National University of Computer & Emerging Sciences
Karachi Campus

Multivariable Calculus (MT2008) — Final Exam

Date: $May\ 31^{st}, 2024$
Time: 9:00 am - 12:00 noon
Course Instructor(s)
Dr. Fahad Riaz, Dr. Nazish Kanwal
Ms. Fareeha Sultan, MS. Alishba, & Ms. Uzma.

Total Time: 3 Hours
Total Marks: 100
Total Questions: 07

Student Name ______ **Roll No** ______ **Section** ______ **Student Signature** ______

Attempt all questions. There are 7 questions and 2 page.

CLO #1: Understand the basic concepts and know the basic techniques of differential and integral calculus of functions of several variables.

Question 1 [15 marks]

(a) [6 points] Suppose the temperature at (x, y, z) is given by $T = xy + sin(yz)$. In what direction should you go from the point $(1, 1, 0)$ to decrease the temperature as quickly as possible? What is the rate of change of temperature in this direction?

(b) [3 points] Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value **1**.

(c) [6 points] Evaluate the integral given below:

$$\int_0^2 \int_{x^2}^1 \int_0^{xz} (y^2 - 6z)\, dy\, dz\, dx$$

CLO #2: Apply the theory to calculate the gradients, directional derivatives, arc length of curves, area of surfaces, and volume of solids. .

Question 2 [10 marks]

(a) [6 points] Use a total differential to approximate the change in the values of f from P to Q. Compare your estimate with the actual change in f.

$$f(x, y, z) = xe^{yz}; \quad P(-1, 1, 1), \quad Q(-0.99, 0.99, 1.01)$$

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(b) [4 points] Show that the function $f(x, y) = \frac{2x^2y}{x^4+y^2}$ has no limit when (x, y) approaches to $(0, 0)$).

CLO #2: Apply the theory to calculate the gradients, directional derivatives, arc length of curves, area of surfaces, and volume of solids. .

Question 3 [15 marks]

(a) [7 points] Evaluate the integral $\iint_R \sqrt{y+1}\, dA$, where R be the trapezoid with the vertices $(0, 0), (0, 1)$, $(1, 1)$ and $(2, 0)$.

(b) [5 points] Evaluate the integral by reversing the order of integration:

$$\int_0^4 \int_{\sqrt{x}}^2 x^{\frac{-2}{3}} \sqrt{y^{\frac{5}{3}} + 1}\, dy\, dx$$

(c) [3 points] Rewrite the double integral as an iterated integral in polar coordinates. (Do not Evaluate.)

$$\iint_R x^2\, dA; \quad \text{where } R \text{ is the right half of the ring} \quad 4 \leq x^2 + y^2 \leq 9.$$

CLO #2: Apply the theory to calculate the gradients, directional derivatives, arc length of curves, area of surfaces, and volume of solids.

Question 4 **[10 marks]**

(a) $\boxed{\text{5 points}}$ Let $\phi(x, y, z) = 2x^3y^2z^4$, find $div(grad(\phi))$ and $curl(grad(\phi))$.

(b) $\boxed{\text{5 points}}$ Verify the following, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$(i)\ curl(\mathbf{r}) = 0 \qquad (ii)\ \bigtriangledown \parallel \mathbf{r} \parallel = \frac{\mathbf{r}}{\|\mathbf{r}\|}$

CLO #2: Apply the theory to calculate the gradients, directional derivatives, arc length of curves, area of surfaces, and volume of solids.

Question 5 **[15 marks]**

(a) $\boxed{\text{8 points}}$ Evaluate the line integral with respect to $\mathbf{s}$ along the line segment C from P to Q.

$$\int_C z\cos(xy)\, ds; \quad P(1, 0, 1),\ Q(2, 2, 3).$$

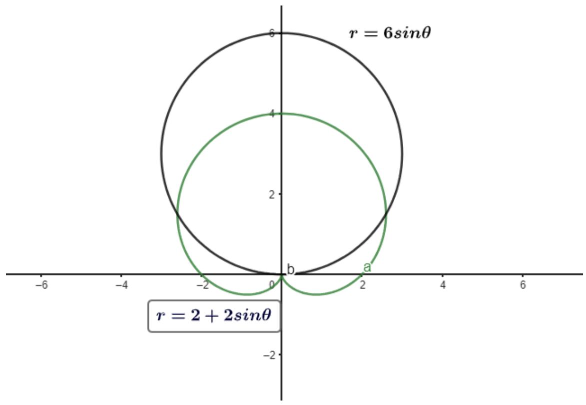
(b) $\boxed{\text{7 points}}$ Evaluate $\int_C F \cdot dr$ where $F(x, y) = e^{x-1}\ \mathbf{i} + xy\ \mathbf{j}$ and C is given by $r(t) = t^2\ \mathbf{i} + t^3\ \mathbf{j}$, $0 \le t \le 1$.

CLO #3: Solve problems involving maxima and minima, line integral and surface integral, and vector calculus.

Question 6 **[20 marks]**

(a) $\boxed{\text{6 points}}$ Find the surface area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xy-plane and inside the cylinder $x^2 + y^2 = 4y$.

(b) $\boxed{\text{6 points}}$ Use double integral to find the area of the shaded region above the x-axis, between the cardioid $r = 2 + 2\sin\theta$ and circle $r = 6\sin\theta$.



(c) $\boxed{\text{8 points}}$ Use **Green's Theorem** to evaluate the line integral along the curve C:

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy,$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$, oriented counter-clockwise.

CLO #3: Solve problems involving maxima and minima, line integral and surface integral, and vector calculus.

Question 7 **[15 marks]**

(a) $\boxed{\text{4 points}}$ Show that the function $x^4 + y^4 + z^4 + x^2 - xy + y^2 + yz + z^2$ is convex.

(b) $\boxed{\text{6 points}}$ Find the absolute extrema of the $f(x, y) = xy^2$ on the closed and bounded set R, where R is the region that satisfies the inequalities $x \ge 0,\ y \ge 0$ and $x^2 + y^2 \le 1$.

(c) $\boxed{\text{5 points}}$ Use **Lagrange multipliers** to find the maximum and minimum values of the function **f** subject to the given constraint. Also find the points at which these values occurs.

$$f(x, y, z) = xyz; \quad \text{subject to the constraint} \quad x^2 + 2y^2 + 3z^2 = 6.$$