National University of Computer and Emerging Sciences
Karachi Campus

Multivariable Calculus (MT2008)-Spring 2025

Solution Final Exam

Rubrics:

- Full marks require correct steps and calculations.
- Partial credit for incorrect computations but correct setup.
- Clear notation and justification are encouraged for full marks. Deduct one mark for wrong notation in a question.

Do not write below this line

Attempt ALL questions. There are 4 questions in 2 pages.

CLO #1: *Understand the basic concepts and know the basic techniques of differential & integral calculus of functions of several variables.*

Q.1

a)

[6 Marks]

Function is defined at (0,0). Now, **(01 Mark)**

$$\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x^2+y^2} \quad \left(\frac{0}{0}\right) \quad f(0,0)=0\text{, function is defined.}$$

let $x=0 \Rightarrow \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1$

$y=0 \Rightarrow \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$

$\lim_{x\to 0^+} \frac{1}{x} = +\infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\Rightarrow$ limit does not exist.

$\Rightarrow$ function is not continuous at $(0,0)$.

(04 Mark)

Since the limit DNE, implies function is not continuous at (0,0) **(01 Mark)**

b) Find and sketch the domain of the following function:

[6 Marks]
[3+3]

$$f(x,y) = \frac{\ln(x-y)}{\sqrt{4-x^2-y^2}}$$

(area inside the circle in red color is including in domain)

$x-y>0 \Rightarrow x>y$ or $y<x$, $\quad 4-x^2-y^2 \geq 0 \Rightarrow -x^2-y^2 \geq -4$ or $x^2+y^2 \leq 4$

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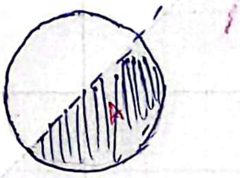

c) Use the fact that if all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point. Show that $f(x,y) = xe^{xy}$ is differentiable at (1,0). Also, find its local linear approximation at (1,0). Then use it to approximate $f(1.1, 0.1)$. [6 Marks]

Solution:

[1+5]

The partial derivatives are

$$f_x(x,y) = e^{xy} + xye^{xy} \qquad f_y(x,y) = x^2e^{xy}$$

$$f_x(1,0) = 1 \qquad f_y(1,0) = 1$$

(02 Mark)

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$z_0 + f_x(x-x_0) + f_y(y-y_0)$

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

$$= 1 + 1(x-1) + 1\cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

(03 Marks)

so

$$f(1.1, -0.1) \approx 1.1 + 0.1 = 1.2$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$ (01 Mark)

1.2...

$f_x(x,y) = e^{xy} + xye^{xy}$, $f_x(1,0) = e^0 = 1$, $f(1,0) = e^0 = 1$

$f_y(x,y) = x^2e^{xy}$, $f_y(1,0) = e^0 = 1$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$= 1 + 1(x-1) + 1(y-0) = 1 + x - 1 + y = x + y$

$L(1.1, 0.1) = 1.1 + 0.1 = 1.2$

$f(1.1, 0.1) = (1.1)e^{(1.1)(0.1)} = 1.22790$

error $= 1.22790 - 1.2 = 0.027$

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.2

a) If $\mathbf{F} = x\mathbf{i} - z\mathbf{j} + y^2x\mathbf{k}$ and $\mathbf{G} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$, find $\nabla.(\mathbf{F}\times\mathbf{G})$ and $\nabla\times(\mathbf{F}\times\mathbf{G})$. [6 Marks]

$$\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & -z & y^2x \\ xz & -y^2 & 2x^2y \end{vmatrix}$$

Compute determinant:

$$\mathbf{F}\times\mathbf{G} = \mathbf{i}\left((-z)(2x^2y) - (y^2x)(-y^2)\right) - \mathbf{j}\left(x(2x^2y) - (y^2x)(xz)\right) + \mathbf{k}\left(x(-y^2) - (-z)(xz)\right)$$ (02 Mark)

So,

$$\mathbf{F}\times\mathbf{G} = (-2x^2yz + xy^4)\,\mathbf{i} + (-2x^3y + x^2y^2z)\,\mathbf{j} + (-xy^2 + x^2z^2)\,\mathbf{k}$$

$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (-4xyz + y^4) + (-2x^3 + 2x^2yz) + 2x^2z$$

$$= y^4 - 2x^3 - 2xyz + 4x^2z$$ (02 Mark)

