LECTURE ON INTERSECTION THEORY (V)

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Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.
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Contents

In this lecture, we talk about the following topics.

- Properties of Chern class.
- Intersection with zero-section of vector bundles.

1. Chern class of vector bundles (continued)

1.1. **Properties of Chern class.** Recall our set-up first: let X be a scheme and $\pi : E \to X$ a vector bundle over X of rank $r = e+1$, we can associate it a projective bundle $p : \mathbb{P}(E) \to X$ over X of rank e.

$$\begin{tikzcd}
& \mathcal{O}_E(-1) \arrow[dl, hook'] \arrow[d] \\
p^*(E) \arrow[r] & \mathbb{P}(E) \arrow[d, "p"] \\
E \arrow[u, dashed] \arrow[r, "\pi"] & X
\end{tikzcd}$$

Proposition 1.1. *Here are some basic properties of Chern class.*

(1) $c_i(E) = 0$ *if* $i > r = \operatorname{rank}(E)$.

(2) (Whitney sum) *If* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of vector bundles over* X, *then*

$$c_t(E) = c_t(E')c_t(E'')$$

i.e.,

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$$

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(3) (Grothendieck's formula) *Denote by* $\xi := c_1(\mathcal{O}_E(1))$, *then*
$$\xi^r + c_1(p^*E)\xi^{r-1} + c_2(p^*E)\xi^{r-2} + \cdots + c_r(p^*E) = 0$$

Proof. By Splitting construction (regarding the injective part), we may assume that there is a filtration
$$E = E_r \supseteq \cdots \supseteq E_1 \supseteq E_0 = 0$$
such that $E_i/E_{i-1} =: L_i$ are line bundles.

Lemma 1.2. *Suppose E is as above and $s : X \to E$ is a section. Let*
$$Z := \{x \in X : s(x) = 0\} \subset X$$
a closed subset. Then for any $\alpha \in \mathrm{CH}_k(X)$, there exists $\beta \in \mathrm{CH}_{k-r}(Z)$, such that
$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \beta \text{ holds in } \mathrm{CH}_{k-r}(X)$$
In particular, if $Z = \emptyset$, i.e., s nowhere vanish, then
$$\prod_{i=1}^{r} c_1(L_i) = 0$$

Remark 1.3. If E itself is a line bundle, then we can explicitly construct such β as follows
$$c_1(E) \cap \alpha = j_*(D.\alpha) \in \mathrm{CH}_{k-1}(X)$$
where $D = (E, Z, s)$ is a persodu-divisor on X and $j : Z \hookrightarrow X$ is the inclusion.

Re-consider the diagram
$$\begin{tikzcd}
 & \mathcal{O}_E(-1) \arrow[dl, hook'] \arrow[d] \\
p^*(E) \arrow[r] & \mathbb{P}(E) \arrow[d, "p"] \\
E \arrow[u, dashed] \arrow[r, "\pi"] & X
\end{tikzcd}$$
and tensoring with $\mathcal{O}_E(1)$ on both sides of $\mathcal{O}_E(-1) \hookrightarrow p^*(E)$, we get
$$\mathcal{O}_E \hookrightarrow p^*(E) \otimes \mathcal{O}_E(1)$$

(1) $p^*(E) \otimes \mathcal{O}_E(1)$ has a nowhere vanish section and
(2) there is a natural filtration
$$p^*(E) \otimes \mathcal{O}_E(1) = p^*(E_r) \otimes \mathcal{O}_E(1) \supseteq \cdots \supseteq p^*(E_0) \otimes \mathcal{O}_E(1) = 0$$
such that
$$p^*(E_i) \otimes \mathcal{O}_E(1)/p^*(E_{i-1}) \otimes \mathcal{O}_E(1) = p^*(L_i) \otimes \mathcal{O}_E(1)$$
are line bundles.

Apply Lemma 1.2 to $p^*(E) \otimes \mathcal{O}_E(1)$ and the nowhere vanish section yields that
$$\prod_{i=1}^{r} c_1(p^*(L_i) \otimes \mathcal{O}_E(1)) = 0$$
i.e.,
$$\prod_{i=1}^{r} [c_1(p^*(L_i)) + \xi] = 0$$
hence
$$\xi^r + p^*(\sigma_1)\xi^{r-1} + p^*(\sigma_2)\xi^{r-2} + \cdots + p^*(\sigma_r) = 0 \tag{1.1}$$

where σ_k is the k-th elementary symmetric polynomial in $c_1(L_i)$. Multiplied equation (1.1) by some power of ξ and then push-forward, we obtain

$$(1+\sigma_1 t+\sigma_2 t^2+\cdots+\sigma_r t^r)s_t(E)=1$$

so $\sigma_i = c_i(E)$ by definition, which is equivalent to say

$$c_t(E)=\prod_{i=1}^r [\underbrace{1+c_1(L_i)t}_{c_t(L_i)}]$$

Hence (1) and (3) are both okay. □

Remark 1.4 (Splitting principal)**.** We have a few words about the splitting principal, which will be frequently used in our course. In particular, we can use this principal to do the following.

