2010*) **Ans.** (*b*)

11.12 EXACT DIFFERENTIAL EQUATION

An exact differential equation is formed by directly differentiating its primitive (solution) without any other process

$$Mdx+Ndy=0$$

is said to be an exact differential equation if it satisfies the following condition

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

where $\frac{\partial M}{\partial y}$ denotes the differential co-efficient of M with respect to y keeping x constant and $\frac{\partial N}{\partial x}$, the differential co-efficient of N with respect to x, keeping y constant.

Method for Solving Exact Differential Equations

Step I. Integrate M w.r.t. x keeping y constant

Step II. Integrate w.r.t. y, only those terms of N which do not contain x.

Step III. Result of I + Result of II = Constant.

Example 27. *Solve* :

$$(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$$

Solution. Here, $M = 5x^4 + 3x^2y^2 - 2xy^3$, $\quad N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now $\int M\,dx + \int (\text{terms of } N \text{ is not containing } x)\,dy = C$ (y constant)

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right)dx + \int -5y^4 dy = C$$

$$\Rightarrow \quad x^5 + x^3y^2 - x^2y^3 - y^5 = C$$ **Ans.**

Example 28. *Solve*: $\left\{2xy\cos x^2 - 2xy + 1\right\}dx + \left\{\sin x^2 - x^2 + 3\right\}dy = 0$

(*Nagpur University, Summer 2000*)

Solution. Here we have

$$\left\{2xy\cos x^2 - 2xy + 1\right\}dx + \left\{\sin x^2 - x^2 + 3\right\}dy = 0 \quad \text{... (1)}$$

$$M\,dx + N\,dy = 0 \quad \text{... (2)}$$

Comparing (1) and (2), we get

$$M = 2xy\cos x^2 - 2xy + 1 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$$

$$N = \sin x^2 - x^2 + 3 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$$

Here, $\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So the given differential equation is exact differential equation.

Hence solution is $\int\limits_{y \text{ as const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 3\,dy = C$$

$$\Rightarrow \quad \int [y(2x\cos x^2) - y(2x) + 1]\,dx + 3\int dy = C$$

$$\Rightarrow \quad y\int 2x\cos x^2 dx - y\int 2x\,dx + \int 1\,dx + 3\int y\,dy = C$$

Put $x^2 = t$ so that $2x\,dx = dt$

$$y\int \cos t\,dt - 2y\frac{x^2}{2} + x + 3y = C$$

$$\Rightarrow \quad y\sin t - x^2y + x + 3y = C$$

$$y\sin x^2 - yx^2 + x + 3y = C$$ **Ans.**

Example 29. *Solve* :

$$(1+e^{x/y})+e^{x/y}\left(1-\frac{x}{y}\right)\frac{dy}{dx}=0$$

(*Nagpur University, Summer 2008, A.M.I.E.T.E. June, 2009*)

Solution. We have,

$$\left(1+e^{\frac{x}{y}}\right)+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\frac{dy}{dx}=0 \quad\Rightarrow\quad \left(1+e^{\frac{x}{y}}\right)dx+\left(e^{\frac{x}{y}}-e^{\frac{x}{y}}\frac{x}{y}\right)dy=0$$

$$M=1+e^{\frac{x}{y}} \quad\Rightarrow\quad \frac{\partial M}{\partial y}=-\frac{x}{y^2}e^{\frac{x}{y}}$$

$$N=e^{\frac{x}{y}}-e^{\frac{x}{y}}\frac{x}{y} \quad\Rightarrow\quad \frac{\partial N}{\partial x}=\frac{1}{y}e^{\frac{x}{y}}-\frac{1}{y}e^{\frac{x}{y}}-\frac{x}{y^2}e^{\frac{x}{y}}=-\frac{x}{y^2}e^{\frac{x}{y}}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$\therefore$ Given equation is exact.

Its solution is $\int\left(1+e^{\frac{x}{y}}\right)dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$

$$\Rightarrow \quad \int\left(1+e^{\frac{x}{y}}\right)dx+\int 0\,dy=C \quad\Rightarrow\quad x+ye^{\frac{x}{y}}=C$$ **Ans.**

Example 30. *Solve* : $x\,dx+y\,dy=\dfrac{a^2(x\,dy-y\,dx)}{x^2+y^2}$ (*U.P. Second Semester Summer 2005*)

Solution. We have, $x\,dx+y\,dy=\dfrac{a^2(x\,dy-y\,dx)}{x^2+y^2}$

$$\Rightarrow \quad \left(x+\frac{a^2y}{x^2+y^2}\right)dx+\left(y-\frac{(a^2x)}{x^2+y^2}\right)dy=0$$

Here, $M=x+\dfrac{a^2y}{x^2+y^2}, \quad N=y-\dfrac{a^2x}{x^2+y^2}$

Now, $\dfrac{\partial M}{\partial y}=\dfrac{a^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}, \quad \dfrac{\partial N}{\partial x}=\dfrac{a^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$

Since, $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

Therefore equation is exact. Hence,

$$\int\left(x+\frac{a^2y}{x^2+y^2}\right)dx+\int y\,dy=C$$

$$\Rightarrow \quad \frac{x^2}{2}+a^2y.\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)+\frac{y^2}{2}=C$$

$$\Rightarrow \quad \left(\frac{x^2+y^2}{2}\right)+a^2\tan^{-1}\left(\frac{x}{y}\right)=C$$ **Ans.**

Example 31. *Solve*: $[1+\log(x\,y)]\,dx+\left[1+\frac{x}{y}\right]dy=0$ (*Nagpur University, Winter 2003*)

Solution. $[1+\log x\,y]\,dx+\left[1+\frac{x}{y}\right]dy=0$

$\therefore \quad [1+\log x+\log y]\,dx+\left[1+\frac{x}{y}\right]dy=0$

which is in the form

$$M\,dx+N\,dy=0$$

$$M=[1+\log x+\log y] \quad \text{and} \quad N=1+\frac{x}{y}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y}=\frac{1}{y} \quad \text{and} \quad \Rightarrow \quad \frac{\partial N}{\partial x}=\frac{1}{y}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence the given differential equation is exact.

$\therefore$ Solution is $\quad \int_{y\text{ constant}} M\,dx+\int N\,(\text{terms not containing } x)\,dy=C$

y constant

$$\therefore \quad \int(1+\log x+\log y)\,dx+\int dy=C$$

$$\Rightarrow \quad x+\int \log x\,dx+\int \log y\,dx+y=C \qquad \dots (1)$$

Now, $\int \log x\,dx=\int \log x.(1)\,dx=(\log x)x-\int\left[\frac{d}{dx}(\log x)x\right]dx=x\log x-\int \frac{1}{x}.x\,dx$

$$=x\log x-\int dx=x\log x-x=x[\log x-1]$$

$\therefore$ Equation (1) becomes $\Rightarrow \quad x+x\log x-x+x\log y+y=C$

$$x\,[\log x+\log y]+y=C \Rightarrow x\log xy+y=C \qquad \textbf{Ans.}$$

Example 32. *Find the value of* λ, *for the differential equation*

$$(xy^2+\lambda x^2y)dx+(x+y)x^2dy=0 \text{ is exact}$$

Solve the equation for this value of λ.

(*Uttarakhad, II Summer 2010, Nagpur University, Summer 2002*)

Solution. Here $\quad (xy^2+\lambda x^2y)\,dx+(x+y)x^2dy=0 \qquad \dots (1)$

which is of the form $M\,dx+N\,dx=0$

Where $M=xy^2+\lambda\,x^2y$ and $N=(x+y)\,x^2=x^3+x^2y$

$$\frac{\partial M}{\partial y}=2xy+\lambda x^2, \qquad \frac{\partial N}{\partial x}=3x^2+2xy$$

Condition of to be exact is

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$2xy+\lambda x^2=3x^2+2xy$$

$$\Rightarrow \qquad \lambda x^2 = 3x^2 \qquad \Rightarrow \qquad \lambda = 3$$

If $\lambda = 3$ then (1) becomes an excat differential equation.

$$(xy^2 + 3x^2y)\, dx + (x + y)\, x^2\, dy = 0$$

Its solution is given by

$$\underset{(y\,=\,\text{constant})}{\int M\, dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = \text{constant}$$

$$\Rightarrow \qquad \int (xy^2 + 3x^2y)dx + \int 0\, dy = C$$

$$\Rightarrow \qquad \frac{x^2y^2}{2} + \frac{3x^3y}{3} = C \qquad \Rightarrow \qquad \frac{x^2y^2}{2} + x^3y = C$$

$$x^2y^2 + 2x^3y = C_1$$ **Ans.**

EXERCISE 11.7

Solve the following differential equation (1 – 12).

1. $(x + y - 10)\, dx + (x - y - 2)\, dy = 0$ **Ans.** $\dfrac{x^2}{2} + xy - 10x - \dfrac{y^2}{2} - 2y = C$
2. $(y^2 - x^2)\, dx + 2x\, y\, dy = 0$ **Ans.** $\dfrac{x^3}{3} = x\,y^2 + C$
3. $\left(1 + 3e^{x/y}\right)dx + 3e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$ (*R.G.P.V. Bhopal, Winter 2010*) **Ans.** $x + 3y\, e^{x/y} = C$
4. $(2x - y)\, dx = (x - y)\, dy$ **Ans.** $xy = x^2 + \dfrac{y^2}{2} + C$
5. $(y \sec^2 x + \sec x \tan x)\, dx + (\tan x + 2y)\, dy = 0$ **Ans.** $y \tan x + \sec x + y^2 = C$
6. $(ax + hy + g)\, dx + (hx + by + f)\, dy = 0$ **Ans.** $ax^2 + 2h\, xy + by^2 + 2gx + 2fy + C = 0$
7. $(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0$ **Ans.** $\dfrac{x^5}{5} - x^2y^2 + xy^4 + \cos y = C$
8. $(2xy + e^y)\, dx + (x^2 + xe^y)\, dy = 0$ **Ans.** $x^2y + xe^y = C$
9. $(x^2 + 2ye^{2x})\, dy + (2xy + 2y^2e^{2x})\, dx = 0$ **Ans.** $x^2y + y^2\, e^{2x} = C$
10. $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\, \sin y)\, dy = 0$ (*M.D.U., 2010*)
    Ans. $y\,(x + \log x) + x \cos y = C$
11. $(x^3 - 3xy^2)\, dx + (y^3 - 3x^2y)\, dy = 0,\ y(0) = 1$ **Ans.** $x^4 - 6x^2y^2 + y^4 = 1$
12. The differential equation $M\,(x, y)\, dx + N\,(x, y)\, dy = 0$ is an exact differential equation if
    (*a*) $\dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial x} = 0$ (*b*) $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 0$ (*c*) $\dfrac{\partial M}{\partial y} \times \dfrac{\partial N}{\partial x} = 1$ (*d*) None of the above
    (*A.M.I.E.T.E. Dec. 2010, Dec 2006*) **Ans.** (*b*)

11.13 EQUATIONS REDUCIBLE TO THE EXACT EQUATIONS

Sometimes a differential equation which is not exact may become so, on multiplication by a suitable function known as the integrating factor.

Rule 1. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f\,(x)$, then $\text{I.F.} = e^{\int f(x)dx}$

Example 33. *Solve* $(2x \log x - xy)\, dy + 2y\, dx = 0$

Solution. $M = 2y, \qquad N = 2x \log x - xy$ ... (1)

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2(1+\log x) - y$$

Here, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-2-2\log x + y}{2x\log x - xy} = \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x} = f(x)$$

$$\text{I.F.} = e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

On multiplying the given differential equation (1) by $\frac{1}{x}$, we get

$$\frac{2y}{x}dx + (2\log x - y)dy = 0 \quad \Rightarrow \quad \int \frac{2y}{x}dx + \int -y\, dy = c$$

$$\Rightarrow \qquad 2y\log x - \frac{1}{2}y^2 = c$$ **Ans.**

EXERCISE 11.8

Solve the following differential equations:

1. $(y \log y)\, dx + (x - \log y)\, dy = 0$ **Ans.** $2x \log y = c + (\log y)^2$
2. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(1+y^2\right)x\, dy = 0$ **Ans.** $\frac{yx^4}{4} + \frac{y^3x^4}{12} + \frac{x^6}{12} = c$
3. $(y - 2x^3)\, dx - x\,(1 - xy)\, dy = 0$ **Ans.** $-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$
4. $(x \sec^2 y - x^2 \cos y)\, dy = (\tan y - 3x^4)\, dx$ **Ans.** $-\frac{1}{x}\tan y - x^3 + \sin y = c$
5. $(x - y^2)\, dx + 2xy\, dy = 0$ **Ans.** $y^2 = cx - x \log x$

Rule II. If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y alone, say $f(y)$, then

$$\text{I.F.} = e^{\int f(y)dy}$$

Example 34. *Solve* $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

Solution. Here $M = y^4 + 2y; \qquad N = xy^3 + 2y^4 - 4x$ ...(1)

$$\therefore \qquad \frac{\partial M}{\partial y} = 4y^3 + 2; \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \qquad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(y^3-4)-(4y^3+2)}{y^4+2y} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y} = f(y)$$

$$\text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

On multiplying the given equation (1) by $\frac{1}{y^3}$ we get the exact differential equation.

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=c \quad \Rightarrow \quad x\left(y+\frac{2}{y^2}\right)+y^2=c \qquad \textbf{Ans.}$$

EXERCISE 11.9

Solve the following differential equations:

1. $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$ **Ans.** $x^3y^2+\dfrac{x^2}{y}=c$

2. $(xy^3+y)\,dx+2(x^2y^2+x+y^4)\,dy=0$ **Ans.** $\dfrac{x^2y^4}{2}+xy^2+\dfrac{y^6}{3}=c$

3. $y(x^2y+e^x)dx-e^xdy=0$ **Ans.** $\dfrac{x^3}{3}+\dfrac{e^x}{y}=c$

4. $(2x^4y^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0$ **Ans.** $x^2e^y+\dfrac{x^2}{y}+\dfrac{x}{y^3}=c$

Rule III. If M is of the form $M=y\,f_1(xy)$ and N is of the form $N=x\,f_2(xy)$

Then $$\text{I.F.}=\frac{1}{M.x-N.y}$$

Example 35. *Solve* $y\,(xy+2x^2y^2)\,dx+x\,(xy-x^2y^2)\,dy=0$

Solution. $$y\,(xy+2x^2y^2)\,dx+x\,(xy-x^2y^2)\,dy=0 \qquad \dots(1)$$

Dividing (1) by xy, we get

$$y\,(1+2xy)\,dx+x\,(1-xy)\,dy=0 \qquad \dots(2)$$

$$M=y\,f_1(xy), \quad N=x\,f_2(xy)$$

$$\text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{xy(1+2xy)-xy(1-xy)}=\frac{1}{3x^2y^2}$$

On multiplying (2) by $\dfrac{1}{3x^2y^2}$, we have an exact differential equation

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0 \quad \Rightarrow \quad \int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int-\frac{1}{3y}dy=c$$

$$\Rightarrow \quad -\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c \quad \Rightarrow \quad -\frac{1}{xy}+2\log x-\log y=b \qquad \textbf{Ans.}$$

EXERCISE 11.10

Solve the following differential equations

1. $(y-xy^2)\,dx-(x+x^2y)\,dy=0$ **Ans.** $\log\left(\dfrac{x}{y}\right)-xy=A$

2. $y\,(1+xy)\,dx+x(1-xy)\,dy=0$ **Ans.** $xy\log\left(\dfrac{y}{x}\right)=c\,xy-1$

3. $y\,(1+xy)\,dx+x\,(1+xy+x^2y^2)\,dy=0$ **Ans.** $\dfrac{1}{2x^2y^2}+\dfrac{1}{xy}-\log y=c$

4. $(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0$ **Ans.** $y \cos xy = cx$

Rule IV. For of this type of $x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a'\,y\,dx + b'\,x\,dy) = 0,$ the integrating factor is $x^h y^k$.

where $$\frac{m+h+1}{a} = \frac{n+k+1}{b}, \quad \text{and} \quad \frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

Example 36. *Solve* $(y^3 - 2x^2y)\, dx + (2xy^2 - x^3)\, dy = 0$

Solution. $(y^3 - 2x^2y)\, dx + (2xy^2 - x^3)\, dy = 0$

$$y^2\,(ydx + 2xdy) + x^2\,(-2ydx - xdy) = 0$$

Here $m = 0,\ h = 2,\ a = 1,\ b = 2, \quad m' = 2,\ n' = 0,\ a' = -2,\ b' = -1$

$$\frac{0+h+1}{1} = \frac{2+k+1}{2} \quad \text{and} \quad \frac{2+h+1}{-2} = \frac{0+k+1}{-1}$$

$\Rightarrow$ $2h + 2 = 2 + k + 1$ and $h + 3 = 2k + 2$

$\Rightarrow$ $2h - k = 1$ and $h - 2k = -1$

On solving $h = k = 1$. Integrating Factor $= xy$

Multiplying the given equation by xy, we get

$$(xy^4 - 2x^3y^2)\, dx + (2x^2y^3 - x^4y)\, dy = 0$$

which is an exact differential equation.

$$\int (xy^4 - 2x^3y^2)dx = C \quad \Rightarrow \quad \frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$$

$\Rightarrow$ $x^2y^4 - x^4y^2 = C' \quad \Rightarrow \quad x^2\,y^2\,(y^2 - x^2) = C'$ **Ans.**

Example 37. Solve $(3y - 2xy^3)\, dx + (4x - 3x^2y^2)\, dy = 0$. (*U.P., II Semester, June 2007*)

Solution. $(3y - 2xy^3)\, dx + (4x - 3x^2y^2)\, dy = 0$

$\Rightarrow$ $(3y\, dx + 4x\, dy) + xy^2(-2y\, dx - 3x\, dy) = 0$ ...(1)

Comparing the coefficients of (1) with

$x^m y^n (a\,y\,dx + b\,x\,dy) + x^{m'} y^{n'} (a'\,y\,dx + b'\,x\,dy) = 0,$ we get

$m = 0,\ n = 0,\ a = 3,\ b = 4$

$m' = 1,\ n' = 2,\ a' = -2,\ b' = -3$

To find the integrating factor $x^h y^k$

$$\frac{m+h+1}{a} = \frac{n+k+1}{b} \quad \text{and} \quad \frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

$$\frac{0+h+1}{3} = \frac{0+k+1}{4} \quad \text{and} \quad \frac{1+h+1}{-2} = \frac{2+k+1}{-3}$$

$\Rightarrow$ $\dfrac{h+1}{3} = \dfrac{k+1}{4}$ and $\dfrac{h+2}{2} = \dfrac{k+3}{3} \quad \Rightarrow \quad 4h - 3k + 1 = 0$ ... (2)

and $3h - 2k = 0 \quad \Rightarrow \quad h = \dfrac{2k}{3}$ ... (3)

Putting the value of h from (3) in (2), we get

$$\frac{8k}{3} - 3k + 1 = 0 \quad \Rightarrow \quad -\frac{k}{3} + 1 = 0 \quad \Rightarrow \quad k = 3$$

Putting $k = 3$ in (2), we get $h = \frac{2k}{3} = \frac{2 \times 3}{3} = 2$

$$\text{I.F.} = x^h y^k = x^2 y^3$$

On multiplying the given differential equation by x^2y^3, we get

$$x^2y^3\,(3y - 2xy^3)dx + x^2y^3(4x - 3x^2y^2)\,dy = 0$$

$$(3x^2y^4 - 2x^3y^6)\,dx + (4x^3y^3 - 3x^4y^5)\,dy = 0$$

This is the exact differential equation.

Its solution is $\int (3x^2y^4 - 2x^3y^6)dx = 0 \quad \Rightarrow \quad x^3y^4 - \frac{x^4}{2}y^6 = C$ **Ans.**

EXERCISE 11.11

Solve the following differential equations.

1. $(2y\,dx + 3x\,dy) + 2xy\,(3y\,dx + 4x\,dy) = 0$ **Ans.** $x^2y^3\,(1 + 2xy) = c$

2. $(y^2 + 2yx^2)\,dx + (2x^3 - xy)\,dy = 0$ **Ans.** $4(xy)^{1/2} - \frac{2}{3}\left(\frac{y}{x}\right)^{3/2} = c$

3. $(3x + 2y^2)y\,dx + 2x\,(2x + 3y^2)\,dy = 0$ **Ans.** $x^2y^4\,(x + y^2) = c$

4. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$ **Ans.** $\frac{7}{5}x^{10/7}y^{-5/7} - \frac{7}{4}x^{-4/7}y^{-12/7} = c$

5. $x\,(3y\,dx + 2x\,dy) + 8y^4\,(y\,dx + 3x\,dy) = 0$ **Ans.** $x^3y^2 + 4x^2y^6 = c$

Rule V.

If the given equation $M\,dx + N\,dy = 0$ is homogeneous equation and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an integrating factor.

Example 38. *Solve* $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$

Solution. $(x^3 + y^3)\,dx - (xy^2)\,dy = 0$ ... (1)

Here $M = x^3 + y^3, \qquad N = -xy^2$

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x(x^3 + y^3) - xy^2(y)} = \frac{1}{x^4}$$

Multiplying (1) by $\frac{1}{x^4}$ we get $\frac{1}{x^4}(x^3 + y^3)dx + \frac{1}{x^4}(-xy^2)dy = 0$

$\Rightarrow \quad \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$, which is an exact differential equation.

$$\int \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = c \quad \Rightarrow \quad \log x - \frac{y^3}{3x^3} = c$$ **Ans.**

EXERCISE 11.12

Solve the following differential equations:

1. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $-\frac{x^3}{3y^3} + \log y = c$

2. $(y^3 - 3xy^2)\,dx + (2x^2y - xy^2)\,dy = 0$ **Ans.** $\frac{y}{x} + 3\log x - 2\log y = c$

3. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ **Ans.** $\frac{x}{y} - 2\log x + 3\log y = c$

4. $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$ **Ans.** $x^2y^4 - x^4y^2 = c$

11.14 DIFFERENTIAL EQUATIONS REDUCIBLE TO EXACT FORM (BY INSPECTION)

The following differentials, which commonly occur, help in selecting the suitable integrating factor.

(*i*) $y\,dx + x\,dy = d\,[xy]$ (*ii*) $\frac{x\,dy - y\,dx}{x^2} = d\left[\frac{y}{x}\right]$

(*iii*) $\frac{y\,dx - x\,dy}{y^2} = d\left[\frac{x}{y}\right]$ (*iv*) $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$

(*v*) $\frac{x\,dy - y\,dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$ (*vi*) $\frac{x\,dx - y\,dy}{x^2 + y^2} = d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right]$

(*vii*) $\frac{x\,dy - y\,dx}{x^2 - y^2} = d\left[\frac{1}{2}\log\frac{x+y}{x-y}\right]$ (*viii*) $\frac{x\,dy + y\,dx}{x^2y^2} = d\left[-\frac{1}{xy}\right]$

11.15 EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

The differential equations will involve $\frac{dy}{dx}$ in higher degree and $\frac{dy}{dx}$ will be denoted by p. The differential equation will be of the form $f(x, y, p) = 0$.

Case 1. Equations solvable for p.

Example 39. *Solve* : $x^2 = 1 + p^2$

Solution. $x^2 = 1 + p^2 \quad \Rightarrow \quad p^2 = x^2 - 1$

$$\Rightarrow \quad p = \pm\sqrt{x^2 - 1} \quad \Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{x^2 - 1} \quad \Rightarrow \quad dy = \pm\sqrt{x^2 - 1}\,dx$$

which gives on integration $y = \pm\frac{x}{2}\sqrt{x^2 - 1} \mp \frac{1}{2}\log\left(x + \sqrt{x^2 - 1}\right) + c$ **Ans.**

Case II. Equations solvable for y.

(*i*) Differentiate the given equation w.r.t. "x".

(*ii*) Eliminate p from the given equation and the equation obtained as above.

(*iii*) The eliminant is the required solution.

Example 40. *Solve*: $y = (x - a)\,p - p^2$.

Solution. $y = (x - a)\,p - p^2$ ... (1)

Differentiating (1) w.r.t. "x" we obtain

$$\frac{dy}{dx} = p + (x - a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$p = p + (x - a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = (x - a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = \frac{dp}{dx}[x - a - 2p] \quad \Rightarrow \quad \frac{dp}{dx} = 0$$

On integration, we get $p = c$.

Putting the value of p in (1), we get

$$y = (x - a)\, c - c^2 \qquad \textbf{Ans.}$$

Case III. Equations solvable for *x*

(*i*) Differentiate the given equation w.r.t. "y".
(*ii*) Solve the equation obtained as in (1) for p.
(*iii*) Eliminate p, by putting the value of p in the given equation.
(*iv*) The eliminant is the required solution.

Example 41. *Solve*: $\quad y = 2px + yp^2$

Solution. $\quad y = 2px + yp^2 \qquad \ldots (1)$

$$\Rightarrow \quad 2px = y - yp^2 \quad \Rightarrow \quad 2x = \frac{y}{p} - yp \qquad \ldots (2)$$

Differentiating (2) w.r.t. "y" we get

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy} \quad \Rightarrow \quad \frac{1}{p} + p = -\frac{y}{p^2}\frac{dp}{dy} - y\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} + p = -y\left(\frac{1}{p^2} + 1\right)\frac{dp}{dy} \quad \Rightarrow \quad \frac{1+p^2}{p} = -y\frac{1+p^2}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \quad 1 = -\frac{y}{p}\frac{dp}{dy} \quad \Rightarrow \quad -\frac{dy}{y} = \frac{dp}{p} \quad \Rightarrow \quad -\log y = \log p + \log c'$$

$$\Rightarrow \quad \log p\,y = \log c \quad \Rightarrow \quad p\,y = c \quad \Rightarrow \quad p = \frac{c}{y}$$

Putting the value of p in (1), we get

$$y = 2\left(\frac{c}{y}\right)x + y\left(\frac{c^2}{y^2}\right) \quad \Rightarrow \quad y^2 = 2\,cx + c^2$$

$$\Rightarrow \quad y^2 = c(2x + c) \qquad \textbf{Ans.}$$

Class IV. Clairaut's Equation.

The equation $y = px + f(p)$ is known as Clairaut's equation. $\qquad \ldots (1)$

Differentiating (1) w.r.t. "x", we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \Rightarrow \quad 0 = x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad [x + f'(p)]\frac{dp}{dx} = 0 \quad \Rightarrow \quad \frac{dp}{dx} = 0 \quad \Rightarrow \quad p = a \quad \text{(constant)}$$

Putting the value of p in (1), we have

$$y = ax + f(a)$$

which is the required solution.

Method. In the Clairaut's equation, on replacing p by a (constant), we get the solution of the equation.

Example 42. *Solve* : $p = \log\,(p\,x - y)$

Solution. $\quad p = \log\,(p\,x - y)$ or $e^p = p\,x - y \quad$ or $\quad y = p\,x - e^p$

Which is Clairaut's equation.

Hence its solution is $y = a\,x - e^a$ **Ans.**

EXERCISE 11.13

Solve the following differential equations.