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x^2yz + xy^4 & -2x^3y + x^2y^2z & -xy^2 + x^2z^2 \end{vmatrix}$$

$$\boxed{\nabla\times(\mathbf{F}\times\mathbf{G}) = (-2xy - x^2y^2)\mathbf{i} + (y^2 - 2xz^2 - 2x^2y)\mathbf{j} + (-6x^2y + 2xy^2z + 2x^2z - 4xy^3)\mathbf{k}}$$ (02 Mark)

b) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region R in the xy −plane bounded by the line $y = 2x$ and the parabola $y = x^2$. [7 Marks]

i. Evaluate the double integral viewing R as a type I region, and

ii. Write only the limits of double integral viewing R as a type II region

[2+2+2]

2(a)

$$F\times G = (-2x^2yz + xy^4)\hat{i} + (-2x^3y + x^2y^2z)\hat{j} + \hat{k}(+xy^2 + xz^2)$$

$$\nabla\cdot(F\times G) = -4xyz + y^4 - 2x^3 + 2x^2yz + 2xz$$

$$\nabla\times(F\times G) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x^2yz + xy^4 & -2x^3y + x^2y^2z & -xy^2 + xz^2 \end{vmatrix}$$

$$= \hat{i}(-2xy - x^2y^2) - \hat{j}(-y^2 + z^2 + 2x^2y) + \hat{k}(-6x^2y + 2xy^2z + 2x^2z - 4xy^3)$$

vation

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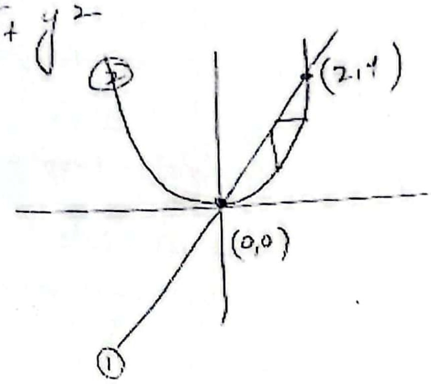
MVC Mid-2

①

b(i) (5)
(ii) (2)

[redacted]/ 2(b) $Z = x^2 + y^2$



① $Y = 2x$

② $Y = x^2$

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0 \qquad (0,0) \quad (2,4)$

$x = 0 \quad , \quad x = 2$

$Y = 0 \quad , \quad Y = 4$

④

region - I $\qquad x^2 \le y \le 2x$

$\qquad 0 \le x \le 2$

(02 Mark)

$$\int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left(x^2 y + \frac{y^3}{3} \right) \Big|_{x^2}^{2x} dx$$

$$= \int_0^2 \left[x^2(2x - x^2) + \frac{1}{3}\left(8x^3 - x^6\right) \right] dx$$

$$= \int_0^2 \left(2x^3 - x^4 + \frac{8}{3}x^3 - \frac{x^6}{3} \right) dx$$

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(2)

$$= \int_0^2 \left(\frac{14}{3}x^3 - x^4 - \frac{x^6}{3}\right) dx$$

$$= \left(\frac{14}{3}\frac{x^4}{4} - \frac{x^5}{5} - \frac{x^7}{21}\right)\Big|_0^2$$

$$= \frac{7}{6}(16-0) - \frac{1}{5}(32-0) - \frac{1}{21}(128-0)$$

$$= \frac{56}{3} - \frac{32}{5} - \frac{128}{21} = \frac{216}{35} = 6.1714$$

(03 Mark)

(b) region – II

$$\frac{y}{2} \le x \le \sqrt{y}$$

$$0 \le y \le 4$$

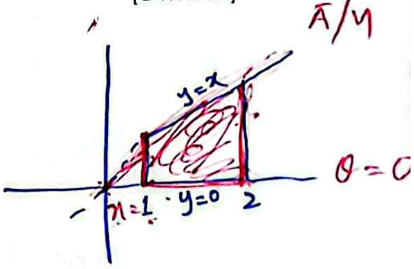
$$\int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)\, dx\, dy$$

(02 Mark)

c) Convert the following integral to polar coordinates then evaluate the integral in polar coordinates.

