LECTURE ON INTERSECTION THEORY (VI)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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A brief review

(1) Given a scheme X, so far we have defined

$$\left\{\begin{matrix} \text{cycle } Z_k(X) \\ \text{rational equivalence } \sim_{\text{rat}} \end{matrix}\right\} \Rightarrow \mathrm{CH}_k(X)$$

and

- proper$_*$, flat*, localization sequence.
- persudo divisor D. and $c_1(L) \cap -$.

(2) Given a vector bundle over X of rank r

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ X \arrow[u, bend left, "0"] \end{tikzcd}$$

and

- Segre class $s_i(E)$, Chern class $c_i(E)$, Chern character $\mathrm{ch}(E)$, Todd class $\mathrm{td}(E)$: via splitting principal.
- Intersection with zero-sections:

$$\pi^* : \mathrm{CH}_{k-r}(X) \xrightarrow{\sim} \mathrm{CH}_k(E)$$
$$\Downarrow$$
$$0^* : \mathrm{CH}_k(E) \xrightarrow{\sim} \mathrm{CH}_{k-r}(X)$$

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1. Application: cubic surface in $\mathbb{P}^3$

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface over $\Bbbk = \overline{\Bbbk}$. There is a well-known result in classical algebraic geometry.

Theorem 1.1. *X contains exactly 27 lines.*

Remark 1.2. This theorem is easy to prove if one know the fact that

$$X = \mathfrak{Bl}_{(6 \text{ general pts})}\mathbb{P}^2$$

where 'general' means

(1) no 3 points are collinear.
(2) not all 6 points lying on a conic.

Then one can verify that all lines in X are given by

- exceptional curves: 6.
- strict transforms of line through 2 points:
$$\binom{6}{2} = 15$$
- strict transforms of conic through 5 points:
$$\binom{6}{5} = 6$$

Hence we obtain

$$27 = 6 + \binom{6}{2} + \binom{6}{5}.$$

In this section we give a 'proof'[1] of this theorem via intersection theory. Recall that the Grassmaninn $\mathrm{Gr}(r, n)$ is given by

$$\mathrm{Gr}(r, n) = r\text{-dim subspaces in an } n\text{-dim linear space}$$

and $\dim(\mathrm{Gr}(r, n)) = r(n - r)$.

Proof. Consider the set

$$\{\text{lines in } \mathbb{P}^3\} = \mathrm{Gr}(2, 4) =: G$$

then $\dim(G) = 4$. Now there is a tautological rank 2 subbundle of $\mathcal{O}^{\oplus 4}$ over G

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] & \text{universal rank 2 subbundle} \arrow[l] \\ G \end{tikzcd}$$

whose fiber over point $[L] \in G$ is the corresponding 2-dim subspace L. Taking the associated projective bundle we obtain

$$\begin{tikzcd} \mathcal{O}_{\mathcal{E}}(1) \arrow[d] \\ \mathbb{P}(\mathcal{E}) \arrow[d, "p"] & \text{universal line bundle} \arrow[l] \\ G \end{tikzcd}$$

[1] this proof only works for general smooth cubic surface.

Let $\mathcal{F} := \mathcal{R}^0 p_* \mathcal{O}_{\mathcal{E}}(3)$, then we get

$$\begin{tikzcd} \mathcal{F} \arrow[d, "q"] & \arrow[l] \text{rank 4 vector bundle} \\ G \end{tikzcd}$$

In details, the fiber over $[L] \in G$ is nothing but $H^0(L, \mathcal{O}_L(3))$ and

$$\dim(H^0(L, \mathcal{O}_L(3))) = 4$$

For a smooth cubic surface X in $\mathbb{P}^3$, we can write

$$X = X_f := \{f = 0\} \subset \mathbb{P}^3$$

for some degree equation f. Notice that f gives rise to a section of q by restriction, denoted by s_f , i.e.,

$$\begin{tikzcd} \mathcal{F} \arrow[d, "q"] \\ G \arrow[u, bend left, "s_f"] \end{tikzcd}$$

where $s_f : [L] \mapsto f|_L \in H^0(L, \mathcal{O}_L(3))$. Then

$$L \subset X_f \Leftrightarrow s_f([L]) = 0$$

therefore

$$\{\text{lines in } X_f\} = \{[L] \in G : s_f([L]) = 0\} =: Z_f$$

in particular

$$\#\{\text{lines in } X_f\} = \#Z_f$$

Lemma 1.3. *For general f, the locus Z_f is of dimension 0 and reduced.*

Corollary 1.4. *Z_f consists of finitely many single points and*

$$\#Z_f = \deg(Z_f)$$

Proof. Consider the section

$$s_f : G \to \mathcal{F}$$

since $\dim(G) = \mathrm{rank}(\mathcal{F}) = 4$, then for general f, the locus of s_f is of dimension 0 and reduced. As desired. $\square$

