LECTURE ON INTERSECTION THEORY (XVII)

ZHANG

ABSTRACT. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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1. NOTATIONS AND RESULTS

In this lecture, we focus on the following problem:

Problem 1.1 (Grothendieck's dream)**.** Category of mixed motives

$$\begin{array}{l}\mathsf{MM} : \text{abelian category of mixed motive over } \Bbbk = \mathbb{C} \\ \quad\cup \\ \mathsf{M}_{\mathrm{num}} : \text{the full subcategory consisting of semi-simple objects}\end{array}$$

together with the mixed motive functor:

$$h : \mathsf{Var} \to \mathsf{MM} \text{ mapping } X \mapsto h(X)$$

It's solved by the work of Voeviosky, which say

Theorem 1.2 (Voeviosky)**.** *Construct a candidate of (the triangled category of mixed motive)*

$$D\mathsf{M} := D(\mathsf{MM})$$

together with the mixed motive functor

$$\mathcal{M} : \mathsf{Var} \to D\mathsf{M} \text{ mapping } X \mapsto \mathcal{M}(X)$$

Hereafter, we fix the following notations:

(1) Var : category of all varieties.

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(2) Sm : category of all smooth varieties.
(3) SmProj : category of all smooth projective varieties.

2. Geometric motive

2.1. Transition category SmCorr.

Definition 2.1 (Finite Correspondence)**.** For any $X, Y \in \mathsf{Var}$, let

$$\mathrm{Corr}^{\mathrm{finite}}(X, Y) \subset Z(X \times Y)$$

be generated by $V \subset X \times Y$ finite over X and dominate a component of X.

Using finite correspondence, one can

Definition 2.2. Define the following categories and functors:

(1) the category SmCorr consisting of
 (a) Object: Sm.
 (b) Morphism: $\mathrm{Hom}(X, Y) := \mathrm{Corr}^{\mathrm{finite}}(X, Y)$.
(2) the triangled category $\mathbb{H}^b(\mathsf{SmCorr})$: the homotopy category[1] of the bounded complexes in SmCorr.
(3) the functors between

$$\begin{aligned} \wp : \mathsf{Sm} &\to \mathsf{SmCorr} \\ X &\mapsto X \\ f : X \to Y &\mapsto \Gamma_f \in \mathrm{Corr}^{\mathrm{finite}}(X, Y) \end{aligned}$$

2.2. Geometric motive.

At first, we construct the category of effective geometric motives, denoted by

$$D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$$

Such construction is divided into two steps (category and functor):

(1) (Verdier) Localize $\mathbb{H}^b(\mathsf{SmCorr})$ with respect to
 (a) ($\mathbb{A}^1$-homotopy) any $\mathbb{A}^1$-projection $p_X : X \times \mathbb{A}^1 \to X$.
 (b) (Mager-Vietoris sequence) any sequence of the form

$$U \cap V \to U \oplus V = U \sqcup V \to X$$

 for any $X \in \mathsf{Sm}$ and $U, V \subset X$ open subsets with $X = U \cup V$.
 to obtain the desired category $D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$.
(2) Take pseudo-abelian hull via adding projectors.

$$\mathcal{M} : \mathsf{Sm} \xrightarrow{\wp} \mathsf{SmCorr} \xrightarrow{\deg=0} \mathbb{H}^b(\mathsf{SmCorr}) \xrightarrow{\text{Localization}} D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$$

 by mapping

$$\begin{aligned} X &\mapsto [X] =: \mathcal{M}(X) \\ f : X \to Y &\mapsto [\Gamma_f] \end{aligned}$$

 to obtain the desired functor $\mathcal{M} : \mathsf{Sm} \to D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$.

Definition 2.3 (Tate object)**.** The *Tate object* of $D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$ is given by

$$\mathbb{Z}(1) := [\mathbb{P}^1 \to pt] \in D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$$

[1]Recall: a homotopy equivalence is of the form

$$\begin{tikzcd} \cdots \arrow[r] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"'] & C_i \arrow[r] \arrow[d, "f_i"'] \arrow[dl, dashed] & C_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] \arrow[dl, dashed] & \cdots \\ \cdots \arrow[r] & D_{i-1} \arrow[r] & D_i \arrow[r] & D_{i+1} \arrow[r] & \cdots \end{tikzcd}$$

Definition 2.4 (Geometric motive)**.** The category of geometric motive $D\mathsf{M}_{\mathrm{gm}}$ is obtained by formally inverting $\mathbb{Z}(1)$ in $D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$, i.e.,

$$D\mathsf{M}_{\mathrm{gm}} := D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}[\mathbb{Z}(1)^{-1}]$$

3. Problems and embeddings

In last section we construct the category $D\mathsf{M}_{\mathrm{gm}}$, however we find that

Problem 3.1. There is no enough structure in $D\mathsf{M}_{\mathrm{gm}}$.

