National University of Computer and Emerging Sciences
Karachi Campus

# Multivariable Calculus (MT1008)

Date: April 10, 2025
Time: 11:00 am - 12:00 pm

## Solution Sessional-II Exam

Time Allowed: 1 Hour
Max. Marks: 30
Total Questions: 03

**Rubrics:**

- Full marks require correct steps and calculations.
- Partial credit for incorrect computations but correct setup.
- Clear notation and justification are encouraged for full marks. Deduct one mark for wrong notation in a question.

***CLO #1**: Understand the basic concepts and know the basic techniques of differential & integral calculus of functions of several variables.*

***Q.1*** **[13 Marks]**

a) Evaluate the double integral over the rectangular region R by first identifying it as a volume of a solid. [4 Marks]

$$\iint_R xy\, e^{y^2 x} dA\,;\ R = \{(x,y)|\ 1 \le x \le 4, 0 \le y \le 1\}$$

Q.1a $V = \iint_R xy\, e^{y^2x}\, dA$ — (01 Mark)

$= \frac{1}{2}\int_1^4 \int_0^1 2xy\, e^{y^2x}\, dy\, dx$ — (01 Mark)

$= \frac{1}{2}\int_1^4 e^{y^2x}\Big|_0^1 dx$

$= \frac{1}{2}\int_1^4 (e^x - 1) dx$

$= \frac{1}{2}(e^4 - e^1 - 4 + 1)$ // — (02 Marks)

b) Find an equation of tangent plane and a vector equation for the normal line to the surface at the given point P. [5 Marks]

$$z = 3x^2y^{-1} + x^{-1}y^2\,; P(1, -1 - 2).$$

**Solution:**

Find partial derivatives: $f_x = 6xy^{-1} - x^{-2}y^2$ $f_y = -3x^2y^{-2} + 2x^{-1}y$ At $(1,-1)$: $f_x(1,-1) = -6 - 1 = -7, \quad f_y(1,-1) = -3 - 2 = -5$ (2 Marks)

Equation of tangent plane: $z + 2 = -7(x-1) - 5(y+1)$ $z = -7x - 5y + 10 - 2$ $z = -7x - 5y + 8$ (2 Marks)

Equation of normal line: $x = 1 + 7t, \quad y = -1 + 5t, \quad z = -2 + t$ (1 Mark)

c) Given a vector field. Find $div(curl\ \mathbf{V})$. [4 Marks]

$$\mathbf{V}(x,y,z) = \sin x\ \mathbf{i} + \cos(x-y)\mathbf{j} + z\mathbf{k}$$

Q.c $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V})$

$$\nabla\times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin x & \cos(x-y) & z \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-\sin(x-y)) \text{ — (02 Marks)}$$

Now,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-\sin(x-y))$$

$$\therefore \boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0} \text{ — (02 Marks)}$$

***CLO #3:*** *Solve problems involving maxima and minima, line integral and surface integral, and vector calculus.*

---

***Q.2*** **[11 Marks]**

a) Given a function $f(x,y,z) = 3x^2 + 6xz + y^2 - 4yz + 8z^2$. Then, [3+3=6 Marks]

i) Compute the Hessian matrix of the function given.

ii) Is $f$ convex or concave?

**Solution:**

Hessian Matrix: $H = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 2 & -4 \\ 6 & -4 & 16 \end{bmatrix}$ (02 Marks)

**Step 1: Compute Leading Principal Minors** $D_1 = 6$, $D_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12$, $D_3 = \begin{vmatrix} 6 & 0 & 6 \\ 0 & 2 & -4 \\ 6 & -4 & 16 \end{vmatrix} = 56$ (03 Marks) (01 Mark)

Since all minors are positive, it is convex.

b) In a data clustering problem, we want to find the closest point on a given decision boundary to a reference point. Suppose a data point (2,3) represents a new observation, and we need to find the closest point $(x, y)$ on the decision boundary, $x + 2y = 5$. Find the point $(x, y)$ on the decision boundary that minimizes the distance between the observation (2,3) and any point $(x, y)$ on the decision boundary using the Lagrange multiplier method. [5 Marks]

Q.2b A/c to the question, the objective function is

$$d(x,y) = (x-2)^2 + (y-3)^2 \quad \text{--- (1)} \quad \text{(01 Mark)}$$

constraint: $g(x,y) = x + 2y - 5 = 0$ --- (2)

Lagrange Multiplier method,

$$\nabla d = \lambda \nabla g$$

$$\Rightarrow \begin{bmatrix} 2(x-2) \\ 2(y-3) \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Comparing, (01 Mark)

$$2(x-2) = \lambda \quad \text{--- (3)}$$

$$2(y-3) = 2\lambda \Rightarrow y - 3 = \lambda \quad \text{--- (4)}$$

eq (3) & (4) $\Rightarrow 2(x-2) = y-3$

$\Rightarrow 2(5-2y-2) = y-3$ (from (2))

$\Rightarrow 10 - 4y - 4 = y - 3$

$\Rightarrow 9 = 5y$

$\Rightarrow \boxed{y = 9/5}$

$\therefore$ (1) $\Rightarrow x + 2\left(\frac{9}{5}\right) - 5 = 0$ (02 Mark)

$\Rightarrow \boxed{x = \frac{7}{5}}$

$\therefore$ point $= \left(\frac{7}{5}, \frac{9}{5}\right) \Rightarrow (\text{distance})^2 = \frac{9}{5}$ (1)

$\therefore$ The closest point on the decision boundary $x + 2y = 5$ to (2,3) is $\left(\frac{7}{5}, \frac{9}{5}\right)$ (01 Mark)

***CLO #4:*** *Apply gradient and derivative for solving various problems arising in sciences.*

***Q*.3** **[6 Marks]**

A neural network model has two hyperparameters, *learning rate* ($x$) and *regularization parameter* ($y$), which affect the model's performance. The loss function is given by:

$$L(x, y) = (x - 4)^2 + 2(y + 3)^2$$

Perform two iterations of gradient-descent algorithm with learning rate, $\alpha = 0.1$ and starting point $(8, -1)$.

**Solution:**

Gradient:

$\nabla L = \begin{bmatrix} 2(x-4) \\ 4(y+3) \end{bmatrix}$ At (8, -1): $\nabla L = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$ (2 Marks)

First Iteration: $(x, y) = (8, -1) - 0.1 \cdot (8, 8) = (7.2, -1.8)$ (2 Marks)

Second Iteration: $\nabla L(7.2, -1.8) = \begin{bmatrix} 6.4 \\ 4.8 \end{bmatrix}$ $(x, y) = (7.2, -1.8) - 0.1 \cdot (6.4, 4.8) = (6.56, -2.28)$ (2 Marks)

---------------------------------------------