LECTURE ON INTERSECTION THEORY (II)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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Contents

Unless stated specifically, we will adapt the following conventions throughout this course.

- $\Bbbk$: the ground field we are working on. For simplicity we may assume that $\Bbbk = \bar{\Bbbk}$, but it's not essential in the subsequent discussion.
- Scheme: separated and finite type over $\Bbbk$. Use symbol $X, Y, \ldots$ to denote.
- Variety: irreducible and reduced (= integral) scheme.
- Subvariety: closed subscheme that is a variety. Use symbol $V, W, \ldots$ to denote.
- Point: closed point. Use symbol $P, Q, \ldots$ to denote.

1. Rational equivalence

Given a variety X. Recall the group of algebraic k-cycle of X is defined by

$$Z_k(X) := \mathbb{Z} \cdot \{\text{subvar of } X \text{ of } \dim k\}$$

so each element $\alpha \in Z_k(X)$ is of the form $\alpha = \sum n_i V_i$ for some $n_i \in \mathbb{Z}$ and V_i subvarieties of X of dimension k.

In this part, we want to define rational equivalence between algebraic cycles $Z_*(X)$ and hence obtain the Chow group $\mathrm{CH}_*(X)$ of X. Roughly say, rational equivalence means 'move along $\mathbb{P}^1$'.

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1.1. **Idea case.** Suppose $V \subset X \times \mathbb{P}^1$ be a subvariety dominating $\mathbb{P}^1$, i.e., the image of V under the projection $\pi : X \times \mathbb{P}^1 \to \mathbb{P}^1$ contains an Zariski-open subset of $\mathbb{P}^1$. For any point $Q \in \mathbb{P}^1$, we denote by $V(Q) := \pi^{-1}(Q)$ its fiber.

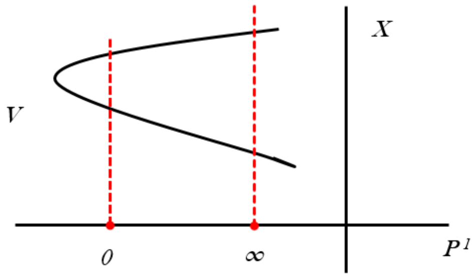

FIGURE 1. Idea case

We want to say $V(0) \sim_{\text{rat}} V(\infty)$[1], here $V(0)$ and $V(\infty)$ are viewed as schemes. But sometimes we may need to handle the following 'terrible' situation. which

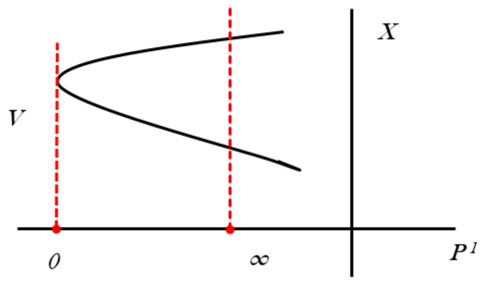

FIGURE 2. 'Terrible' case

inspires us to consider multiplicity. And naturally the tool of Commutative Algebra is introduced. For this purpose, we introduce the notation *cycle of a scheme*, from which we can record the multiplicity information from scheme.

Definition 1.1 (Cycle of a scheme)**.** Let Y be a scheme and $Y_1, \ldots, Y_n$ its irreducible components (reduced). The *multiplicity* of Y_i in Y is defined by

$$m_i := \text{length}_{\mathcal{O}_{Y_i,Y}} \mathcal{O}_{Y_i,Y}$$

where $\mathcal{O}_{Y_i,Y}$ are Artinian local ring of dimension 0. Then the *cycle* of Y is defined by

$$\sum_{i=1}^{n} m_i Y_i$$

With this concept at hand, we can safely say that rational equivalence is an equivalence relation generated by $V(0)-V(\infty)$. More precisely, we give the following two alternative definitions.

[1]notice that $V(0)$ and $V(\infty)$ are of the same dimension since any morphims to $\mathbb{P}^1$ is flat.

1.2. Two alternative definitions.

Definition 1.2 (1^{st} definition)**.** An element $\alpha \in Z_k(X)$ is *rational equivalent* to 0, denoted by $\alpha \sim_{\text{rat}} 0$, if there exists $\beta \in Z_{k+1}(X \times \mathbb{P}^1)$ whose components dominating $\mathbb{P}^1$ such that

$$\alpha = \beta(0) - \beta(\infty)$$

as algebraic cycles. Here we view $\beta(0)$ and $\beta(\infty)$ as schemes and $\beta(0) - \beta(\infty)$ as cycle of scheme.

