LECTURE ON INTERSECTION THEORY (XIV)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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From this lecture, we always work over $\mathbb{Q}$ instead of $\mathbb{Z}$, i.e., the Chow rings and some other related objects involved are defined over $\mathbb{Q}$. Let $\Bbbk$ be a field with char $\Bbbk = 0$. For simplicity, one may simply assume $\Bbbk = \mathbb{C}$.

Denoted by $\mathsf{Var}/\Bbbk$ the category of nonsingular projective variety over $\Bbbk$.

1. (Pure) Motive

1.1. Motivation: formation of cohomology.

For each object $X \in \mathsf{Var}/\Bbbk$, we already have the following cohomology theories

(1) (Singular Cohomology) $H^*(X, \mathbb{Q})$.
(2) (Algebraic de Rham Cohomology) $H^*(X, \Omega_X^\bullet)$.
(3) (ℓ-adic Cohomology) $H^*_{\acute{e}t}(X, \mathbb{Q}_\ell)$.
(4) $\cdots$ et al.

In general

Definition 1.1 (Weil cohomology theory)**.** A cohomology theory can be regarded as a functor

$$\mathscr{H} : (\mathsf{Var}/\Bbbk)^{\mathrm{op}} \to (\mathsf{Vect}/\Bbbk)$$

and a 'good' cohomology theory should satisfy some certain natural axioms:

(1) cup product $\cup$ on $\mathscr{H}^*(X)$ such that

$$\beta \cup \alpha = (-1)^{\deg(\alpha)\cdot\deg(\beta)}(\alpha \cup \beta)$$

(2) Poincáre duality.
(3) Künneth formula.
(4) cycle class map $\mathrm{cl} : \mathrm{CH}^*(X) \to \mathscr{H}^*(X)$.

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(5) Weak Lefschetz: for any nonsingular hypersurface Y and

$$i : Y \hookrightarrow X \text{ with } \dim(X) = d$$

the pull-back

$$i^* : \mathscr{H}^k(X) \to \mathscr{H}^k(Y) \text{ is } \begin{cases} \text{isomorphic} & \text{if } k < d-1 \\ \text{injective} & \text{if } k = d-1 \end{cases}$$

(6) Hard Lefschetz:

$$L^{n-k} : \mathscr{H}^k(X) \xrightarrow{\sim} \mathscr{H}^{2n-k}(X)$$

Remark 1.2. In Singular Cohomology case, the property (5) is called *Lefschetz hypersurface theorem*.

The goal of motive is to find a suitable category $\mathcal{M}$ such that all formations of cohomology theory factor through it. Such $\mathcal{M}$ is called a *motive*.

Ideal: the category $\mathcal{M}$ should be

(1) abelian and semi-simple.
(2) Tanrakan ($\Rightarrow$ motivate Galois group).

In picture, such $\mathcal{M}$ can be fit into the following diagram

$$\begin{tikzcd}
 & & (\mathsf{Vect}) \\
 & & (\mathsf{Vect}) \\
(\mathsf{Var})^{\mathrm{op}} \arrow[r, dashed] \arrow[uurr, "\mathscr{H}_1"] \arrow[urr, "\mathscr{H}_2"] \arrow[drr, "\mathscr{H}_{n-1}"'] \arrow[ddrr, "\mathscr{H}_n"'] & \mathcal{M} \arrow[r] \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots \\
 & & (\mathsf{Vect}) \\
 & & (\mathsf{Vect})
\end{tikzcd}$$

Remark 1.3. The only possible candidate of $\mathcal{M}$ is to use algebraic cycles.

1.2. **Construction of Chow motive.** Recall the notion *correspondence* first.

Definition 1.4. For any $X, Y \in (\mathsf{Var})$, we have

(1) Correspondence:

$$\mathrm{Corr}(X,Y) := \mathrm{CH}^*(X \times Y)$$

with grading

$$\mathrm{Corr}^r(X,Y) := \mathrm{CH}^{\dim(X)+r}(X \times Y)$$

and composition

$$\begin{aligned} \mathrm{Corr}^r(X,Y) \times \mathrm{Corr}^s(Y,Z) &\to \mathrm{Corr}^{r+s}(X,Z) \\ (\Gamma, \Gamma') &\mapsto (p^r_{X,Z})_*[(p^r_{X,Y})^*(\Gamma) \cdot (p^s_{Y,Z})^*(\Gamma')] \end{aligned}$$

where

$$
\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "p_{X,Y}"'] \arrow[d, "p_{X,Z}"] \arrow[dr, "p_{Y,Z}"] & \\
\Gamma \in X \times Y & X \times Z & Y \times Z \ni \Gamma'
\end{tikzcd}
$$

(2) Any element $\Gamma \in \mathrm{Corr}(X, Y)$ induces a morphism

$$
\begin{aligned}
\Gamma_* : \mathrm{CH}^*(X) &\to \mathrm{CH}^*(Y) \\
\alpha &\mapsto (p_Y)_*[(p_X)^*(\alpha) \cdot \Gamma]
\end{aligned}
$$

where

$$
\begin{tikzcd}
 & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X & & Y
\end{tikzcd}
$$

(3) Projector: an element $p \in \mathrm{Corr}^0(X, X)$ s.t. $p \circ p = p$ a *projector* of X.

