LECTURE ON INTERSECTION THEORY (XVIII)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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Contents

Some useful and important references:

(1) The Standard Conjectures by S. Kleiman, see [Kle94].
(2) On the Chow Ring of K3 Surface by A. Beauville and C. Voisin, see [BV04].

1. Work of Beauville and Voisin

In this section, let S be a K3 surface over $k = \mathbb{C}$.

Theorem 1.1 ([BV04]). *There exists a distinguished element*

$$0_S \in \mathrm{CH}_0(S) = \mathrm{CH}^2(S)$$

represented by a point on a rational curve on S, such that

(1) $\mathrm{CH}^1(S) \times \mathrm{CH}^1(S) \xrightarrow{\cdot} \mathbb{Z} \cdot 0_S \subset \mathrm{CH}^2(S)$.
(2) $c_2(\mathscr{T}_S) = 24 \cdot 0_S \in \mathrm{CH}^2(S)$.

Remark 1.2. That is, all points of X which lie on some (possibly singular) rational curve have the same class $0_S \in \mathrm{CH}_0(S)$.

If one introduces

$$\mathrm{CH}^2(S) = \mathbb{Z} \cdot 0_S \oplus \mathrm{CH}^2(S)_{\mathrm{hom}} = \mathrm{CH}^2_{(0)}(S) \oplus \mathrm{CH}^2_{(2)}(S)$$
$$\mathrm{CH}^1(S) = \mathrm{NS}(S) = \mathrm{CH}^1_{(0)}(S)$$

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$$\mathrm{CH}^0(S) = \mathbb{Z}[S] = \mathrm{CH}^0_{(0)}(S)$$

then there is another way to re-formulate this result

Theorem 1.3 (Variation of Theorem 1.1)**.**

(1) *as rings:*

$$(\mathrm{CH}^*(S), \cdot) = \left(\bigoplus_{i,k} \mathrm{CH}^k_{(i)}(S), \cdot \right) \quad \textit{bi-graded ring}$$

(2) $c_2(\mathscr{T}_X) \in \mathrm{CH}^2_{(0)}(S)$.

Remark 1.4. Compare (1) with abelian variety: let A be an abelian variety, then

$$(\mathrm{CH}^*(A)_{\mathbb{Q}}, \cdot) = \left(\bigoplus_{i,k} \mathrm{CH}^k_{(i)}(A), \cdot \right) \quad \text{bi-graded ring}$$

where

$$\mathrm{CH}^k_{(i)}(A) := \{\alpha \in \mathrm{CH}^k(A)_{\mathbb{Q}} : N^*\alpha = N^{2k-i}\alpha \text{ for all } N \in \mathbb{Z}\}$$

In this case, there is a candidate of the Bloch-Beilinson filtration

$$F^i\mathrm{CH}^k(A)_{\mathbb{Q}} := \bigoplus_{j \geq i} \mathrm{CH}^k_{(j)}(A)$$

Conjecture 1.5 (Beauville)**.** *Let A be an abelian variety, then*

(1) $\mathrm{CH}^*_{(<0)}(A) = 0$.
(2) $\mathrm{CH}^*_{(0)}(A) \hookrightarrow H^{2*}(A, \mathbb{Q})$.

For example, $\mathrm{CH}^1(A)_{\mathbb{Q}} = \mathrm{CH}^1_{(0)}(A) \oplus \mathrm{CH}^1_{(1)}(A)$ where

$$\mathrm{CH}^1_{(0)}(A) = \mathrm{NS}(A)_{\mathbb{Q}} \text{ and } \mathrm{CH}^1_{(1)}(A) = \mathrm{Pic}^0(A)_{\mathbb{Q}} \cong \widehat{A}_{\mathbb{Q}}$$

2. Generalization of K3 surface

Question: what should be the 'right' generalization of K3 surface ?
Answer: there are two directions.

2.1. **Calabi-Yau varieties.**

Definition 2.1. Let X be a n-dim'l smooth projective variety, then X is called a *Calabi-Yau variety* if

(1) simply connected.
(2) (Canonical bundle) $K_X := \Omega^n_X \cong \mathcal{O}_X$.
(3) (Middle-Vanishing) $H^0(X, \Omega^i_X) = 0$ for $i \neq 0, n$.