(1) Use split case to test

$$c_i(E^\vee), c_i(E_1\otimes E_2), c_i(\wedge^k E)$$

(2) Formally we can consider Chern class as

$$c_t(E)=\prod_{i=1}^r(1+\alpha_i t)$$

where α_i's are the so-called *Chern roots.* Then any symmetric polynomials in α_i can be written as polynomials in Chern class.

1.2. Grothendieck's Riemann-Roch.

1.2.1. *Chern class and Todd class.* Let X be a scheme and $\pi : E \to X$ a vector bundle over X of rank r.

Definition 1.5 (Chern character)**.** The *Chern character* of E is defined as

$$\begin{aligned}\mathrm{ch}(E) :&= \sum_{i=1}^r \exp(\alpha_i)\\ &= r + c_1(E) + \frac{1}{2}[c_1(E)^2 - c_2(E)] + \cdots\end{aligned}$$

with $\mathbb{Q}$-coefficients.

Remark 1.6. The Chern character $\mathrm{ch}(-)$ defines a ring homomorphism

$$\mathrm{ch}(-) : K_.(X) \to \mathbb{Q}[c_i(-)]$$

i.e., we have

$$\begin{aligned}\mathrm{ch}(E\oplus E') &= \mathrm{ch}(E)+\mathrm{ch}(E')\\ \mathrm{ch}(E\otimes E') &= \mathrm{ch}(E)\cdot\mathrm{ch}(E')\end{aligned}$$

Definition 1.7 (Todd class)**.** The *Todd class* of E is defined as

$$\begin{aligned}\mathrm{td}(E) :&= \prod_{i=1}^r \frac{\alpha_i}{1-\exp(-\alpha_i)}\\ &= 1 + \frac{1}{2}c_1(E) + \frac{1}{12}[c_1^2(E)+c_2(E)] + \frac{1}{24}c_1(E)c_2(E) + \cdots\end{aligned}$$

with $\mathbb{Q}$-coefficients.

Proposition 1.8. *If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on X, then*

$$\mathrm{td}(E) = \mathrm{td}(E')\cdot\mathrm{td}(E'')$$

Recall that under the same condition, $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$

1.2.2. *Statement of G–R–R.* Let X be a non-singular variety, we write

$$c_i(E) := c_i(E) \cap [X] \in \mathrm{CH}_{n-i}(X) =: \mathrm{CH}^i(X)$$

Definition 1.9. The *Grothendieck ring* of vector bundles on X is defined by

$$K.(X) := \mathbb{Z}\{\text{vector bundles on } X\}/[E] = [E'] + [E'']$$

where $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on X. One can similarly define the Grothendieck ring $K^{\cdot}(X)$ of coherent sheaves on X.

Fact 1.10 (Functoriality of $K.(X)$)**.** Let $f : X \to Y$ be a proper morphism between non-singular varieties, then we obtain

$$\begin{aligned} f_! : K.(X) &\to K.(Y) \\ \mathcal{F} &\mapsto \sum_i (-1)^i [\mathcal{R}^i f_*(\mathcal{F})] \end{aligned}$$

Theorem 1.11 (Grothendieck's Riemann-Roch)**.** *We have the following commutative diagram:*

$$\begin{tikzcd} K.(X) \arrow[r, "f_!"] \arrow[d, "\mathrm{ch}(-).\mathrm{td}(T_X)"'] & K.(Y) \arrow[d, "\mathrm{ch}(-).\mathrm{td}(T_Y)"] \\ \mathrm{CH}_*(X)_{\mathbb{Q}} \arrow[r, "f_*"] & \mathrm{CH}_*(Y)_{\mathbb{Q}} \end{tikzcd}$$

i.e., for any $\mathcal{F} \in K.(X)$ a coherent sheaf $\mathcal{F}$ on X

$$\mathrm{ch}(f_!\mathcal{F}).\mathrm{td}(T_Y) = f_*(\mathrm{ch}(\mathcal{F}).\mathrm{td}(T_X))$$

