Characterizing open substacks of algebraic stacks that admit good moduli spaces

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Abstract

Given a moduli problem described by an algebraic stack, usually the algebraic stack itself does not admit a (proper) good moduli space, but some of its open substacks would do. In this thesis we provide two examples where all such open substacks can be characterized:

1. Moduli stack of rank 2 vector bundles over a curve.

2. Quotient stack by $\mathbb{G}_m$.

A major ingredient of the proof is the recent work of Jarod Alper, Daniel Halpern-Leistner and Jochen Heinloth on the existence of good moduli spaces for algebraic stacks.

Keywords: algebraic stack; good moduli space; vector bundle; quotient stack.

Zusammenfassung

Bei einem Moduliproblem, das durch einen algebraischen Stack beschrieben wird, ist es oft gar nicht so schwer zu zeigen, dass ein offener Unterstack des algebraischen Stacks einen guten Modulraum zulässt. In dieser Arbeit liefern wir zwei Beispiele, in denen alle offenen Unterstacks charakterisiert werden können:

1. Modulstacks von Rang 2-Vektorbündeln über einer Kurve.

2. Quotientenstacks durch $\mathbb{G}_m$.

Ein wesentlicher Bestandteil des Beweises ist die Arbeit von Jarod Alper, Daniel Halpern-Leistner und Jochen Heinloth über die Existenz guter Modulräume für algebraische Stacks.

Keywords: algebraischer Stack; guter Modulräum; Vektorbündel; Quotientenstack.

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– Voyez comme on danse, Jean d'Ormesson.

Per Aspera Ad Astra. Formally my Ph.D. career is coming to an end. As of today, I still feel that I am wandering outside the door of mathematics, having never really taken a step into it. But compared to three years ago, I have grown up a lot. To be frank, I am neither too ambitious nor fully confident about how far I can survive in academia; but I also understand that it is enough to leave at least one useful nail or one intact tile in the academic edifice along the way. Therefore we should not be overly fearful and anxious about the future, for its end is already lit by a mission that we are capable of and will be capable of.

I once saw a sentence in one of J.S. Milne's books when he mentioned a result from someone's thesis, he said this may come from his supervisor. I try to quote the exact words but unfortunately I couldn't find the reference. Please forgive me if my memory goes into the wrong direction. Now I am on a position to finish my thesis, I finally have a better understanding of J.S. Milne's comments. So first I would like to express my gratitude and appreciation to my supervisor Jochen Heinloth, whose endless patience and continuous guidance over the past three years has made this thesis smoothly come to you: a lot of his insights and ideas are condensed in this thesis (of course all mistakes are mine). I really like the two problems he proposed to me as my Ph.D. project, from which I have received a variety of training and I am truly grateful. I still hope that someday we could figure out a complete answer to each of them. Along the journey, he is always there to help and every single (even tiny) contribution from me is recognized by him in time. This gives me a lot of encouragement and also a feeling of working *with* him. As a mathematician, the way he understands and explains mathematics has a great impact on me. Apart from mathematics, he also sets an example of how to treat others with care, in many specific events. Hanc marginis exiguitas non caperet (thanks Pierre de Fermat). I could not have imagined having a better supervisor. By the way, he is probably the only person in our faculty who comes to work in a suit every day, which is quite impressive. Finally, an apology is to him for suffering through previous awful versions of this thesis.

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Table of Contents

Introduction

Chapter 1

Introduction

- Why do algebraic geometers love moduli spaces?
- It is just like with people, if you want to get to know someone, go to their family reunion.
– Angela Gibney

1.1 Guiding problem

For an algebraic stack arising from a moduli problem in algebraic geometry, usually one can find some open substack that admits a proper (sometimes even projective) good moduli space (in the sense of [Alp13, Definition 4.1]), via introducing suitable stability criteria with the help of educated-guessing. However, it would be also interesting to find all open substacks of the algebraic stack that admit proper good moduli spaces and then explore which objects are identified when passing to the corresponding good moduli spaces. In this thesis, we succeed in two examples.

1.1.1 Moduli stack of vector bundles over a curve

The first example is the moduli stack of vector bundles over a curve. Let C be a smooth projective connected curve over an algebraically closed field k of characteristic 0 and denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C. The guiding problem is:

Problem 1. Characterize open[1] substacks of $\mathscr{B}un_n^d$ that admit proper good moduli spaces.

Remark 1.1.1. Several candidates to this problem include:

- the open substack $\mathscr{B}un_n^{d,s} \subset \mathscr{B}un_n^d$ of stable vector bundles, which admits a quasi-projective good moduli space.

[1] There are many closed substacks of $\mathscr{B}un_n^d$ that admit even projective good moduli spaces, see [BPMN09] and also [Jac18, GBK].

- the open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ of semi-stable vector bundles, which admits a projective good moduli space.

- the open substack $\mathscr{B}un_n^{d,simple} \subset \mathscr{B}un_n^d$ of simple vector bundles, which admits a (in general non-separated) good moduli space.

In the first part of this thesis (i.e., Part I), we give a complete answer to Problem 1 in rank 2 case. The main result is

Theorem A. *Let C be a smooth projective connected curve of genus $g(C) > 1$ over an algebraically closed field k of characteristic 0. Denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C.*

1. *If $(2, d) = 1$, i.e., d is odd, then the open substack $\mathscr{B}un_2^{d,simple} \subset \mathscr{B}un_2^d$ of simple vector bundles is the unique maximal open substack that admits a good moduli space.*

   In other words, if $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack that admits a good moduli space, then $\mathscr{U} \subset \mathscr{B}un_2^{d,simple}$.

2. *If $(2, d) \neq 1$, i.e., d is even, then the open substack $\mathscr{B}un_2^{d,simple} \subset \mathscr{B}un_2^d$ of simple vector bundles and the open substack $\mathscr{B}un_2^{d,ss} \subset \mathscr{B}un_2^d$ of semi-stable vector bundles are the only maximal open substacks that admit good moduli spaces.*

   In other words, if $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack that admits a good moduli space, then $\mathscr{U} \subset \mathscr{B}un_2^{d,simple}$ or $\mathscr{U} \subset \mathscr{B}un_2^{d,ss}$.

3. *The open substack $\mathscr{B}un_2^{d,ss} \subset \mathscr{B}un_2^d$ of semi-stable vector bundles is the unique maximal open substack that admits a separated good moduli space.*

   In other words, if $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack that admits a separated good moduli space, then $\mathscr{U} \subset \mathscr{B}un_2^{d,ss}$.

Remark 1.1.2. In general, if $\mathscr{U} \subset \mathscr{B}un_n^d$ is an open substack that admits a (resp., separated) good moduli space U, then the base-change $\mathscr{U}^\circ := \mathscr{U} \times_U U^\circ \subset \mathscr{U}$, for any open subset $U^\circ \subset U$, is an open substack of $\mathscr{B}un_n^d$ that admits a (resp., separated) good moduli space. This gives rise to a lot of open substacks of $\mathscr{B}un_n^d$ that admit (resp., separated) good moduli spaces.

To give just one class of examples: for any pair (i, j) of non-negative integers such that $\max\{i(n-1)+j, i+j(n-1)\} < (n-1)(g-1)$, the substack $\mathscr{B}un_n^d(i, j) \subset \mathscr{B}un_n^d$ of (i, j)-stable vector bundles (see Definition 5.1.2 later) is open dense and satisfies

$$\mathscr{B}un_n^d(i,j) \subset \mathscr{B}un_n^{d,s} \subset \mathscr{B}un_n^{d,simple} \cap \mathscr{B}un_n^{d,ss}$$

Moreover, it admits a separated good moduli space by [MGN17, Theorem 2.5].

In higher rank case, we construct an explicit example indicating that Theorem A no longer holds. To be precise

Theorem B (Theorem 5.3.7)**.** *Let C be a smooth projective connected curve of genus $g(C) > 2$ over an algebraically closed field k of characteristic 0. Denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C.*

There exists an open dense substack $\mathscr{U} \subset \mathscr{B}un_3^2$ that admits a separated (but non-proper) good moduli space, and is not contained in the open substack $\mathscr{B}un_3^{2,ss} \subset \mathscr{B}un_3^2$ of semi-stable vector bundles.

This example can be easily generalized to arbitrary higher rank case, showing that in general separatedness alone cannot help us to identify the semi-stable locus.

Motivated by our results in rank 2 case and example in rank 3 case, we propose the following question.

Question 1.1.3. Is $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ the unique maximal open substack that admits a *proper* good moduli space?

Remark 1.1.4. Theorem A answers this question affirmatively in rank 2 case and Theorem B implies that the condition "proper" in this question cannot be weakened to "separated".

Question 1.1.3 is true if we replace "proper" by "quasi-projective". Indeed

Proposition 1.1.5 (Proposition 6.0.1)**.** *Let C be a smooth projective connected curve of genus $g(C) > 1$ over an algebraically closed field k of characteristic 0. Denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C.*

The open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ of semi-stable vector bundles is the unique maximal open substack that admits a quasi-projective good moduli space.

In addition, the open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ is also maximal in the following sense.

Proposition 1.1.6 (Proposition 6.0.4)**.** *Let C be a smooth projective connected curve of genus $g(C) > 1$ over an algebraically closed field k of characteristic 0. Denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C. Let $\mathscr{B}un_n^{d,ss} \to \mathcal{M}^{ss}(n,d)$ be the good moduli space of the open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ of semi-stable vector bundles.*

If $\mathscr{U} \subset \mathscr{B}un_n^d$ is an open substack containing $\mathscr{B}un_n^{d,ss}$ that admits a separated good moduli space $\mathcal{U}$, then there exists a surjection $\mathcal{M}^{ss}(n,d) \to \mathcal{U}$ such that under the good moduli space morphism $\mathscr{U} \to \mathcal{U}$ every unstable vector bundle is identified with a semi-stable vector bundle.

1.1.2 Quotient stack by $\mathbb{G}_m$

The second example is the quotient stack by $\mathbb{G}_m$. Let an e-dimensional torus $\mathbb{T} \cong \mathbb{G}_m^e$ act on a geometrically normal, proper and geometrically irreducible scheme X over a field k.

The guiding problem is to characterize open substacks of $[X/\mathbb{T}]$ that admit proper good moduli spaces, or equivalently

Problem 2 ([BBS87a], Problem)**.** Characterize $\mathbb{T}$-invariant open subsets $U \subset X$ such that the quotient stacks $[U/\mathbb{T}]$ admit proper good moduli spaces.

Remark 1.1.7. Previous work on Problem 2 consists of:

- If X is smooth, then as an application of geometric invariant theory (GIT) there exists an elegant description of the maximal (with respect to inclusion) open subsets with quasi-projective good moduli spaces (see [Mum65, Converse 1.13] and also [Alp13, Theorem 11.14 (ii)]): they are subsets of semi-stable points of X with respect to some $\mathbb{T}$-linearized line bundle.

- For $e = 1$. If $k = \mathbb{C}$, then a complete answer is provided in [BBS83] (if the open subsets have no fixed points) and complemented by [Gro84], using analytic methods. This is also proved for $k = \bar{k}$ in [BBS82] using algebraic methods.

- For $e = 2$. If $k = \mathbb{C}$ and X is a compact Kähler manifold, then a complete answer is provided in [BBS85] (if the open subsets have no fixed points), using analytic methods.

- For arbitrary e. If X is a projective space (or a product of them), then a complete answer is provided in [BB02a], see also [BBS96] for a slightly general result.

Building on these results, A. Białynicki-Birula and A. J. Sommese proposed the following so-called moment measure conjecture.

Conjecture 1.1.8 ([BBS87a], Conjecture 1.4)**.** *Let X be a geometrically normal, proper and geometrically irreducible scheme X over a field k with a $\mathbb{T}$-action. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\text{Moment measures on } X \\ \text{(see Definition 7.4.3).}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\mathbb{T}\text{-invariant open dense subsets } U \subset X \\ \text{such that the quotient stacks } [U/\mathbb{T}] \text{ admit} \\ \text{proper good moduli spaces.}\end{matrix}\right\}$$

Remark 1.1.9. If $k = \mathbb{C}$ and X is projective, then the direction from moment measures to open subsets in Conjecture 1.1.8 is proved in [BBS87b, Theorem 2.1] (see also [BBS87a, Theorem 1.3.1]). For more details on this topic see [BB02b, Chapter 11].

In the second part of this thesis (i.e., Part II), we provide a purely algebro-geometric approach to Problem 2 in the case $e = 1$, where moment measures are in one-to-one correspondence with semi-sections (see Example 7.4.5). The main result is

Theorem C (Theorem 9.0.1 and Theorem 10.0.1)**.** *Let X be a geometrically normal, proper and geometrically irreducible scheme over a field k with a $\mathbb{G}_m$-action. Suppose the fixed point locus $X^{\mathbb{G}_m}$ has r connected components. Then there is a one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Semi-sections of } \{1,\ldots,r\} \\ \textit{(see Definition 7.3.11).}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\mathbb{G}_m\textit{-invariant open dense subsets } U \subset X \\ \textit{such that the quotient stacks } [U/\mathbb{G}_m] \textit{ admit} \\ \textit{proper good moduli spaces.}\end{array}\right\}$$

1.2 Key ingredient

Both parts of this thesis can be seen as applications of the following criteria for an algebraic stack to admit a (separated or proper) good moduli space.

Theorem 0 ([AHLH18], Theorem A)**.** *Let $\mathscr{X}$ be an algebraic stack, locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S. Then $\mathscr{X}$ admits a good moduli space X if and only if*

1. *$\mathscr{X}$ is locally linearly reductive (see Definition 2.1.1).*

2. *$\mathscr{X}$ is Θ-reductive (see Definition 2.2.1).*

3. *$\mathscr{X}$ has unpunctured inertia (see Definition 2.3.1).*

The good moduli space X is separated if and only if $\mathscr{X}$ is S-complete (see Definition 2.4.1), and proper if and only if $\mathscr{X}$ is S-complete and satisfies the existence part of the valuative criterion for properness.

Assume in addition that S is of characteristic 0 and $\mathscr{X}$ is quasi-compact. If $\mathscr{X}$ is S-complete, then (1) and (3) hold automatically. In particular, $\mathscr{X}$ admits a separated good moduli space if and only if $\mathscr{X}$ is Θ-reductive and S-complete.

This theorem reduces the existence of the good moduli space to checking several local conditions of the algebraic stack in question. A successful application of Theorem 0 requires effective translations, in various specific settings, of the (slightly) abstract notions appearing therein. In this thesis, we achieve this in the aforementioned two examples.

1.3 Leitfaden

The structure of the thesis is as follows. In Chapter §2, following [AHLH18], we give a detailed account of the conditions in Theorem 0 and this will be used frequently throughout the thesis. No originality is claimed is this chapter, except for Theorem 2.4.4, asserting that S-completeness of certain algebraic stacks can be checked on their dense substacks.

1.3.1 Part I

In Part I, we present our results on the moduli stack of vector bundles over a curve.

- In Chapter §3, we provide a first translation of the conditions in Theorem 0 for open substacks of $\mathscr{B}un_n^d$.

  The only satisfactory part so far is S-completeness (see Proposition 3.3.17). Along the way, we give a quick proof that $\mathscr{B}un_n^{d,ss}$ admits a proper good moduli space (see Proposition 3.4.1).

- In Chapter §4, we provide more specific consequences of the conditions in Theorem 0 in rank 2 case, which are sufficient to prove Theorem A.

  Let us briefly explain why Theorem A holds: let $\mathscr{U} \subset \mathscr{B}un_2^d$ be an open substack, then

  – If $\mathscr{U}$ is locally linearly reductive and Θ-reductive, then it is contained in $\mathscr{B}un_2^{d,ss} \cup \mathscr{B}un_2^{d,simple}$ (see Lemma 4.4.1).

  – If $\mathscr{U} \not\subseteq \mathscr{B}un_2^{d,ss}$ and $\mathscr{U} \not\subseteq \mathscr{B}un_2^{d,simple}$, then it supports a degeneration of a strictly semi-stable vector bundle to an unstable vector bundle, which is not allowed due to Θ-reductivity (see Lemma 3.2.17).

  i.e., to obtain a good moduli space, $\mathscr{U} \subset \mathscr{B}un_2^{d,ss}$ or $\mathscr{U} \subset \mathscr{B}un_2^{d,simple}$. Thus Theorem A (1) and (2) follow. Moreover

  – If $\mathscr{U}$ is Θ-reductive, then it cannot contain *any* direct sum of line bundles of different degrees (see Proposition 4.2.1).

  – If $\mathscr{U}$ is S-complete, then it contains *some* direct sum of line bundles of different degrees, provided it contains an unstable vector bundle (see Proposition 4.3.1).

  i.e., to obtain a separated good moduli space, $\mathscr{U}$ cannot contain any unstable vector bundles. Thus Theorem A (3) follows.

- In Chapter §5, we construct the promised example in rank 3 case (see Theorem 5.3.7).

- In Chapter §6, we prove several results concerning the maximality of the open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ in certain sense.

1.3.2 Part II

In Part II, we present our results on the quotient stack by $\mathbb{G}_m$.

- In Chapter §7, we collect some materials on the geometry of $\mathbb{G}_m$-schemes.

  The key notions are:

  - Smoothable maximal chain of orbits (see Definition 7.2.1 and 7.2.2), which is a test object for separatedness or properness of good moduli spaces whenever exist (see Theorem 8.3.7).
  - Semi-section (see Definition 7.3.11), which is a one-dimensional reformulation of the moment measure appearing in Conjecture 1.1.8 (see Example 7.4.5).

- In Chapter §8, we provide two sets of characterization when a good moduli space is separated or proper.

  - The first set of characterization comes from Theorem 0. This is summarized in Theorem 8.3.7, which is motivated by [BBS83, Lemma 1.2] and [Gro84, Lemma 2.7].

    By translating the conditions in Theorem 0 into geometric ones on open subsets in question, we prove, among other things, that for a $\mathbb{G}_m$-invariant open subset to have a separated (resp., proper) good moduli space, it is sufficient and necessary that this open subset intersects every smoothable maximal chain of orbits in at most (resp., exactly) one (partial) orbit.

  - The second set of characterization is a purely topological one. This is summarized in Proposition 8.4.1, which is motivated by [BBS85, Theorem 1.3].

    We prove that, for a separated good moduli space of a $\mathbb{G}_m$-invariant open subset to be proper, it is sufficient and necessary that the complement of this open subset has two connected components. This turns out to be a quite useful tool to read off semi-section from the open subset.

- In Chapter §9 and §10, we establish the injectivity and surjectivity part of Theorem C respectively, using the two sets of characterization established in Chapter §8.

- In Chapter §11, we prove, as an application of Theorem C, that over a perfect field any geometrically normal and projective $\mathbb{G}_m$-scheme can be covered by $\mathbb{G}_m$-invariant open subsets with proper good moduli spaces (see Proposition 11.0.1).

  This can be compared with our result on the rank 2 vector bundles over a curve (see Lemma 4.4.3), where only semi-stable vector bundles could admit such open neighbourhoods.

Chapter 2

Existence of good moduli spaces for algebraic stacks

In this chapter, we give a brief collection of the conditions in Theorem 0 together with examples and non-examples.

Throughout we will fix a base S that will be a quasi-separated algebraic space, but of course the most interesting case will be when $S = \mathrm{Spec}(k)$ is the spectrum of a field.

2.1 Local linear reductivity

Definition 2.1.1 ([AHLH18], Definition 2.1)**.** An algebraic stack $\mathscr{X}$ over S with affine stabilizers is *locally linearly reductive* if every point specializes to a closed point and every closed point of $\mathscr{X}$ has a linearly reductive automorphism group.

Example 2.1.2 ([AHLH18], Paragraph after Definition 2.1)**.** For a quasi-compact[1] quotient stack $[X/G]$, its closed points correspond to the closed G-orbits on X. Then $[X/G]$ is locally linearly reductive if and only if the points of X with closed G-orbits have linearly reductive stabilizers.

2.2 Θ-reductivity

Denote by

- $\mathrm{B}\mathbb{G}_m := [0/\mathbb{G}_m]$ the classifying stack of the multiplicative group $\mathbb{G}_m$.

  A vector bundle over $\mathrm{B}\mathbb{G}_m$ (i.e., a morphism $\mathrm{B}\mathbb{G}_m \to \mathrm{BGL}$) is the same as a vector space equipped with a $\mathbb{G}_m$-action. As the $\mathbb{G}_m$-action on a vector space is the same as a grading, it is the same as a vector space equipped with a grading. Therefore we often think of a morphism $\mathrm{B}\mathbb{G}_m \to \mathscr{X}$ as a point of $\mathscr{X}$ equipped with a grading.

[1] Recall that in a quasi-compact scheme every point has a closed point in its closure, i.e., every point specializes to a closed points.

- $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ the quotient stack defined by the standard contracting action of the multiplicative group $\mathbb{G}_m$ on the affine line $\mathbb{A}^1$.

  It has two geometric points $0, 1$ which are the images of the points of the same name in $\mathbb{A}^1$, one corresponding to the closed $\mathbb{G}_m$-orbit of 0 in $\mathbb{A}^1$ and the other corresponding to the open $\mathbb{G}_m$-orbit of 1 in $\mathbb{A}^1$. In particular, Θ has a unique closed point $\mathrm{B}\mathbb{G}_m = [0/\mathbb{G}_m]$.

  A vector bundle over Θ (i.e., a morphism $\Theta \to \mathrm{BGL}$) is the same as a $\mathbb{G}_m$-invariant vector bundle over $\mathbb{A}^1$. As the vector bundles over $\mathbb{A}^1$ are all trivial, it is the same as a vector space equipped with a filtration. Therefore we often think of a morphism $f : \Theta \to \mathscr{X}$ as a point $f(1) \in \mathscr{X}$ equipped with a filtration which has $f(0)$ as the associated graded object.

