LECTURE ON INTERSECTION THEORY (XII)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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In this lecture, we mainly focus on the Hodge structure and abelian variety.

1. Hodge structure

1.1. Alternative definitions.

Definition 1.1 (Hodge structure)**.** A *Hodge structure of weight k*, denoted by

$$\{V_{\mathbb{Z}}, V^{p,q}\}$$

consists of the following data

(1) a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$.
(2) a decomposition of complex vector spaces

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that

$$\overline{V^{p,q}} = V^{q,p}$$

Definition 1.2 (Hodge filtration)**.** Given a Hodge structure $\{V_{\mathbb{Z}}, V^{p,q}\}$ of weight k, its corresponding *Hodge filtration*, denoted by

$$\{V_{\mathbb{Z}}, F^pV_{\mathbb{C}}\}$$

is given by

$$\cdots \subset F^pV_{\mathbb{C}} \subset F^{p-1}V_{\mathbb{C}} \subset \cdots$$

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where

$$F^pV_{\mathbb{C}} := \bigoplus_{p' \geq p} V^{p',q}$$

and satisfying

$$F^pV_{\mathbb{C}} \oplus \overline{F^qV_{\mathbb{C}}} = V_{\mathbb{C}} \text{ for any } p+q = k+1$$

Similarly we can define $\mathbb{Q}, \mathbb{R}$-Hodge structure.

Remark 1.3. Giving a Hodge structure is equivalent to giving a Hodge filtration.

(1) Giving a Hodge structure $\{V_{\mathbb{Z}}, V^{p,q}\}$, one define

$$F^pV_{\mathbb{C}} := \bigoplus_{p' \geq p} V^{p',q}$$

(2) Giving a Hodge filtration $\{V_{\mathbb{Z}}, F^pV_{\mathbb{C}}\}$, one define

$$V^{p,q} := F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}$$

Example 1.4. Here are some examples of Hodge structure.

(1) Tate structure:

$$\mathbb{Z}_{(1)} := (2\pi i)\mathbb{Z} \subset \mathbb{C}$$

of weight -2 and of type $(-1,-1)$

And in fact it appears in the exceptional sequence

$$0 \to \mathbb{Z}_{(1)} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

(2) From old to new: let V, W be two Hodge structures, we can construct some new Hodge structure upon them:
   - $V \otimes W$.
   - $\mathrm{Hom}(V, W)$.
   - $V^* := \mathrm{Hom}(V, \mathbb{Z})$ with $\mathbb{Z}$ viewed as the trivial Hodge structure.

   In particular, from Tate structure one can get

Table 1. Tate structure

Tate structure	weight	type
$\mathbb{Z}_{(k)} := \mathbb{Z}_{(1)}^{\otimes k}$	$-2k$	$(-k,-k)$
$\mathbb{Z}_{(-1)} := \mathbb{Z}_{(1)}^*$	2	$(1,1)$
$\mathbb{Z}_{(-k)} := \mathbb{Z}_{(-1)}^{\otimes k}$	$2k$	(k,k)
$\mathbb{Z}_{(0)} := \mathbb{Z}$	0	$(0,0)$

Remark 1.5. Also in this language, we have an algebraic version of cycle class map

$$\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X, \mathbb{Z}_{(k)})$$

1.2. **Generalized Hodge structure.** Let X be a nonsingular projective variety over $\mathbb{C}$ of dimension n, then we have

$$H^k(X, \mathbb{Z})$$

For any $Y \hookrightarrow X$ a subvariety of codimension p, taking a resolution of singularities for Y yields

$$f : \widetilde{Y} \to Y \hookrightarrow X$$

then by functoriality of f and Poincaré duality, one get

$$(\dagger) \qquad H^{k-2p}(\widetilde{Y}, \mathbb{Z}) \to H^k(X, \mathbb{Z}) \cap F^pH^k(X, \mathbb{C})$$

which preserves the Hodge structure on both sides

$$H^{*,*}(\widetilde{Y}) \to H^{*+p,*+p}(X)$$

Conjecture 1.6 (Generalized Hodge)**.** *All sub-Hodge structures in*

$$H^k(X,\mathbb{Z}) \cap F^p H^k(X,\mathbb{C})$$

are supported on a subvariety of X of codimension p.

