LECTURE ON INTERSECTION THEORY (III)

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Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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Approach	Transform this problem to a problem of vector bundles
Today	Special case: intersection with divisors/line bundles
Later	From divisors case to general case

1. Preliminaries: Divisors

Let X be a variety of dimension n. In this section, we recall the definitions of two kinds of divisors on X, i.e., Weil divisor and Cartier divisor. For more information on this topic, the readers are invited to consult Chapter II.6 of [Har77].

1.1. Weil divisor.

Definition 1.1 (Weil divisor)**.** A *Weil divisor* on X is nothing but an element of $Z_{n-1}(X)$, i.e., an $(n-1)$-algebraic cycle on X.

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1.2. **Cartier divisor.** Let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$ and $\mathcal{K}_X^*$ (of multiplicative groups) the sheaf of invertible elements in $\mathcal{K}_X$. Similarly $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Definition 1.2 (Cartier divisor)**.** A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$.

Let $\mathrm{Div}(X)$ denote the set of Cartier divisors on X, then

$$\mathrm{Div}(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$$

Thinking of the properties of quotient sheaves, a Cartier divisor $D \in \mathrm{Div}(X)$ can be alternatively described by

$$\{(U_\alpha, f_\alpha)\}_\alpha$$

where

(1) $\{U_\alpha\}_\alpha$ is an open covering of X and
(2) each $f_\alpha \in \mathcal{K}(U_\alpha)^* = \mathcal{K}(X)^*$ and on $U_\alpha \cap U_\beta$, $f_\alpha/f_\beta \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$, i.e., f_α/f_β is regular and nowhere vanish on $U_\alpha \cap U_\beta$.

Remark 1.3. For a Cartier divisor D represented by $\{(U_\alpha, f_\alpha)\}_\alpha$, we define its associated Weil divisor $\mathcal{A}(D)$ to be

$$\mathcal{A}(D) = \sum_\alpha (\text{zeros of } f_\alpha - \text{ poles of } f_\alpha)$$

in which zeros and poles are counted with multiplicity.

Definition 1.4. For each $D \in \mathrm{Div}(X)$, we associate to it

(1) *Support* of D, denoted by $|D|$, is the union of all subvarieties appearing in $\mathcal{A}(D)$ with non-zero coefficients.
(2) *Line bundle* $\mathcal{O}_X(D)$: sheaf of sections, i.e., $\mathcal{O}_X$-subsheaf of $\mathcal{K}_X$ generated by f_α^{-1} on U_α.

Remark 1.5. Roughly say, we have

$$\text{Cartier divisor } D$$
$$\Updownarrow$$
$$\text{Line bundle } \mathcal{O}_X(D) + \text{regular meromorphic section } s_D, \text{ such that } s_D|_{X-|D|} = 1$$

Definition 1.6 (Principal divisor)**.** Each $f \in \mathcal{K}(X)^*$ defines a Cartier divisor

$$\mathrm{div}(f) := \{(U_\alpha, f|_{U_\alpha})\}_\alpha$$

which will be called the *principal divisor* on X.

Definition 1.7 (Linear equivalence)**.** For $D, D' \in \mathrm{Div}(X)$, we say $D \sim_{\mathrm{lin}} D'$ iff $D - D' = \mathrm{div}(f)$ for some $f \in \mathcal{K}(X)^*$.

Fact 1.8. $\underbrace{\mathrm{Div}(X)/\sim_{\mathrm{lin}}}_{\text{forget section } s_D} \cong \mathrm{Pic}(X)$, the Picard group of X.

1.3. **Weil divisor v.s. Cartier divisor.** In some scenes, Cartier divisor looks like cohomology and hence 'good', easy to pull-back; while Weil divisor looks like 'homology', easy to push-forward.

By assigning to each Cartier divisor D its associated Weil divisor $\mathcal{A}(D)$, we get a map

$$\mathcal{A} : \mathrm{Div}(X) \to Z_{n-1}(X)$$

By definition, we get

$$\mathcal{A} : \mathrm{Div}(X)/\sim_{\mathrm{lin}} \to Z_{n-1}(X)/\sim_{\mathrm{rat}} = \mathrm{CH}_{n-1}(X)$$

In general, $\mathcal{A}$ is neither injective nor surjective, see the following examples

Example 1.9. The first one shows that $\mathcal{A}$ is not necessarily injective while the second one shows that $\mathcal{A}$ is not necessarily surjective.

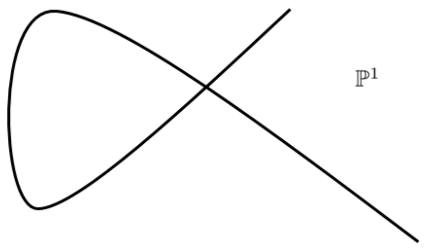

FIGURE 1. Nodal $\mathbb{P}^1/\mathbb{C}$: not injective

(1) In this case, $\mathrm{CH}_0(X) \cong \mathbb{Z}$ and $\ker(\mathcal{A}) = \mathbb{C}$.