Both stacks are defined over $\mathrm{Spec}(\mathbb{Z})$ and therefore pull back to any base S.

For any DVR R with fraction field K and residue field κ, let $\Theta_R := \Theta \times \mathrm{Spec}(R)$, then $0 := [0/\mathbb{G}_m] \times \mathrm{Spec}(\kappa) \in \Theta \times \mathrm{Spec}(R) = \Theta_R$ is its unique closed point.

Definition 2.2.1 ([AHLH18], Definition 3.10)**.** A locally noetherian algebraic stack $\mathscr{X}$ over S is Θ-*reductive* if for every DVR R, any commutative diagram

$$\begin{tikzcd} \Theta_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & S \end{tikzcd}$$

of solid arrows can be uniquely filled in.

Remark 2.2.2. Observe that $\Theta_R - \{0\}$ is covered by the two open substacks

$$[\mathbb{A}^1 - \{0\}/\mathbb{G}_m] \times \mathrm{Spec}(R) \cong \mathrm{Spec}(R) \text{ and } [\mathbb{A}^1/\mathbb{G}_m] \times \mathrm{Spec}(K) = \Theta_K$$

glued along $\mathrm{Spec}(K)$. Therefore a morphism $\Theta_R - \{0\} \to \mathscr{X}$ is the data of morphisms $\mathrm{Spec}(R) \to \mathscr{X}$ and $\Theta_K \to \mathscr{X}$ together with an isomorphism of their restrictions to $\mathrm{Spec}(K)$.

Remark 2.2.3. Suppose S is noetherian. If $\mathscr{X}$ is an algebraic stack, locally of finite type and quasi-separated over S, with affine stabilizers, then $\mathscr{X}$ is Θ-reductive if and only if the evaluation morphism $\mathrm{ev}_1 : \underline{\mathrm{Map}}_S(\Theta, \mathscr{X}) \to \mathscr{X}$ satisfies the valuative criterion for properness, i.e., for every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & \underline{\mathrm{Map}}_S(\Theta, \mathscr{X}) \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathscr{X} \end{tikzcd}$$

of solid arrows can be uniquely filled in (see [AHLH18, Remark 3.11 and 3.12]).

Moreover, if $\mathscr{X}$ is a quotient stack, then we have an explicit description of the mapping stack $\underline{\mathrm{Map}}_S(\Theta, \mathscr{X})$ (see [HL14, Theorem 1.37]) and this commutative diagram can be therefore simplified into a commutative diagram in the category of algebraic spaces (see [AHLH18, Proposition 3.13]).

Example 2.2.4. If X is an affine scheme of finite type over a field k and G is a reductive group over k, then the quotient stack $[X/G]$ is Θ-reductive (see [AHLH18, Example 3.15]). However, the quotient stack $[\mathbb{P}^1/\mathbb{G}_m]$ is not Θ-reductive since the evaluation morphism $\mathrm{ev}_1 : \underline{\mathrm{Map}}_k(\Theta, [\mathbb{P}^1/\mathbb{G}_m]) \to [\mathbb{P}^1/\mathbb{G}_m]$ doesn't satisfy the valuative criterion for properness as

$$\underline{\mathrm{Map}}_k(\Theta, [\mathbb{P}^1/\mathbb{G}_m]) = [\mathbb{P}^1/(\mathbb{G}_m, \lambda_0)] \sqcup \bigsqcup_{n\in\mathbb{Z}-\{0\}} [\mathbb{P}^1 - \{0\}/(\mathbb{G}_m, \lambda_n)] \sqcup [\mathbb{P}^1 - \{\infty\}/(\mathbb{G}_m, \lambda_n)].$$

where the $(\mathbb{G}_m, \lambda_n)$-action is given via the cocharacter $\lambda_n \in X_*(\mathbb{G}_m) \cong \mathbb{Z}$ corresponding the integer n.

2.3 Unpunctured inertia

Definition 2.3.1 ([AHLH18], Definition 3.53)**.** A noetherian algebraic stack $\mathscr{X}$ over S has *unpunctured inertia* if for any closed point $x \in |\mathscr{X}|$ and versal deformation $p : (U, u) \to (\mathscr{X}, x)$, where U is the spectrum of a local ring with closed point u, each connected component of the inertia group scheme $\mathrm{Aut}_{\mathscr{X}}(p) \to U$ has non-empty intersection with the fiber over u.

Example 2.3.2. Suppose S is separated. If X is an affine scheme of finite type over S and G is a geometrically reductive group scheme over S, then the quotient stack $[X/G]$ has unpunctured inertia (see [AHLH18, Proposition 5.7 and Theorem 5.2]). Moreover, if $\mathscr{X}$ is a noetherian algebraic stack with connected stabilizer groups, then $\mathscr{X}$ has unpunctured inertia (see [AHLH18, Proposition 3.56]).

2.4 S-completeness

For any DVR R with fraction field K, residue field κ and uniformizer $\pi \in R$, as in [Hei17, 2.B], the "separatedness test scheme" is defined as the quotient stack

$$\overline{\mathrm{ST}}_R := [\mathrm{Spec}(R[x,y]/xy - \pi)/\mathbb{G}_m]$$

where x, y have $\mathbb{G}_m$-weights $1, -1$ respectively. This can be viewed as a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$, where $\mathbb{A}^2$ has coordinates x, y with $\mathbb{G}_m$-weights $1, -1$ respectively. To be precise, $\overline{\mathrm{ST}}_R$ is the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1 = \mathrm{Spec}(k[xy])$ along the morphism $\mathrm{Spec}(R) \to \mathbb{A}^1 = \mathrm{Spec}(k[xy])$ corresponding to $xy \to \pi$, i.e., we have the following commutative diagram with Cartesian squares

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Spec}(K) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spec}(R[x,y]/xy-\pi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^2 = \mathrm{Spec}(k[x,y]) \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
\mathrm{Spec}(K) \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathrm{ST}}_R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[d, "\mathrm{gms}"] \\
\mathrm{Spec}(K) \arrow[r, hook, "\mathrm{open}"'] & \mathrm{Spec}(R) \arrow[r, "xy\to\pi"'] & \mathbb{A}^1 = \mathrm{Spec}(k[xy])
\end{tikzcd} \tag{2.4.1}$$

Note that the open locus where $x \neq 0$ (resp., $y \neq 0$) in $[\mathbb{A}^2/\mathbb{G}_m]$ is isomorphic to $\mathbb{A}^1$:

$$[\mathrm{Spec}(k[x,y]_x)/\mathbb{G}_m] \cong \mathrm{Spec}(k[xy]) \cong \mathbb{A}^1 =: \mathbb{A}^1_x$$
$$\Big(\text{resp., } [\mathrm{Spec}(k[x,y]_y)/\mathbb{G}_m] \cong \mathrm{Spec}(k[xy]) \cong \mathbb{A}^1 =: \mathbb{A}^1_y\Big)\,.$$

Let $0 := \mathrm{B}\mathbb{G}_{m,\kappa} = [\mathrm{Spec}(\kappa)/\mathbb{G}_m] \in \overline{\mathrm{ST}}_R$ be the unique closed point defined by the vanishing of x and y. Similarly the locus where $x \neq 0$ (resp., $y \neq 0$) in $\overline{\mathrm{ST}}_R$ is isomorphic to $\mathrm{Spec}(R)$:

$$[\mathrm{Spec}(R[x,y]_x/(y-\pi/x))/\mathbb{G}_m] \cong [\mathrm{Spec}(R[x]_x)/\mathbb{G}_m] \cong \mathrm{Spec}(R) =: \mathrm{Spec}(R_x)$$
$$(\text{resp., } [\mathrm{Spec}(R[x,y]_y/(x-\pi/y))/\mathbb{G}_m] \cong [\mathrm{Spec}(R[y]_y)/\mathbb{G}_m] \cong \mathrm{Spec}(R) =: \mathrm{Spec}(R_y))\,.$$

Definition 2.4.1 ([AHLH18], Definition 3.37)**.** A locally noetherian algebraic stack $\mathscr{X}$ over S is *S-complete* if for every DVR R, any commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\
\overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists !"'] & S
\end{tikzcd}$$

of solid arrows can be uniquely filled in.

Remark 2.4.2. Observe that $\overline{\mathrm{ST}}_R - \{0\}$ is covered by the two open substacks

$$(\overline{\mathrm{ST}}_R - \{0\})|_{x\neq 0} \cong \mathrm{Spec}(R) \text{ and } (\overline{\mathrm{ST}}_R - \{0\})|_{y\neq 0} \cong \mathrm{Spec}(R)$$

glued along $\mathrm{Spec}(K)$, i.e.,

$$\overline{\mathrm{ST}}_R - \{0\} \cong \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R)$$

is the non-separated union. Therefore a morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$ is the data of two morphisms $\mathrm{Spec}(R) \to \mathscr{X}$ together with an isomorphism of their restrictions to $\mathrm{Spec}(K)$.

Example 2.4.3. Suppose S is a locally noetherian scheme. If X is a locally noetherian affine scheme over S and G is a geometrically reductive group scheme over S, then the quotient stack $[X/G]$ is S-complete (see [AHLH18, Proposition 3.42 (2)]).

If G is an algebraic group over a field k, then $\mathrm{B}G$ is S-complete if and only G is geometrically reductive (see [AHLH18, Proposition 3.45]).

2.4.1 S-completeness can be checked on dense substacks

In this subsection, we prove that S-completeness of an algebraic stack, in several cases, can be checked on its dense substacks. Since S-completeness of an algebraic stack is closely related to separateness of its good moduli space (if exists) by [AHLH18, Proposition 3.47 (2)], this result may be compared with [Sta21, Tag 0CM4 and Tag 0CM5].

Theorem 2.4.4 (S-completeness can be checked on dense substacks)**.** *Let $\mathscr{X}$ be an algebraic stack of finite type over a field k and $\mathscr{Y} \subset \mathscr{X}$ be a dense substack. Suppose one of the following holds:*

(A) $\mathscr{X}$ admits a good moduli space.

(B) $\mathscr{X}$ is a quotient stack by $\mathbb{G}_m$.

Then the following are equivalent:

1. *$\mathscr{X}$ is S-complete, i.e., for every DVR R, any commutative diagram*

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd} \tag{2.4.2}$$

   of solid arrows can be uniquely filled in.

2. *$\mathscr{X}$ is S-complete relative to $\mathscr{Y}$, i.e., for every DVR R with fraction field K, any commutative diagram*

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

   of solid arrows such that $\mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$ factors through $\mathscr{Y} \subset \mathscr{X}$ can be uniquely filled in.

In both cases only the implication (2) $\Rightarrow$ (1) requires a proof.

Proof of Theorem 2.4.4: Case A

In this case we could prove the following slightly more general statement:

Proposition 2.4.5. *Let $\mathscr{X}$ be an algebraic stack of finite type over a field k and $h : \mathscr{Y} \to \mathscr{X}$ be a morphism of algebraic stacks, of finite type and with dense image. If $\mathscr{X}$ admits a good moduli space, then the following are equivalent:*

1. *$\mathscr{X}$ is S-complete, i.e., for every DVR R, any commutative diagram*

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows can be uniquely filled in.

2. *$\mathscr{X}$ is S-complete relative to $\mathscr{Y}$, i.e., for every DVR R with fraction field K, any commutative diagram*

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows such that $\mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$ factors through $h : \mathscr{Y} \to \mathscr{X}$ can be uniquely filled in.

Proof. Let $\mathscr{X} \to X$ be its good moduli space. Take a dense subset $Y \subset X$ contained in the image of $\mathscr{Y}$ under the composition $\mathscr{Y} \xrightarrow{h} \mathscr{X} \xrightarrow{\text{gms}} X$ and this can be done since both morphisms are dominant. Then the following assertions are equivalent:

(a) The algebraic stack $\mathscr{X}$ is S-complete.

(b) The good moduli space X of $\mathscr{X}$ is separated.

(c) For every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & Y \arrow[r, hook, "\text{dense}"] & X \arrow[d] \\ \mathrm{Spec}(R) \arrow[rr] \arrow[urr, dashed] & & \mathrm{Spec}(k) \end{tikzcd} \tag{2.4.3}$$

of solid arrows admits at most one dotted arrow filling in.

Indeed, (a) $\Leftrightarrow$ (b) is established in [AHLH18, Proposition 3.47 (2)] and (b) $\Leftrightarrow$ (c) is established in [Sta21, Tag 0CM4]. Thus it suffices to show (2) $\Rightarrow$ (c). The argument is taken from [AHLH18, Proof of Proposition 3.47 (2)]. Given a commutative diagram (2.4.3) of solid arrows. Suppose there exist two dotted arrows filling in

$$f_1, f_2 : \mathrm{Spec}(R) \to X \text{ such that } u := (f_1)_K = (f_2)_K : \mathrm{Spec}(K) \to Y,$$

then we need to show $f_1 = f_2$. Note that this equality can be checked up to any finite extension of DVR's.

Since $\mathscr{X} \to X$ is universally closed (see [Alp13, Theorem 4.16 (2)]), up to a finite extension of DVR's we may choose liftings (by [Sta21, Tag 0CLW])

- $\tilde{u} : \mathrm{Spec}(K) \to \mathscr{Y}$ of u.
- $\tilde{f}_1, \tilde{f}_2 : \mathrm{Spec}(R) \to \mathscr{X}$ of f_1, f_2 respectively such that $(\tilde{f}_1)_K = (\tilde{f}_2)_K = \tilde{u}$.

Therefore $\tilde{f}_1 \cup \tilde{f}_2$ defines a commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r, "\tilde{f}_1 \cup \tilde{f}_2"] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd} \tag{2.4.4}$$

of solid arrows and $(\tilde{f}_1 \cup \tilde{f}_2)_K = \tilde{u} : \mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$ factors through $h : \mathscr{Y} \to \mathscr{X}$. Then by (2) there exists a unique dotted arrow $\overline{\mathrm{ST}}_R \to \mathscr{X}$ filling in (2.4.4). As $\overline{\mathrm{ST}}_R \to \mathrm{Spec}(R)$ is the good moduli space and hence universal for maps to algebraic spaces (see [Alp13, Theorem 6.6]), the morphism $\overline{\mathrm{ST}}_R \to \mathscr{X}$ descends to a unique morphism $\mathrm{Spec}(R) \to X$ which is necessarily equal to both f_1 and f_2. $\square$

Proof of Theorem 2.4.4: Case B

Suppose that $\mathscr{X} = [X/\mathbb{G}_m]$ is a quotient stack. In this case $\mathscr{Y} \cong [Y/\mathbb{G}_m]$ for some $\mathbb{G}_m$-invariant dense subset $Y \subset X$. Hereafter, without loss of generality we may (and will) assume that X is irreducible since the lifting property in (2.4.2) actually happens in one of its irreducible components and $Y \subset X$ is open dense.

The idea to prove Theorem 2.4.4 in this case is to reformulate S-completeness as a lifting property and then use [Sta21, Tag 0CM4 and Tag 0CM5] to conclude that it can checked on dense substacks. This is done by a series of reformulations presented in Lemma 2.4.6, 2.4.9 and 2.4.12 respectively.

- 1^{st} reformulation

Denote by $\underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$ the mapping stack over $\mathbb{A}^1 = \mathrm{Spec}(k[xy])$ such that for any scheme $T \to \mathbb{A}^1$,

$$\underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})(T) := \mathrm{Maps}([\mathbb{A}^2/\mathbb{G}_m] \times_{\mathbb{A}^1} T, \mathscr{X}).$$

This mapping stack is an algebraic stack, locally of finite type over $\mathbb{A}^1$, with quasi-separated diagonal (see [AHR20, Theorem 5.10]). The good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1 = \operatorname{Spec}(k[xy])$ has two sections s_x, s_y

$$\begin{tikzcd} & & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[d, "\mathrm{gms}"] & & \\ {[\operatorname{Spec}(k[x,y]_x)/\mathbb{G}_m] \cong \mathbb{A}^1_x} \arrow[urr, "s_x"] & & \mathbb{A}^1 \arrow[ll, "\sim"'] \arrow[rr, "\sim"] & & \mathbb{A}^1_y \cong [\operatorname{Spec}(k[x,y]_y)/\mathbb{G}_m] \arrow[ull, "s_y"'] \end{tikzcd}$$

corresponding to two $\mathbb{G}_m$-equivariant morphisms

$$s_x : k[x,y] \to k[x,y]_x \text{ mapping } x \mapsto x, y \mapsto y$$
$$s_y : k[x,y] \to k[x,y]_y \text{ mapping } x \mapsto x, y \mapsto y$$

and this equips $\underline{\operatorname{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$ with a morphism

$$p = (s_x, s_y) : \underline{\operatorname{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X}) \to (\mathscr{X} \times \mathbb{A}^1) \times_{\mathbb{A}^1} (\mathscr{X} \times \mathbb{A}^1) \cong \mathbb{A}^1 \times \mathscr{X} \times \mathscr{X}.$$

The key observation is:

Lemma 2.4.6. *The quotient stack $\mathscr{X} = [X/\mathbb{G}_m]$ is S-complete if and only if the following holds: For every DVR R over k with fraction field K and uniformizer π, any commutative diagram*

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "v"] \arrow[d, hook] & \underline{\operatorname{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X}) \arrow[d, "p"] \\ \operatorname{Spec}(R) \arrow[ur, dashed, "\exists !"'] \arrow[r, "(xy \mapsto \pi, u_1, u_2)"'] & \mathbb{A}^1 \times \mathscr{X} \times \mathscr{X} \end{tikzcd} \tag{2.4.5}$$

of solid arrows can be uniquely filled in.

Proof. Only If Part: Given a commutative diagram (2.4.5) of solid arrows, i.e., morphisms

$$u_1 : \operatorname{Spec}(R) \to \mathscr{X} \text{ and } u_2 : \operatorname{Spec}(R) \to \mathscr{X},$$
$$v : [\mathbb{A}^2/\mathbb{G}_m] \times_{\mathbb{A}^1} \operatorname{Spec}(K) \cong \operatorname{Spec}(K) \to \mathscr{X} \text{ by (2.4.1)}$$

such that $(u_1)_K \cong (u_2)_K \cong v : \operatorname{Spec}(K) \to \mathscr{X}$, we obtain a morphism

$$u_1 \cup u_2 : \operatorname{Spec}(R) \bigcup_{\operatorname{Spec}(K)} \operatorname{Spec}(R) \cong \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}.$$

By assumption there exists a unique dotted arrow $u : \overline{\mathrm{ST}}_R \to \mathscr{X}$ filling in (2.4.2). This defines a dotted arrow $\operatorname{Spec}(R) \to \underline{\operatorname{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$ filling in (2.4.5), given by

$$[\mathbb{A}^2/\mathbb{G}_m] \times_{\mathbb{A}^1} \operatorname{Spec}(R) \cong \overline{\mathrm{ST}}_R \xrightarrow{u} \mathscr{X} \text{ by (2.4.1)}.$$

IF PART: Given a commutative diagram (2.4.2) of solid arrows, i.e., a morphism $u : \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$, we obtain a commutative diagram (2.4.5) of solid arrows by setting

$$u_1 : \mathrm{Spec}(R) \xrightarrow{u|_{x\neq 0}} \mathscr{X} \text{ and } u_2 : \mathrm{Spec}(R) \xrightarrow{u|_{y\neq 0}} \mathscr{X},$$
$$v : [\mathbb{A}^2/\mathbb{G}_m] \times_{\mathbb{A}^1} \mathrm{Spec}(K) \cong \mathrm{Spec}(K) \xrightarrow{u|_{\mathrm{Spec}(K)}} \mathscr{X} \text{ by (2.4.1)}.$$

By assumption there exists a unique dotted arrow $\tilde{u} : \mathrm{Spec}(R) \to \underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$ filling in (2.4.5), i.e., a morphism $\tilde{u} : [\mathbb{A}^2/\mathbb{G}_m] \times_{\mathbb{A}^1} \mathrm{Spec}(R) \to \mathscr{X}$. This defines a dotted arrow $\overline{\mathrm{ST}}_R \to \mathscr{X}$ filling in (2.4.2), given by

$$\overline{\mathrm{ST}}_R \cong [\mathbb{A}^2/\mathbb{G}_m] \times_{\mathbb{A}^1} \mathrm{Spec}(R) \xrightarrow{\tilde{u}} \mathscr{X} \text{ by (2.4.1)}.$$

$\square$

In spirit of [Sta21, Tag 0CM5], we expect that the lifting property in (2.4.5) is equivalent to the one in the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "v"] \arrow[d, hook] & \underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{Y}) \arrow[r] & \underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X}) \arrow[d, "p"] \\
\mathrm{Spec}(R) \arrow[urr, dashed, "\exists!"] \arrow[rr, "(xy\mapsto\pi, u_1, u_2)"'] & & \mathbb{A}^1 \times \mathscr{X} \times \mathscr{X}
\end{tikzcd}$$

and this will conclude the proof. The problem is that usually the morphism

$$\underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{Y}) \to \underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$$

does not have dense image. This forces us to cook up a further reformulation.

