LECTURE ON INTERSECTION THEORY (VIII)

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Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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1. Final remarks

1.1. **Refined Gysin homomorphism.** Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d and $f : Y' \to Y$ a morphism with Y' pure of dimension ℓ. Consider the following fiber square[1]

$$\begin{tikzcd} X' \arrow[r, hook, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

one can obtain/define

(1) The *intersection product* of X and Y' can be defined as

$$\begin{aligned} X \cdot Y' &:= 0^*[C_{X'}Y'] \in \mathrm{CH}_{\ell-d}(X') \\ &= i^!([Y']) \text{ in the language to come} \end{aligned}$$

where

$$\begin{tikzcd} C_{X'}Y' \arrow[rr, hook] \arrow[dr] & & g^*(N_XY) \arrow[dl] \\ & X' \arrow[ur, bend right, "0"'] & \end{tikzcd}$$

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[1]notice that i' is not necessarily regular.

(2) The *refined Gysin homomorphism* can be defined as
$$i^! : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k-d}(X')$$
by the formula
$$[V] \mapsto X \cdot V$$
and then extend it by linearity. A *prior* $X \cdot V \in \mathrm{CH}_{k-d}(X' \cap V)$, and we can view it an element in $\mathrm{CH}_{k-d}(X')$.

Clearly there are a lot of things to check before obtaining the well-definedness. But we have an alternative way to handle with this: a variant definition of $i^!$ can be given by the composition
$$i^! : \mathrm{CH}_k(Y') \xrightarrow{\sigma} \mathrm{CH}_k(C_{X'}Y') \to \mathrm{CH}_k(g^* N_X Y) \xrightarrow{0^*} \mathrm{CH}_{k-d}(X')$$
where
- σ is the specialization to normal cone.
- the second map is induced by the inclusion $C_{X'}Y' \subset g^*(N_X Y)$.
- 0^* is the Gysin pull-back of the 0-section of X' in $g^*(N_X Y)$.

One already know σ passes through rational equivalence, so that $i^!$ is well-defined.

Remark 1.1. If $Y' = Y, f = \mathrm{id}_Y$, then
$$i^! = i^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k-d}(X)$$
i.e., we are back to the usual Gysin homomorphism.

Example 1.2. Suppose $\mathcal{Y} \to T$ is a family of objects parametrized by T. For any point $t \in T$, consider the fiber diagram
$$\begin{tikzcd} Y_t \arrow[r, hook] \arrow[d] & \mathcal{Y} \arrow[d] \\ t \arrow[r, hook, "i_t"'] & T \end{tikzcd}$$
then one has
$$\alpha \in \mathrm{CH}_k(\mathcal{Y}) \Rightarrow \alpha_t := \alpha|_{Y_t} = i_t^!([\alpha])$$

1.2. Properties of refined Gysin.

Proposition 1.3. *The basic properties of refined Gysin homomorphism are summarized as follows.*

(1) *Consider a fiber diagram*
$$\begin{tikzcd} X'' \arrow[r, hook, "i''"] \arrow[d, "q"'] & Y'' \arrow[d, "p"] \\ X' \arrow[r, hook, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$
with $i : X \hookrightarrow Y$ a regular embedding of codimension d.

(a) (Push-forward) *If p is proper, then for any $\alpha \in \mathrm{CH}_k(Y'')$, we have*
$$i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathrm{CH}_{k-d}(X')$$

(b) (Pull-back) *If p is flat of relative dim n, then for any $\beta \in \mathrm{CH}_k(Y')$, we have*
$$i^! p^*(\beta) = q^* i^!(\beta) \in \mathrm{CH}_{k+n-d}(X'')$$

(c) (Excess intersection) *If i' is also regular of codimension d' (so necessarily one have $d' \leq d$), we have*

(c–1) *there is a canonical embedding $N_{X'}Y' \hookrightarrow g^*(N_X Y)$ and the resulting quotient bundle*

$$E := g^*(N_X Y)/N_{X'}Y'$$

is a vector bundle of rank $d - d'$ on X'. We call E the excess normal bundle of the lower fiber square.

(c–2) *for any $\alpha \in \mathrm{CH}_k(Y'')$, we have*

$$i^!(\alpha) = c_{d-d'}(E) \cap (i')^!(\alpha) \in \mathrm{CH}_{k-d}(X'')$$

As a special case, we have

(c–3) (Compatibility) *If $d' = d$, then for any $\alpha \in \mathrm{CH}_k(Y'')$, we have*

$$i^!(\alpha) = (i')^!(\alpha) \in \mathrm{CH}_{k-d}(X'')$$

(2) (Commutativity) *Consider the fiber square*

$$\begin{tikzcd}
X'' \arrow[r, hook, "i''"] \arrow[d, "q"'] & Y'' \arrow[r] \arrow[d, "p"] & S \arrow[d, hook, "j"] \\
X' \arrow[r, hook, "i'"] \arrow[d, "g"'] & Y' \arrow[r] \arrow[d, "f"] & T \\
X \arrow[r, hook, "i"] & Y &
\end{tikzcd}$$