Chow motive is one of the classical constructions of motive satisfying the pre-described properties. Its construction is divided into 3 steps.

Step-1: enlarge the class of morphism by letting

$$\mathrm{Hom}(Y, X) := \mathrm{Corr}(X, Y) = \mathrm{CH}^*(X \times Y)$$

such that one can add up morphisms (*hence can talk about abelian*).

Step-2: cut objects into pieces.

$$\forall p \in \mathrm{End}(X) = \mathrm{Hom}(X, X) \text{ with } p \circ p = p$$

one want to define $\ker(p)$ and $\mathrm{Im}(p)$ (*pseudo-abelian hull*).

Step-3: invert certain objects to take care of grading and polarization (*Tate twist*).

Formally say, we have

Definition 1.5 (Chow Motive)**.** The *Chow motive* $\mathcal{M}_{\mathrm{rat}}$ is the category consisting of the following data

(1) Object: each object $M \in \mathcal{M}_{\mathrm{rat}}$ is a triple

$$M = (X, p, m)$$

where $X \in (\mathsf{Var}/\Bbbk)$, p is a projector of X and $m \in \mathbb{Z}$.

(2) Morphism: for any two objects $M = (X, p, m)$ and $N = (Y, q, n)$

$$\mathrm{Hom}(M, N) := q \circ \mathrm{Corr}^{n-m}(X, Y) \circ p$$

Remark 1.6. For any objective $M = (X, p, m) \in \mathcal{M}_{\mathrm{rat}}$, one has

(1) M is said to be *effective* if $m = 0$.

(2) the identity morphism $\mathrm{id}_M \in \mathrm{Hom}(M, M) =: \mathrm{End}(M)$ is by definition

$$\mathrm{id}_M = p \circ \Delta_X \circ p = p \circ p = p$$

(3) the Chow ring/cohomology of M are defined[1] by

$$
\begin{aligned}
\mathrm{CH}^*(M) &:= p_*(\mathrm{CH}^*(X)) \\
H^*(M) &:= p_*(H^*(X))
\end{aligned}
$$

i.e., via those of X and with grading

$$\mathrm{CH}^k(M) := p_*(\mathrm{CH}^{k+m}(X))$$

[1] the well-definedness follows from the axioms of Weil cohomology theory.

$$H^k(M) := p_*(H^{k+2m}(X))$$

Remark 1.7. Attached to the Chow motive $\mathcal{M}$ are the following related concepts.

(1) any element $\Gamma \in \mathrm{Corr}(X, Y)$ induces a morphism
$$\Gamma_* : H^*(X) \to H^*(Y)$$
given by
$$\alpha \mapsto (p_Y)_*[(p_X)^*(\alpha) \cup \mathrm{cl}(\alpha)]$$
where
$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

(2) there is a natural functor
$$\begin{aligned} h : (\mathsf{Var})^{\mathrm{op}} &\longrightarrow \mathcal{M}_{\mathrm{rat}} \\ X &\mapsto (X, \Delta_X, 0) \\ (Y \xrightarrow{f} X) &\mapsto (h(X) \xrightarrow{h(f)} h(Y)) \end{aligned}$$
where
$$h(f) := \Delta_Y \circ [\Gamma_f^t] \circ \Delta_X = [\Gamma_f^t] \in \mathrm{Corr}^0(X, Y)$$
is the transportation of graph of f.

(3) call any functor a realization of a cohomology theory
$$\mathscr{H} : \mathcal{M}_{\mathrm{rat}} \to (\mathsf{Vect})$$

1.3. **Basic properties and examples.**

(1) the category $\mathcal{M}_{\mathrm{rat}}$ admits
$$\oplus, \otimes, (-)^\vee$$
In fact, given any two objects $M = (X, p, m), N = (Y, q, n) \in \mathcal{M}_{\mathrm{rat}}$, one can simply let
$$\begin{aligned} M \oplus N &:= (X \sqcup Y, p \sqcup q, -) \\ M \otimes N &:= (X \times Y, p \times q, m + n) \\ M^\vee &:= (X, p^t, \dim(X) - m) \end{aligned}$$

(2) for any element $f \in \mathrm{End}(M)$ satisfying $f \circ f = f$, one can define its *image* and *kernel* as follows[2]
$$\mathrm{Im}(f) := (X, f, m) \text{ and } \ker(f) := (X, p - f, m)$$
and then
$$M = \ker(f) \oplus \mathrm{Im}(f)$$
since $f \circ (p - f) = (p - f) \circ f = 0$.
$$(1) + (2) \Rightarrow \text{"}\textit{rigid, tensor and pseudo-abel category}\text{"}.$$

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[2]one can check that
$$(p - f) \circ (p - f) = p + f - f \circ p - p \circ f = p - f$$
since $f \in p \circ \mathrm{Corr}^0(X, X) \circ p$, then $p \circ f = f \circ p = f$.

