LECTURE ON INTERSECTION THEORY (XIV)

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ABSTRACT. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

 $Instructor:\ Qizheng\ YIN\ [BICMR,\ Peking\ University]$

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From this lecture, we always work over $\mathbb Q$ instead of $\mathbb Z$, i.e., the Chow rings and some other related objects involved are defined over $\mathbb Q$. Let $\mathbb k$ be a field with char $\mathbb k=0$. For simplicity, one may simply assume $\mathbb k=\mathbb C$.

Denoted by Var/k the category of nonsingular projective variety over k.

- 1.1. Motivation: formation of cohomology. For each object $X \in \mathsf{Var}/\mathbb{k}$, we already have the following cohomology theories
 - (1) (Singular Cohomology) $H^*(X, \mathbb{Q})$.
 - (2) (Algebraic de Rham Cohomology) $H^*(X, \Omega_X^{\bullet})$.
 - (3) (ℓ -adic Cohomology) $H_{\acute{e}t}^*(X, \mathbb{Q}_{\ell})$.
 - (4) ··· et al.

In general

Definition 1.1 (Weil cohomology theory). A cohomology theory can be regarded as a functor

$$\mathscr{H}: (\mathsf{Var}/\Bbbk)^{\mathrm{op}} \to (\mathsf{Vect}/\Bbbk)$$

and a 'good' cohomology theory should satisfy some certain natural axioms:

(1) cup product \cup on $\mathcal{H}^*(X)$ such that

$$\beta \cup \alpha = (-1)^{\deg(\alpha) \cdot \deg(\beta)} (\alpha \cup \beta)$$

- (2) Poincáre duality.
- (3) Künneth formula.
- (4) cycle class map cl : $CH^*(X) \to \mathcal{H}^*(X)$.

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(5) Weak Lefschetz: for any nonsingular hypersurface Y and

$$i: Y \hookrightarrow X$$
 with $\dim(X) = d$

the pull-back

$$i^*: \mathscr{H}^k(X) \to \mathscr{H}^k(Y)$$
 is $\begin{cases} \text{isomorphic} & \text{if } k < d-1 \\ \text{injective} & \text{if } k = d-1 \end{cases}$

(6) Hard Lefschetz:

$$L^{n-k}: \mathscr{H}^k(X) \xrightarrow{\sim} \mathscr{H}^{2n-k}(X)$$

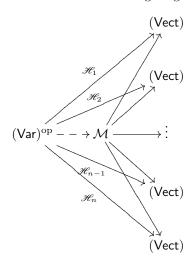
Remark 1.2. In Singular Cohomology case, the property (5) is called *Lefschetz hypersurface theorem*.

The goal of motive is to find a suitable category \mathcal{M} such that all formations of cohomology theory factor through it. Such \mathcal{M} is called a *motive*.

Ideal: the category \mathcal{M} should be

- (1) abelian and semi-simple.
- (2) Tanrakan (\Rightarrow motivate Galois group).

In picture, such \mathcal{M} can be fit into the following diagram



Remark 1.3. The only possible candidate of \mathcal{M} is to use algebraic cycles.

1.2. Construction of Chow motive. Recall the notion correspondence first.

Definition 1.4. For any $X, Y \in (Var)$, we have

(1) Correspondence:

$$Corr(X, Y) := CH^*(X \times Y)$$

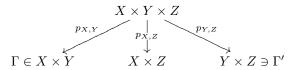
with grading

$$\operatorname{Corr}^r(X,Y) := \operatorname{CH}^{\dim(X)+r}(X \times Y)$$

and composition

$$\operatorname{Corr}^{r}(X,Y) \times \operatorname{Corr}^{s}(Y,Z) \to \operatorname{Corr}^{r+s}(X,Z)$$
$$(\Gamma,\Gamma') \mapsto (p_{X|Z}^{r})_{*}[(p_{X|Y}^{r})^{*}(\Gamma) \cdot (p_{Y|Z}^{s})^{*}(\Gamma')]$$

where

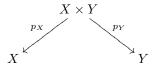


(2) Any element $\Gamma \in Corr(X, Y)$ induces a morphism

$$\Gamma_* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

 $\alpha \mapsto (p_Y)_*[(p_X)^*(\alpha) \cdot \Gamma]$

where



(3) Projector: an element $p \in \operatorname{Corr}^0(X, X)$ s.t. $p \circ p = p$ a projector of X.

Chow motive is one of the classical constructions of motive satisfying the predescribed properties. Its construction is divided into 3 steps.

Step-1: enlarge the class of morphism by letting

$$\operatorname{Hom}(Y, X) := \operatorname{Corr}(X, Y) = \operatorname{CH}^*(X \times Y)$$

such that one can add up morphisms (hence can talk about abelian).

