LECTURE ON INTERSECTION THEORY (XV)

ZHANG

ABSTRACT. This is a private note taken from the course 'Intersection Theory'. The note taker is responsible for any inaccuracies.

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1. MANINN PRINCIPAL

1.1. **General phenomenon.** Consider two Chow motives in $\mathcal{M}_{\text{rat}}$, denoted by

$$M = (X, p, m), N = (Y, q, n)$$

and any morphism $f : M \to N$ of Chow motives induces

$$f_* : \mathrm{CH}^*(M) \to \mathrm{CH}^*(N)$$

Remark 1.1. One knows $f_* = 0 \not\Rightarrow f = 0$. For example, let C be a curve and take two points $a, b \in C$ such that $[a] \neq [b] \in \mathrm{CH}^1(C)$. Now

$$0 \neq f := [a \times C - b \times C] \in \mathrm{Corr}^0(C, C)$$

while $f_* = 0 : \mathrm{CH}^*(C) \to \mathrm{CH}^*(C)$.

But on the other hand, there is the so-called *Maninn principal.* Notice that any object $T \in \mathsf{Var}$ induces

$$f_T : h(T) \otimes M \to h(T) \otimes N$$

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and hence

$$\begin{tikzcd}
\mathrm{CH}^*(h(T)\otimes M) \arrow[r, "(f_T)_*"] \arrow[d, hook] & \mathrm{CH}^*(h(T)\otimes N) \arrow[d, hook] \\
\mathrm{CH}^*(T\times X) \arrow[r] \arrow[d, equal] & \mathrm{CH}^*(T\times Y) \arrow[d, equal] \\
\mathrm{Corr}(T,X) \arrow[r, "f\circ -", "(*)"'] & \mathrm{Corr}(T,Y)
\end{tikzcd}$$

since $f\in \mathrm{Hom}(M,N) = q\circ \mathrm{Corr}^{n-m}(X,Y)\circ p \subset \mathrm{Corr}(X,Y)$.

Theorem 1.2 (Mannin principal)**.** *Let $f,g: M\to N$ be two morphisms of Chow motives, then*

$$\begin{gathered}
f = g \\
\Updownarrow \\
(f_T)_* = (g_T)_* \textit{ for any } T\in \mathsf{Var} \\
\Updownarrow \\
(f_X)_* = (g_X)_*
\end{gathered}$$

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). For (3) $\Rightarrow$ (1), just letting $T=X$ in $(*)$. $\square$

1.2. **Applications.**

1.2.1. $\mathbb{P}^r$*-bundle.* Let E be a vector bundle of rank $r+1$ over X and hence

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[d] \\
X
\end{tikzcd}$$

Previously we already know

$$\begin{aligned}
\mathrm{CH}^*(\mathbb{P}(E)) &= \mathrm{CH}^*(X)[\xi]/\langle \xi^{r+1} = \cdots \rangle \\
&= \mathrm{CH}^*(X)\cdot\{1,\xi,\cdots,\xi^r\}
\end{aligned}$$

where $\xi := c_1(\mathcal{O}_E(1))$. By Maninn principal

$$h(\mathbb{P}(E)) = \bigoplus_{i=0}^{r} h(X)\otimes \mathbb{L}(i) = \bigoplus_{i=0}^{r} h(X)(-i)$$

1.2.2. *Blow-up $\mathfrak{Bl}_Y X$.* Let $X\in \mathsf{Var}$ and $Y\subset X$ a smooth projective variety of codim $r+1$, then

$$\mathrm{CH}^*(\mathfrak{Bl}_Y X) = [\mathrm{CH}^*(X)\oplus \mathrm{CH}^*(Y)]\cdot\{\xi,\cdots,\xi^r\}$$

where $\xi := c_1(\mathcal{O}_{N_Y X}(1))$. By Maninn principal

$$h(\mathfrak{Bl}_Y X) = h(X)\oplus \bigoplus_{i=1}^{r} h(Y)\otimes \mathbb{L}(i) = h(X)\oplus \bigoplus_{i=1}^{r} h(Y)(-i)$$