[8 Marks]

$$\int_1^2 \int_0^x (x^2+y^2)\,dy\,dx$$



$y=0,\ y=x,\ x=1,\ x=2$

$0 \le \theta \le \pi/4$, $r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta} = \sec\theta$

$r\cos\theta = 2 \Rightarrow r = \frac{2}{\cos\theta} = 2\sec\theta$

(3)

$$\int_0^{\pi/4}\int_{\sec\theta}^{2\sec\theta} r^3\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\right]_{\sec\theta}^{2\sec\theta} d\theta = \frac{1}{4}\int_0^{\pi/4}\left[16\sec^4\theta - \sec^4\theta\right]d\theta = \frac{1}{4}\int_0^{\pi/4} 15\sec^4\theta\,d\theta$$

(2)

$$= \frac{15}{4}\int_0^{\pi/4} \sec^2\theta\,(1+\tan^2\theta)\,d\theta = \frac{15}{4}\int_0^{\pi/4} \sec^2\theta\,d\theta + \sec^2\theta\tan^2\theta\,d\theta = \frac{15}{4}\left[\tan x + \frac{\tan^3 x}{3}\right]_0^{\pi/4}$$

$$\frac{15}{4}\left[1+\frac{1}{3}\right] = \frac{15}{4}\left[\frac{4}{3}\right] = 5.$$

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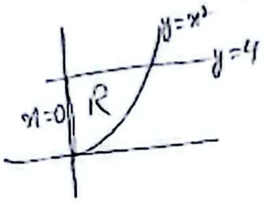
Q2d



Type II $0 \le y \le 4$, $0 \le x \le \sqrt{y}$

$$\iint_R \frac{x}{\sqrt{1+y^2}}\, dA = \int_0^4 \int_0^{\sqrt{y}} \frac{x}{\sqrt{1+y^2}}\, dx\, dy$$

$$= \frac{1}{2}\int_0^4 y(1+y^2)^{-1/2}\, dy$$

$$= \frac{1}{4}\, \frac{(1+y^2)^{1/2}}{1/2}\Bigg|_0^4$$

$$= \frac{\sqrt{17}-1}{2} \quad // \quad = 1.561$$

e) Evaluate the following triple integral [8 Marks]

$$\iiint_B xy^2z\,dV \text{ ; where } B \text{ is cuboid as shown in figure}$$

$$\int_0^1\int_y^1\int_0^{xy} e^{z/y}\, dz\, dx\, dy \qquad (3)$$

$$= \int_0^1\int_y^1 y\, e^{z/y}\Big|_0^{xy}\, dx\, dy$$

$$= \int_0^1\int_y^1 y\,[e^x - 1]\, dx\, dy$$

$$= \int_0^1 (ye^x - xy)\Big]_y^1\, dy$$

$$= \int_0^1 [y(e-e^y) - y(1-y)]\, dy$$

$$= \int_0^1 (ye - ye^y - y + y^2)\, dy$$

$$\frac{ey^2}{2}\Big|_0^1 - \frac{y^2}{2}\Big|_0^1 + \frac{y^3}{3}\Big|_0^1$$

$$= \frac{1}{2}e - \frac{1}{2} + \frac{1}{3} - 1 = \frac{1}{2}e - \frac{7}{6} = 0.19247$$

~~0.19247~~

$$\int_0^1 ye^y\, dy = ye^y\Big]_0^1 - \int_0^1 e^y\, dy$$

$$= e - e^y\Big]_0^1 = e - e + 1 = 1$$

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CLO #3: *Solve problems involving maxima and minima, line integral and surface integral, and vector calculus.*

Q.3

a) Find all local maximum and minimum values and saddle point(s) of the function $f(x,y) = xe^{-2x^2-2y^2}$, if exist. [8 Marks]

~~$f(x,y) = xe^{-2x^2-2y^2} \Rightarrow f_x = (1-4x^2)e^{-2x^2-2y^2}$, $f_y = -4xye^{-2x^2-2y^2}$, $f_{xx} = (16x^2-12)xe^{-2x^2-2y^2}$, $f_{xy} = (16x^2-4)ye^{-2x^2-2y^2}$, $f_{yy} = (16y^2-4)xe^{-2x^2-2y^2}$. Then $f_x = 0$ implies $1-4x^2 = 0 \Rightarrow x = \pm\frac{1}{2}$, and substitution into $f_y = 0 \Rightarrow -4xy = 0$ gives $y = 0$, so the critical points are $(\pm\frac{1}{2}, 0)$. Now~~