- 2^{nd} reformulation

The mapping stack $\underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$ turns out to be too large to deal with so we would like to find something smaller where the lifting property in (2.4.5) actually happens. Note that any morphism $[\mathbb{A}^2/\mathbb{G}_m] \to \mathscr{X}$ produces a commutative diagram with Cartesian squares

$$\begin{tikzcd}
\mathbb{A}^2 \times \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
\mathbb{A}^2 \arrow[r, "\mathbb{G}_m\text{-tor}"'] & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[r] & \mathscr{X}
\end{tikzcd} \tag{2.4.6}$$

where $\mathbb{A}^2 \times \mathbb{G}_m$ has coordinates x, y and t with $\mathbb{G}_m$-weights $1, -1$ and d respectively, for some integer d. Since X is a scheme, the morphism $[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m] \to X$ factors through the good moduli space of $[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]$, which is computed as follows:

1. If $d \geq 0$, then the good moduli space of $[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]$ is given by

$$\mathrm{Spec}(k[x,y,t^{\pm}]^{\mathbb{G}_m}) = \mathrm{Spec}(k[xy, x^d t^{-1}, y^d t]) =: \mathrm{Spec}(k[\varpi, x_d, y_d]/x_d y_d - \varpi^d)$$

   where $\varpi := xy$ and $x_d := y^d t, y_d := x^d t^{-1}$.

2. If $d \leq 0$, then the good moduli space of $[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]$ is given by

$$\mathrm{Spec}(k[x,y,t^{\pm}]^{\mathbb{G}_m}) = \mathrm{Spec}(k[xy, x^{-d} t, y^{-d} t^{-1}]) =: \mathrm{Spec}(k[\varpi, x_d, y_d]/x_d y_d - \varpi^{-d})$$

   where $\varpi := xy$ and $x_d := y^{-d} t^{-1}, y_d := x^{-d} t$.

i.e., the good moduli space of $[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]$ is given by

$$\mathbb{A}^2_d := \mathrm{Spec}(k[\varpi, x_d, y_d]/x_d y_d - \varpi^{|d|}).$$

Therefore the morphism $[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m] \to X$ in (2.4.6) factors through a unique $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^2_d \to X$ with respect to the $\mathbb{G}_m$-action on $\mathbb{A}^2_d$ such that x_d, y_d and ϖ have $\mathbb{G}_m$-weights $|d|, -|d|$ and 0 respectively. This computation leads us to consider the following $\mathbb{G}_m$-equivariant mapping stack over $\mathbb{A}^1 = \mathrm{Spec}(k[\varpi])$:

$$\mathbb{X}_d := \underline{\mathrm{Map}}_{\mathbb{A}^1}(\mathbb{A}^2_d, X)^{\mathbb{G}_m}$$

It admits a canonical morphism to the mapping stack encountered in the 1^{st} reformulation

$$\mathbb{X}_d := \underline{\mathrm{Map}}_{\mathbb{A}^1}(\mathbb{A}^2_d, X)^{\mathbb{G}_m} \to \underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$$

by precomposing with the $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^2 \to \mathbb{A}^2_d$ corresponding to

$$k[\varpi, x_d, y_d]/x_d y_d - \varpi^{|d|} \to k[x,y] \text{ mapping } x_d \mapsto x^{|d|}, y_d \mapsto y^{|d|}, \varpi \mapsto xy$$

The observation is that all morphisms to $\underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$ in (2.4.5) factor through $\mathbb{X}_d$ for some uniquely determined integer d. Thus we turn to consider the mapping stack $\mathbb{X}_d$. Similarly the good quotient $\mathbb{A}^2_d \to \mathbb{A}^1 = \mathrm{Spec}(k[\varpi])$ has two sections

$$\begin{tikzcd} & \mathbb{A}^2_d \arrow[d, "\mathrm{gms}"] & \\ {[\mathrm{Spec}(k[x,y]_x)/\mathbb{G}_m] \cong \mathbb{A}^1_x} \arrow[ur, "\sigma_x"] & \mathbb{A}^1 \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathbb{A}^1_y \cong [\mathrm{Spec}(k[x,y]_y)/\mathbb{G}_m] \arrow[ul, "\sigma_y"'] \end{tikzcd}$$

defined as follows: Restricting (2.4.6) to the open locus where $x \neq 0$ gives

$$\begin{tikzcd}
{[\mathrm{Spec}(k[x,y,t^{\pm}]_x)/\mathbb{G}_m]} \arrow[r, hook] \arrow[d] & {[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]} \arrow[d] \\
\mathbb{A}^1_x = {[\mathrm{Spec}(k[x,y]_x)/\mathbb{G}_m]} \arrow[r, hook, "x\neq 0"'] \arrow[u, bend left, "\imath_x"] & {[\mathbb{A}^2/\mathbb{G}_m]}
\end{tikzcd}$$

Let $\imath_x : \mathbb{A}^1_x = [\mathrm{Spec}(k[x,y]_x)/\mathbb{G}_m] \to [\mathrm{Spec}(k[x,y,t^{\pm}]_x)/\mathbb{G}_m]$ be the section corresponding to the $\mathbb{G}_m$-equivariant morphism

$$\imath_x : k[x,y,t^{\pm}]_x \to k[x,y]_x \text{ mapping } x \mapsto x, y \mapsto y, t \mapsto x^d$$

and the section $\sigma_x : \mathbb{A}^1_x = [\mathrm{Spec}(k[x,y]_x)/\mathbb{G}_m] \to \mathbb{A}^2_d$ is defined by the composition

$$\sigma_x : \mathbb{A}^1_x = [\mathrm{Spec}(k[x,y]_x)/\mathbb{G}_m] \xrightarrow{\imath_x} [\mathrm{Spec}(k[x,y,t^{\pm}]_x)/\mathbb{G}_m] \hookrightarrow [\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m] \xrightarrow{\text{gms}} \mathbb{A}^2_d$$

corresponding to the $\mathbb{G}_m$-equivariant morphism

$$k[\varpi, x_d, y_d]/x_d y_d - \varpi^{|d|} \to k[x,y,t^{\pm}] \to k[x,y,t^{\pm}]_x \to k[x,y]_x$$
$$\text{mapping } \varpi \mapsto xy, x_d \mapsto (xy)^{|d|}, y_d \mapsto 1.$$

Similarly we have a section $\sigma_y : \mathbb{A}^1_y := [\mathrm{Spec}(k[x,y]_y)/\mathbb{G}_m] \to \mathbb{A}^2_d$ corresponding to the $\mathbb{G}_m$-equivariant morphism

$$k[\varpi, x_d, y_d]/x_d y_d - \varpi^{|d|} \to k[x,y,t^{\pm}] \to k[x,y,t^{\pm}]_y \to k[x,y]_y$$
$$\text{mapping } \varpi \mapsto xy, x_d \mapsto 1, y_d \mapsto (xy)^{|d|}.$$

These two sections together equips $\mathbb{X}_d$ with a morphism

$$p := (\sigma_x, \sigma_y) : \mathbb{X}_d \to (X \times \mathbb{A}^1) \times_{\mathbb{A}^1} (X \times \mathbb{A}^1) \cong \mathbb{A}^1 \times X \times X.$$

Remark 2.4.7. By construction we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k[\varpi]) = \mathbb{A}^1 \arrow[rr, "\sim"] & & \mathbb{A}^1_x \arrow[lldd, bend right, "\sigma_x"'] \arrow[d, "\imath_x"] \arrow[rrdd, bend left, "s_x"] & & \mathbb{A}^1 = \mathrm{Spec}(k[xy]) \arrow[ll, "\sim"'] \\
& & {[\mathrm{Spec}(k[x,y,t^{\pm}]_x/\mathbb{G}_m)]} \arrow[d, hook] & & \\
\mathbb{A}^2_d \arrow[uu, "\text{gms}"] \arrow[d] & & {[\mathbb{A}^2 \times \mathbb{G}_m/\mathbb{G}_m]} \arrow[ll, "\text{gms}"'] \arrow[rr] & & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[uu, "\text{gms}"'] \arrow[d] \\
X \arrow[rrrr] & & & & \mathscr{X}
\end{tikzcd}$$

relating the section σ_x of $\mathbb{X}_d$ and the section s_x of $\underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X})$. A similar commutative diagram relates σ_y and s_y. As a result we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{X}_d = \underline{\mathrm{Map}}_{\mathbb{A}^1}(\mathbb{A}^2_d, X)^{\mathbb{G}_m} \arrow[r] \arrow[d, "(\sigma_x,\sigma_y)"'] & \underline{\mathrm{Map}}_{\mathbb{A}^1}([\mathbb{A}^2/\mathbb{G}_m], \mathscr{X}) \arrow[d, "(s_x,s_y)"] \\
X \times X \arrow[r] & \mathscr{X} \times \mathscr{X}
\end{tikzcd}$$

Lemma 2.4.8. *The morphism $p : \mathbb{X}_d \to \mathbb{A}^1 \times X \times X$ induces an isomorphism between $\mathbb{G}_m \times_{\mathbb{A}^1} \mathbb{X}_d$ and the graph Γ of the map*

$$\mathbb{G}_m \times X \xrightarrow{(t \mapsto t^d, \mathrm{id}_X)} \mathbb{G}_m \times X \xrightarrow{\mathrm{act}} X,$$

i.e., we have a commutative diagram with Cartesian squares

$$\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{X}_d \arrow[d, "p"] \\
\mathbb{G}_m \times X \times X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1 \times X \times X \arrow[d] \\
\mathrm{Spec}(k[t^{\pm}]) = \mathbb{G}_m \arrow[r, hook, "\varpi \mapsto t"'] & \mathbb{A}^1 = \mathrm{Spec}(k[\varpi])
\end{tikzcd}$$

Proof. We compute the fiber product as

$$\begin{aligned}
\mathbb{G}_m \times_{\mathbb{A}^1} \mathbb{X}_d &= \mathbb{G}_m \times_{\mathbb{A}^1} \underline{\mathrm{Map}}_{\mathbb{A}^1}(\mathbb{A}^2_d, X)^{\mathbb{G}_m} \\
&\cong \underline{\mathrm{Map}}_{\mathbb{G}_m}(\mathbb{A}^2_d \times_{\mathbb{A}^1} \mathbb{G}_m, X)^{\mathbb{G}_m} \\
&\cong \underline{\mathrm{Map}}_{\mathbb{G}_m}(\mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|}, X)^{\mathbb{G}_m}.
\end{aligned}$$

Using the two sections σ_x and σ_y of $\mathbb{A}^2_d$ the affine scheme $\mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|})$ is covered by the two open subsets

$$\begin{aligned}
\sigma_x &: \mathrm{Spec}(k[\varpi^{\pm}, x_d^{\pm}] \hookrightarrow \mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|} \\
\sigma_y &: \mathrm{Spec}(k[\varpi^{\pm}, y_d^{\pm}] \hookrightarrow \mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|}
\end{aligned}$$

Therefore a $\mathbb{G}_m$-equivariant morphism $f : \mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|} \to X$ is equivalent to two $\mathbb{G}_m$-equivariant morphisms

$$\begin{aligned}
f_x &: \mathrm{Spec}(k[\varpi^{\pm}, x_d^{\pm}] \xrightarrow{\sigma_x} \mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|} \to X \\
f_y &: \mathrm{Spec}(k[\varpi^{\pm}, y_d^{\pm}] \xrightarrow{\sigma_y} \mathrm{Spec}(k[\varpi^{\pm}, x_d, y_d]/x_d y_d - \varpi^{|d|} \to X
\end{aligned}$$

such that $\lambda^d.f_x(1) = f_y(1)$, i.e., f is identified with a point $(\lambda, x_1, x_2) \in \mathbb{G}_m \times X \times X$ such that $\lambda^d.x_1 = x_2$. This finishes the proof. $\square$

The key observation is:

Lemma 2.4.9. *The quotient stack $\mathscr{X} = [X/\mathbb{G}_m]$ is S-complete if and only if the following holds: For every integer d, every DVR R with fraction field K and uniformizer π, any commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "v"] \arrow[d, hook] & \mathbb{X}_d \arrow[d, "p"] \\ \mathrm{Spec}(R) \arrow[r, "(\varpi\mapsto\pi, u_1, u_2)"'] \arrow[ur, dashed, "\exists!"] & \mathbb{A}^1 \times X \times X \end{tikzcd} \tag{2.4.7}$$

of solid arrows can be uniquely filled in.

Proof. Similar to (2.4.1), we have a commutative diagram with Cartesian squares:

$$\begin{tikzcd} \mathbb{G}_m \times \mathrm{Spec}(K) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spec}(R[x_d, y_d]/x_d y_d - \pi^{|d|}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^2_d \arrow[d, "\mathbb{G}_m\text{-tor}"] \\ \mathrm{Spec}(K) \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & \left[\mathrm{Spec}(R[x_d, y_d]/x_d y_d - \pi^{|d|})/\mathbb{G}_m\right] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & [\mathbb{A}^2_d/\mathbb{G}_m] \arrow[d, "\mathrm{gms}"] \\ \mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) \arrow[r, "\varpi\mapsto\pi"'] & \mathbb{A}^1 = \mathrm{Spec}(k[\varpi]) \end{tikzcd} \tag{2.4.8}$$

and this computes that

$$\mathbb{X}_d(\mathrm{Spec}(K)) = \mathrm{Maps}(\mathbb{A}^2_d \times_{\mathbb{A}^1} \mathrm{Spec}(K), X)^{\mathbb{G}_m} \cong \mathrm{Maps}(\mathbb{G}_m \times \mathrm{Spec}(K), X)^{\mathbb{G}_m} \cong X(K).$$

Thus we can view a morphism $\mathrm{Spec}(K) \to \mathbb{X}_d$ as a K-point of X and the commutativity of (2.4.7) is then equivalent to (by Lemma 2.4.8)

$$v = (u_1)_K \text{ and } \pi^d.(u_1)_K = (u_2)_K.$$

IF PART: Given a commutative diagram (2.4.2) of solid arrows, i.e., a morphism $f : \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$, we obtain two morphisms

$$f_x : \mathrm{Spec}(R) \xrightarrow{f|_{x\neq 0}} \mathscr{X},\ f_y : \mathrm{Spec}(R) \xrightarrow{f|_{y\neq 0}} \mathscr{X}$$

together with an isomorphism $\phi_K : (f_x)_K \xrightarrow{\sim} (f_y)_K$. Since $X \to \mathscr{X}$ is a $\mathbb{G}_m$-torsor, we can lift f_x, f_y to $\tilde{f}_x, \tilde{f}_y : \mathrm{Spec}(R) \to X$ respectively and the isomorphism ϕ_K becomes an element $g_K \in \mathbb{G}_m(K)$ such that $g_K.(\tilde{f}_x)_K = (\tilde{f}_y)_K$. Using the Cartan decomposition

$\mathbb{G}_m(K) = \mathbb{G}_m(R)\mathbb{G}_m(K)\mathbb{G}_m(R)$ we can write

$$g_K = a_x \pi^d a_y \text{ for some } a_x, a_y \in \mathbb{G}_m(R).$$

Replacing $\tilde{f}_x, \tilde{f}_y$ by $a_x\tilde{f}_x, a_y^{-1}\tilde{f}_y$ respectively, we may assume that $\pi^d.(\tilde{f}_x)_K = (\tilde{f}_y)_K$. This defines a commutative diagram (2.4.7) of solid arrows by setting

$$u_1 : \operatorname{Spec}(R) \xrightarrow{\tilde{f}_x} X, u_2 : \operatorname{Spec}(R) \xrightarrow{\tilde{f}_y} X$$
$$v : \operatorname{Spec}(K) \to \mathbb{X}_d \text{ given by } (\tilde{f}_x)_K : \operatorname{Spec}(K) \to X.$$

By assumption there exists a unique dotted arrow $\operatorname{Spec}(R) \to \mathbb{X}_d$ filling in (2.4.7), i.e., a $\mathbb{G}_m$-equivariant morphism

$$\ell : \mathbb{A}^2_d \times_{\mathbb{A}^1} \operatorname{Spec}(R) \cong \operatorname{Spec}(R[x_d, y_d]/x_d y_d - \pi^{|d|}) \to X \text{ by (2.4.8)}$$

and it defines a dotted arrow $\ell' : \overline{\mathrm{ST}}_R \to \mathscr{X}$ filling in (2.4.2) as follows:

$$\begin{tikzcd}
\operatorname{Spec}(R[x,y]/xy-\pi) \arrow[r, "\varepsilon_d"] \arrow[d] & \operatorname{Spec}(R[x_d,y_d]/x_d y_d - \pi^{|d|}) \arrow[r, "\ell"] \arrow[d] & X \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
\overline{\mathrm{ST}}_R \arrow[r] \arrow[rr, bend right, "\ell'"'] & \left[\operatorname{Spec}(R[x_d,y_d]/x_d y_d - \pi^{|d|})/\mathbb{G}_m\right] \arrow[r] & \mathscr{X}
\end{tikzcd}$$

where the morphism ε_d corresponding to the $\mathbb{G}_m$-equivariant morphism

$$\varepsilon_d : R[x_d, y_d]/x_d y_d - \pi^{|d|} \to R[x,y]/xy - \pi \text{ mapping } x_d \mapsto x^{|d|}, y_d \mapsto y^{|d|}, \pi \mapsto \pi.$$

Only If Part: Given a commutative diagram (2.4.7) of solid arrows, i.e., morphisms

$$u_1, u_2 : \operatorname{Spec}(R) \to X \text{ and } v : \operatorname{Spec}(K) \to X \text{ such that } v = (u_1)_K \text{ and } \pi^d.(u_1)_K = (u_2)_K,$$

then

$$\overline{u_1 \cup u_2} : \operatorname{Spec}(R) \bigcup_{\operatorname{Spec}(K)} \operatorname{Spec}(R) \cong \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$$

defines a commutative diagram (2.4.2) of solid arrows. By assumption there exists a unique dotted arrow $\overline{\mathrm{ST}}_R \to \mathscr{X}$ filling in (2.4.2). Then we have a commutative diagram with

Cartesian squares

$$
\begin{tikzcd}
\mathrm{Spec}(K)\times\mathbb{G}_m \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spec}(R[x,y]/xy-\pi)\times\mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{L}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
\mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R[x,y]/xy-\pi) \arrow[r, "\mathbb{G}_m\text{-tor}"'] & \overline{\mathrm{ST}}_R \arrow[r] & \mathscr{X}
\end{tikzcd}
\tag{2.4.9}
$$

where

$$
\mathrm{L}_d \cong [(\mathrm{Spec}(R[x,y]/xy-\pi)\times\mathbb{G}_m)/\mathbb{G}_m] = [(\mathrm{Spec}(R[x,y,t^{\pm}])/xy-\pi)/\mathbb{G}_m]
$$

such that x, y and t have $\mathbb{G}_m$-weights $1, -1$ and d (as the coordinate t on $\mathrm{Spec}(K)\times\mathbb{G}_m = K[t^{\pm}]$ has $\mathbb{G}_m$-weight d) respectively. Similar as $[\mathbb{A}^2\times\mathbb{G}_m/\mathbb{G}_m]$, the good moduli space of L_d is $\mathrm{Spec}(R[x_d,y_d]/x_dy_d-\pi^{|d|})$.

Since X is a scheme, the morphism $\mathrm{L}_d\to X$ in (2.4.9) factors through a unique $\mathbb{G}_m$-equivariant morphism $\zeta : \mathrm{Spec}(R[x_d,y_d]/x_dy_d-\pi^{|d|})\to X$ with respect to the $\mathbb{G}_m$-action on $\mathrm{Spec}(R[x_d,y_d]/x_dy_d-\pi^{|d|})$ such that x_d, y_d and π have $\mathbb{G}_m$-weights $|d|, -|d|$ and 0 respectively. This defines a dotted arrow $\mathrm{Spec}(R)\to\mathbb{X}_d$ filling in (2.4.7), given by

$$
\mathbb{A}^2_d\times_{\mathbb{A}^1}\mathrm{Spec}(R)\cong\mathrm{Spec}(R[x_d,y_d]/x_dy_d-\pi^{|d|})\xrightarrow{\zeta} X \text{ by (2.4.8).}
$$

$\square$

In spirit of [Sta21, Tag 0CM5], we expect that the lifting property in (2.4.7) is equivalent to the one in the following diagram

$$
\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "v"] \arrow[d, hook] & \mathbb{Y}_d \arrow[r] & \mathbb{X}_d \arrow[d, "p"] \\
\mathrm{Spec}(R) \arrow[rru, dashed, "\exists!"'] \arrow[rr, "(\varpi\mapsto\pi, u_1, u_2)"'] & & \mathbb{A}^1\times X\times X
\end{tikzcd}
$$

and this will conclude the proof. The problem is still that a priori we do not know whether the morphism

$$
\mathbb{Y}_d\to\mathbb{X}_d
$$

has dense image. However, the situation now is improved, at least we know it is an open immersion.

Lemma 2.4.10. *Let $Y\subset X$ be a $\mathbb{G}_m$-invariant subset. Consider the following commutative diagram*

$$
\begin{tikzcd}
\mathbb{Y}_d \arrow[r] \arrow[d, "p_Y"'] & \mathbb{X}_d \arrow[d, "p_X"] \\
\mathbb{A}^1\times Y\times Y \arrow[r, hook] & \mathbb{A}^1\times X\times X
\end{tikzcd}
\tag{2.4.10}
$$

1. *If $Y \subset X$ is closed, then (2.4.10) is Cartesian.*

   In particular, the morphism $\mathbb{Y}_d \to \mathbb{X}_d$ is a closed immersion.

2. *If $Y \subset X$ is open, then (2.4.10) identifies $\mathbb{Y}_d$ with an open subset of*

$$\mathbb{X}_d \times_{\mathbb{A}^1 \times X \times X} (\mathbb{A}^1 \times Y \times Y).$$

   In particular, the morphism $\mathbb{Y}_d \to \mathbb{X}_d$ is an open immersion.

