LECTURE ON INTERSECTION THEORY (XI)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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Dedicated to Joseph Ayoub.

Contents

1. More on Bloch–Srinivas' theorem

Let X be a non-singular projective variety over $\mathbb{C}$ of dimension n. Recall

Theorem 1.1 (Bloch–Srinivas)**.** *If there exists $j : Y \hookrightarrow X$ such that*

$$j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$

is surjective, then there exists $N \in \mathbb{N}$ such that

$$N \cdot [\Delta_X] = \Gamma_1 + \Gamma_2 \in \mathrm{Corr}^0(X, X) = \mathrm{CH}^n(X \times X)$$

with $\Gamma_i \in \mathrm{Corr}^0(X, X)$ and

1. *Γ_1 is supported on $Y \times X$.*
2. *Γ_2 is supported on $X \times D$, where D is a divisor on X.*

Remark 1.2. A few words about Bloch–Srinivas' theorem.

1. The integer N is essential: there are many examples where $[\Delta_X]$ itself doesn't admit such an decomposition.
2. To get ride of the integer N, one is led to consider the $\mathbb{Q}$-coefficient. This gives a phenomenon that $\mathbb{Z}$-coefficient Chow group doesn't behave well.
3. Both Y and D in the theorem may be not irreducible.
4. Bloch–Srinivas's theorem holds even if X is proper.

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1.1. **Applications.** Here we give some applications of Bloch–Srinivas' theorem.

Corollary 1.3. *If* $\mathrm{CH}_0(X) \cong \mathbb{Z}$*, then* $H^{p,0}(X) = 0$ *for any* $p > 0$*.*

In picture, its Hodge diamond looks like

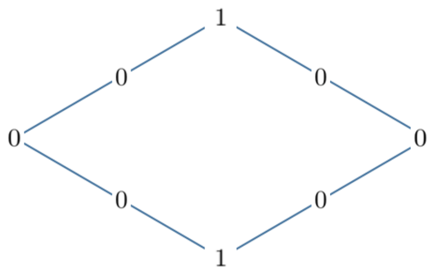

FIGURE 1. Hodge diamond for trivial $\mathrm{CH}_0(X)$

Corollary 1.4. *If there exists* $j : Y \hookrightarrow X$ *with* $\dim(Y) = 3$ *such that*

$$j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$

is surjective, then codim 2 *Hodge conjecture holds for* X*.*

Proof. Since $\mathrm{Hdg}^2(X)_{\mathbb{Q}} = H^4(X, \mathbb{Q}) \cap H^{2,2}(X)$. As usual, consider

$$(N \cdot [\Delta_X])_* = (\Gamma_1)_* + (\Gamma_2)_* \text{ on } \mathrm{Hdg}^2(X)_{\mathbb{Q}}$$

(LHS) $(N \cdot [\Delta_X])_* = \mathrm{id}_{\mathrm{Hdg}^2(X)_{\mathbb{Q}}}$ since we are in $\mathbb{Q}$-coefficient.

(RHS) 1^{st} term: $(\Gamma_1)_*$ factors through $\mathrm{Hdg}^2(\widetilde{Y})_{\mathbb{Q}}$ with $\dim(\widetilde{Y}) = 3$, then

$$(\Gamma_1)_*(\mathrm{Hdg}^2(\widetilde{Y})_{\mathbb{Q}}) \text{ comes from algebra}$$

(RHS) 2^{nd} term: $(\Gamma_2)_*$ factors through (by projection formula) $\mathrm{Hdg}^1(\widetilde{D})_{\mathbb{Q}}$, then by Lefschetz $(1,1)$-theorem

$$(\Gamma_2)_*(\mathrm{Hdg}^2(X)_{\mathbb{Q}}) \text{ comes from algebra}$$

Thus (RHS) comes from algebra, implying that

$$\mathrm{Hdg}^2(X)_{\mathbb{Q}} \text{ comes from algebra}$$

□

Corollary 1.5. *For* $n = 4, 5$*, if there exists* $j : Y \hookrightarrow X$ *with* $\dim(Y) = 3$ *such that*

$$j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$

is surjective, then Hodge conjecture holds for X*.*

Proof. Due to Remark 2.11 in Lecture (IX), in the case $n = 4, 5$ it's remaining to prove the case $k = 2$, which is exactly Corollary 1.4. □

Remark 1.6. This applies to uniruled[1] 4-folds OR rationally connected 4,5-folds.