2. Intersection with 0 of vector bundle

Theorem 2.1. *Let X be a scheme, $\pi : E \to X$ a vector bundle over X of rank $r = e + 1$ and $p : \mathbb{P}(E) \to X$ its associated projective bundle, then*

(1) $\pi^* : \mathrm{CH}_{k-r}(X) \xrightarrow{\sim} \mathrm{CH}_k(E)$ *is an isomorphism.*

(2) *The morphism*

$$\theta : \bigoplus_{i=0}^{e} \mathrm{CH}_{k-e+i}(X) \to \mathrm{CH}_k(\mathbb{P}(E))$$

given by

$$\theta[(\alpha_i)_i] = \sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*(\alpha_i)$$

is an isomorphism.

Remark 2.2. This theorem allows us to define the intersection with zero-section of vector bundles via

$$0^* : \mathrm{CH}_k(E) \to \mathrm{CH}_{k-r}(E)$$

by $0^* = (\pi^*)^{-1}$. It's the so-called *Gysin pull-back* along the zero section.

Proof. The injectivity of (1) relies on that of (2).

(1)
- Surjective: Already proved before (using localization sequence).
- Injective: If $\pi^*(\alpha) = 0$, i.e., $j^*(q^*(\alpha)) = 0$. By exactness of the first row

$$q^*(\alpha) = i_* \left(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*(\alpha_i) \right)$$

then

$$q^*(\alpha) = \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^{i+1} \cap q^*(\alpha_i)$$

but this implies that $\alpha = 0$ by the injectivity of (2).

(2) – Surjective: similarly to that of (1), we can reduce to the trivial bundle case and then to trivial line bundle case, i.e.,

$$E = X \times \mathbb{A}^1$$

So we only need to prove that

$$\textit{if } F = E \oplus I, \textit{ then } \theta_E \textit{ surjective } \Rightarrow \theta_F \textit{ surjective.}$$

Consider the following diagram

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "i"] \arrow[dr, "p"'] & \mathbb{P}(F) \arrow[d, "q"] & E \arrow[l, "j"'] \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

and then by the functoriality of Chow group, we obtain

$$\begin{tikzcd} \mathrm{CH}_k(\mathbb{P}(E)) \arrow[r, "i_*"] & \mathrm{CH}_k(\mathbb{P}(F)) \arrow[r, "j^*"] & \mathrm{CH}_k(E) \arrow[r] & 0 \\ & \mathrm{CH}_{k-i}(X) \arrow[ul, "p^*"] \arrow[u, "q^*"] \arrow[ur, "\pi^*"'] & & \end{tikzcd}$$

where only the right triangle is commutative but the left is in general not. Then for each $\alpha \in \mathrm{CH}_{k-i}(X)$

$$i_*(p^*\alpha) = c_1(\mathcal{O}_F(1)) \cap q^*(\alpha)$$

But for each $\beta \in \mathrm{CH}_k(\mathbb{P}(F))$, we have

$$j^*(\beta) = \pi^*(\alpha) = j^*q^*(\alpha)$$

for some $\alpha \in \mathrm{CH}_{k-i}(X)$: π^* is surjective, then

$$j^*(\beta - q^*(\alpha)) = 0$$

so by exactness of first row

$$\beta - q^*(\alpha) = i_*\left(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*(\alpha_i)\right)$$

and therefore

$$\begin{aligned} \beta &= q^*(\alpha) + i_*\left(\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*(\alpha_i)\right) \\ &= q^*(\alpha) + i_*\left(\sum_{i=0}^{e} c_1(i^*\mathcal{O}_F(1))^i \cap p^*(\alpha_i)\right) \\ &\underbrace{=}_{\text{proj formula}} q^*(\alpha) + \sum_{i=0}^{e} [c_1(\mathcal{O}_F(1))^i \cap i_*p^*(\alpha_i)] \\ &= q^*(\alpha) + \sum_{i=0}^{e} [c_1(\mathcal{O}_F(1))^i \cap c_1(\mathcal{O}_F(1)) \cap q^*(\alpha_i)] \\ &= q^*(\alpha) + \sum_{i=0}^{e} [c_1(\mathcal{O}_F(1))^{i+1} \cap q^*(\alpha_i)] \end{aligned}$$

- Injective: If

$$\sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*(\alpha_i) = 0$$

multiplied by some power of $c_1(\mathcal{O}_E(1))$ and then push-forward, we obtain that $\alpha_i = 0$.

$\square$

References

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