LECTURE ON INTERSECTION THEORY (XIII)

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Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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In this lecture, we will mainly talk about various equivalence of cycles and discuss their relations. Finally, we will state the celebrated Standard Conjecture, which gives a more precise relations between them.

1. Adequate equivalences of cycles

Let X be a nonsingular projective variety over $\mathbb{C}$ of dimension n.

1.1. Definition.

Definition 1.1. An equivalence relation $\sim$ on $Z^*(X)$ should satisfy the following natural requirements:

1. compatible with grading and addition.
2. if $\alpha \sim 0$ on X, then $\alpha \times Y \sim 0$ on $X \times Y$ for any nonsingular projective variety Y over $\mathbb{C}$.
3. if $\alpha \sim 0$ on $X \times Y$ for any nonsingular projective variety Y over $\mathbb{C}$, then $(p_X)_*(\alpha) \sim 0$ on Y.
4. if $\alpha \sim 0$ on X and α, β properly intersects, then $\alpha \cdot \beta \sim 0$ on X.
5. (moving lemma) for any α, β on X, there exists $\alpha' \sim \alpha$ such that α', β properly intersects.

1.2. **Some examples of $\sim$.** Here are some examples of equivalence on the Chow ring $Z^*(X)$.

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1.2.1. *Rational equivalence.* For any $\alpha \in Z^*(X)$, say $\alpha \sim_{\text{rat}} 0$ if there exists

$$\beta \in Z^*(X \times \mathbb{P}^1) \text{ dominating } \mathbb{P}^1$$

such that

$$\alpha = \beta_0 - \beta_\infty$$

1.2.2. *Algebraic equivalence.* For any $\alpha \in Z^*(X)$, say $\alpha \sim_{\text{alg}} 0$ if there exists

$$\begin{gathered}\text{an nonsingular curve } C\\ \beta \in Z^*(X \times C) \text{ dominating } C\\ a, b \in C\end{gathered}$$

such that

$$\alpha = \beta_a - \beta_b$$

Remark 1.2. The only non-trivial part for well-definedness is to check the compatibility with addition. Indeed, if $\alpha_i \sim_{\text{alg}} 0$ on X is realized by a nonsingular curve C_i for $i = 1, 2$. If we want $\alpha_1 + \alpha_2 \sim_{\text{alg}} 0$ on X, one need to find a nonsingular curve C realizing it. By the knowledge of geometry of curves, we can actually find such one curve on $C_1 \times C_2$.

1.2.3. *Homological equivalence.* Recall the cycle class map

$$\text{cl} : Z^k(X) \to H^{2k}(X, \mathbb{Z})$$

For any $\alpha \in Z^*(X)$, say $\alpha \sim_{\text{homo}} 0$ if $\text{cl}(\alpha) = 0$.

1.2.4. *Numerical equivalence.* For any $\alpha \in Z^*(X)$, say $\alpha \sim_{\text{num}} 0$ if

$$\deg([\alpha] \cdot [\beta]) = 0$$

for any $\beta \in Z^{n-*}(X)$.

Remark 1.3. Notice that we cannot see torsion in $Z^*(X)_{\text{num}}$, i.e., any torsion is numerically equivalent to zero.

1.2.5. *Smash-nilpotent equivalence.* (Voevodsky) For any $\alpha \in Z^*(X)$, say $\alpha \sim_\otimes 0$ if there exists $N \in \mathbb{N}$ such that

$$\underbrace{\alpha \times \cdots \times \alpha}_{N \text{ copies}} \sim_{\text{rat}} 0 \text{ on } X^N$$

1.3. **Relations among $\sim$.** <u>Notion</u>: In all cases, we denoted by

$$\begin{gathered}Z^*(X)_\square := \{\alpha \in Z^*(X) : \alpha \sim_\square 0\}\\ Z^*(X)_\square^\tau := \{\alpha \in Z^*(X) : \exists N \in \mathbb{N} \text{ such that } N\alpha \sim_\square 0\}\end{gathered}$$

and clearly

$$Z^*(X)_\square \subset Z^*(X)_\square^\tau$$

Proposition 1.4 (Relation–1)**.**

$$Z^*(X)_{\text{rat}} \subset Z^*(X)_{\text{alg}} \subset Z^*(X)_{\text{homo}} \subset Z^*(X)_{\text{num}}$$

Proof. Explain one by one.

(1) the first inclusion: follows from definition.
(2) the second inclusion: if $\alpha = \beta_a - \beta_b$, then $\text{cl}(\alpha) = \beta_*(\text{cl}(a - b)) = 0$
(3) the third inclusion: follows from definition.

