LECTURE ON INTERSECTION THEORY (IV)

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Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.
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Contents

1. Answer to the Main problem 1
1.1. Basic properties of $D.\alpha$ 1
1.2. Commutativity 2
2. Chern class of vector bundles 3
2.1. Preliminaries: Associated projective bundle 3
2.2. Properties of first Chern class 4
2.3. Algebraic Chern classes: Definition 4
References 7

In this lecture, we will mainly cover the following topics

- The commutativity of intersection with divisors.
- Definition of Chern class of vector bundles.

1. Answer to the Main problem

In this section we will affirm the Main problem proposed by left in last lecture. For this purpose we first investigate some basic properties of $D.\alpha$, the intersection with divisors.

1.1. Basic properties of $D.\alpha$.

Let X be a scheme. Recall that we have already defined the intersection product

$$D.\alpha \in \mathrm{CH}_{k-1}(|D| \cap |\alpha|)$$

for any persodu-divisor D on X and $\alpha \in Z_k(X)$.

Proposition 1.1. *Here are some basic properties of $D.\alpha$.*

(1) (Linearity) *For any persodu-divisors D, D' on X and $\alpha, \alpha' \in Z_k(X)$, we have*

$$D.(\alpha + \alpha') = D.\alpha + D.\alpha' \in \mathrm{CH}_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|))$$
$$(D + D').\alpha = D.\alpha + D'.\alpha \in \mathrm{CH}_{k-1}((|D| \cup |D'|) \cap |\alpha|)$$

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(2) (Projection formula) *If $f : X' \to X$ is a proper morphism of schemes, then for any persodu-divisor D on X and $\alpha' \in Z_k(X')$, we have*

$$f_*(f^*(D).\alpha') = D.f_*(\alpha') \in \mathrm{CH}_{k-1}(|D| \cap f(|\alpha'|))$$

(3) (Flat pull-back) *If $f : X' \to X$ is a flat morphism of schemes with relative dimension n, then for any persodu-divisor D on X and $\alpha \in Z_k(X)$, we have*

$$f^*(D).f^*(\alpha) = f^*(D.\alpha) \in \mathrm{CH}_{n+k-1}(f^{-1}(|D|) \cap |\alpha|)$$

(4) *If $\mathcal{O}_X(D) \to X$ is a trivial line bundle on X, then*

$$D.\alpha = 0 \in \mathrm{CH}_{k-1}(|\alpha|)$$

for any $\alpha \in Z_k(X)$.

Proof. Part (1) is trivial by definition. Let $\dagger \in \{\emptyset, \ '\}$, then by (1) all the remaining proofs can be easily reduced to the case that $\alpha^\dagger = V^\dagger$ with $V^\dagger \subset X^\dagger$ a closed subvariety of X of dimension k. Moreover, one may further assume that $V^\dagger = X^\dagger$. Now there is nothing to prove. □

1.2. **Commutativity.** In this subsection, we will give a positive answer to the commutativity of intersection with divisors, which is raised as a Main problem in last lecture.

Theorem 1.2. *Let X be a variety of dimension n and D, D' two Cartier divisors on X, then*

$$D.\mathcal{A}(D') = D'.\mathcal{A}(D) \in \mathrm{CH}_{n-2}(|D| \cap |D'|) \tag{1.1}$$

Sketch of the proof. This proof is divided into 2 cases.

(1) Effective case.

(1-1) If D, D' are both effective and properly intersect, then (1.1) can be concluded by a multiplicity check.[1]

(1-2) If D, D' are both effective and non properly intersect, then we introduce a notation to measure excess of D and D'

$$\vartheta(D, D') := \max_{V \subseteq X \ \text{codim-1}} \{(\mathrm{mult}_D V) \cdot (\mathrm{mult}_{D'} V)\}$$

Here both D and D' are viewed as Weil divisors.[2] Then we can separate D and D' by blowing up $D \cap D'$

$$\pi : \mathfrak{Bl}_{D \cap D'} X \to X$$

such that

$$\pi^*(D) = E + C \text{ and } \pi^*(D') = E + C'$$

with E the exceptional divisor. One verify immediately that

(a) $C \cap C' = \emptyset$ and $\vartheta(C, E), \vartheta(C', E) < \vartheta(D, D')$.

(b) Commutativity for $(E, E), (E, C'), (C, E), (C, C')$, and hence commutativity for (D, D').