To count lines in X_f, we now want to compute the degree of Z_f. Since

$$[Z_f] = c_4(\mathcal{F})(\cap[G]) \in \mathrm{CH}^4(G) = \mathrm{CH}_0(G)$$

it suffices to compute degree of $c_4(\mathcal{F})$ or more general, $\mathrm{ch}(\mathcal{F})$. This can be done using Grothendieck's Riemann-Roch. Apply G-R-R to

$$\begin{tikzcd} \mathcal{O}(3) \arrow[d] \\ \mathbb{P}(\mathcal{E}) \arrow[d, "p"] \\ G \end{tikzcd}$$

yields that

$$\mathrm{ch}(p_! \mathcal{O}(3)) \cdot \mathrm{td}(T_G) = p_*(\mathrm{ch}(\mathcal{O}(3)) \cdot \mathrm{td}(T_{\mathbb{P}(\mathcal{E})}))$$

Since $\mathcal{F} = p_! \mathcal{O}(3)$, then

$$\mathrm{ch}(\mathcal{F}) = p_*(\mathrm{ch}(\mathcal{O}(3)) \cdot \mathrm{td}(T_p))$$

where T_p is the relatively tangent bundle[2]. So one consider $\mathrm{ch}(\mathcal{O}(3))$ and $\mathrm{td}(T_p)$ respectively. Notice that

(1) $\mathrm{ch}(\mathcal{O}(3))$: we can handle.
(2) $\mathrm{td}(T_p)$: using the Euler sequence

$$0 \to \mathcal{O} \to p^*\mathcal{E} \otimes \mathcal{O}(1) \to T_p \to 0$$

and the property of Todd class, we can also handle $\mathrm{td}(T_p)$.

From what have been discussed above, we can express $c_i(\mathcal{F})$ only in terms of $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$. To finish the proof, we need to know the degree of $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$: this is given by *Schubert calculus*.

Fact 1.5 (Schubert calculus)**.** We have

$$\deg \begin{cases} c_1^4(\mathcal{E}) = 2 \\ c_1^2(\mathcal{E})c_2(\mathcal{E}) = 1 \\ c_2^2(\mathcal{E}) = 1 \end{cases}$$

If one compute all these correctly, we can finally obtain the number 27. □

Example 1.6. Via the same method, one can also try

$$\#\{\text{lines in a general smooth quintic 3-fold}^3\}$$

Answer: 2875.

Remark 1.7. Why we want to count line in quintic 3-fold? Since the expected number of lines in quintic 3-fold is *finite*. To see this, one only need to notice that in the corresponding version of Lemma 1.3, dim = rank(= 6), so the expected dimension of locus is 0, hence finite.

2. Deformation to the normal cone

2.1. **Recall: purpose and idea.** Let X be a scheme and V, W two closed subschemes of X. Denote by $Z := W \cap V$ the scheme theoretical intersection of V and W. In general we want to define the intersection

$$V.W \in \mathrm{CH}_*(Z)$$

Firstly we assume one of the embeddings, say $V \hookrightarrow X$, is 'good' for the moment.

$$\begin{tikzcd} Z = W \cap V \arrow[r, hook] \arrow[d, hook] & W(\text{arbitrary}) \arrow[d, hook] \\ V(\text{good}) \arrow[r, hook] & X \end{tikzcd}$$

The idea of the approach is:

(1) 'replace' $V \hookrightarrow X$ by the normal bundle

$$\begin{tikzcd} N_V X \arrow[d] \\ V \arrow[u, bend left, "0"] \end{tikzcd}$$

(2) 'replace' W by the normal cone $C_Z W$.

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[2]there is an exact sequence defining the relatively tangent bundle

$$0 \to p^*T_{\mathbb{G}} \to T_{\mathbb{P}(\mathcal{E})} \to T_p \to 0$$

[3]i.e., $X \subset \mathbb{P}^4$ smooth hypersurface of degree 5.

then the diagram turns into

$$\begin{tikzcd} C_Z W \arrow[rr, hook] \arrow[dr] & & N_V X|_Z \arrow[dl] \\ & Z \arrow[ur, bend right, "0"'] & \end{tikzcd}$$

and we can define the intersection as

$$V.W := 0^*([C_Z W]) \in \mathrm{CH}_*(Z)$$

2.2. **Normal cone.** Let X be a scheme and

$$S^\bullet = \bigoplus_{n\geq 0} S^n$$

a graded sheaf of $\mathcal{O}_X$-algebra such that

(1) the natural morphism $\mathcal{O}_X \xrightarrow{\sim} S^0$ is an isomorphism.
(2) $S^\bullet$ is locally finitely generated by S^1.

Definition 2.1. Here are two kinds of cone associated to $S^\bullet$.

(1) The *cone* of $S^\bullet$ is defined to be

$$\begin{tikzcd} C := \mathrm{Spec}(S^\bullet) \arrow[d, "\pi"] \\ X \arrow[u, bend left, "0"] \end{tikzcd}$$

Locally, $C \subset X \times \mathbb{A}^{n+1}$.

(2) The *projective cone* of $S^\bullet$ is defined to be

$$\begin{tikzcd} \mathbb{P}(C) := \mathrm{Proj}(S^\bullet) \arrow[d, "p"] \\ X \arrow[u, bend left, "0"] \end{tikzcd}$$

Locally, $\mathbb{P}(C) \subset X \times \mathbb{P}^n$.