$\square$

Proposition 1.5 (Relation–2)**.**

$$Z^*(X)_{\text{rat}} \subset Z^*(X)_\otimes \subset Z^*(X)_{\text{homo}}$$

Proof. The second inclusion follows from Künneth formula. □

Proposition 1.6 (Voevodsky & Voisin)**.** *$Z^*(X)^\tau_{\mathrm{alg}} \subset Z^*(X)^\tau_\otimes$ with $\mathbb{Q}$-coefficient.*

Proof. If $\alpha \sim_{\mathrm{alg}} 0$, then there exists $\beta \in Z^*(C \times X)$ and $a, b \in C$ such that

$$\alpha = \beta_a - \beta_b$$

then

$$[\alpha] = [\beta]_*([a-b])$$

here we view $[\beta]$ as a correspondence.

$$[\alpha \times \cdots \times \alpha] = [\beta \times \cdots \times \beta]_* \left([(a-b) \times \cdots \times (a-b)]\right)$$

Suffice to prove: $\exists N \gg 0$

$$[\underbrace{(a-b) \times \cdots \times (a-b)}_{N \text{ copies}}] = 0 \in \mathrm{CH}_0(C^N)^{S_N}_{\mathbb{Q}} \cong \mathrm{CH}_0(C^{[N]})_{\mathbb{Q}}$$

where $C^{[N]} := C^N/S_N$ is a nonsingular projective variety (only for curve). Let

$$\pi : C^N \to C^{[N]}$$

be the quotient map, then

$$\pi^* : \mathrm{CH}^*(C^{[N]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}^*(C^N)^{S_N}_{\mathbb{Q}}$$

From

$$\mathrm{aj} : (C, P) \to (\mathrm{Jac}(C), 0)$$

we get

$$\begin{tikzcd} C^N \arrow[rr] \arrow[dr, "\pi"'] & & \mathrm{Jac}(C) \\ & C^{[N]} \arrow[ur, "\mathrm{ab}"'] & \end{tikzcd}$$

with fibers of ab are liner systems. By Riemann-Roch, for $N \gg 0$ (in fact, $N > 2g-2$), $C^{[N]}$ is a projective bundle over $\mathrm{Jac}(C)$, then

$$\mathrm{aj} : \mathrm{CH}_0(C^{[N]})_{\mathbb{Q}} \cong \mathrm{CH}_0(\mathrm{Jac}(C))_{\mathbb{Q}} \text{ holds even for } \mathbb{Z}$$

Suffices to show

$$\mathrm{aj}_*[(a-b) \times \cdots \times (a-b)] = 0$$

then

$$\mathrm{LHS} = \underbrace{[\mathrm{aj}_*([a-b])]}_{\in \mathrm{CH}_{0,\geq 1}(\mathrm{Jac}(C))_{\mathbb{Q}}}{}^{*N} = 0 \quad \forall N > g$$

dues to Barue's decomposition. Recall

$$\mathrm{Jac}(C) \times \mathrm{Jac}(C) \xrightarrow{\mu} \mathrm{Jac}(C)$$
$$\alpha * \beta := \mu_*(\alpha \times \beta)$$

□

Theorem 1.7 (Summary–1)**.**

$$Z^*(X)^\tau_{\mathrm{rat}} \subset Z^*(X)^\tau_{\mathrm{alg}} \subset Z^*(X)^\tau_{\mathrm{homo}} \subset Z^*(X)^\tau_{\mathrm{num}} = Z^*(X)_{\mathrm{num}}$$

Theorem 1.8 (Summary–2)**.** *With $\mathbb{Q}$-coefficient, we have*

$$Z^*(X)_{\mathrm{rat}} \subset Z^*(X)_{\mathrm{alg}} \subset Z^*(X)_\otimes \subset Z^*(X)_{\mathrm{homo}} \subset Z^*(X)_{\mathrm{num}}$$

One will see examples in the next section to show the first and second inclusion are very far from equality.

1.4. **Differences among $\sim$.** In this section, we will give some examples to show how far or how close between these equivalences.

1.4.1. *Codimension* 1. By exceptional sequence, we obtain

$$\begin{cases} Z^1(X)_{\mathrm{alg}} = Z^1(X)_{\mathrm{homo}} \\ Z^1(X)^{\tau}_{\mathrm{alg}} = Z^1(X)^{\tau}_{\mathrm{homo}} = Z^1(X)_{\mathrm{num}} \end{cases}$$

then

$$\begin{cases} Z^1(X)_{\mathrm{alg}}/Z^1(X)_{\mathrm{rat}} \cong \mathrm{Pic}^0(X) \\ Z^*(X)^{\tau}_{\mathrm{alg}}/Z^*(X)_{\mathrm{alg}} = H^2(X,\mathbb{Z})_{\mathrm{tor}} \end{cases}$$

1.4.2. *Dimension* 0. One has

$$Z^1(X)_{\mathrm{alg}} = Z^1(X)_{\mathrm{homo}} = Z^1(X)_{\mathrm{num}}$$

Remark 1.9. In general, the *Griffith group* of X

$$Z^*(X)_{\mathrm{homo}}/Z^*(X)_{\mathrm{alg}} = \mathrm{Griffth}^*(X)$$

can be very big.