Notice that up to blowing up, (a) allows us to strictly decrease the excess of D and D' until $\vartheta(D, D')$, i.e., they properly intersect; and (b) implies that we only need to deal with the commutativity in that reduced case. Now everything goes back to (1-1), done.

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[1]Indeed, in this case both sides of (1.1) are equal to the scheme theoretic intersection $D \cap D'$.

[2]Under this convention, we have

$$D, D' \text{ properly intersect} \Leftrightarrow \vartheta(D, D') = 0$$

(2) Non-effective case. If, for example, D is non-effective, then we can write D as a difference of two effective Cartier divisors by blowing up the poles
$$\pi : \mathfrak{Bl}_{\text{poles}} X \to X$$
such that
$$\pi^*(D) = C - E$$
for two effective Cartier divisors C, E. Then everything reduces to the effective case (1).

$\square$

The commutativity has the following pair of important corollaries.

Corollary 1.3 (Commutativity)**.** *Let X be a scheme, D, D' two persodu-divisors on X and $\alpha \in Z_k(X)$, then*
$$D.(D'.\alpha) = D'.(D.\alpha) \in \mathrm{CH}_{k-2}(|D| \cap |D'| \cap |\alpha|)$$

Corollary 1.4. *Let X be a scheme, D a persodu-divisor on X, then*
$$\alpha \sim_{\text{rat}} 0 \in Z_k(X) \Rightarrow D.\alpha \sim_{\text{rat}} 0 \in Z_{k-1}(|D|)$$
i.e.,
$$\alpha = 0 \in \mathrm{CH}_k(X) \Rightarrow D.\alpha = 0 \in \mathrm{CH}_{k-1}(|D|)$$

Remark 1.5. Upon this corollary, we see that the intersection with divisors $D.$ can descend to the level of Chow groups
$$D. : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(|D|)$$

Proof. Assume that $\alpha = \mathrm{div}(f)$ for some $f \in \mathcal{R}(V)^*$ with $V \subset X$ a closed subvariety of X of dimension $k+1$. We may further assume that $V = X$ and D is a Cartier divisor on X, then
$$D.\alpha = D.\mathrm{div}(f) = \mathrm{div}(f).D = 0 \in \mathrm{CH}_{k-1}(|D|)$$
since $\mathrm{div}(f)$ corresponds to a trivial line bundle. $\square$

2. Chern class of vector bundles

In this section, we will define the Chern class of vector bundles.

2.1. **Preliminaries: Associated projective bundle.** Let X be a scheme and $\pi : E \to X$ a vector bundle over X of rank r. By definition there exists an open covering $\{U_i\}$ of X such that
$$\varphi_i : E|_{\pi^{-1}(U_i)} \cong U_i \times \mathbb{A}^r$$
and therefore we obtain
$$g_{ij} := \varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \to \mathrm{GL}(\mathbb{C}, r)$$

Remark 2.1. Recall that we have the equivalence of categories.
$$(\mathsf{Vector\ bundle}/X) \longleftrightarrow (\mathsf{Locally\ free\ sheaf}/X)$$
$$(E \to X) \dashrightarrow \mathcal{E} : \text{the sheaf of sections on } E$$
$$\mathrm{Spec}\left(\bigoplus_{i \geq 0} \mathrm{Sym}^i(\mathcal{E}^\vee)\right) \dashleftarrow \mathcal{E}$$

Given a vector bundle $\pi : E \to X$ over X of rank $r = e + 1$, we can associate it a projective bundle

$$\begin{tikzcd} \mathbb{P}(E) := \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E}^\vee) \arrow[d, "p"] \\ X \end{tikzcd}$$

which is of rank e over X and over each fiber $x \in X$

$$\mathbb{P}(E)_x := \mathbb{P}(E_x)$$

Moreover, one have a tautological subbundle $\mathcal{O}_E(-1)$ of $p^*(E)$ over $\mathbb{P}(E)$

$$p^*(E) \supseteq \mathcal{O}_E(-1) \to \mathbb{P}(E)$$

whose fiber over $(x, [L])$ is L itself. In picture we have

$$\begin{tikzcd} & \mathcal{O}_E(-1) \arrow[dl, hook'] \arrow[d] \\ p^*(E) \arrow[r] & \mathbb{P}(E) \arrow[d, "p"] \\ E \arrow[u, dashed] \arrow[r, "\pi"] & X \end{tikzcd}$$

Hereafter, denoted by $\mathcal{O}_E(1) = \mathcal{O}_E(-1)^\vee$.