Step-2: cut objects into pieces.

$$\forall p \in \operatorname{End}(X) = \operatorname{Hom}(X, X) \text{ with } p \circ p = p$$

one want to define ker(p) and Im(p) (pseudo-abelian hull).

Step-3: invert certain objects to take care of grading and polarization (Tate twist).

Formally say, we have

Definition 1.5 (Chow Motive). The *Chow motive* \mathcal{M}_{rat} is the category consisting of the following data

(1) Object: each object $M \in \mathcal{M}_{rat}$ is a triple

$$M = (X, p, m)$$

where $X \in (Var/\mathbb{k})$, p is a projector of X and $m \in \mathbb{Z}$.

(2) Morphism: for any two objects M = (X, p, m) and N = (Y, q, n)

$$\operatorname{Hom}(M,N) := q \circ \operatorname{Corr}^{n-m}(X,Y) \circ p$$

Remark 1.6. For any objective $M = (X, p, m) \in \mathcal{M}_{rat}$, one has

- (1) M is said to be effective if m = 0.
- (2) the identity morphism $id_M \in \text{Hom}(M, M) =: \text{End}(M)$ is by definition

$$id_M = p \circ \Delta_X \circ p = p \circ p = p$$

(3) the Chow ring/cohomology of M are defined by

$$\mathrm{CH}^*(M) := p_*(\mathrm{CH}^*(X))$$

$$H^*(M) := p_*(H^*(X))$$

i.e., via those of X and with grading

$$CH^{k}(M) := p_{*}(CH^{k+m}(X))$$

¹the well-definedness follows from the axioms of Weil cohomology theory.

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$$H^k(M) := p_*(H^{k+2m}(X))$$

Remark 1.7. Attached to the Chow motive \mathcal{M} are the following related concepts.

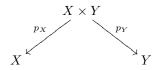
(1) any element $\Gamma \in \text{Corr}(X, Y)$ induces a morphism

$$\Gamma_*: H^*(X) \to H^*(Y)$$

given by

$$\alpha \mapsto (p_Y)_*[(p_X)^*(\alpha) \cup \operatorname{cl}(\alpha)]$$

where



(2) there is a natural functor

$$\begin{split} h: (\mathsf{Var})^{\mathrm{op}} &\longrightarrow \mathcal{M}_{\mathrm{rat}} \\ X &\mapsto (X, \Delta_X, 0) \\ (Y \xrightarrow{f} X) &\mapsto (h(X) \xrightarrow{h(f)} h(Y)) \end{split}$$

where

$$h(f) := \Delta_Y \circ [\Gamma_f^t] \circ \Delta_X = [\Gamma_f^t] \in \operatorname{Corr}^0(X, Y)$$

is the transportation of graph of f.

(3) call any functor a realization of a cohomology theory

$$\mathscr{H}: \mathcal{M}_{\mathrm{rat}} \to (\mathsf{Vect})$$

1.3. Basic properties and examples.

(1) the category $\mathcal{M}_{\rm rat}$ admits

$$\oplus, \otimes, (-)^{\vee}$$

In fact, given any two objects $M=(X,p,m), N=(Y,q,n)\in\mathcal{M}_{\mathrm{rat}},$ one can simply let

$$M \oplus N := (X \sqcup Y, p \sqcup q, -)$$

$$M \otimes N := (X \times Y, p \times q, m + n)$$

$$M^{\vee} := (X, p^{t}, \dim(X) - m)$$

(2) for any element $f \in \text{End}(M)$ satisfying $f \circ f = f$, one can define its *image* and kernel as follows²

$$Im(f) := (X, f, m) \text{ and } \ker(f) := (X, p - f, m)$$

and then

$$M=\ker(f)\oplus \operatorname{Im}(f)$$

since
$$f \circ (p - f) = (p - f) \circ f = 0$$
.

 $(1) + (2) \Rightarrow$ "rigid, tensor and pseudo-abel category".

$$(p-f)\circ(p-f)=p+f-f\circ p-p\circ f=p-f$$

since $f \in p \circ Corr^0(X, X) \circ p$, then $p \circ f = f \circ p = f$.

