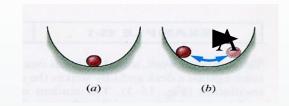# *Physics For Engineers NS (1001)*
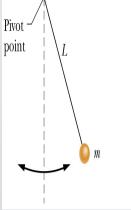
Oscillations

Oscillatory motion

Motion which is periodic in time, that is, motion that repeats itself in time.
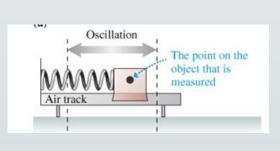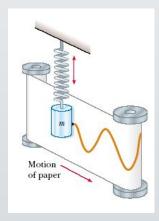
Examples:

- Power line oscillates when the wind blows past it
- Earthquake oscillations move buildings

Sometimes the oscillations are so severe, that the system exhibiting oscillations break apart.

Simple Harmonic Motion

- *Simple harmonic motion* (SHM) refers to a certain kind of oscillatory, or wave-like motion that describes the behavior of many physical phenomena:
  - a pendulum
  - a bob attached to a spring
  - low amplitude waves in air (sound), water, the ground
  - the electromagnetic field of laser light
  - vibration of a plucked guitar string
  - the electric current of most AC power supplies

Simple Harmonic Motion

When the block is displaced a small distance x from equilibrium, the spring exerts on the block a force that is proportional to the displacement and given by Hoo ke's law
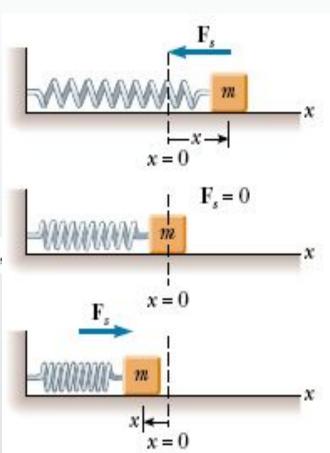
$$F_s = -kx$$

Applying Newton's second law to the motion of the block,
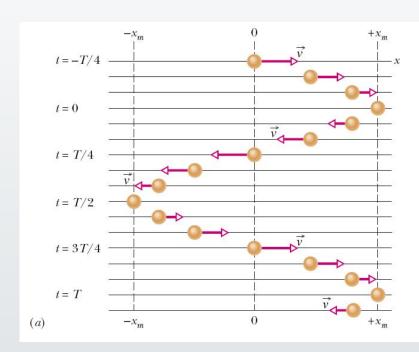
$$F_s = -kx = ma$$

$$a = -\frac{k}{m}x$$

An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.

In the figure snapshots of a simple oscillatory system is shown. A particle repeatedly moves back and forth about the point x=0.

The time taken for one complete oscillation is the period, T. In the time of one T, the system travels from x=+x$_m$, to –x$_m$, and then back to its original position x$_m$.

The velocity vector arrows are scaled to indicate the magnitude of the speed of the system at different times. At x=±x$_m$, the velocity is zero.

Frequency of oscillation is the number of oscillations that are completed in each second.

The symbol for frequency is f, and the SI unit is the hertz (abbreviated as Hz).

It follows that

$$T = \frac{1}{f}$$

Any motion that repeats itself is periodic or harmonic.

If the motion is a sinusoidal function of time, it is called simple harmonic motion (SHM).

Mathematically SHM can be expressed as:
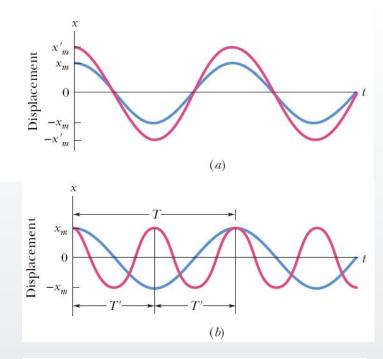
$$x(t) = x_m \cos(\omega t + \phi)$$

Here,
- x_m is the amplitude (maximum displacement of the system)
- t is the time
- ω is the angular frequency, and
- φ is the phase constant or phase angle
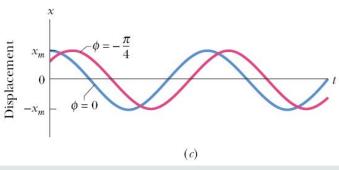
Simple Harmonic Motion

(a)

(b)

(c)

Figure a plots the displacement of two SHM systems that are different in amplitudes, but have the same period.