Idea. Embed $D\mathsf{M}_{\mathrm{gm}}^{\mathrm{eff}}$ into a larger triangled category of motivic complexes. □

3.1. Motivic complexes.

Recall presheaves with transfers:

$$F : \mathsf{SmCorr}^{\mathrm{op}} \to \mathsf{Ab}$$

Definition 3.2. F is called *homotopy invariant* if

$$F(X) \cong F(X \times \mathbb{A}^1)$$

for any $X \in \mathsf{Sm}$.

3.2. Nisnevid topology.

Nisnevid topology is a kind of Grothendieck topology lies between Zariski topology and étale topology.

Definition 3.3 (Nisnevid covering)**.** A family of étale morphisms $\{p_i : U_i \to X\}$ is called a *Nisnevid covering* of X if for any (scheme theoretically) $x \in X$, there exists U_i and $u \in U_i$ such that

$$p_i(u) = x \text{ and } k(x) \xrightarrow{\sim} k(u) \text{ an isomorphism}$$

Thus Nisnevid coverings form a Grothendieck topology, hence

(1) $\mathsf{Nis}^{\mathrm{tr}}$: the category of Nisnevid sheaves with transfers (NS+PWT).
(2) $D^{-}(\mathsf{Nis}^{\mathrm{tr}})$: the bounded above category of Nisnevid sheaves with transfers.

Definition 3.4 (Category of effective motivic complexes)**.** The *category of effective motivic complexes* $D\mathsf{M}^{-,\mathrm{eff}} \subset D^{-}(\mathsf{Nis}^{\mathrm{tr}})$ is the full subcategory of complexes with homotopy invariant cohomology sheaves.

3.3. Embeddings and properties.

For our purpose, we introduce the so-called Suslin Complex.

Definition 3.5. For any presheaf $F : \mathsf{Sm}^{\mathrm{op}} \to \mathsf{Ab}$, we can define the associated *Suslin complex* as

$$C_*(F) : \quad \cdots \to C_{n+1}(F) \to \underbrace{C_n(F)}_{\text{presheaf}} \to C_{n-1}(F) \to \cdots \to C_0(F)$$

Here for any $U \in \mathsf{Sm}$, we have $C_n(F)(U) := F(U \times \Delta^n)$ where

$$\Delta^n := \mathrm{Spec}\ \Bbbk[t_0, \ldots, t_n] / \sum_{i=0}^{n} t_i - 1$$

Fact 3.6. If $F \in \mathsf{Nis}^{\mathrm{tr}}$, then $C_*(F)$ has homotopy invariant cohomology and hence

$$C_* : \mathsf{Nis}^{\mathrm{tr}} \to D\mathsf{M}^{-,\mathrm{eff}}$$

For any $X \in \mathsf{Var}$, we define the presheaf with transfers $\mathbb{Z}^{\mathrm{tr}}(X)$ as

$$\mathbb{Z}^{\mathrm{tr}}(X)(U) := \mathrm{Corr}^{\mathrm{finite}}(U, X)$$

for any $U \in \mathsf{Sm}$. Here we view any variety as a presheaf on smooth varieties.

Fact 3.7. $\mathbb{Z}^{\mathrm{tr}}(X) \in \mathsf{Nis}^{\mathrm{tr}}$.

Remark 3.8. In particular, we obtain

$$\mathbb{Z}^{\mathrm{tr}} : \mathbb{H}^b(\mathsf{SmCorr}) \to D^-(\mathsf{Nis}^{\mathrm{tr}})$$

and also define

$$C_*(X) := C_*(\mathbb{Z}^{\mathrm{tr}}(X)) \in D\mathsf{M}^{-,\mathrm{eff}}$$

Theorem 3.9 (Voevosky)**.** *With the notations as above, we have*

(1) (localization) *the functor C_* extends to*

$$RC_* : D^-(\mathsf{Nis}^{\mathrm{tr}}) \to D\mathsf{M}^{-,\mathrm{eff}}$$

In fact, RC_* is identified with the localization with respect to

$$\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}(X) \quad \forall X \in \mathsf{Sm}$$

(2) (embedding) *there is a commutative diagram*

$$\begin{tikzcd} \mathbb{H}^b(\mathsf{SmCorr}) \arrow[r, "\mathbb{Z}^{\mathrm{tr}}"] \arrow[d, "\mathcal{M}"'] & D^-(\mathsf{Nis}^{\mathrm{tr}}) \arrow[d, "RC_*"] \\ D\mathsf{M}^{\mathrm{eff}}_{\mathrm{gm}} \arrow[r, hook, "i", "\textit{full embedding}"'] & D\mathsf{M}^{-,\mathrm{eff}} \end{tikzcd}$$

where both vertical arrows are localizations, i.e., for any $X \in \mathsf{Sm}$

$$i(\mathcal{M}(X)) = RC_*(\mathbb{Z}^{\mathrm{tr}}(X)) = C_*(X)$$

Remark 3.10. The category $D\mathsf{M}^{-,\mathrm{eff}}$ has more structure, for example, it's with homotopy T-structure and $\otimes$-structure.

3.4. **Consequence on $D\mathsf{M}_{\mathrm{gm}}$.** Here we list some consequence on $D\mathsf{M}_{\mathrm{gm}}$ via the embedding constructed in Theorem 3.9.