²one can check that

(3) Tate object: set

$$\mathbb{I}(i) := (pt, \Delta_{pt}, i)$$

and its dual (sometimes called Lefchtze motive)

$$\mathbb{L} := \mathbb{I}(-1) = (pt, \Delta_{pt}, -1) \cong (\mathbb{P}^1, [\mathbb{P}^1 \times pt], 0)$$

then for any object M = (X, p, m), its twist is therefore defined by

$$M(i) := M \otimes \mathbb{I}(i) = (X, p, m+i)$$

Now Tate object or motive is nothing but

$$\mathbb{I}(1) := \mathbb{L}^{\vee}$$

(4) in this language, one can write Chow ring of $M=(X,p,m)\in\mathcal{M}_{\mathrm{rat}}$ in terms of morphism

$$\mathrm{CH}^k(M) := \mathrm{Hom}(\mathbb{L}^k, M) = \mathrm{Hom}(\mathbb{I}, M(k))$$

(5) for any element $X \in (Var)$ of dimension d and a point $x \in X$, one define

$$h^0(X) := (X, [x \times X], 0) \cong \mathbb{I} = (pt, \Delta_{pt}, 0)$$

$$h^{2d}(X) := (X, [X \times x], 0) \cong \mathbb{L}^d = \mathbb{I}(-d)$$

due to semi-simple, one obtain a decomposition

$$h(X) = \mathbb{I} \oplus h'(X) \oplus \mathbb{L}^d$$

where

$$h'(X) = (X, \Delta_X - [x \times X] - [X \times x], 0)$$

2. Other options for motive

The construction of Chow motive $\mathcal{M}_{\rm rat}$ allows us to, somehow, replace the rational equivalence $\sim_{\rm rat}$ by any other algebraic equivalence \sim_{\square} on cycles, in which context one can get other kinds of motives \mathcal{M}_{\square} .

Recall: there are 5 known kinds of equivalence on cycles.

(1) rational equivalence $V \subset \mathbb{P}^1 \times X$

$$Z_{\rm rat}(X) := \mathbb{Q}\langle V_0 - V_{\infty} \rangle$$

(2) algebraic equivalence $V \subset C \times X$

$$Z_{\rm alg}(X) := \mathbb{Q}\langle V_a - V_b \rangle$$

(3) \otimes -equivalence

$$Z_{\otimes}(X) := \{\alpha : \exists N \in \mathbb{N} \text{ s.t. } \underbrace{\alpha \times \cdots \times \alpha}_{N \text{ copies}} \sim_{\mathrm{rat}} 0 \text{ on } X^N \}$$

(4) homological equivalence

$$Z_{\text{hom}}(X) := \{ \alpha : \text{cl}(\alpha) = 0 \}$$

(5) numerical equivalence

$$Z_{\text{num}}(X) := \{\alpha : \deg(\alpha \cdot \beta) = 0 \text{ for any } \beta \text{ of opposite dimension} \}$$

One already know the following inclusions

$$Z_{\mathrm{rat}}(X) \subsetneq Z_{\mathrm{alg}}(X) \subsetneq Z_{\otimes}(X) \subset Z_{\mathrm{hom}}(X) \subset Z_{\mathrm{num}}(X)$$

Remark 2.1. Some words about the inclusions.

- (1) the Vosoasky conjecture implies the third inclusion is indeed an equality.
- (2) the D(X) conjecture (Standard Conjecture) implies the last inclusion is also an equality.

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The following result implies that, surprisingly, if we want to obtain an abelian and semi-simple motive, it's necessary and sufficient to use numerical equivalence.

Theorem 2.2 (Jansen). \mathcal{M}_{\square} is abelian and semi-simple iff \sim_{\square} is numerical equivalence.

Although with this theorem, we still don't know whether \mathcal{M}_{num} has a realization, unless one prove the Standard Conjecture, since we only has realization up to Z_{hom} .

Hereafter we use

$$A^*(X)_{\sim} := Z^*(X)/Z^*(X)_{\sim}$$

Proof. \Rightarrow Consider the Tate object $\mathbb{I} = (pt, \Delta_{pt}, 0)$ in \mathcal{M}_{\square} , we know

$$\operatorname{End}(\mathbb{I})=\mathbb{Q}$$

- (1) by semi-simple, \mathbb{I} is a simple object.
- (2) by the abelian, one has

$$\forall f \neq 0 : \mathbb{I} \rightarrow h(X)(i) \Rightarrow f \text{ is monomorphism}$$

(3) by semi-simple, one has

$$\exists g: h(X)(i) \to \mathbb{I} \text{ such that } g \circ f = \mathrm{id}_{\mathbb{I}}$$

that is to say

$$(2) + (3)$$

$$\uparrow$$

$$\forall \alpha \neq 0 \in A^i(X)_{\sim}, \ \exists \beta \in A^{\dim(X)-i}(X)_{\sim} \text{ s.t. } \deg(\alpha \cdot \beta) = 1$$

which is exactly the context of numerical equivalence.

References

Institute of Mathematica, Academy of Mathematics and System Sciences, Chinese Academy of Science, Beijing 100190, China

E-mail address: zhangxucheng15@mails.ucas.cn