Figure b plots the displacement of two SHM systems which are different in periods but have the same amplitude.

The value of the phase constant term, φ, depends on the value of the displacement and the velocity of the system at time t = 0.

Figure c plots the displacement of two SHM systems having the same period and amplitude, but different phase constants.

For an oscillatory motion with period T,

$$x(t) = x(t + T)$$

The cosine function also repeats itself when the argument increases by 2π. Therefore,

$$\omega(t + T) = \omega t + 2\pi$$

$$\rightarrow \omega T = 2\pi$$

$$\rightarrow \omega = \frac{2\pi}{T} = 2\pi f$$

Here, ω is the angular frequency, and measures the angle per unit time. Its SI unit is radians/second. To be consistent, φ then must be in radians.

Simple Harmonic Motion

The velocity of SHM:

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right]$$

$$\rightarrow v(t) = -\omega x_m \sin(\omega t + \phi)$$

The maximum value (amplitude) of velocity is ωx_m. The phase shift of the velocity is $\pi/2$, making the cosine to a sine function.
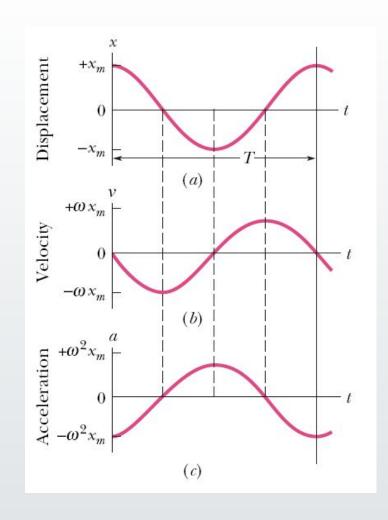
The acceleration of SHM is:

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[-\omega x_m \sin(\omega t + \phi)\right]$$

$$\rightarrow a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

$$\rightarrow a(t) = -\omega^2 x(t)$$

The acceleration amplitude is $\omega^2 x_m$.

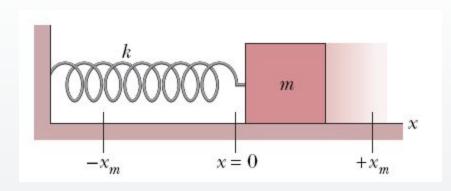*In SHM a(t) is proportional to the displacement but opposite in sign.*

From Newton's 2nd law:

$$F = ma = -(m\omega^2)x = -kx$$

SHM is the motion executed by a system subject to a force that is proportional to the displacement of the system but opposite in sign.
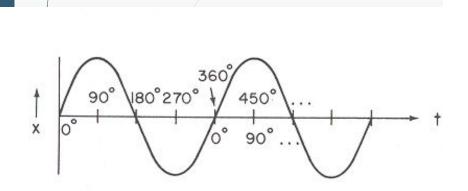
The block-spring system shown on the right forms a linear SHM oscillator.

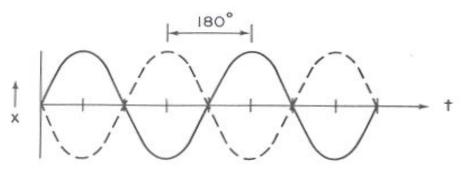The spring constant of the spring, k, is related to the angular frequency, ω, of the oscillator:
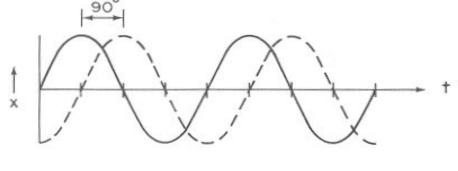
$$\omega = \sqrt{\frac{k}{m}} \rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$
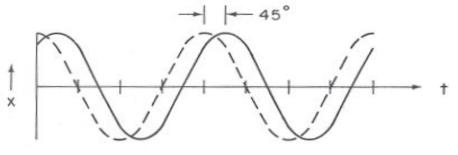
Phase (Time) Phase Diffference