1. $xp^2 + x = 2yp$ **Ans.** $2cy = c^2x^2 + 1$

2. $x(1 + p^2) = 1$ **Ans.** $y - c = \sqrt{(x - x^2)} - \tan^{-1}\sqrt{\frac{1-x}{x}}$

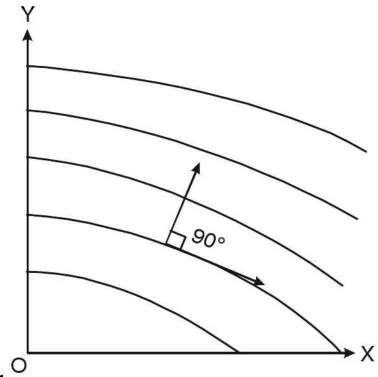
3. $x^2p^2 + xyp - 6y^2 = 0$ **Ans.** $y = \frac{c}{x^3}, y = c_1x^2$

4. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ **Ans.** $xy = c,\ x^2 - y^2 = c$

5. $y = px + p^3$ **Ans.** $y = ax + a^3$

6. $x^2(y - px) = yp^2$ **Ans.** $y^2 = cx^2 + c^2$

11.16 ORTHOGONAL TRAJECTORIES



Two families of curves are such that every curve of either family cuts each curve of the other family at right angles. They are called orthogonal trajectories of each other.

Orthogonal trajectories are very useful in engineering problems.

For example:

(*i*) The path of an electric field is perpendicular to equi-potential curves.

(*ii*) In fluid flow, the stream lines and equipotential lines are orthogonal trajectories.

(*iii*) The lines of heat flow is perpendicular to isothermal curves.

Working rule to find orthogonal trajectories of curves

Step 1. By differenciating the equaton of curves find the differential equations in the form $f\left(x, y, \frac{dy}{dx}\right) = 0$

Step 2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ $(M_1 . M_2 = -1)$

Step 3. Solve the differential equation of the orthogonal trajectories *i.e.*, $f\left(x, y - \frac{dx}{dy}\right) = 0$

Self-orthogonal. A given family of curves is said to be 'self-orthogonal' if the family of orthogonal trajectory is the same as the given family of curves.

Example 43. Find the orthogonal trajectories of the family of curves $xy = c$.

Solution. Here, we have

$$xy = c \qquad \text{... (1)}$$

Differentiating (1), w.r.t., "x", we get

$$y + x\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

On replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$\Rightarrow \quad -\frac{dx}{dy} = -\frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx \qquad \text{... (2)}$$

Integrating (2), we get $\frac{y^2}{2} = \frac{x^2}{2} + c$

$\Rightarrow \qquad y^2 - x^2 = 2c$ **Ans.**

Example 44. Find the orthogonal trajectories of $x^p + cy^p = 1$, p = constant.

Solution. Here we have $x^p + cy^p = 1$ ... (1)

Differentiating (1), we get $px^{p-1} + pcy^{p-1}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^{p-1}}{cy^{p-1}}$ ... (2)

Putting the value of $\frac{1}{c}$ in (2), we get $\frac{dy}{dx} = -\frac{x^{p-1}}{y^{p-1}}\frac{y^p}{1-x^p} \Rightarrow \frac{dy}{dx} = -\frac{x^{p-1}y}{1-x^p}$ ... (3)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ for orthogonal trajectory in (3), we get

$$-\frac{dx}{dy} = -\frac{x^{p-1}y}{1-x^p} \Rightarrow \frac{dy}{dx} = \frac{1-x^p}{x^{p-1}y} \quad \text{... (4)}$$

$$\Rightarrow \quad y\,dy = \frac{1-x^p}{x^{p-1}}dx \quad \Rightarrow \quad \int y\,dy = \int x^{1-p}dx - \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^{1-p+1}}{1-p+1} - \frac{x^2}{2} + c \Rightarrow y^2 = \frac{2x^{2-p}}{2-p} - x^2 + 2c$$ **Ans.**

Putting $p = 2$ in (4), we get $\frac{dy}{dx} = \frac{1-x^2}{xy} \Rightarrow y\,dy = \frac{1-x^2}{x}dx$

$$y\,dy = \left(\frac{1}{x} - x\right)dx \quad \Rightarrow \quad \frac{y^2}{2} = \log x - \frac{x^2}{2} + \log c$$

$$\log x + \log c = \frac{x^2+y^2}{2} \quad \Rightarrow \quad 2\log x + 2\log c = x^2 + y^2$$

$$x^2c^2 = e^{x^2+y^2}$$

$$c_1x^2 = e^{x^2+y^2} \quad [C_1 = C^2]$$ **Ans.**

Example 45. *Show that the family of porabolas $y^2 = 2cx + c^2$ is "self-orthogonal."*

Solution. Here we have

$$y^2 = 2cx + c^2 \quad \text{... (1)}$$

Differentiating (1), we get $2y\frac{dy}{dx} = 2c \quad \Rightarrow \quad c = y\frac{dy}{dx}$

Putting the value of c in (1), we have $y^2 = 2\left(y\frac{dy}{dx}\right)x + \left(y\frac{dy}{dx}\right)^2$ ... (2)

Putting $\frac{dy}{dx} = p$ in (2), we get

$$y^2 = 2ypx + y^2p^2 \quad \text{... (3)}$$

This is differential equation of give n family of parabolas.

For orthogonal trajectories we put $-\frac{1}{p}$ for p in (3)

$$y^2 = 2y\left(-\frac{1}{p}\right)x + y^2\left(-\frac{1}{p}\right)^2 \quad \Rightarrow \quad y^2 = -\frac{2yx}{p} + \frac{y^2}{p^2}$$

$$\Rightarrow \qquad y^2p^2 = -2pyx + y^2$$

Rewriting, we get

$$\Rightarrow \qquad y^2 = 2ypx + y^2p^2$$

Which is same as equation (3). Thus (2) is D.E. for the given family and its orthogonal trajectories.

Hence, the given family is self-orthogonal. **Proved.**

Example 46. *Show that the system of confocal conics*

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

Where λ is a parameter, is self orthogonal.

Solution. Here we have $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ ... (1)

Differentiating (1), we get

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0 \qquad \left(\text{Put } \frac{dy}{dx} = p\right)$$

$$\Rightarrow \qquad \frac{x}{a^2+\lambda} + \frac{yp}{b^2+\lambda} = 0$$

$$\Rightarrow \qquad x(b^2+\lambda) + py(a^2+\lambda) = 0 \quad \Rightarrow \quad \lambda(x+py) = -b^2x - a^2yp$$

$$\Rightarrow \qquad \lambda = \frac{-(b^2x + a^2yp)}{x+py}$$

Now $\quad a^2 + \lambda = a^2 - \dfrac{b^2x + a^2yp}{x+py} = \dfrac{a^2x + a^2py - b^2x - a^2yp}{x+py} = \dfrac{(a^2-b^2)x}{x+py}$

Again $\quad b^2 + \lambda = b^2 - \dfrac{b^2x + a^2yp}{x+py} = \dfrac{b^2x + b^2py - b^2x - a^2yp}{x+py} = \dfrac{-(a^2-b^2)yp}{x+py}$

Eliminating λ by putting the value of $a^2+\lambda$ and $b^2+\lambda$ in (1), we get

$$\frac{x^2(x+py)}{(a^2-b^2)x} + \frac{y^2(x+py)}{-(a^2-b^2)yp} = 1 \qquad \Rightarrow \qquad \frac{x(x+py)}{(a^2-b^2)} - \frac{y(x+py)}{(a^2-b^2)p} = 1$$

$$\frac{x+py}{a^2-b^2}\left[x - \frac{y}{p}\right] = 1 \;\Rightarrow\; \frac{(x+py)\left(x-\dfrac{y}{p}\right)}{a^2-b^2} = 1 \;\Rightarrow\; (x+py)\left(x - \frac{y}{p}\right) = a^2 - b^2 \qquad \text{... (2)}$$

Equation (2) is the differential equation of (1),
To get the differential equation of orthogonal trajectory

Replace p by $-\dfrac{1}{p}$ in (2) $\left(x - \dfrac{1}{p}y\right)(x+py) = a^2 - b^2$ ...(3)

Equation (3) is the same as eq. (2).

Thus the differential equation of the family of the orthogonal trajectory is the same as the differential equation of the family of the given curves.

Hence it is a self orthogonal family of curves. **Ans.**

EXERCISE 11.14

Find the orthogonal trajectories of the following family of curves:

1. $y^2 = cx^3$ **Ans.** $(x+1)^2 + y^2 = a^2$ **2.** $x^2 - y^2 = cx$ **Ans.** $y\,(y^2 + 3x^2) = c$

3. $x^2 - y^2 = c$ **Ans.** $xy = c$

4. $(a + x)\,y^2 = x^2\,(3a - x)$ **Ans.** $(x^2 + y^2)^5 = cy^3\,(5x^2 + y^2)$

5. $y = ce^{-2x} + 3x$, passing through the point $(0, 3)$

Ans. $9x - 3y + 5 = -4e^{6(3-y)}$

6. $16x^2 + y^2 = c$ **Ans.** $y^{16} = kx$

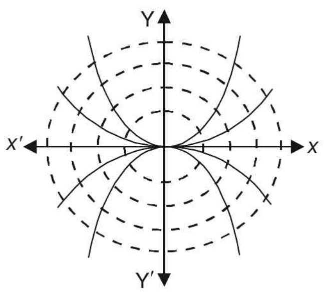
7. $y = \tan x + c$ **Ans.** $2x + 4y + \sin 2x = k$

8. $y = ax^2$ **Ans.** $x^2 + 2y^2 = c$

9. $x^2 + (y - c)^2 = c^2$ **Ans.** $x^2 + y^2 = cx$

10. $x^2 + y^2 + 2gx + 2fy + c = 0$ **Ans.** $x^2 + y^2 + 2fy - c = 0$

11. Family of parabolas through origin and focii on y-axis.



Ans. Ellipses with centre at the origin and focii on x-axis.

12. Show that the system of rectangular hyperbola $x^2 - y^2 = c^2$ and $xy = c^2$ are mutually orthogonal trajectories.

13. Show that the family of curves $y^2 = 4c(c + x)$ is self orthogonal.

11.17 POLAR EQUATION OF THE FAMILY OF CURVES

Let the polar equation of the family of curves be $f(r, \theta, c) = 0$ ... (1)

Working Rule

Step 1. On differentiating and eliminating the arbitrary constant c between (1) and $f'(r, \theta, c) = 0$ we get the differential equation of (1) *i.e.*,

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \qquad \text{... (2)}$$

Step 2. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (2). Here we will get the differential equation of orthogonal trajectory *i.e.*,

$$F\left(r, \theta - r^2\frac{d\theta}{dr}\right) = 0 \qquad \text{... (3)}$$

Step 3. Integrating (3) to get the equation of the orthogonal trajectory.

Example 47. *Find the orthogonal trajectory of the cardioids $r = a\,(1 - \cos\theta)$.*

Solution. We have, $r = a(1 - \cos\theta)$ ... (1)

Differentiating (1) w.r.t. θ, we get $\dfrac{dr}{d\theta} = a\sin\theta$ ... (2)

Dividing (2) by (1) to eliminate a, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-1+2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2} \qquad \text{... (3)}$$

which is the differential equation of (1).

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (3), we get $\dfrac{1}{r}\left(-r^2\dfrac{d\theta}{dr}\right) = \cot\dfrac{\theta}{2}$

$$r\frac{d\theta}{dr} = -\cot\frac{\theta}{2}$$

Separating the variables we get $\dfrac{dr}{r} = -\tan\dfrac{\theta}{2}\,d\theta$ ... (4)

Integrating (4), we get $\log r = 2\log\cos\dfrac{\theta}{2} + \log c \quad = \log c\cos^2\dfrac{\theta}{2}$

$\Rightarrow \qquad r = c\cos^2\dfrac{\theta}{2} \qquad \Rightarrow \qquad r = \dfrac{c}{2}\,(1+\cos\theta)$

Which is the required trajectory. **Ans.**

Example 48. *Find the orthogonal trajectory the family of curves*

$$r^2 = c\,\sin 2\theta$$

Solution. We have

$$r^2 = c\,\sin 2\theta \qquad \text{... (1)}$$

Differentiating (1), we get $2r\dfrac{dr}{d\theta} = 2c\cos 2\theta$ ... (2)

Dividing (2) by (1), to eliminate 'c' we get $\dfrac{2}{r}\dfrac{dr}{d\theta} = 2\cot 2\theta$ ... (3)

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (3), we have $\dfrac{2}{r}\left(-r^2\dfrac{d\theta}{dr}\right) = 2\cot 2\theta$

$$-2r\frac{d\theta}{dr} = 2\cot 2\theta \qquad \text{... (4)}$$

Separating the variables of (4), we obtain $\dfrac{dr}{r} = -\tan 2\theta\,d\theta$ ... (5)

Integrating (5), we get $\log r = \dfrac{1}{2}\log\cos 2\theta + \log c$

$$2\log r = \log c\cos\theta$$

$$r^2 = c\,\cos 2\theta$$

which is the required trajectory **Ans.**

Example 49. *Find the orthogonal trajectory of the family of curves*

$$r = c\,(\sec\theta + \tan\theta)$$

Solution. We have $r = c\,(\sec\theta + \tan\theta)$ ... (1)

Differentiating (1) w.r.t. 'θ' we get $\dfrac{dr}{d\theta} = c\,(\sec\theta\tan\theta + \sec^2\theta)$ ...(2)

$$\frac{dr}{d\theta} = c\sec\theta\ (\tan\theta + \sec\theta)$$

Dividing (2) by (1), we get $\frac{1}{r}\frac{dr}{d\theta} = \sec\theta$ ... (3)

Separating the variables of (3), we have $\frac{1}{r}\frac{dr}{d\theta} \equiv \sec\theta$ ... (4)

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \sec\theta \quad \Rightarrow \quad -r\frac{d\theta}{dr} = \sec\theta \qquad \text{... (5)}$$

Separating the variables of (5), we obtain $\frac{dr}{r} = -\cos\theta\ d\theta$ ... (6)

Integrating (6), we get $\log r = -\sin\theta + c \quad \Rightarrow \quad r = c'e^{-\sin\theta}$

which is the required orthogonal trajectory. **Ans.**

EXERCISE 11.15

Find the orthogonal trajectory of the following families of the curves:

1. $r = ce^{\theta}$ **Ans.** $r = ke^{-\theta}$
2. $r = c\theta^2$ **Ans.** $r = ke^{-\frac{\theta^2}{4}}$
3. $r = a(1+\cos\theta)$ **Ans.** $r = c(1-\cos\theta)$
4. $r^n \sin n\theta = a^n$ **Ans.** $r^n \cos n\theta = c^n$
5. $r = a\cos^2\theta$ **Ans.** $r^2 = c\sin\theta$
6. $r = 2a(\sin\theta+\cos\theta)$ **Ans.** $r = 2c(\sin\theta-\cos\theta)$
7. $r = c(1+\sin^2\theta)$ **Ans.** $r^2 = k\cos\theta.\cot\theta$
8. $r = \frac{a}{1+2\cos\theta}$ **Ans.** $r^2\sin^3\theta = (1+\cos\theta)$

CHAPTER

12

LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

12.1 LINEAR DIFFERENTIAL EQUATIONS

If the degree of the dependent variable and all derivatives is one, such differential equations are called *linear differential equations e.g.*

(1) $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=x^2+x+1$ (2) $2\frac{d^2x}{dt^2}-\frac{dx}{dt}-3x=f(t)$

12.2 NON LINEAR DIFFERENTIAL EQUATIONS

If the degree of the dependent variable and / or its derivatives are of greater than 1 such differential equations are called one-linear differential equations.

(1) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y^2=\sin x$ (2) $\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+y^2=e^x$ (3) $\left(\frac{d^2x}{dt^2}\right)^2+\frac{dx}{dt}+x=f(t)$

The order of a differential equation is the highest order of the derivative involved. All the above differential equations are of second order.

Fourier and Laplace transforms are mathematical tools to solve the differential equations.

12.3 LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENTS

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

where P and Q are constants and R is a function of x or constant.

Differential operator. Symbol D stands for the operation of differential *i.e.*,

$$Dy=\frac{dy}{dx},\quad D^2y=\frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ can be written in the operator form.

$$D^2y+P\,Dy+Q\,y=R \qquad \Rightarrow \qquad (D^2+PD+Q)\,y=R$$

12.4 COMPLETE SOLUTION = COMPLEMENTARY FUNCTION + PARTICULAR INTEGRAL

Let us consider a linear differential equation of the first order

$$\frac{dy}{dx}+Py=Q \qquad ...(1)$$

Its solution is $\quad ye^{\int P dx}=\int (Q\,e^{\int P dx})\,dx+C$

$$\Rightarrow \quad y=Ce^{-\int P dx}+e^{-\int P dx}\int (Qe^{\int P dx})\,dx$$

$$\Rightarrow \quad y = cu + v \text{ (say)} \qquad ...(2)$$

where $u=e^{-\int P dx}$ and $v=e^{-\int P dx}\int Q\,e^{\int P dx}\,dx$

(*i*) Now differentiating $u=e^{-\int P dx}$ w.r.t. x, we get $\dfrac{du}{dx}=-Pe^{-\int P dx}=-Pu$

$$\Rightarrow \quad \frac{du}{dx}+Pu=0 \quad \Rightarrow \quad \frac{d(cu)}{dx}+P(cu)=0$$

which shows that $y = c.u$ is the solution of $\dfrac{dy}{dx}+Py=0$

(*ii*) Differentiating $v=e^{-\int P dx}\int (Qe^{\int P dx}\,dx$ with respect to x, we get

$$\frac{dv}{dx}=-Pe^{\int P dx}\int (Qe^{\int P dx})\,dx+e^{-\int P dx}Qe^{\int P dx} \quad \Rightarrow \quad \frac{dv}{dx}=-Pv+Q$$

$\Rightarrow \quad \dfrac{dv}{dx}+Pv=Q$ which shows that $y = v$ is the solution of $\boxed{\dfrac{dy}{dx}+Py=Q}$

Solution of the differential equation (1) is (2) consisting of two parts *i.e.* cu and v. cu is the solution of the differential equation whose R.H.S. is zero. cu is known as *complementary function*. Second part of (2) is v free form any arbitrary constant and is known as *particular integral.*

Complete Solution = Complementary Function + Particular Integral.

$$\Rightarrow \quad \boxed{y = C.F. + P.I.}$$

12.5 METHOD FOR FINDING THE COMPLEMENTARY FUNCTION

(1) In finding the complementary function, R.H.S. of the given equation is replaced by zero.

(2) Let $y = C_1\, e^{mx}$ be the C.F. of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \qquad ...(1)$$

Putting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1) then $C_1e^{mx}\,(m^2 + Pm + Q) = 0$

$\Rightarrow \quad m^2 + Pm + Q = 0$. It is called **Auxiliary equation.**

(3) Solve the auxiliary equation :

Case I : Roots, Real and Different. If m_1 and m_2 are the roots, then the C.F. is

$$y=C_1e^{m_1x}+C_2e^{m_2x}$$

Case II : Roots, Real and Equal. If both the roots are m_1, m_1 then the C.F. is

$$y=(C_1+C_2x)\,e^{m_1x}$$

Proof. Equation (1) can be written as

$$(D - m_1)(D - m_1)y = 0 \qquad \text{... (2)}$$

Replacing $(D - m_1)y = v$ in (2), we get

$$(D - m_1)v = 0 \qquad \text{... (3)}$$

$$\frac{dv}{dx} - m_1 v = 0 \quad \Rightarrow \quad \frac{dv}{v} = m_1 dx \quad \Rightarrow \quad \log v = m_1 x + \log c_2 \quad \Rightarrow \quad v = c_2 e^{m_1 x}$$

$$v = c_2 e^{m_1 x}$$

From (3) $\quad (D - 1)y = c_2 e^{m_1 x}$

This is the linear differential equation.

$$\text{I.F.} = e^{-m_1 \int dx} = e^{-m_1 x}$$

Solution is

$$y.e^{-m_1 x} = \int (c_2 e^{m_1 x})(e^{-m_1 x})\, dx + c_1 = \int c_2\, dx + c_1 = c_2 x + c_1$$

$$y = (c_2 x + c_1) e^{m_1 x}$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{m_1 x}$$

Example 1. *Solve:* $\dfrac{d^2 y}{dx^2} - 8\dfrac{dy}{dx} + 15y = 0.$

Solution. Given equation can be written as

$$(D^2 - 8D + 15)\, y = 0$$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$\Rightarrow \qquad (m - 3)(m - 5) = 0 \qquad \therefore\ m = 3, 5$

Hence, the required solution is

$$y = C_1\, e^{3x} + C_2 e^{5x}$$ **Ans.**

Example 2. *Solve*: $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

Solution. Given equation can be written as

$$(D^2 - 6D + 9)\, y = 0$$

A.E. is $m^2 - 6m + 9 = 0 \quad \Rightarrow \quad (m - 3)^2 = 0 \quad \Rightarrow \quad m = 3, 3$

Hence, the required solution is

$$y = (C_1 + C_2 x)\, e^{3x}$$ **Ans.**

Example 3. *Solve*: $\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0,$

$$y = 2 \text{ and } \frac{dy}{dx} = \frac{d^2 y}{dx^2} \text{ when } x = 0.$$

Solution. Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its root are $-2 \pm i$

The complementary function is

$$y = e^{-2x}\, (A \cos x + B \sin x) \qquad \text{...(1)}$$

On putting $y = 2$ and $x = 0$ in (1), we get

$$2 = A$$

On putting $A = 2$ in (1), we have

$$y = e^{-2x}\,[2\cos x + B\sin x] \qquad \ldots(2)$$

On differentiating (2), we get

$$\frac{dy}{dx} = e^{-2x}[-2\sin x + B\cos x] - 2e^{-2x}[2\cos x + B\sin x]$$

$$= e^{-2x}\,[(-2B-2)\sin x + (B-4)\cos x]$$

$$\frac{d^2y}{dx^2} = e^{-2x}[(-2B-2)\cos x - (B-4)\sin x]$$

$$-2e^{-2x}\,[(-2B-2)\sin x + (B-4)\cos x]$$

$$= e^{-2x}\,[(-4B+6)\cos x + (3B+8)\sin x]$$

But $\qquad \dfrac{dy}{dx} = \dfrac{d^2y}{dx^2}$

$$e^{-2x}\,[(-2B-2)\sin x + (B-4)\cos x] = e^{-2x}\,[(-4B+6)\cos x + (3B+8)\sin x]$$

On putting $x = 0$, we get

$$B - 4 = -4B + 6 \quad \Rightarrow \quad B = 2$$

(2) becomes, $\qquad y = e^{-2x}\,[2\cos x + 2\sin x]$

$$y = 2e^{-2x}\,[\sin x + \cos x] \qquad \textbf{Ans.}$$

Example 4. *The general solution of the differential equation*

$$\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0 \text{ is given by} \qquad \textit{(U.P. II Semester, 2009)}$$

Solution. Here, we have

$$\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$$

or $\qquad D^5y - D^3y = 0 \quad \Rightarrow \quad (D^5 - D^3)y = 0 \quad \Rightarrow \quad D^3(D^2-1)y = 0$

A.E. is $m^3(m^2-1) = 0 \quad \Rightarrow \quad m = 0, 0, 0, 1, -1$

Here the solution is

$$y = (C_1 + C_2x + C_3x^2) + C_3e^x + C_5e^{-x} \qquad \textbf{Ans.}$$

Case III: Roots Imaginary. If the roots are $\alpha \pm i\beta$, then the solution will be

$$y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x} = e^{\alpha x}.[C_1e^{i\beta x} + C_2e^{-i\beta x}]$$

$$= e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(C_1+C_2)\cos\beta x + i(C_1-C_2)\sin\beta x]$$

$$= e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

EXERCISE 12.1

Solve the following equations :

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ **Ans.** $y = C_1e^x + C_2e^{2x}$ 2. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 30y = 0$ **Ans.** $y = C_1e^{5x} + C_2e^{-6x}$

3. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = 0$ **Ans.** $y = (C_1 + C_2x)\,e^{4x}$

4. $\frac{d^2y}{dx^2}+\mu^2 y=0$ **Ans.** $y=C_1\cos\mu x+C_2\sin\mu x$

5. $(D^2+2D+2)\,y=0,\ y(0)=0,\ y'(0)=1$ (*A.M.I.E.T.E., June 2006*) **Ans.** $y=e^{-x}\sin x$

6. $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}+4\frac{dy}{dx}-8y=0$ **Ans.** $y=C_1e^{2x}+C_2\cos 2x+C_3\sin 2x$

7. $\frac{d^4y}{dx^4}-32\frac{d^2y}{dx^2}+256=0$ (*A.M.I.E.T.E., Dec. 2004*) **Ans.** $y=(C_1+x)\cos 4x+(C^3+C^4x)\sin 4x$

8. $\frac{d^4y}{dx^4}-4\frac{d^3y}{dx^3}+8\frac{d^2y}{dx^2}-8\frac{dy}{dx}+4y=0$ **Ans.** $y=e^x\,[(C_1+C_2x)\cos x+(C_3+C_4x)\sin x]$

9. $\frac{d^4y}{dx^4}+\frac{d^2y}{dx^2}=0,\ \ y(0)=y'(0)=y''(0)=0,\ y'''(0)=1$ **Ans.** $y=x-\sin x$

10. The equation for the bending of a strut is EI $\frac{d^2y}{dx^2}+Py=0$

If $y=0$ when $x=0$, and $y=a$ when $x=\frac{1}{2}$, find y. **Ans.** $y=\dfrac{a\sin\sqrt{\frac{P}{EI}}\,x}{\sin\sqrt{\frac{P}{EI}}\,\frac{1}{2}}$

11. $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+12\frac{dy}{dx}+8y=0,\ y(0)=0$, and $y'(0)=0$ and $y''(0)=2$

(*A.M.I.E.T.E. Dec. 2008*) **Ans.** $y=x^2e^{-2x}$

12. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{4dy}{dx}+4y=0,\ y(0)=0,\ y'(0)=0,\ y''(0)=-5,$ **Ans.** $y=-e^x+\cos 2x-\frac{1}{2}\sin 2x$

13. $(D^8+6D^6-32D^2)\,y=0$ (*A.M.I.E.T.E., Summer 2005*)

Ans. $y=C_1+C_2x+C_3e^{\sqrt{2}x}+C_4e^{-\sqrt{2}x}+C_5\cos 2x+C_6\sin 2x$

14. Show that non-trivial solutions of the boundary value problem $y^{(iv)}-w^4y=0,\ y(0)=0=y''(0),$ $y(L)=0,\ y''(L)=0$ are $y(x)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi x}{L}\right)$ where D_n are constants.

(*A.M.I.E.T.E. Dec. 2005*)

15. Solve the initial value problem $y'''+6y''+11y'+6y=0,\ y(0)=0,\ y'(0)=1,\ \ y''(0)=-1.$

(*A.M.I.E.T.E., Dec. 2006*) **Ans.** $y=2e^{-x}-3e^{-2x}+e^{-3x}$.