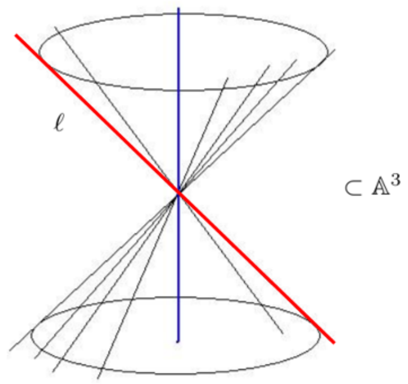

FIGURE 2. Cone in $\mathbb{A}^3$: not surjective

(2) In this case, $\mathrm{CH}_1(X) \cong \mathbb{Z}/2\mathbb{Z}$ with the generator $[\ell]$; while $\mathrm{Div}(X)/\sim_{\mathrm{lin}} = 0$.

Remark 1.10. All in all, we have the following results.

(1) $\mathcal{A}$ is injective if X normal.
(2) $\mathcal{A}$ is injective & surjective if X is locally factorial (e.g. X is non-singular).

2. Intersection with divisors

Let X be a variety of dimension n, and let $D \in Z_{n-1}(X), \alpha \in Z_k(X)$. In this section, we want to define the intersection product $D.\alpha$. By linearity, it suffices to define $D.\alpha$ for $\alpha = V$ with $V \subset X$ a closed subvariety of dimension k.

2.1. Approaches.

2.1.1. *Before approach.* Move D if possible, so as to intersect α properly, then we can get

$$(\dagger) \qquad D.\alpha \in \mathrm{CH}_{k-1}(X)$$

2.1.2. *Now approach.* Represent D by Cartier divisor if possible, then the problem reduces to define $D.V$ for $D \in \mathrm{Div}(X)$ a Cartier divisor and $V \subset X$ a closed subvariety of dimension k. There are two possible cases:

(1) Case 1: if $V \nsubseteq |D|$, i.e., D and V intersect 'properly'. In this case we can do the pull-back safely: notice that $D|_V$ is always a Cartier divisor on V, via the restriction
$$(U_\alpha, f_\alpha) \to (U_\alpha \cap V, f_\alpha|_{U_\alpha \cap V})$$
So we define
$$D.V := \mathcal{A}(D|_V) \in Z_{k-1}(|D| \cap V)$$
It's an algebraic cycle in $|D|$[1].

(2) Case 2: if $V \subseteq |D|$. In this case we can not pull-back[2], but in the line bundle level we can always write
$$\mathcal{O}_X(D)|_V = \mathcal{O}_V(C)$$
for some Cartier divisor C on V. Then
$$D.V := \mathcal{A}(C) \in \mathrm{CH}_{k-1}(V \cap |D|)$$

Both cases lead to an intersection map
$$D. : Z_k(X) \to \mathrm{CH}_{k-1}(|D|)$$
which will descend to (will be proved later)
$$D. : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(|D|)$$
So we are done.

2.2. **Better-formation: via pensedo-divisors.** The procedure above motives us to 'enlarge' the category of divisors, such that in both cases we can always do the pull-back freely. For this purpose, we introduce the concept of pesedo-divisor, it behaves like 'Cartier divisor' under favourable situation and like 'line bundle' under unfavourable situation (see Example 2.2).

Definition 2.1. Let X be a scheme. A *pesedo-divisor* on X is a triple (L, Z, s) such that

(1) L is a line bundle on X.
(2) $Z \subset X$ is a closed subset (something like 'support')[3].
(3) s is a nowhere vanishing section on $X - Z$.

Clearly we can always do pull-back for pesedo-divisors: for morphism $f : Y \to X$, we can just define
$$f^*(L, Z, s) := (f^*(L), f^{-1}(Z), f^*(s))$$

Example 2.2. Let X be a variety of dimension n.

(1) take $Z = X$, then a pesedo-divisor on X is nothing but a line bundle $L \in \mathrm{Pic}(X)$.
(2) for each $D \in \mathrm{Div}(X)$, we get a pesedo-divisor $(\mathcal{O}_X(D), |D|, s_D)$.

Definition 2.3. We say a pesedo-divisor (L, Z, s) is *represented* by a Cartier divisor $D \in \mathrm{Div}(X)$ if

(1) $|D| \subseteq Z$ and

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[1]Compared to (†), where we only obtain an algebraic cycle in X. In this scenes, we have defined a 'finer' intersection here.

[2]Indeed, the pull-back of D to V cannot be a Cartier divisor on V

[3]In fact, this is the main modification to Cartier divisors: we allow the support of a divisor to be the whole scheme.

(2) $\mathcal{O}_X(D) \cong L$ and this isomorphism maps $s_D \mapsto s$.

Lemma 2.4. *Let X be a variety. Then every pesendo-divisor (L, Z, s) can be represented by a Cartier divisor, unique if $Z \neq X$.*

Taking count of this lemma, if no abuse of notation, hereafter we always write D for (L, Z, s) if it's represented by D, which indicates that $\mathcal{O}_X(D)$ for L, $|D|$ for Z and s_D for s.