Proof. The proof is identical to that of [DG14, Proposition 2.3.2] and we reproduce the proof here for the convenience of readers. For any scheme $S \to \mathbb{A}^1$ and $\mathbb{G}_m$-equivariant morphism $f : \mathbb{A}^2_d \times_{\mathbb{A}^1} S \to X$, we need to show that if f maps two sections of $\mathbb{A}^2_d \times_{\mathbb{A}^1} S \to S$ (induced by the two sections σ_x and σ_y of $\mathbb{A}^2_d \to \mathbb{A}^1$) to $Y \subset X$, then

1. If $Y \subset X$ is closed, then $f(\mathbb{A}^2_d \times_{\mathbb{A}^1} S) \subset Y$.

2. If $Y \subset X$ is open, then the subset $\{s \in S : (\mathbb{A}^2_d \times_{\mathbb{A}^1} S)_s \subset f^{-1}(Y)\} \subset S$ is open.

By $\mathbb{G}_m$-equivariance we have $f((\mathbb{A}^2_d - \{0\}) \times_{\mathbb{A}^1} S) \subset Y$.

1. If $Y \subset X$ is closed, then $f(\mathbb{A}^2_d \times_{\mathbb{A}^1} S) \subset Y$ since $(\mathbb{A}^2_d - \{0\}) \times_{\mathbb{A}^1} S \subset \mathbb{A}^2_d \times_{\mathbb{A}^1} S$ is dense.

2. The complement of the subset $\{s \in S : (\mathbb{A}^2_d \times_{\mathbb{A}^1} S)_s \subset f^{-1}(Y)\} \subset S$ equals $\mathrm{pr}_S(\mathbb{A}^2_d \times_{\mathbb{A}^1} S - f^{-1}(Y))$, where $\mathrm{pr}_S : \mathbb{A}^2_d \times_{\mathbb{A}^1} S \to S$ is the projection. The complement is closed in S since $\mathbb{A}^2_d \times_{\mathbb{A}^1} S - f^{-1}(Y) \subset \mathbb{A}^2_d \times_{\mathbb{A}^1} S - (\mathbb{A}^2_d - \{0\}) \times_{\mathbb{A}^1} S$ is closed (using $Y \subset X$ is open) and the restricted projection $\mathrm{pr}_S : \mathbb{A}^2_d \times_{\mathbb{A}^1} S - (\mathbb{A}^2_d - \{0\}) \times_{\mathbb{A}^1} S \to S$ is closed.

□

Remark 2.4.11. Starting from Lemma 2.4.10 we can finally show that $\mathbb{X}_d$ is a scheme of finite type over k, using the same argument as [DG14, Theorem 2.4.2].

In case where $\mathbb{X}_d$ is irreducible, the open immersion $\mathbb{Y}_d \hookrightarrow \mathbb{X}_d$ has dense image and we are done. This happens, e.g., if the structure morphism $\mathbb{X}_d \to \mathbb{A}^1$ is flat, but this is not the case in general (even if X is irreducible, see the example in [DG14, Remark 2.5.4]). However, if X is smooth, then the structure morphism $\mathbb{X}_d \to \mathbb{A}^1$ is smooth, using the same arguments in the proof of [DG14, Proposition 2.5.2]. This forces us to cook up a further reformulation.

- 3^{rd} reformulation

It is now clear that we need to find some irreducible subset of $\mathbb{X}_d$ where the lifting property in (2.4.7) actually happens. Such a subset is constructed using the description of $\mathbb{X}_d$ in Lemma 2.4.8. Indeed, let $\mathbb{X}^\flat_d \subset \mathbb{X}_d$ be the closed subset given by the closure of the graph Γ of the map

$$\mathbb{G}_m \times X \xrightarrow{(t \mapsto t^d, \mathrm{id}_X)} \mathbb{G}_m \times X \to X \text{ inside } \mathbb{A}^1 \times X \times X$$

then the induced structure morphism $\mathbb{X}_d^\flat \to \mathbb{A}^1$ is flat since $\Gamma \to \mathbb{G}_m \times X \times X \to \mathbb{G}_m$ is, in particular open. Thus $\mathbb{X}_d^\flat$ is irreducible and the open immersion $\mathbb{Y}_d^\flat \hookrightarrow \mathbb{X}_d^\flat$ has dense image.

The key observation is:

Lemma 2.4.12. *The quotient stack $\mathscr{X} = [X/\mathbb{G}_m]$ is S-complete if and only if the following holds: For every integer d, every DVR R with fraction field K and uniformizer π, any commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[rr, "v"] \arrow[d, hook] & & \mathbb{X}_d^\flat \arrow[d, "p"] \\ \mathrm{Spec}(R) \arrow[rru, dashed, "\exists!"] \arrow[r, "{(\varpi\mapsto\pi, u_1, u_2)}"'] & \mathbb{A}^1 \times X \times X & \end{tikzcd} \tag{2.4.11}$$

of solid arrows can be uniquely filled in.

Proof. According to Lemma 2.4.8, all morphisms to $\mathbb{X}_d$ in (2.4.7) factor through the closed subset $\mathbb{X}_d^\flat \subset \mathbb{X}_d$. $\square$

By [Sta21, Tag 0CM5], the lifting property in (2.4.11) is equivalent to the one in the following diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "v"] \arrow[d, hook] & \mathbb{Y}_d^\flat \arrow[r, hook, "\text{dense}"] & \mathbb{X}_d^\flat \arrow[d, "p"] \\ \mathrm{Spec}(R) \arrow[rru, dashed, "\exists!"] \arrow[rr, "{(\varpi\mapsto\pi, u_1, u_2)}"'] & & \mathbb{A}^1 \times X \times X \end{tikzcd}$$

Tracing back to (2.4.2), this means we can assume that $\mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$ factors through $\mathscr{Y} \subset \mathscr{X}$. This finishes the proof of Theorem 2.4.4 in Case B.

2.5 Existence part of valuative criterion for properness

Similar to S-completeness, existence part of valuative criterion for properness of an algebraic stack can also be checked on its dense substacks.

Theorem 2.5.1 ([Sta21], Tag 0CQM and [Rom13], Lemma 4.1.1)**.** *Let $\mathscr{X}$ be an algebraic stack of finite type over a field k and $h : \mathscr{Y} \to \mathscr{X}$ be a morphism of algebraic stacks, of finite type and with dense image (e.g., $\mathscr{Y} \subset \mathscr{X}$ is a dense substack and h is the immersion). The following are equivalent:*

1. *$\mathscr{X}$ satisfies the existence part of valuative criterion for properness, i.e., for every DVR R with fraction field K, any commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \mathrm{Spec}(R) \arrow[ru, dashed] \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows can be filled in.

2. *$\mathscr{X}$ satisfies the existence part of valuative criterion for properness relative to $h : \mathscr{Y} \to \mathscr{X}$, i.e., for every DVR R with fraction field K, any commutative diagram*

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & \mathscr{Y} \arrow[r, "h"] & \mathscr{X} \arrow[d] \\
\mathrm{Spec}(R) \arrow[rr] \arrow[urr, dashed] & & \mathrm{Spec}(k)
\end{tikzcd}$$

of solid arrows, up to a finite extension of the fraction field K, can be filled in.

Part I

Moduli stack of vector bundles over a curve

Throughout this part, let

- $k = \bar{k}$ be an algebraically closed field of characteristic 0 and
- C be a smooth projective connected curve of genus $g := g(C) > 1$ over k.
- $\mathscr{B}un_n^d$ be the moduli stack of vector bundles of rank n and degree d over C.

  It is a smooth irreducible algebraic stack, locally of finite type with affine diagonal over k (see, e.g., [Neu09, Theorem 2.67]). Then any open substack of $\mathscr{B}un_n^d$ is locally of finite type with affine diagonal over k and hence we are eligible to apply Theorem 0 to check whether it admits a (separated or proper) good moduli space.
- $\mathscr{C}oh_n^d$ be the moduli stack of coherent sheaves of rank n and degree d over C.

  It is a smooth irreducible algebraic stack, locally of finite type with affine diagonal over k (see, e.g., [Hof10, Proposition A.1] and [Sta21, Tag 0DLX]).

Chapter 3

Preliminaries

In this chapter, we give the translations of local linear reductivity, Θ-reductivity and S-completeness in Theorem 0 for open substacks of $\mathscr{B}un_n^d$. Since the automorphism groups of vector bundles (or general, coherent sheaves) over C are connected[1], any open substack of $\mathscr{B}un_n^d$ has unpunctured inertia by [AHLH18, Proposition 3.56].

To warm-up, we recall the Rees construction. It can be viewed as a flat degeneration of a filtered sheaf to its associated graded sheaf and will be used frequently in this paper.

Lemma 3.0.1 (Rees construction, [Hei17], Lemma 1.10 and [HL14], Proposition 1.0.1)**.** *Let X be a locally noetherian scheme over a field $\mathbb{K}$. For any coherent sheaf $\mathcal{E}$ over X together with an ascending filtration $\mathcal{E}^\bullet$ of $\mathcal{E}$ such that $\mathcal{E}^i = \mathcal{E}$ for $i \gg 0$ and $\mathcal{E}^i = 0$ for $i \ll 0$, there exists a coherent sheaf $\mathcal{E}_{\mathbb{A}^1}$ over $X \times \mathbb{A}^1$, flat over $\mathbb{A}^1$ and $\mathbb{G}_m$-equivariant for the action defined on the coordinate parameter such that*

$$\mathcal{E}_{\mathbb{A}^1}|_{X\times\mathbb{G}_m} \cong \mathcal{E} \times \mathbb{G}_m \text{ and } \mathcal{E}_{\mathbb{A}^1}|_{X\times\{0\}} \cong \mathrm{gr}(\mathcal{E}^\bullet).$$

Proof. Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{K}[t])$. Define an $\mathcal{O}_X[t]$-module

$$\mathrm{Rees}(\mathcal{E}^\bullet) = \bigoplus_{i\in\mathbb{Z}} t^i\mathcal{E}^i \subset \mathcal{E}[t, t^{-1}]$$

where the action of t sends each summand to the next. Note that as a $\mathbb{K}[t]$-module $\mathrm{Rees}(\mathcal{E}^\bullet)$ is flat since it is torsion-free (see [Eis95, Corollary 6.11]). This shows that $\mathrm{Rees}(\mathcal{E}^\bullet)$ defines a coherent sheaf $\mathcal{E}_{\mathbb{A}^1}$ over $X \times \mathbb{A}^1$, flat over $\mathbb{A}^1$ and $\mathbb{G}_m$-equivariant for the action defined

[1]For any vector bundle $\mathcal{E}$ over C, its automorphism group $\mathrm{Aut}(\mathcal{E}) \subset \mathrm{End}(\mathcal{E})$ is the Zariski open dense subset defined by non-vanishing of the determinant and hence connected, where $\mathrm{End}(\mathcal{E}) = H^0(C, \mathscr{E}nd(\mathcal{E}))$ is the space of the global sections of the sheaf of endomorphisms of $\mathcal{E}$.

on the coordinate parameter. Moreover, we compute that

$$\mathcal{E}_{\mathbb{A}^1}|_{X\times\{0\}} = \mathrm{Rees}(\mathcal{E}^\bullet)/t.\mathrm{Rees}(\mathcal{E}^\bullet) \cong \bigoplus_{i\in\mathbb{Z}} t^i(\mathcal{E}^i/\mathcal{E}^{i-1}) \cong \mathrm{gr}(\mathcal{E}^\bullet)$$
$$\mathcal{E}_{\mathbb{A}^1}|_{X\times\mathbb{G}_m} = \mathrm{Rees}(\mathcal{E}^\bullet)\otimes_{\mathcal{O}_X[t]}\mathcal{O}_X[t,t^{-1}] = \bigoplus_{i\in\mathbb{Z}} t^i\mathcal{E} = \mathcal{E}[t,t^{-1}] = \mathcal{E}\times\mathbb{G}_m.$$

This finishes the proof. □

3.1 Consequence of local linear reductivity

For local linear reductivity, the first step is to understand points in $\mathscr{B}un_n^d$ with linearly reductive automorphism groups.

Lemma 3.1.1. *If $\mathcal{E}\in\mathscr{B}un_n^d(k)$ is indecomposable and $\mathrm{Aut}(\mathcal{E})$ is linearly reductive, then $\mathcal{E}$ is simple.*

Recall that $\mathcal{E}$ is indecomposable if it has no non-trivial decomposition $\mathcal{E}=\mathcal{E}_1\oplus\mathcal{E}_2$.

Proof. For any endomorphism $\phi\in\mathrm{End}(\mathcal{E})$, the characteristic polynomial of $\phi_x:\mathcal{E}_x\to\mathcal{E}_x$ remains unchanged when varying $x\in C$ since C is connected. As $\mathcal{E}$ is indecomposable, all roots of ϕ_x are equal. This shows that $\mathrm{End}(\mathcal{E}) = k\cdot\mathrm{id}_{\mathcal{E}}\oplus\mathrm{Nil}(\mathcal{E})$ as k-vector spaces, where $\mathrm{Nil}(\mathcal{E})\subset\mathrm{End}(\mathcal{E})$ consists of nilpotent endomorphisms of $\mathcal{E}$ (see also [Ati57, Proposition 16]). Therefore

$$\mathrm{Aut}(\mathcal{E}) = k^*\cdot\mathrm{id}_{\mathcal{E}}\oplus\mathrm{Nil}(\mathcal{E})$$

and the assignment $\phi\mapsto\mathrm{id}_{\mathcal{E}}+\phi$ realizes $\mathrm{Nil}(\mathcal{E})$ as a normal unipotent subgroup of $\mathrm{Aut}(\mathcal{E})$, then $\mathrm{Nil}(\mathcal{E})$ must vanish since $\mathrm{Aut}(\mathcal{E})$ is linearly reductive. We are done. □

Remark 3.1.2. In general, if $\mathcal{E}\cong\bigoplus_{i=1}^s\mathcal{E}_i^{\oplus n_i}\in\mathscr{B}un_n^d(k)$ such that all $\mathcal{E}_i$ are simple and $\mathrm{Hom}(\mathcal{E}_i,\mathcal{E}_j)=0$ for any $i\neq j$, then

$$\mathrm{Aut}(\mathcal{E}) = \prod_{i=1}^s \mathrm{GL}_{n_i}$$

and it is linearly reductive.

As a warming-up, we give some (non-)examples of open substacks of $\mathscr{C}oh_n^d$ that are locally linearly reductive.

Lemma 3.1.3 ([AHLH18], Lemma 8.4)**.** *The stack $\mathscr{B}un_n^{d,ss}$ is locally linearly reductive.*

Proof. Every point in $\mathscr{B}un_n^{d,ss}$ specializes to the unique closed polystable point in its S-equivalence class by applying Rees construction to one of its Jordan-Hölder filtration. And any polystable point in $\mathscr{B}un_n^{d,ss}$ has a linearly reductive automorphism group by Remark 3.1.2. □

Lemma 3.1.4. *Both stacks $\mathscr{B}un_n^d$ (if $n > 1$) and $\mathscr{C}oh_n^d$ are not locally linearly reductive.*

Proof. We show that there is no closed point in $\mathscr{B}un_n^d$ (if $n > 1$) or $\mathscr{C}oh_n^d$. Indeed, for any field $\mathbb{K}/k$, we have

1. any point $\mathcal{E} \in \mathscr{B}un_n^d(\mathbb{K})$ (if $n > 1$) admits a sub-line bundle $\mathcal{L}$ with $\mathcal{E} \ncong \mathcal{L} \oplus \mathcal{E}/\mathcal{L}$.
2. any point $\mathcal{E} \in \mathscr{C}oh_n^d(\mathbb{K})$ admits a subsheaf $\mathcal{L}$ with $\mathcal{E} \ncong \mathcal{L} \oplus \mathcal{E}/\mathcal{L}$.

then the Rees construction provides a (non-trivial) degeneration from $\mathcal{E}$ to $\mathcal{L} \oplus \mathcal{E}/\mathcal{L}$, so no point in $\mathscr{B}un_n^d$ (if $n > 1$) or $\mathscr{C}oh_n^d$ is closed. □

By Theorem 0, neither $\mathscr{B}un_n^d$ (if $n > 1$) nor $\mathscr{C}oh_n^d$ admits a good moduli space.

3.2 Consequence of Θ-reductivity

To investigate Θ-reductivity for open substacks of $\mathscr{B}un_n^d$, it will be useful to enlarge the target a little bit and classify morphisms $\Theta \to \mathscr{C}oh_n^d$. By definition, they correspond to coherent sheaves over $C \times \mathbb{A}^1$, flat over $\mathbb{A}^1$ and $\mathbb{G}_m$-equivariant for the action defined on the coordinate parameter.

Lemma 3.2.1 ([Aso06], Theorem 1.1 and [Hei17], Lemma 1.10)**.** *For any open substack $\mathscr{U} \subset \mathscr{C}oh_n^d$, there is an equivalence*

$$\mathrm{Maps}(\Theta, \mathscr{U}) \cong \left\langle (\mathcal{E}, \mathcal{E}^\bullet) : \begin{array}{c} \mathcal{E} \in \mathscr{C}oh_n^d,\ \mathcal{E}^i \subset \mathcal{E}^{i+1} \subset \mathcal{E} \\ \mathcal{E}^i = \mathcal{E} \text{ for } i \gg 0,\ \mathcal{E}^i = 0 \text{ for } i \ll 0,\ \mathrm{gr}(\mathcal{E}^\bullet) \in \mathscr{U} \end{array} \right\rangle,$$

i.e., a morphism $\Theta \to \mathscr{U}$ is equivalent to a coherent sheaf together with an ascending filtration such that the associated graded sheaf is in $\mathscr{U}$.

Proof. For a coherent sheaf $\mathcal{E}$ over C together with an ascending filtration $\mathcal{E}^\bullet$ such that $\mathrm{gr}(\mathcal{E}^\bullet) \in \mathscr{U}$, the Rees construction gives a coherent sheaf $\mathcal{E}_{\mathbb{A}^1}$ over $C \times \mathbb{A}^1$, flat over $\mathbb{A}^1$ and $\mathbb{G}_m$-equivariant for the action defined on the coordinate parameter. By definition this defines a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathscr{C}oh_n^d$ and it factors through $\mathscr{U}$ since $\mathrm{gr}(\mathcal{E}^\bullet) \in \mathscr{U}$ and $\mathscr{U} \subset \mathscr{C}oh_n^d$ is open.

Conversely, a morphism $f : \Theta \to \mathscr{U}$ induces a morphism $\mathbb{G}_m \to \mathrm{Aut}_{\mathscr{U}}(f(0))$, i.e., a grading on the coherent sheaf $f(0) \in \mathscr{U}$ such that the corresponding filtration lifts canonically to the coherent sheaf $f(1)$. This gives rise to $\mathcal{E}$ and $\mathcal{E}^\bullet$. □

By Remark 2.2.2 and Lemma 3.2.1, we immediately have the following description of Θ-reductivity for open substacks of $\mathscr{C}oh_n^d$.

Proposition 3.2.2. *An open substack $\mathscr{U} \subset \mathscr{C}oh_n^d$ is Θ-reductive if and only if for*

- *every DVR R over k with fraction field K and residue field κ,*

- *every family $\mathcal{E}_R \in \mathscr{U}(R)$,*

any filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ with $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$ can be uniquely extended to a filtration $\mathcal{E}_R^\bullet$ of $\mathcal{E}_R$ and $\mathrm{gr}(\mathcal{E}_R^\bullet) \in \mathscr{U}(R)$.

Remark 3.2.3. A few words about Proposition 3.2.2.

1. Any filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ can be uniquely extended to a filtration $\mathcal{E}_R^\bullet$ of $\mathcal{E}_R$ because $\mathscr{C}oh_n^d$ is Θ-reductive (see Lemma 3.2.4), so the only (honest) condition for Θ-reductivity of $\mathscr{U}$ is $\mathrm{gr}(\mathcal{E}_R^\bullet) \in \mathscr{U}(R)$, or equivalently, the induced filtration $\mathcal{E}_\kappa^\bullet$ of $\mathcal{E}_\kappa$ satisfies $\mathrm{gr}(\mathcal{E}_\kappa^\bullet) \in \mathscr{U}(\kappa)$.

2. The condition $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$ in Proposition 3.2.2 forces $\mathcal{E}_K^\bullet$ to be a filtration of subbundles, while the induced filtration $\mathcal{E}_\kappa^\bullet$ need not be. The so-called Θ-testing family in §3.2.1 provides such an example. But there are some situations where the induced filtration $\mathcal{E}_\kappa^\bullet$ of $\mathcal{E}_\kappa$ is still a filtration of subbundles, see, e.g., Lemma 3.2.7.

As before, we provide some examples that are Θ-reductive.

Lemma 3.2.4. *The stack $\mathscr{C}oh_n^d$ is Θ-reductive.*

Proof. For every DVR R, any commutative diagram

$$\begin{tikzcd}
\Theta_R - \{0\} \arrow[r] \arrow[d, hook, "j"'] & \mathscr{C}oh_n^d \arrow[d] \\
\Theta_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k)
\end{tikzcd}$$

of solid arrows, we need to show there exists a unique dotted arrow filling in.

First we show the uniqueness. If $\mathcal{E}$ is the coherent sheaf over $C \times (\Theta_R - \{0\})$ defined by the morphism $\Theta_R - \{0\} \to \mathscr{C}oh_n^d$, then the extension $\Theta_R \to \mathscr{C}oh_n^d$, if exists, is necessarily defined by the coherent sheaf $(\mathrm{id}_C \times j)_* \mathcal{E}$ over $C \times \Theta_R$ since $C \times \Theta_R$ is normal for any DVR R and $\mathrm{codim}_{C \times \Theta_R}(C \times \Theta_R - C \times (\Theta_R - \{0\})) \geq 2$. This implies the uniqueness.