2. Decomposition of $h(\mathsf{Var})$

Let $X \in \mathsf{Var}$ of dimension d and $x \in X$. Define

$$\pi^0 := [x \times X] \in \mathrm{Corr}^0(X, X)$$
$$\pi^{2d} := [X \times x] \in \mathrm{Corr}^0(X, X)$$
$$\Downarrow$$
$$h^0(X) := (X, \pi^0, 0) = \mathbb{I}$$
$$h^{2d}(X) := (X, \pi^{2d}, 0) = \mathbb{L}^d \cong \mathbb{I}(-d)$$

and this leads to a decomposition

$$h(X) = h^0(X) \oplus h'(X) \oplus h^{2d}(X)$$

Here $h'(X) := (X, \Delta_X - \pi^0 - \pi^{2d}, 0)$ is the remaining part. In summary, we have

Table 1. Known correspondence so far

	$h^0(X)$	$h^{2d}(X)$
H	$H^0(X)$	$H^{2d}(X)$
CH	$\mathbb{Q} \cdot [X]$	$\mathbb{Q} \cdot [x]$

2.1. **Curve cases.** If $X = C$ is a curve, then

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C)$$

and we can completely determine the correspondence.

Table 2. Correspondence in curve cases

	$h^0(C)$	$h^1(C)$	$h^2(C)$
H	$H^0(C)$	$H^1(C)$	$H^2(C)$
CH	$\mathbb{Q} \cdot [C]$	$\mathrm{CH}^1(C)_{\deg 0} = \mathrm{Jac}(C)$	$\mathbb{Q} \cdot [a]$

Finally we recall a result of A. Weil.

Theorem 2.1 (A. Weil). *Let C, C' be two curves, then*

$$\mathrm{Hom}_{\mathcal{M}}(h^1(C), h^1(C')) = \mathrm{Hom}_{\mathrm{Jac}}(\mathrm{Jac}(C), \mathrm{Jac}(C'))_{\mathbb{Q}}$$

and moreover

$$\mathrm{CH}^1(C \times C') = \mathrm{CH}^1(C) \oplus \mathrm{CH}^1(C') \oplus \mathrm{Hom}_{\mathrm{Jac}}(\mathrm{Jac}(C), \mathrm{Jac}(C'))_{\mathbb{Q}}$$

2.2. **Abelian variety cases.**

Theorem 2.2 (Deninger-Morie). *Let A be an d-dim'l abelian variety, then*

(1) *there is a decomposition*

$$\Delta_A = \sum_{i=0}^{2d} \pi^i \in \mathrm{Corr}^0(A, A)$$

such that

  (a) $\pi^i \circ \pi^j = \delta_{ij} \circ \pi^i$.

  (b) $(\mathrm{id} \times [N])^* \pi^i = N^i \cdot \pi^i$ *for any* $N \in \mathbb{Z}$.

(2) $h^i(A) = (A, \pi^i, 0)$ *and* $h(A) = \bigoplus_{i=0}^{2d} h^i(A)$.

(3) $h^i(A) = \mathrm{Sym}^i(h^1(A))$.[1]

[1]Algebra: $a \cdot b = b \cdot a$; Topology: $a \cup b = (-1)^{\deg(a)\deg(b)} b \cup a$.

(4) *$H^*(h^i(A)) = H^i(A)$ and*

$$\begin{array}{c}\mathrm{CH}^k_{(i)}(A) = \mathrm{CH}^k(h^{2k-i}(A)) \\ \| \\ \{\alpha \in \mathrm{CH}^k(A) : [N]^*\alpha = N^{2k-i}\alpha \text{ for any } N \in \mathbb{Z}\}\end{array}$$

(5) *In particular, if $A = C$ is curve, then $h^1(C) = h^1(J(C))$.*

Proof. Use Fourier transformation. In particular, (3) holds since the cohomology of abelian variety is determined by its H^1:

$$H^i = \wedge^i H^1$$

□

2.3. **Surface cases.** Let $X \in \mathsf{Var}$ and $\dim X = d$.