(3) Tate object: set

$$\mathbb{I}(i) := (pt, \Delta_{pt}, i)$$

and its dual (sometimes called *Lefchtze motive*)

$$\mathbb{L} := \mathbb{I}(-1) = (pt, \Delta_{pt}, -1) \cong (\mathbb{P}^1, [\mathbb{P}^1 \times pt], 0)$$

then for any object $M = (X, p, m)$, its twist is therefore defined by

$$M(i) := M \otimes \mathbb{I}(i) = (X, p, m + i)$$

Now *Tate object* or *motive* is nothing but

$$\mathbb{I}(1) := \mathbb{L}^\vee$$

(4) in this language, one can write Chow ring of $M = (X, p, m) \in \mathcal{M}_{\mathrm{rat}}$ in terms of morphism

$$\mathrm{CH}^k(M) := \mathrm{Hom}(\mathbb{L}^k, M) = \mathrm{Hom}(\mathbb{I}, M(k))$$

(5) for any element $X \in (\mathsf{Var})$ of dimension d and a point $x \in X$, one define

$$h^0(X) := (X, [x \times X], 0) \cong \mathbb{I} = (pt, \Delta_{pt}, 0)$$
$$h^{2d}(X) := (X, [X \times x], 0) \cong \mathbb{L}^d = \mathbb{I}(-d)$$

due to semi-simple, one obtain a decomposition

$$h(X) = \mathbb{I} \oplus h'(X) \oplus \mathbb{L}^d$$

where

$$h'(X) = (X, \Delta_X - [x \times X] - [X \times x], 0)$$

2. Other options for motive

The construction of Chow motive $\mathcal{M}_{\mathrm{rat}}$ allows us to, somehow, replace the rational equivalence $\sim_{\mathrm{rat}}$ by any other algebraic equivalence $\sim_\Box$ on cycles, in which context one can get other kinds of motives $\mathcal{M}_\Box$.

Recall: there are 5 known kinds of equivalence on cycles.

(1) rational equivalence $V \subset \mathbb{P}^1 \times X$

$$Z_{\mathrm{rat}}(X) := \mathbb{Q}\langle V_0 - V_\infty \rangle$$

(2) algebraic equivalence $V \subset C \times X$

$$Z_{\mathrm{alg}}(X) := \mathbb{Q}\langle V_a - V_b \rangle$$

(3) $\otimes$-equivalence

$$Z_\otimes(X) := \{\alpha : \exists N \in \mathbb{N} \text{ s.t } \underbrace{\alpha \times \cdots \times \alpha}_{N \text{ copies}} \sim_{\mathrm{rat}} 0 \text{ on } X^N\}$$

(4) homological equivalence

$$Z_{\mathrm{hom}}(X) := \{\alpha : \mathrm{cl}(\alpha) = 0\}$$

(5) numerical equivalence

$$Z_{\mathrm{num}}(X) := \{\alpha : \deg(\alpha \cdot \beta) = 0 \text{ for any } \beta \text{ of opposite dimension}\}$$

One already know the following inclusions

$$Z_{\mathrm{rat}}(X) \subsetneq Z_{\mathrm{alg}}(X) \subsetneq Z_\otimes(X) \subset Z_{\mathrm{hom}}(X) \subset Z_{\mathrm{num}}(X)$$

Remark 2.1. Some words about the inclusions.

(1) the Vosoasky conjecture implies the third inclusion is indeed an equality.
(2) the $D(X)$ conjecture (Standard Conjecture) implies the last inclusion is also an equality.

The following result implies that, surprisingly, if we want to obtain an abelian and semi-simple motive, it's necessary and sufficient to use numerical equivalence.

Theorem 2.2 (Jansen). *$\mathcal{M}_\square$ is abelian and semi-simple iff $\sim_\square$ is numerical equivalence.*

Although with this theorem, we still don't know whether $\mathcal{M}_{\text{num}}$ has a realization, unless one prove the Standard Conjecture, since we only has realization up to Z_{hom}.

Hereafter we use

$$A^*(X)_\sim := Z^*(X)/Z^*(X)_\sim$$

Proof. $\Rightarrow$ Consider the Tate object $\mathbb{I} = (pt, \Delta_{pt}, 0)$ in $\mathcal{M}_\square$, we know

$$\text{End}(\mathbb{I}) = \mathbb{Q}$$

(1) by semi-simple, $\mathbb{I}$ is a simple object.
(2) by the abelian, one has

$$\forall f \neq 0 : \mathbb{I} \to h(X)(i) \Rightarrow f \text{ is monomorphism}$$

(3) by semi-simple, one has

$$\exists g : h(X)(i) \to \mathbb{I} \text{ such that } g \circ f = \text{id}_{\mathbb{I}}$$

that is to say

$$(2) + (3)$$

$$\Updownarrow$$

$$\forall \alpha \neq 0 \in A^i(X)_\sim,\ \exists \beta \in A^{\dim(X)-i}(X)_\sim \text{ s.t. } \deg(\alpha \cdot \beta) = 1$$

which is exactly the context of numerical equivalence. $\square$

References

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