Then we show the existence using Proposition 3.2.2. For $\mathscr{C}oh_n^d$ the condition on gr is superfluous, so it suffices to show that for every DVR R over k with fraction field K and every family $\mathcal{E}_R \in \mathscr{C}oh_n^d(R)$, any filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ can be extended to a filtration $\mathcal{E}_R^\bullet$ of $\mathcal{E}_R$. This is done by induction on the length of $\mathcal{E}_K^\bullet$ and using the properness of Quot-scheme. The arguments below are adapted from [Sha77, Proposition 9].

1. If the filtration $\mathcal{E}_K^\bullet$ has length two $0 \subset \mathcal{E}_K^1 \subset \mathcal{E}_K$, then the coherent quotient $\mathcal{E}_K \twoheadrightarrow \mathcal{E}_K/\mathcal{E}_K^1$ of $\mathcal{E}_K$ defines a section $\sigma_K : \mathrm{Spec}(K) \to \mathrm{Quot}_{C_R}(\mathcal{E}_R)$ of the Quot-scheme $\mathrm{Quot}_{C_R}(\mathcal{E}_R)$ over $\mathrm{Spec}(R)$ parametrizing coherent quotients of $\mathcal{E}_R$, i.e.,

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "\sigma_K"] \arrow[d, hook] & \mathrm{Quot}_{C_R}(\mathcal{E}_R) \arrow[d, "\text{proper}"] \\
\mathrm{Spec}(R) \arrow[r, equal] \arrow[ur, dashed, "\exists! \sigma_R"'] & \mathrm{Spec}(R)
\end{tikzcd}$$

Since $\mathrm{Quot}_{C_R}(\mathcal{E}_R) \to \mathrm{Spec}(R)$ is proper (see [Gro95, Lemme 3.7]), the section σ_K admits a unique extension $\sigma_R : \mathrm{Spec}(R) \to \mathrm{Quot}_{C_R}(\mathcal{E}_R)$ and it corresponds to a coherent quotient $\mathcal{E}_R \twoheadrightarrow \mathcal{E}_R/\mathcal{E}_R^1$ of $\mathcal{E}_R$. This gives the desired extension $\mathcal{E}_R^\bullet : 0 \subset \mathcal{E}_R^1 \subset \mathcal{E}_R$ of $\mathcal{E}_K^\bullet$.

2. Suppose that any filtration $\mathcal{E}_K^\bullet$ of length $\leq m-1$ can be extended. For a filtration $\mathcal{E}_K^\bullet$ of length m, we split it into two parts:

$$0 \subset \mathcal{E}_K^1 \subset \mathcal{E}_K^2 \text{ and } 0 \subset \mathcal{E}_K^2 \subset \cdots \subset \mathcal{E}_K^m = \mathcal{E}_K$$

By induction hypothesis, using $\mathcal{E}_R$ the second filtration has an extension $0 \subset \mathcal{E}_R^2 \subset \cdots \subset \mathcal{E}_R^m = \mathcal{E}_R$ and then using $\mathcal{E}_R^2$ the first filtration has an extension $0 \subset \mathcal{E}_R^1 \subset \mathcal{E}_R^2$. Patching these two filtrations together gives an extension $\mathcal{E}_R^\bullet$ of $\mathcal{E}_K^\bullet$.

$\square$

Remark 3.2.5. Lemma 3.2.4 simplifies the verification of Θ-reductivity for open substacks $\mathscr{U} \subset \mathscr{C}oh_n^d$. Indeed, if we complete the set-up of Θ-reductivity for $\mathscr{U}$ as

$$\begin{tikzcd} \Theta_R - \{0\} \arrow[r] \arrow[d, hook] \arrow[dr, "\circ"] & \mathscr{U} \arrow[d, hook] \\ \Theta_R \arrow[r, dashed, "\exists!"'] & \mathscr{C}oh_n^d \end{tikzcd}$$

then there exists a unique dotted arrow filling in. So to check whether $\mathscr{U}$ is Θ-reductive, it suffices to check whether the image of $0 \in \Theta_R$ in $\mathscr{C}oh_n^d$ lies in $\mathscr{U}$.

This is summarized, in the down-to-earth term, in the following proposition.

Proposition 3.2.6. *An open substack $\mathscr{U} \subset \mathscr{C}oh_n^d$ is Θ-reductive if and only if for*

- *every DVR R over k with fraction field K and residue field κ,*
- *every family $\mathcal{E}_R \in \mathscr{U}(R)$,*
- *every filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ with $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$,*

the induced filtration $\mathcal{E}_\kappa^\bullet$ of $\mathcal{E}_\kappa$ (given by Lemma 3.2.4) satisfies $\mathrm{gr}(\mathcal{E}_\kappa^\bullet) \in \mathscr{U}(\kappa)$.

Lemma 3.2.7 ([AHLH18], Lemma 8.5)**.** *The stack $\mathscr{B}un_n^{d,ss}$ is Θ-reductive.*

Proof. This is proved by using Proposition 3.2.6. Given a DVR R over k with fraction field K and residue field κ, a family $\mathcal{E}_R \in \mathscr{B}un_n^{d,ss}(R)$ and a filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ with $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{B}un_n^{d,ss}(K)$, first we see that each $\mathcal{E}_K^i/\mathcal{E}_K^{i-1}$ is semi-stable of the same slope as $\mathcal{E}_K$. Due to the semi-stability of $\mathcal{E}_\kappa$, the induced filtration $\mathcal{E}_\kappa^\bullet$ of $\mathcal{E}_\kappa$ must be a filtration of subbundles and each $\mathcal{E}_\kappa^i/\mathcal{E}_\kappa^{i-1}$ is semi-stable (otherwise there would be a subbundle of $\mathcal{E}_\kappa$ with strictly larger slope) of the same slope as $\mathcal{E}_\kappa$ (since $\mu(\mathcal{E}_\kappa^i/\mathcal{E}_\kappa^{i-1}) = \mu(\mathcal{E}_K^i/\mathcal{E}_K^{i-1}) = \mu(\mathcal{E}_K) = \mu(\mathcal{E}_\kappa)$). This means that $\mathrm{gr}(\mathcal{E}_\kappa^\bullet) \in \mathscr{B}un_n^{d,ss}(\kappa)$. $\square$

3.2.1 Θ-testing family

In this subsection we define the so-called Θ-testing family. This is motivated by Proposition 3.2.6: the condition $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$ forces $\mathcal{E}_K^\bullet$ to be a filtration of subbundles, while the uniquely induced filtration $\mathcal{E}_\kappa^\bullet$ need not to be. The Θ-testing family we are going to construct provides such an example. By its nature, any open substack of $\mathscr{B}un_n^d$ supporting a Θ-testing family (see Lemma 3.2.12 below) cannot be Θ-reductive and this explains the terminology.

Roughly speaking, any degeneration of a vector bundle to a torsion sheaf can be upgraded, as a quotient, to a degeneration of vector bundles and this gives rise to a Θ-testing family.

Proposition-Definition 3.2.8 (Θ-testing family)**.** Let R be a DVR over k with fraction field K and residue field κ. Given the following data:

- two pairs of integers $(n_1, d_1), (n_2, d_2) \in \mathbb{N} \times \mathbb{Z}$ such that $(n, d) = (n_1, d_1) + (n_2, d_2)$.
- two families $\mathcal{G}_{1,R} \in \mathscr{C}oh_{n_1}^{d_1}(R)$ and $\mathcal{G}_{2,R} \in \mathscr{C}oh_{n_2}^{d_2}(R)$ such that

$$\mathcal{G}_{2,K} \in \mathscr{B}un_{n_2}^{d_2}(K) \text{ and } \mathcal{G}_{2,\kappa} \in (\mathscr{C}oh_{n_2}^{d_2} - \mathscr{B}un_{n_2}^{d_2})(\kappa).$$

- an element $[\varrho] \in \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$.

Such a triple $(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho]) \in \mathscr{C}oh_{n_1}^{d_1}(R) \times \mathscr{C}oh_{n_2}^{d_2}(R) \times \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$ is called *weird*. Then there exists a family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ such that

- the generic fiber $\mathcal{G}_K$ fits into a short exact sequence $0 \to \mathcal{G}_{1,K} \to \mathcal{G}_K \to \mathcal{G}_{2,K} \to 0$.
- the special fiber $\mathcal{G}_\kappa$ fits into the short exact sequence corresponding to $[\varrho] : 0 \to \mathcal{G}_{1,\kappa} \to \mathcal{G}_\kappa \to \mathcal{G}_{2,\kappa} \to 0$.

Such a family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ is said to be a Θ-*testing family* attached to the weird triple $(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho])$.

Put it another way, a Θ-testing family is a family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ such that $\mathcal{G}_K$ admits a subbundle that only extended to a subsheaf of $\mathcal{G}_\kappa$.

Proposition-Definition 3.2.8 follows from a general statement that we can always lift extensions on special fiber to the entire family (see Proposition 3.2.11 below). Technically this can done via the so-called $\mathscr{E}xt$-stack classifying extensions of coherent sheaves.

Definition 3.2.9 ($\mathscr{E}xt$-stack)**.** Let $t_1 = (n_1, d_1), t_2 = (n_2, d_2) \in \mathbb{N} \times \mathbb{Z}$ be two pairs of integers such that $(n_1, d_1) + (n_2, d_2) = (n, d)$. Let $\mathscr{E}xt(t_2, t_1)$ be the stack of extensions of coherent sheaves over C of type (t_2, t_1), i.e., for any k-scheme T, we have

$$\mathscr{E}xt(t_2, t_1)(T) = \langle 0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0 : \mathcal{E}_i \in \mathscr{C}oh_{n_i}^{d_i}(T) \text{ for } i = 1, 2 \rangle.$$

Proposition 3.2.10. *One has*

1. *The stack $\mathscr{E}xt(t_2, t_1)$ is a smooth irreducible algebraic stack, locally of finite type over k.*

2. *The assignment $[0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0] \mapsto (\mathcal{E}_1, \mathcal{E}_2)$ defines a smooth morphism of algebraic stacks*
$$\mathrm{pr}_{13} : \mathscr{E}xt(t_2, t_1) \to \mathscr{C}oh_{n_1}^{d_1} \times \mathscr{C}oh_{n_2}^{d_2}$$
which is a generalized vector bundle[2].

3. *The assignment $[0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0] \mapsto \mathcal{E}$ defines a morphism of algebraic stacks*
$$\mathrm{pr}_2 : \mathscr{E}xt(t_2, t_1) \to \mathscr{C}oh_n^d.$$

Proof. All statements here, except that the morphism pr_{13} is a generalized vector bundle, are covered in [Hof10, Appendix A]. So we only add a proof for this.

Indeed, the fiber of pr_{13} over any $(\mathcal{G}_1, \mathcal{G}_2) \in \mathscr{C}oh_{n_1}^{d_1}(k) \times \mathscr{C}oh_{n_2}^{d_2}(k)$ is the stack classifying extensions of $\mathcal{G}_2$ by $\mathcal{G}_1$ and it is a generalized vector bundle
$$[\mathrm{Ext}^1(\mathcal{G}_2, \mathcal{G}_1)/\mathrm{Hom}(\mathcal{G}_2, \mathcal{G}_1)] \text{ over } \mathrm{Spec}(k)$$
where the algebraic group $\mathrm{Hom}(\mathcal{G}_2, \mathcal{G}_1)$ acts trivially on the affine space $\mathrm{Ext}^1(\mathcal{G}_2, \mathcal{G}_1)$.

In general, for every affine scheme $\mathrm{Spec}(R)$ of finite type over k, any morphism $\mathrm{Spec}(R) \to \mathscr{C}oh_{n_1}^{d_1} \times \mathscr{C}oh_{n_2}^{d_2}$ produces a Cartesian diagram
$$\begin{tikzcd}
\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{E}xt(t_2, t_1) \arrow[d, "\mathrm{pr}_{13}"] \\
\mathrm{Spec}(R) \arrow[r, "(\mathcal{G}_{1,R}, \mathcal{G}_{2,R})"'] & \mathscr{C}oh_{n_1}^{d_1} \times \mathscr{C}oh_{n_2}^{d_2}
\end{tikzcd}$$
where $\mathcal{G}_{i,R} \in \mathscr{C}oh_{n_i}^{d_i}(R)$ is the coherent sheaf over $C \times \mathrm{Spec}(R)$ corresponding to the morphism $\mathrm{Spec}(R) \to \mathscr{C}oh_{n_i}^{d_i}$ for $i = 1, 2$ and $\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R})$ is the stack classifying extensions over C of the forms
$$0 \to \mathcal{G}_{1,R} \to \square \to \mathcal{G}_{2,R} \to 0.$$

[2] Recall that (see [BF97, §2]) if $\delta : \mathcal{E}_0 \to \mathcal{E}_1$ is a morphism of vector bundles over a scheme X, then the quotient stack $[\mathcal{E}_1/\mathcal{E}_0] \to X$ is a *generalized vector bundle* over X, where $\mathcal{E}_0$ acts on $\mathcal{E}_1$ via δ. Up to equivalence, the generalized vector bundle $[\mathcal{E}_1/\mathcal{E}_0]$ depends only on the class of the complex $[\mathcal{E}_0 \xrightarrow{\delta} \mathcal{E}_1] \in \mathscr{D}(X)$. In particular, if we can choose a complex such that
$$[\mathcal{E}'_0 = 0 \xrightarrow{0} \mathcal{E}'_1] = [\mathcal{E}_0 \xrightarrow{\delta} \mathcal{E}_1] \in \mathscr{D}(X)$$
then $[\mathcal{E}_1/\mathcal{E}_0] \cong [\mathcal{E}'_1/\mathcal{E}'_0] = \mathcal{E}'_1$ is a honest vector bundle over X.

Then we need to show that the base-change $\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \to \mathrm{Spec}(R)$ is a generalized vector bundle. By [EGA$_{\mathrm{III}}$] the object $\mathbf{R}p_*\mathscr{H}om(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \in \mathscr{D}(\mathrm{Spec}(R))$, where $p : C \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is the projection, can be represented by a length-one complex $[\mathcal{V}_0 \to \mathcal{V}_1]$ of vector bundles over $\mathrm{Spec}(R)$. By [Hei04, Example after Lemma 0.1], we have $\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \cong [\mathcal{V}_1/\mathcal{V}_0]$, as desired. □

Proposition 3.2.11 (Extension family)**.** *Let R be a DVR over k with fraction field K and residue field κ. Given the following data:*

- *two families $\mathcal{G}_{1,R} \in \mathscr{C}oh_{n_1}^{d_1}(R)$ and $\mathcal{G}_{2,R} \in \mathscr{C}oh_{n_2}^{d_2}(R)$.*
- *an element $[\varrho] \in \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$.*

then there exists an extension family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ of $\mathcal{G}_{2,R}$ by $\mathcal{G}_{1,R}$ with the special fiber $[\varrho]$, i.e.,

- *the generic fiber $\mathcal{G}_K$ fits into a short exact sequence $0 \to \mathcal{G}_{1,K} \to \mathcal{G}_K \to \mathcal{G}_{2,K} \to 0$.*
- *the special fiber $\mathcal{G}_\kappa$ fits into the short exact sequence corresponding to $[\varrho] : 0 \to \mathcal{G}_{1,\kappa} \to \mathcal{G}_\kappa \to \mathcal{G}_{2,\kappa} \to 0$.*

Proof. As before we have a Cartesian diagram

$$\begin{tikzcd}
\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{E}xt(t_2, t_1) \arrow[d, "\mathrm{pr}_{13}"] \\
\mathrm{Spec}(R) \arrow[r, "(\mathcal{G}_{1,R}, \mathcal{G}_{2,R})"'] \arrow[u, dashed, bend left, "{[s]}"] & \mathscr{C}oh_{n_1}^{d_1} \times \mathscr{C}oh_{n_2}^{d_2}
\end{tikzcd}$$

If there exists a section $[s] : \mathrm{Spec}(R) \to \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R})$ mapping $\mathrm{Spec}(\kappa) \mapsto [\varrho]$, then the composition

$$\mathrm{Spec}(R) \xrightarrow{[s]} \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \to \mathscr{E}xt(t_2, t_1) \xrightarrow{\mathrm{pr}_2} \mathscr{C}oh_n^d$$

defines a family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$, which is an extension of $\mathcal{G}_{2,R}$ by $\mathcal{G}_{1,R}$ with the special fiber $[\varrho]$. So it remains to show the existence of such a section.

By Proposition 3.2.10 (2) there exists a length-one complex $[\mathcal{F}_R \to \mathcal{E}_R]$ of vector bundles over $\mathrm{Spec}(R)$ such that $\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \cong [\mathcal{E}_R/\mathcal{F}_R]$. Then we choose a lifting $\varrho \in \mathcal{E}_\kappa$ of $[\varrho]$ as

$$\begin{tikzcd}
\varrho \in \mathcal{E}_\kappa \arrow[r, hook] \arrow[d, two heads] & \mathcal{E}_R \arrow[d, two heads, "\sigma"] \\
{[\varrho] \in \mathscr{E}xt(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})} \arrow[r, hook] \arrow[d] & \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \cong [\mathcal{E}_R/\mathcal{F}_R] \arrow[d] \\
\mathrm{Spec}(\kappa) \arrow[r, hook] & \mathrm{Spec}(R)
\end{tikzcd}$$

and up to an extension of DVR's, there exists a section $s : \mathrm{Spec}(R) \to \mathcal{E}_R$ mapping $\mathrm{Spec}(\kappa) \mapsto \varrho$. So we simply take $[s] := \sigma \circ s : \mathrm{Spec}(R) \to \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R})$. □

Lemma 3.2.12. *Let $\mathscr{U} \subset \mathscr{B}un_n^d$ be an open substack. Let R be a DVR over k with fraction field K and residue field κ. If there exists a weird triple*

$$(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho]) \in \mathscr{C}oh_{n_1}^{d_1}(R) \times \mathscr{C}oh_{n_2}^{d_2}(R) \times \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$$

such that $\mathcal{G}_{1,K} \oplus \mathcal{G}_{2,K} \in \mathscr{U}(K)$ and $[\varrho] \in \mathscr{U}(\kappa)$, then $\mathscr{U}$ is not Θ-reductive.

In this case, we say that the open substack $\mathscr{U} \subset \mathscr{B}un_n^d$ supports a Θ-testing family.

Proof. By Proposition-Definition 3.2.8, choose a Θ-testing family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ attached to the weird triple $(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho])$. Since $\mathscr{U} \subset \mathscr{B}un_n^d$ is open, $\mathcal{G}_\kappa = [\varrho] \in \mathscr{U}(\kappa)$ implies that $\mathcal{G}_R \in \mathscr{U}(R)$. By construction the filtration $\mathcal{G}_K^\bullet$ of $\mathcal{G}_K$ satisfies $\mathrm{gr}(\mathcal{G}_K^\bullet) = \mathcal{G}_{1,K} \oplus \mathcal{G}_{2,K} \in \mathscr{U}(K)$ while the uniquely induced filtration $\mathcal{G}_\kappa^\bullet$ of $\mathcal{G}_\kappa$ satisfies $\mathrm{gr}(\mathcal{G}_\kappa^\bullet) = \mathcal{G}_{1,\kappa} \oplus \mathcal{G}_{2,\kappa} \notin \mathscr{B}un_n^d(\kappa)$ since $\mathcal{G}_{2,\kappa} \notin \mathscr{B}un_{n_2}^{d_2}(\kappa)$. Then $\mathscr{U}$ is not Θ-reductive by Proposition 3.2.6. $\square$

Corollary 3.2.13. *The stack $\mathscr{B}un_n^d$ (if $n > 1$) is not Θ-reductive.*

By Theorem 0, this gives another proof that $\mathscr{B}un_n^d$ (if $n > 1$) doesn't admit a good moduli space.

Proof. Since Θ-testing family always exists in $\mathscr{B}un_n^d$ if $n > 1$. $\square$

3.2.2 A special Θ-testing family

For our application, the following special case of Θ-testing family will be sufficient, where in the weird triple $(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho]) \in \mathscr{C}oh_{n_1}^{d_1}(R) \times \mathscr{C}oh_{n_1}^{d_1}(R) \times \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$

1. the first family $\mathcal{G}_{1,R} \in \mathscr{C}oh_{n_1}^{d_1}(R)$ is a constant family given by a vector bundle $\mathcal{E}_1 \in \mathscr{B}un_{n_1}^{d_1}(k)$.
2. the second family $\mathcal{G}_{2,R} \in \mathscr{C}oh_{n_2}^{d_2}(R)$ is a degeneration family given by a length-two filtration $\mathcal{E}_2^\bullet$ of a vector bundle $\mathcal{E}_2 \in \mathscr{B}un_{n_2}^{d_2}(k)$ such that the quotient is a torsion sheaf, via Rees construction.

Such a length-two filtration is built from the so-called elementary modification.

Definition 3.2.14 (Elementary modification)**.** Let $\mathcal{E}$ be a vector bundle over C and D an effective divisor on C. Then any surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_D$ induces a short exact sequence

$$0 \to \mathcal{E}^D \to \mathcal{E} \to \mathcal{O}_D \to 0.$$

and the resulting vector bundle $\mathcal{E}^D$ is called an *elementary modification* of $\mathcal{E}$ along $D \subset C$.

Remark 3.2.15. A few words about Definition 3.2.14.

1. There always exists a surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_D$. Indeed, any choice of trivialization $\mathcal{E}|_D \cong \mathcal{O}_D^{\oplus \mathrm{rk}(\mathcal{E})}$ gives rise to a surjection via

$$\begin{tikzcd}
 & \mathcal{E}^D \arrow[dr, hook] & & & \\
0 \arrow[r] & \mathcal{E}(-D) \arrow[u, hook] \arrow[r] & \mathcal{E} \arrow[r] \arrow[dr] & \mathcal{O}_D^{\oplus \mathrm{rk}(\mathcal{E})} \arrow[r] \arrow[d, two heads, "\text{any quot}"] & 0. \\
 & & & \mathcal{O}_D &
\end{tikzcd}$$

2. If $\mathcal{E}$ is a line bundle, then $\mathcal{E}^D \cong \mathcal{E}(-D)$.

3. By Rees construction, the vector bundle $\mathcal{E}$ can degenerate to the torsion-sheaf $\mathcal{E}^D \oplus \mathcal{O}_D$ and in this way we produce the second family $\mathcal{G}_{2,R}$.