By definition, the Hodge diamond of Calabi-Yau varieties is of the form
Such configuration is important in mirror symmetry.

2.2. **Hyperkalher varieties/Irreducible holomorphic sympletic varieties.**

Definition 2.2. Let X be a smooth projective variety, then X is called a *hyperkalher varieties* if

(1) simply connected.
(2) (Sympletic structure) $H^0(X, \Omega^2_X) = \mathbb{C} \cdot \sigma$ where σ is a nonwhere degenerate sympletic 2-form.

This implies that

(1) dim $X = 2n$.

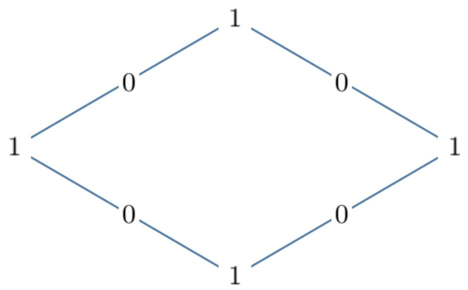

FIGURE 1. Hodge diamond of Calabi-Yau variety

(2) the canonical line bundle $K_X := \Omega_X^{2n} \cong \mathcal{O}_X$ as $H^0(X, \Omega_X^{2n}) \neq 0$
(3) the Hodge diamond of hyperkalher varieties is of the form

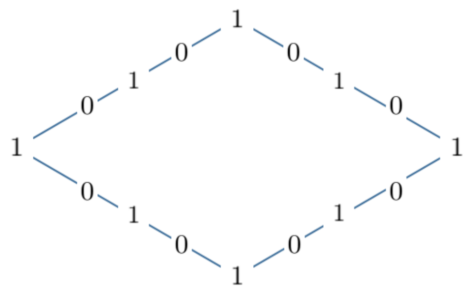

FIGURE 2. Hodge diamond of Hyperkalher variety

Theorem 2.3 (Beauville-Bogomolov decomposition theorem). *Let X be a n-dim'l smooth projective variety. Assume*

$$K_X = c_1(\Omega_X^n) \sim_{\text{num}} 0$$

then there exists a finite étale covering

$$\tilde{X} \to X$$

such that $\tilde{X}$ is the product of abelian varieties, Calabi-Yau varieties and hyperkalher varieties.

Example 2.4 (of hyperkalher variety). Only a few are known.

(1) (Beauville) Start with

$$\begin{gathered} S\text{: a K3 surface} \\ \Downarrow \\ S^n : n\text{-fold products} \\ \Downarrow \\ S^{(n)} = S^n/S_n \text{ maybe singular} \\ \Downarrow \\ S^{[n]}\text{: Hilbert scheme of length } n \text{ subscheme on } S \end{gathered}$$

Fact 2.5 (Fagarty). For any surface S, $S^{[n]}$ is smooth.

(2) (Beauville) Start with A any abelian variety.

$$\pi : A^{[n+1]} \xrightarrow{\text{forget scheme structure}} A^{(n+1)} \xrightarrow{+} A$$

Define the so-called *generalized Kummer varieties*

$$K_n(A) := \pi^{-1}(0_A)$$

In the case $n = 1$, we obtain the so-called 'Kummer K3 surface'

$$\pi : A^{[2]} \to A \Rightarrow K_1(A) = \mathfrak{Bl}_{16 \text{ pts}} A/\{\pm 1\}$$

(3) Deformation of (1) & (2), denoted by (1') and (2').
(4) (O' Grady) two of such examples: (OG10) and (OG6)
   (a) (OG10) exists only at dim 10

$$\begin{array}{c}\text{singular moduli space of sheaves on K3}\\ \uparrow \\ \text{sympletic resolution}\end{array}$$

   (b) (OG6) exists only at dim 6: similar for abelian surface A.
(5) Deformation of (4), denoted by (4').
(6) (Beauville-Donagi) can be viewed as (1"): for $Y \subset \mathbb{P}^5$ a smooth cubic fourfold[1], then the Fano variety of lines in Y, say $F(Y)$. It's also a locally complete family.
(7) (Lazz-Saecc-Voisin) can be viewed as (3"): bigger family of deformation of (OG10).