One the other hand, we know that rational equivalence can be viewed as a generalization of linear equivalence of divisors, so we may also define it in terms of rational functions. Let X be a variety of dimension $k+1$ and $\mathcal{R}(X)$ its field of rational functions. For each $f \in \mathcal{R}(X)^*$, we recall its divisor is defined by

$$\begin{aligned}\operatorname{div}(f) :&= \text{zeros} - \text{poles} \\ &= \underbrace{f^{-1}(0)}_{\text{of } \dim k} - \underbrace{f^{-1}(\infty)}_{\text{of } \dim k} \in Z_k(X)\end{aligned}$$

Definition 1.3 (2^{nd} definition)**.** An element $\alpha \in Z_k(X)$ is *rational equivalent* to 0, denoted by $\alpha \sim_{\text{rat}} 0$, if there exists subvarieties $W_1, \ldots, W_n$ of X dimension $k+1$ and $f_i \in \mathcal{R}(W_i)^*$ such that

$$\alpha = \sum_{i=1}^{n} \operatorname{div}(f_i)$$

Having defined the rational equivalence, we obtain the *Chow group* of k-cycle

$$\mathrm{CH}_k(X) := Z_k(X)/\sim_{\text{rat}}$$

Example 1.4. Suppose X is pure of dimension d, then by Definition 1.3

$$\mathrm{CH}_d(X) = Z_d(X) = \mathbb{Z} \cdot \{\text{irred comp of } X\}$$

1.3. **Equivalence of two definitions.** One can easily check the equivalence of these two definition. Indeed, if $Y \subset X$ is a closed subvariety, then for each $f \in \mathcal{R}(Y)^*$, i.e., $f : Y \dashrightarrow \mathbb{P}^1$, let V be the closure of graph of f, then $V \subset X \times \mathbb{P}^1$ and one can easily check that $\operatorname{div}(f) = V(0) - V(\infty)$.

2. Functoriality of $\mathrm{CH}_i(-)$

Roughly say, $\mathrm{CH}_i(X)$ looks like some kind of some 'homology' of X, so we can image that it's easy to push-forward but hard to pull-back. In this part, we discuss the behaviour of Chow group under some appropriate morphism.

2.1. **Proper push-forward.** Let $f : X \to Y$ be a proper (separated + universally closed) morphism and $V \subset X$ a subvariety, hence define an element in $Z_*(X)$. Our aim is to define an element $f_*(V) \in Z_*(Y)$, i.e., push-forward of V via the proper morphism f.

A natural (and maybe unique) choice of $f_*(V)$ is the subvariety $W := f(V) \subset Y$, but to make W an element in $Z_*(Y)$, we need to make some modification, i.e., again need to consider multiplicity. For this, we introduce the concept of *degree*.

Definition 2.1. By properness of f, we get a field extension $f^{\#} : \mathcal{R}(W) \hookrightarrow \mathcal{R}(V)$, so define

$$\deg(V/W) := \begin{cases} 0 & \dim V > \dim W \\ [\mathcal{R}(V) : \mathcal{R}(W)] < \infty & \dim V = \dim W \end{cases}$$

With this concept, we can define

$$f_*(V) := \deg(V/W)W$$

and therefore get a map

$$f_* : Z_k(X) \to Z_k(Y)$$

A natural question is whether this map can descend to the level of Chow group or not. This is validated by the following theorem.

Theorem 2.2. *If $\alpha \sim_{\text{rat}} 0$, then $f_*(\alpha) \sim_{\text{rat}} 0$.*

Hence we finally get the desired map

$$f_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$$

Proof. The idea of this proof is illustrated in the following picture.

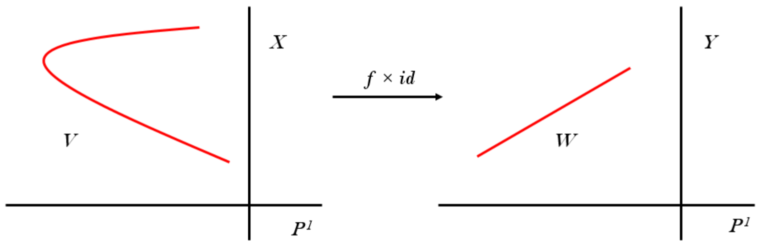

FIGURE 3. Idea of the proof

Then

$$f_*(V(0) - V(\infty)) = \# \cdot [f(V(0)) - f(V(\infty))]$$

where # denotes some integer. □

Example 2.3. Let $p : X \to \mathrm{Spec}(\Bbbk)$ be a complete scheme (proper over $\Bbbk$). For each 0-cycle of X

$$\alpha = \sum n_i P_i \in Z_0(X)$$

where $P_i \in X$ are closed points, we can define

$$\begin{aligned} \deg(\alpha) :&= \sum n_i [\mathcal{R}(P_i) : \Bbbk] \\ &= p_*(\alpha) \end{aligned}$$