with i (resp. j) a regular embedding of codimension d (resp. e), then for any $\alpha \in \mathrm{CH}_k(Y')$, we have

$$j^! i^!(\alpha) = i^! j^!(\alpha) \in \mathrm{CH}_{k-d-e}(X'')$$

(3) (Functoriality) *Consider the fiber square*

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d] & Y' \arrow[r, hook, "j'"] \arrow[d] & Z' \arrow[d] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & Z
\end{tikzcd}$$

with i (resp. j) a regular embedding of codimension d (resp. e), then

(a) *$j \circ i$ is a regular embedding of codimension $d + e$*

(b) *for any $\alpha \in \mathrm{CH}_k(Z')$, we have*

$$i^! j^!(\alpha) = (j \circ i)^!(\alpha) \in \mathrm{CH}_{k-d-e}(X').$$

Proof. All follows from definition. □

Remark 1.4. We explain the phenomenons appearing in Proposition 1.3 (c–2) and (c–3) in details. If

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\
X \arrow[r, hook, "i"] & Y
\end{tikzcd}$$

is a fiber square with i, i' regular embedding of codimension d. As an important consequence of Proposition 1.3 (c–3), we have for any $\alpha \in \mathrm{CH}_k(Y')$

$$i^!(\alpha) = (i')^*(\alpha) \in \mathrm{CH}_{k-d}(X')$$

If i' is not a regular embedding, or if i' is a regular embedding of codimension $\neq d$, then $i^!(\alpha)$ depends on i, not just i' (cf. Proposition 1.3 (c–2)). This means Gysin pull-back depends heavily on the base regular embedding i.

Example 1.5 (to Proposition 1.3 (c–2))**.** Consider the fiber square

$$\begin{tikzcd} X \arrow[r, hook, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "i"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

with $i : X \hookrightarrow Y$ regular embedding of codimension d, then Proposition 1.3 (c–2) reads as

$$i^* i_*(\alpha) = c_d(N_X Y) \cap \alpha$$

for any $\alpha \in \mathrm{CH}_k(X)$, which we have already seen.

Example 1.6 (to Proposition 1.3 (3))**.** Let

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ X \arrow[u, bend left=60, "s"] \arrow[u, bend right=60, "0"'] \end{tikzcd}$$

be a vector over X of rank d and s any section of π, then Proposition 1.3 (3) implies

(1) s is a regular embedding and
(2) s^* is independent of the choice of section s. Indeed,

$$0^*\pi^* = (\pi \circ 0)^* = \mathrm{id}$$
$$s^*\pi^* = (\pi \circ s)^* = \mathrm{id}$$

since π^* is isomorphism, then

$$s^* = 0^*$$

2. Intersection theory on non-singular variety

2.1. **Chow ring: Definition.** Let X be a non-singular variety of dimension n so the diagonal embedding

$$\Delta_X : X \hookrightarrow X \times X$$

is regular. For any $\alpha \in \mathrm{CH}_k(X), \beta \in \mathrm{CH}_\ell(X)$, we have defined their intersection product as

$$\alpha \cdot \beta := (\Delta_X)^*(\alpha \times \beta) \in \mathrm{CH}_{k+\ell-n}(X)$$

It turns out that this product makes $\mathrm{CH}^*(X)$ into a commutative, graded and associate ring with unit $[X]$.

Remark 2.1. One can have a 'refine' version of intersection. Consider the fiber square

$$\begin{tikzcd} {|\alpha| \cap |\beta|} \arrow[r, hook] \arrow[d] & {|\alpha| \times |\beta|} \arrow[d] \\ X \arrow[r, hook, "\Delta_X"] & X \times X \end{tikzcd}$$

then

$$\alpha \cdot \beta := (\Delta_X)^!(\alpha \times \beta) \in \mathrm{CH}_{k+\ell-n}(|\alpha| \cap |\beta|)$$

2.2. **Gysin pull-back.** If $f : X \to Y$ is a morphism with Y non-singular, then the graph morphism

$$\Gamma_f : X \to X \times Y$$

is a regular embedding of codimension $= \dim(Y)$. In this case we can define the *Gysin pull-back* as follows: for any $\alpha \in \mathrm{CH}^*(Y)$, its Gysin pull-back $f^*(\alpha)$ is defined via the formula

$$f^*(\alpha) \cap \beta := \Gamma_f^*(\beta \times \alpha)$$

for any $\beta \in \mathrm{CH}_*(X)$. It turns out that this product makes $\mathrm{CH}_*(X)$ into a graded $\mathrm{CH}^*(Y)$-module. In addition, if X is non-singular, setting

$$f^*(\alpha) := f^*(\alpha) \cap [X]$$

defines a homomorphism of graded rings:

$$f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$$

or precisely

$$f^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+\dim(X)-\dim(Y)}(X)$$

Remark 2.2. Similarly one can also have a 'refine' version. Consider

$$\begin{tikzcd} {|\beta| \cap f^{-1}(|\alpha|)} \arrow[r, hook] \arrow[d] & {|\beta| \times |\alpha|} \arrow[d] \\ X \arrow[r, hook, "\Gamma_f"] & X \times Y \end{tikzcd}$$

then

$$f^*(\alpha) \cap \beta := (\Gamma_f)^!(\alpha \times \beta) \in \mathrm{CH}_{k+\ell-n}(|\beta| \cap f^{-1}(|\alpha|))$$

The properties of Gysin pull-back are summarized as follows.