Example 3.2.16 (A special Θ-testing family)**.** Given the following data:

- two pairs $(n_1, d_1), (n_2, d_2) \in \mathbb{N} \times \mathbb{Z}$ of integers such that $(n, d) = (n_1, d_1) + (n_2, d_2)$.
- two vector bundles $\mathcal{E}_1 \in \mathscr{B}un_{n_1}^{d_1}(k)$ and $\mathcal{E}_2 \in \mathscr{B}un_{n_2}^{d_2}(k)$.
- an effective divisor D on C.
- an element $[\varrho] \in \mathrm{Ext}^1(\mathcal{E}_2^D \oplus \mathcal{O}_D, \mathcal{E}_1)$.

Such a quadruple

$$(\mathcal{E}_1, \mathcal{E}_2, D, [\varrho]) \in \mathscr{B}un_{n_1}^{d_1}(k) \times \mathscr{B}un_{n_2}^{d_2}(k) \times \mathrm{Div}^{\mathrm{eff}}(C) \times \mathrm{Ext}^1(\mathcal{E}_2^D \oplus \mathcal{O}_D, \mathcal{E}_1)$$

is said to be *weird*. Then for any DVR R over k with fraction field K and residue field κ, there exist two families $\mathcal{G}_{1,R} \in \mathscr{C}oh_{n_1}^{d_1}(R)$ and $\mathcal{G}_{2,R} \in \mathscr{C}oh_{n_2}^{d_2}(R)$ such that

- $\mathcal{G}_{1,K} \cong \mathcal{E}_{1,K}$ and $\mathcal{G}_{1,\kappa} \cong \mathcal{E}_{1,\kappa}$.

  For example, we can take $\mathcal{G}_{1,R} := p^*\mathcal{E}_1$ where $p : C \times \mathrm{Spec}(R) \to C$ is the projection.

- $\mathcal{G}_{2,K} \cong \mathcal{E}_{2,K}$ and $\mathcal{G}_{2,\kappa} \cong (\mathcal{E}_2^D \oplus \mathcal{O}_D)_\kappa$.

  For example, we can apply Rees construction to any length-two filtration $0 \subset \mathcal{E}_2^D \subset \mathcal{E}_2$.

Now the triple $(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho]_\kappa) \in \mathscr{C}oh_{n_1}^{d_1}(R) \times \mathscr{C}oh_{n_2}^{d_2}(R) \times \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$ is weird in the sense of Proposition-Definition 3.2.8 and any Θ-testing family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ attached to it is also called a Θ-testing family attached to the weird quadruple $(\mathcal{E}_1, \mathcal{E}_2, D, [\varrho])$.

In this special case, Lemma 3.2.12 has the following form.

Lemma 3.2.17. *Let $\mathscr{U} \subset \mathscr{B}un_n^d$ be an open substack. If there exists a weird quadruple*

$$(\mathcal{E}_1, \mathcal{E}_2, D, [\varrho]) \in \mathscr{B}un_{n_1}^{d_1}(k) \times \mathscr{B}un_{n_2}^{d_2}(k) \times \mathrm{Div}^{\mathrm{eff}}(C) \times \mathrm{Ext}^1(\mathcal{E}_2^D \oplus \mathcal{O}_D, \mathcal{E}_1)$$

such that $\mathcal{E}_1 \oplus \mathcal{E}_2 \in \mathscr{U}(k)$ and $[\varrho] \in \mathscr{U}(k)$, then $\mathscr{U}$ is not Θ-reductive.

3.3 Consequence of S-completeness

To investigate S-completeness for open substacks of $\mathscr{B}un_n^d$, it will be useful to enlarge the target a little bit and classify morphisms $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$. By definition, they correspond to coherent sheaves over $C \times \overline{\mathrm{ST}}_R$, flat over $\overline{\mathrm{ST}}_R$.

Lemma 3.3.1 ([AHLH18], Corollary 7.13)**.** *Let R be a DVR over k with uniformizer π. A quasi-coherent sheaf $\mathcal{G}$ over $C \times \overline{\mathrm{ST}}_R$ corresponds to a $\mathbb{Z}$-graded coherent sheaf $\bigoplus_{i\in\mathbb{Z}} \mathcal{G}_i$ over $C \times \mathrm{Spec}(R)$ together with a diagram*

$$\begin{tikzcd} \cdots \arrow[r, bend right, "x"'] & \mathcal{G}_{i-1} \arrow[l, bend right, "y"'] \arrow[r, bend right, "x"'] & \mathcal{G}_i \arrow[l, bend right, "y"'] \arrow[r, bend right, "x"'] & \mathcal{G}_{i+1} \arrow[l, bend right, "y"'] \arrow[r, bend right, "x"'] & \cdots \arrow[l, bend right, "y"'] \end{tikzcd}$$

such that $xy = yx = \pi$. Moreover

1. *$\mathcal{G}$ is coherent if each $\mathcal{G}_i$ is coherent, $x : \mathcal{G}_{i-1} \to \mathcal{G}_i$ is an isomorphism for $i \gg 0$ and $y : \mathcal{G}_i \to \mathcal{G}_{i-1}$ is an isomorphism for $i \ll 0$.*

2. *$\mathcal{G}$ is flat over $\overline{\mathrm{ST}}_R$ if and only if the maps x and y are injective, and the induced map $y : \mathcal{G}_{i+1}/x\mathcal{G}_i \to \mathcal{G}_i/x\mathcal{G}_{i-1}$ is injective.*

Lemma 3.3.2. *The stack $\mathscr{C}oh_n^d$ is S-complete.*

Proof. For every DVR R, any commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook, "j"'] & \mathscr{C}oh_n^d \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows, we need to show there exists a unique dotted arrow filling in.

First we show the uniqueness. If $\mathcal{E}$ is the coherent sheaf over $C \times (\overline{\mathrm{ST}}_R - \{0\})$ defined by the morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$, then the extension $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$, if exists, is necessarily defined by the coherent sheaf $(\mathrm{id}_C \times j)_*\mathcal{E}$ over $C \times \overline{\mathrm{ST}}_R$, since $\mathrm{codim}_{C\times\overline{\mathrm{ST}}_R}(C \times \overline{\mathrm{ST}}_R - C \times (\overline{\mathrm{ST}}_R - \{0\})) \geq 2$ and $C \times \overline{\mathrm{ST}}_R$ is normal for any DVR R. This implies the uniqueness.

Then we show the existence using Lemma 3.3.1. Let K be the fraction field of R and $\pi \in R$ a uniformizer. By Remark 2.4.2 the morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$ is equivalent to two families $\mathcal{E}_R, \mathcal{E}'_R \in \mathscr{C}oh_n^d(R)$ together with an isomorphism $\lambda_K : \mathcal{E}_K \xrightarrow{\sim} \mathcal{E}'_K$, i.e., $\mathcal{E}_R \otimes_R R[\pi^{-1}] = \mathcal{E}_K \cong \mathcal{E}'_K = \mathcal{E}'_R \otimes_R R[\pi^{-1}]$, let $a, b \in \mathbb{N}$ be the minimal integers such that $\mathcal{E}'_R \subset \pi^{-b}\mathcal{E}_R$ and $\mathcal{E}_R \subset \pi^{-a}\mathcal{E}'_R$ and they are also the minimal integers such that $\pi^a\lambda_K$ and $\pi^b\lambda_K^{-1}$ define morphisms $\mathcal{E}_R \to \mathcal{E}'_R$ and $\mathcal{E}'_R \to \mathcal{E}_R$ extending λ_K and λ_K^{-1} respectively.

By Lemma 3.3.1, we can define a coherent sheaf over $C \times \overline{\mathrm{ST}}_R$, flat over $\overline{\mathrm{ST}}_R$, as follows

$$\mathcal{G}_i := \begin{cases} \mathcal{E}'_R & \text{if } i \leq -b \\ \pi^i\mathcal{E}_R \cap \mathcal{E}'_R & \text{if } -b < i < 0 \\ \mathcal{E}_R \cap \pi^{-i}\mathcal{E}'_R & \text{if } 0 \leq i < a \\ \mathcal{E}_R & \text{if } i \geq a \end{cases}$$

and let $x_i : \mathcal{G}_i \to \mathcal{G}_{i+1}$ and $y_i : \mathcal{G}_{i+1} \to \mathcal{G}_i$ act via

$$\begin{tikzcd}
\cdots \arrow[r, bend right, "\pi"'] & \mathcal{E}'_R \arrow[l, bend right, "\mathrm{id}"'] \arrow[r, bend right, "\pi"'] & \mathcal{G}_{-(b-1)} \arrow[l, bend right, "\mathrm{icl}"'] \arrow[r, bend right, "\pi"'] & \cdots \arrow[l, bend right, "\mathrm{icl}"'] \arrow[r, bend right, "\pi"'] & \mathcal{E}_R \cap \mathcal{E}'_R \arrow[l, bend right, "\mathrm{icl}"'] \arrow[r, bend right, "\mathrm{icl}"'] & \cdots \arrow[l, bend right, "\pi"'] \arrow[r, bend right, "\mathrm{icl}"'] & \mathcal{G}_{a-1} \arrow[l, bend right, "\pi"'] \arrow[r, bend right, "\mathrm{icl}"'] & \mathcal{E}_R \arrow[l, bend right, "\pi"'] \arrow[r, bend right, "\mathrm{id}"'] & \cdots \arrow[l, bend right, "\pi"'] \\
\cdots & -b & -(b-1) & \cdots & 0 & \cdots & a-1 & a & \cdots
\end{tikzcd}$$

Then we claim that this diagram defines a morphism $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$ extending $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$. According to Lemma 3.3.1, it remains to check that the induced map $x_i : \mathcal{G}_i/y_i\mathcal{G}_{i+1} \to \mathcal{G}_{i+1}/y_{i+1}\mathcal{G}_{i+2}$ is injective for all $i \in \mathbb{Z}$, i.e.,

$$\cdots \to 0 \to \frac{\mathcal{E}'_R}{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R} \xrightarrow{\pi} \frac{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi^{-(b-2)}\mathcal{E}_R \cap \mathcal{E}'_R} \xrightarrow{\pi} \cdots \xrightarrow{\pi} \frac{\pi^{-1}\mathcal{E}_R \cap \mathcal{E}'_R}{\mathcal{E}_R \cap \mathcal{E}'_R} \xrightarrow{\pi} \frac{\mathcal{E}_R \cap \mathcal{E}'_R}{\pi\mathcal{E}_R \cap \mathcal{E}'_R}$$
$$\xrightarrow{\mathrm{icl}} \frac{\mathcal{E}_R \cap \pi^{-1}\mathcal{E}'_R}{\pi\mathcal{E}_R \cap \pi^{-1}\mathcal{E}'_R} \xrightarrow{\mathrm{icl}} \cdots \xrightarrow{\mathrm{icl}} \frac{\mathcal{E}_R \cap \pi^{-(a-1)}\mathcal{E}'_R}{\pi\mathcal{E}_R} \xrightarrow{\mathrm{icl}} \mathcal{E}_k \xrightarrow{\mathrm{id}} \cdots$$

is injective in each degree. Indeed, if $i < 0$, then the induced map $x_i : \mathcal{G}_i/y_i\mathcal{G}_{i+1} \to \mathcal{G}_{i+1}/y_{i+1}\mathcal{G}_{i+2}$ is injective since in the following commutative diagram the above square is Cartesian

$$\begin{tikzcd}
\mathcal{G}_{i+1} = \pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R \arrow[r, hook, "x_{i+1}=\pi"] \arrow[d, hook, "y_i=\mathrm{icl}"'] \arrow[dr, phantom, "\ulcorner"] & \pi^{-(i+2)}\mathcal{E}_R \cap \mathcal{E}'_R = \mathcal{G}_{i+2} \arrow[d, hook, "y_{i+1}=\mathrm{icl}"] \\
\mathcal{G}_i = \pi^{-i}\mathcal{E}_R \cap \mathcal{E}'_R \arrow[r, hook, "x_i=\pi"'] \arrow[d, two heads] & \pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R = \mathcal{G}_{i+1} \arrow[d, two heads] \\
\dfrac{\mathcal{G}_i}{\mathcal{G}_{i+1}} = \dfrac{\pi^{-i}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R} \arrow[r, hook, dashed, "x_i=\pi"'] & \dfrac{\pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi^{-(i+2)}\mathcal{E}_R \cap \mathcal{E}'_R} = \dfrac{\mathcal{G}_{i+1}}{\mathcal{G}_{i+2}}
\end{tikzcd}$$

Similar for the case $i > 0$. $\square$

Proposition-Definition 3.3.3 (Canonical filtrations)**.** Given the following data:

- a DVR R over k with fraction field K and residue field κ.
- two families $\mathcal{E}_R, \mathcal{E}'_R \in \mathscr{C}oh_n^d(R)$ such that $\mathcal{E}_K \cong \mathcal{E}'_K$.

then there exist filtrations $\mathcal{E}_\kappa^{\mathrm{can}}, (\mathcal{E}'_\kappa)^{\mathrm{can}}$ of $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ of the same length m such that

1. There exists an isomorphism $\mathcal{E}_\kappa^{\mathrm{can},i}/\mathcal{E}_\kappa^{\mathrm{can},i-1} \cong (\mathcal{E}'_\kappa)^{\mathrm{can},m-i+1}/(\mathcal{E}'_\kappa)^{\mathrm{can},m-i}$ for $i = 1, \ldots, m$. In particular $\mathrm{gr}(\mathcal{E}_\kappa^{\mathrm{can}}) \cong \mathrm{gr}((\mathcal{E}'_\kappa)^{\mathrm{can}})$.
2. If $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$ is the morphism defined by $\mathcal{E}_R \cup \mathcal{E}'_R$ and $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$ is its unique extension (given by Lemma 3.3.2), then the image of $0 \in \overline{\mathrm{ST}}_R$ in $\mathscr{C}oh_n^d$ is given by $\mathrm{gr}(\mathcal{E}_\kappa^{\mathrm{can}}) \cong \mathrm{gr}((\mathcal{E}'_\kappa)^{\mathrm{can}})$.

Such filtrations $\mathcal{E}_\kappa^{\mathrm{can}}, (\mathcal{E}'_\kappa)^{\mathrm{can}}$ are called the *canonical filtrations* of $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$.

Proof. Keep the construction and notation in the proof of Lemma 3.3.2. According to [AHLH18, Corollary 7.13], the restriction of $\mathcal{G}_\bullet$ to $x = 0$ (resp., $y = 0$) gives a (finite) filtration $\mathcal{E}_\kappa^{\mathrm{can}}$ (resp., $(\mathcal{E}'_\kappa)^{\mathrm{can}}$) of subsheaves of $\mathcal{E}_\kappa$ (resp., $\mathcal{E}'_\kappa$)

$$\mathcal{E}_\kappa^{\mathrm{can}} : \cdots \to 0 \to \frac{\mathcal{E}'_R}{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R} \to \cdots \to \frac{\mathcal{E}_R \cap \pi^{-(a-1)}\mathcal{E}'_R}{\pi\mathcal{E}_R} \to \mathcal{E}_\kappa \to \cdots$$

$$\left(\text{resp., } \cdots \leftarrow \mathcal{E}'_\kappa \leftarrow \frac{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi\mathcal{E}'_R} \leftarrow \cdots \leftarrow \frac{\mathcal{E}_R}{\mathcal{E}_R \cap \pi^{-(a-1)}\mathcal{E}'_R} \leftarrow 0 \leftarrow \cdots : (\mathcal{E}'_\kappa)^{\mathrm{can}}\right)$$

and they satisfy

1. There exists an isomorphism $\mathcal{E}_\kappa^{\mathrm{can},i}/\mathcal{E}_\kappa^{\mathrm{can},i-1} \cong (\mathcal{E}'_\kappa)^{\mathrm{can},m-i+1}/(\mathcal{E}'_\kappa)^{\mathrm{can},m-i}$ for $i = 1, \ldots, m$.
2. The filtration $\mathcal{E}_\kappa^{\mathrm{can}}$ (resp., $(\mathcal{E}'_\kappa)^{\mathrm{can}}$) corresponds to the morphism

$$\overline{\mathrm{ST}}_R|_{x=0} = \Theta_\kappa \to \mathscr{C}oh_n^d \quad \left(\text{resp., } \overline{\mathrm{ST}}_R|_{y=0} = \Theta_\kappa \to \mathscr{C}oh_n^d\right).$$

This finishes the proof. $\square$

Remark 3.3.4. Lemma 3.3.2 simplifies the verification of S-completeness for open substacks $\mathscr{U} \subset \mathscr{C}oh_n^d$. Indeed, if we complete the set-up of S-completeness for $\mathscr{U}$ as

$$\begin{tikzcd}
\overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] \arrow[dr, "\circ"] & \mathscr{U} \arrow[d, hook] \\
\overline{\mathrm{ST}}_R \arrow[r, dashed, "\exists !"'] & \mathscr{C}oh_n^d
\end{tikzcd}$$

then there exists a unique dotted arrow filling in. So to check whether $\mathscr{U}$ is S-complete, it suffices to check whether the image of $0 \in \overline{\mathrm{ST}}_R$ in $\mathscr{C}oh_n^d$ lies in $\mathscr{U}$.

Combined with Proposition-Definition 3.3.3, this gives a first description of S-completeness for open substacks of $\mathscr{C}oh_n^d$.

Proposition 3.3.5. *An open substack $\mathscr{U} \subset \mathscr{C}oh_n^d$ is S-complete if and only if for*

1. *every DVR R over k with fraction field K and residue field κ,*

2. *every two families $\mathcal{E}_R, \mathcal{E}'_R \in \mathscr{U}(R)$ such that $\mathcal{E}_K \cong \mathcal{E}'_K$,*

we have $\mathrm{gr}(\mathcal{E}_\kappa^{\mathrm{can}}) \cong \mathrm{gr}((\mathcal{E}'_\kappa)^{\mathrm{can}}) \in \mathscr{U}(\kappa)$, where $\mathcal{E}_\kappa^{\mathrm{can}}, (\mathcal{E}'_\kappa)^{\mathrm{can}}$ are the canonical filtrations of $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ (given by Proposition-Definition 3.3.3).

3.3.1 Special fibers in S-complete family

Proposition 3.3.5 hints us to have a closer look at the special fibers $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ in an S-complete family. A priori we have two descriptions of them and they turn out to be equivalent. This equivalence leads to the promised characterization of S-completeness for open substacks of $\mathscr{C}oh_n^d$ (see Proposition 3.3.17).

Non-separated points in $\mathscr{C}oh_n^d$

Definition 3.3.6. Two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{C}oh_n^d(k)$ are said to be *non-separated* if there exist

1. a DVR R over k with fraction field K and residue field κ.

2. two families $\mathcal{G}_R, \mathcal{G}'_R \in \mathscr{C}oh_n^d(R)$ such that $\mathcal{G}_K \cong \mathcal{G}'_K$ and $\mathcal{G}_\kappa = \mathcal{E}_\kappa \ncong \mathcal{E}'_\kappa = \mathcal{G}'_\kappa$.

and we say that the non-separation of $\mathcal{E} \ncong \mathcal{E}'$ is *realized* by $\mathcal{G}_R, \mathcal{G}'_R \in \mathscr{C}oh_n^d(R)$.

By definition, the special fibers $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ appearing in an S-complete family are non-separated points in $\mathscr{C}oh_n^d$.

Remark 3.3.7. This notion of non-separation can be seen as an algebraic analogy of non-Hausdorff separation in the usual sense. If two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{C}oh_n^d(k)$ are non-separated, then any open neighbourhoods of $\mathcal{E}$ and $\mathcal{E}'$ intersect non-trivially. In this way we recover the usual non-Hausdorff separation.

Indeed, if the non-separation of $\mathcal{E} \ncong \mathcal{E}'$ is realized by $\mathcal{G}_R, \mathcal{G}'_R \in \mathscr{C}oh_n^d(R)$, then for any open neighbourhood $\mathscr{U} \subset \mathscr{C}oh_n^d$ of $\mathcal{E}$ and $\mathscr{U}' \subset \mathscr{C}oh_n^d$ of $\mathcal{E}'$, it follows that $\mathcal{G}_R \in \mathscr{U}(R), \mathcal{G}'_R \in \mathscr{U}'(R)$ since $\mathscr{U}, \mathscr{U}' \subset \mathscr{C}oh_n^d$ are open. Then $\mathcal{G}_K \cong \mathcal{G}'_K \in (\mathscr{U} \cap \mathscr{U}')(K)$ implies that $\mathscr{U} \cap \mathscr{U}' \neq \emptyset$.

Lemma 3.3.8. *For any non-separated points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{C}oh_n^d(k)$, there exist*

1. *a field κ/k.*

2. *an isomorphism $\det(\mathcal{E}_\kappa) \cong \det(\mathcal{E}'_\kappa)$.*

3. *non-zero morphisms $h_\kappa : \mathcal{E}_\kappa \to \mathcal{E}'_\kappa, h'_\kappa : \mathcal{E}'_\kappa \to \mathcal{E}_\kappa$ such that $h_\kappa \circ h'_\kappa = h'_\kappa \circ h_\kappa = 0$.*

Proof. Suppose the non-separation of $\mathcal{E} \ncong \mathcal{E}'$ is realized by $\mathcal{G}_R, \mathcal{G}'_R \in \mathscr{C}oh_n^d(R)$ for some DVR R over k with fraction field K and residue field κ. This gives (1).