Our Goal: define projectors

$$\pi^1 = (X, \pi^1, 0) \in \mathrm{Corr}^0(X, X) \text{ (Picard motive)}$$
$$\pi^{2d-1} = (X, \pi^{2d-1}, 0) \in \mathrm{Corr}^0(X, X) \text{ (Albnase motive)}$$

Table 3. Known correspondence so far

	$h^1(X)$	$h^{2d-1}(X)$
H	$H^1(X)$	$H^{2d-1}(X)$
CH	$\mathrm{CH}^1(X)_{\mathrm{hom}} \cong \mathrm{Pic}^0(X)_{\mathbb{Q}}$	$\mathrm{CH}^{2d}(X)_{\mathrm{hom}}/\ker(\mathrm{alb}) \cong \mathrm{Alb}(X)_{\mathbb{Q}}$

Construction: take $j : C := X \cap H_1 \cap \cdots \cap H_{d-1} \to X$ a smooth curve.

Theorem 2.3 (Weil)**.** *The morphism*

$$\phi : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(C) = \mathrm{Jac}(C) = \mathrm{Alb}(C) \to \mathrm{Alb}(X)$$

is an isogeny (i.e., surjective and has finite kernel) between abelian varieties. If C is ample, there exists $\psi : \mathrm{Alb}(X) \to \mathrm{Pic}^0(X)$ such that

$$\psi \circ \phi = [N]$$

for some integer $N \in \mathbb{Z}$.

From $\psi \dashrightarrow \tilde{\psi} \in \mathrm{CH}^1(X \times X)$ and $\tilde{\psi}^t = \psi$.

Definition 2.4. Define two items as

$$\pi^1 := \frac{1}{N}\tilde{\psi} \circ [\Gamma_j] \circ [\Gamma_j^t] \in \mathrm{Corr}^0(X, X)$$

and

$$\pi^{2d-1} := (\pi^1)^t = \frac{1}{N}[\Gamma_j] \circ [\Gamma_j^t] \circ \tilde{\psi} \in \mathrm{Corr}^0(X, X)$$

Theorem 2.5 (Murre)**.** *π^1, π^{2d-1} are projectors and*

$$\pi^1 \circ \pi^{2d-1} = \pi^{2d-1} \circ \pi^1 = 0$$

All these constructions lead to a refine decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \underbrace{h''(X)}_{\text{hard}} \oplus h^{2d-1}(X) \oplus h^{2d}(X)$$

If $X = S$ is a surface, then

$$h(S) = h^0(S) \oplus h^1(S) \oplus h^2(S) \oplus h^3(S) \oplus h^4(S)$$

Here $h^2(S) = (X, \pi^2, 0)$. Furthermore

$$h^2(S) = h^2_{\mathrm{alg}}(S) \oplus h^2_{\mathrm{tr}}(S) \text{ and } \pi^2 = \pi^2_{\mathrm{alg}} + \pi^2_{\mathrm{tr}}$$

Remark 2.6. If we choose an orthonormal basis $\{e_j\}$ of $\mathrm{NS}(S)$, then

$$\pi^2_{\mathrm{alg}} := \sum \frac{1}{\deg(e_i^2)}([e_i] \times [e_i])$$

Table 4. Correspondence in surface cases

	$h^0(S)$	$h^1(S)$	$h^2_{\mathrm{alg}}(S)$	$h^2_{\mathrm{tr}}(S)$	$h^3(S)$	$h^4(S)$
H	$H^0(S)$	$H^1(S)$	$H^2_{alg}(S)$	$H^2_{tr}(S)$	$H^3(S)$	$H^4(S)$
CH^0	$\mathbb{Q} \cdot [S]$					
CH^1		$\mathrm{CH}^1(S)_{\mathrm{hom}} =$ $\mathrm{Pic}^0(S)$	$\mathrm{CH}^1(S)/\mathrm{CH}^1(S)_{\mathrm{hom}} =$ $\mathrm{NS}(S)$			
CH^2				$\ker(\mathrm{alb})_{\mathbb{Q}}$	$\mathrm{CH}^2(X)_{\mathrm{hom}}/\ker(\mathrm{alb}) =$ $\mathrm{Alb}(S)_{\mathbb{Q}}$	$\mathbb{Q} \cdot [a]$

Conjecture 2.7 (Bloch)**.**

$$\begin{gathered} H^{2,0}(S) = 0 \Leftrightarrow H^2_{\mathrm{tr}}(S) = 0 \\ \Downarrow \\ \mathrm{CH}^*(h^2_{\mathrm{tr}}(S)) = 0 \Leftrightarrow \ker(\mathrm{alb}) = 0 \Leftrightarrow h^2_{\mathrm{tr}}(S) = 0 \end{gathered}$$

References

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