Consider H^{2k} and $p = k$, we are back to the original Hodge conjecture.

Remark 1.7. The words 'sub-Hodge structures' doesn't appear in the original version of generalized Hodge conjecture, and this modification version above dues to Grothendieck, who in [Gro69] first gives a counterexample to show that

$$H^k(X,\mathbb{Z}) \cap F^p H^k(X,\mathbb{C})$$

may not be a Hodge structure, while the image of (†) is necessarily a Hodge structure.

1.3. **Polarization.**

Definition 1.8. A *polarization* on a weight k Hodge structure $\{V_{\mathbb{Z}}, V^{p,q}\}$ is a non-degenerate, bilinear form on $V_{\mathbb{Z}}$

$$Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$$

such that

$$Q(-,-) := \begin{cases} \text{symmetric} & \text{if } k \text{ even} \\ \text{anti-symmetric} & \text{if } k \text{ odd} \end{cases}$$

and its extension to $V_{\mathbb{C}}$ satisfies the two *Hodge-Riemann bilinear relation*

(1) $Q(V^{p,q}, V^{p',q'}) = 0$ unless $p + p' = k$ and $q + q' = k$.
(2) over each part $V^{p,q}$, the so-defined form

$$\begin{aligned} H : V^{p,q} \times V^{p,q} &\longrightarrow \mathbb{C} \\ (a,b) &\mapsto (2\pi\sqrt{-1})^k(\sqrt{-1})^{p-q} Q(a,\bar{b}) \end{aligned}$$

is Hermitian, symmetric and positive-definite.

In this case, we call $\{V_{\mathbb{Z}}, V^{p,q}, Q\}$ is a *polarized Hodge structure.*

Equipped with polarization, we can get the semistability of the category of polarized Hodge structure.

Fact 1.9. Let V be a polarized Hodge structure and $W \subset V$ a sub-Hodge structure, then there is a decomposition

$$V = W \oplus W^{\perp}$$

as Hodge structure.

Example 1.10. Let X be a nonsingular projective variety over $\mathbb{C}$ of dimension n and L an ample line bundle on X, we have already known that on $H^k(X,\mathbb{Z})$

$$L : H^k(X,\mathbb{Z}) \xrightarrow{\cup c_1(L)} H^{k+2}(X,\mathbb{Z})$$

By Hard Lefschetz, we have an isomorphism for $k \leq n$

$$L^{n-k} : H^k(X,\mathbb{Q}) \xrightarrow{\sim} H^{2n-k}(X,\mathbb{Q})$$

then one can form a diagram

$$\begin{tikzcd} H^k(X,\mathbb{Q}) \arrow[r, "L^{n-k}", "\sim"'] & H^{2n-k}(X,\mathbb{Q}) \\ H^{k-2}(X,\mathbb{Q}) \arrow[u, "L"] \arrow[r, "L^{n-k+2}", "\sim"'] & H^{2n-k+2}(X,\mathbb{Q}) \arrow[u, dashed, "\Lambda"'] \end{tikzcd}$$

therefore

$$(L, \Lambda) \text{ determines a representation of } \mathfrak{sl}_2$$

From this fact one can find some evidence to verify the following statement.

Fact 1.11. For any $k \leq n$, there is a decomposition

$$H^k(X,\mathbb{Q}) = \bigoplus_{i=0}^{[\frac{k}{2}]} L^i H^{k-2i}_{\mathrm{prim}}(X,\mathbb{Q})$$

where

$$H^j_{\mathrm{prim}}(X,\mathbb{Q}) := \ker(L^{n-j+1})$$

Over each part $H^j_{\mathrm{prim}}(X,\mathbb{Q})$, the so-defined form

$$Q(a,b) = \int c_1(L)^{n-k} \cup a \cup b$$

is a polarization.