Properties of simple harmonic motion
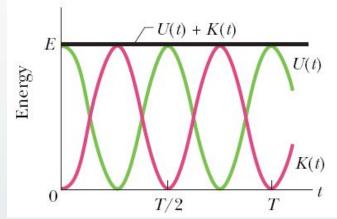
- The acceleration of the particle is proportional to the displacement but is in the opposite direction. This is the *necessary and sufficient condition for simple harmonic motion*, as opposed to all other kinds of vibration.
- The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase
- The frequency and the period of the motion are independent of the amplitude.

Energy in SHM

The potential energy of a linear oscillator is associated entirely with the spring.

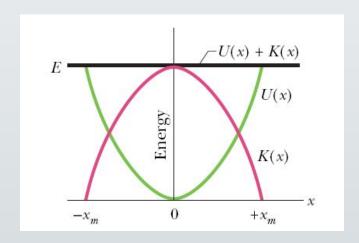$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m{}^2\cos^2(\omega t + \phi)$$

The kinetic energy of the system is associated entirely with the speed of the block.
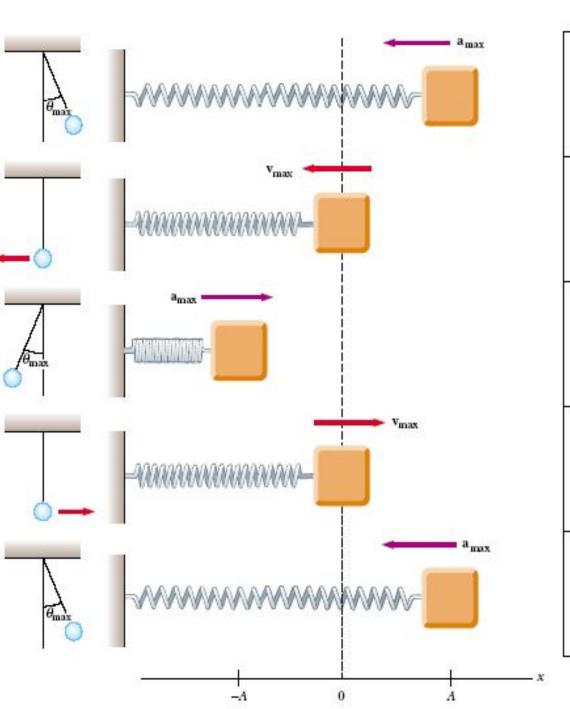
$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m{}^2\sin^2(\omega t + \phi) = \frac{1}{2}kx_m{}^2\sin^2(\omega t + \phi)$$
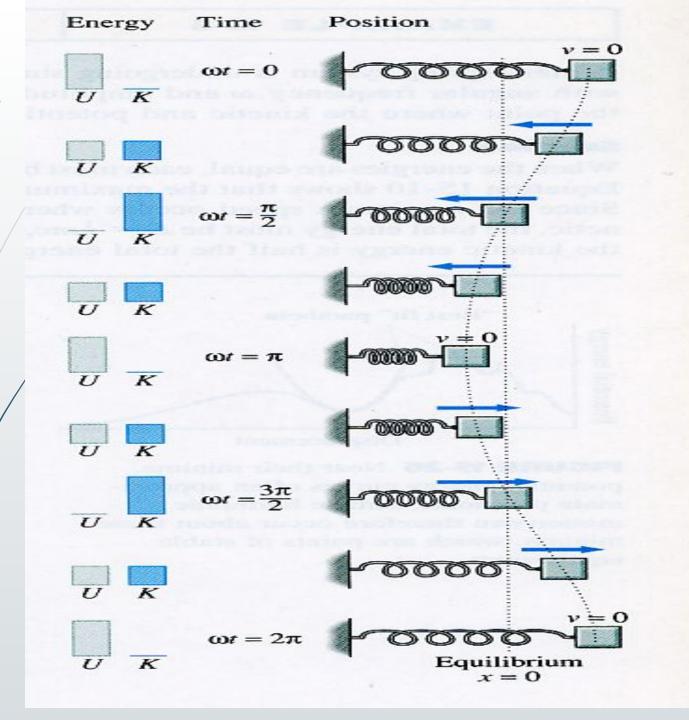
The total mechanical energy of the system:

$$E = U + K = \frac{1}{2}kx_m{}^2$$

t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Energy	Time	Position

U $\overline{K}$ $\omega t = 0$ $v = 0$

$\overline{U}$ K $\omega t = \dfrac{\pi}{2}$

U K

U $\overline{K}$ $\omega t = \pi$ $v = 0$

U K

$\overline{U}$ K $\omega t = \dfrac{3\pi}{2}$

U K

U $\overline{K}$ $\omega t = 2\pi$

Equilibrium
$x = 0$

SHM and uniform circular motion

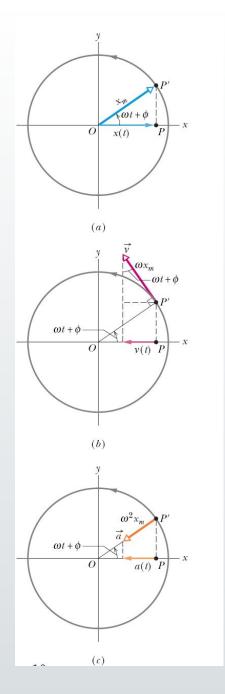Consider a reference particle P' moving in uniform circular motion with constant angular speed (w).

The projection of the particle on the x-axis is a point P, describing motion given by:

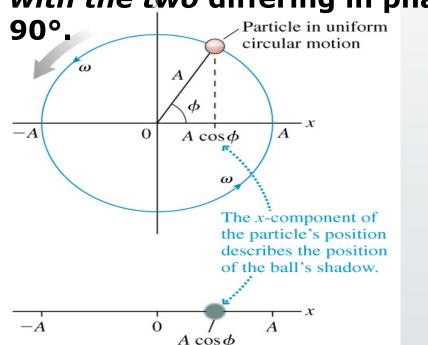$$x(t) = x_m \cos(\omega t + \phi).$$

This is the displacement equation of SHM.

SHM, therefore, is the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs.

(a)

(b)

(c)

COMPARING SIMPLE HARMONIC MOTION WITH UNIFORM CIRCULAR MOTION

Uniform circular motion can be considered a combination of two simple harmonic motions, one along the *x axis and one along the y axis, with the two* differing in phase by 90°.

(a)

Light from projector

Turntable

Circular motion of ball

Ball

Particle in uniform circular motion

ω

A

ϕ

$-A$ 0 $A \cos\phi$ A x

ω

The *x*-component of the particle's position describes the position of the ball's shadow.

$-A$ 0 $A \cos\phi$ A x

Shadow

Screen

Oscillation of ball's shadow

(b)   Simple harmonic motion of block

http://surendranath.tripod.com/Applets/Oscillations/SHM/SHM.html

simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

Pendulums

(a)

(b)

In a *simple pendulum*, a particle of mass m is suspended from one end of an unstretchable massless string of length L that is fixed at the other end.

