① $f(x,y) = 20 - x^2 - y^2$ [3 marks]

$\nabla f(x,y) = -2x\hat{i} - 2y\hat{j}$, $\nabla f(-1,-3) = 2\hat{i} + 6\hat{j}$

$\hat{u} = \dfrac{\nabla f}{|\nabla f|} = -\dfrac{(\hat{i} + 3\hat{j})}{\sqrt{10}}$ ⟶ [1.5 marks]

The rate of change in that direction is

$-|\nabla f(-1,-3)| = -2\sqrt{10}$ ⟶ [1.5 marks]

② $f(x,y) = x^4 + y^4 - 4xy + 1$ ⟶ [4 marks]

$f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$

$\Rightarrow x^3 - y = 0 \rightarrow (1)$, $y^3 - x = 0 \rightarrow (2)$

$\Rightarrow y = x^3$ into (2) $\Rightarrow (x^3)^3 - x = 0 \Rightarrow x^9 - x = 0$ [2 marks]

$\Rightarrow x(x^8 - 1) = 0$

$\Rightarrow x = 0, 1, -1$

critical points are $(0,0), (1,1), (-1,-1)$

$f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$

Points	f_{xx}	f_{yy}	f_{xy}	D
$(0,0)$	0	0	-4	$-16 < 0$, saddle point
$(1,1)$	12	12	-4	$128 > 0$, local minima
$(-1,-1)$	12	12	-4	$128 > 0$, local minima

[2 marks]

$f(1,1) = f(-1,-1) = -1$ is local minimum.

③ $f(x,y) = e^{xy}$, $g(x,y) = x^3 + y^3 = 16$ [3 marks]

$\nabla f = \lambda \nabla g \Rightarrow \langle ye^{xy}, xe^{xy}\rangle = \lambda\langle 3x^2, 3y^2\rangle$

$\Rightarrow ye^{xy} = 3\lambda x^2$, $xe^{xy} = 3\lambda y^2$

$\Rightarrow \lambda = \dfrac{ye^{xy}}{3x^2}$, $\lambda = \dfrac{xe^{xy}}{3y^2} \Rightarrow \dfrac{ye^{xy}}{3x^2} = \dfrac{xe^{xy}}{3y^2} \Rightarrow x^3 = y^3 \Rightarrow x = y$

$x^3 + y^3 = 16 \Rightarrow x^3 + x^3 = 16 \Rightarrow 2x^3 = 16 \Rightarrow x = 2 \Rightarrow y = 2$

$(2,2)$, $f(2,2) = e^4$