16. Let y_1, y_2 be two linearly independent solutions of the differential equation $yy''-(y')^2=0$. Then, $c_1y_1+c_2y_2$, where c_1, c_2 are constants is a solution of this differential equation for
(*a*) $c_1=c_2=0$ only. (*b*) $c_1=0$ or $c_2=0$ (*c*) no value of c_1, c_2. (*d*) all real c_1, c_2

(*A.M.I.E.T.E., Dec. 2004*)

17. The solution of the differential equation $\frac{d^2y}{dx^2}+y=0$ satisfying the initial conditions $y(0)=1$, $y\left(\frac{\pi}{2}\right)=2$ is

(*a*) $y=2\cos(x)+\sin(x)$ (*b*) $y=\cos(x)+2\sin(x)$
(*c*) $y=\cos(x)+\sin(x)$ (*d*) $y=2\cos(x)+2\sin(x)$ (*AMIETE, Dec. 2009*) **Ans.** (*b*)

18. Find the complementary function of $(D-2)^2 = 8\,(e^{2x} + \sin 2x - x^2)$

(*a*) $(C_1 + C_2 e^{2x})\, x$ (*b*) $(C_1 + C_2 x)\, e^{2x}$

(*c*) $(C_1 x + C_2 x^2)\, e^{2x}$ (*d*) $(C_1 x + C_2 e x^2)\,.\, 2^{2x}$ (*AMIETE, Dec. 2010*) **Ans.** (*b*)

19. Solution of $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$ is

(*a*) $y = C_1 e^{x} + C_2 e^{2x}$ (*b*) $y = C_1 + (C_2 + C_3 x)\, e^{-x}$

(*c*) $y = (C_1 + C_2 x + C_3 x^2)\, e^{-x}$ (*d*) $y = C_1 + C_2 e^{-x}$ (*AMIETE, June 2009*) **Ans.** (*b*)

12.6 RULES TO FIND PARTICULAR INTEGRAL

(*i*) $\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$

If $f(a) = 0$ then $\dfrac{1}{f(D)} \cdot e^{ax} = x \cdot \dfrac{1}{f'(a)} \cdot e^{ax}$

If $f'(a) = 0$ then $\dfrac{1}{f(D)} \cdot e^{ax} = x^2 \dfrac{1}{f''(a)} \cdot e^{ax}$

(*ii*) $\dfrac{1}{f(D)} x^n = [f(D)]^{-1} x^n$ Expand $[f(D)]^{-1}$ and then operate.

(*iii*) $\dfrac{1}{f(D^2)} \sin ax = \dfrac{1}{f(-a^2)} \sin ax$ and $\dfrac{1}{f(D^2)} \cos ax = \dfrac{1}{f(-a^2)} \cos ax$

If $f(-a^2) = 0$ then $\dfrac{1}{f(D^2)} \sin ax = x \cdot \dfrac{1}{f'(-a^2)} \cdot \sin ax$

(*iv*) $\dfrac{1}{f(D)} e^{ax} \cdot \phi(x) = e^{ax} \cdot \dfrac{1}{f(D+a)} \phi(x)$

(*v*) $\dfrac{1}{D+a} \phi(x) = e^{-ax} \displaystyle\int e^{ax} \cdot \phi(x)\, dx$

12.7 $\boxed{\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}.}$

We know that, $D.e^{ax} = a.e^{ax}$, $D^2 e^{ax} = a^2.e^{ax}, \ldots\ldots\ldots\ldots, D^n e^{ax} = a^n e^{ax}$

Let $f(D)\, e^{ax} = (D^n + K_1 D^{n-1} + \ldots + K_n)\, e^{ax} = (a^n + K_1 a^{n-1} + \ldots + K_n) e^{ax} = f(a)\, e^{ax}$.

Operating both sides by $\dfrac{1}{f(D)}$

$$\frac{1}{f(D)} \cdot f(D) e^{ax} = \frac{1}{f(D)} \cdot f(a) e^{ax}$$

$$\Rightarrow \qquad e^{ax} = f(a) \frac{1}{f(D)} \cdot e^{ax} \quad \Rightarrow \quad \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

If $f(a) = 0$, then the above rule fails.

Then $\dfrac{1}{f(D)} e^{ax} = x \cdot \dfrac{1}{f'(D)} e^{ax} = x \dfrac{1}{f'(a)} e^{ax} \quad \Rightarrow \quad \boxed{\dfrac{1}{f(D)} e^{ax} = x \,.\, \dfrac{1}{f'(a)} e^{ax}}$

If $f'(a) = 0$ then $\boxed{\dfrac{1}{f(D)} e^{ax} = x^2 \dfrac{1}{f''(a)} e^{ax}}$

Example 5. *Solve the differential equation*

$$\frac{d^2x}{dt^2}+\frac{g}{t}x=\frac{g}{l}L$$

where g, l, L are constants subject to the conditions,

$$x = a,\ \frac{dx}{dt}=0 \text{ at } t = 0.$$

Solution. We have, $\dfrac{d^2x}{dt^2}+\dfrac{g}{t}x=\dfrac{g}{l}L \quad \Rightarrow \quad \left(D^2+\dfrac{g}{l}\right)x=\dfrac{g}{l}L$

A.E. is $\quad m^2+\dfrac{g}{l}=0 \quad \Rightarrow \quad m=\pm\, i\sqrt{\dfrac{g}{l}}$

$$\text{C.F.} = C_1\cos\sqrt{\frac{g}{l}}\,t+C_2\sin\sqrt{\frac{g}{l}}\,t$$

$$\text{P.I.} = \frac{1}{D^2+\frac{g}{l}}\cdot\frac{g}{l}L=\frac{g}{l}L\frac{1}{D^2+\frac{g}{l}}e^{0t}=\frac{g}{l}L\frac{1}{0+\frac{g}{l}}=L \qquad [D=0]$$

$\therefore$ General solution is = C.F. + P.I.

$$x=C_1\cos\left(\sqrt{\frac{g}{l}}\right)t+C_2\sin\left(\sqrt{\frac{g}{l}}\right)t+L \qquad \text{...(1)}$$

$$\frac{dx}{dt}=-C_1\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\right)t+C_2\sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}\right)t$$

Put $t = 0$ and $\dfrac{dx}{dt}=0$

$$0=C_2\sqrt{\frac{g}{l}} \qquad \therefore \quad C_2=0$$

(1) becomes $\quad x=C_1\cos\sqrt{\dfrac{g}{l}}\,t+L \qquad \text{...(2)}$

Put $x = a$ and $t = 0$ in (2), we get

$$a=C_1+L \quad \text{or} \quad C_1=a-L$$

On putting the value of C_1 in (2), we get $x=(a-L)\cos\left(\sqrt{\dfrac{g}{l}}\right)t+L$ **Ans.**

Example 6. *Solve* : $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y=5e^{3x}$

Solution. $\quad (D^2 + 6D + 9)y = 5e^{3x}$

Auxiliary equation is $m^2 + 6m + 9 = 0 \quad \Rightarrow \quad (m + 3)^2 = 0 \quad \Rightarrow \quad m = -3, -3,$

$$\text{C.F.} = (C_1 + C_2x)\, e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2+6D+9}.5.e^{3x}=5\frac{e^{3x}}{(3)^2+6(3)+9}=\frac{5e^{3x}}{36}$$

The complete solution is $\quad y=(C_1+C_2x)e^{-3x}+\dfrac{5e^{3x}}{36}$ **Ans.**

Example 7. *Solve* : $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=6e^{3x}+7e^{-2x}-\log 2$

Solution. $(D^2-6D+9)y = 6e^{3x}+7e^{-2x}-\log 2$

A.E. is $(m^2-6m+9)=0 \quad \Rightarrow \quad (m-3)^2=0, \quad \Rightarrow \quad m=3, 3$

$$\text{C.F.} = (C_1+C_2x)e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2-6D+9}6e^{3x}+\frac{1}{D^2-6D+9}7e^{-2x}+\frac{1}{D^2-6D+9}(-\log 2)$$

$$= x\frac{1}{2D-6}6e^{3x}+\frac{1}{4+12+9}7e^{-2x}-\log 2\frac{1}{D^2-6D+9}e^{0x}$$

$$= x^2\frac{1}{2}\cdot 6\cdot e^{3x}+\frac{7}{25}e^{-2x}-\log 2\left(\frac{1}{9}\right)=3x^2e^{3x}+\frac{7}{25}e^{-2x}-\frac{1}{9}\log 2$$

Complete solution is $y=(C_1+C_2\,x)e^{3x}+3x^2e^{3x}+\frac{7}{25}e^{-2x}-\frac{1}{9}\log 2$ **Ans.**

EXERCISE 12.2

Solve the following differential equations:

1. $[D^2+5D+6]\,[y]=e^x$ **Ans.** $C_2\,e^{-2x}+C_2e^{-3x}+\frac{e^x}{12}$

2. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{3x}$ **Ans.** $C_1e^x+C_2e^{2x}+\frac{e^{3x}}{2}$
(*A.M.I.E.T.E. June 2010, 2007*)

3. $(D^3+2D^2-D-2)\,y=e^x$ **Ans.** $C_1e^x+C_2e^{-x}+C_3e^{-2x}+\frac{x}{6}e^x$

4. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=\sinh x$ **Ans.** $e^{-x}[C_1\cos x+C_2\sin x]+\frac{e^x}{10}-\frac{e^{-x}}{2}$

5. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh x$ **Ans.** $e^{-2x}(C_1\cos x+C_2\sin x)-\frac{1}{10}e^x-\frac{e^{-x}}{2}$

6. $(D^3-2D^2-5D+6)\,y=e^{3x}$ **Ans.** $C_1e^x+C_2e^{-2x}+C_3e^{3x}+\frac{x.e^{3x}}{10}$

7. $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+4\frac{dy}{dx}-4y=e^x$ **Ans.** $C_1e^x+C_2\cos 2x+C_3\sin 2x+\frac{x\,e^x}{5}$

8. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=e^{3x}$ **Ans.** $(C_1+C_2x)e^{3x}+\frac{x^2}{2}e^{3x}$

9. $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=e^{-x}$ **Ans.** $(C_1+C_2x+C_3x^2)e^{-x}+\frac{x^3}{6}e^{-x}$

10. $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=e^x\cosh 2x$ **Ans.** $C_1e^{3x}+C_2e^{-2x}+\frac{1}{10}xe^{3x}-\frac{1}{8}e^{-x}$

11. $(D-2)\,(D+1)^2\,y=e^{2x}+e^x$ **Ans.** $C_1e^{2x}+(C_2+C_3x)e^{-x}+\frac{x}{9}e^{2x}-\frac{e^x}{4}$

12. $(D-1)^3\,y=16\,e^{3x}$ **Ans.** $(C_1+C_2x+C_3\,x^2)\,e^x+2e^{3x}$

13. The particular integral (PI) of differential equation $[D^2+5D+6]\,y=e^x$ is

(*a*) $\frac{e^x+x}{12}$ (*b*) $\frac{e^x-x}{12}$ (*c*) $\frac{e^{-x}}{12}$ (*d*) $\frac{e^x}{12}$ (*AMIETE, June 2010*) **Ans.** (*d*)

12.8 $\boxed{\dfrac{1}{f(D)}x^n = [f(D)]^{-1}x^n.}$

Expand $[f(D)]^{-1}$ by the Binomial theorem in ascending powers of D as far as the result of operation on x^n is zero.

Example 8. *Solve the differential equation* $\dfrac{d^2y}{dx^2}+a^2y=\dfrac{a^2R}{p}(l-x)$

where a, R, p and l are constants subject to the conditions $y = 0$, $\dfrac{dy}{dx}=0$ *at* $x = 0$.

Solution. $\dfrac{d^2y}{dx^2}+a^2y=\dfrac{a^2}{p}R(l-x) \quad \Rightarrow \quad (D^2+a^2)y=\dfrac{a^2}{p}R(l-x)$

A.E. is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

C.F. $= C_1 \cos ax + C_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2+a^2}\frac{a^2}{p}R(l-x) = \frac{a^2R}{p}\frac{1}{a^2}\left[\frac{1}{1+\frac{D^2}{a^2}}\right](l-x)=\frac{R}{p}\left[1+\frac{D^2}{a^2}\right]^{-1}(l-x)$$

$$= \frac{R}{p}\left[1-\frac{D^2}{a^2}\right](l-x)=\frac{R}{p}(l-x)$$

$$y = C_1\cos ax+C_2\sin ax+\frac{R}{p}(l-x) \qquad ...(1)$$

On putting $y = 0$, and $x = 0$ in (1), we get $0 = C_1+\dfrac{R}{p}l \Rightarrow \quad C_1=-\dfrac{R\,l}{p}$

On differentiating (1), we get $\dfrac{dy}{dx}=-a\,C_1\sin ax+a\,C_2\cos ax-\dfrac{R}{p}$ ...(2)

On putting $\dfrac{dy}{dx}=0$ and $x = 0$ in (2), we have

$$0 = a\,C_2-\frac{R}{p} \quad \Rightarrow \quad C_2=\frac{R}{a.p}$$

On putting the values of C_1 and C_2 in (1), we get

$$y = -\frac{R}{p}l\cos a.x+\frac{R}{a.p}\sin a.x+\frac{R}{p}(l-x) \Rightarrow y=\frac{R}{p}\left[\frac{\sin ax}{a}-l\cos a.x+l-x\right] \qquad \textbf{Ans.}$$

EXERCISE 12.3

Solve the following equations :

1. $(D^2 + 5D + 4)\,y = 3 - 2x$ **Ans.** $C_1e^{-x}+C_2e^{-4x}+\dfrac{1}{8}(11-4x)$

2. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=x$ **Ans.** $(C_1 + C_2\,x)\,e^{-x} + x - 2$

3. $(2D^2 + 3D + 4)\,y = x^2 - 2x$ **Ans.** $e^{-\frac{3}{4}x}[A\cos\dfrac{\sqrt{23}}{4}x+B\sin\dfrac{\sqrt{23}}{4}x]+\dfrac{1}{32}[8x^2-28x+13]$

4. $(D^2 - 4D + 3)\,y = x^3$ **Ans.** $C_1e^{x}+C_2e^{3x}+\dfrac{1}{27}(9x^3+36x^2+78x+80).$

5. $5\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}=1+x^2.$ **Ans.** $A+Be^{-2x}+Ce^{3x}-\dfrac{1}{36}\left(2x^3-x^2+\dfrac{25}{3}x\right)$

6. $\dfrac{d^4y}{dx^4}+4y=x^4$ **Ans.** $e^x(C_1\cos x+C_2\sin x)+e^{-x}(C_3\cos x+C_4\sin x)+\dfrac{1}{4}(x^4-6)$

7. $\dfrac{d^2y}{dx^2}+2p\dfrac{dy}{dx}+(p^2+q^2)y=e^{cx}+p.q\,x^2$

Ans. $e^{-px}[C_1\cos qx+C_2\sin qx]+\dfrac{e^{Cx}}{(p+C)^2+q^2}+\dfrac{pq}{p^2+q^2}\left[x^2-\dfrac{4px}{p^2+q^2}+\dfrac{6p^2-2q^2}{(p^2+q^2)^2}\right]$

8. $D^2(D^2+4)y=96x^2$ **Ans.** $C_1+C_2x+C_3\cos 2x+C_4\sin 2x+2x^2(x^2-3)$

12.9 $\boxed{\dfrac{1}{f(D^2)}\sin ax=\dfrac{\sin ax}{f(-a^2)}}$ $\boxed{\dfrac{1}{f(D^2)}\cdot\cos ax=\dfrac{\cos ax}{f(-a^2)}}$

$D(\sin ax)=a.\cos ax$, $D^2(\sin ax)=D(a\cos ax)=-a^2.\sin ax$

$D^4(\sin ax)=D^2.D^2(\sin ax)=D^2(-a^2\sin ax)=(-a^2)^2\sin ax$

$(D^2)^n\sin ax=(-a^2)^n\sin ax$

Hence, $f(D^2)\sin ax=f(-a^2)\sin ax$

$$\frac{1}{f(D^2)}\cdot f(D^2)\sin ax=\frac{1}{f(D^2)}.f(-a^2).\sin ax$$

$$\sin ax=f(-a^2)\frac{1}{f(D^2)}\sin ax \quad\Rightarrow\quad \frac{1}{f(D^2)}\cdot\sin ax=\frac{\sin ax}{f(-a^2)}$$

Similarly, $$\frac{1}{f(D^2)}\cos ax=\frac{\cos ax}{f(-a^2)}$$

If $f(-a^2)=0$ then above rule fails.

$$\frac{1}{f(D^2)}\sin ax=x\frac{\sin ax}{f'(-a^2)}$$

If $f'(-a^2)=0$ then, $\dfrac{1}{f(D^2)}\sin ax=x^2\dfrac{\sin ax}{f''(-a^2)}$

Example 9. *Solve* : $(D^2+4)y=\cos 2x$

(*R.G.P.V., Bhopal June, 2008, A.M.I.E.T.E. Dec 2008*)

Solution. $(D^2+4)y=\cos 2x$

Auxiliary equation is $m^2+4=0$

$m=\pm 2i$, C.F. $=A\cos 2x+B\sin 2x$

$$\text{P.I.}=\frac{1}{D^2+4}\cos 2x = x.\frac{1}{2D}\cos 2x=\frac{x}{2}\left(\frac{1}{2}\sin 2x\right)=\frac{x}{4}\sin 2x$$

Complete solution is $y=A\cos 2x+B\sin 2x+\dfrac{x}{4}\sin 2x$ **Ans.**

Example 10. *Solve* : $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^3}+4\dfrac{dy}{dx}-2y=e^x+\cos x$ (*U.P., II Semester, Summer* 2006, 2001)

Solution. Given $(D^3-3D^2+4D-2)y=e^x+\cos x$

A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$\Rightarrow$ $(m-1)(m^2-2m+2) = 0$, *i.e.*, $m = 1,\ 1 \pm i$

$\therefore$ C.F. $= C_1e^x + e^x(C_2 \cos x + C_3 \sin x)$

$$\text{P.I.} = \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{(-1)D-3(-1)+4D-2}\cos x$$

$$= \frac{1}{(D-1)}e^x + \frac{1}{3D+1}\cos x = x\frac{1}{1}e^x + \frac{3D-1}{9D^2-1}\cos x$$

$$= e^x.x + \frac{(-3\sin x - \cos x)}{-9-1} = e^x.x + \frac{1}{10}(3\sin x + \cos x)$$

Hence, complete solution is

$$y = C_1e^x + e^x(C_2\cos x + C_3 \sin x) + x\,e^x + \frac{1}{10}(3\sin x + \cos x) \qquad \textbf{Ans.}$$

Example 11. *Solve* : $(D^4 - 3D^2 - 4)y = 5\sin 2x - e^{-2x}$ (*Nagpur University*, *Summer* 2001)

Solution. Auxiliary equation is

$m^4 - 3m^2 - 4 = 0$

$(m^2+1)(m^2-4) = 0$

$\Rightarrow$ $m^2 + 1 = 0 \Rightarrow m^2 - 4 = 0 \Rightarrow m = \pm i \Rightarrow m = \pm 2$

$\therefore$ C.F. $= C_1e^{2x} + C_2e^{-2x} + C_3\cos x + C_4 \sin x$

$$\text{P.I.} = \frac{1}{D^4-3D^2-4}(5\sin 2x - e^{-2x}) = 5.\frac{1}{D^4-3D^2-4}\sin 2x - \frac{1}{D^4-3D^2-4}e^{-2x}$$

$$= 5\frac{1}{(-2^2)^2-3(-2^2)-4}\sin 2x - \frac{1}{16-12-4}e^{-2x} \qquad \text{(Rule fails)}$$

$$= \frac{5}{24}\sin 2x - x\frac{1}{4D^3-6D}e^{-2x}$$

$$= \frac{5}{24}\sin 2x - x\frac{1}{4(-2)^3-6(-2)}e^{-2x} = \frac{5}{24}\sin 2x + \frac{xe^{-2x}}{20} \qquad \textbf{Ans.}$$

Example 12. *Solve* : $(D^3+1)y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$ (*Nagpur University, Summer* 2004)

Solution. $(D^3+1)y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$

A.E. is $m^3 + 1 = 0$

$(m+1)(m^2-m+1) = 0 \Rightarrow m = -1$

or $m = \frac{-(-1)\pm\sqrt{1-4}}{2} = \frac{1\pm i\sqrt{3}}{2} \Rightarrow m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\therefore \quad \text{C.F.} = C_1e^{-x} + e^{\frac{x}{2}}\left[C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^3+1}\left[\cos^2\left(\frac{x}{2}\right) + e^{-x}\right] = \frac{1}{D^3+1}\cos^2\left(\frac{x}{2}\right) + \frac{1}{D^3+1}e^{-x} \qquad [\text{Put } D = -1]$$

$$= \frac{1}{D^3+1}\left(\frac{1+\cos x}{2}\right)+\frac{1}{3D^2+1}e^{-x}$$

$$= \frac{1}{2}\frac{1}{D^3+1}e^{0x}+\frac{1}{2}\frac{1}{D^3+1}\cos x+\frac{1}{3(-1)^2+1}e^{-x} = \frac{1}{2}+\frac{1}{2}\frac{1}{-D+1}\cos x+\frac{1}{4}e^{-x}$$

$$= \frac{1}{2}-\frac{1}{2}\frac{(D+1)\cos x}{(D-1)(D+1)}+\frac{1}{4}e^{-x} = \frac{1}{2}-\frac{1}{2}\frac{(-\sin x+\cos x)}{(D^2-1)}+\frac{1}{4}e^{-x}$$

$$= \frac{1}{2}+\frac{1}{2}\frac{\sin x}{(D^2-1)}-\frac{1}{2}\frac{1}{(D^2-1)}\cos x+\frac{1}{4}e^{-x}$$

Put $D^2 = -1$

$$= \frac{1}{2}+\frac{1}{2}\frac{\sin x}{(-1-1)}-\frac{1}{2}\frac{1}{(-1-1)}\cos x+\frac{1}{4}e^{-x} = \frac{1}{2}-\frac{\sin x}{4}+\frac{\cos x}{4}+\frac{1}{4}e^{-x}$$

$$\text{P.I.} = \frac{1}{2}+\frac{1}{4}(\cos x-\sin x+e^{-x})$$

Hence, the complete solution is

$$y = C_1e^{-x}+e^{\frac{x}{2}}\left[C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right]+\frac{1}{2}+\frac{1}{4}(\cos x-\sin x+e^{-x})$$ **Ans.**

Example 13. *Solve the differential equation*

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = \sinh x+\sin\sqrt{2}\,x.$$ (*Nagpur University, Winter* 2001)

Solution. A.E. is $m^2 - 2m + 2 = 0$

$$\therefore \quad m = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2} = 1\pm i$$

$$\text{C.F.} = e^x (C_1 \cos x + C_2 \sin x) \qquad \left(\sinh x = \frac{e^x-e^{-x}}{2}\right)$$

$$\text{P.I.} = \frac{1}{D^2-2D+2}\sinh x+\frac{1}{D^2-2D-2}\sin\sqrt{2}x$$

$$\text{P.I.} = \frac{1}{D^2-2D+2}\left(\frac{e^x-e^{-x}}{2}\right)+\frac{1}{D^2-2D+2}\sin\sqrt{2}x$$

$$= \frac{1}{2}\left[\frac{1}{1-2+2}e^x-\frac{1}{1+2+2}e^{-x}\right]+\frac{1}{-2-2D+2}\sin\sqrt{2}x$$

$$= \frac{1}{2}e^x-\frac{1}{10}e^{-x}+\frac{1}{2\sqrt{2}}\cos\sqrt{2}x \qquad \left(\frac{1}{D}\sin\sqrt{2}x = \int\sin\sqrt{2}\,x\,dx\right)$$

Hence the solution is,

$$y = e^x(C_1\cos x+C_2\sin x)+\frac{1}{2}e^x-\frac{1}{10}e^{-x}+\frac{1}{2\sqrt{2}}\cos\sqrt{2}x.$$ **Ans.**

EXERCISE 12.4

Solve the following differential equations :

1. $\frac{d^2y}{dx^2}+6y = \sin 4x$ **Ans.** $C_1\cos\sqrt{6}x+C_2\sin\sqrt{6}x-\frac{1}{10}\sin 4x$

2. $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+3x=\sin t$ **Ans.** $e^{-t}[A\cos\sqrt{2}t+B\sin\sqrt{2}t]-\dfrac{1}{4}(\cos t-\sin t)$

3. $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+5x=\sin 2t$, given that when $t = 0$, $x = 3$ and $\dfrac{dx}{dt}=0$

Ans. $e^{-t}\left[\dfrac{55}{17}\cos 2t+\dfrac{53}{34}\sin 2t\right]-\dfrac{1}{17}(4\cos 2t-\sin 2t)$

4. $\dfrac{d^2y}{dx^2}-7\dfrac{dy}{dx}+6y=2\sin 3x$, given that $y = 1$, $\dfrac{dy}{dx}=0$ when $x = 0$.