[1] for more information on this topic, see Appendix A

1.2. **Generalization and corollaries.**

Theorem 1.7 (Generalized Bloch–Srinivas)**.** *If there exists $Y_i \hookrightarrow X$ such that for each i, the $\mathrm{CH}_i(X)$ is supported on Y_i, then $\exists N \in \mathbb{N}$ such that*

$$N \cdot [\Delta_X] = \sum \Gamma_i \in \mathrm{CH}^n(X \times X)$$

where Γ_i is supported on $Y_i \times W^i$ with $W^i \subset X$ of codim i.

Corollary 1.8. *If* $\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}(X, \mathbb{Q})$ *is injective, then*

1. $H^{\mathrm{odd}}(X) = 0$.
2. $\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}(X, \mathbb{Q})$ *is surjective.*

Proof. (1) is clear. For (2), under this condition

$$N \cdot [\Delta_X] = \sum \Gamma_i = \sum a_i [\underbrace{Y_i}_{\dim i} \times \underbrace{W^i}_{\mathrm{codim}\ i}]$$

Consider

$$(N \cdot [\Delta_X])_* = \sum (\Gamma_i)_* \text{ on } H^*(X, \mathbb{Q})$$

then for any $\alpha \in H^*(X, \mathbb{Q})$

$$(N \cdot [\Delta_X])_*(\alpha) = \sum (\Gamma_i)_*(\alpha)$$

i.e.,

$$N\alpha = \sum a_i \left(\langle \alpha, \mathrm{cl}(Y_i) \rangle \cdot \ \mathrm{cl}(W^i) \right)$$

then $\alpha \in \mathrm{Hdg}^*(X)_{\mathbb{Q}}$. $\square$

Corollary 1.9. *If* $\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}(X, \mathbb{Q})$ *is injective for $i \leq k$, then the Hodge diamond of X looks like*

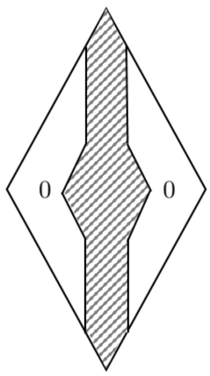

Figure 2. Hodge diamond

2. Bloch's conjecture: the inverse of Mumford

In this section, let $X = S$ be a surface. Recall

Theorem 2.1 (Mumford)**.**

$$\begin{tikzcd} \ker(\mathrm{alb}) = 0 \arrow[rr, Rightarrow] \arrow[dr, dashed] & & H^{2,0}(S) = 0 \\ & \mathrm{CH}_0(S) \arrow[ur, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{CH}_0(C)_0 \arrow[r, "\text{A-J}", "\sim"'] \arrow[d, two heads] & \mathrm{Jac}(C) \arrow[d, two heads] \\ \mathrm{CH}_0(S)_0 \arrow[r, "\text{alb}", "\sim"'] & \mathrm{Alb}(S) \end{tikzcd}$$

Bloch conjectured that the inverse of Mumford is also true, i.e.,

Conjecture 2.2 (Bloch)**.** $H^{2,0}(S) = 0 \Rightarrow \ker(\mathrm{alb}) = 0$.

Known cases: Suffice to prove this conjecture for the minimal models of surface.

(1) (Bloch–Kas–Lidenman) S not of general type[2].

Example 2.3 (Enriques surface)**.** Let S be an Enriques surface, then $\kappa(S) = 0$ and the canonical bundle $K_S \neq \mathcal{O}_S$ but $K_S^{\otimes 2} = \mathcal{O}_S$. Its Hodge diamond has the form

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Since Enriques surfaces are all projective and elliptic surfaces of genus 0, there exists an *elliptic fibration*

$$\begin{tikzcd} S \arrow[d] \\ \mathbb{P}^1 \arrow[u, bend left, "\times"] \end{tikzcd}$$

i.e., general fibers are elliptic curves, with no canonical choice of origin in each fiber. Hence there is no sections for this fibration. Taking Jacobian of each fiber (which is an elliptic curve) yields a *Jacobian fibration*

$$\begin{tikzcd} J \arrow[d] \\ \mathbb{P}^1 \arrow[u, bend left, "s"] \end{tikzcd}$$

then the result follows from the following pair of facts.

(a) $\mathrm{CH}_0(S) \cong \mathrm{CH}_0(J)$.

(b) J is a rational surface (i.e., $J \cong \mathbb{P}^2$), then $\mathrm{CH}_0(J) = \mathbb{Z}$.