3. Cycles aspect: Beauville-Voisin Conjecture

Let X be a hyperkalher variety of dimension $2n$. We expect

$$(\text{Ideally}) \ (\mathrm{CH}^*(X)_{\mathbb{Q}}, \cdot) = \left(\bigoplus_{i,k} \mathrm{CH}^k_{(i)}(X)_{\mathbb{Q}}, \cdot \right) \ \text{bi-graded}$$

as rings such that

$$F^i \mathrm{CH}^k(X)_{\mathbb{Q}} = \bigoplus_{j \geq i} \mathrm{CH}^k_{(j)}(X)$$

gives the Bloch-Beilnson filtration.

multiplicity splitting of the Bloch-Beilnson filtration

Too abstract, more down-to-earth, we have the following Beauville-Voisin Conjecture: the following (3.1)+(3.2) is called Beauville-Voisin Conjecture.

3.1. **Beauville Conjecture.** $\mathrm{CH}^1(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$.

$$\text{Expect: } \mathrm{CH}^1(X)_{\mathbb{Q}} = \mathrm{CH}^1_{(0)}(X)$$

Define $\mathrm{DCH}^*(X)_{\mathbb{Q}} \subset \mathrm{CH}^*(X)_{\mathbb{Q}}$ to be the subring generated by $\mathrm{CH}^1(X)_{\mathbb{Q}}$.

$$\text{Expect: } \mathrm{DCH}^*(X)_{\mathbb{Q}} \subset \mathrm{CH}^*_{(0)}(X)$$
$$\text{Expect: } \mathrm{CH}^*_{(0)}(X) \hookrightarrow H^{2*}(X, \mathbb{Q})$$

Conjecture 3.1 (Beauville)**.** $\mathrm{DCH}^*(X)_{\mathbb{Q}} \hookrightarrow H^{2*}(X, \mathbb{Q})$.

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[1] # of such Y: in coordinates $x_0, \ldots, x_5$, there are

$$\binom{8}{3} = 56 \text{ choices}$$

hence in the projective space, one obtain $56 - 1 = 55$. Finally quotient the automorphism of $\mathbb{P}^5$, i.e., the action of PGL(6), which is of dimension 35, we get $55 - 35 = 20$.

As a comparison, let A be an abelian variety and $\mathrm{DCH}^*_{(0)}(A) \subset \mathrm{CH}^*(A)_{\mathbb{Q}}$ the subring generated by $\mathrm{CH}^1_{(0)}(A)$, then

Theorem 3.2 (Ancona, Monner, O' Sllon)**.** $\mathrm{DCH}^*_{(0)}(A) \hookrightarrow H^{2*}(A, \mathbb{Q})$.

3.2. Voisin Conjecture.

$$\text{Expect: } c_i(\mathscr{T}_X) \in \mathrm{CH}^i_{(0)}(X) \text{ for each } i$$

Define $\tilde{\mathrm{D}}\mathrm{CH}^*(X)_{\mathbb{Q}} \subset \mathrm{CH}^*(X)_{\mathbb{Q}}$ to be the subring generated by $\mathrm{CH}^1(X)_{\mathbb{Q}}$ and $\{c_i(\mathscr{T}_X)\}$.

Conjecture 3.3 (Voisin)**.** $\tilde{\mathrm{D}}\mathrm{CH}^*(X)_{\mathbb{Q}} \hookrightarrow H^{2*}(X, \mathbb{Q})$.

3.3. Known cases.

(1) (Lie Fu) true for $K_n(A)$: reduce to $\mathrm{DCH}^*_{(0)}(A^m) \hookrightarrow H^{2*}(A^m, \mathbb{Q})$.
(2) (Beauville and Voisin) true for $S^{[n]}$ for n small[2].
(3) (Voisin) true for $F(Y)$: Fano variety of lines on Y.