Ans. $-\dfrac{13}{75}e^{6x}+\dfrac{27}{25}e^{x}+\dfrac{1}{75}(7\cos 3x-\sin 3x)$

5. $(D^3 + 1)\,y = 2\cos^2 x$

Ans. $C_1e^{-x}+e^{\frac{1}{2}x}\left(C_2\cos\dfrac{\sqrt{3}}{2}x+C_3\sin\dfrac{\sqrt{3}}{2}x\right)+1+\dfrac{1}{65}(-8\sin 2x+\cos 2x)$

6. $(D^2 + a^2)\,y = \sin ax$ (*A.M.I.E.T.E., June 2009*) (**Ans.** $C_1\cos ax+C_2\sin ax-\dfrac{x}{2a}\cos ax$

7. $(D^4 + 2a^2D^2 + a^4)\,y = 8\cos ax$ **Ans.** $(C_1+C_2x+C_3\cos ax+C_4\sin ax)-\dfrac{x^2}{a^2}\cos ax$

8. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=\sin 2x$ (*A.M.I.E.T.E., Summer* 2002)

Ans. $C_1e^{-x}+C_2e^{-2x}-\dfrac{1}{20}(3\cos 2x+\sin 2x)$

9. $\dfrac{d^2y}{dx^2}+y=\sin 3x\cos 2x$ **Ans.** $C_1\cos x+C_2\sin x+\dfrac{1}{48}[-\sin 5x-12x\cos x]$

10. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}-3y=2e^{2x}+10\sin 3x$ given that $y(0) = 2$ and $y'(0) = 4$

Ans. $\dfrac{29}{12}e^{3x}-\dfrac{1}{12}e^{-x}-\dfrac{2}{3}e^{2x}+\dfrac{1}{3}[\cos 3x-2\sin 3x]$

11. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=4\cos^2 x$ (*R.G.P.V., Bhopal, I Semester, June 2007*)

Ans. $C_1e^{-x}+C_2e^{-2x}-e^{2x}+\dfrac{1}{10}(3\sin 2x-\cos 2x)+1$

12. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+3y=\cos x+x^2$

Ans. $e^{x}[C_1\cos\sqrt{2}\,x+C_2\sin\sqrt{2}\,x]+\dfrac{1}{4}(\cos x-\sin x)+\dfrac{1}{3}(x^2+\dfrac{4}{3}x+\dfrac{2}{9})$

13. $(D^3-3D^2+4D-2)y=e^{x}+\cos x$

Ans. $(C_1+C_2\cos x+C_3\sin x)e^{x}+\dfrac{1}{10}(3\sin x+\cos x)$

14. $(D^3 - 4D^2 + 13D)\,y = 1 + \cos 2x$

Ans. $C_1+e^{2x}(C_2\cos 3x+C_3\sin 3x)+\dfrac{1}{290}(9\sin 2x+8\cos 2x)+\dfrac{x}{13}$

15. $(D^2 - 4D + 4)\,y = e^{2x} + x^3 + \cos 2x$

Ans. $(C_1+C_2x)e^{2x}+\dfrac{1}{2}x^2e^{2x}+\dfrac{1}{8}(2x^3+6x^2+9x+6)-\dfrac{1}{8}\sin 2x$

16. $\dfrac{d^2y}{dx^2}+n^2y=h\sin px$ $(P\neq n)$

where h, p and n are constants satisfying the conditions

$y = a$, $\dfrac{dy}{dx}=b$ for $x = 0$ **Ans.** $a\cos nx+\left(\dfrac{b}{n}-\dfrac{ph}{n(n^2-p^2)}\right)\sin nx+\dfrac{h\sin px}{(n^2-p^2)}$

17. $y''+y'-2y=-6\sin 2x-18\cos 2x$, $y(0) = 2$, $y'(0) = 2$ **Ans.** $-e^{-2x} + 3\cos 2x$

12.10 $\boxed{\dfrac{1}{f(D)}.e^{ax}\cdot\phi(x)=e^{ax}.\dfrac{1}{f(D+a)}.\phi(x)}$

$$D[e^{ax}\phi(x)]=e^{ax}D\phi(x)+ae^{ax}\phi(x)=e^{ax}(D+a)\phi(x)$$

$$D^2[e^{ax}\phi(x)]=D[e^{ax}(D+a)\phi(x)]=e^{ax}(D^2+aD)\phi(x)+ae^{ax}(D+a)\phi(x)$$

$$=e^{ax}(D^2+2aD+a^2)\phi(x)=e^{ax}(D+a)^2\phi(x)$$

Similarly, $\quad D^n[e^{ax}\phi(x)]=e^{ax}(D+a)^n\phi(x)$

$$f(D)[e^{ax}\phi(x)]=e^{ax}f(D+a)\phi(x)$$

$$e^{ax}\phi(x)=\frac{1}{f(D)}.[e^{ax}f(D+a)\phi(x)] \qquad ...(1)$$

Put $f(D+a)\phi(x)=X$, so that $\phi(x)=\dfrac{1}{f(D+a)}.X$

Substituting these values in (1), we get

$$e^{ax}\frac{1}{f(D+a)}X=\frac{1}{f(D)}[e^{ax}.X] \Rightarrow \frac{1}{f(D)}[e^{ax}.\phi(x)]=e^{ax}\frac{1}{f(D+a)}\phi(x)$$

Example 14. *Obtain the general solution of the differential equation*

$y''-2y'+2y=x+e^x\cos x.$ (*U.P. II Semester Summer*, 2002)

Solution. $\quad y''-2y'+2y=x+e^x\cos x$

A.E. is $\quad m^2-2m+2=0 \quad \Rightarrow m=1\pm i$

C.F. $=e^x(A\cos x+B\sin x)$

$$\text{P.I.}=\frac{1}{D^2-2D+2}x+\frac{1}{D^2-2D+2}e^x\cos x$$

where $\quad I_1=\dfrac{1}{D^2-2D+2}x=\dfrac{1}{2\left[1-D+\dfrac{D^2}{2}\right]}x=\dfrac{1}{2\left[1-\left(D-\dfrac{D^2}{2}\right)\right]}x$

$$=\frac{1}{2}\left[1-\left(D-\frac{D^2}{2}\right)\right]^{-1}x=\frac{1}{2}\left[1+\left(D-\frac{D^2}{2}\right)+\ldots\right]x=\frac{1}{2}\left[x+Dx-\frac{D^2}{2}x+\ldots\right]=\frac{1}{2}[x+1]$$

and, $\quad I_2=\dfrac{1}{D^2-2D+2}e^x\cos x=e^x\dfrac{1}{(D+1)^2-2(D+1)+2}\cos x=e^x\dfrac{1}{D^2+1}\cos x=e^x.x\dfrac{1}{2D}\cos x$

$$\left[\text{If } f(-a^2)=0,\text{ then } \frac{1}{f(D^2)}\cos ax=x\frac{1}{f'(-a^2)}\cos ax\right]$$

$$=\frac{1}{2}xe^x\sin x$$

$y=$ C.F. + P.I.

$$=e^x(A\cos x+B\sin x)+\frac{1}{2}(x+1)+\frac{1}{2}xe^x\sin x.$$ **Ans.**

Example 15. *Solve* : $(D^2 - 4D + 4)\, y = x^3\, e^{2x}$

Solution. $(D^2 - 4D + 4)\, y = x^3\, e^{2x}$

A.E. is $m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0 \quad \Rightarrow \quad m = 2, 2$

C.F. $= (C_1 + C_2 x)\, e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} x^3 \cdot e^{2x} = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \frac{1}{D^2} x^3 = e^{2x} \cdot \frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x} \cdot \frac{x^5}{20}$$

The complete solution is $y = (C_1 + C_2 x)\, e^{2x} + e^{2x} \cdot \frac{x^5}{20}$ **Ans.**

Example 16. *Solve* :

$$\frac{d^4 y}{dx^4} - y = \cos x.\cosh x$$ (*Nagpur University*, *Summer* 2003)

Solution. We have, $(D^4 - 1)\, y = \cos x \cosh x$

A.E. is $m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0 \quad \Rightarrow \quad m = \pm 1, \pm i$

C.F. $= C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$

$$\text{P.I.} = \frac{1}{D^4 - 1} \cos x \cosh x = \frac{1}{D^4 - 1} \cos x \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4 - 1} e^x \cos x + \frac{1}{D^4 - 1} e^{-x} \cos x\right] = \frac{1}{2}\left[e^x \frac{1}{(D+1)^4 - 1} \cos x + e^{-x} \frac{1}{(D-1)^4 - 1} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x + e^{-x} \frac{1}{D^4 - 4D^3 + 6D^2 - 4D} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1)^2 + 4D(-1) + 6(-1) + 4D} \cos x + e^{-x} \frac{1}{(-1)^2 - 4D(-1) + 6(-1) - 4D} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-5} \cos x + e^{-x} \frac{1}{-5} \cos x\right] = -\frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right) \cos x = -\frac{1}{5} \cosh x \cos x$$

Hence, the complete solution is

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x - \frac{1}{5} \cos x \cosh x$$ **Ans.**

Example 17. *Solve the differential equation* :

$$\frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 10\frac{dy}{dx} = e^{2x} \sin x$$ (*AMIETE, June 2010, Nagpur University*, *Summer* 2005)

Solution. $$\frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 10\frac{dy}{dx} = e^{2x} \sin x$$

$\Rightarrow \quad D^3 y - 7D^2 y + 10\, Dy = e^{2x} \sin x$

A.E. is

$m^3 - 7m^2 + 10\, m = 0 \quad \Rightarrow \quad (m-2)(m^2 - 5m) = 0$

$\Rightarrow \quad m\,(m-2)(m-5) = 0 \quad \Rightarrow \quad m = 0, 2, 5$

C.F $= C_1 e^{0x} + C_2 e^{2x} + C_3 e^{5x}$

$$\text{P.I.} = \frac{1}{D^3 - 7D^2 + 10D} e^{2x} \sin x = e^{2x} \frac{1}{(D+2)^3 - 7\,(D+2)^2 + 10(D+2)} . \sin x$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 12D + 8 - 7D^2 - 28D - 28 + 10D + 20} . \sin x$$

$$= e^{2x} \frac{1}{D^3 - D^2 - 6D} \sin x \quad = e^{2x} \frac{1}{(-1^2)\, D - (-1^2) - 6D} \sin x$$

$$= e^{2x} \frac{1}{-D + 1 - 6D} \sin x \;= e^{2x} \frac{1}{1 - 7D} \sin x \;= e^{2x} \frac{1 + 7D}{1 - 49D^2} \sin x \;= e^{2x} \frac{1 + 7D}{1 - 49(-1^2)} \sin x$$

$$= e^{2x} \frac{1 + 7D}{50} \sin x \;= \frac{e^{2x}}{50} (\sin x + 7 \cos x)$$

Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad y = C_1 + C_2\, e^{2x} + C_3\, e^{5x} + \frac{e^{2x}}{50} (\sin x + 7 \cos x) \qquad \textbf{Ans.}$$

Example 18. *A body executes damped forced vibrations given by the equation*

$$\frac{d^2 x}{dt^2} + 2k \frac{dx}{dt} + b^2 x = e^{-kt} \sin \omega t.$$

Solve the differential equation for both the cases when $\omega^2 \neq b^2 - k^2$ *and when* $\omega^2 = b^2 - k^2$.

(*U.P., II Semester, Summer* 2002)

Solution. The given equation is $(D^2 + 2kD + b^2)\, x = e^{-kt} \sin \omega\, t$, ...(1)

which is a linear differential equation with constant coefficients.

A.E. is $m^2 + 2km + b^2 = 0$ or $m = \dfrac{-2k \pm \sqrt{(4k^2 - 4b^2)}}{2} = -k \pm \sqrt{(k^2 - b^2)}$

As the given problem is on vibration, we must have $k^2 < b^2$

$$m = -k \pm \sqrt{-(b^2 - k^2)} = -k \pm i \sqrt{(b^2 - k^2)}$$

$$\text{C.F.} = e^{-kt} \left\{ C_1 \cos \sqrt{(b^2 - k^2)}\, t + C_2 \sin \sqrt{(b^2 - k^2)}\, t \right\}$$

$$\text{P.I.} = \frac{1}{D^2 + 2kD + b^2} e^{-kt} \sin \omega t = e^{-kt} \frac{1}{(D - k)^2 + 2k(D - k) + b^2} \sin \omega t$$

$$= e^{-kt} \frac{1}{D^2 + (b^2 - k^2)} \sin \omega t = e^{-kt} \frac{1}{-\omega^2 + (b^2 - k^2)} \sin \omega t, \text{ if } \omega^2 \neq b^2 - k^2 \quad \text{... (2)}$$

If $\omega^2 = b^2 - k^2$, then $\quad \text{P.I.} = e^{-kt} . t \dfrac{1}{2D} \sin \omega t = e^{-kt} \left(-\dfrac{t}{2\omega} \cos \omega t \right)$, ...(3)

Case. I. If $\omega^2 \neq b^2 - k^2$, the complete solution of (1) is

$$x = e^{-kt} \left\{ C_1 \cos \sqrt{(b^2 - k^2)}\, t + C_2 \sin \sqrt{(b^2 - k^2)}\, t \right\} + \frac{e^{-kt}}{(b^2 - k^2) - \omega^2} \sin \omega t \quad \text{[From (2)]}$$

Case II. If $\omega^2 = b^2 - k^2$, the complete solution of (1) is

$$x = e^{-kt} \{ C_1 \cos \omega t + C_2 \sin \omega t \} - \frac{e^{-kt} t \cos \omega t}{2\omega} \qquad \text{[From (3)]} \qquad \textbf{Ans.}$$

Example 19. *Solve* $(D^2 + 6D + 9) y = \dfrac{e^{-3x}}{x^3}$.

(*Nagpur University, Summer 2002, A.M.I.E.T.E., June 2009*)

Solution A.E. is $m^2 + 6m + 9 = 0$

$(m + 3)^2 = 0 \qquad \therefore \qquad m = -3, -3$

C.F. $= (C_1 + C_2x)\, e^{-3x}$

$$\text{P.I.} = \frac{1}{D^2+6D+9}\frac{e^{-3x}}{x^3} = e^{-3x}\frac{1}{(D-3)^2+6(D-3)+9}\frac{1}{x^3}$$

$$= e^{-3x}\frac{1}{D^2-6D+9+6D-18+9}\frac{1}{x^3} \qquad = e^{-3x}\frac{1}{D^2}(x^{-3})$$

$$= e^{-3x}\frac{1}{D}\left(\frac{x^{-2}}{-2}\right) = e^{-3x}\frac{x^{-1}}{(-2)\,(-1)} = \frac{e^{-3x}x^{-1}}{2} = \frac{e^{-3x}}{2x}$$

Hence, the solution is $y = (C_1+C_2x)e^{-3x} + \dfrac{e^{-3x}}{2x}$ **Ans.**

Example 20. *Solve* $(D^2 - 4D + 3)\, y = 2x\, e^{2x} + 3e^x \cos 2x$

Solution. The auxiliary equation is

$m^2 - 4\, m + 3 = 0$ which gives $m = 1, 3$

C.F. $= C_1e^x + C_2e^{3x}$

$$\text{P.I.} = \frac{1}{D^2-4D+3}2xe^{3x} + \frac{1}{D^2-4D+3}3e^x\cos 2x$$

$$= 2e^{3x}.\frac{1}{(D+3)^2-4(D+3)+3}x + 3e^x.\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x$$

$$= 2e^{3x}.\frac{1}{D^2+2D}x + 3e^x.\frac{1}{D^2-2D}\cos 2x = 2e^{3x}.\frac{1}{2D(1+D/2)}x + 3e^x.\frac{1}{-4-2D}\cos 2x$$

$$= e^{3x}.\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1} x - \frac{3e^x}{2}.\frac{1}{2+D}\cos 2x = e^{3x}.\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}\ldots\right)x - \frac{3e^x}{2}.\frac{2-D}{4-D^2}\cos 2x$$

$$= e^{3x}.\left(\frac{1}{D}-\frac{1}{2}+\frac{D}{4}\ldots\right)x - \frac{3e^x}{2}.\frac{2-D}{4+4}\cos 2x = e^{3x}.\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right) - \frac{3e^x}{16}(2\cos 2x + 2\sin 2x)$$

The complete solution is

$$y = C_1e^x + C_2e^{3x} + e^{3x}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{3e^x}{8}(\cos 2x + \sin 2x)$$

The term $\dfrac{e^{3x}}{4}$ has been omitted from the P.I., since C_2e^{3x} is present in the C.F. **Ans.**

Example 21. *Find the complete solution of* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

(*U.P. II Semester* 2003)

Solution. The auxiliary equation is

$$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad m^2 - 2m - m + 2 = 0$$

$$\Rightarrow \quad (m-2)\,(m-1) = 0 \quad \Rightarrow \quad m = 1, 2$$

C.F. $= C_1e^x + C_2e^{2x}$

$$\text{P.I.} = \frac{1}{D^2-3D+2}(xe^{3x} + \sin 2x) = \frac{1}{D^2-3D+2}xe^{3x} + \frac{1}{D^2-3D+2}\sin 2x$$

$$= e^{3x}\frac{1}{(D+3)^2-3(D+3)+2}x + \frac{1}{-4-3D+2}\sin 2x$$

$$= e^{3x}\frac{1}{D^2+6D+9-3D-9+2}x + \frac{1}{-3D-2}\sin 2x = e^{3x}\frac{1}{D^2+3D+2}x - \frac{1}{3D+2}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1+\left(\frac{3D+D^2}{2}\right)\right]^{-1} x - \frac{(3D-2)}{9D^2-4}\sin 2x = \frac{e^{3x}}{2}\left[1-\left(\frac{3D+D^2}{2}\right)+\ldots\right]x - \frac{(3D-2)}{9(-4)-4}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[x-\left(\frac{3D+D^2}{2}\right)x+\ldots\right] - \frac{(3D-2)}{-36-4}\sin 2x = \frac{e^{3x}}{2}\left[x-\frac{3}{2}\right]+\frac{3D-2}{40}\sin 2x$$

$$\Rightarrow \qquad \text{P.I.} = \frac{e^{3x}}{4}(2x-3)+\frac{1}{40}(6\cos 2x - 2\sin 2x) = \frac{e^{3x}}{4}(2x-3)+\frac{3}{20}\cos 2x - \frac{1}{20}\sin 2x$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad y = c_1 e^x + c_2 e^{2x} + \frac{e^{3x}}{4}(2x-3)+\frac{3}{20}\cos 2x - \frac{1}{20}\sin 2x \qquad \textbf{Ans.}$$

Example 22. *Solve* : $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x$

(*R.G.P.V., Bhopal, 2001, Nagpur University, Winter* 2002)

Solution. The given equation is $(D^2 - 2D + 1)\, y = xe^x \sin x$

A.E. is $m^2 - 2m + 1 = 0$ $\quad \therefore\ m = 1, 1$

C.F. $= (C_1 + C_2 x)\, e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x . x\sin x = e^x \frac{1}{(D+1-1)^2} x\sin x = e^x \frac{1}{D^2} x \sin x = e^x . \frac{1}{D}\int x \sin x\, dx$$

Integrating by parts

$$= e^x \frac{1}{D}[x(-\cos x) - \int(-\cos x)\,dx] = e^x . \frac{1}{D}(-x\cos x + \sin x)$$

$$= e^x \int(-x\cos x + \sin x)\,dx = e^x \left\{-x\sin x + \int 1.\sin x\, dx - \cos x\right\}$$

$$= e^x\, [-x \sin x - \cos x - \cos x] \qquad = -e^x\,(x \sin x + 2\cos x)$$

Hence, the complete solution is

$$y = (C_1 + C_2 x)\, e^x - e^x\,(x \sin x + 2 \cos x). \qquad \textbf{Ans.}$$

Example 23. *Solve* $(D^2 + 5D + 6)\, y = e^{-2x} \sec^2 x\, (1 + 2 \tan x)$ (*A.M.I.E.T.E., Summer 2003*)

Solution. $(D^2 + 5D + 6)\, y = e^{-2x} \sec^2 x\,(1 + 2\tan x)$

Auxiliary Equation is $m^2 + 5m + 6 = 0$

$$\Rightarrow \quad (m+2)(m+3) = 0 \quad \Rightarrow \quad m = -2, \text{ and } m = -3$$

Hence, complementry function (C.F.) $= C_1 e^{-2x} + C_2 e^{-3x}$

$$\text{P.I.} = \frac{1}{D^2+5D+6} e - 2x \sec 2x\,(1+2\tan x) = e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}\sec^2 x\,(1+2\tan x)$$

$$= e^{-2x}\frac{1}{D^2-4D+4+5D-10+6}\sec^2 x(1+2\tan x)$$

$$= e^{-2x}\frac{1}{D^2+D}\sec^2 x(1+2\tan x) = e^{-2x}\left[\frac{\sec^2 x}{D^2+D}+\frac{2\tan x\sec^2 x}{D^2+D}\right]$$

$$= e^{-2x}\frac{1}{D(D+1)}\sec^2 + \frac{1}{D(D+1)}2\tan x \sec^2 x$$

$$= e^{-2x}\left[\left(\frac{1}{D}-\frac{1}{D+1}\right)\sec^2 x + \right]\left(\frac{1}{D}-\frac{1}{D+1}\right)2\tan x \sec^2 x$$

$$= e^{-2x}\left[\frac{1}{D}\sec^2 x - \frac{1}{D+1}\sec^2 x + \frac{1}{D}2\tan x\sec^2 x - \frac{1}{D+1}2\tan x\sec^2 x\right]$$

$$= e^{-2x}\left[\tan x - e^{-x}\int ex.\sec 2x\,dx + \tan^2 x - e^{-x}\int 2e^x \tan x\sec 2x\,dx\right]$$

Now, $= e^{-2x}\int e^x \sec^2 x\,dx = e^x\sec^2 x - \int e^x.2\sec x\sec x\tan x.dx$

$= e^x\sec^2 x - 2\int ex\sec^2 x.\tan x\,dx$

$\therefore$ P.I. $= e^{-2x}\left[\tan x^{-e} - x.e^x\sec^2 x + 2e^{-x}\int e^x\sec x\tan x\,dx + \tan^2 x - 2e^{-x}\int e^x\sec^2 x\tan x\,dx\right]$

$= e^{-2x}[\tan x - \sec^2 x + \tan^2 x] = e^{-2x}[\tan x - (\sec^2 x - \tan 2x] = e^{-2x}(\tan x - 1)$

$\therefore$ Complete solution is

$\Rightarrow\ y = C.F. + P.I. = C_1e^{-2x} + C_2e^{-3x} + e^{-2x}(\tan x - 1)$

Example 24. *Solve the differential equation* $(D^2 - 4D + 4)\,y = 8x^2\,e^{2x}\sin 2x$

(U.P. II Semester, Summer 2008, Uttrakhand 2007, 2005, 2004; Nagpur University June 2008)

Solution. $(D^2 - 4D + 4)\,y = 8x^2\,e^{2x}\sin 2x$

A.E. is $(m^2 - 4m + 4) = 0 \quad\Rightarrow\quad (m-2)^2 = 0 \quad\Rightarrow\quad m = 2, 2$

C.F. $= (C_1 + C_2x)\,e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}8x^2e^{2x}\sin 2x = 8\frac{1}{(D-2)^2}x^2e^{2x}\sin 2x$$

$$= 8e^{2x}\frac{1}{(D-2+2)^2}x^2\sin 2x = 8e^{2x}\frac{1}{D^2}x^2\sin 2x$$

$$= 8e^{2x}\frac{1}{D}\left[x^2\frac{(-\cos 2x)}{2} - 2x\left(-\frac{\sin 2x}{4}\right) + 2\frac{\cos 2x}{8}\right] = 8e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 8e^{2x}\left[-\frac{x^2}{2}\left(\frac{\sin 2x}{2}\right) - \left(\frac{-2x}{2}\right)\left(-\frac{\cos 2x}{4}\right) + (-1)\left(-\frac{\sin 2x}{8}\right) + \frac{x}{2}\left(-\frac{\cos 2x}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sin 2x}{4}\right) + \frac{\sin 2x}{8}\right]$$

$= e^{2x}[-2x^2\sin 2x - 2x\cos 2x + \sin 2x - 2x\cos 2x + \sin 2x + \sin 2x]$

$= e^{2x}[-2x^2\sin 2x - 4x\cos 2x + 3\sin 2x] = -e^{2x}[4x\cos 2x + (2x^2 - 3)\sin 2x]$

Complete solution is, y = C.F. + P.I.

$y = (C_1 + C_2x)\,e^{2x} - e^{2x}[4x\cos 2x + (2x^2 - 3)\sin 2x]$ **Ans.**

EXERCISE 12.5

Solve the following equations :

1. $(D^2 - 5D + 6)\,y = e^x\sin x$ **Ans.** $y = C_1e^{2x} + C_2e^{3x} + \frac{e^x}{10}(3\cos x + \sin x)$
2. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = e^{2x}\sin x$ **Ans.** $y = C_1e^{2x} + C_2e^{5x} + \frac{e^{2x}}{10}(3\cos x - \sin x)$
3. $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x\cos x$ **Ans.** $y = C_1e^{-2x} + e^x(C_2\cos x + C_3\sin x) + \frac{xe^x}{20}(3\sin x - \cos x)$
4. $(D^2 - 4D + 3)\,y = 2xe^{3x} + 3e^{3x}\cos 2x$

   Ans. $y = C_1e^x + C_2e^{3x} + \frac{1}{2}e^{3x}(x^2 - x) + \frac{3}{8}e^{3x}(\sin 2x - \cos 2x)$

5. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=\dfrac{e^{-x}}{x^2}$ **Ans.** $y=(C_1+C_2x)e^{-x}-e^{-x}\log x$

6. $(D^2-4)y=x^2e^{3x}$ **Ans.** $y=C_1e^{2x}+C_2e^{-2x}+\dfrac{e^{3x}}{5}\left[x^2-\dfrac{12x}{5}+\dfrac{62}{25}\right]$

7. $(D^2-3D+2)y=2x^2e^{4x}+5e^{3x}$ **Ans.** $y=C_1e^x+C_2e^{2x}+\dfrac{e^{4x}}{54}[18x^2-30x+19]+\dfrac{5}{2}e^{3x}$

8. $\dfrac{d^2y}{dx^2}-4y=x\sinh x$ **Ans.** $y=C_1e^{2x}+C_2e^{-2x}-\dfrac{x}{3}\sinh x-\dfrac{2}{9}\cosh x$

9. $\dfrac{d^2y}{dt^2}+2h\dfrac{dy}{dt}+(h^2+p^2)y=ke^{-ht}\cos pt$ **Ans.** $y=e^{-ht}[A\cos pt+B\sin pt]+\dfrac{k}{2p}te^{-ht}\sin pt$

12.11 TO FIND THE VALUE OF $\dfrac{1}{f(D)}x^n\sin ax$.

Now $\dfrac{1}{f(D)}x^n(\cos ax+i\sin ax)=\dfrac{1}{f(D)}x^ne^{iax}=e^{iax}\dfrac{1}{f(D+ia)}x^n$

$$\boxed{\frac{1}{f(D)}\cdot x^n\sin ax=\text{Imaginary part of } e^{iax}\frac{1}{f(D+ia)}\cdot x^n}$$

$$\boxed{\frac{1}{f(D)}\cdot x^n\cos ax=\text{Real part of } e^{iax}\frac{1}{f(D+ia)}\cdot x^n}$$

Example 25. *Solve* $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x\sin x$

Solution. Auxiliary equation is $m^2-2m+1=0$ or $m=1,1$

C.F. $=(C_1+C_2x)e^x$

P.I. $=\dfrac{1}{D^2-2D+1}x\cdot\sin x$ $(e^{ix}=\cos x+i\sin x)$

$=$ Imaginary part of $\dfrac{1}{D^2-2D+1}x(\cos x+i\sin x)$ $=$ Imaginary part of $\dfrac{1}{D^2-2D+1}x\cdot e^{ix}$

$=$ Imaginary part of $e^{ix}\dfrac{1}{(D+i)^2-2(D+i)+1}\cdot x$

$=$ Imaginary part of $e^{ix}\dfrac{1}{D^2-2(1-i)D-2i}\cdot x$

$=$ Imaginary part of $e^{ix}\dfrac{1}{-2i}\left[1-(1+i)D-\dfrac{1}{2i}D^2\right]^{-1}\cdot x$

$=$ Imaginary part of $(\cos x+i\sin x)\left(\dfrac{i}{2}\right)[1+(1+i)D]x$

$=$ Imaginary part of $\dfrac{1}{2}(i\cos x-\sin x)[x+1+i]$

P.I. $=\dfrac{1}{2}x\cos x+\dfrac{1}{2}\cos x-\dfrac{1}{2}\sin x$

Complete solution is $y=(C_1+C_2x)e^x+\dfrac{1}{2}(x\cos x+\cos x-\sin x)$ **Ans.**

EXERCISE 12.6

Solve the following differential equations :

1. $(D^2 + 4)y = 3x \sin x$ **Ans.** $C_1 \cos 2x + C_2 \sin 2x + x \sin x - \frac{2}{3} \cos x$

2. $\frac{d^2y}{dx^2} - y = x \sin 3x + \cos x$ **Ans.** $C_1 e^x + C_2 e^{-x} - \frac{1}{10}\left[\frac{3}{5}\cos 3x + x \sin 3x + 5\cos x\right]$

3. $\frac{d^2y}{dx^2} - y = x \sin x + e^x + x^2 e^x$ **Ans.** $C_1 e^x + C_2 e^{-x} - \frac{1}{2}[x \sin x + \cos x] + \frac{x}{12} e^x (2x^2 - 3x + 9)$

4. $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$

Ans. $(C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x + \frac{1}{12} x^3 \sin x - \frac{1}{48}(x^4 - 9x^2)\cos x$