(1) (a) Homotopy invariant: $\mathcal{M}(X \times \mathbb{A}^1) = \mathcal{M}(X)$.
  (b) MV sequence: $\mathcal{M}(U \cap V) \to \mathcal{M}(U) \oplus \mathcal{M}(V) \to \mathcal{M}(X) \xrightarrow{+1} \cdots$
  (c) Kunneth formula: $\mathcal{M}(X \times Y) = \mathcal{M}(X) \otimes \mathcal{M}(Y)$.
  (d) Gysin sequence: $Z \subset X$ closed with $Z \in \mathsf{Sm}$ of codim c, then

$$\mathcal{M}(X \setminus Z) \to \mathcal{M}(X) \to \mathcal{M}(Z)(c)[2c] \xrightarrow{+1} \cdots$$

  (e) Blow-up, Proj-bundle, etc.

(2) Motive for all varieties: for any $X \in \mathsf{Var}$, we have

$$\begin{aligned} C_*(X) &\in D\mathsf{M}^{-,\mathrm{eff}} \\ &\in i(D\mathsf{M}^{\mathrm{eff}}_{\mathrm{gm}}) \end{aligned}$$

and hence get

$$\mathcal{M} : \mathsf{Var} \to D\mathsf{M}^{\mathrm{eff}}_{\mathrm{gm}} \text{ mapping } X \mapsto \mathcal{M}(X)$$

(3) Motives with compact support: for any $X \in \mathsf{Var}$, we construct

$$\mathcal{M}^c(X) \in D\mathsf{M}^{\mathrm{eff}}_{\mathrm{gm}}$$

such that if X proper, then $\mathcal{M}^c(X) = \mathcal{M}(X)$. Indeed, can construct via

$$\mathbb{Z}^{\mathrm{tr},c}(X) \in \mathsf{Nis}^{\mathrm{tr}}$$

defined by

$$\mathbb{Z}^{\mathrm{tr},c}(X)(U) := \mathrm{Corr}^{\mathrm{quasi-finite}}(U, X)$$

for any $Z \subset X$ closed. Then use Gysin sequence

$$\mathcal{M}^c(Z) \to \mathcal{M}^c(X) \to \mathcal{M}^c(X \setminus Z) \xrightarrow{+1} \cdots$$

(4) Dual action:

$$D : D\mathsf{M}_{\mathrm{gm}}^{\mathrm{op}} \to D\mathsf{M}_{\mathrm{gm}}$$

such that

$$\mathrm{Hom}(\mathcal{M}(X), \mathcal{M}(Y)) = \mathrm{Hom}(\mathcal{M}(X) \otimes D(\mathcal{M}(Y)), \mathbb{Z})$$

In particular, if $X \in \mathsf{Sm}$ of dimension d, then

$$D(\mathcal{M}(X)) = \mathcal{M}^c(X) \underbrace{(-d)}_{\text{twist}} \underbrace{[-2d]}_{\text{shift}}$$

(5) (a) Connection with Chow groups: for any $X \in \mathsf{Var}$, we have

$$\mathrm{Hom}(\mathbb{Z}(i)[j], \mathcal{M}^c(X)) = \mathrm{CH}_i(X, j - 2i)$$

where $\mathrm{CH}_i(X, j-2i)$ is the *Bloch higher Chow group*. If $j = 2i$, then it's nothing but $\mathrm{CH}_i(X)$.

(b) Motivic cohomology:

$$H^j_M(X, \mathbb{Z}(i)) = \mathrm{Hom}(\mathcal{M}(X), \mathbb{Z}(i)[j])$$

In particular, if $X \in \mathsf{Sm}$, then

$$H^{2i}_M(X, \mathbb{Z}(i)) = \mathrm{CH}^i(X)$$

(6) Embedding:

$$\underbrace{\mathcal{M}_{\mathrm{rat}}^{\mathrm{op}}}_{\mathbb{Z}\text{-coeff}} \hookrightarrow D\mathsf{M}_{\mathrm{gm}}$$

Proof. For any $X, Y \in \mathsf{SmProj}$, then (Hom are all computed in $D\mathsf{M}_{\mathrm{gm}}$)

$$\begin{aligned}
\mathrm{Hom}(\mathcal{M}(X), \mathcal{M}(Y)) &= \mathrm{Hom}(\mathcal{M}(X) \otimes D(\mathcal{M}(Y)), \mathbb{Z}) \\
&= \mathrm{Hom}(\mathcal{M}(X) \otimes \mathcal{M}^c(Y)(-d_Y)[-2d_Y], \mathbb{Z}) \\
&= \mathrm{Hom}(\mathcal{M}(X) \otimes \mathcal{M}^c(Y), \mathbb{Z}(d_Y)[2d_Y]) \\
&= \mathrm{Hom}(\mathcal{M}(X) \otimes \mathcal{M}(Y), \mathbb{Z}(d_Y)[2d_Y]) \\
&= \mathrm{CH}^{d_Y}(X \times Y) \\
&= \mathrm{CH}^{d_Y}(Y \times X) \\
&= \mathrm{Corr}^0(Y, X)
\end{aligned}$$

□

References

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