Example 1.12 ((Polarized) Hodge structure of weight 1)**.** One can give a complete characterization of Hodge structure of weight one: there is a bijection

$$\left\{ \text{free effective H.S. of weight 1: } \begin{array}{l} * V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1} \\ * \dim(V^{1,0}) = g \end{array} \right\} \leftrightarrow \{\text{complex tori of dim } g\}$$

$$\{V_{\mathbb{Z}}, V^{p,q}\} \mapsto V^{1,0}/V_{\mathbb{Z}}$$

$$H_1(\mathbb{C}^g/\Lambda, \mathbb{Z}) \leftarrow \mathbb{C}^g/\Lambda$$

inducing a bijection

$$\{\text{polarization free effective H.S. of weight 1}\} \leftrightarrow \{\text{abelian variety}\}$$

where the word *effective* mentioned above means all $p, q \geq 0$.

2. Abelian variety

2.1. **Basic notations and facts.**

Definition 2.1. An *abelian variety* is a proper variety with a group structure.

It turns out that abelian variety is commutative and projective. Let A be an abelian variety of dimension g, one has

(1) <u>Basic operations on A</u>
  (1a) additivity (with $0 \in A$):

$$\mu : A \times A \to A$$

  (1b) multiplication by $N \in \mathbb{Z}$:

$$N : A \longrightarrow A$$
$$a \mapsto Na := \underbrace{a + \cdots + a}_{n \text{ items}}$$

  and in particular $N^*|_{H^k(A,\mathbb{Z})} = N^k \mathrm{id}_{H^k(A,\mathbb{Z})}$.

(1c) translation by $a \in A$:

$$\begin{aligned} t_a : A &\to A \\ x &\mapsto x + a \end{aligned}$$

(2) Dual abelian variety

$$\widehat{A} := \mathrm{Pic}^0(A) \text{ and } \widehat{\widehat{A}} \cong A$$

(3) Polarization: let L be an ample line bundle on A, then we obtain a finite and surjective morphism

$$\begin{aligned} \Phi_L : A &\to \widehat{A} = \mathrm{Pic}^0(A) \\ a &\mapsto t_a^*(L) \otimes L^{-1} \end{aligned}$$

Definition 2.2. One define the *polarization degree* of $A = \deg \Phi_L$. If

$$\deg \Phi_L = 1$$

then (A, L) is called a *principally polarized abelian variety* (p.p.a.v).

(4) Hodge structure on $H^k(A, \mathbb{Z})$

$$H^k(A, \mathbb{Z}) = \wedge^k H^1(A, \mathbb{Z})$$

this implies Hodge diamond of A is nonzero everywhere.

2.2. **Chow ring $\mathrm{CH}^*(A)_{\mathbb{Q}}$ of abelian variety.**

Theorem 2.3 (Beauville)**.**

$$\mathrm{CH}^k(A)_{\mathbb{Q}} = \bigoplus_{i=k-g}^{k} \mathrm{CH}^k_{(i)}(A)_{\mathbb{Q}}$$

where

$$\mathrm{CH}^k_{(i)}(A)_{\mathbb{Q}} := \{\alpha \in \mathrm{CH}^k(A)_{\mathbb{Q}} : N^*(\alpha) = N^{2k-i}(\alpha) \textit{ for all } N \in \mathbb{Z}\}$$

Remark 2.4. Theorem 2.3 gives an candidate filtration for B–S conjecture

$$F_i = \bigoplus_{j \geq i} \mathrm{CH}^k_{(j)}(A)_{\mathbb{Q}}$$

Proof. (Use Fourier transformation) Let

$$\begin{tikzcd} & P \arrow[d] & \\ A & A \times \widehat{A} \arrow[l, "p_A"'] \arrow[r, "p_{\widehat{A}}"] \arrow[d, equal] & \widehat{A} \\ & \mathrm{Pic}^0(\widehat{A}) \times \mathrm{Pic}^0(A) & \end{tikzcd}$$

be the Poincaré line bundle over $A \times \widehat{A}$, i.e., P is an universal object such that

$$P|_{A \times a} = \text{line bundle corresponding to } a \in \widehat{A} = \mathrm{Pic}^0(A)$$

$$P|_{b \times \widehat{A}} = \text{line bundle corresponding to } b \in A = \mathrm{Pic}^0(\widehat{A})$$

then we have a group isomorphism

$$\begin{aligned} \mathcal{F} : \mathrm{CH}^*(A)_{\mathbb{Q}} &\xrightarrow{\sim} \mathrm{CH}^*(\widehat{A})_{\mathbb{Q}} \\ \alpha &\mapsto (p_{\widehat{A}})_*[p_A^*(\alpha) \cdot \mathrm{ch}(P)] \end{aligned}$$

Then the conclusion follows from the following properties of $\mathcal{F}$.