12.12 GENERAL METHOD OF FINDING THE PARTICULAR INTEGRAL OF ANY FUNCTION ϕ (*x*)

$$\text{P.I.} = \frac{1}{D-a}\phi(x) = y \qquad ...(1)$$

or $$(D-a)\frac{1}{D-a}\cdot\phi(x) = (D-a)\cdot y$$

$$\phi(x) = (D-a)y \qquad \text{or} \qquad \phi(x) = Dy - ay$$

$$\frac{dy}{dx} - ay = \phi(x) \text{ which is the linear differential equation.}$$

Its solution is $ye^{-\int a\,dx} = \int e^{-\int a\,dx}\cdot\phi(x)\,dx$ or $ye^{-ax} = \int e^{-ax}\cdot\phi(x)\,dx$

$$y = e^{ax}\int e^{-ax}\cdot\phi(x)\,dx \qquad \boxed{\frac{1}{D-a}\cdot\phi(x) = e^{ax}\int e^{-ax}\cdot\phi(x)\,dx}$$

Example 26. *Solve* $\frac{d^2y}{dx^2} + 9y = \sec 3x.$

Solution. Auxiliary equation is $m^2 + 9 = 0$ or $m = \pm 3i$,

$$\text{C.F.} = C_1 \cos 3x + C_2 \sin 3x$$

$$\text{P.I.} = \frac{1}{D^2+9}\cdot\sec 3x = \frac{1}{(D+3i)(D-3i)}\cdot\sec 3x = \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right]\cdot\sec 3x$$

$$= \frac{1}{6i}\cdot\frac{1}{D-3i}\cdot\sec 3x - \frac{1}{6i}\frac{1}{D+3i}\cdot\sec 3x \qquad ...(1)$$

Now, $\frac{1}{D-3i}\sec 3x = e^{3ix}\int e^{-3ix}\sec 3x\,dx$ $\quad\left[\frac{1}{D-a}\phi(x) = e^{ax}\int e^{-ax}\phi(x)\,dx\right]$

$$= e^{3ix}\int\frac{\cos 3x - i\sin 3x}{\cos 3x}dx = e^{3ix}\int(1 - i\tan 3x)\,dx = e^{3ix}\left(x + \frac{i}{3}\log\cos 3x\right)$$

Changing i to $-i$, we have $\frac{1}{D+3i}\sec 3x = e^{-3ix}\left(x - \frac{i}{3}\log\cos 3x\right)$

Putting these values in (1), we get

$$\text{P.I.} = \frac{1}{6i}\left[e^{3ix}\left(x + \frac{i}{3}\log\cos 3x\right) - e^{-3ix}\left(x - \frac{i}{3}\log\cos 3x\right)\right]$$

$$= \frac{x}{6i}e^{3ix} + \frac{e^{3ix}\log\cos 3x}{18} - \frac{xe^{-3ix}}{6i} + \frac{e^{-3ix}}{18}\log\cos 3x$$

$$= \frac{x}{3}\frac{e^{3ix}-e^{-3ix}}{2i}+\frac{1}{9}\cdot\frac{e^{3ix}+e^{-3ix}}{2}\log\cos 3x = \frac{x}{3}\sin 3x+\frac{1}{9}\cos 3x\cdot\log\cos 3x$$

Hence, complete solution is $y = C_1\cos 3x + C_2\sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\cdot\log\cos 3x$ **Ans.**

Example 27. *Solve* $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y = e^{-x}\sec^3 x.$ *(Nagpur, Winter 2000)*

Solution. Here we have

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y = e^{-x}\sec^3 x$$

$$(D^2+2D+2)y = e^{-x}\sec^3 x$$

A.E. is $m^2+2m+2=0 \quad \Rightarrow m = \frac{-2\pm\sqrt{4-8}}{2} = -1+i$

$C.F. = e^{-x}(C_1\cos x + C_2\sin x)$

$$\text{P.I.} = \frac{1}{D^2+2D+2}e^{-x}\sec^3 x = e^{-x}\frac{1}{(D-1)^2+2(D-1)+2}\sec^{3x}$$

$$= e^{-x}\frac{1}{D^2+1}\sec^{3x} = \frac{1}{(D+i)(D-i)}\sec^3 x = e^{-x}\frac{1}{2i}\left[\frac{1}{D-i}-\frac{1}{D+i}\right]\sec^3 x \qquad ...(1)$$

Now $\frac{1}{D-i}\sec^3 x = e^{ix}\int e^{-ix}\sec^3 x\,dx \qquad \left[\frac{1}{D-a}\phi(x) = e^{ax}\int e^{-ax}\phi(x)\,dx\right]$

$$= e^{ix}\int(\cos x - i\sin x)\sec^3 x\,dx = e^{ix}\int[\sec^2 x - i\tan x\sec^2 x]dx = e^{ix}\left[\tan x - \frac{i\tan^2 x}{2}\right] \quad ...(2)$$

Similarly $\frac{1}{D+i}\sec^3 x = e^{-ix}\left[\tan x + i\frac{\tan^2 x}{2}\right]$ ...(3) [changing i to $-i$]

Putting the values from (2) and (3) in (1), we get

$$P.I. = \frac{e^{-x}}{2i}\left[e^{ix}\left(\tan x - \frac{i\tan^2 x}{2}\right) - e^{-ix}\left(\tan x + i\frac{\tan^2 x}{2}\right)\right]$$

$$= e^{-x}\left[\tan x\frac{e^{ix}-e^{-ix}}{2i} - i\frac{\tan^2 x}{2}\left(\frac{e^{ix}+e^{-ix}}{2i}\right)\right] = e^{-x}\left[\tan x\sin x - \frac{\tan^2 x}{2}\cos x\right]$$

$$= e^{-x}\left[\tan x.\sin x - \frac{\tan x}{2}\frac{\sin x}{\cos x}\cos x\right] = e^{-x}\left[\tan x\sin x - \frac{\tan x}{2}\sin x\right] = e^{-x}\left(\frac{1}{2}\tan x\sin x\right)$$

Complete solution = C.F. + P.I.

$$= e^{-x}(c_1\cos x + c_2\sin x) + \frac{e^{-x}}{2}\tan x\sin x = e^{-x}\left[C_1\cos x + C_2\sin x + \frac{\sin x\tan x}{2}\right] \textbf{ Ans.}$$

Example 28. *Solve* $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = e^x\tan x.$

Solution. Here we have $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = e^x\tan x$

$$(D^2-2D+2)y = e^x\tan x$$

A.E. is $m^2-2m+2=0 \quad \Rightarrow m = \frac{2\pm\sqrt{4-8}}{2} = 1\pm i$

$C.F. = e^x(C_1\cos x + C_2\sin x)$

$$P.I. = \frac{1}{D^2 - 2D + 2} e^{-x} \tan x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} \tan x$$

$$= e^x \frac{1}{D^2 + 1} \tan x = e^x \frac{1}{(D+i)(D-i)} \tan x = \frac{e^x}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \tan x \qquad \ldots(1)$$

Now $\frac{1}{D-i} \tan x = e^{ix} \int e^{-ix} \tan x \, dx$

$$= e^{ix} \int (\cos x - i \sin x) \tan x \, dx = e^{ix} \int \left(\sin x - i \frac{\sin^2 x}{\cos x}\right) dx$$

$$= e^{ix} \int \left[\sin x - \frac{i(1 - \cos^2 x)}{\cos x}\right] dx = e^{ix} \int (\sin x - i \sec x + i \cos x] dx$$

$$= e^{ix}[-\cos x + i \sin x - i \log (\sec x + \tan x)] \qquad \ldots(2)$$

Similarly $\frac{1}{D+i} \tan x = e^{-ix}[-\cos x - i \sin x + i \log (\sec x + \tan x)] \qquad \ldots(3)$

On putting the values from (2) and (3) in (1), we get

$$P.I. = \frac{e^x}{2i}[e^{ix}(-\cos x + i \sin x - i \log (\sec x + \tan x) - e^{-ix}(-\cos x - i \sin x + i \log (\sec x + \tan x)]$$

$$= e^x \left[-\cos x \frac{e^{ix} - e^{-ix}}{2i} + \sin x \frac{e^{ix} + e^{-ix}}{2} - \log (\sec x + \tan x) \frac{e^{ix} + e^{-ix}}{2}\right]$$

$$= e^x[-\cos x \sin x + \sin x \cos x - \cos x \log (\sec x + \tan x)]$$

$$= -e^x \cos x \log (\sec x + \tan x)$$

Complete solution $=$ C.F. + P.I.

$= e^x (C_1 \cos x + C_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$ **Ans.**

EXERCISE 12.7

Solve the following differential equations :

1. $\frac{d^2y}{dx^2} + a^2 y = \sec ax$ (*R.G.P.V., Bhopal April, 2010*)

Ans. $C_1 \cos ax + C_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \cdot \log \cos ax$

2. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ **Ans.** $C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$

3. $(D^2 + 4) y = \tan 2x$ **Ans.** $C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \cos 2x \log (\sec 2x + \tan 2x)$

4. $\frac{d^2y}{dx^2} + y = (x - \cot x)$ (*A.M.I.E. Winter* 2002)

Ans. $C_1 \cos x + C_2 \sin x - x \cos^2 x - \sin x \log (\operatorname{cosec} x - \cot x)$

CHAPTER

13

CAUCHY – EULER EQUATIONS, METHOD OF VARIATION OF PARAMETERS

13.1 CAUCHY EULER HOMOGENEOUS LINEAR EQUATIONS

$$a_n x_n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{d x^{n-1}} + \dots + a_0 y = \phi(x) \qquad \dots (1)$$

where $a_0, a_1, a_2, \dots$ are constants, is called a homogeneous equation.

Put $\quad x = e^z, \quad z = \log_e x, \quad \frac{d}{dz} \equiv D$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = Dy$$

Again,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{1}{x} = \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}(D^2 - D)\,y$$

$$x^2 \frac{d^2 y}{dx^2} = (D^2 - D)\,y$$

or

$$\boxed{\boldsymbol{x^2 \frac{d^2 y}{dx^2} = D(D-1)\,y}}$$

Similarly.

$$\boxed{\boldsymbol{x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)\,y}}$$

The substitution of these values in (1) reduces the given homogeneous equation to a differential equation with constant coefficients.

Example 1. *Solve*: $\quad x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \qquad$ (*A.M.I.E. Summer* 2000)

Solution. We have, $\quad x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \qquad \dots (1)$

Putting $x = e^z$, $\quad D \equiv \frac{d}{dz}, \; x\frac{dy}{dx} = Dy, \; x^2\frac{d^2 y}{dx^2} = D(D-1)\,y$ in (1), we get

$$D(D-1)\,y - 2\,Dy - 4y = e^{4z} \quad \text{or} \quad (D^2 - 3D - 4)\,y = e^{4z}$$

A.E. is $m^2 - 3m - 4 = 0 \quad \Rightarrow \quad (m-4)(m+1) = 0 \quad \Rightarrow \quad m = -1, 4$

$$\text{C.F.} = C_1 e^{-z} + C_2 e^{4z}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D - 4} e^{4z} \qquad \text{[Rule Fails]}$$

$$= z \frac{1}{2D-3} e^{4z} = z \frac{1}{2(4)-3} e^{4z} = \frac{z e^{4z}}{5}$$

Thus, the complete solution is given by

$$y = C_1 e^{-z} + C_2 e^{4z} + \frac{z e^{4z}}{5} \quad \Rightarrow \quad y = \frac{C_1}{x} + C_2 x^4 + \frac{1}{5} x^4 \log x \quad \textbf{Ans.}$$

Example 2. *Solve:* $x^2 \dfrac{d^2 y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = e^x$

(*U.P., II Semester 2005; Nagpur University, Summer* 2001)

Solution. Given equation is $x^2 \dfrac{d^2 y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = e^x$ ...(1)

To solve (1), we put $x = e^z$ or $z = \log x$ and $D \equiv \dfrac{d}{dz}$

$$x \frac{dy}{dx} = Dy \quad \text{and} \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Substituting these values in (1), it reduces

$$[D(D-1) + 4D + 2]\, y = e^{e^z} \quad \Rightarrow \quad (D^2 + 3D + 2)\, y = e^{e^z}$$

$\therefore$ It's *A.E.* is $m^2 + 3m + 2 = 0$

$\therefore$ $(m+1)(m+2) = 0 \quad \therefore \quad m = -1, -2$

$\therefore$ $\text{C.F.} = C_1 e^{-z} + C_2 e^{-2z} = \dfrac{C_1}{x} + \dfrac{C_2}{x^2}$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2} e^{e^z} = \frac{1}{(D+1)(D+2)} e^{e^z}$$

$$= \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z} = \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z}$$

$$= e^{-z} \int e^{e^z} . e^z \, dz - e^{-2z} \int e^{e^z} . e^{2z} \, dz \qquad \left[\text{Since } \frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx\right]$$

Put $e^z = t$ so that $e^z dz = dt$

$$\text{P.I.} = e^{-z} \int e^t \, dt - e^{-2z} \int e^t . t \, dt = e^{-z} e^t - e^{-2z} (te^t - e^t)$$

$$= e^{-z} e^{e^z} - e^{-2z} (e^z e^{e^z} - e^{e^z})$$

$$= e^{-z} e^{e^z} - e^{-z} e^{e^z} + e^{-2z} e^{e^z} = e^{-2z} e^{e^z} = x^{-2} e^x = \frac{e^x}{x^2} \qquad (\because x = e^z)$$

Hence, the C.S. of (1) is $y = \text{C.F.} + \text{P.I.}$

$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{e^x}{x^2} \qquad \textbf{Ans.}$$

Example 3. *Solve* $(x^3 D^3 + x^2 D^2 - 2)\, y = x - \dfrac{1}{x^3}$ (*Nagpur University, Summer* 2000)

Solution. Put $x = e^z$, $z = \log x$

Let $D_1 \equiv \dfrac{d}{dz}$

Then $x^2 \dfrac{d^2 y}{dx^2} = D_1 (D_1 - 1)\, y \quad \Rightarrow \quad x^3 \dfrac{d^3 y}{dx^3} = D_1 (D_1 - 1)(D_1 - 2)\, y$

Substituting in the given equation, we get

$$[\, D_1 (D_1 - 1)(D_1 - 2) + D_1 (D_1 - 1) - 2\,]\, y = e^z + e^{-3z}$$

$$\Rightarrow \quad [\, D_1^3 - 3D_1^2 + 2D_1 + D_1^2 - D_1 - 2\,]\, y = e^z + e^{-3z}$$

$$\Rightarrow \quad (\, D_1^3 - 2D_1^2 + D_1 - 2\,)\, y = e^z + e^{-3z}$$

A.E. is $m^3 - 2m^2 + m - 2 = 0$

i.e. $(m - 2)(m^2 + 1) = 0$ *i.e.* $m = 2, \ \pm i$.

$$\text{C.F.} = C_1 e^{2z} + C_2 \cos z + C_3 \sin z$$

$$\text{P.I.} = \frac{1}{D_1^3 - 2D_1^2 + D_1 - 2} e^z + \frac{1}{D_1^3 - 2D_1^2 + D_1 - 2} e^{-3z}$$

$$= \frac{1}{1 - 2 + 1 - 2} e^z + \frac{1}{-27 - 18 - 3 - 2} e^{-3z} \qquad \begin{bmatrix} D_1 = 1 \\ D_1 = -3 \end{bmatrix}$$

$$= -\frac{e^z}{2} - \frac{1}{50} e^{-3z}$$

$$\therefore \quad y = C_1 e^{2z} + C_2 \cos z + C_3 \sin z - \frac{1}{2} e^z - \frac{1}{50} e^{-3z}$$

$$= C_1 x^2 + C_2 \cos (\log x) + C_3 \sin (\log x) - \frac{1}{2} x - \frac{1}{50} \cdot \frac{1}{x^3}$$ **Ans.**

Example 4. *Solve* $x^2 \dfrac{d^2 y}{dx^2} + x \dfrac{dy}{dx} + y = \sin (\log x^2)$ (*Nagpur University, Summer* 2005)

Solution. We have, $x^2 \dfrac{d^2 y}{dx^2} + x \dfrac{dy}{dx} + y = \sin (\log x^2)$ ... (1)

Let $x = e^z$, so that $z = \log x$, $D \equiv \dfrac{d}{dz}$

(1) becomes

$$D (D - 1)\, y + Dy + y = \sin (2z) \quad \Rightarrow \quad (D^2 + 1)\, y = \sin 2z$$

A.E. is $m^2 + 1 = 0$ or $m = \pm i$

$$C.F. = C_1 \cos z + C_2 \sin z$$

$$\text{P.I} = \frac{1}{D^2 + 1} \sin 2z = \frac{1}{-4 + 1} \sin 2z = -\frac{1}{3} \sin 2z$$

$$y = \text{C.F.} + \text{P.I.} = C_1 \cos z + C_2 \sin z - \frac{1}{3} \sin 2z$$

$$= C_1 \cos (\log x) + C_2 \sin (\log x) - \frac{1}{3} \sin (\log x^2)$$ **Ans.**

Example 5. *Solve* $x^3 \dfrac{d^3y}{dx^3} + 3x^2 . \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x + \log x$

(*AMIETE, June 2010, U.P., II Semester, Summer,* 2001)

Solution. Putting $x = e^z$ or $z = \log x$ and denoting $\dfrac{d}{dz}$ by D the equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\, y = e^z + z$$

$$\Rightarrow \qquad [D^3 + 1]\, y = e^z + z$$

$\therefore$ A.E. is $\qquad m^3 + 1 = 0$

$$\Rightarrow \qquad (m+1)(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\text{C.F.} = C_1 e^{-z} + e^{\frac{1}{2}z} \left\{ C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z \right\}$$

$$\text{P.I.} = \frac{1}{D^3+1}\{e^z + z\}$$

$$= \frac{1}{D^3+1} e^z + \frac{1}{D^3+1} z = \frac{e^z}{1+1} + (1 + D^3)^{-1} z$$

$$= \frac{1}{2} e^z + (1 - D^3 + \ldots)\, z = \frac{1}{2} e^z + z.$$

$\therefore$ Complete solution is

$$y = C_1 e^{-z} + e^{z/2} \left\{ C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z \right\} + \frac{1}{2} e^z + z$$

$$\Rightarrow \qquad y = C_1 x^{-1} + \sqrt{x} \left\{ C_2 \cos\left(\frac{\sqrt{3}}{2} \log x\right) + C_3 \sin\left(\frac{\sqrt{3}}{2} \log x\right) \right\} + \frac{1}{2} x + \log x \qquad \textbf{Ans.}$$

Example 6. *Solve:* $x^2 \dfrac{d^3y}{dx^3} + 3x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 \log x$ (*Nagpur University, Summer* 2003)

Solution. We have, $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = x^3 \log x$

Let $x = e^z$ so that $z = \log x$, $D \equiv \dfrac{d}{dz}$

The equation becomes after substitution

$$[D(D-1)(D-2) + 3D(D-1) + D]\, y = z\, e^{3z} \qquad \Rightarrow \qquad D^3 y = z e^{3z}$$

Auxiliary equation is $m^3 = 0 \Rightarrow \quad m = 0, 0, 0.$

C.F. $= C_1 + C_2 z + C_3 z^2 = C_1 + C_2 \log x + C_3 (\log x)^2$

$$\text{P.I.} = \frac{1}{D^3} . z e^{3z} = e^{3z} . \frac{1}{(D+3)^3} . z$$

$$= e^{3z} \frac{1}{27}\left(1 + \frac{D}{3}\right)^{-3} z = \frac{e^{3z}}{27}(1 - D)\, z = \frac{e^{3z}}{27}(z - 1) = \frac{x^3}{27}(\log x - 1)$$

Complete solution is $\qquad y = C_1 + C_2 \log x + C_3 (\log x)^2 + \dfrac{x^3}{27}(\log x - 1)$ **Ans.**

Example 7. *Solve the differential equation*

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y=x\log x.$$ (*Nagpur University, Winter 2002, Summer* 2000)

Solution. Putting $x = e^z$ or $z = \log x$ and $D\equiv\frac{d}{dz}$

On putting $x\frac{dy}{dx}=Dy$ and $x^2\frac{d^2y}{dx^2}=D(D-1)\,y$ in the given equation, we get

$$[D(D-1)-3D+5]\,y=ze^z$$

i.e. $(D^2-4D+5)\,y=ze^z$

It's A.E. is $m^2-4m+5=0$

$$\therefore \quad m=\frac{4\pm\sqrt{16-20}}{2}=\frac{4\pm 2i}{2}=2\pm i$$

$$\text{C.F.}=e^{2z}(C_1\cos z+C_2\sin z)$$

$$\text{P.I.}=\frac{1}{D^2-4D+5}ze^z$$

$$=e^z\cdot\frac{1}{(D+1)^2-4(D+1)+5}z \qquad \text{(by replacing } D \text{ by } D+1)$$

$$=e^z\cdot\frac{1}{D^2-2D+2}z=\frac{e^z}{2}\cdot\frac{1}{1+\left(\frac{D^2-2D}{2}\right)}z$$

$$=\frac{e^z}{2}\left[1+\left(\frac{D^2-2D}{2}\right)\right]^{-1}z=\frac{e^z}{2}\left[1-\left(\frac{D^2-2D}{2}\right)+\ldots\right]z$$

$$=\frac{e^z}{2}[z+Dz]=\frac{e^z}{2}(z+1).$$

Hence, the solution is

$$y=e^{2z}[C_1\cos z+C_2\sin z]+\frac{e^z}{2}(z+1)$$

$$y=x^2\,[C_1\cos(\log x)+C_2\sin(\log x)]+\frac{x}{2}(1+\log x)$$ **Ans.**

Example 8. *Solve the homogeneous linear differential equation.*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=(\log x)\sin(\log x)$$

(*Nagpur University, Winter* 2002, *U.P. II Semester, Summer* 2002)

Solution. Since given equation is homogeneous,

Put $x=e^z \Rightarrow \log x=z$

Also, $x\frac{dy}{dx}=Dy,\quad x^2\frac{d^2y}{dx^2}=D(D-1)\,y,\quad D\equiv\frac{d}{dz}$

The transformed equation is

$$D(D-1)\,y+Dy+y=z\sin z$$

$$(D^2-D+D+1)\,y=z\sin z$$

$$(D^2+1)\,y=z\sin z$$

A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$$\text{C.F.} = C_1 \cos z + C_2 \sin z$$

$$P.I. = \frac{1}{D^2+1} z \sin z$$

$$= \text{Imaginary part of } \frac{1}{D^2+1} z (\cos z + i \sin z)$$

$$= \text{Imaginary part of } \frac{1}{D^2+1} ze^{iz} \qquad = \text{Imaginary part of } e^{iz} \frac{1}{(D+i)^2+1} z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{D^2+2iD-1+1} z \quad = \text{Imaginary part of } e^{iz} \frac{1}{D^2+2iD} z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{2iD} \frac{1}{\left(1+\frac{D}{2i}\right)} z \qquad = \text{Imaginary part of } e^{iz} \frac{1}{2iD}\left(1-\frac{D}{2i}\right) z$$

$$= \text{Imaginary part of } e^{iz} \frac{1}{2iD}\left(z-\frac{1}{2i}\right) \qquad = \text{Imaginary part of } e^{iz} \frac{1}{2i}\left(\frac{z^2}{2}-\frac{z}{2i}\right)$$

$$= \text{Imaginary part of } \frac{1}{2i} (\cos z + i \sin z) \left(\frac{z^2}{2}-\frac{z}{2i}\right)$$

$$= \text{Imaginary part of } (\cos z + i \sin z) \left(\frac{z^2}{4i}+\frac{z}{4}\right)$$

$$= \text{Imaginary part of } (\cos z + i \sin z) \left(-i\frac{z^2}{4}+\frac{z}{4}\right) \quad = -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

Complete solution is, y = C.F. + P.I.

$$y = C_1 \cos z + C_2 \sin z - \frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

$$y = C_1 \cos (\log x) + C_2 \sin (\log x) - \frac{1}{4} (\log x)^2 \cos (\log x) + \frac{1}{4} (\log x) \sin (\log x) \quad \textbf{Ans.}$$

Example 9. *The radial displacement in a rotating disc at a distance r from the axis is given by* $r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0,$ *where k is a constant. Solve the equation under the conditions u = 0 when r = 0, u = 0 when r = a.* (*Nagpur University, Summer* 2008)

Solution. Here, we have

$$r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0 \qquad \text{... (1)}$$

On putting $r = e^z$, $\quad r\frac{du}{dr} = Dz$, $\quad r^2 \frac{d^2u}{dr^2} = D(D-1)z \quad$ in (1), we get

$$D(D-1)\, u + Du - u = -ke^{3z} \qquad \left[D \equiv \frac{d}{dz}\right]$$

$\Rightarrow \qquad (D^2 - D + D - 1)\, u = -ke^{3z} \quad \Rightarrow \quad (D^2 - 1)u = -k\, e^{3z}$

A.E. is $\quad m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$$\text{C.F.} = C_1 e^z + C_2 e^{-z} = C_1\, r + \frac{C_2}{r}$$

$$\text{P.I.} = \frac{1}{D^2 - 1}(-ke^{3z}) = -k\,\frac{1}{(3)^2 - 1}\, e^{3z} = -\frac{k}{8}\, e^{3z} = \frac{-k}{8}\, r^3$$

C.S. = C.F. + P.I.

$$u = C_1\, r + \frac{C_2}{r} - \frac{k}{8} r^3 \qquad \text{... (2)}$$

Putting $\quad u = 0, \quad r = 0 \quad$ in (2), we get

$$0 = C_2$$

Putting $\; C_2 = 0$ in (2), we get

$$u = C_1\, r - \frac{k}{8} r^3 \qquad \text{... (3)}$$

Putting $u = 0$, $r = a$ in (3), we get

$$0 = C_1\, a - \frac{k}{8} a^3 \quad \Rightarrow \quad C_1 = \frac{k}{8} a^2$$

Putting $\quad C_1 = \dfrac{k}{8} a^2$ in (3), we get

$$u = \frac{k}{8} a^2 r - \frac{k}{8} r^3$$

$$u = \frac{kr}{8}(a^2 - r^2) \qquad \textbf{Ans.}$$

13.2 LEGENDRE'S HOMOGENEOUS DIFFERENTIAL EQUATIONS

A linear differential equation of the form

$$(a + bx)^n \frac{d^n y}{dx^n} + a_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = X \qquad \text{... (1)}$$

where a, b, a_1, a_2, ... a_n are constants and X is a function of x, is called Legendre's linear equation.

Equation (1) can be reduced to linear differential equation with constant coefficients by the substitution.

$$a + bx = e^z \quad \Rightarrow \quad z = \log(a + bx)$$

so that
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a + bx} \cdot \frac{dy}{dz}$$

$\Rightarrow$
$$(a + bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\, Dy, \quad D \equiv \frac{d}{dz} \quad \Rightarrow \quad (a + bx)\frac{dy}{dx} = b\, Dy$$

where

Again
$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a + bx} \cdot \frac{dy}{dz}\right)$$

$$= -\frac{b^2}{(a + bx)^2}\frac{dy}{dz} + \frac{b}{(a + bx)} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\cdot\frac{b}{(a+bx)}$$

$$\Rightarrow \quad (a+bx)^2\frac{d^2y}{dx^2} = -b^2\frac{dy}{dz} + b^2\frac{d^2y}{dz^2}$$

$$= b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = b^2(D^2y - Dy) = b^2D(D-1)y$$

$$\Rightarrow \quad (a+bx)^2\frac{d^2y}{dx^2} = b^2 D(D-1)$$

Similarly, $(a+bx)^3\dfrac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y$

....

$$(a+bx)^n\frac{d^ny}{dx^n} = b^n D(D-1)(D-2)\ldots\ldots(D-n+1)y$$

Substituting these values in equation (1), we get a linear differential equation with constant coefficients, which can be solved by the method given in the previous section.