Remark 3.4. In all 3 previous cases

(1) Little is known for the deformation.
(2) there exists a candidate of

$$(\mathrm{CH}^*(X)_{\mathbb{Q}}, \cdot) = \left(\bigoplus_{i,k} \mathrm{CH}^k_{(i)}(X)_{\mathbb{Q}}, \cdot \right)$$

3.4. Something new: Voisin.
Notice that

$$\text{bi-graded } \Leftrightarrow 2 \text{ opposite filtrations}$$

$$\bigoplus_{i,k} \mathrm{CH}^k_{(i)}(X)_{\mathbb{Q}} \rightharpoonup \begin{cases} F^i\mathrm{CH}^k(X)_{\mathbb{Q}} := \bigoplus_{j \geq i} \mathrm{CH}^k_{(j)}(X) \\ S_i\mathrm{CH}^k(X)_{\mathbb{Q}} := \bigoplus_{j \leq i} \mathrm{CH}^k_{(j)}(X) \end{cases}$$

$$\mathrm{CH}^k_{(i)}(X)_{\mathbb{Q}} := F^i \cap S_i \leftharpoondown F^i\mathrm{CH}^k(X)_{\mathbb{Q}} + S_i\mathrm{CH}^k(X)_{\mathbb{Q}}$$

<u>C. Voisin</u>: a candidate for S_i for 0-cycle. Recall the two filtrations

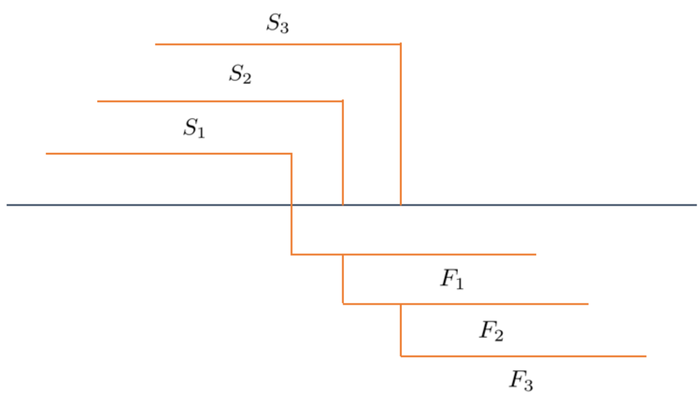

FIGURE 3. Configuration of two filtrations

<u>Idea:</u> for any $x \in X$, let

$$O_x := \{y \in X : [y] = [x] \in \mathrm{CH}_0(X)\}$$

[2]For all n, it's a consequence of Kinnura Finiteness Conjecture of S.

then O_x is the countable unions of Zariski closed subsets of X and

$$\dim O_x := \max \dim \text{ of the component} \leq n$$

Consider

$$S_{2i}(X) := \{x \in X : \dim O_x \geq n - i\}$$
$$S_{2i}\mathrm{CH}_0(X) := \langle [x] : x \in S_{2i}(X) \rangle \subset \mathrm{CH}_0(X)$$

Conjecture 3.5 (Voisin)**.**

$$\begin{tikzcd}
\dim S_{2i}(X) = n+i \arrow[d, Rightarrow, "hyperkalher"'] & \\
(n+1)\text{-}dim\ Z \arrow[r, hook] \arrow[d, "fibres\ are\ (n-i)\text{-}dim\ orbits"'] & S_{2i}(X) \\
2i\text{-}dim\ B \arrow[d, Rightarrow] & \\
S_{2i}\mathrm{CH}_0(X)_{\mathbb{Q}} \arrow[r, two heads] & \mathrm{CH}_0(X)_{\mathbb{Q}}/F^{2i+2}\mathrm{CH}_0(X)_{\mathbb{Q}}
\end{tikzcd}$$

In picture, one should have

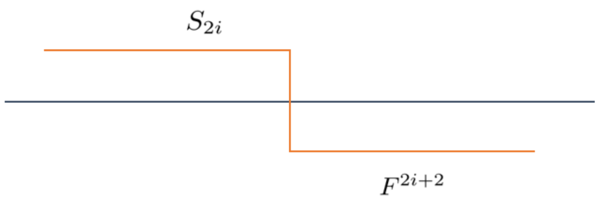

Figure 4. Illustration of configuration

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