Total Differentiation

Differentiation of Composite Functions :-

① If $u = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

② If $u = f(x,y)$ where $x = \phi(t,z)$, $y = \psi(t,z)$

then $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial z}$$

③ If $f(x,y) = c$ where $y = f(x)$

then $$\frac{dy}{dx} = \frac{-f_x}{f_y}$$ (Here f is composite fun. of x only)

$$\left[\because \frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \Rightarrow \frac{df}{dx} = f_x + f_y\cdot\frac{dy}{dx}\right]$$

since $f(x,y) = c$

$$\Rightarrow \frac{df}{dx} = 0 = f_x + f_y\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$$

16. If $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = 2 \cos 3u \sin u$.

17. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$, show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

18. If $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, then evaluate $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2}$.

19. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(13 + \tan^2 u)$.

Answers

4. 0 **18.** $2u$.

6 COMPOSITE FUNCTIONS

(*i*) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$

en u is called a composite function of (the **single variable**) t and we can find $\dfrac{du}{dt}$.

(*ii*) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$

en z is called a composite function of (**two variables**) u and v so that we can find and $\dfrac{\partial z}{\partial v}$.

7 DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $\psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Proof. Here $u = f(x, y)$ ...(1)

Let δt be an increment in t and δx, δy, δu the corresponding increments in x, y and espectively. Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y) \quad ...(2)$$

Subtracting (1) from (2), we get

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$\frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta x}{\delta t} \quad ...(3)$$

As $\delta t \to 0$, δx and δy both $\to 0$, so that

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt}, \quad \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \quad \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

$$\lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$$

and
$$\lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$$

∴ From (1),
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\frac{du}{dt}$ is called the **total derivative** of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Cor. 1. If $u = f(x, y, z)$ and x, y, z are function of t, then y is a composite function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

Cor. 2. If $z = f(x, y)$ and x, y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u};\quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

Cor. 3. If $u = f(x, y)$ where $y = \phi(x)$ then since $x = \psi(x)$, u is a composite function of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

Cor. 4. If we are given an **implicit function $f(x, y) = c$,** then $u = f(x, y)$ where $u = c$

Using Cor. 3, we have $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

But $\frac{du}{dx} = 0$ ∴ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0$ or $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Hence the differential coefficient of $f(x, y)$ w.r.t. x is $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}.$

Cor. 5. If $f(x, y) = c$, then by Cor. 4, we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

Differentiating again w.r.t. x, we get

$$\frac{d^2u}{dx^2} = -\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{f_y^2} = -\frac{f_y\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx}\right] - f_x\left[\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\cdot\frac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\cdot\frac{f_x}{f_y}\right]}{f_y^2} = -\frac{f_{xx}f_y^2 - f_xf_yf_{xy} - f_xf_yf_{xy} - f_{yy}f_x^2}{f_y^3}$$

Hence $$\boldsymbol{\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that* $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$.

Sol. The given equations define u as a composite function of t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}.3 + \frac{1}{\sqrt{1-(x-y)^2}}(-1).12t^2$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}.$$

Example 2. *If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm per second when it passes through the value $x = 3$ cm, show that if y is passing through the value $y = 1$ cm, y must be decreasing at the rate of $2\frac{2}{15}$ cm per second, in order that z shall remain constant.*

Sol. Given : $z = 2xy^2 - 3x^2y$ and $\dfrac{dx}{dt} = 2$ cm/sec when $x = 3$ cm, we have to find $\dfrac{dy}{dt}$ when $y = 1$ cm.

Now $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = (2y^2 - 6xy)\frac{dx}{dt} + (4xy - 3x^2)\frac{dy}{dt}$$

Since z remains constant, $\dfrac{dz}{dt} = 0$

$$\therefore \quad 0 = (2y^2 - 6 \times 3 \times y) \times 2 + (4 \times 3 \times y - 3 \times 3^2)\frac{dy}{dt}$$

$$\Rightarrow \quad 0 = (4y^2 - 36y) + (12y - 27)\frac{dy}{dt}$$

When $y = 1$ cm, we have

$$0 = (4 - 36) + (12 - 27)\frac{dy}{dt}$$

$$\Rightarrow \qquad \frac{dy}{dt} = -\frac{32}{15} \text{ which is negative}$$

$\therefore$ When $y = 1$ cm, y is decreasing at the rate of $2\frac{2}{15}$ cm/sec.