Definition 2.5. Let X be a variety of dimension n and D a pesedo-divisor. The associated Weil divisor class $\mathcal{A}_p(D)$ of D is defined to be

$$\mathcal{A}_p(D) := \mathcal{A}(D') \in \mathrm{CH}_{n-1}(|D|)$$

where D' is any Cartier divisor representing D.

With these definitions, we can define the intersection product in the language of pensedo-divisor.

Definition 2.6. Let X be a scheme, D a pensedo-divisor on X and $V \subset X$ closed subvariety of dimension k. Let $j : V \hookrightarrow X$ denote the inclusion, then

$$D.V := \mathcal{A}_p(j^{-1}(D)) \in \mathrm{CH}_{k-1}(|D| \cap V)$$

3. Two important applications

3.1. **First Chern class of line bundles.** The main idea is that: Chern class of a vector bundle looks like 'cohomology objects', so we can view them as operator on 'homology', e.g., Chow groups. In this way we can detect their information.

Let X be a scheme, L a line bundle on X and $V \subset X$ a closed subvariety of dimension k. So

$$L|_V = \mathcal{O}_V(C)$$

for some Cartier divisor C in V. Then we define

$$c_1(L) \cap V := [\mathcal{A}(C)] \in \mathrm{CH}_{k-1}(X)$$

and hence get a map

$$c_1(L) \cap : Z_k(X) \to \mathrm{CH}_{k-1}(X)$$

which will descend to

$$c_1(L) \cap : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(X)$$

If $L = \mathcal{O}_X(D)$ for some pensdo-divisor D, then

$$c_1(L) \cap (-) = D.(-)$$

3.2. **Gysin pull-back for divisors.** Let X be a scheme, D an effective Cartier divisors (subscheme, locally defined by 1 equation) on X and $i : D \hookrightarrow X$ the inclusion. The Gysin pull-back is defined by

$$\begin{aligned} i^* : \mathrm{Z}_k(X) &\to \mathrm{CH}_{k-1}(D) \\ \alpha &\mapsto D.\alpha \end{aligned}$$

which descends to

$$i^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(D)$$

4. Main problem/property

The definition of intersection with divisors given above leaves us the following main problem or property (if we can fix it): Let X be a variety of dimension n and $D, D' \in \mathrm{Div}(X)$ Cartier divisors on X, then we have two ways to define the intersection $D.D'$

$$D.D' = \underbrace{D}_{\text{viewed Cartier}} .\mathcal{A}(D') \in \mathrm{CH}_{n-2}(|D| \cap |D'|)$$

and

$$D.D' = \underbrace{D'}_{\text{viewed Cartier}} .\mathcal{A}(D) \in \mathrm{CH}_{n-2}(|D| \cap |D'|)$$

We are wondering if

$$\underbrace{D}_{\text{viewed Cartier}} .\mathcal{A}(D') \cong \underbrace{D'}_{\text{viewed Cartier}} .\mathcal{A}(D)$$

in $\mathrm{CH}_{n-2}(|D| \cap |D'|)$.

Remark 4.1. Notice that this does not hold in $Z_{n-2}(|D| \cap |D'|)$ even if it is well-defined, see the following examples.

Example 4.2. Consider $\mathfrak{Bl}_O\mathbb{A}^2$, the blow-up of $\mathbb{A}^2$ at O.

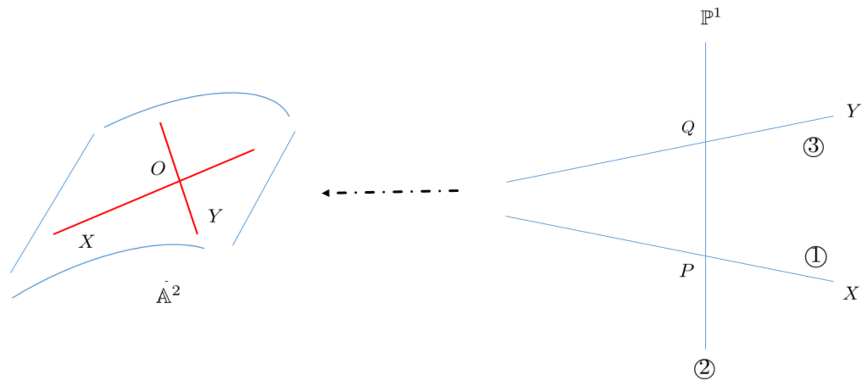

Figure 3. Blow-up of $\mathbb{A}^2$ at O

Let $D :=$①+② and $D' =$②+③, then

$$D.D' = D.② + D.③ = 0 + Q = Q$$

and similarly

$$D'.D = P$$

so they are not equal in $Z_0(\mathbb{P}^1)$. But $P \sim_{\mathrm{rat}} Q$ since they are connected by $\mathbb{P}^1$, i.e.,

$$[P] = [Q] \in \mathrm{CH}_0(\mathbb{P}^1).$$

References

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