1.4.3. *Cerese.* Let C be a general curve with $g := g(C) \geq 3$. Consider the *Abel-Jacobi* map of C

$$\mathrm{aj} : (C, P) \hookrightarrow (\mathrm{Jac}(C), 0)$$

Consider

$$(-1) : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$$

then we have an element

$$(C - (-1)^*C) \nsim_{\mathrm{alg}} 0 \text{ and } \notin Z_1(\mathrm{Jac}(C))^{\tau}_{\mathrm{alg}} \text{ and } \sim_{\mathrm{homo}} 0$$

2. Standard Conjecture

2.1. **A complete list.** There are series of conjectures related to the relations among these equivalences, called the *Standard Conjecture.*

Conjecture 2.1 ($D(X)$)**.** $Z^*(X)_{\mathrm{homo}} = Z^*(X)_{\mathrm{num}}$*, where*

$$Z^*(X)_{\mathrm{homo}} = \{\alpha \in Z^k(X) : \mathrm{cl}(\alpha) \cup \beta = 0 \textit{ for any } \beta \in \mathrm{Hdg}^{n-k}(X)\}$$
$$Z^*(X)_{\mathrm{num}} = \{\alpha \in Z^k(X) : \mathrm{cl}(\alpha) \cup \mathrm{cl}(\beta) = 0 \textit{ for any } \beta \in Z^{n-k}(X)\}$$

Remark 2.2. In character 0, Hodge conjecture $\Rightarrow$ Standard Conjecture.

Conjecture 2.3 (Voevodsky)**.** $Z^*(X)_{\otimes} = Z^*(X)_{\mathrm{homo}}$.

Fact 2.4. Bloch–Bailisen + $D(X) \Rightarrow$ Voevodsky $\Rightarrow$ Bloch conjecture for surfaces.

Recall: for $k \leq n$, we have

$$\begin{tikzcd} H^k(X,\mathbb{Q}) \arrow[r, "L^{n-k}", "\sim"'] & H^{2n-k}(X,\mathbb{Q}) \\ H^{k-2}(X,\mathbb{Q}) \arrow[u, "L"] \arrow[r, "L^{n-k+2}"] & H^{2n-k+2}(X,\mathbb{Q}) \arrow[u, "\Lambda"'] \end{tikzcd}$$

then

$$\begin{aligned} \Lambda &\in \mathrm{Hom}(H^{2n-k+2}(X,\mathbb{Q}), H^{2n-k}(X,\mathbb{Q})) \\ &= (H^{2n-k+2}(X,\mathbb{Q}))^* \otimes H^{2n-k}(X,\mathbb{Q}) \\ &= H^{k-2}(X,\mathbb{Q}) \otimes H^{2n-k}(X,\mathbb{Q}) \subset H^{2n-2}(X \times X,\mathbb{Q}) \end{aligned}$$

where the last inclusion follows from Künneth formula.

Conjecture 2.5 ($B(X)$ or Lefschetz)**.** Λ *is algebraic.*

Remark 2.6. The $B(X)$ conjecture is true for abelian variety.

Recall: $\Delta_X \subset X \times X$ and the cycle class map

$$\begin{aligned}\mathrm{cl}(\Delta_X) &\in H^{2n}(X \times X, \mathbb{Q}) \\ &= \bigoplus_{k=0}^{2n} H^{2n-k}(X, \mathbb{Q}) \otimes H^k(X, \mathbb{Q}) \\ &= \sum_{k=0}^{2n} \pi_k\end{aligned}$$

with $\pi_k \in H^{2n-k} \otimes H^k \subset H^{2n}(X \times X, \mathbb{Q})$ and in fact

$$\pi_k \in \mathrm{Hdg}^n(X \times X, \mathbb{Q})$$

Conjecture 2.7 ($C(X)$ or Künneth)**.** π_k *is algebraic.*

Let $Y \subset X$ be a closed subvariety of codimension d.

Conjecture 2.8 ($V(X)$ or Voison)**.** *For $\alpha \in Z^k(X)_{\mathbb{Q}}$ such that*

$$\mathrm{cl}(\alpha) \in \mathrm{Im}[H^{2k-2d}(\widetilde{Y}, Q) \to H^{2k}(X, \mathbb{Q})]$$

is supported on Y, then there exists $\beta \in Z^k(X)_{\mathbb{Q}}$ supported on Y with

$$\mathrm{cl}(\alpha) = \mathrm{cl}(\beta)$$

Remark 2.9. The $V(X)$ conjecture is true for $k = 2$.

2.2. **Implications.** In character 0, $B(X)$ is the strongest one.

TABLE 1. Implications among Standard Conjectures

$B(X)$ + Hodge Index Theorem	$\Rightarrow$	$D(X)$
$D(X)$ for any X	$\Rightarrow$	$B(X)$ for any X
$B(X)$	$\Rightarrow$	$C(X)$
$C(X) + V(X)$ for any X	$\Leftrightarrow$	$B(X)$ for any X

REFERENCES

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