Example 3. *If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Sol. Here z is a composite function of u and v.

$$\therefore \qquad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

Subtracting, $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$

Example 4. *If $u = f(y - z, z - x, x - y)$, prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Sol. Here $\qquad u = f(X, Y, Z) \quad$ where $X = y - z$, $Y = z - x$, $Z = x - y$

$\therefore \quad u$ is a composite function of x, y and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Example 5. *If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Sol. The given equations define w as a composite function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\cdot\sin\theta$$

or

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad \ldots(1) \quad [\because \; w = f(x, y)]$$

Also

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

or

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Example 6. *If u is a homogeneous function of nth degree in x, y, z, prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Sol. Since u is a homogeneous function of degree n in x, y, z, let

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$$

or

$$u = x^n f(t, s) \quad \text{where} \quad t = \frac{y}{x}, \quad s = \frac{z}{x}$$

Here f is a composite function of x, y, z.

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f(t, s) + x^n\left(\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}\right)$$

$$= nx^{n-1} f(t, s) + x^n\left[\frac{\partial f}{\partial t}\cdot\left(-\frac{y}{x^2}\right) + \frac{\partial f}{\partial s}\left(-\frac{z}{x^2}\right)\right]$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = nx^n f(t,s) - yx^{n-1}\frac{\partial f}{\partial t} - zx^{n-1}\frac{\partial f}{\partial s} \quad \ldots(1)$$

$$\frac{\partial u}{\partial y} = x^n\left(\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y}\right) = x^n\left[\frac{\partial f}{\partial t}\cdot\frac{1}{x} + \frac{\partial f}{\partial s}\cdot 0\right]$$

$$\Rightarrow \quad y\frac{\partial u}{\partial y} = yx^{n-1}\frac{\partial f}{\partial t} \quad \ldots(2)$$

$$\frac{\partial u}{\partial z} = x^n\left[\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z}\right] = x^n\left[\frac{\partial f}{\partial t}\cdot 0 + \frac{\partial f}{\partial s}\cdot\frac{1}{x}\right]$$

$$\Rightarrow \quad z\frac{\partial u}{\partial z} = zx^{n-1}\frac{\partial f}{\partial s} \quad \ldots(3)$$

Adding (1), (2) and (3), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nx^n f(t, s) = nu$$

Example 7. *If by the substitution* $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, *show that*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right).$$

Sol. Here $\quad f(x, y) = \theta(u, v) \quad$ and $\quad u = x^2 - y^2$, $v = 2xy$

⇒ f is a function of u, v and u, v are functions of x, y

⇒ f is a composite function of x, y.

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial u}\cdot 2x + \frac{\partial \theta}{\partial v}\cdot 2y \qquad [\because f = \theta \text{ (given)}]$$

$$= 2\left(x\frac{\partial \theta}{\partial u} + y\frac{\partial \theta}{\partial v}\right) = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\theta$$

$$\Rightarrow \quad \frac{\partial}{\partial x} = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right) \qquad \ldots(1)$$

Also

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial u}\cdot(-2y) + \frac{\partial \theta}{\partial v}\cdot 2x \qquad [\because f = \theta \text{ (given)}]$$

$$= 2\left(-y\frac{\partial \theta}{\partial u} + x\frac{\partial \theta}{\partial v}\right) = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right)\theta$$

$$\Rightarrow \quad \frac{\partial}{\partial y} = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right) \qquad \ldots(2)$$