The restoring torque acting on the mass when its angular displacement is θ, is:
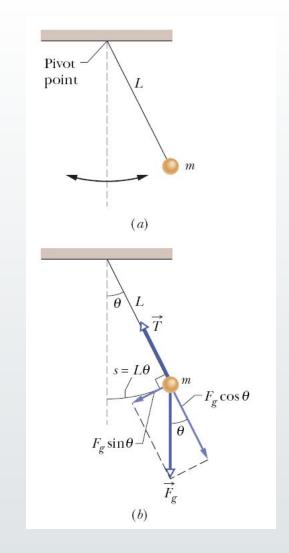
$$\tau = -L(F_g \sin\theta) = I\alpha$$

α is the angular acceleration of the mass. Finally,

$$\alpha = -\frac{mgL}{I}\theta, and$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

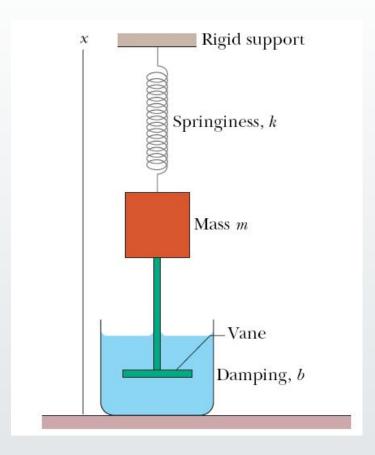This is true for *small angular displacements*, θ.

Damped Oscillations

In a damped oscillation, the motion of the oscillator is reduced by an external force.

Example: A block of mass m oscillates vertically on a spring with spring constant k.
From the block a rod extends to a vane which is submerged in a liquid.
The liquid provides the external damping force, F_d.

Damped Oscillations

Often the damping force, F_d, is proportional to the 1st power of the velocity v. That is,

$$F_d = -\gamma v$$

$$m\frac{d^2 x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$
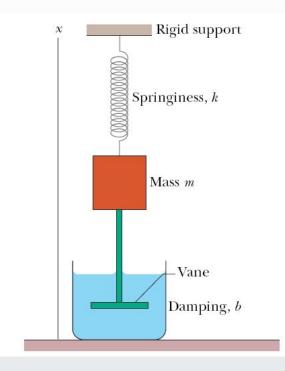
The solution is:

$$x(t) = x_0 e^{\frac{-\gamma t}{2m}} \cos(\omega' t + \varphi)$$

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Damped SHM

Often the damping force, F_d, is proportional to the 1st power of the velocity v. That is,

$$F_d = -bv$$

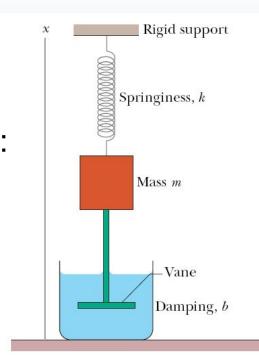From Newton's 2nd law, the following DE results:

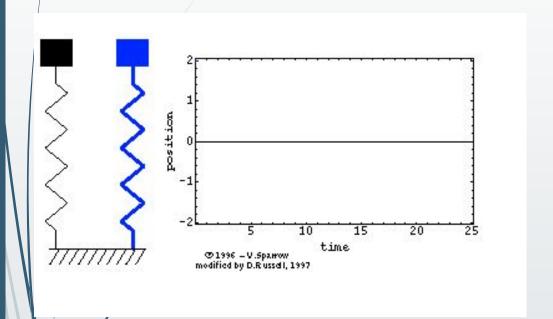$$m\frac{d^2 x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The solution is:

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)$$
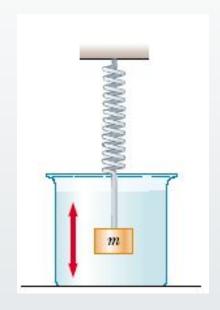
Here ω' is the angular frequency, and is given by:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