The Picard scheme $\mathrm{Pic}^d(C)$ is separated, then $\det(\mathcal{G}_K) \cong \det(\mathcal{G}'_K)$ implies that $\det(\mathcal{G}_R) \cong \det(\mathcal{G}'_R)$. In particular $\det(\mathcal{G}_\kappa) \cong \det(\mathcal{G}'_\kappa)$, i.e., $\det(\mathcal{E}_\kappa) \cong \det(\mathcal{E}'_\kappa)$. This proves (2).

For (3) we keep the construction and notation in the proof of the existence part of Lemma 3.3.2. Composing the maps x_i's and y_i's in the obvious way gives rise to two morphisms $h_R : \mathcal{G}_R \to \mathcal{G}'_R$ and $h'_R : \mathcal{G}'_R \to \mathcal{G}_R$ such that

$$h_R \circ h'_R = h'_R \circ h_R = \pi^{a+b} \text{ (using } x_i y_i = y_i x_i = \pi),$$

then $h_\kappa \circ h'_\kappa = h'_\kappa \circ h_\kappa = 0$. Both h_κ and h'_κ are non-zero because a and b are the minimal integers such that $\pi^a \lambda_K$ and $\pi^b \lambda_K^{-1}$ define morphisms $\mathcal{G}_R \to \mathcal{G}'_R$ and $\mathcal{G}'_R \to \mathcal{G}_R$ extending λ_K and λ_K^{-1} respectively, for some (and hence any) fixed isomorphism $\lambda_K : \mathcal{G}_K \xrightarrow{\sim} \mathcal{G}'_K$. $\square$

Remark 3.3.9. If $k = \mathbb{C}$ and $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^d(k)$, then Lemma 3.3.8 (2) is proved in [Nor79, Proposition 2] using analytic methods. Here we give an alternative argument (independent of the construction in the proof of Lemma 3.3.2) in this case to show that there exists a non-zero morphism $h_\kappa : \mathcal{E}_\kappa \to \mathcal{E}'_\kappa$ as this is the statement we really use in the sequel. The argument essentially comes from [Hei10, Proof of Proposition 6.5] and uses semi-continuity of sheaf cohomology. Indeed, consider the Cartesian diagram

$$\begin{tikzcd} C \times \mathrm{Spec}(R) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

and the vector bundle $\mathcal{G}_R \otimes (\mathcal{G}'_R)^* \otimes q^*\omega_C$ over $C \times \mathrm{Spec}(R)$. Applying the functor $\mathbf{R}^1 p_*$ we obtain by base change a coherent sheaf over $\mathrm{Spec}(R)$ whose fiber over $\mathrm{Spec}(K)$ is

$$H^1(C_K, \mathcal{G}_K \otimes (\mathcal{G}'_K)^* \otimes q^*\omega_C) \cong H^0(C_K, \mathcal{G}_K^* \otimes \mathcal{G}'_K)^* = \mathrm{Hom}(\mathcal{G}_K, \mathcal{G}'_K)^* \neq 0.$$

Then its fiber over $\mathrm{Spec}(\kappa)$ is also non-zero by semi-continuity of sheaf cohomology, i.e., $\mathrm{Hom}(\mathcal{E}_\kappa, \mathcal{E}'_\kappa)^* = \mathrm{Hom}(\mathcal{G}_\kappa, \mathcal{G}'_\kappa)^* \neq 0$. This gives a non-zero morphism from $\mathcal{E}_\kappa$ to $\mathcal{E}'_\kappa$.

Example 3.3.10. Non-separation does NOT occur in $\mathscr{B}un_n^{d,s}$. There are several ways to see this:

1. For any $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^{d,s}(k)$, we have $\mathrm{Hom}(\mathcal{E}_\kappa, \mathcal{E}'_\kappa) = 0$ for any field κ/k and then we can apply Lemma 3.3.8 (2).

2. Any two families $\mathcal{G}_R, \mathcal{G}'_R \in \mathscr{B}un_n^{d,s}(R)$ coinciding on generic fiber must be isomorphic on special fiber by Langton's theorem A (see [Lan75]).

Points in $\mathscr{C}oh_n^d$ with opposite filtrations

Definition 3.3.11. Two points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{C}oh_n^d(\mathbb{K})$ for some field $\mathbb{K}/k$ are said to have *opposite filtrations* if there exist filtrations of subsheaves (of the same length)

$$\mathcal{E}^\bullet : 0 = \mathcal{E}^0 \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^m = \mathcal{E} \text{ and } (\mathcal{E}')^\bullet : 0 = (\mathcal{E}')^0 \subset (\mathcal{E}')^1 \subset \cdots \subset (\mathcal{E}')^m = \mathcal{E}'$$

such that $\mathcal{E}^i/\mathcal{E}^{i-1} \cong (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i}$ for $i = 1, \ldots, m$. In this case, $\mathrm{gr}(\mathcal{E}^\bullet) \cong \mathrm{gr}((\mathcal{E}')^\bullet)$ and $\det(\mathcal{E}) \cong \det(\mathcal{E}')$.

By Proposition-Definition 3.3.3, the special fibers $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ in an S-complete family have opposite filtrations.

Proposition 3.3.12. *Let $\mathscr{U} \subset \mathscr{B}un_n^d$ be an S-complete open substack. If $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(k)$ are non-separated and $\mathcal{E}$ is unstable, then $\mathscr{U}$ contains an unstable direct sum of vector bundles (of rank strictly less than n).*

Proof. By Proposition 3.3.5, the canonical filtrations $\mathcal{E}_\kappa^{\mathrm{can}}, (\mathcal{E}'_\kappa)^{\mathrm{can}}$ of $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ (for some field κ/k) satisfy

$$\mathrm{gr}(\mathcal{E}_\kappa^{\mathrm{can}}) \cong \mathrm{gr}((\mathcal{E}'_\kappa)^{\mathrm{can}}) \in \mathscr{U}(\kappa).$$

Then both filtrations are filtrations of subbundles and cannot be trivial as $\mathcal{E}_\kappa \not\cong \mathcal{E}'_\kappa$. Since $\mathcal{E}_\kappa$ is unstable, $\mathrm{gr}(\mathcal{E}_\kappa^{\mathrm{can}})$ is again unstable and it is a direct sum of vector bundles. □

Equivalence of two notions

By Proposition-Definition 3.3.3, non-separated points in $\mathscr{C}oh_n^d$ have canonical filtrations which are opposite. The interesting part is that the converse is also true, i.e.,

Theorem 3.3.13. *For any points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{C}oh_n^d(k)$, the following are equivalent:*

1. *$\mathcal{E}$ and $\mathcal{E}'$ are non-separated.*
2. *$\mathcal{E}$ and $\mathcal{E}'$ have opposite filtrations.*

Proof. It remains to show (2) $\Rightarrow$ (1). Suppose $\mathcal{E}$ and $\mathcal{E}'$ have opposite filtrations, i.e., there exist filtrations of subsheaves (of the same length)

$$\mathcal{E}^\bullet : 0 = \mathcal{E}^0 \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^m = \mathcal{E} \text{ and } (\mathcal{E}')^\bullet : 0 = (\mathcal{E}')^0 \subset (\mathcal{E}')^1 \subset \cdots \subset (\mathcal{E}')^m = \mathcal{E}'$$

such that $\mathcal{E}^i/\mathcal{E}^{i-1} \cong (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i}$ for $i = 1, \ldots, m$. Let R be a complete DVR over k with fraction field K, residue field κ and uniformizer π (e.g., $R = k[[\pi]]$), let

$$A := R[x, y]/xy - \pi$$

then A is a complete noetherian ring with respect to the π-adic topology by [Sta21, Tag 0DYC]. Let $\mathbb{G}_m$ act on A such that x, y have $\mathbb{G}_m$-weights $1, -1$ respectively. We claim that the two opposite filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$ define a family over the coordinate cross $\mathrm{Spec}(A/\pi) \cong \mathrm{Spec}(\kappa[x,y]/xy)$. Indeed, the Rees construction applied to $\mathcal{E}^\bullet$ (resp., $(\mathcal{E}')^\bullet$) gives a $\mathbb{G}_m$-invariant family over $\mathrm{Spec}(\kappa[x]) = \mathrm{Spec}(A/(\pi, y))$ (resp., $\mathrm{Spec}(\kappa[y]) = \mathrm{Spec}(A/(\pi, x))$) and the isomorphisms $\mathcal{E}^i/\mathcal{E}^{i-1} \cong (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i}$ for $i = 1, \ldots, m$ are employed to glue these two families at the origin $\mathrm{Spec}(\kappa) = \mathrm{Spec}(A/(\pi, x, y))$ to obtain a $\mathbb{G}_m$-invariant family over

$$\mathrm{Spec}(\kappa[x]) \cup_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa[y]) \cong \mathrm{Spec}(\kappa[x,y]/xy) \cong \mathrm{Spec}(A/\pi).$$

Upshot, we have a $\mathbb{G}_m$-invariant coherent sheaf $\mathcal{F}_1$ over $C \times \mathrm{Spec}(A/\pi)$, flat over $\mathrm{Spec}(A/\pi)$, thus inducing a morphism

$$f_1 : [\mathrm{Spec}(A/\pi)/\mathbb{G}_m] \to \mathscr{C}oh_n^d$$

and we want to lift it to a morphism $f_2 : [\mathrm{Spec}(A/\pi^2)/\mathbb{G}_m] \to \mathscr{C}oh_n^d$, i.e.,

$$\begin{tikzcd}
{[\mathrm{Spec}(A/\pi)/\mathbb{G}_m]} \arrow[r, "f_1"] \arrow[d, hook] & \mathscr{C}oh_n^d \arrow[d] \\
{[\mathrm{Spec}(A/\pi^2)/\mathbb{G}_m]} \arrow[r] \arrow[ur, dashed, "f_2"'] & \mathrm{Spec}(k)
\end{tikzcd}$$

Since $\mathscr{C}oh_n^d$ is smooth over k, such a lifting f_2 always exists (see [Ill04, Chapter 3, Theorem 4.7 and Corollary 4.8]). Let $\mathcal{F}_2$ be the $\mathbb{G}_m$-invariant coherent sheaf over $C \times \mathrm{Spec}(A/\pi^2)$ corresponding to f_2 and there is a $\mathbb{G}_m$-equivariant isomorphism $\mathcal{F}_2|_{C \times \mathrm{Spec}(A/\pi)} \cong \mathcal{F}_1$.

By induction, we obtain a $\mathbb{G}_m$-invariant coherent sheaf $\mathcal{F}_i$ over $C \times \mathrm{Spec}(A/\pi^i)$, flat over $\mathrm{Spec}(A/\pi^i)$ and there is a $\mathbb{G}_m$-equivariant isomorphism $\mathcal{F}_i|_{C \times \mathrm{Spec}(A/\pi^{i-1})} \cong \mathcal{F}_{i-1}$ for each $i \geq 1$, i.e.,

$$\begin{tikzcd}
\mathrm{Spec}(A/\pi) \arrow[r, "f_1"] \arrow[d, hook] & \mathscr{C}oh_n^d \\
\mathrm{Spec}(A/\pi^2) \arrow[ur, "f_2"'] \arrow[d, hook] & \\
\vdots \arrow[d, hook] & \\
\mathrm{Spec}(A/\pi^i) \arrow[uuur, "f_i"'] \arrow[d, hook] & \\
\vdots &
\end{tikzcd}$$

By coherent completeness (see [HR19, Corollary 1.5]), the natural morphism

$$\mathscr{C}oh_n^d(A) \to \varprojlim \mathscr{C}oh_n^d(A/\pi^i)$$

is an equivalence of groupoids. Hence the $\mathbb{G}_m$-invariant system $\{\mathcal{F}_i \in \mathscr{C}oh_n^d(A/\pi^i)\}_{i\in\mathbb{N}}$ corresponds to a $\mathbb{G}_m$-invariant coherent sheaf $\mathcal{F}$ over $C \times \operatorname{Spec}(A)$, flat over $\operatorname{Spec}(A)$ such that

$$\mathcal{F}|_{x\neq 0} = \mathcal{E}, \mathcal{F}|_{y\neq 0} = \mathcal{E}' \text{ and } \mathcal{F}|_{x=0} = \mathcal{E}^{\bullet}, \mathcal{F}|_{y=0} = (\mathcal{E}')^{\bullet}.$$

In other words, this $\mathbb{G}_m$-invariant coherent sheaf $\mathcal{F}$ induces a morphism

$$\overline{\mathrm{ST}}_R = [\operatorname{Spec}(A)/\mathbb{G}_m] \to \mathscr{C}oh_n^d$$

such that when restricted to $\overline{\mathrm{ST}}_R - \{0\} = \operatorname{Spec}(R) \cup_{\operatorname{Spec}(K)} \operatorname{Spec}(R)$, we obtain two families $\mathcal{G}_R, \mathcal{G}'_R \in \mathscr{C}oh_n^d(R)$ such that $\mathcal{G}_K \cong \mathcal{G}'_K$ and $\mathcal{G}_\kappa = \mathcal{E}_\kappa \not\cong \mathcal{E}'_\kappa = \mathcal{G}'_\kappa$. Then $\mathcal{E}$ and $\mathcal{E}'$ are non-separated by definition. $\square$

Remark 3.3.14. By Theorem 3.3.13, it follows that in the definition of non-separation (Definition 3.3.6), it is equivalent to require that the residue field is k.

Remark 3.3.15. The $\mathbb{G}_m$-invariant coherent sheaf $\mathcal{F}_1$ over $C \times \operatorname{Spec}(A/\pi)$ which is flat over $\operatorname{Spec}(A/\pi)$ can also be written down explicitly using Lemma 3.3.1.

Indeed, fixing an isomorphism $\mathcal{E}^i/\mathcal{E}^{i-1} \cong (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i}$ for each i we can identify both sides and denote by

$$\mathcal{Q}_i := \mathcal{E}^i/\mathcal{E}^{i-1} \cong (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i} \text{ for } i = 1, \ldots, m.$$

Set

$$\mathcal{G}_i := \begin{cases} \mathcal{E}' & \text{if } i \leq 0 \\ \mathcal{G}_i & \text{if } 0 < i < m-1 \\ \mathcal{E} & \text{if } i \geq m-1 \end{cases}$$

where the connecting coherent sheaves $\mathcal{G}_i$ (for $0 < i < m-1$) over C are defined by $\mathcal{G}_i := \ker(\theta_i)$ and

$$\theta_i : (\mathcal{E}')^{m-i} \oplus \mathcal{E}^{i+1} \twoheadrightarrow ((\mathcal{E}')^{m-i}/(\mathcal{E}')^{m-i-1}) \oplus (\mathcal{E}^{i+1}/\mathcal{E}^i) \cong \mathcal{Q}_{i+1}^{\oplus 2} \twoheadrightarrow \mathcal{Q}_{i+1}$$

Let x_i and y_i act via

$$\begin{tikzcd}[column sep=small]
\mathcal{G}_\bullet : & \cdots \arrow[r, bend right, "0"'] & \mathcal{E}' \arrow[l, bend right, "\mathrm{id}"'] \arrow[r, bend right, "0"'] & \mathcal{E}' \arrow[l, bend right, "\mathrm{id}"'] \arrow[r, bend right, "x_0"'] & \mathcal{G}_1 \arrow[l, bend right, "y_0"'] \arrow[r, bend right, "x_1"'] & \cdots \arrow[l, bend right, "y_1"'] \arrow[r, bend right, "x_{m-3}"'] & \mathcal{G}_{m-2} \arrow[l, bend right, "y_{m-3}"'] \arrow[r, bend right, "x_{m-2}"'] & \mathcal{E} \arrow[l, bend right, "y_{m-2}"'] \arrow[r, bend right, "\mathrm{id}"'] & \mathcal{E} \arrow[l, bend right, "0"'] \arrow[r, bend right, "\mathrm{id}"'] & \cdots \arrow[l, bend right, "0"'] \\
& \cdots & -1 & 0 & 1 & \cdots & m-2 & m-1 & m & \cdots
\end{tikzcd}$$

where the connecting morphisms x_i and y_i (for $0 \le i \le m-2$) are defined respectively, by

1. $x_0 : \mathcal{E}' \to \mathcal{G}_1$ is induced by the morphism

$$\mathcal{E}' \twoheadrightarrow \mathcal{E}'/(\mathcal{E}')^{m-1} \cong \mathcal{E}^1 \hookrightarrow \mathcal{E}^2 \hookrightarrow (\mathcal{E}')^{m-1} \oplus \mathcal{E}^2$$

   which factors through $\mathcal{G}_1$ since its composition with θ_1 is zero.

2. $y_0 : \mathcal{G}_1 \to \mathcal{E}'$ is given by the composition

$$\mathcal{G}_1 \hookrightarrow (\mathcal{E}')^{m-1} \oplus \mathcal{E}^2 \xrightarrow{\begin{pmatrix} \mathrm{icl} & 0 \\ 0 & 0 \end{pmatrix}} \mathcal{E}'$$

3. $x_i : \mathcal{G}_i \to \mathcal{G}_{i+1}$ (for $0 < i < m-2$) is induced by the morphism

$$\mathcal{G}_i \hookrightarrow (\mathcal{E}')^{m-i} \oplus \mathcal{E}^{i+1} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{icl} \end{pmatrix}} (\mathcal{E}')^{m-i-1} \oplus \mathcal{E}^{i+2}$$

   which factors through $\mathcal{G}_{i+1}$ since its composition with θ_{i+1} is zero.

4. $y_i : \mathcal{G}_{i+1} \to \mathcal{G}_i$ (for $0 < i < m-2$) is induced by the morphism

$$\mathcal{G}_{i+1} \hookrightarrow (\mathcal{E}')^{m-i-1} \oplus \mathcal{E}^{i+2} \xrightarrow{\begin{pmatrix} \mathrm{icl} & 0 \\ 0 & 0 \end{pmatrix}} (\mathcal{E}')^{m-i} \oplus \mathcal{E}^{i+1}$$

   which factors through $\mathcal{G}_i$ since its composition with θ_i is zero.

5. $x_{m-2} : \mathcal{G}_{m-2} \to \mathcal{E}$ is given by the composition

$$\mathcal{G}_{m-2} \hookrightarrow (\mathcal{E}')^2 \oplus \mathcal{E}^{m-1} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{icl} \end{pmatrix}} \mathcal{E}$$

6. $y_{m-2} : \mathcal{E} \to \mathcal{G}_{m-2}$ is induced by the morphism

$$\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}^{m-1} \cong (\mathcal{E}')^1 \hookrightarrow (\mathcal{E}')^2 \hookrightarrow (\mathcal{E}')^2 \oplus \mathcal{E}^{m-1}$$

   which factors through $\mathcal{G}_{m-2}$ since its composition with θ_{m-2} is zero.

In this diagram, we can check that

1. $x_i y_i = y_i x_i = 0$ for any $i \in \mathbb{Z}$, as the composition of the matrices given above is zero.

2. $\mathcal{G}_i$ is finitely presentable for any $i \in \mathbb{Z}$.

3. the restriction of $\mathcal{G}_\bullet$ to $\{x \neq 0\}$ is the colimit over the $\mathbb{Z}$-sequence of maps

$$x_i : \mathcal{G}_i \to \mathcal{G}_{i+1} \Rightarrow \mathcal{E}$$

4. the restriction of $\mathcal{G}_\bullet$ to $\{y \neq 0\}$ is the colimit over the $\mathbb{Z}$-sequence of maps

$$y_i : \mathcal{G}_{i+1} \to \mathcal{G}_i \Rightarrow \mathcal{E}'$$

5. the restriction of $\mathcal{G}_\bullet$ to $\{x = 0\}$ is the (generalized) $\mathbb{Z}$-filtration

$$\cdots \longleftarrow \mathcal{G}_i/x_{i-1}\mathcal{G}_{i-1} \xleftarrow[y_i]{} \mathcal{G}_{i+1}/x_i\mathcal{G}_i \longleftarrow \cdots$$

Claim 3.3.16. This $\mathbb{Z}$-filtration is nothing but the filtration $\mathcal{E}'_\bullet$

$$\cdots \longleftarrow \mathcal{E}' \longleftarrow (\mathcal{E}')^{m-1} \longleftarrow \cdots \longleftarrow (\mathcal{E}')^2 \longleftarrow (\mathcal{E}')^1 \longleftarrow 0 \longleftarrow \cdots$$

In particular, the connecting morphisms are injective.