Now

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial \theta}{\partial u} + 2y\frac{\partial \theta}{\partial v}\right) \qquad \text{[Using (1)]}$$

$$= 4\left(x^2\frac{\partial^2 \theta}{\partial u^2} + xy\frac{\partial^2 \theta}{\partial u \partial v} + yx\frac{\partial^2 \theta}{\partial v \partial u} + y^2\frac{\partial^2 \theta}{\partial v^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 4\left(x^2\frac{\partial^2 \theta}{\partial u^2} + 2xy\frac{\partial^2 \theta}{\partial u \partial v} + y^2\frac{\partial^2 \theta}{\partial v^2}\right) \qquad \ldots(3)$$

$$\left[\because \frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial^2 \theta}{\partial v \partial u}\right]$$

and

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial \theta}{\partial u} + 2x\frac{\partial \theta}{\partial v}\right) \qquad \text{[Using (2)]}$$

$$= 4\left(y^2\frac{\partial^2 \theta}{\partial u^2} - yx\frac{\partial^2 \theta}{\partial u \partial v} - xy\frac{\partial^2 \theta}{\partial v \partial u} + x^2\frac{\partial^2 \theta}{\partial v^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = 4\left(y^2\frac{\partial^2 \theta}{\partial u^2} - 2xy\frac{\partial^2 \theta}{\partial u \partial v} + x^2\frac{\partial^2 \theta}{\partial v^2}\right) \qquad \ldots(4) \quad \left[\because \frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial^2 \theta}{\partial v \partial u}\right]$$

Adding (3) and (4), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\left[(x^2 + y^2)\frac{\partial^2 \theta}{\partial u^2} + (y^2 + x^2)\frac{\partial^2 \theta}{\partial v^2}\right]$$

$$= 4\,(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right).$$

Example 8. *If* $x + y = 2e^{\theta}\cos\phi$ *and* $x - y = 2ie^{\theta}\sin\phi$, *show that*

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}, \ \left(i = \sqrt{-1}\right).$$

Sol. Here $\quad x + y = 2e^{\theta} \cos \phi$ and $x - y = 2ie^{\theta} \sin \phi$

Adding $\quad 2x = 2e^{\theta} (\cos \phi + i \sin \phi)$

$\Rightarrow \quad x = e^{\theta} \cdot e^{i\phi} = e^{\theta + i\phi}$ (By Euler's Theorem)

Subtracting $\quad 2y = 2e^{\theta} (\cos \phi - i \sin \phi)$

$\Rightarrow \quad y = e^{\theta} \cdot e^{-i\phi} = e^{\theta - i\phi}$

Now u is a function of x, y and x, y are functions of θ, ϕ

$\Rightarrow$ u is a composite function of θ, ϕ.

$$\therefore \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} . e^{\theta + i\phi} + \frac{\partial u}{\partial y} \cdot e^{\theta - i\phi}$$

$$= x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) u$$

$$\Rightarrow \quad \frac{\partial}{\partial \theta} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \ldots(1)$$

Also,
$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x} \cdot ie^{\theta + i\phi} + \frac{\partial u}{\partial y} \cdot (-ie^{\theta - i\phi})$$

$$= ix \frac{\partial u}{\partial x} - iy \frac{\partial u}{\partial y} = \left(ix \frac{\partial}{\partial x} - iy \frac{\partial}{\partial y} \right) u$$

$$\Rightarrow \quad \frac{\partial}{\partial \phi} = ix \frac{\partial}{\partial x} - iy \frac{\partial}{\partial y} \quad \ldots(2)$$

$$\therefore \quad \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial \theta} \right) = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \quad \text{[Using (1)]}$$

$$= x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} + yx \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial \theta^2} = x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \quad \ldots(3) \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$

nd
$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi} \left(\frac{\partial u}{\partial \phi} \right)$$

$$= \left(ix \frac{\partial}{\partial x} - iy \frac{\partial}{\partial y} \right) \left(ix \frac{\partial u}{\partial x} - iy \frac{\partial u}{\partial y} \right) \quad \text{[Using (2)]}$$

$$= i^2 \left(x^2 \frac{\partial^2 u}{\partial x^2} - xy \frac{\partial^2 u}{\partial x \partial y} - yx \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} \right)$$

$$= -\left(x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right) \quad \ldots(4) \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$