2.2. **Properties of first Chern class.** Let X be a scheme, D a persodu-divisor on X and $L \to X$ a line bundle over X. We have similar properties (refer to Proposition 1.1 and we omit the statements here) for the first Chern class $c_1(L)$

$$c_1(L) \cap\ : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(X)$$

and the inclusion map $i : D \hookrightarrow X$

$$i^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(D)$$

And in addition, we have

(1) For any line bundles L, L' over X and $\alpha \in Z_k(X)$, we have

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$
$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha$$

(2) For any $\alpha \in Z_k(X)$, we have

$$i_*(i^*(\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha$$
$$i^*(i_*(\alpha) = c_1(N) \cap \alpha$$

where $N = \mathcal{O}_X(D)|_D$ is the normal bundle.

Example 2.2. Recall that $\mathrm{CH}_k(\mathbb{P}^n) = \mathbb{Z}[L_k]$ for some vector subspace L_k of dimension k. For the line bundle $\mathcal{O}_{\mathbb{P}^n}(1) \to \mathbb{P}^n$, we have

$$c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cap [L_k] = [L_{k-1}].$$

2.3. **Algebraic Chern classes: Definition.** Let X be a scheme and $\pi : E \to X$ a vector bundle over X of rank r. In this subsection we want to define the i-th Chern class $c_i(E)$ of E as

$$c_i(E) \cap : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-i}(X)$$

Usually there are 2 approaches towards the definition of $c_i(E)$. And recall that we have already defined the first Chern class of line bundle in last lecture.

2.3.1. *Topologist's approach.* We expect that Chern class $c_i(-)$ should satisfy *Whitney sum formula* (or shortly *Whitney formula*), i.e., if

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles over X, then we expect that

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'') \tag{2.1}$$

Remark 2.3. If we set

$$c_t(E) := 1 + c_1(E)t + c_2(E)t^2 + \cdots$$

then Whitney formula (2.1) can be alternatively written as

$$c_t(E) = c_t(E')c_t(E'')$$

In topological way to define the Chern class of a vector bundle, the key point is the following so-call *splitting construction.* It essentially allows us to pass everything into the line bundle, over which we have defined the first Chern class already.

Theorem 2.4 (Splitting construction)**.** *Let X be a scheme and $\pi : E \to X$ a vector bundle over X of rank r, then there exists a flat morphism of schemes $f : X' \to X$ such that*

1. *$f^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(X')$ is injective and*
2. *$f^*(E)$ admits a filtration*
$$f^*(E) = E_r \supset E_{r-1} \supset \ldots \supset E_1 \supset E_0 = 0$$
such that each $L_i := E_i/E_{i-1}$ is a line bundle.

Remark 2.5. We make a few remarks here before we give the proof.

1. The requirement (1) in this theorem guarantees that we don't lose any information when applying splitting construction.
2. Via this splitting construction, there is an natural way to define (we omit the details here) the Chern class of a vector bundle satisfying
   – Whitney sum formula.
   – functoriality.
   – coincide with the first Chern class of line bundles defined before.

   And in this case, our definition can be viewed as a generalization of the first Chern class of line bundle we have defined before. Moreover, these data implies the uniqueness of $c_i(-)$.

Proof of splitting construction. Induction on the rank of the vector bundle. The theorem is trivial if E is a line bundle (i.e., $r = 1$). Suppose the splitting construction holds for any vector bundle of rank less or equal to $r - 1$. Now for vector bundle $\pi : E \to X$ over X of rank r, we consider its associated projective bundle

$$\begin{tikzcd}
& \mathcal{O}_E(-1) \arrow[dl, hook'] \arrow[d] \\
p^*(E) \arrow[r] & \mathbb{P}(E) \arrow[d, "p"] \\
E \arrow[u, dashed] \arrow[r, "\pi"] & X
\end{tikzcd}$$

then we have

1. $p^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(\mathbb{P}(E))$ is injective (see later Proposition 2.7 (1)).
2. $r(p^*(E)/\mathcal{O}_E(-1)) = r - 1$.

Thus by induction hypothesis, there exists a flat morphism of schemes

$$q : X' \to \mathbb{P}(E)$$

such that

(1) $q^* : \mathrm{CH}_*(\mathbb{P}(E)) \to \mathrm{CH}_*(X')$ is injective and
(2) $q^*(p^*(E)/\mathcal{O}_E(-1))$ admits a filtration with line bundles quotients.

