LECTURE ON INTERSECTION THEORY (XVI)

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ABSTRACT. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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1. TWO CONJECTURES AND PROGRESS

We report two important conjectures and some known results.

Conjecture 1.1 (Bloch-Beilnson). *Let $X \in \mathsf{Var}$, then there exists a functorial filtration on $\mathrm{CH}^k(X)$, denoted by*

$$\mathrm{CH}^k(X) = F^0\mathrm{CH}^k(X) \supset F^1\mathrm{CH}^k(X) \supset \cdots$$

such that

(1) (Setup)
$$F^1\mathrm{CH}^k(X) = \mathrm{CH}^k(X)_{\mathrm{hom}}$$

(2) (Bounded)
$$F^{k+1}\mathrm{CH}^k(X) = 0$$

(3) (Cohomology controls Chow group)
$$\mathrm{Gr}^i_F\mathrm{CH}^k(X) := F^i\mathrm{CH}^k(X)/F^{i+1}\mathrm{CH}^k(X)$$
is controlled by $H^{2k-i}(X)$.

Conjecture 1.2 (Murre). *Let $X \in \mathsf{Var}$ of dim $2d$ and*

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X)$$

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(1) (Chow-Kunneth decomposition)[1]

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X) \text{ with } h^i(X) = (X, \pi^i, 0)$$

(2)

$$\pi^i_* = 0 \text{ on } \mathrm{CH}^k(X) \text{ for } i \in \{0, \dots, k-1\} \cup \{2k+1, \dots, 2d\}$$

or equivalently

$$\pi^i_* \neq 0 \text{ on } \mathrm{CH}^k(X) \text{ can only for } i \in \{k, \dots, 2k\}$$

(3) (Independent of choices)

$$F^i\mathrm{CH}^k(X) = \ker \pi^{2k}_* \cap \ker \pi^{2k-1}_* \cap \cdots \cap \ker \pi^{2k-i+1}_*$$

(4) $F^1\mathrm{CH}^k(X) = \mathrm{CH}^k(X)_{\hom}$.

The two conjectures are equivalent due to the following theorem.

Theorem 1.3 (Jannsen)**.** *Upon standard conjecture, we have*

Bloch-Beilnson conjecture 1.1

$$\Updownarrow$$

Murre conjecture 1.2

2. Finiteness result

Theorem 2.1 (Kimure finiteness)**.** *For any motive $M = (X, p, m)$, $H^*(M)$ is a finite-dim'l vector space.*

As a consequence, $\wedge^N H^*(M) = 0$ for $N \gg 0$.
Goal: work out a motive version.

Definition 2.2. For any motive $M = (X, p, m)$ and integer $n \in \mathbb{N}$, we define

$$M^{\otimes n} := (X^n, p^{\times n}, m \times n) \in \mathrm{Corr}^0(X^n, X^n)$$

$$S^n(M) := \left(X^n, \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(p^{\times n}), m \times n \right)$$

$$\wedge^n(M) := \left(X^n, \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma(p^{\times n}), m \times n \right)$$

and M is said to be

(1) *even* (resp., *odd*) if there is $N \in \mathbb{N}$ such that

$$\wedge^N M = 0 \quad (\text{resp., } S^N(M) = 0)$$

(2) *finite-dim'l* if there is a decomposition

$$M = M^{\text{even}} \oplus M^{\text{odd}}$$

such that M^{even} is even and M^{odd} odd. Let $\mathcal{M}_{\text{rat}}^{<\infty} \subset \mathcal{M}_{\text{rat}}$ be the category consisting of finite-dim'l motives.

Example 2.3. For any curve C, we have $h(C) \in \mathcal{M}_{\text{rat}}^{<\infty}$. In fact

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C)$$

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[1] Assume standard conjecture $C(X)$: the projectors

$$\pi^i \in H^{2d-i}(X) \otimes H^i(X) \subset H^{2d}(X \times X)$$

lifts to $\pi^i \in \mathrm{CH}^d(X \times X)$ such that $\pi^i \circ \pi^j = \delta_{ij} \circ \pi^i$.

(1) even part
$$h^0(C) \oplus h^2(C) =: \mathbb{I} \oplus \mathbb{L}$$
(2) odd part
$$h^1(C) = h^1(J(C)) = \bigoplus_{i=0}^{2g} h^i(J(C))$$
where $h^i(J(C)) = S^i(h^1(J(C)))$ and $g = g(C)$. Hence $S^{2g+1}(h^1(J(C))) = 0$.

Conjecture 2.4 (Kimure)**.** $\mathcal{M}_{\mathrm{rat}}^{<\infty} = \mathcal{M}_{\mathrm{rat}}$

Fact 2.5. $\mathcal{M}_{\mathrm{rat}}^{<\infty} \subset \mathcal{M}_{\mathrm{rat}}$ is a sub-tensor category.

Let $\mathcal{M}_{\mathrm{rat}}^{ab} \subset \mathcal{M}_{\mathrm{rat}}^{<\infty}$ be the subcategory generated by the motive of curves.

Fact 2.6. $\mathcal{M}_{\mathrm{rat}}^{ab}$ is very small in $\mathcal{M}_{\mathrm{rat}}$.

Remark 2.7. Deligne: general surface in $\mathbb{P}^3$ of degree ≥ 5 (via Hodge structure).
(1) $\deg = 2, 3$: O.K.;
(2) $\deg = 4$: unknown (K3 surface O.K. if some conjecture holds).

Fact 2.8. Not a single example known in $\mathcal{M}_{\mathrm{rat}}^{<\infty}$ outside $\mathcal{M}_{\mathrm{rat}}^{ab}$.