Proposition 2.5. *Here are some properties about $\mathcal{F}$.*

(1) $\mathcal{F} \circ N^* = \widehat{N}_* \circ \mathcal{F}$.
(2) $\mathcal{F}(\mathrm{CH}^k_{(i)}(A)_{\mathbb{Q}}) = \mathrm{CH}^{g-k+i}_{(i)}(\widehat{A})_{\mathbb{Q}}$.
(3) $\widehat{\mathcal{F}} \circ \mathcal{F} = (-1)^g(-1)^*$.
(4) (Pontryagin product) *Consider the Pontryagin product*

$$\mathrm{CH}^*(A) \times \mathrm{CH}^*(A) \xrightarrow{*} \mathrm{CH}^*(A)$$
$$(\alpha, \beta) \mapsto \mu_*(\alpha \times \beta)$$

then

$$\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$$

This also explains why the notion 'Fourier transformation'.

□

Example 2.6 (dim = 5)**.** We arrange $\mathrm{CH}^k_{(i)}(\mathbb{Q})$ in the following way.

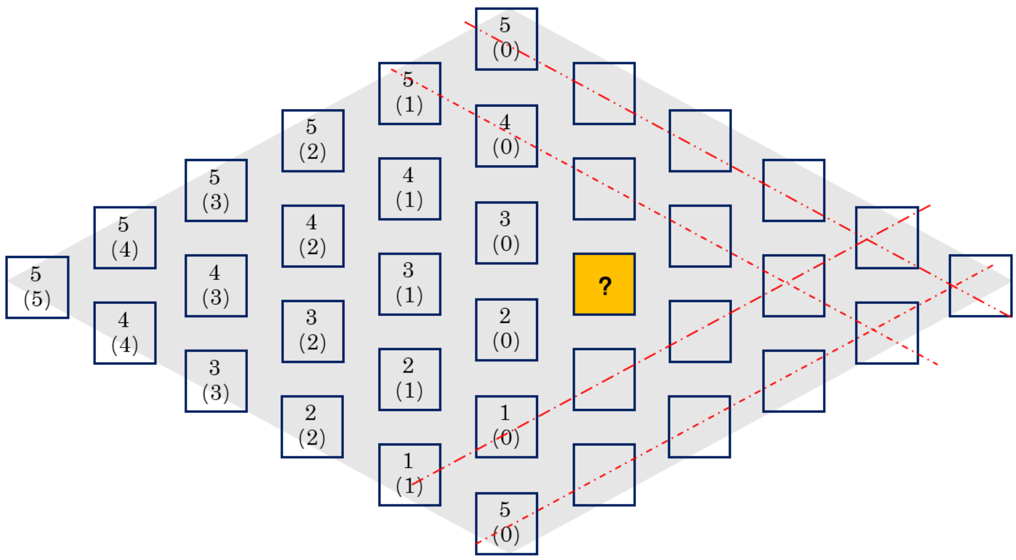

FIGURE 1. $\mathrm{CH}^k_{(i)}(\mathbb{Q})$ in dimension 5

The Fourier transformation $\mathcal{F}$ gives the symmetry from top to bottom. So the only unknown part in this diagram is

$$\mathrm{CH}^2_{(-1)}(A)_{\mathbb{Q}} = \{\alpha \in \mathrm{CH}^2(A)_{\mathbb{Q}} : N^*(\alpha) = N^5\alpha\}$$

which is illustrated in the figure with coloured orange box .

Conjecture 2.7 (Beauivalle)**.** *One has*

(1) $\mathrm{CH}^k_{(i)}(A)_{\mathbb{Q}} = 0$ *for any* $i < 0$.
(2) $\mathrm{cl} : \mathrm{CH}^k_{(0)}(A)_{\mathbb{Q}} \hookrightarrow H^{2k}(A, \mathbb{Q})$ *is injective.*

REFERENCES

[Gro69] A. Grothendieck. Hodge's general conjecture is false for trivial reasons. *Topology*, 8:299–303, 1969.

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