[2] Clearly that $\mathrm{CH}_0(\mathrm{Spec}(\Bbbk)) = Z_0(\mathrm{Spec}(\Bbbk)) \cong \mathbb{Z}$. So by Theorem 2.2 we get a *degree* map

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$$

[2]Here the factor $[\mathcal{R}(P_i) : \Bbbk]$ is added in case $\Bbbk$ is not algebraically closed. If $\Bbbk$ is algebraically closed, then

$$\deg(\alpha) = \sum n_i$$

2.2. **Flat pull-back.** Let $f : X \to Y$ be a morphism and $V \subset Y$ a subvariety, hence define an element in $Z_*(Y)$. Our aim is to define an element $f^*(V) \in Z_*(X)$, i.e., pull-back of V via the morphism f.

Remark 2.4. In general, it's hard to define pull-back. In fact,

$$\text{pull-back} \Longleftrightarrow \text{intersection}$$

(1) If we have defined pull-back, then the intersection of algebraic cycles can be defined by
$$\alpha.\beta := \Delta_X^*(\alpha \times \beta)$$
where $\Delta_X : X \to X \times X$ is the diagonal embedding.
(2) If we have defined intersection, then the pull-back of algebraic cycles can be defined by
$$f^*(\alpha) := (p_X)_*[(X \times \alpha).\Gamma_f]$$
where $f : X \to Y$ is proper, $\Gamma_f \in Z_*(X \times Y)$ the graph of f and $p_X : X \times Y \to X$ the projection.

In this section, we only consider the pull-back for *flat* morphism. Let $X \to Y$ be a flat (Geometrically, this means the fibers range 'continuously' with the same dimension) morphism of relative dimension n.

Remark 2.5. We list some example of (non-)flatness.

(1) Examples of not flat: blow-up, etc.
(2) Examples of flat: vector bundle, open immersion, morphism to non-singular curves, etc.

As before, let $V \subset Y$ be a subvariety of dimension k. Define
$$f^*(V) := \text{cycle of } f^{-1}(V), \text{ as a scheme}$$
and we get a map
$$f^* : Z_k(Y) \to Z_{k+n}(X)$$
A natural question is whether this map can descend to the level of Chow group or not. This is validated by the following theorem.

Theorem 2.6. *If $\alpha \sim_{\text{rat}} 0$, then $f^*(\alpha) \sim_{\text{rat}} 0$.*

Hence we finally get the desired map
$$f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+n}(X)$$

Proof. The idea of this proof is illustrated in the following picture. □

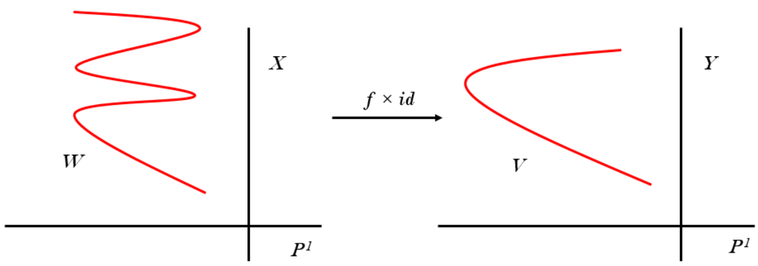

FIGURE 4. Idea of the proof

Example 2.7 (Flat+Proper)**.** Consider the following Cartesian diagram in which f is proper and g is flat. Since properness and flatness are preserved under base change, f' is proper and g' is flat.

$$\begin{tikzcd} X' \arrow[r, "g'", "\text{flat}"'] \arrow[d, "f'", "\text{proper}"'] & X \arrow[d, "f"', "\text{proper}"] \\ Y' \arrow[r, "g", "\text{flat}"'] & Y \end{tikzcd}$$

Then for each $\alpha \in Z_*(X)$, we have

$$g^* f_*(\alpha) = (f')_*(g')^*(\alpha) \in Z_*(X').$$

Example 2.8. Let $f : X \to Y$ be finite and flat of degree d. Then for each $\alpha \in Z_*(Y)$, we have

$$f_* f^*(\alpha) = d\alpha \in Z_*(Y).$$

3. Localization sequence

In this part, we introduce localization sequence, which is a useful tool in the following discussion.