$f_x = (1-4x^2)e^{-2x^2-2y^2}$, $f_y = -4xye^{-2x^2-2y^2}$, $f_{xx} = (16x^2-12)xe^{-2x^2-2y^2}$, $f_{xy} = (16x^2-4)ye^{-2x^2-2y^2}$, $f_{yy} = (16y^2-4)xe^{-2x^2-2y^2}$

At $(\frac{1}{2}, 0)$:

1. Evaluate each term:

- $f_{xx} = (16\cdot\frac{1}{4} - 12)\cdot\frac{1}{2}\cdot e^{-2\cdot\frac{1}{4}} = (-8)\cdot\frac{1}{2}\cdot e^{-0.5} = -4e^{-0.5}$
- $f_{yy} = (16\cdot 0 - 4)\cdot\frac{1}{2}\cdot e^{-0.5} = -2e^{-0.5}$
- $f_{xy} = (16\cdot\frac{1}{4} - 4)\cdot 0\cdot e^{-0.5} = 0$

2. Compute the determinant:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (-4e^{-0.5})(-2e^{-0.5}) - 0 = 8e^{-1} > 0$$

Since $D > 0$ and $f_{xx} < 0$, this is a local ~~maximum~~ max at $(\frac{1}{2}, 0)$.

At $(-\frac{1}{2}, 0)$:

- $f_{xx} = (+4e^{-0.5})$ (same as before, since x^2 is the same and odd power of x gives negative sign)
- $f_{yy} = (+2e^{-0.5})$
- $f_{xy} = 0$

$f_{xx} > 0$

So again:

$$D = (-4e^{-0.5})(-2e^{-0.5}) = 8e^{-1} > 0, \quad f_{xx} < 0$$

So this is also a local ~~maximum~~ min at $(-\frac{1}{2}, 0)$.

Ans.

3(b)

$F(r(t)) = (3t^2+6t^2)\hat{i} - 14t^2 \cdot t^3\hat{j} + 20t(t^3)^2\hat{k}$

$= 9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k}$

$r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $r'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

$F(r(t)) \cdot r'(t) = (9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$

$= 3t^3 - 4t^7 + 6t^{10}\Big]_0^1 = 3 - 4 + 6 = 5$

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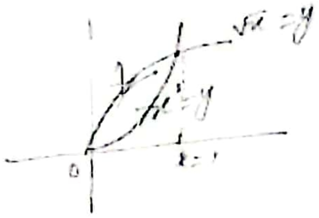
c) **Verify Green's theorem** to evaluate the line integral $\oint_C x^2y\,dx + (y + xy^2)\,dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$. [8 Marks]

[5+3]

Q.3c Green's theorem,

$$\oint_C F\,dx + G\,dy = \iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dA \quad \text{— (1)}$$

$$\text{RHS} = \iint_R (G_x - F_y)\,dA$$



Here $G = y + xy^2 \Rightarrow G_x = y^2$
$F = x^2y \Rightarrow F_y = x^2$

Type I $0 \le x \le 1,\ x^2 \le y \le \sqrt{x}$

$F = x^2y$
$F_y = x^2$
$G = y + xy^2$
$G_x = y^2$

$$\text{R.H.S} = \int_0^1 \int_{x^2}^{\sqrt{x}} (y^2 - x^2)\,dy\,dx$$

$$\int_0^1 \left(\frac{y^3}{3} - x^2y\right)\Big|_{x^2}^{\sqrt{x}} dx$$

$$\int_0^1 \left(\frac{x^{3/2}}{3} - x^{5/2} + x^4 - \frac{x^6}{3}\right) dx$$

$$= \left[\frac{2x^{5/2}}{15} - \frac{2x^{7/2}}{7} + \frac{x^5}{5} - \frac{x^7}{21}\right]_0^1 = \frac{2}{15} - \frac{2}{7} + \frac{1}{5} - \frac{1}{21} = 0$$