(2) (Italians) S with many automorphisms. The philosophy is that: for two automorphisms

$$\sigma, \tau : S \to S$$

we have

$$S/\sigma, S/\tau, S/(\sigma \circ \tau) \text{ `nice'} \ \Rightarrow \mathrm{CH}_0(S) \text{ `nice'}$$

[2]Recall that if $\dim(X) = n$, then the canonical bundle $K_X = \Omega_X^n$. If

$$H^0(X, K_X^{\otimes r}) \neq 0$$

then we get

$$\Phi_r : X \dashrightarrow \mathbb{P}^{h^0(X, K_X^{\otimes r})-1}$$

the *Kodaira dimension* of X is defined by

$$\kappa(X) := \begin{cases} -\infty & \text{if } H^0(X, K_X^{\otimes r}) = 0 \text{ for } \forall r \\ \max_{r>0}\{\dim(\mathrm{Im}(\Phi_r))\} & \text{else} \end{cases}$$

and X is called *general type* if $\kappa(X) = \dim(X)$. In the case $X = S$ is a surface, $\kappa(S) = -\infty, 0, 1, 2$. So if $\kappa(S) = 2$, then S is called of *general type*.

(3) (Voisin) S with 'nice' moduli space. If

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & \mathcal{S} \arrow[d] \\ 0 \arrow[r, hook] & B \end{tikzcd}$$

is a family of surfaces such that

$$\overline{\mathcal{S} \times_B \mathcal{S}} \text{ is rationally connected } \textsc{Or } \mathrm{CH}_0(\overline{\mathcal{S} \times_B \mathcal{S}}) = \mathbb{Z}$$

Example 2.4. Take $S = Y/\mathbb{Z}_5$ where $Y \subset \mathbb{P}^3$ of degree 5.

(4) (Kimirmm) S dominated by a product of curves $C_1 \times C_2$.

$$C_1 \times C_2 \dashrightarrow S$$

(5) Most difficult case: *fake* projective planes (or *Mumford surface*)

$$H^*(S, \mathbb{Q}) \cong H^*(\mathbb{P}^2, \mathbb{Q})$$

Generalized Bloch conjecture gives an inverse of Corollary 1.8.

Conjecture 2.5 (Generalized Bloch)**.** *If*

(1) $H^{\text{odd}}(X) = 0$ *and*
(2) $\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}(X, \mathbb{Q})$ *is surjective*

then $\mathrm{cl} : \mathrm{CH}^*(X)_{\mathbb{Q}} \to H^{2*}(X, \mathbb{Q})$ *is injective.*

3. From geometry to Chow ring

3.1. **Back to surface.** Let $X = S$ be a surface. Consider its Hodge diamond

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

And we already know

Table 1. Algebra $\Leftarrow$ Geom/Topo

Algebra	Geom/Topo
$\mathrm{CH}^0(S) \cong \mathbb{Z}$	$h^{0,0}$
$\mathrm{CH}^1(S) \xrightarrow{\mathrm{cl}} H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$	$h^{1,1}$
$\cup$	$\downarrow$
$\ker(\mathrm{cl}) \cong \mathrm{Pic}^0(S)$	$h^{0,1} = h^{1,0}$
$\mathrm{CH}^2(S) \xrightarrow{\mathrm{cl}} H^4(S,\mathbb{Z}) \cap H^{2,2}(S)$	$h^{2,2}$
$\cup$	$\downarrow$
$\ker(\mathrm{cl}) \xrightarrow{\mathrm{alb}} \mathrm{Alb}(S)$	$h^{2,1} = h^{1,2}$
$\cup$	$\downarrow$
$\ker(\mathrm{alb})$	$h^{2,0} = h^{0,2}$ (conjectured by Bloch)

3.2. **Conjecture in general.**

Conjecture 3.1 (Bloch–Bailinson)**.** *Let X be a nonsingular projective variety of dimension n, then for any $k \in \mathbb{N}$ there is a filtration*

$$\mathrm{CH}^k(X)_{\mathbb{Q}} = F^0\mathrm{CH}^k(X)_{\mathbb{Q}} \supset F^1\mathrm{CH}^k(X)_{\mathbb{Q}} \supset F^2\mathrm{CH}^k(X)_{\mathbb{Q}} \supset \cdots$$

denoted by $F^\bullet\mathrm{CH}^k(X)_{\mathbb{Q}}$, such that

(1) (Functorial) $F^\bullet$ *is compatible with pull-back, push-forward, intersection*[3] *and correspondence, etc.*