2.1. **Important consequence.**

Theorem 2.9 (Kimure)**.** *Let* $M = (X, p, m) \in \mathcal{M}_{\mathrm{rat}}^{<\infty}$ *be a motive and* $f \in \mathrm{End}(M) \subset \mathrm{Corr}^0(X, X)$*, then*
$$f \sim_{\mathrm{num}} 0 \Rightarrow \underbrace{f \circ \cdots \circ f}_{N} = 0 \text{ for } N \gg 0$$

Recall we have the following conjecture

Conjecture 2.10 (Voevodsky)**.**
$$f \sim_{\mathrm{num}} 0 \Rightarrow \underbrace{f \times \cdots \times f}_{N} = 0$$

and we know
$$\underbrace{f \times \cdots \times f}_{N} = 0 \Rightarrow \underbrace{f \circ \cdots \circ f}_{N} = 0$$

Proof. Let $M = (X, p, m)$ be an even motive and
$$\mathrm{pr}_i : X^N \times X^N \to X \times X$$
$$((x_1, \ldots, x_N), (y_1, \ldots, y_N)) \mapsto (x_i, y_i)$$
Now the formula $\wedge^N M = 0$ implies that
$$\mathrm{pr}_1^* \left[\frac{1}{N!} \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma)\sigma(p^{\times N}) \cdot \mathrm{pr}_2^*(f) \cdots \mathrm{pr}_N^*(f) \right] = 0$$
on $X^N \times X^N$, since
$$\frac{1}{N!} \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma)\sigma(p^{\times N}) = 0$$
On the other hand, we express
$$\mathrm{LHS} = \underbrace{\text{constant}}_{\neq 0} \cdot \underbrace{(f \circ \cdots \circ f)}_{N} + \underbrace{\text{terms containing } \deg(p \cdot f)}_{=0}$$
as
(1) the first term corresponding to $\sigma = (12 \cdots N)$.

(2) the second term corresponding to $\sigma \neq (12\cdots N)$ and it's 0 since $f \sim_{\text{num}} 0$.

□

Corollary 2.11. *If $M \in \mathcal{M}_{\text{rat}}^{<\infty}$, then*

$$H^*(M) = 0 \Rightarrow M = 0$$

Proof. One writes $M = (X, p, m)$, then

$$\begin{aligned} H^*(M) = 0 &\Leftrightarrow p \in \mathrm{CH}^*(X \times X)_{\text{hom}} \\ &\Rightarrow p \sim_{\text{num}} 0 \\ &\Rightarrow \underbrace{p \circ \cdots \circ p}_{N} = 0 \text{ for some } N \text{ (by Theorem 2.9)} \\ &\Rightarrow p = 0 \text{ (since } p \text{ is a projector)} \end{aligned}$$

□

Remark 2.12. Recall: if $X = S$ is a surface, then

$$h(S) = \underbrace{h^0(S)}_{\text{even}} \oplus \underbrace{h^1(S)}_{\text{odd}} \oplus \underbrace{h^2_{\text{alg}}(S)}_{\mathbb{L}^{\dim \mathrm{NS}(S)}\text{: even}} \oplus \underbrace{h^2_{\text{tr}}(S)}_{?} \oplus \underbrace{h^3(S)}_{\text{odd}} \oplus \underbrace{h^4(S)}_{\mathbb{L}\text{: even}}$$

At least we know

$$H^*(h^2_{\text{tr}}(S)) = H^2_{\text{tr}}(S) \text{ and } \mathrm{CH}^*(h^2_{\text{tr}}(S)) = \ker(\text{alb})$$

then

$$H^{2,0}(S) = 0 \Leftrightarrow H^2_{\text{tr}}(S) = 0 \xrightarrow{(*)} h^2_{\text{tr}}(S) = 0 \Rightarrow \ker(\text{alb}) = 0$$

where $(*)$ holds by Kimure theorem if S is finite (for example, covered by product of curves).

2.2. Conservetity conjecture.

Conjecture 2.13 (Torelli-type)**.** *Let $M, N \in \mathcal{M}_{\text{rat}}$ be two motives and $f : M \to N$. If $f_* : H^*(M) \xrightarrow{\sim} H^*(N)$, then $f : M \xrightarrow{\sim} N$.*

3. Naïve mixed motive

Definition 3.1. Define

$$K_0(\mathsf{All\ Var}) := \mathbb{Z} \cdot \{\text{varieties}\} / \langle [X] = [X \setminus Z] + [Z] \rangle$$

for any $Z \subset X$ closed subvariety.

There is a well-defined map

$$\begin{aligned} \chi : K_0(\mathsf{All\ Var}) &\to \mathbb{Z} \\ [X] &\mapsto \chi(X) \end{aligned}$$

Theorem 3.2 (Looijeage-Bitternar, holds for char $= 0$ since resolution of singularities)**.** *There is a surjective map*

$$K_0(\mathsf{SmProVar}) \twoheadrightarrow K_0(\mathsf{All\ Var})$$

and the kernel generated by

$$[\mathfrak{Bl}_Y X] - [X] - [E] + [Y]$$

This implies a well-defined map

$$\begin{aligned} \underbrace{K_0(\mathsf{All\ Var})}_{\text{bad}} &\to \underbrace{K_0(\mathcal{M}_{\text{rat}})}_{\text{good}} \\ \underbrace{[X]}_{\text{sm proj}} &\mapsto [h(X)] \end{aligned}$$

Remark 3.3. Let $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathsf{All\ Var})$, does there exist $0 \neq \alpha \in K_0(\mathsf{All\ Var})$ such that

$$\mathbb{L} \cdot \alpha = 0?$$

The answer is: yes, so the following map isn't injective

$$K_0(\mathsf{All\ Var}) \to K_0(\mathsf{All\ Var})[\mathbb{L}^{-1}]$$

References

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