Example 10. *Solve* $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin 2\{\log(1+x)\}$

Solution. We have, $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin 2\{\log(1+x)\}$

Put $1 + x = e^z$ or $\log(1+x) = z$

$(1+x)\dfrac{dy}{dx} = Dy$ and $(1+x)^2\dfrac{d^2y}{dx^2} = D(D-1)y$, where $D \equiv \dfrac{d}{dz}$

Putting these values in the given differential equation, we get

$$D(D-1)y + Dy + y = \sin 2z \quad \text{or} \quad (D^2 - D + D + 1)y = \sin 2z$$

$$(D^2+1)y = \sin 2z$$

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F. = A\cos z + B\sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\sin 2z = \frac{1}{-4+1}\sin 2z = -\frac{1}{3}\sin 2z$$

Now, complete solution is y = C.F. + P.I.

$$\Rightarrow \quad y = A\cos z + B\sin z - \frac{1}{3}\sin 2z$$

$$\Rightarrow \quad y = A\cos\{\log(1+x)\} + B\sin\{\log(1+x)\} - \frac{1}{3}\sin 2\{\log(1+x)\}$$ **Ans.**

Example 11. *Solve:* $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

(*Uttarakhand II, Semester, June* 2007)

Solution. Let $3x + 2 = e^z \Rightarrow z = \log(3x+2)$ $\left[x = \dfrac{e^z - 2}{3}\right]$

So that $(3x+2)\dfrac{dy}{dx} = 3Dy$ and $(3x+2)^2\dfrac{d^2y}{dx^2} = 9D(D-1)y$ where $D \equiv \dfrac{d}{dz}$

Putting these values in the given differential equation, we get

$$9D\,(D-1)\,y+9Dy-36y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

$$\therefore\qquad (9D^2-36)\,y=\frac{1}{3}(e^{2z}-4\,e^z+4)+\frac{4}{3}e^z-\frac{8}{3}+1=\frac{e^{2z}}{3}-\frac{1}{3}$$

A.E. is $9m^2-36=0$

$m^2-4=0 \qquad \therefore \quad m=\pm 2$

$$\text{C.F.}=C_1\,e^{2z}+C_2\,e^{-2z}$$

$$\text{P.I.}=\frac{1}{9D^2-36}\left[\frac{e^{2z}}{3}-\frac{1}{3}\right]=\frac{1}{27}\frac{1}{D^2-4}e^{2z}-\frac{1}{3}\frac{1}{9D^2-36}e^{0z}$$

$$=\frac{1}{27}z\frac{1}{2D}e^{2z}-\frac{1}{3}\frac{1}{0-36}=\frac{1}{27}z\left(\frac{e^{2z}}{4}\right)+\frac{1}{108}=\frac{1}{108}[\,ze^{2z}+1\,]$$

Complete Solution is

$$y=C.F.+P.I.=C_1\,e^{2z}+C_2\,e^{-2z}+\frac{1}{108}(ze^{2z}+1)$$

$$y=C_1\,(3x+2)^2+\frac{C_2}{(3x+2)^2}+\frac{1}{108}[\,(3x+2)^2\log(3x+2)+1]$$ **Ans.**

EXERCISE 13.1

Solve the following differential equations:

1. $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=\dfrac{42}{x^4}$ **Ans.** $C_1\,x^2+C_2\,x^3+\dfrac{1}{x^4}$

2. $(x^2\,D^2-3xD+4)\,y=2x^2$ **Ans.** $(C_1+C_2\log x)\,x^2+x^2\,(\log x)^2$

3. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=\log x$ (*AMIETE, June 2010*) **Ans.** $(C_1+C_2\log x)\,x+\log x+2$

4. $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=\dfrac{12\log x}{x^2}$ **Ans.** $C_1+C_2\log x+2\,(\log x)^3$

5. $(x^2D^2-x\,D-3)\,y=x^2\log x$ (*A.M.I.E. Winter* 2001, *Summer* 2001)

   Ans. $\dfrac{C_1}{x}+C_2\,x^3-\dfrac{x^2}{3}\left(\log x+\dfrac{2}{3}\right)$

6. $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=x^2+\sin(5\log x)$

   Ans. $c_1\,x+c_2\,x^2+x^2\log x+\dfrac{1}{754}[15\cos(5\log x)-23\sin(5\log x)\,]$

7. $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+y=\log x\,\dfrac{\sin(\log x)+1}{x}$ (*AMIETE, Dec. 2009*)

   Ans. $y+C_1\,x^{2+\sqrt{3}}+C_2\,x^{2-\sqrt{3}}+\dfrac{1}{x}\left[\dfrac{382}{61}\cos\log x+\dfrac{54}{61}\sin(\log x)+6\log x\cos(\log x)+5\log x\sin(\log x)\right]+\dfrac{1}{6x}$

8. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2 \sin \log (1+x)$

Ans. $y = C_1 \cos \log (1+x) + C_2 \sin \log (1+x) - \log (1+x) \cos \log (1+x)$

9. Which of the basis of solutions are for the differential equation $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$

(*a*) $x, x\, I_n\, x,$ (*b*) $I_n\, x, e^x$ (*c*) $\dfrac{1}{x}, \dfrac{1}{x^2},$ (*d*) $\dfrac{1}{x^2}e^x, x\, I_n\, x$

(*A.M.I.E., Winter* 2001) **Ans.** (*a*)

10. The general solution of $x^2 \dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} + 9y = 0$ is

(*a*) $(C_1 + C_2x)\, e^{3x}$ (*b*) $(C_1 + C_{2n}\, x)\, x^3$ (*c*) $(C_1 + C_2x)\, x^3$ (*d*) $(C_1 + C_2\, l_n\, x)\, e^{x^3}$

(*AMIETE, Dec. 2009*) **Ans.** (*b*)

11. To transform $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \dfrac{1}{x}$ into a linear differential equation with constant coefficients, the required substitution is

(*a*) $x = \sin t$ (b) $x = t^2 + 1$ (*c*) $x = \log t$ (*d*) $x = e^t$

(*AMIETE, June 2010*) **Ans.** (*d*)

13.3 METHOD OF VARIATION OF PARAMETERS

To find particular integral of

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = X \qquad \dots (1)$$

Let complementary function $= Ay_1 + By_2$, so that y_1 and y_2 satisfy

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0 \qquad \dots (2)$$

Let us assume particular integral $y = uy_1 + vy_2$, ... (3)

where u and v are unknown functions of x.

Differentiation (3) *w.r.t.* x, we have $y' = uy_1' + vy_2' + u'y_1 + v'y_2$ assuming that u,v satisfy the equation

$$u'y_1 + v'y_2 = 0 \qquad \dots (4)$$

then $\quad y' = uy_1' + vy_2' \qquad \dots (5)$

Differentiating (5) *w.r.t.x*, we have $\quad y'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$

Substituting the values of y, y' and y'' in (1), we get

$$(uy_1'' + u'y_1' + vy_2'' + v'\,y_2') + b\,(uy_1' + vy_2') + c\,(uy_1 + vy_2) = X$$

$$\Rightarrow \quad u(y_1'' + by_1' + cy_1) + v(y_2'' + by_2' + cy_2) + (u'\,y_1' + v'\,y_2') = X \qquad \dots (6)$$

y_1 and y_2 will satisfy equation (1)

$$\therefore \quad y_1'' + by_1' + cy_1 = 0 \qquad \dots (7)$$

$$\text{and} \quad y_2'' + by_2' + cy_2 = 0 \qquad \dots (8)$$

Putting the values of expressions from (7) and (8) in (6), we get

$$\Rightarrow \quad u'\,y_1' + v'y_2' = X \qquad \dots (9)$$

Solving (4) and (9), we get

$$u' = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \frac{-y_2 X}{y_1 y_2' - y_1' y_2}$$

$$v' = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \frac{y_1 X}{y_1 y_2' - y_1' y_2}$$

$$\boxed{u = \int \frac{-y_2 X}{y_1 y_2' - y_1' y_2}\, dx}$$

$$\boxed{v = \int \frac{y_1 X}{y_1 y_2' - y_1' y_2}\, dx}$$

General solution = complementary function + particular integral.

Working Rule

Step 1. Find out the *C.F. i.e.,* $A\, y_1 + B\, y_2$

Step 2. Particular integral = $u\, y_1 + v\, y_2$

Step 3. Find u and v by the formulae

$$u = \int \frac{-y_2 X}{y_1 y_2' - y_1' y_2}\, dx, \qquad v = \int \frac{y_1 X}{y_1 y_2' - y_1' y_2}\, dx$$

Example 12. *Solve* $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x.$ (*Nagpur University, Summer* 2005)

Solution. $(D^2 + 1)\, y = \operatorname{cosec} x$

A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

C.F. $= A \cos x + B \sin x$

Here $y_1 = \cos x, \qquad y_2 = \sin x$

P.I. $= y_1\, u + y_2\, v$

where
$$u = \int \frac{-y_2 \,.\, \operatorname{cosec} x\, dx}{y_1 \,.\, y'_2 - y'_1 \,.\, y_2} = \int \frac{-\sin x \,.\, \operatorname{cosec} x\, dx}{\cos x\,(\cos x) - (-\sin x)\,(\sin x)}$$

$$= \int \frac{-\sin x \,.\, \dfrac{1}{\sin x}\, dx}{\cos^2 x + \sin^2 x} = -\int dx = -x$$

$$v = \int \frac{y_1 \,.\, X\, dx}{y_1 \,.\, y'_2 - y'_1\, y_2} = \int \frac{\cos x \,.\, \operatorname{cosec} x\, dx}{\cos x\,(\cos x) - (-\sin x)\,(\sin x)}$$

$$= \int \frac{\cos x \,.\, \dfrac{1}{\sin x}}{\cos^2 x + \sin^2 x}\, dx = \int \frac{\cot x\, dx}{1} = \log \sin x$$

$$P.I. = uy_1 + vy_2 = -x \cos x + \sin x\, (\log \sin x)$$

General solution = C.F. + P.I.

$$y = A \cos x + B \sin x - x \cos x + \sin x\, .(\log \sin x)$$ **Ans.**

Example 13. *Apply the mehtod of variation of parameters to solve*

$$\frac{d^2y}{dx^2} + y = \tan x \qquad (A.\ M.\ I.\ E.\ T.\ E.,\ Dec.\ 2010,\ Winter\ 2001,\ Summer\ 2000)$$

Solution. We have, $\frac{d^2y}{dx^2} + y = \tan x$

$(D^2 + 1)y = \tan x$

A.E. is $m^2 = -1$ or $m = \pm i$

C. F. $y = A\cos x + B\sin x$

Here, $y_1 = \cos x,$ $y_2 = \sin x$

$$y_1 \cdot y'_2 - y'_1 \cdot y_2 = \cos x\,(\cos x) - (-\sin x)\sin x = \cos^2 x + \sin^2 x = 1$$

P. I. $= u.\,y_1 + v.\,y_2$ where

$$u = \int \frac{-y_2 \tan x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\,dx = -\int \frac{\sin x \tan x}{1}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \int (\cos x - \sec x)\,dx = \sin x - \log(\sec x + \tan x)$$

$$v = \int \frac{y_1 \tan x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\,dx = \int \frac{\cos x \cdot \tan x}{1}\,dx = \int \sin x\,dx = -\cos x$$

P. I. $= u.\,y_1 + v.\,y_2$

$= [\sin x - \log(\sec x + \tan x)]\cos x - \cos x \sin x$

$= -\cos x \log(\sec x + \tan x)$

Complete solution is

$$y = A\cos x + B\sin x - \cos x \log(\sec x + \tan x)$$ **Ans.**

Example 14. *Use variation of parameters method to solve* $y'' + y = \sec x$

(*Nagpur University, Winter,* 2002, 2001, *U. P. Second Semester* 2002, *AMIETE, June 2010,* 2004)

Solution. We have, $$\frac{d^2y}{dx^2} + y = \sec x \qquad \ldots (1)$$

A. E. is $(m^2 + 1) = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

Complementary Function of (1) is

C. F. $= A\cos x + B\sin x$

Here $y_1 = \cos x,$ $y_2 = \sin x$

$$\text{P. I.} = uy_1 + vy_2 \qquad \ldots (2)$$

where $u = \int \frac{-y_2 \sec x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\,dx$ $\left[\begin{array}{r} y_1 y'_2 - y_1' y_2 = \cos x \cos x - (-\sin x)\sin x \\ = \cos^2 x + \sin^2 x = 1 \end{array}\right]$

On putting the values of y_2 and $y_1 y_2' - y_1' y_2$, we get

$$u = \int \frac{-\sin x \sec x}{1}\,dx = -\int \tan x\,dx = \log\cos x$$

$$v = \int \frac{y_1 \sec x}{y_1 y_2' - y_1' y_2} dx$$

On putting the values of y_1 and $y_1 y_2' - y_1' y_2$, we get

$$v = \int \frac{\cos x \cdot \sec x}{1} dx = \int dx = x$$

Putting the values of u and v in (2), we get

$$P.\ I. = \cos x \cdot \log \cos x + x \sin x$$

Complete solution is $y = C.\ F. + P.\ I.$

$\Rightarrow \qquad y = A \cos x + B \sin x + \cos x \log \cos x + x \sin x$ **Ans.**

Example 15. *Obtain general solution of the differential equation* $x^2 y'' + xy' - y = x^3 e^x$.
(*Nagpur University, Summer* 2004, *U. P. II Semester, Summer* 2002)

Solution. The given differential equation is $x^2 y'' + xy' - y = x^3 e^x$

$$\Rightarrow \qquad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^3 e^x \qquad \ldots (1)$$

Putting $x = e^z$ $\Rightarrow$ $D = \dfrac{d}{dz}$, $x \dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2 y}{dx^2} = D(D-1)\,y$, in (1), we get

$$D(D-1)\,y + D\,y - y = e^{3z} e^{e^z} \quad \Rightarrow \quad (D^2 - 1)y = e^{3z} e^{e^z}$$

A. E. is $m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$\therefore \qquad$ C. F. $= c_1 e^z + c_2 e^{-z}$

$= uy_1 + vy_2$, where $y_1 = e^{-z}$, $y = \left[y_1 = x_1\ y_2 = \dfrac{1}{x} \right]$

P.I. $= uy_1 + vy_2$

Also $u = -\displaystyle\int \frac{y_2 z}{y_1 y_2' - y_1' y_2} dz = -\int \frac{e^{-z} . e^{3z} . e^{e^z}}{e^z(-e^{-z}) - e^z(e^{-z})} dz = -\int \frac{e^{2z} e^{e^z}}{-1-1} dz$ $\quad \left[x = e^z, dx = e^z dz \quad dz = \dfrac{dx}{e^z} = \dfrac{dx}{x} \right]$

$$= \frac{1}{2} \int e^{2z} e^{e^z} dz = \int x^2 e^x \frac{dx}{x} = \frac{1}{2} \int x e^x dx$$

$$= \frac{1}{2} \left[xe^x - (1)\,e^x \right] = \frac{1}{2}(xe^x - e^x)$$

and $v = \displaystyle\int \frac{y_1 z}{y_1 y_2' - y_1' y_2} dz = \int \frac{e^z . e^{3z} . e^{e^z}}{e^z(-e^{-z}) - e^z(e^{-z})} dz = \int \frac{e^{4z} e^{e^z}}{-1-1} dz = \int \frac{x^4 e^x}{-2} \frac{dx}{x} = -\frac{1}{2} \int x^3 e^x dx$

$$= -\frac{1}{2}[x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x]$$

P.I. $= uy_1 + vy_2 = \dfrac{1}{2}(xe^x - e^x)\,x - \dfrac{1}{2}(x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x)\dfrac{1}{x}$

$$= \frac{1}{2}\left[x^2 - x - x^2 + 3x - 6 + \frac{6}{x} \right] e^x = \frac{1}{2}\left(2x - 6 + \frac{6}{x} \right) e^x = \left(x - 3 + \frac{3}{x} \right) e^x$$

Complete solution = C.F. + P.I.

$$y = (c_1 e^z + c_2 e^{-z}) + \left(x - 3 + \frac{3}{x} \right) e^x$$

$$= c_1 x + \frac{c_2}{x} + \left(x - 3 + \frac{3}{x} \right) e^x$$ **Ans.**

$$= \frac{e^x}{2}\left[x^4 - 3x^3 + 6x^2 - 6x - x^4 + 5x^3 - 20x^2 + 60x - 120 + \frac{120}{x}\right]$$

$$= \frac{e^x}{2}\left[2x^3 - 14x^2 + 54x - 120 + \frac{120}{x}\right]$$

$$y = C.\,F. + P.\,I.$$

$$= C_1\, x + \frac{C_2}{x} + (x^3 - 7x^2 + 27x - 60 + \frac{60}{x})\, e^x$$ **Ans.**

Example 16. *Solve by method of variation of parameters:*

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

(*Uttarakhand, II Semester, June* 2007, *A.M.I.E.T.E., Summer* 2001)

(*Nagpur University, Summer* 2001)

Solution. $$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

A. E. is $(m^2 - 1) = 0$

$m^2 = 1, \quad m = \pm 1$

$C.\,F. = C_1\, e^x + C_2\, e^{-x}$

$\therefore$ $P.I. = uy_1 + vy_2$

Here, $y_1 = e^x, \quad y_2 = e^{-x}$

and $y_1 \cdot y'_2 - y'_1 \cdot y_2 = -e^x\,.\,e^{-x} - e^x.e^{-x} = -2$

$$u = \int \frac{-y_2 X}{y_1.y'_2 - y'_1 . y_2}\,dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x}\,dx$$

$$= \int \frac{e^{-x}}{1+e^x}\,dx = \int \frac{dx}{e^x\,(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$

$$= \int e^{-x}\,dx - \int \frac{e^{-x}}{e^{-x}+1}\,dx = -e^{-x} + \log\,(e^{-x}+1)$$

$$v = \int \frac{y_1 X}{y_1.y'_2 - y'_1.y_2}\,dx = \int \frac{e^x}{-2}\,\frac{2}{1+e^x}\,dx = -\int \frac{e^x}{1+e^x}\,dx = -\log\,(1+e^x)$$

$$\text{P.I.} = u.\,y_1 + v.\,y_2 = [-e^{-x} + \log\,(e^{-x}+1)]\,e^x - e^{-x}\,\log\,(1+e^x)$$

$$= -1 + e^x\,\log\,(e^{-x}+1) - e^{-x}\,\log\,(e^x+1)$$

Complete solution = $y = C_1 e^x + C_2 e^{-x} - 1 + e^x\,\log\,(e^{-x}+1) - e^{-x}\,\log\,(e^x+1)$ **Ans.**

Example 17. *Apply the method of variation of parameters to solve*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$

(*U.P. II Semester Summer* 2005)

Solution. Given equation is $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \dfrac{e^x}{1+e^x}$

Auxiliary equation is $m^2 - 3m + 2 = 0 \quad \Rightarrow \quad (m-1)\,(m-2) = 0 \quad \Rightarrow \quad m = 1, 2$

C. F. $= C_1 e^x + C_2 e^{2x}$

$= C_1y_1 + C_2y_2$, Here $y_1 = e^x$, $y_2 = e^{2x}$

P. I. $= uy_1 + vy_2$,

$$u = \int \frac{-y_2\, X\, dx}{y_1y_2' - y_2y_1'} = \int \frac{-e^{2x}\dfrac{e^x}{1+e^x}}{e^x\,(2e^{2x}) - e^{2x}\,(e^x)}\, dx = \int \frac{\dfrac{-e^{3x}}{1+e^x}}{2e^{3x} - e^{3x}}\, dx$$

$$= \int \frac{-e^{3x}}{e^{3x}\,(1+e^x)}\, dx = -\int \frac{1}{1+e^x}\, dx = -\int \frac{e^{-x}}{e^{-x}+1}\, .\, dx \qquad \text{[Dividing by } e^x\text{]}$$

$= \log (e^{-x} + 1)$

Now, $$v = \int \frac{y_1X\, dx}{y_1y_2' - y_2y_1'} = \int \frac{e^x\left(\dfrac{e^x}{1+e^x}\right)dx}{e^x\,(2e^{2x}) - e^{2x}\,(e^x)} = \int \frac{\dfrac{e^{2x}}{1+e^x}}{2e^{3x} - e^{3x}}\, dx = \int \frac{e^{2x}}{e^{3x}\,(1+e^x)}\, dx$$

$$= \int \frac{1}{e^x\,(1+e^x)}\, dx = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx \qquad \text{[By Partial fraction]}$$

$$= \int \left(e^{-x} - \frac{e^{-x}}{e^{-x}+1}\right) dx = -e^{-x} + \log (e^{-x}+1)$$

P.I. $= uy_1 + vy_2$

P.I. $= e^x \log (e^{-x} + 1) + e^{2x} \{- e^{-x} + \log (e^{-x} + 1)\}$

P.I. $= e^x \log (e^{-x} + 1) - e^x + e^{2x} \log (e^{-x} + 1)$

Complete solution is

$y =$ C. F. + P. I.

$= C_1 e^x + C_2e^{2x} + e^x \log (e^{-x} + 1) - e^x + e^{2x} \log (e^{-x} + 1)$ **Ans.**

Example 18. *Solve by method of variation of parameters.*

$$\frac{d^2y}{dx^2} - y = \left(1 + \frac{1}{e^x}\right)^{-2} \qquad \text{(Nagpur University, Summer 2000)}$$

Solution. $$\frac{d^2y}{dx^2} - y = \left(\frac{e^x+1}{e^x}\right)^{-2} = \frac{e^{2x}}{(e^x+1)^2}$$

A. E. is $m^2 - 1 = 0$ $\therefore$ $m = \pm 1$

C. F. $= C_1e^{-x} + C_2e^x$

Let P. I. $= uy_1 + vy_2$, where $y_1 = e^{-x}$, $y_2 = e^x$

$y_1y_2' - y_1'y_2 = e^{-x} . e^x + e^{-x} . e^x = 2$

$$u = \int \frac{-y_2\, X}{y_1y_2' - y_1'\, y_2}\, dx = \frac{1}{2}\int - e^x . \frac{e^{2x}}{(1+e^x)^2}\, dx = -\frac{1}{2}\int \frac{e^{2x}}{(1+e^x)^2}\, e^x dx$$

Putting $t = 1 + e^x$, $dt = e^x\, dx$, we get

$$u = -\frac{1}{2}\int (t-1)^2\, \frac{dt}{t^2}$$

$$u = -\frac{1}{2}\int\left(1-\frac{2}{t}+t^{-2}\right)dt = -\frac{1}{2}(t-2\log t - t^{-1})$$

$$u = -\frac{1}{2}(1+e^x)+\log(1+e^x)+\frac{1}{2}(1+e^x)^{-1}$$

Now, $$v = \int\frac{y_1X}{y_1y_2'-y_1'y_2}\,dx = \frac{1}{2}\int\frac{e^{-x}.\,e^{2x}}{(1+e^x)^2}\,dx = \frac{1}{2}\int\frac{e^x\,dx}{(1+e^x)^2}$$

$$= \frac{1}{2}\int\frac{1}{t^2}\,dt = -\frac{1}{2}\frac{1}{t} = -\frac{1}{2}(1+e^x)^{-1}$$ where $t = 1 + e^x$

P.I. $= uy_1 + vy_2$

$$\text{P.I.} = e^{-x}\left[-\frac{1}{2}(1+e^x)+\log(1+e^x)+\frac{1}{2}(1+e^x)^{-1}\right]-\frac{1}{2}e^x(1+e^x)^{-1}$$

$$= -\frac{1}{2}(1+e^x)^{-1}\{e^x-e^{-x}\}+e^{-x}\left(-\frac{1}{2}\right)[(1+e^x)+e^{-x}\log(1+e^x)]$$

$$= -(1+e^x)^{-1}\sinh x - \frac{1}{2}e^{-x}(1+e^x)+e^{-x}\log(1+e^x)$$

Hence the solution is

$$y = C_1e^{-x}+C_2e^x-(1+e^x)^{-1}\sinh x-\frac{1}{2}e^{-x}(1+e^x)+e^{-x}\log(1+e^x)$$ **Ans.**

Example 19. *Solve by method of variation of parameters*

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx} = e^x\sin x$$ (*U.P. II Semester,* 2003)

Solution. $$\frac{d^2y}{dx^2}-2\frac{dy}{dx} = e^x\sin x$$

$\Rightarrow$ $(D^2 - 2D)y = e^x \sin x$

A. E. is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$

C. F. $= C_1 + C_2 e^{2x}$

P. I. $= uy_1 + vy_2$ where, $y_1 = 1$, $y_2 = e^{2x}$

$\therefore$ $$u = \int\frac{-y_2X\,dx}{y_1y_2'-y_1'y_2} = \int\frac{-e^{2x}.\,e^x\sin x}{1(2e^{2x})-0(e^{2x})}\,dx$$

$$= -\frac{1}{2}\int e^x\sin x\,dx \qquad \left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= -\frac{1}{2}\frac{e^x}{1+1}\{\sin x-\cos x\} = -\frac{e^x}{4}(\sin x-\cos x)$$

$$v = \int\frac{y_1X\,dx}{y_1y_2'-y_1'y_2} = \int\frac{1.\,e^x\sin x\,dx}{1(2e^{2x})-0(e^{2x})} = \frac{1}{2}\int e^{-x}\sin x\,dx$$

$$= \frac{1}{2}\frac{e^{-x}}{(-1)^2+(1)^2}[-\sin x-\cos x]$$

$$= -\frac{1}{2}\frac{e^{-x}}{2}(\sin x+\cos x) = -\frac{e^{-x}}{4}(\sin x+\cos x)$$

$$\text{P. I.} = uy_1 + vy_2 = -\frac{e^x}{4}(\sin x - \cos x)\,1 - \frac{e^{-x}}{4}(\sin x + \cos x)\,e^{2x}$$

$$= -\frac{e^x}{4}[\sin x - \cos x + \sin x + \cos x] = -\frac{e^x}{2}\sin x$$

The complete solution is

$$y = C.\,F. + P.\,I.$$

$$\Rightarrow \qquad y = C_1 + C_2 e^{2x} - \frac{e^x}{2}\sin x$$ **Ans.**

Example 20. *Solve* : $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin(e^x)$ (*Nagpur University, Winter* 2003)