Definition 2.2 (Normal cone)**.** Let $X \hookrightarrow Y$ be a closed embedding with the corresponding ideal sheaf $\mathcal{I}$, then

$$S^\bullet := \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$$

is a graded sheaf of $\mathcal{O}_X$-algebra. The *normal cone* of X in Y is defined by

$$C_X Y = \mathrm{Spec}\left(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

and the associated *projective normal cone*

$$\mathbb{P}(C_X Y) = \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

Roughly say, the normal cone $C_X Y$ describes the behaviour of Y around X, something like the 'tubular neighbourhood' in A.T. See the example below.

Example 2.3. (1) Normal cone of a point = tangent cone.
(2) Let $Y = \{(x,y) : y^2 = x^3 + x^2\} \subset \mathbb{A}^2$ and $X = (0,0)$. Then

$$C_X Y = \{(u,v) : u^2 = v^2\}$$

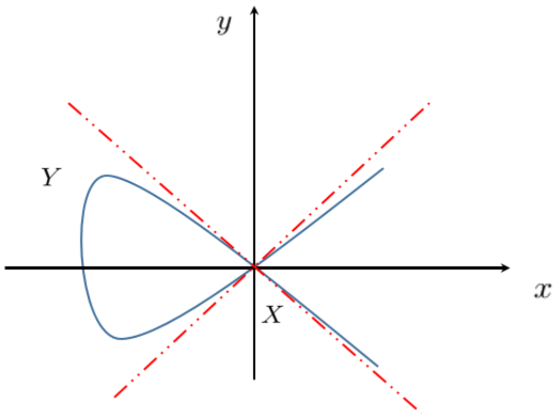

FIGURE 1. Normal cone at $X = (0,0)$

Remark 2.4. Compared to the tangent bundle $T_X Y$ at X, which is the whole space, we see that the tangent cone $C_X Y$ at X is more 'refiner'.

Proposition 2.5 (Key property)**.** *If Y is of pure dimension, then*

$$\dim(C_X Y) = \dim(Y)$$

i.e., it's independent of X.

2.3. **Regular embedding & normal bundle.**

Definition 2.6 (Regular embedding)**.** A closed embedding $X \hookrightarrow Y$ with corresponding ideal sheaf $\mathcal{I}$ is called *regular* of codimension d if $\mathcal{I}$ is locally generated by a regular sequence of length d.

Fact 2.7. In the case of regular embedding, we have

(1) the co-normal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank d.
(2) the normal bundle $N_X Y$ and the normal cone $C_X Y$ coincide. In fact, the normal bundle is given by

$$N_X Y = \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2))$$

and in the regular case

$$\mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2) \xrightarrow{\sim} \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$$

this implies that

$$C_X Y = N_X Y$$

Remark 2.8. If $X \hookrightarrow Y$ is regular, then we have

$$\underbrace{0 \to}_{\text{if regular}} \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_Y \to \Omega^1_X \to 0$$

by taking dual

$$0 \to T_X \to i^*T_Y \to N_X Y \to 0$$

Example 2.9 (Blowing-up)**.** Let $X \hookrightarrow Y$ be a closed embedding, then the blowing-up of Y along X is defined by

$$\pi : \tilde{Y} = \mathfrak{Bl}_X(Y) := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathcal{I}^n\right) \longrightarrow Y$$

(1) the exceptional divisor is given by
$$E := \operatorname{Proj}\left(\bigoplus_{n\geq 0}\mathcal{I}^n\otimes_{\mathcal{O}_Y}\mathcal{O}_X\right) = \operatorname{Proj}\left(\bigoplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1}\right) = \mathbb{P}(C_XY)$$
(2) the corresponding vector bundle
$$\begin{tikzcd} \mathcal{O}(1) \arrow[d] \\ \tilde{Y} \end{tikzcd}$$
with $\mathcal{O}(1) = \mathcal{I}_E$, the ideal sheaf of exceptional divisor.

(3)
$$N_E\tilde{Y} = \underbrace{\mathcal{O}_{\tilde{Y}}(E)}_{(\mathcal{I}_E)^\vee}|_E = \mathcal{O}(-1)|_E = \mathcal{O}_C(-1)$$

Appendix A. Topics for the Mid-term

- Moving Lemma (easy part). Reference is Appendix A in [EH11].
- Grothendieck's Riemann-Roch. Reference is Chapter 15 in [Ful98].
- 27 & 2875. Reference are everywhere.
- Virtual fundamental class. Let $\mathcal{E}$ be a vector bundle over X of rank r and two sections $s, 0$ of π.
$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] \\ X \arrow[u, bend left=60, "s"] \arrow[u, bend right=60, "0"'] \end{tikzcd}$$
with $\dim(X) = n$. Define
$$Z := \{x \in X : s(x) = 0\} = s \cap 0$$
We know that $0^*([s(X)]) \in \mathrm{CH}_{n-r}(X)$. But we want to define
$$0^!(s[X]) \in \mathrm{CH}_{n-r}(Z)$$
This is prototype of 'virtual fundamental class'. Reference is [Sie04].

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