Now take $f := p \circ q : X' \to X$. As desired. □

2.3.2. *Grothendieck's formula.* Another way to define the Chern class of vector bundle is via Segre class, which can be defined only upon the first Chern class of line bundle we have just defined. It turns out in some scenes, Chern class is an 'inverse' to Segre class.

Let X be a scheme and $\pi : E \to X$ a vector bundle over X of rank $r = e + 1$. Consider the following diagram

$$\begin{tikzcd}
& \mathcal{O}_E(-1) \arrow[dl, hook] \arrow[d] \\
p^*(E) \arrow[r] & \mathbb{P}(E) \arrow[d, "p"] \\
E \arrow[u, dashed] \arrow[r, "\pi"] & X
\end{tikzcd}$$

Set $\xi := c_1(\mathcal{O}_E(1))$, then as before one expect that Chern class should satisfy the following *Grothendieck's formula*[3]

$$\xi^r + c_1(p^*E)\xi^{r-1} + c_2(p^*E)\xi^{r-2} + \cdots + c_r(p^*E) = 0 \tag{2.2}$$

over $\mathbb{P}(E)$.

Definition 2.6 (Segre Class)**.** The i-th Segre class

$$s_i(E) \cap : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-i}(X)$$

is given by

$$s_i(E) \cap \alpha := p_*[c_1(\mathcal{O}_E(1))^{e+i} \cap p^*(\alpha)]$$

Proposition 2.7. *Here are some basic properties of Segre class.*

(1) *For any $\alpha \in \mathrm{CH}_k(X)$, we have*

$$s_i(E) \cap \alpha = \begin{cases} 0 & i < 0 \\ \alpha & i = 0 \end{cases}$$

In particular, the second formula implies that p^* is injective since

$$p_*[c_1(\mathcal{O}_E(1))^e \cap p^*(\alpha)] = \alpha$$

(2) (Communcativity & Functorial) *If E and F are two vector bundles over X, then for any $\alpha \in \mathrm{CH}_k(X)$, we have*

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha)$$

(3) (Projection formula) *If $f : X' \to X$ is a proper morphism of schemes, then for any vector bundle E over X and $\alpha \in \mathrm{CH}_k(X')$, we have*

$$f_*(s_i(f^*(E)) \cap \alpha) = s_i(E) \cap f_*(\alpha)$$

[3] In fact, this formula not only gives the uniqueness of Chern class but also advices a way to construct the Chern class.

(4) (Flat pull-back) *If $f : X' \to X$ is a flat morphism of schemes, then for any vector bundle E over X and $\alpha \in \mathrm{CH}_k(X)$, we have*

$$s_i(f^*E) \cap f^*(\alpha) = f^*(s_i(E) \cap \alpha)$$

(5) *If $E \to X$ is a line bundle over X, then for any $\alpha \in \mathrm{CH}_k(X)$, we have*

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha$$

Proof. Omitted. □

Via Segre class, we can reformulate the Grothendieck's formula (2.2) as follows. First multiple (2.2) by some power of ξ and then apply it on any element $\alpha \in \mathrm{CH}_k(X)$. We push-forward the resulting formula by applying p_* and finally get

$$p_*(\xi^\dagger \cdot (2.2) \cap \alpha) = 0$$

One can easily rephrase this formula as

$$\begin{aligned} s_1(E) + c_1(E) &= 0 \\ s_2(E) + s_1(E)c_1(E) + c_2(E) &= 0 \\ &\vdots \\ s_n(E) + s_{n-1}(E)c_1(E) + \cdots + c_n(E) &= 0 \end{aligned}$$

then we can solve out each $c_i(E)$ inductively, for example

$$\begin{aligned} c_1(E) &= -s_1(E) \\ c_2(E) &= -s_2(E) - s_1(E)c_1(E) \\ &\vdots \\ c_n(E) &= -s_n(E) - s_{n-1}(E)c_1(E) - \cdots - s_1(E)c_{n-1}(E) \end{aligned}$$

Remark 2.8. If we write

$$\begin{aligned} c_t(E) &:= 1 + c_1(E)t + c_2(E)t^2 + \cdots \\ s_t(E) &:= 1 + s_1(E)t + s_2(E)t^2 + \cdots \end{aligned}$$

then Grothendieck formula (2.2) says

$$c_t(E)s_t(E) = 1$$

i.e., Segre class and Chern class are inverse to each other.

References

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