(2) (Boundedness) *for any $k \in \mathbb{N}$*

$$F^{k+1}\mathrm{CH}^k(X)_{\mathbb{Q}} = 0$$

(3) (Chow ring controlled by homology) *Set*

$$\mathrm{Gr}^i_F\mathrm{CH}^k(X)_{\mathbb{Q}} := F^i\mathrm{CH}^k(X)_{\mathbb{Q}}/F^{i+1}\mathrm{CH}^k(X)_{\mathbb{Q}}$$

then it's controlled by $H^{k,k-i}(X)$, i.e.,

$$H^{k,k-i}(X) = 0 \Rightarrow \mathrm{Gr}^i_F\mathrm{CH}^k(X)_{\mathbb{Q}} = 0$$

(4) (Initial) $F^1\mathrm{CH}^k(X)_{\mathbb{Q}} = \ker(\mathrm{cl})_{\mathbb{Q}} \quad [=: \mathrm{CH}^k(X)_{\mathbb{Q},\mathrm{hom}}]$.

In picture we have the control for codimension k cycles

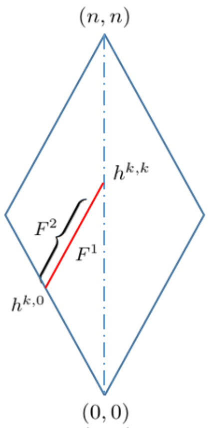

FIGURE 3. Control for codimension k cycles

Example 3.2. In the case $X = S$ is a surface. Know $F^2\mathrm{CH}^2(X)_{\mathbb{Q}} = \ker(\mathrm{alb})_{\mathbb{Q}}$.

TABLE 2. Bloch–Bailinson conjecture for surface

codim	F^0/F^1	F^1/F^2	F^2/F^3
0	$\mathbb{Q}$		
1	$\mathrm{NS}(S)_{\mathbb{Q}}$	$\mathrm{Pic}^0(S)_{\mathbb{Q}}$	
2	$\mathbb{Q}$	$\mathrm{Alb}(S)_{\mathbb{Q}}$	$\ker(\mathrm{alb})_{\mathbb{Q}}$

The remaining unknown part is to whether $\ker(\mathrm{alb})_{\mathbb{Q}}$ is controlled by $H^{2,0}(X)$.

[3] $F^i \cdot F^j \subset F^{i+j}$.

Appendix A. Ruled variety

Definition A.1. (1) A variety over a field $\Bbbk$ is *ruled* if it is birational to the product of $\mathbb{P}^1$ with some variety over $\Bbbk$.

(2) A variety is *uniruled* if it is covered by a family of rational curves. More precisely, a variety X is *uniruled* if there is a variety Y and a dominant rational map
$$\mathbb{P}^1 \times Y \dashrightarrow X$$
which does not factor through the projection to Y.

A list of properties of ruled/uniruled variety.

(1) (char $\Bbbk = 0$) Every uniruled variety has Kodaira dimension $-\infty$. The converse is a conjecture which is known in dimension at most 3.

(2) (char $\Bbbk = 0$)

Theorem A.2 (Boucksom, Demailly, Păun and Peternell). *Let X be a nonsingular projective variety over $\Bbbk$, then*
$$X \text{ is uniruled iff } K_X \text{ is not pseudo-effective}$$

Corollary A.3. *A nonsingular hypersurface of degree d in $\mathbb{P}^n$ is uniruled iff $d \leq n$.*

Proof. By the adjunction formula. In fact, a nonsingular hypersurface of degree $d \leq n$ in $\mathbb{P}^n$ is a Fano variety and hence is rationally connected, which is stronger than being uniruled. □

(3) ($\overline{\Bbbk} = \Bbbk$ and $\#\Bbbk > \aleph_0$) A variety X is uniruled iff there is a rational curve passing though every $\Bbbk$-point of X.

Remark A.4. The assumption $\#\Bbbk > \aleph_0$ is necessary: there are varieties over the algebraic closure $\Bbbk$ of a finite field which are not uniruled but have a rational curve through every $\Bbbk$-point. The Kummer variety of any non-supersingular abelian surface over $\mathbb{F}_p$ with p odd has these properties.

(4) Uniruledness is a geometric property[4], whereas ruledness is not.

References

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[4] it is unchanged under field extensions.