3.1. **Statement.** Let X be a scheme and $Y \hookrightarrow X$ a closed subscheme. Denoted by $U := X - Y$ the open subscheme, then we have

$$Y \xrightarrow{i} X \xleftarrow{j} U$$

where

(1) i is a closed immersion, hence proper, we can define
$$i_* : Z_k(Y) \to Z_k(X)$$
(2) j is an open immersion, hence flat, we can define
$$j^* : Z_k(X) \to Z_k(U).$$

Composing these two maps together, it turns out that

Lemma 3.1. *The sequence of algebraic cycles*

$$(\dagger) \qquad 0 \to Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \to 0$$

is exact.

Proof. We prove the exactness place by place.

(1) Exactness at (1): trivial.
(2) Exactness at (2): for each algebraic k-cycle $V \in Z_k(X)$, if $j^*(V) = V|_U = 0$, then V is supported in Y and hence an element of $Z_k(Y)$.
(3) Exactness at (3): for each algebraic k-cycle $V \in Z_k(U)$, its closure $\bar{V} \in Z_k(X)$ and therefore $j^*(\bar{V}) = V$.

□

The exactness of algebraic cycles can descend to level of Chow group, but with injectivity missing. One can easily find such counter-examples.

Proposition 3.2. *The sequence of Chow groups*

$$\mathrm{CH}_k(Y) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \to 0$$

is exact.

Proof. We prove the exactness place by place.

(1) Exactness at (1): if $j^*(\alpha) \sim_{\text{rat}} 0$, then there exist $W_1, \dots, W_n$ subvarieties of U and $f_i \in \mathcal{R}(W_i)^*$ such that

$$j^*(\alpha) = \sum \operatorname{div}(f_i)$$

Write $\bar{f}_i \in \mathcal{R}(\bar{W}_i)^* = \mathcal{R}(W_i)^*$ corresponds to the element $f_i \in \mathcal{R}(W_i)^*$, then we have

$$j^*(\alpha - \sum \operatorname{div}(\bar{f}_i)) = 0$$

as cycles. By exactness of (†), there exists $\beta \in \mathrm{CH}_*(Y)$ such that

$$\alpha - \sum \operatorname{div}(\bar{f}_i) = i_*(\beta)$$

so $\alpha \sim_{\text{rat}} i_*(\beta)$.

(2) Exactness at (2): trivial.

□

3.2. **Application.** Consider the vector bundle E of rank r over X

$$\begin{tikzcd} E \arrow[d, "p"] \\ X \arrow[u, bend left, "0"] \end{tikzcd}$$

Clearly p is flat, so we get

$$p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$$

Recall in last lecture, we want to establish an isomorphism

$$p^* : \mathrm{CH}_k(X) \xrightarrow{\sim} \mathrm{CH}_{k+r}(E)$$

in order to define intersection with zero section. By Proposition 3.2, we can prove

Corollary 3.3. *p^* is surjective.*

Proof. (Only a sketch) Choose Y a closed subscheme of X such that

$$E|_{U:=X-Y} = U \times \mathbb{A}^r.$$

Consider the following commutative digram given by Proposition 3.2,

$$\begin{tikzcd} \mathrm{CH}_*(Y) \arrow[r] \arrow[d, "(1)"'] & \mathrm{CH}_*(X) \arrow[r] \arrow[d, "(2)"', "p^*"] & \mathrm{CH}_*(U) \arrow[r] \arrow[d, "(3)"'] & 0 \\ \mathrm{CH}_*(E|_Y) \arrow[r] & \mathrm{CH}_{*+r}(E) \arrow[r] & \mathrm{CH}_*(E|_U) \arrow[r] & 0 \end{tikzcd}$$

By 5-Lemma, (1) & (3) surjective $\Rightarrow$ (2) surjective. By noetherian induction, it suffices to prove the statement for (3), i.e., for the case of trivial bundle $E = X \times \mathbb{A}^r$. By further induction on r, it suffices to prove the case $E = X \times \mathbb{A}^1$, which use some knowledge of AG. □

Corollary 3.4. *We have*

$$\mathrm{CH}_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k < n \end{cases}$$

and

$$\mathrm{CH}_k(\mathbb{P}^n) = \mathbb{Z}[L_k]$$

where L_k is a vector space of dimension k.

Proof. The case of $\mathbb{A}^n$ is easy. For $\mathbb{P}^n$, use the decomposition $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$ and by induction. □

References

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