Proposition 2.3. *Let $f : X \to Y$ be a morphism between non-singular varieties.*

(1) $(\mathrm{CH}^*(X), \cdot)$ *is a commutative, graded and associate ring with unit* $[X]$.

(2) (Pull-back) *The Gysin-pull back*

$$f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$$

satisfying

$$f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta)$$

hence a ring homomorphism.

(3) (Projection formula) *If f is proper, then for any $\alpha \in \mathrm{CH}^*(Y), \beta \in \mathrm{CH}^*(X)$, we have*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

2.3. **Chow ring: Examples.** From now on, we are interested in $(\mathrm{CH}^*(X), \cdot)$ for

$$X = \text{smooth projective variety}/\mathbb{k} = \bar{\mathbb{k}}$$

or we can simply assume $\mathbb{k} = \mathbb{C}$.

(1) (Projective space) $\mathbb{P}^n$. Already know.

(2) (Projective bundle) $\mathbb{P}(E)$ where $E \xrightarrow{\pi} X$ is a vector bundle over X of rank $n+1$. Already know

$$\mathrm{CH}_k(\mathbb{P}(E)) = \bigoplus_{i=0}^{n} \mathrm{CH}_{k-n+i}(X)$$

hence

$$\mathrm{CH}^*(\mathbb{P}(E)) = \mathrm{CH}^*(X)[\xi]/\sim$$

where $\xi := c_1(\mathcal{O}_E)$ and $\sim$ is the nothing but Grothendieck relation

$$\xi^{n+1} + c_1(E)\xi^n + \cdots + c_{n+1}(E) = 0$$

(3) (Blowing-up) $\widetilde{Y} := \mathfrak{Bl}_X Y$ where $i : X \hookrightarrow Y$ is an embedding of codim d with both X, Y are non-singular. Consider the fiber square

$$\begin{tikzcd} \mathbb{P}(N_X Y) \arrow[r, equal] & E \arrow[r, hook, "j"] \arrow[d, "g"'] & \widetilde{Y} \arrow[d, "f"] \\ & X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

where E is the exceptional divisor of the blowing-up. In this case, we have two localization sequences

$$\mathrm{CH}_*(E) \to \mathrm{CH}_*(\widetilde{Y}) \to \mathrm{CH}_*(U) \to 0$$
$$\mathrm{CH}_*(E) \to \mathrm{CH}_*(Y) \to \mathrm{CH}_*(U) \to 0$$

where $U = \widetilde{Y} \setminus E \cong Y \setminus X$. Hence the Chow ring $\mathrm{CH}^*(\widetilde{Y})$ is given by

$$\begin{aligned} \mathrm{CH}_k(\widetilde{Y}) &= f^*(\mathrm{CH}_k(Y)) + j_*(\mathrm{CH}_k(E)) \\ &= \mathrm{CH}_k(Y) \oplus \bigoplus_{i=0}^{d-2} \mathrm{CH}_{k-(d-1)+i}(X) \end{aligned}$$

and the Chow ring structure is given by

$$\begin{cases} f^*(\alpha) \cdot f^*(\beta) = f^*(\alpha \cdot \beta) \\ j_*(\gamma) \cdot j_*(\delta) = j_*(-\xi \cdot \gamma \cdot \delta) \text{ where } \xi := c_1(\mathcal{O}_{N_X Y}(1)) \\ f^*(\alpha) \cdot j_*(\gamma) = j_*(g^* i^* \alpha \cdot \gamma) \end{cases}$$

notice that it's completely determined by that on $\mathrm{CH}^*(Y)$ and $\mathrm{CH}^*(E)$.

(4) (Curve) Let $X = C$ be a curve, then

$$\mathrm{CH}^*(C) = \mathrm{CH}^0(C) \oplus \mathrm{CH}^1(C)$$

where

$$\mathrm{CH}^0(C) = \mathbb{Z}[C]$$
$$\mathrm{CH}^1(C) = \mathrm{CH}_0(C) \xrightarrow{\deg} \mathbb{Z} \qquad (\dagger)$$

Recall in this case

$$\mathrm{CH}_0(C)_0 := \ker(\deg) = \mathrm{Jac}(C) : \text{the Jacobian of } C$$

In addition, if C has a $\Bbbk$-rational points, then $(\dagger)$ is surjective and we can therefore determine $\mathrm{CH}^1(C)$.

References

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