DAMPED OSCILLATIONS

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modified by D.Russell, 1997

http://www.lon-capa.org/~mmp/applist/damped/d.htm
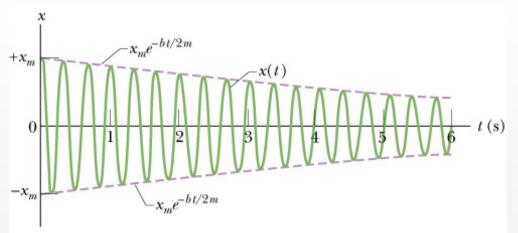
$$x(t) = x_0 e^{\frac{-\gamma t}{2m}} \cos(\omega' t + \varphi)$$
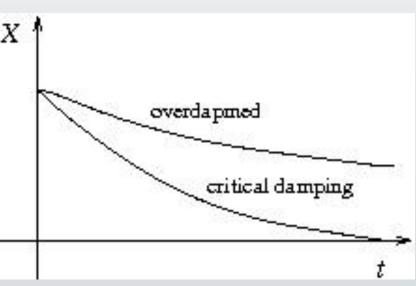
The above figure shows the displacement function $x(t)$ for the damped oscillator described before.

The amplitude decreases as $x_0 \exp(-\gamma t / 2m)$ with time.

The above is for $\gamma < 2m\omega_0$ (underdapmed).

For $\gamma > 2m\omega_0$ (overdapmed) and $\gamma = 2m\omega_0$ (critical damping), the oscillation goes like the right figure.

DAMPED OSCILLATIONS

In many real systems, dissipative forces, such as friction, retard the motion.
Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be Damped.
Retarding force $\mathbf{R} = -b\mathbf{v}$
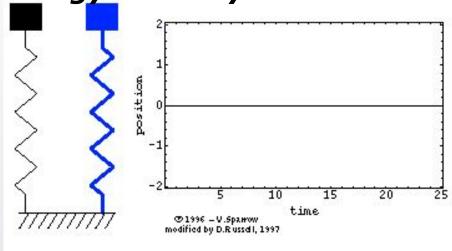
, we can write Newton's second law as

$$\sum F_x = -kx - bv = ma_x$$
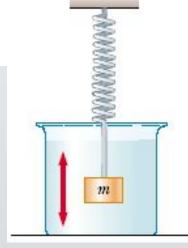
$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

The solution of this equation $x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$
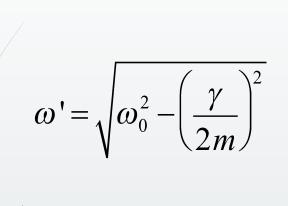
where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

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modified by D.R ussell, 1997
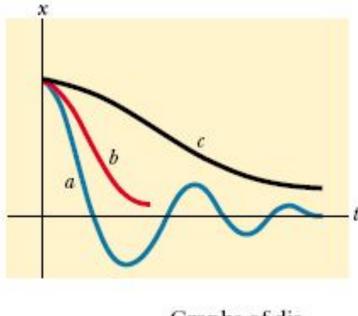
http://www.lon-capa.org/~mmp/applist/damped/d.htm

DAMPED OSCILLATIONS

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

where $\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency**

Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

Forced Oscillations and Resonance

When the oscillator is subjected to an external force that is periodic, the oscillator will exhibit <u>forced/driven oscillations</u>.

There are two frequencies involved in a forced oscillator:

I. w_0, the natural angular frequency of the oscillator, without the presence of any external force, and

II. w_e, the angular frequency of the applied external force.

The equation of motion is like the following:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F_0\cos(\omega_e t)$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = F_0\cos(\omega_e t)$$

The *steady state* solution is

$$x(t) = A\cos(\omega_e t + \delta)$$

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_e^2)^2 + \left(\frac{\gamma}{m}\omega_e\right)^2}}$$

$$\tan\delta = \frac{\gamma}{m}\frac{\omega_e}{\omega_0^2 - \omega_e^2} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$
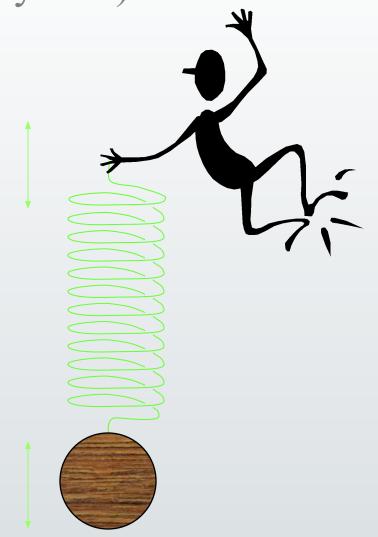
Example (Mass-Spring System)

Periodic driving force of freq. f

Oscillating with natural freq. f_o

Resonance

When a system is disturbed by a periodic driving force which frequency is *equal to the natural frequency (fo)* of the system, the system will oscillate with *LARGE amplitude*.