Solution. The given equation can be written as

$$(D^2 + 3D + 2)y = \sin e^x$$

A. E. is $(m^2 + 3m + 2) = 0 \quad \Rightarrow \quad (m+1)(m+2) = 0$

$m = -1, -2,$

$\therefore$ $\quad$ C. F. $= C_1 e^{-x} + C_2 e^{-2x}$

P. I. $= uy_1 + vy_2$

where $\quad y_1 = e^{-x}, \quad y_2 = e^{-2x}$

$y_1' = -e^{-x}, \quad y_2' = -2e^{-2x}$

$$y_1 y_2' - y_2 y_1' = e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x}) = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$u = \int \frac{-y_2 X}{y_1 y'_2 - y'_1 y_2}\,dx = \int \frac{-e^{-2x}}{-e^{-3x}}\sin(e^x)\,dx$$

$$= \int e^x \sin(e^x)\,dx = -\cos(e^x)$$

$$v = \int \frac{y_1 X}{y_1 y'_2 - y'_1 y_2}\,dx = \int \frac{e^{-x}\sin(e^x)\,dx}{-e^{-3x}}$$

$$= -\int e^{2x}\sin(e^x)\,dx = -\int t\sin t\,dt \qquad [t = e^x \text{ so that } dt = e^x\,dx]$$

$$= t\cos t - \sin t = e^x\cos(e^x) - \sin(e^x)$$

Putting the values of u, v, y_1 and y_2 in (1), we get

$$\text{P.I.} = e^{-x}[-\cos(e^x)] + e^{-2x}[e^x\cos(e^x) - \sin(e^x)] = -e^{-2x}\sin(e^x)$$

The solution is $y = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x}\sin(e^x).$ **Ans.**

Example 21. *Solve by method of variation of parameters:*

$$\frac{d^2y}{dx^2} + y = (x - \cot x)$$ (*Nagpur University, Winter* 2001)

Solution. Here A. E. is $m^2 + 1 = 0 \qquad \therefore \qquad m = \pm i$

$$C.\,F. = C_1\cos x + C_2\sin x = C_1 y_1 + C_2 y_2$$

where $\quad y_1 = \cos x, \qquad y_2 = \sin x$

$y'_1 = -\sin x, \qquad y'_2 = \cos x$

$y_1 y'_2 - y_2 y'_1$ = cos x .cos x + sin x. sin x = 1

Let P. I. = $uy_1 + vy_2$ ... (1)

Where $u = \int \frac{-y_2 X}{y_1 y_2' - y_2 y'_1} dx = \int \frac{-\sin x (x - \cot x)}{1} dx$

$= \int \cos x \, dx - \int x \sin x \, dx = \sin x - \{x(-\cos x) + \sin x\} = x \cos x$

$v = \int \frac{y_1 X}{y_1 y'_2 - y_2 y'_1} dx = \int \cos x \,(x - \cot x)\, dx$

$= \int x \cos x \, dx - \int \frac{\cos^2 x}{\sin x} dx = \int x \cos x \, dx - \int \frac{1 - \sin^2 x}{\sin x} dx$

$= \int x \cos x \, dx - \int \operatorname{cosec} x \, dx + \int \sin x \, dx$

$= x \sin x + \cos x - \log(\operatorname{cosec} x - \cot x) - \cos x$

$= x \sin x - \log(\operatorname{cosec} x - \cot x)$

Putting the values of u, v, y_1, y_2 in (1), we get

P. I. = cos x. x cos x + sin x {x sin x – log (cosec x – cot x)}

$= x \cos^2 x + x \sin^2 x - \sin x \log(\operatorname{cosec} x - \cot x)$

$= x - \sin x \log(\operatorname{cosec} x - \cot x)$

Hence, complete solution is

$y = C_1 \cos x + C_2 \sin x + x - \sin x \log(\operatorname{cosec} x - \cot x)$ **Ans.**

Example 22. *Solve* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = e^{-x} \sec^3 x.$ (*Nagpur University, Winter* 2000)

Solution. A. E. is $m^2 + 2m + 2 = 0$

$$m = \frac{-2 + \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 + i$$

C. F. = $e^{-x}(C_1 \cos x + C_2 \sin x) = C_1 y_1 + C_2 y_2$

where $y_1 = e^{-x} \cos x \Rightarrow y_1' = -e^{-x} \sin x - e^{-x} \cos x$

$y_2 = e^{-x} \sin x \Rightarrow y'_2 = -e^{-x} \sin x + e^{-x} \cos x$

$y_1 y'_2 - y'_1 y_2 = e^{-x} \cos x(-e^{-x} \sin x + e^{-x} \cos x) - (-e^{-x} \cos x - e^{-x} \sin x)\, e^{-x} \sin x$

$= e^{-2x} (\sin^2 x + \cos^2 x - \sin x \cos x + \sin x \cos x)$

$= e^{-2x} (\sin^2 x + \cos^2 x) = e^{-2x}$

Let P.I. = $uy_1 + vy_2$

where $u = -\int \frac{y_2 X}{y_1 y'_2 - y'_1 y_2} dx = -\int \frac{e^{-x} \sin x . e^{-x} \sec^3 x}{e^{-2x}} dx$

$= -\int \sin x . \sec^3 x \, dx = -\int \tan x \cdot \sec^2 x \, dx = -\frac{\tan^2 x}{2}$

$v = \int \frac{y_1 X}{y_1 y'_2 - y'_1 y_2} dx = \int \frac{e^{-x} \cos x . e^{-x} \sec^3 x}{e^{-2x}} dx = \int \cos x . \sec^3 x \, dx = \int \sec^2 x \, dx = \tan x$

P. I. $= uy_1 + vy_2$

$= \frac{-\tan^2 x}{2} e^{-x} \cos x + \tan x . e^{-x} \sin x = e^{-x} \left[\frac{-\tan^2 x}{2} \cos x + \tan x . \sin x \right]$

$$= e^{-x}\left[\frac{-\sin x \,.\tan x}{2} + \tan x \,.\sin x\right] = \frac{1}{2}e^{-x}\sin x \tan x$$

Complete solution is

$$y = C.\,F. + P.\,I.$$

$$= e^{-x}(C_1 \cos x + C_2 \sin x) \; + \frac{1}{2}e^{-x}\sin x \tan x \qquad \textbf{Ans.}$$

Example 23. *Apply the method of variation of parameters to solve*

$$\frac{d^2y}{dx^2} - y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$$ (*Nagpur University, Summer* 2002)

Solution. The auxiliary equation is $m^2 - 1 = 0$

$$\therefore \quad m = \pm 1$$

$$\therefore \quad C.\,F. = C_1 e^x + C_2 e^{-x} = C_1 y_1 + C_2 y_2$$

where $y_1 = e^x$ and $y_2 = e^{-x}$

Let P. I. $= uy_1 + vy_2$

where $$u = \int \frac{-y_2 X}{y_1 y'_2 - y'_1 y_2}\,dx = \int -\frac{e^{-x}\{e^{-x}\sin(e^{-x}) + \cos(e^{-x})\}}{e^x(-e^{-x}) - e^x e^{-x}}\,dx$$

i.e. $$u = \frac{1}{2}\int [e^{-2x}\sin(e^{-x}) + e^{-x}\cos(e^{-x})]\,dx \quad (\text{Put } e^{-x} = t,\ -e^{-x}\,dx = dt)$$

$$= -\frac{1}{2}\int (t\sin t + \cos t)\,dt = -\frac{1}{2}\{-t\cos t + \sin t + \sin t\}$$

$$= \frac{1}{2}[t\cos t - 2\sin t] = \frac{1}{2}[e^{-x}\cos(e^{-x}) - 2\sin(e^{-x})]$$

$$v = \int \frac{y_1 X}{y_1 y'_2 - y'_1 y_2}\,dx = \int \frac{e^x\{e^{-x}\sin(e^{-x}) + \cos(e^{-x})\}}{e^x(-e^{-x}) - e^x e^{-x}}\,dx$$

$$= -\frac{1}{2}\int \{\sin(e^{-x}) + e^x\cos(e^{-x})\}dx$$

$$\left[e^{-x} = t \text{ and } -e^{-x}\,dx = dt \text{ i.e. } dx = -\frac{dt}{t}\right]$$

$$= \frac{1}{2}\int\left(\frac{\sin t}{t} + \frac{\cos t}{t^2}\right)dt = \frac{1}{2}\left\{\frac{-\cos t}{t} - \int\frac{\cos t}{t^2}\,dt + \int\frac{\cos t}{t}\,dt\right\}$$

$$= -\frac{1}{2}\frac{\cos t}{t} = -\frac{1}{2}e^x\cos(e^{-x})$$

$$P.\,I. = e^x\left\{\frac{1}{2}e^{-x}\cos(e^{-x}) - \sin(e^{-x})\right\} - \frac{1}{2}e^{-x}e^x\cos(e^{-x})$$

$$= -e^x\sin(e^{-x})$$

$\therefore$ Required solution is y = C. F. + P. I.

i.e. $$y = C_1 e^x + C_2 e^{-x} - e^x\sin(e^{-x}) \qquad \textbf{Ans.}$$

Example 24. *Solve* $(D^2 + 2D + 1)\,y = 4e^{-x}\log x$ *by method of variation of parameters.* (*Nagpur University, Winter* 2004)

Solution. $(D^2 + 2D + 1)\,y = 4e^{-x}\log x$

A. E. is $m^2 + 2m + 1 = 0$

$$(m+1)^2 = 0$$
$$m = -1, -1$$
$$\therefore \quad C.F. = (C_1 + C_2 x)e^{-x} \Rightarrow C.F. = C_1 e^{-x} + C_2 x e^{-x} = C_1 y_1 + C_2 y_2$$
$$y_1 = e^{-x} \Rightarrow y'_1 = -e^{-x}$$
$$y_2 = xe^{-x} \Rightarrow y'_2 = -xe^{-x} + e^{-x}$$
$$y_1 y'_2 - y'_1 y_2 = e^{-x}(-xe^{-x} + e^{-x}) + e^{-x}(x e^{-x})$$
$$= -xe^{-2x} + e^{-2x} + xe^{-2x} = e^{-2x}$$

Let $\quad$ P. I. $= uy_1 + vy_2$ ... (1)

where $\quad u = -\int \dfrac{y_2 X}{y_1 y'_2 - y'_1 y_2}\,dx$

$$u = -\int \frac{xe^{-x}\, 4e^{-x} \log x}{e^{-2x}}\,dx = -4\int x \log x\,dx = -4\left[\log x\,\frac{x^2}{2} - \int \frac{1}{x}\,\frac{x^2}{2}\,dx\right]$$
$$= -4\left[\frac{x^2}{2}\log x - \frac{1}{2}\,\frac{x^2}{2}\right] = -2x^2 \log x + x^2$$
$$u = x^2(1 - 2\log x)$$

and $\quad v = \int \dfrac{y_1 X}{y_1 y'_2 - y'_1 y_2}\,dx$

$$v = \int \frac{e^{-x}\,.\,4e^{-x}\log x}{e^{-2x}}\,dx = 4\int \log x\,dx = 4\int 1\cdot \log x\,dx = 4\left[(\log x)\,x - \int \frac{1}{x}\,.\,x\,dx\right]$$
$$= 4[x \log x - x]$$
$$v = 4x(\log x - 1)$$

Putting the values of u, v, y_1 and y_2 in (1), we get

$$\text{P. I.} = x^2(1 - 2\log x)e^{-x} + 4x(\log x - 1)x e^{-x} = x^2 e^{-x}[1 - 2\log x + 4\log x - 4]$$
$$\text{P.I.} = x^2 e^{-x}[2\log x - 3]$$
$$\text{C. S.} = \text{C. F.} + \text{P. I.}$$
$$y = (C_1 + C_2 x)e^{-x} + x^2 e^{-x}[2\log x - 3]$$ **Ans.**

EXERCISE 13.2

Solve the following equations by variation of parameters method.

1. $\dfrac{d^2y}{dx^2} - 4y = e^{2x}$ **Ans.** $y = C_1 e^{2x} + C_2 e^{-2x} + \dfrac{x}{4}e^{2x} - \dfrac{e^{2x}}{16}$
2. $\dfrac{d^2y}{dx^2} + y = \sin x$ **Ans.** $y = C_1 \cos x + C_2 \sin x - \dfrac{x}{2}\cos x + \dfrac{1}{4}\sin x$
3. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \sin x$ **Ans.** $y = C_1 e^{x} + C_2 e^{2x} + \dfrac{1}{10}(3\cos x + \sin x)$
4. $\dfrac{d^2y}{dx^2} + y = \sec x \tan x$ **Ans.** $y = C_1 \cos x + C_2 \sin x + x\cos x + \sin x \log \sec x - \sin x$
5. $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$ (*AMIETE, June 2010, 2009*) **Ans.** $y = (C_1 + x\,C_2)e^{3x} - e^{3x}\log x$

CHAPTER

14

Simultaneous Linear Differential Equations

14.1 SIMULTANEOUS DIFFERENTIAL EQUATIONS

If two or more dependent variables are functions of a single independent variable, the equations involving their deriavatives are called simultaneous equations, *e.g.*

$$\frac{dx}{dt} + 4y = t$$

$$\frac{dy}{dt} + 2x = e^t$$

The method of solving these equations is based on the process of elimination, as we solve algebraic simultaneous equations.

Example 1. *The equations of motions of a particle are given by*

$$\frac{dx}{dt} + \omega y = 0$$

$$\frac{dy}{dt} - \omega x = 0$$

Find the path of the particle and show that it is a circle.

(*R.G.P.V. Bhopal, Feb. 2006, U.P. II Semester summer* 2009)

Solution. On putting $\frac{d}{dt} \equiv \mathrm{D}$ in the equations, we have

$$Dx + \omega y = 0 \qquad \text{...(1)}$$

$$-\omega x + Dy = 0 \qquad \text{...(2)}$$

On multiplying (1) by w and (2) by D, we get

$$\omega Dx + \omega^2 y = 0 \qquad \text{...(3)}$$

$$-\omega Dx + D^2 y = 0 \qquad \text{...(4)}$$

On adding (3) and (4), we obtain

$$\omega^2 y + D^2 y = 0 \quad \Rightarrow \quad (D^2 + \omega^2)\, y = 0 \qquad \text{...(5)}$$

Now, we have to solve (5) to get the value of y.

A.E. is $m^2 + \omega^2 = 0 \quad \Rightarrow \quad m^2 = -\omega^2 \quad \Rightarrow \quad m = \pm i\omega$

$$\therefore \quad y = A \cos \omega t + B \sin \omega t \qquad \text{...(6)}$$

$\Rightarrow Dy = -A\, \omega \sin \omega t + B\omega \cos wt$

On putting the value of Dy in (2), we get

$-\omega x - A\omega \sin \omega t + B\omega \cos \omega t = 0$

$$\Rightarrow \quad \omega x = -A\omega \sin \omega t + B\omega \cos \omega t$$

$\Rightarrow \qquad x = -A \sin \omega t + B \cos \omega t \qquad ...(7)$

On squaring (6) and (7) and adding , we get

$$x^2 + y^2 = A^2(\cos^2\omega t + \sin^2\omega t) + B^2(\cos^2\omega t + \sin^2\omega t)$$

$$\Rightarrow \qquad x^2 + y^2 = A^2 + B^2$$

This is the equation of circle. **Proved.**

Example 2. *Solve the following differential equation*

$$\frac{dx}{dt} = y + 1, \quad \frac{dy}{dt} = x + 1$$

(*U.P. II Semester, 2009*)

Solution. Here, we have

$$Dx - y = 1 \qquad ...(1)$$

$$-x + Dy = 1 \qquad ...(2)$$

Multiplying (1) by D, we get

$$D^2x - Dy = D.1 \qquad ...(3)$$

Adding (2) and (3), we get

$$(D^2 - 1)\, x = 1 + D.1$$

$$\Rightarrow \qquad (D^2 - 1)\, x = 1 \text{ or } (D^2 - 1)x = e^0 \qquad [D.(1) = 0]$$

A.E. is $\qquad m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$\therefore \qquad \text{C.F.} = c_1e^t + c_2\, e^{-t}$

$$\text{P.I.} = \frac{1}{D^2 - 1} . e^0 = \frac{1}{0 - 1} e^0 = -1$$

$\therefore \qquad x = \text{C.F.} + \text{P.I.} = c_1e^t + c_2\, e^{-t} - 1$

From (1), $y = \dfrac{dx}{dt} - 1$

$$\Rightarrow \qquad y = \frac{d}{dt}(c_1e^t + c_2e^{-t} - 1) - 1$$

$$\Rightarrow \qquad y = c_1e^t - c_2\, e^{-t} - 1$$

and $\qquad x = c_1e^t + c_2e^{-t} - 1$ **Ans.**

Example 3. *Solve:*

$$\frac{dx}{dt} + y = \sin t$$

$$\frac{dy}{dt} + x = \cos t \qquad \text{where } y(0) = 0, \quad x(0) = 2$$

(*R.G.P.V., Bhopal, I Semester, April, 2010 June 2007*)

Solution. We have,

$$\frac{dx}{dt} + y = \sin t \qquad \Rightarrow \qquad Dx + y = \sin t \qquad ...(1)$$

$$\frac{dy}{dt} + x = \cos t \qquad \Rightarrow \qquad Dy + x = \cos t \qquad ...(2)$$

Multiplying (2) by D, we get

$$D^2y + Dx = D\cos t$$

$$D^2y + Dx = -\sin t \qquad ...(3)$$

Subtracting (1) from (3), we have

$$D^2y - y = -2\sin t$$

$$\Rightarrow \qquad (D^2 - 1)\, y = -2\sin t$$

A.E. is $m^2 - 1 = 0 \quad \Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = \pm 1$

$$\text{C.F.} = C_1 e^{t} + C_2 e^{-t}$$

$$\text{P.I.} = \frac{1}{D^2 - 1}(-2 \sin t)$$

$$\Rightarrow \quad \text{P.I.} = \frac{1}{-1-1}(-2 \sin t) = \sin t$$

Complete solution = C.F. + P.I.

$$y = C_1 e^{t} + C_2 e^{-t} + \sin t \qquad \text{... (4)}$$

Putting $y = 0$ and $t = 0$ in (4), we get

$$0 = C_1 + C_2 \quad \text{or} \quad C_2 = -C_1$$

On putting $C_2 = -C_1$ in (4), we get

$$y = C_1 e^{t} - C_1 e^{-t} + \sin t$$

On putting the value of y in (2), we get

$$D(C_1 e^{t} - C_1 e^{-t} + \sin t) + x = \cos t$$

$$C_1 e^{t} + C_1 e^{-t} + \cos t + x = \cos t$$

$$x = -C_1 e^{t} - C_1 e^{-t} \qquad \text{...(5)}$$

On putting $x = 2$, $t = 0$ in (5), we get

$$2 = -C_1 - C_1 \quad \Rightarrow \quad C_1 = -1$$

Putting the value of C_1 in (5) and (4), we have

$$x = e^{t} + e^{-t}$$

$$y = -e^{t} + e^{-t} + \sin t$$

Which is the required solution. **Ans.**

Example 4. *Solve:* $\dfrac{dx}{dt} + 4x + 3y = t$

$$\frac{dy}{dt} + 2x + 5y = e^t$$

[*U.P. II Semester, 2006*]

Solution. Here, we have

$$(D + 4)x + 3y = t \qquad \text{...(1)}$$

$$2x + (D + 5)y = e^t \qquad \text{...(2)} \quad \left(D \equiv \frac{d}{dt}\right)$$

To eliminate y, operating (1) by $(D + 5)$ and multiplying (2) by 3 then subtracting, we get

$$(D + 5)(D + 4)x + 3(D + 5)y - 3(2x) - 3(D + 5)y = (D + 5)t - 3e^t$$

$$[(D + 4)(D + 5) - 6]x = (D + 5)t - 3e^t$$

$$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

Auxiliary equation is

$$m^2 + 9m + 14 = 0 \Rightarrow m = -2, -7$$

$$\therefore \qquad \text{C.F.} = c_1 e^{-2t} + c_2 e^{-7t}$$

$$\text{P.I.} = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t)$$

$$= \frac{1}{D^2+9D+14}e^{0t} + 5\frac{1}{D^2+9D+14}t - 3\frac{1}{D^2+9D+14}e^t$$

$$= \frac{1}{0^2+9(0)+14}e^{0t} + 5.\ \frac{1}{14\left(1+\frac{9D}{14}+\frac{D^2}{14}\right)}t - 3\frac{1}{1^2+9(1)+14}e^t \quad \text{... (5)}$$

$$= \frac{1}{14} + \frac{5}{14}\left[1+\left(\frac{9D}{14}+\frac{D^2}{14}\right)\right]^{-1} t - \frac{1}{8}e^t = \frac{1}{14} + \frac{5}{14}\left[1-\left(\frac{9D}{14}+\frac{D}{14}\right)+\cdots\right]t - \frac{1}{8}e^t$$

$$= \frac{1}{14} + \frac{5}{14}\left(t-\frac{9}{14}\right) - \frac{1}{8}e^t = \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

$$x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

$$3y = t - \frac{dx}{dt} - 4x \qquad \text{[From (1)]}$$

$$= t - \frac{d}{dt}\left[c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t\right] - 4\left[c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{e^t}{8}\right]$$

$$3y = t + 2c_1e^{-2t} + 7c_2e^{-7t} - \frac{5}{14} + \frac{1}{8}e^t - 4c_1e^{-2t} - 4c_2e^{-7t} - \frac{10}{7}t + \frac{31}{49} + \frac{1}{2}e^t$$

$$\therefore\ y = \frac{1}{3}\left[-2c_1e^{-2t} + 3c_2e^{-7t} - \frac{3}{7}t + \frac{27}{98} + \frac{5}{8}e^t\right]$$

Hence,
$$x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

$$y = -\frac{2}{3}c_1e^{-2t} + c_2e^{-7t} - \frac{1}{7}t + \frac{9}{98} + \frac{5}{24}e^t \qquad \textbf{Ans.}$$

Example 5. *Solve:* $\frac{dx}{dt} + 5x + y = e^t,\ \frac{dy}{dt} + x + 5y = e^{5t}.$ (*R.G.P.V. Bhopal, 2003*)

Solution. Here, we have

$$(D+5)\,x + y = e^t \qquad \text{...(1)}$$

$$x + (D+5)y = e^{5t} \qquad \text{...(2)}\left(D \equiv \frac{d}{dt}\right)$$

Multiplying (1) by $(D + 5)$, we get

$$(D+5)^2x + (D+5)y = (D+5)\,e^t \qquad \text{...(3)}$$

Subtracting (3) form (2), we get

$$\left\{1-(D+5)^2\right\}x = e^{5t} - (D+5)e^t$$

$$\Rightarrow \quad [\,1 - D^2 - 10D - 25]x = e^{5t} - e^t - 5e^t$$

$$\Rightarrow \quad (D^2 + 10D + 24)x = 6e^t - e^{5t}$$

Auxiliary equation is

$$m^2 + 10m + 24 = 0 \quad \Rightarrow \quad (m+4)(m+6) = 0$$

$$\Rightarrow \quad m = -4, -6$$

$$\therefore \quad \text{C.F.} = c_1 e^{-4t} + c_2 e^{-6t}$$

$$\text{P.I.} = \frac{1}{D^2 + 10D + 24}(6e^t - e^{5t})$$

$$= \frac{6}{1^2 + 10(1) + 24}e^t - \frac{1}{(5)^2 + 10(5) + 24}e^{5t} = \frac{6e^t}{35} - \frac{e^{5t}}{99}$$

Thus $\quad x = \text{C.F.} + \text{P.I.} = c_1 e^{-4t} + c_2 e^{-6t} + \dfrac{6e^t}{35} - \dfrac{e^{5t}}{99}$

From (1),

$$y = e^t - (D+5)x = e^t - \frac{dx}{dt} - 5x$$

$$= e^t - \frac{d}{dt}\left(c_1 e^{-4t} + c_2 e^{-6t} + \frac{6e^t}{35} - \frac{e^{5t}}{99}\right) - 5\left(c_1 e^{-4t} + c_2 e^{-6t} + \frac{6e^t}{35} - \frac{e^{5t}}{99}\right)$$

$$= e^t + 4c_1 e^{-4t} + 6c_2 e^{-6t} - \frac{6e^t}{35} + \frac{5e^{5t}}{99} - 5c_1 e^{-4t} - 5c_2 e^{-6t} - \frac{30e^t}{35} + \frac{5e^{5t}}{99}$$

$$y = -\frac{1}{35}e^t - c_1 e^{-4t} + \frac{10}{99}e^{5t} + c_2 e^{-6t}$$

$$x = c_1 e^{-6t} + c_2 e^{-4t} + \frac{6e^t}{35} - \frac{e^{5t}}{99} \qquad \textbf{Ans.}$$

Example 6. *Solve the following system of differential equations*

$$Dx + Dy + 3x = \sin t \textit{ and}$$

$$Dx + y - x = \cos t \qquad (\textit{U.P. II Semester, Summer } 2003)$$

Solution. We have, $(D+3)\,x + Dy = \sin t \qquad \text{...(1)}$

$$(D-1)\,x + y = \cos t \qquad \text{...(2)}$$

Operating (2) by D, we get

$$D\,(D-1)\,x + Dy = -\sin t \qquad \text{...(3)}$$

Subtracting (1) from (3), we get

$$\{D\,(D-1) - (D+3)\}\,x = -2\sin t$$

$$\Rightarrow \quad \{D^2 - D - D - 3\}\,x = -2\sin t$$

$$\Rightarrow \quad (D^2 - 2D - 3)\,x = -2\sin t$$

A.E. is $\quad m^2 - 2m - 3 = 0 \quad \Rightarrow \quad (m+1)(m-3) = 0$

$$\Rightarrow \quad (D-3)(D+1) = 0 \quad \Rightarrow \quad m = 3, -1$$

$$\therefore \quad C.F. = c_1 e^{3t} + c_2 e^{-t}$$

$$P.I. = \frac{1}{D^2 - 2D - 3}(-2\sin t) = -2\frac{1}{(-1) - 2D - 3}\sin t$$

$$= 2\cdot\frac{1}{2\,(D+2)}\sin t = \frac{(D-2)}{D^2 - 4}\sin t$$

$$= \frac{D-2}{-1-4}\sin t = \frac{\cos t - 2\sin t}{-5} = \frac{1}{5}(2\sin t - \cos t)$$

$$x = \text{C.F.} + \text{P.I.} = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{5}(2\sin t - \cos t) \quad ...(4)$$

From (2), we get

$$(D-1)\,x + y = \cos t$$

$$\Rightarrow (D-1)\left\{c_1 e^{3t} + c_2 e^{-t} + \frac{1}{5}(2\sin t - \cos t)\right\} + y = \cos t$$

$$\Rightarrow y = \cos t - D\left\{c_1 e^{3t} + c_2 e^{-t} + \frac{1}{5}(2\sin t - \cos t)\right\} + \left\{c_1 e^{3t} + c_2 e^{-t} + \frac{1}{5}(2\sin t - \cos t)\right\}$$

$$= \cos t - 3c_1 e^{3t} + c_2 e^{-t} - \frac{1}{5}(2\cos t + \sin t) + c_1 e^{3t} + c_2 e^{-t} + \frac{1}{5}(2\sin t - \cos t)$$

$$= \cos t - 2c_1 e^{3t} + 2c_2 e^{-t} - \frac{1}{5}[3\cos t - \sin t]$$

$$= \frac{2}{5}\cos t + \frac{1}{5}\sin t + 2\,c_2 e^{-t} - 2\,c_1 e^{3t}$$

Here, $y = \frac{1}{5}(2\cos t + \sin t) - 2c_1 e^{3t} + 2c_2 e^{-t}$ ...(5)