$x = t$, $y = t^2$

$C_1 : y = x^2$; $dy = 2x\,dx$, $0 \le x \le 1$

$$\int_0^1 t^4 + (t^2 + t^5)2t\,dt$$

$$= \int_0^1 (t^4 + 2t^3 + 2t^6)\,dt = \left[\frac{t^5}{5} + \frac{2t^4}{4} + \frac{2t^7}{7}\right]_0^1$$

$$= \frac{1}{5} + \frac{2}{4} + \frac{2}{7} = \frac{69}{70}$$

$C_2 : y = t,\ x = t^2$

$$\int_1^0 t^5 \cdot 2t\,dt + (t + t^2 \cdot t^2)\,dt = \int_1^0 (2t^6 + t + t^4)\,dt$$

$$= \left[\frac{2t^7}{7} + \frac{t^2}{2} + \frac{t^5}{5}\right]_1^0 = 0 - \left(\frac{2}{7} + \frac{1}{2} + \frac{1}{5}\right)$$

$$= -\frac{69}{70}$$

Q.4

a) Using Lagrange Multiplier Method find three positive numbers whose sum is 48 and whose product is as large as possible. [8 Marks]

$f(x,y,z) = xyz \quad , g(x,y,z) = x+y+z-48=0$

$\nabla f = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 1,1,1 \rangle$

$yz = \lambda \rightarrow (1)$

$xz = \lambda \rightarrow (2)$

$xy = \lambda \rightarrow (3)$

$(1),(2) \Rightarrow \quad yz = xz \Rightarrow y = x$ since $z > 0$

$(1),(3) \Rightarrow \quad yz = xy \Rightarrow z = x$ since $y > 0$

$y = x, \; z = x \Rightarrow x = y = z$

substituting into constraint

$x + x + x = 48 \Rightarrow 3x = 48 \Rightarrow x = 16$

$\Rightarrow x = y = z = 16$

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b) In a 3D vision model's latent space, the decision boundary for object detection forms a cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$. Compute the weighted sensitivity metric: [8 Marks]

$$\iint_\sigma y^2 z^2 \, dS$$

Solution. We will apply Formula (8) with

$$z = g(x, y) = \sqrt{x^2 + y^2} \quad \text{and} \quad f(x, y, z) = y^2 z^2$$

Thus,

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

so

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2}$$

(verify), and (8) yields

$$\iint_\sigma y^2 z^2 \, dS = \iint_R y^2 \left(\sqrt{x^2+y^2}\right)^2 \sqrt{2} \, dA = \sqrt{2} \iint_R y^2 (x^2 + y^2) \, dA$$

where R is the annulus enclosed between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ (Figure 15.5.4). Using polar coordinates to evaluate this double integral over the annulus R yields

$$\iint_\sigma y^2 z^2 \, dS = \sqrt{2} \int_0^{2\pi} \int_1^2 r^2 \sin^2\theta \cdot r^2 \cdot r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_1^2 r^5 \sin^2\theta \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left[\frac{r^6}{6} \sin^2\theta\right]_1^2 d\theta = \frac{21}{\sqrt{2}} \int_0^{2\pi} \sin^2\theta \, d\theta = \frac{21}{\sqrt{2}} \int_0^{2\pi} \frac{1 - \cos 2\theta}{2}$$

$$\frac{2^6}{6} - \frac{1}{6} = \frac{64}{6} - \frac{1}{6} = \frac{63}{6} = \frac{21}{2}$$

$$= \frac{21}{\sqrt{2}} \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}}$$

Formula (7), Section 7.3

$$\frac{21}{\sqrt{2}} \left[\frac{2\pi}{2}\right] = \frac{21}{\sqrt{2}}\pi = 46.65$$

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c) Consider a data flow in a 3D network region modeled by the vector field. Use the Divergence Theorem to find the flux of **F** across the surface σ with outward orientation.
$\mathbf{F}(x,y,z) = (x^2 + y)\,\mathbf{i} + z^2\,\mathbf{j} + (e^y - z)\,\mathbf{k}$; σ is the surface of the rectangular solid bounded by the coordinate planes and the planes $x = 3, y = 1$ and $z = 2$. [7 Marks]

Solution:

Q.4c Gauss-Divergence theorem,

$$\iint_\sigma \vec{F}\cdot\hat{n}\,dS = \iiint_G \vec{\nabla}\cdot\vec{F}\,dV$$

we have,

$$\vec{F} = (x^2+y)\hat{i} + z^2\hat{j} + (e^y - z)\hat{k}$$

$\Rightarrow \vec{\nabla}\cdot\vec{F} = 2x - 1$

A/c to question limits are,

$0 \le x \le 3,\ 0 \le y \le 1,\ 0 \le z \le 2$

$$\therefore \iiint_G \vec{\nabla}\cdot\vec{F}\,dV = \int_0^3\int_0^1\int_0^2 (2x-1)\,dz\,dy\,dx$$

$$= 12$$