Resonance is said to occur.

http://www.acoustics.salford.ac.uk/feschools/waves/shm3.htm

Example 1
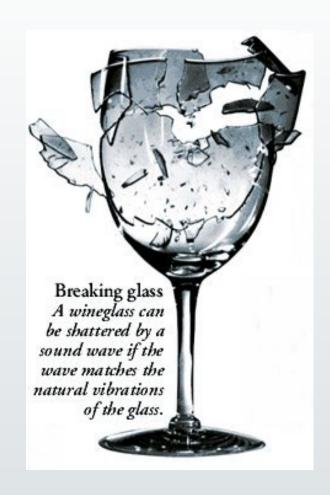
Breaking Glass

System : *glass*

Driving Force
:
 sound wave

Breaking glass
*A wineglass can
be shattered by a
sound wave if the
wave matches the
natural vibrations
of the glass.*

Example 2
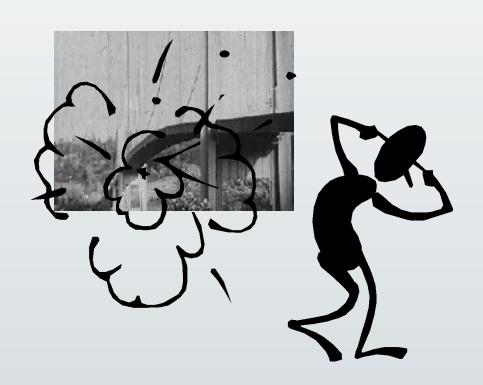
Collapse of the Tacoma Narrows
suspension bridge in America in 1940

System : *bridge*

Driving Force

:

   strong wind

Example

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-12:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65 \text{ N/m}}{0.68 \text{ kg}}} = 9.78 \text{ rad/s}$$

$$\approx 9.8 \text{ rad/s.} \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-5, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78 \text{ rad/s}}{2\pi \text{ rad}} = 1.56 \text{ Hz} \approx 1.6 \text{ Hz.} \quad \text{(Answer)}$$

The period follows from Eq. 15-2, which yields

$$T = \frac{1}{f} = \frac{1}{1.56 \text{ Hz}} = 0.64 \text{ s} = 640 \text{ ms.} \qquad \text{(Answer)}$$

Example

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(b) What is the amplitude of the oscillation?

KEY IDEA With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm.} \qquad \text{(Answer)}$$

Example

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-6.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m})$$
$$= 1.1 \text{ m/s}. \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-4a and 15-4b, where you can see that the speed is a maximum whenever $x = 0$.

Example

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-7.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m})$$
$$= 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path. At those points, the force acting on the block has its maximum magnitude; compare Figs. 15-4a and 15-4c, where you can see that the magnitudes of the displacement and acceleration are maximum at the same times.

Example

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(e) What is the phase constant ϕ for the motion?

An oscillator consists of a block attached to a spring (k=400 N/m). At some time t, the position (measured from the system's equilibrium location), velocity, and acceleration of the block are x =0.100 m, v =13.6 m/s, and a= -123 m/s^2 . Calculate (a) the frequency of oscillation, (b) the mass of the block, and (c) the amplitude of the motion `

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(e) What is the phase constant ϕ for the motion?

Calculations: Equation 15-3 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-3 and canceling x_m give us

$$1 = \cos \phi. \tag{15-14}$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad.} \tag{Answer}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)