(4) and (5) are the required solutions **Ans.**

Example 7. *Solve the simultaneous equations*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t \quad ...(1)$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t \quad ...(2)$$

(*U.P., B. Pharm* 2005, *II Semester, Summer* 2001)

Solution. $Dx + (D-2)\,y = 2\cos t - 7\sin t$ ...(3)

$(D+2)\,x - Dy = 4\cos t - 3\sin t$ ...(4)

Operating (3) by D (4) by $(D-2)$, we get

$$\Rightarrow \begin{array}{ll} D^2x + D(D-2)\,y = -2\sin t - 7\cos t & ...(5) \\ (D^2-4)\,x - D(D-2)\,y = (D-2)\,4\cos t - (D-2)\,3\sin t & ...(6) \end{array}$$

On adding (5) and (6), we get

$$(D^2 + D^2 - 4)\,x = -2\sin t - 7\cos t - 4\sin t - 8\cos t - 3\cos t + 6\sin t$$

$$\Rightarrow (2D^2 - 4)\,x = -18\cos t$$

$$\Rightarrow (D^2 - 2)\,x = -9\cos t$$

A.E. is $m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$, C.F. $= c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$

$$P.I. = \frac{1}{D^2 - 2}(-9\cos t) = \frac{-9}{-1-2}\cos t = 3\cos t$$

$$x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 3\cos t \quad ...(7)$$

Putting the value of x in (2), we get

$$\frac{dy}{dt} = \sqrt{2}\,c_1 e^{\sqrt{2}t} - \sqrt{2}\,c_2 e^{-\sqrt{2}t} - 3\sin t + 2c_1 e^{\sqrt{2}t} + 2c_2 e^{-\sqrt{2}t} + 6\cos t - 4\cos t + 3\sin t$$

$$\frac{dy}{dt} = (2+\sqrt{2})\,c_1 e^{\sqrt{2}t} + (2-\sqrt{2})\,c_2 e^{-\sqrt{2}t} + 2\cos t$$

On integrating, we get

$$\Rightarrow y = (\sqrt{2}+1)\, c_1 e^{\sqrt{2}t} - (\sqrt{2}-1)\, c_2 e^{-\sqrt{2}t} + 2\sin t + c_3 \qquad ...(8)$$

Relations (7) and (8) are the required solutions **Ans.**

Example 8. *Solve* $\dfrac{dx}{dt} = 2y,\ \dfrac{dy}{dt} = 2z,\ \dfrac{dz}{dt} = 2x$ (*Uttarakhand, II Semester, June* 2007)

Solution. Here, we have

$$\frac{dx}{dt} = 2y \quad \Rightarrow \quad Dx = 2y \qquad ...(1)$$

$$\frac{dy}{dt} = 2z \quad \Rightarrow \quad Dy = 2z \qquad ...(2)$$

$$\frac{dz}{dt} = 2x \quad \Rightarrow \quad Dz = 2x \qquad ...(3)$$

From (1), we have $\dfrac{dx}{dt} = 2y$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = \frac{2\,dy}{dt} = 2\,(2z) = 4z \qquad \left[\text{Using (2), } \frac{dy}{dt} = 2z\right]$$

$$\frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 4\,(2x) = 8x \qquad \left[\text{Using (3), } \frac{dz}{dt} = 2x\right]$$

$$\Rightarrow \quad \frac{d^3x}{dt^3} - 8x = 0 \quad \Rightarrow \quad (D^3 - 8)\, x = 0$$

A.E. is $\quad m^3 - 8 = 0 \quad \Rightarrow \quad (m-2)(m^2 + 2m + 4) = 0$

$\Rightarrow \quad m - 2 = 0 \quad \Rightarrow \quad m = 2$

or $\quad m^2 + 2m + 4 = 0 \quad \Rightarrow \quad m = \dfrac{-2 \pm \sqrt{4-16}}{2} = \dfrac{-2 \pm i\sqrt{12}}{2} = -1 \pm i\sqrt{3}$

So the C.F. of x is

$$x = C_1 e^{2t} + e^{-t}(A\cos\sqrt{3}t + B\sin\sqrt{3}t) \qquad ...(4)$$

$$[A = C_2\cos\alpha,\ B = C_2\sin\alpha] \qquad \left[\begin{array}{l}\tan\alpha = \dfrac{B}{A} \\ \alpha = \tan^{-1}\left(\dfrac{B}{A}\right)\end{array}\right]$$

$$x = C_1 e^{2t} + e^{-t}[C_2\cos\alpha\cos\sqrt{3}t + C_2\sin\alpha\sin\sqrt{3}t]$$

$$x = C_1 e^{2t} + e^{-t} C_2\cos(\sqrt{3}t - \alpha) = C_1 e^{2t} + C_2 e^{-t}\cos(\sqrt{3}t - \alpha)$$

From (3), we have $\dfrac{dz}{dt} = 2x$

$$\Rightarrow \frac{dz}{dt} = 2C_1 e^{2t} + 2C_2 e^{-t}\cos(\sqrt{3}t - \alpha) \qquad \text{[On putting the value of } x\text{]}$$

$$z = C_1 e^{2t} + 2C_2 \frac{e^{-t}}{\sqrt{1+3}}\cos(\sqrt{3}t - \alpha - \beta) \qquad \left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx - \beta)\right]$$

$$\Rightarrow \quad z = C_1 e^{2t} + 2C_2 \frac{e^{-t}}{\sqrt{1+3}}\cos\left[\sqrt{3}t - \alpha - \frac{2\pi}{3}\right] \qquad \left[\beta = \tan^{-1}\frac{\sqrt{3}}{-1} = \frac{2\pi}{3}\right]$$

$$\Rightarrow \quad z = C_1 e^{2t} + C_2 e^{-t} \cos\left(\sqrt{3}\,t - \alpha + \frac{4\pi}{3}\right) \qquad ...(5) \qquad \left[-\frac{2\pi}{3} = \frac{4\pi}{3}\right]$$

From (2), we have $\dfrac{dy}{dx} = 2z$

$$\Rightarrow \quad \frac{dy}{dt} = 2C_1 e^{2t} + 2C_2 e^{-t} \cos\left(\sqrt{3}\,t - \alpha + \frac{4\pi}{3}\right) \qquad \text{[On putting the value of } z]$$

$$\Rightarrow \quad y = \int 2C_1 e^{2t}\,dt + 2C_2 \int e^{-t} \cos\left(\sqrt{3}\,t - \alpha + \frac{4\pi}{3}\right) dt \qquad \left(\gamma = \tan^{-1}\frac{\sqrt{3}}{-1} = \frac{2\pi}{3}\right)$$

$$y = C_1 e^{2t} + 2C_2 \frac{e^{-x}}{\sqrt{1+3}} \cos\left(\sqrt{3}\,t - \alpha + \frac{4\pi}{3} - \gamma\right)$$

$$\Rightarrow \quad y = C_1 e^{2t} + 2C_2 \frac{e^{-t}}{\sqrt{1+3}} \cos\left(\sqrt{3}\,t - \alpha + \frac{4\pi}{3} - \frac{2\pi}{3}\right)$$

$$y = C_1 e^{2t} + C_2 e^{-t} \cos\left(\sqrt{3}\,t - \alpha + \frac{2\pi}{3}\right) \qquad ...(6)$$

Relations (4), (5) and (6) are the required solutions. **Ans.**

Example 9. *Solve the following simultaneous equations* :

$$\frac{d^2x}{dt^2} - 3x - 4y = 0, \quad \frac{d^2y}{dt^2} + x + y = 0 \qquad \text{(U.P. II Semester, Summer 2005)}$$

Solution. We have, $\dfrac{d^2x}{dt^2} - 3x - 4y = 0$

$$\frac{d^2y}{dt^2} + x + y = 0$$

$$(D^2 - 3)x - 4y = 0 \qquad ...(1)$$

$$x + (D^2 + 1)\,y = 0 \qquad ...(2)$$

Operating equation (2) by $(D^2 - 3)$, we get

$$(D^2 - 3)x + (D^2 - 3)(D^2 + 1)\,y = 0 \qquad ...(3)$$

Subtracting (3) from (1), we get

$$-4y - (D^2 - 3)(D^2 + 1)\,y = 0 = \quad \Rightarrow \quad -4y - (D^4 - 2D^2 - 3)\,y = 0$$

$$\Rightarrow \quad (D^4 - 2D^2 - 3 + 4)\,y = 0 \quad \Rightarrow \quad (D^4 - 2D^2 + 1)\,y = 0$$

$$\Rightarrow \quad (D^2 - 1)^2\,y = 0$$

A.E. is $\quad (m^2 - 1)^2 = 0 \quad \Rightarrow \quad (m^2 - 1) = 0 \quad \Rightarrow \quad m = \pm 1$

$$y = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t} \qquad ...(4)$$

From (2), we have

$$x = -(D^2 + 1)y$$

$$= -D^2 y - y$$

$$= -D^2[(c_1 + c_2 t)\,e^t + (c_3 + c_4 t)e^{-t}] - [(c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t}]$$

$$= -D\,[\{(c_1 + c_2 t)e^t + c_2 e^t\} + \{(c_3 + c_4 t)(-e^{-t}) + c_4 e^{-t}\}] - [(c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t}]$$

$$= -[(c_1 + c_2 t)\,e^t + c_2 e^t + c_2 e^t + (c_3 + c_4 t)(e^{-t}) - c_4 e^{-t} - c_4 e^{-t}] - [(c_1 + c_2 t)\,e^t + (c_3 + c_4 t)e^{-t}]$$

$$= -[(c_1 + c_2 t + 2c_2 + c_1 + c_2 t)\,e^t + (c_3 + c_4 t - 2c_4 + c_3 + c_4 t)\,e^{-t}]$$

$$= -[(2c_1 + 2c_2 + 2c_2 t)e^t + (2c_3 - 2c_4 + 2c_4 t)e^{-t}] \qquad ...(5)$$

Relations (4) and (5) are the required solutions. **Ans.**

Example 10. *A mechanical system with two degrees of freedom satisfies the equations:*

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$$

$$2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$$

Obtain expressions for x and y in terms of t, given $x, y \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at $t = 0$

(R.G.P.V., Bhopal, I Sem. 2003)

Solution. $2D^2x + 3Dy = 4$ ... (1)

$-3Dx + 2D^2y = 0$ ... (2)

Multiplying (1) by 3 and (2) by $2D$ We get

$6D^2x + 9Dy = 12$ ...(3)

$-6D^2x + 4D^3y = 0$ ...(4)

Adding (3) and (4), we have

$$4D^3y + 9Dy = 12$$

$$\Rightarrow (4D^3 + 9D)\,y = 12$$

A.E. $\Rightarrow 4m^3 + 9m = 0$

$$\Rightarrow m\,(4m^2 + 9) = 0$$

$$m = 0,\ m = \frac{3}{2}i\,;\ -\frac{3}{2}i$$

$$\therefore\ y = c_1 + \left(c_2 \cos\frac{3}{2}t + c_3 \sin\frac{3}{2}t\right)$$

$$\text{P.I.} = \frac{12}{4D^3 + 9D}e^0 = t \cdot \frac{12}{12D^2 + 9}e^0 \quad \Rightarrow \text{P.I.} = t\frac{12}{9} = \frac{12t}{9} = \frac{4t}{3}$$

$\therefore$ G.S = C.F. + P.I.

$$\Rightarrow y = c_1 + c_2 \cos\frac{3}{2}t + c_3 \sin\frac{3}{2}t + \frac{4t}{3}$$

$$y = 0, t = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\therefore\ y = c_1 - c_1 \cos\frac{3}{2}t + c_3 \sin\frac{3}{2}t + \frac{4}{3}t$$

$$\frac{dy}{dt} = \frac{3}{2}c_1 \sin\frac{3}{2}t + \frac{3}{2}c_3 \cos\frac{3}{2}t + \frac{4}{3}$$

$$t = 0,\ \frac{dy}{dt} = 0 \Rightarrow 0 = \frac{3}{2}c_3 + \frac{4}{3} \Rightarrow c_3 = -\frac{8}{9}$$

Now $y = c_1 - c_1 \cos\frac{3}{2}t - \frac{8}{9}\sin\frac{3}{2}t + \frac{4}{3}t$

Putting the value of y in (1), we get

$$\frac{2d^2x}{dt^2} + \frac{3d}{dt}\left(c_1 - c_1\cos\frac{3}{2}t - \frac{8}{9}\sin\frac{3}{2}t + \frac{4t}{3}\right) = 4$$

$$\Rightarrow 2\frac{d^2x}{dt^2} + 3\left[0 + c_1\frac{3}{2}\sin\frac{3}{2}t - \frac{3}{2}\times\frac{8}{9}\cos\frac{3}{2}t + \frac{4}{3}\right] = 4$$

$$\Rightarrow 2\frac{d^2x}{dt^2} + \frac{9}{2}c_1 \sin\frac{3}{2}t - 4\cos\frac{3}{2}t + 4 = 4$$

$$\Rightarrow 2\frac{d^2x}{dt^2} = -\frac{9}{2}c_1 \sin\frac{3}{2}t + 4\cos\frac{3}{2}t$$

Integrating, we get

$$\therefore \quad 2\frac{dx}{dt} = \frac{9}{2}c_1 \times \frac{2}{3}\cos\frac{3}{2}t + 4\times\frac{2}{3}\sin\frac{3}{2}t + c_4$$

$$\therefore \quad t = 0, \frac{dx}{dt} = 0 \Rightarrow 0 = 3c_1 + c_4$$

$$\Rightarrow c_4 = -3c_1$$

Now, $2\dfrac{dx}{dt} = 3c_1\cos\dfrac{3}{2}t + \dfrac{8}{3}\sin\dfrac{3}{2}t - 3c_1$

Again integrating we have

$$\therefore \quad 2x = 3c_1 \times \frac{2}{3}\sin\frac{3}{2}t - \frac{8}{3}\times\frac{2}{3}\cos\frac{3}{2}t - 3c_1 t + c_5$$

$$t = 0,\ x = 0 \Rightarrow 0 = -\frac{16}{9} + c_5 \Rightarrow c_5 = \frac{16}{9}$$

$$\therefore \quad 2x = 2c_1\sin\frac{3}{2}t - \frac{16}{9}\cos\frac{3t}{2} - 3c_1 t + \frac{16}{9}$$

$$\Rightarrow x = c_1\sin\frac{3}{2}t - \frac{8}{9}\cos\frac{3}{2}t - \frac{3}{2}c_1 t + \frac{8}{9}$$

Hence, $x = c_1\sin\dfrac{3}{2}t - \dfrac{8}{9}\cos\dfrac{3}{2}t - \dfrac{3}{2}c_1 t + \dfrac{8}{9}$

$$y = c_1 - c_1\cos\frac{3}{2}t - \frac{8}{9}\sin\frac{3}{2}t + \frac{4}{3}t$$ **Ans**

Example 11. *Solve* : $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 3x = e^{-t}$,

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$$ (*U.P., II Semester, June* 2007)

Solution. We have, $D^2x + Dy + 3x = e^{-t} \quad \Rightarrow \quad (D^2 + 3)\,x + D\,y = e^{-t}$ ...(1)

$D^2y - 4Dx + 3y = \sin 2t \quad \Rightarrow \quad -4Dx + (D^2 + 3)\,y = \sin 2t$ ...(2)

To eliminate y operating (1) by $(D^2 + 3)$ and (2) by D, we get

$(D^2 + 3)^2\,x + D\,(D^2 + 3)\,y = (D^2 + 3)e^{-t}$

$-4D^2\,x + D(D^2 + 3)\,y = D\sin 2t$

$(D^4 + 6D^2 + 9)x + D\,(D^2 + 3)\,y = e^{-t} + 3e^{-t}$ ...(3)

$-4D^2x + D(D^2 + 3)\,y = 2\cos 2t$ ...(4)

Subtracting (4) from (3), we get

$(D^4 + 10\,D^2 + 9)x = 4e^{-t} - 2\cos 2t$

A.E. is $m^4 + 10m^2 + 9 = 0 \quad \Rightarrow (m^2 + 1)\,(m^2 + 9) = 0 \quad \Rightarrow m = \pm\, i,\ m = \pm 3i$

C.F. $= C_1\cos t + C_2\sin t + C_3\cos 3t + C_4\sin 3t$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}4e^{-t} - \frac{1}{D^4 + 10D^2 + 9}(2\cos 2t)$$

$$= \frac{4}{1+10+9}e^{-t} - \frac{1}{(-4)^2+10(-4)+9}(2\cos 2t) = \frac{e^{-t}}{5} + \frac{2}{15}\cos 2t$$

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{e^{-t}}{5} + \frac{2}{15}\cos 2t$$

Putting the value of x in (2), we get

$$-4D\,[C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{e^{-t}}{5} + \frac{2}{15}\cos 2t] + (D^2+3)\,y = \sin 2t$$

$$\Rightarrow \quad -4\,(-C_1 \sin t + C_2 \cos t - 3C_3 \sin 3t + 3C_4 \cos 3t - \frac{e^{-t}}{5} - \frac{4}{15}\sin 2t) + (D^2+3)\,y = \sin 2t$$

$$\Rightarrow \quad (D^2+3)\,y = \sin 2t - 4C_1 \sin t + 4C_2 \cos t - 12\,C_3 \sin 3t + 12C_4 \cos 3t - \frac{4}{5}e^{-t} - \frac{16}{15}\sin 2t$$

A.E. is $m^2 + 3 = 0 \Rightarrow m = \pm i\sqrt{3}$; C.F. $= C_1 \cos\sqrt{3}\,t + C_2 \sin\sqrt{3}\,t$

$$\text{P.I.} = \frac{1}{D^2+3}[\sin 2t - 4C_1 \sin t + 4C_2 \cos t - 12C_3 \sin 3t + 12C_4 \cos 3t - \frac{4}{5}e^{-t} - \frac{16}{15}\sin 2t]$$

$$= \frac{1}{D^2+3}\left(-\frac{1}{15}\sin 2t\right) + \frac{1}{D^2+3}\left(-4C_1 \sin t\right) + \frac{1}{D^2+3}4\,C_2 \cos t + \frac{1}{D^2+3}(-12C_3 \sin 3t) + \frac{1}{D^2+3}(12C_4 \cos 3t) + \frac{1}{D^2+3}\left(\frac{-4}{5}e^{-t}\right)$$

$$= \frac{1}{-4+3}\left(-\frac{1}{15}\sin 2t\right) + \frac{1}{-1+3}(-4C_1 \sin t) + \frac{1}{-1+3}4C_2 \cos t + \frac{1}{-9+3}(-12C_3 \sin 3t) + \frac{1}{-9+3}(12\,C_4 \cos 3t) + \frac{1}{1+3}\left(\frac{-4}{5}e^{-t}\right)$$

$$= \frac{1}{15}\sin 2t - 2C_1 \sin t + 2C_2 \cos t + 2C_3 \sin 3t - 2C_4 \cos 3t - \frac{1}{5}e^{-t}$$

$$y = C.F. + P.I.$$

$$y = C_1 \cos\sqrt{3}\,t + C_2 \sin\sqrt{3}t + \frac{1}{15}\sin 2t - 2C_1 \sin t + 2C_2 \cos t + 2C_3 \sin 3t - 2C_4 \cos 3t - \frac{1}{5}e^{-t}$$

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{e^{-t}}{5} + \frac{2}{15}\cos 2t$$ **Ans.**

Example 12. *Solve the simultaneous differential equations*

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y,$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t$$ (*U.P. II Semester, 2001*)

Solution. Here, we have

$$(D^2 - 4D + 4)x - y = 0 \quad ...(1)$$

$$-25x + (D^2 + 4D + 4)y = 16e^t \quad ...(2) \quad \left(D \equiv \frac{d}{dt}\right)$$

Operating (1) by $D^2 + 4D + 4$ and adding to (2), we get

$$\Rightarrow \quad (D^2 + 4D + 4)\,(D^2 - 4D + 4)\,x - (D^2 + 4D + 4)\,y - 25x + (D^2 + 4D + 4)\,y = 16e^t$$

$$(D^2 - 4D + 4)\,(D^2 + 4D + 4)\,x - 25x = 16e^t \quad \Rightarrow \quad (D^4 - 8D^2 - 9)x = 16e^t$$

Auxiliary equation is

$$m^4 - 8m^2 - 9 = 0 \qquad \Rightarrow (m^2 - 9)\,(m^2 + 1) = 0 \Rightarrow m = \pm\, i, \pm\, 3$$

$$\text{C.F.} = c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t, \quad \text{P.I.} = \frac{1}{D^4 - 8D^2 - 9}(16e^t) = -e^t$$

x = C.F. + P.I.

$$x = c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t - e^t$$

From (1), $y = \dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x$

$$= \frac{d^2}{dt^2}\left(c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t - e^t\right) - 4\frac{d}{dt}\left(c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t - e^t\right)$$
$$+ 4\left(c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t - e^t\right)$$

$$= \frac{d}{dt}\left[3c_1e^{3t} - 3c_2e^{-3t} + c_3\,(-\sin t) + c_4 \cos t - e^t\right]$$
$$- 4\left[3c_1e^{3t} - 3c_2e^{-3t} + c_3\,(-\sin t) + c_4 \cos t - e^t\right]$$
$$+ 4\left[c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t - e^t\right]$$

$$= 9c_1e^{3t} + 9c_2e^{-3t} - c_3 \cos t - c_4 \sin t - e^t +$$
$$\left[-12\, c_1e^{3t} + 12c_2e^{-3t} + 4c_3 \sin t - 4c_4 \cos t + 4\, e^t\right]$$
$$+\left[4c_1e^{3t} + 4c_2e^{-3t} + 4c_3 \cos t + 4c_4 \sin t - 4\, e^t\right]$$

$$y = c_1e^{3t} + 25c_2e^{-3t} + (3c_3 - 4c_4) \cos t + (4c_3 + 3c_4) \sin t - e^t$$

$$x = c_1e^{3t} + c_2e^{-3t} + c_3 \cos t + c_4 \sin t - e^t \quad \text{[From (3)]}$$

Ans.

EXERCISE 14.1

Solve the following simultaneous equations:

1. $\dfrac{dx}{dt} + 2x - 3y = 0$, $\dfrac{dy}{dt} - 3x + 2y = 0$ **Ans.** $x = c_1\, e^t - c_2\, e^{-5t}$, $y = c_1\, e^t + c_2\, e^{-5t}$

2. $\dfrac{d^2y}{dt^2} = x,$ $\dfrac{d^2x}{dt^2} = y$ **Ans.** $x = c_1\, e^t + c_2\, e^{-t} + (c_3 \cos t + c_4 \sin t)$

$y = c_1\, e^t + c_2\, e^{-t} - (c_3 \cos t + c_4 \sin t)$

3. $\dfrac{dx}{dt} + 5x - 2y = t$, $\dfrac{dy}{dt} + 2x + y = 0$ **Ans.** $x = -\dfrac{1}{27}(1 + 6t)\, e^{-3t} + \dfrac{1}{27}(1 + 3t)$

So that $x = y = 0$ when $t = 0$ (*AMIETE, June 2009, U.P., II Semester, June* 2008)

Ans. $y = -\dfrac{2}{27}(2 + 3t)e^{-3t} + \dfrac{2}{27}(2 - 3t)$

4. $\dfrac{dx}{dt} - y = t$, $\dfrac{dy}{dt} = t^2 - x$ **Ans.** $x = c_1 \cos t + c_2 \sin t + t^2 - 1$; $y = -c_1 \sin t + c_2 \cos t + t$

5. $\dfrac{dx}{dt} + 2y + \sin t = 0$

$\dfrac{dy}{dt} - 2x - \cos t = 0$ **Ans.** $x = c_1 \cos 2t + c_2 \sin 2t - \cos t$; $y = c_1 \sin 2t - c_2 \cos 2t - \sin$ t

6. $4\frac{dx}{dt} - \frac{dy}{dt} + 3x = \sin t$

$\frac{dx}{dt} + y = \cos t$ **Ans.** $x = c_1 e^{-t} + c_2 e^{-3t}, y = c_1 e^{-t} + 3 c_2 e^{-3t} + \cos t$

7. $\frac{dy}{dx} = x$ and $\frac{dx}{dt} = y + e^{2t}$ **Ans.** $x = C_1 e^t + C_2 e^{-t} + \frac{2}{3}e^{2t}, y = C_1 e^t - C_2 e^{-t} + \frac{1}{3}e^{2t}$

8. $\frac{dx}{dt} = y + t, \quad \frac{dy}{dx} = -2x + 3y + 1$ **Ans.** $x = c_1 e^t + \frac{1}{2}c_2 e^{2t} - \frac{3}{2}t - \frac{5}{4}, y = c_1 e^t + c_2 e^{2t} - t - \frac{3}{2}$

9. $t\frac{dx}{dt} + y = 0, \quad t\frac{dy}{dx} + x = 0$ **Ans.** $x = c_1 t + c_2 t^{-1}, y = c_2 t^{-1} - c_1 t$

given $x(1) = 1$ and $y(-1) = 0$

10. $\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t$, given that $x = 2, y = 0$ when $t = 0$ (*U.P., II Semester*, 2004)

Ans. $x = e^t + e^{-t}, y = \sin t - e^t + e^{-t}$

11. $(D - 1)x + Dy = 2t + 1$

$(2D + 1)x + 2Dy = t$ **Ans.** $x = -t - \frac{2}{3}, y = \frac{t^2}{2} + \frac{4}{3}t + C$

12. $\frac{dx}{dt} + \frac{2}{t}(x - y) = 1,$ (*U.P., II Semester, Summer* (*C.O.*) 2005)

$\frac{dy}{dt} + \frac{1}{t}(x + 5y) = t$ **Ans.** $x = At^{-4} + Bt^{-3} + \frac{t^2}{15} + \frac{3y}{10}, \ y = -At^{-4} - \frac{1}{2}Bt^{-3} + \frac{2t^2}{15} - \frac{t}{20}$

13. $(D^2 - 1)x + 8Dy = 16e^t$ and $Dx + 3(D^2 + 1)y = 0$ (*Q. Bank U.P.T.U.* 2001)

Ans. $y = c_1 \cos\frac{t}{\sqrt{3}} + c_2 \sin\frac{t}{\sqrt{3}} + c_3 \cosh\sqrt{3}\,t + c_4 \sin h\sqrt{3}\,t + 2e^t$

$x = \sqrt{3}\,c_1 \sin\frac{t}{\sqrt{3}} - \sqrt{3}\,c_2 \cos\frac{t}{\sqrt{3}} - 3\sqrt{3}\,c_3 \sinh\sqrt{3}\,t - 3\sqrt{3}\,c_4 \cosh\sqrt{3}\,t - 6e^t - 3t.$

14. $\frac{dx}{dt} + \frac{2}{t}(x - y) = 1,$ (*U.P. II Semester, 2005*)

$\frac{dy}{dt} + \frac{1}{t}(x + 5y) = t.$ **Ans.** $x = At^{-4} + Bt^{-3} + \frac{t^2}{15} + \frac{3t}{10}, \ y = -At^{-4} - \frac{1}{2}Bt^{-3} + \frac{2t^2}{15} - \frac{t}{20}$