Proof of Claim 3.3.16. By definition, this $\mathbb{Z}$-filtration is of the form

$$\cdots \longleftarrow \mathcal{E}' \xleftarrow[y_0]{} \mathcal{G}_1/x_0\mathcal{E}' \longleftarrow \cdots \longleftarrow \mathcal{G}_{m-2}/x_{m-3}\mathcal{G}_{m-3} \xleftarrow[y_{m-2}]{} \mathcal{E}/x_{m-2}\mathcal{G}_{m-2} \longleftarrow 0 \longleftarrow \cdots$$

Indeed, for $0 < i < m-1$, one has $\mathcal{G}_i/x_{i-1}\mathcal{G}_{i-1} \cong \mathcal{E}'_{m-i}$ which follows from the following commutative diagram and Snake lemma

$$\begin{tikzcd}
 & 0 & & & \\
 & ? \arrow[u] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{G}_i \arrow[r] \arrow[u] & (\mathcal{E}')^{m-i} \oplus \mathcal{E}^{i+1} \arrow[r] \arrow[d, equal] & \mathcal{Q}_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x_{i-1}\mathcal{G}_{i-1} \arrow[r] \arrow[u] & (\mathcal{E}')^{m-i} \oplus \mathcal{E}^{i+1} \arrow[r] & (\mathcal{E}')^{m-i} \oplus \mathcal{Q}_{i+1} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[u] & & (\mathcal{E}')^{m-i} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

For $i = m-2$, we have $x_{m-2}\mathcal{G}_{m-2} \cong \mathrm{Im}((\mathcal{E}')^2 \oplus \mathcal{E}^{m-1}) \cong \mathcal{E}^{m-1} \subset \mathcal{E}$, where the first isomorphism follows from considering the image of the short exact sequence in $\mathcal{E}$

$$0 \to \mathcal{G}_{m-2} \to (\mathcal{E}')^2 \oplus \mathcal{E}^{m-1} \to \mathcal{Q}_2 \to 0$$

and the second one follows from definition. Thus $\mathcal{E}/x_{m-2}\mathcal{G}_{m-2} \cong \mathcal{E}/\mathcal{E}^{m-1} \cong (\mathcal{E}')^1$. $\square$

6. the restriction of $\mathcal{G}_\bullet$ to $\{y = 0\}$ is the (generalized) $\mathbb{Z}$-filtration

$$\cdots \longrightarrow \mathcal{G}_{i-1}/y_{i-1}\mathcal{G}_i \xrightarrow{x_{i-1}} \mathcal{G}_i/y_i\mathcal{G}_{i+1} \longrightarrow \cdots$$

   Similarly, the $\mathbb{Z}$-filtration is nothing but the filtration $\mathcal{E}_\bullet$

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{E}^1 \longrightarrow \mathcal{E}^2 \longrightarrow \cdots \longrightarrow \mathcal{E}^{m-1} \longrightarrow \mathcal{E} \longrightarrow \cdots$$

   In particular, the connecting morphisms are injective.

7. $x_i : \mathcal{G}_i \to \mathcal{G}_{i+1}$ is an isomorphism for $i > m-2$.

8. $y_i : \mathcal{G}_{i+1} \to \mathcal{G}_i$ is an isomorphism for $i < 0$.

This finishes the construction.

Moreover, the proof of (2) $\Rightarrow$ (1) in Theorem 3.3.13 provides the relatively difficult Only If direction in the following characterization for S-completeness.

Proposition 3.3.17. *An open substack $\mathscr{U} \subset \mathscr{C}oh_n^d$ is S-complete if and only if for any field $\mathbb{K}/k$, any points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(\mathbb{K})$ with opposite filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$, we have $\mathrm{gr}(\mathcal{E}^\bullet) = \mathrm{gr}((\mathcal{E}')^\bullet) \in \mathscr{U}(\mathbb{K})$.*

Proof. Only If Part: Suppose $\mathcal{E}, \mathcal{E}' \in \mathscr{U}(\mathbb{K})$ have opposite filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$. By the proof of Theorem 3.3.13 there exists a morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$ for some DVR R with residue field $\mathbb{K}$ such that the image of $0 \in \overline{\mathrm{ST}}_R$ in $\mathscr{C}oh_n^d$ is $\mathrm{gr}(\mathcal{E}^\bullet) = \mathrm{gr}((\mathcal{E}')^\bullet)$. Then $\mathscr{U}$ being S-complete implies that $\mathrm{gr}(\mathcal{E}^\bullet) = \mathrm{gr}((\mathcal{E}')^\bullet) \in \mathscr{U}(\mathbb{K})$.

If Part: This is achieved using Proposition 3.3.5. For every DVR R over k with fraction field K and residue field κ, every two families $\mathcal{E}_R, \mathcal{E}'_R \in \mathscr{U}(R)$ such that $\mathcal{E}_K \cong \mathcal{E}'_K$, since the canonical filtrations $\mathcal{E}_\kappa^{\mathrm{can}}, (\mathcal{E}'_\kappa)^{\mathrm{can}}$ of $\mathcal{E}_\kappa, \mathcal{E}'_\kappa$ are opposite we have $\mathrm{gr}(\mathcal{E}_\kappa^{\mathrm{can}}) = \mathrm{gr}((\mathcal{E}'_\kappa)^{\mathrm{can}}) \in \mathscr{U}(\kappa)$ by assumption, as desired. $\square$

Remark 3.3.18. In view of Proposition 3.3.17, to check S-completeness it is vital to understand when two points in $\mathscr{C}oh_n^d$ have opposite filtrations, or equivalently when two points in $\mathscr{C}oh_n^d$ are non-separated. A sufficient condition for two points in $\mathscr{B}un_n^d$ being non-separated will be given later.

As a first application of Proposition 3.3.17, we show that $\mathscr{B}un_n^{d,ss}$ is S-complete. The starting point is to characterize opposite filtrations of semi-stable vector bundles.

Lemma 3.3.19. *If $\mathbb{K}/k$ is a field and $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^{d,ss}(\mathbb{K})$ have opposite filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$, then $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$ can be refined to be Jordan-Hölder filtrations of $\mathcal{E}, \mathcal{E}'$ respectively. In particular,* $\mathrm{gr}(\mathcal{E}^\bullet) = \mathrm{gr}((\mathcal{E}')^\bullet) \in \mathscr{B}un_n^{d,ss}(\mathbb{K})$.

Proof. Let $\mu := \mu(\mathcal{E}) = \mu(\mathcal{E}')$. By definition, it suffices to show each $\mathcal{E}^i/\mathcal{E}^{i-1}$ is semi-stable of slope μ. A simple computation about slope yields that it is equivalent to show each $\mathcal{E}^i$ is of slope μ (and hence semi-stable). For this, we prove by induction that

$$\mu(\mathcal{E}^i) = \mu = \mu((\mathcal{E}')^{m-i}) \text{ for each } i, \text{ where } m := \mathrm{length}(\mathcal{E}^\bullet) = \mathrm{length}((\mathcal{E}')^\bullet).$$

For $i = 1$, by semi-stability of $\mathcal{E}$ and $\mathcal{E}'$ we have

$$\mu = \mu(\mathcal{E}') \le \mu(\mathcal{E}'/(\mathcal{E}')^{m-1}) = \mu(\mathcal{E}^1) \le \mu(\mathcal{E}) \le \mu(\mathcal{E}/\mathcal{E}^{m-1}) = \mu((\mathcal{E}')^1) \le \mu(\mathcal{E}') = \mu$$

and hence $\mu(\mathcal{E}^1) = \mu = \mu((\mathcal{E}')^{m-1})$. Suppose $\mu(\mathcal{E}^{i-1}) = \mu = \mu((\mathcal{E}')^{m-i+1})$, we need to show $\mu(\mathcal{E}^i) = \mu = \mu((\mathcal{E}')^{m-i})$. For this we consider the following diagram relating these terms

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^{i-1} \arrow[r] & \mathcal{E}^i \arrow[r] & \mathcal{E}^i/\mathcal{E}^{i-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\mathcal{E}')^{m-i} \arrow[r] & (\mathcal{E}')^{m-i+1} \arrow[r] & (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i} \arrow[r] & 0
\end{tikzcd}$$

and the inequality

$$\mu = \mu(\mathcal{E}^{i-1}) \ge \mu(\mathcal{E}^i) \ge \mu(\mathcal{E}^i/\mathcal{E}^{i-1}) = \mu((\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i}) \ge \mu((\mathcal{E}')^{m-i+1}) = \mu$$

gives the desired formula. □

By Proposition 3.3.17, this lemma yields that

Corollary 3.3.20 ([AHLH18], Lemma 8.4)**.** *The stack $\mathscr{B}un_n^{d,ss}$ is S-complete.*

Lemma 3.3.21. *The stack $\mathscr{B}un_n^d$ (if $n > 1$) is not S-complete.*

Proof. Fix a closed point $x \in C(k)$. For any point $\mathcal{E} \in \mathscr{B}un_n^d(k)$, we choose a sub-line bundle $\mathcal{L} \subset \mathcal{E}$ and consider the following filtration of $\mathcal{E}$

$$\mathcal{E}^\bullet : 0 \subset \mathcal{L}(-x) \subset \mathcal{L} \subset \mathcal{E}.$$

Let $\mathcal{L}' := \mathcal{E}/\mathcal{L}$ and choose an extension $\mathcal{E}' \in \mathrm{Ext}^1(\mathcal{L}(-x), \mathcal{L}'(x))$ with $\mathcal{E}' \ncong \mathcal{E}$. Then the following filtration of $\mathcal{E}'$

$$(\mathcal{E}')^\bullet : 0 \subset \mathcal{L}' \subset \mathcal{L}'(x) \subset \mathcal{E}'$$

is, by construction, opposite to $\mathcal{E}^\bullet$. Since $\mathrm{gr}(\mathcal{E}) \cong \mathrm{gr}((\mathcal{E}')^\bullet) \notin \mathscr{B}un_n^d(k)$, this implies that $\mathscr{B}un_n^d$ is not S-complete by Proposition 3.3.17. $\square$

3.3.2 Non-separated points in $\mathscr{B}un_n^d$

In this subsection, as an easy consequence of Theorem 3.3.13, we give a sufficient condition for two points in $\mathscr{B}un_n^d$ being non-separated.

Proposition 3.3.22 ([Nor78], Proposition 2 if $k = \mathbb{C}$)**.** *For any points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^d(k)$, if*

1. *there exists an isomorphism $\lambda : \det(\mathcal{E}) \xrightarrow{\sim} \det(\mathcal{E}')$.*

2. *there exist morphisms $h : \mathcal{E} \to \mathcal{E}'$ of rank r and $h' : \mathcal{E}' \to \mathcal{E}$ of rank $n - r$ such that*

   (a) $h \circ h' = h' \circ h = 0$.

   (b) the following diagram

$$\begin{tikzcd} \wedge^r(\mathcal{E}) \arrow[r, "\wedge^r(h)"] \arrow[d, "\varphi_{\mathcal{E}}"', "\cong"] & \wedge^r(\mathcal{E}') \arrow[d, "\cong"', "\varphi_{\mathcal{E}'}"] \\ \wedge^{n-r}(\mathcal{E})^* \otimes \det(\mathcal{E}) \arrow[r, "\wedge^{n-r}(h')^* \otimes \lambda"'] & \wedge^{n-r}(\mathcal{E}')^* \otimes \det(\mathcal{E}') \end{tikzcd} \tag{3.3.1}$$

   commutes, where $\varphi_\bullet : \wedge^r(\bullet) \xrightarrow{\sim} \wedge^{n-r}(\bullet)^* \otimes \det(\bullet)$ is the canonical isomorphism.

then $\mathcal{E}$ and $\mathcal{E}'$ are non-separated.

The arguments given in [Nor78, Proposition 2] are of analytic nature. In particular [Nor78, Corollary 1 and 2] can be deduced from Theorem 3.3.13.

Proof. By Theorem 3.3.13, it suffices to show that $\mathcal{E}$ and $\mathcal{E}'$ have opposite filtrations.

Suppose there exist morphisms $h : \mathcal{E} \to \mathcal{E}'$ of rank r and $h' : \mathcal{E}' \to \mathcal{E}$ of rank $n - r$ satisfying the conditions above. Then we have the following commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & 0 & & & \\
 & \mathcal{T}_1 \arrow[u] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_2 \arrow[r] \arrow[u] & \mathcal{E} \arrow[r] \arrow[d, shift left, "h"] & \mathcal{E}_1 \arrow[r] \arrow[d, "i"] & 0 \\
0 & \mathcal{E}_1' \arrow[l] \arrow[u, "i'"] & \mathcal{E}' \arrow[l] \arrow[u, shift left, "h'"] & \mathcal{E}_2' \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & 0 \arrow[u] & & \mathcal{T}_2 \arrow[d] & \\
 & & & 0 &
\end{tikzcd}
\tag{3.3.2}
$$

where $\mathcal{E}_1' := \mathrm{Im}(h') \subset \mathcal{E}$ (resp., $\mathcal{E}_1 := \mathrm{Im}(h) \subset \mathcal{E}'$) and $\mathcal{E}_2 \subset \mathcal{E}$ (resp., $\mathcal{E}_2' \subset \mathcal{E}'$) is the subbundle of generated by $\mathcal{E}_1'$ (resp., $\mathcal{E}_1$), $\mathcal{T}_1 := \mathrm{coker}(\mathcal{E}_1' \hookrightarrow \mathcal{E}_2)$ and $\mathcal{T}_2 := \mathrm{coker}(\mathcal{E}_1 \hookrightarrow \mathcal{E}_2')$. Then $\mathcal{E}_1, \mathcal{E}_2'$ are vector bundles of rank r and $\mathcal{E}_1', \mathcal{E}_2$ are vector bundles of rank $n-r$, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are torsion sheaves of rank 0. As for the exactness of the first row, the condition $h \circ h' = 0$ implies that $\mathrm{Im}(h') \subset \ker(h) = \ker(\mathcal{E} \twoheadrightarrow \mathcal{E}_1)$. Since $\ker(\mathcal{E} \twoheadrightarrow \mathcal{E}_1) \subset \mathcal{E}$ is a subbundle, it contains the subbundle of $\mathcal{E}$ generated by $\mathrm{Im}(h')$, i.e., $\mathcal{E}_2$. This shows that $\mathcal{E}_2 \subset \ker(\mathcal{E} \twoheadrightarrow \mathcal{E}_1)$. As both of them are subbundles of $\mathcal{E}$ of the same rank, they must be equal. Similar for the second row.

To finish the proof, we claim that $\mathcal{T}_1 \cong \mathcal{T}_2$ and thus the following filtrations

$$0 \subset \mathcal{E}_1' \subset \mathcal{E}_2 \subset \mathcal{E} \text{ and } 0 \subset \mathcal{E}_1 \subset \mathcal{E}_2' \subset \mathcal{E}'$$

are the opposite filtrations of $\mathcal{E}$ and $\mathcal{E}'$. Indeed, using the factorizations of h, h' in (3.3.2) we can decompose (3.3.1) as follows

$$
\begin{tikzcd}
 & \det(\mathcal{E}_1) \arrow[rr, hook, "\det(i)"] \arrow[dd] & & \det(\mathcal{E}_2') \arrow[dl] \arrow[dd, "\cong"] \\
\wedge^r(\mathcal{E}) \arrow[ur] \arrow[rr, "\cong"] \arrow[dd, "\cong"'] & & \wedge^r(\mathcal{E}') \arrow[dd] & \\
 & \det(\mathcal{E}_2)^* \otimes \det(\mathcal{E}) \arrow[rr, hook, "\det(i')^* \otimes \lambda", "\cong"'] & & \det(\mathcal{E}_1')^* \otimes \det(\mathcal{E}') \arrow[dl] \\
\wedge^{n-r}(\mathcal{E})^* \otimes \det(\mathcal{E}) \arrow[ur] \arrow[rr] & & \wedge^{n-r}(\mathcal{E}')^* \otimes \det(\mathcal{E}') &
\end{tikzcd}
$$

To proceed, the idea is to show that

$$\mathcal{T}_2 \cong \operatorname{coker}(\det(\mathcal{E}_1) \xrightarrow{\det(i)} \det(\mathcal{E}_2'))$$
$$\mathcal{T}_1 \cong \operatorname{coker}(\det(\mathcal{E}_2)^* \otimes \det(\mathcal{E}) \xrightarrow{\det(i')^* \otimes \lambda} \det(\mathcal{E}_1')^* \otimes \det(\mathcal{E}'))$$

and then the commutativity of (3.3.1) implies that $\mathcal{T}_1 \cong \mathcal{T}_2$.

1. From the short exact sequence $0 \to \mathcal{E}_1 \xrightarrow{i} \mathcal{E}_2' \to \mathcal{T}_2 \to 0$ we obtain

$$0 \to \det(\mathcal{E}_1) \xrightarrow{\det(i)} \det(\mathcal{E}_2') \to \mathcal{T}_2 \to 0.$$

2. From the short exact sequence $0 \to \mathcal{E}_1' \xrightarrow{i'} \mathcal{E}_2 \to \mathcal{T}_1 \to 0$ we obtain $0 \to \det(\mathcal{E}_1') \xrightarrow{\det(i')} \det(\mathcal{E}_2) \to \mathcal{T}_1 \to 0$. Applying the functor $\mathcal{H}om(-, \mathcal{O}_C)$ yields

$$0 \to \det(\mathcal{E}_2)^* \xrightarrow{\det(i')^*} \det(\mathcal{E}_1')^* \to \mathcal{T}_1 \to 0$$

and then applying the functor $- \otimes \det(\mathcal{E})$ yields

$$0 \to \det(\mathcal{E}_2)^* \otimes \det(\mathcal{E}) \xrightarrow{\det(i')^* \otimes \mathrm{id}} \det(\mathcal{E}_1')^* \otimes \det(\mathcal{E}) \to \mathcal{T}_1 \to 0$$

where the computation uses the fact that for any torsion sheaf $\mathcal{T}$ of rank 0 and vector bundle $\mathcal{V}$ over C

$$\mathcal{H}om(\mathcal{T}, \mathcal{O}_C) = 0, \mathcal{E}xt^1(\mathcal{T}, \mathcal{O}_C) = \mathcal{T} \text{ and } \mathcal{T} \otimes \mathcal{V} \cong \mathcal{T}^{\oplus \mathrm{rk}(\mathcal{V})}.$$

This gives rise to the following commutative diagram with exact rows

$$\begin{tikzcd}
\det(\mathcal{E}_1) \arrow[r, hook, "\det(i)"] \arrow[d, "\cong"'] & \det(\mathcal{E}_2') \arrow[r, two heads] \arrow[d, "\cong"] & \mathcal{T}_2 \\
\det(\mathcal{E}_2)^* \otimes \det(\mathcal{E}) \arrow[r, hook, "\det(i')^* \otimes \lambda"] \arrow[d, equal] & \det(\mathcal{E}_1')^* \otimes \det(\mathcal{E}') \arrow[r, two heads] & \mathcal{T} \\
\det(\mathcal{E}_2)^* \otimes \det(\mathcal{E}) \arrow[r, hook, "\det(i')^* \otimes \mathrm{id}"'] & \det(\mathcal{E}_1')^* \otimes \det(\mathcal{E}) \arrow[r, two heads] \arrow[u, "\mathrm{id} \otimes \lambda", "\cong"'] & \mathcal{T}_1
\end{tikzcd}$$

and hence $\mathcal{T}_1 \cong \mathcal{T} \cong \mathcal{T}_2$. □

Corollary 3.3.23 ([Nor78], Corollary 1 if $k = \mathbb{C}$)**.** *For any points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^d(k)$, if*

1. *there exists an isomorphism* $\det(\mathcal{E}) \xrightarrow{\sim} \det(\mathcal{E}')$*.*

2. *there exists a morphism* $h : \mathcal{E} \to \mathcal{E}'$ *of rank* $n - 1$*.*

then $\mathcal{E}$ and $\mathcal{E}'$ are non-separated.

Proof. The rank of h being $n-1$ implies that we can define a morphism $h' : \mathcal{E}' \to \mathcal{E}$ of rank 1 using the commutativity of (3.3.1), which satisfies $h \circ h' = h' \circ h = 0$. $\square$

3.4 Proper good moduli space of $\mathscr{B}un_n^{d,ss}$

Proposition 3.4.1 ([AHLH18], Theorem 8.1)**.** *The stack $\mathscr{B}un_n^{d,ss}$ admits a proper good moduli space.*

The proper good moduli space of $\mathscr{B}un_n^{d,ss}$ is actually projective, but a proof of this is beyond Theorem 0.

Proof. Since $\mathscr{B}un_n^{d,ss}$ is locally linear reductive (Lemma 3.1.3), Θ-reductive (Lemma 3.2.7) and has unpunctured inertia, it admits a good moduli space according to Theorem 0. Moreover, this good moduli space is proper since $\mathscr{B}un_n^{d,ss}$ is S-complete (Corollary 3.3.20) and satisfies the existence part of valuative criterion for properness by Langton's theorem B (see [Lan75]). $\square$

Remark 3.4.2. Although $\mathscr{C}oh_n^d$ is S-complete (Lemma 3.3.2) and Θ-reductive (Lemma 3.2.4), it does not admit a good moduli space (let alone separated) as it is not locally linear reductive (Lemma 3.1.4). However, this is not a contradiction to Theorem 0 since $\mathscr{C}oh_n^d$ is not quasi-compact: the family of coherent sheaves over C of rank n and degree d is not bounded. This also indicates that the quasi-compactness assumption in Theorem 0 cannot be dropped.

In summary, we have the following check-list for different stacks towards the conditions in Theorem 0:

	Locally linearly reductive	Θ-reductive	Unpunctured inertia	S-complete
$\mathscr{B}un_n^{d,ss}$	Y (Lemma 3.1.3)	Y (Lemma 3.2.7)	Y	Y (Corollary 3.3.20)
$\mathscr{B}un_n^{d}$	N (Lemma 3.1.4)	N (Corollary 3.2.13)	Y	N (Lemma 3.3.21)
$\mathscr{C}oh_n^{d}$	N (Lemma 3.1.4)	Y (Lemma 3.2.4)	Y	Y (Lemma 3.3.2)

Chapter 4

Rank 2 case

In this chapter, we give some further consequence when open substacks of $\mathscr{B}un_2^d$ are locally linearly reductive, Θ-reductive or S-complete. A combination of these consequences will be sufficient to yield Theorem A.

4.1 More on local linear reductivity

In rank 2 case, we have an explicit description of vector bundles with linearly reductive automorphism groups.

Proposition 4.1.1. *If $\mathcal{E} \in \mathscr{B}un_2^d(k)$ such that $\mathrm{Aut}(\mathcal{E})$ is linearly reductive, then one of the following occurs:*

1. *$\mathcal{E}$ is simple.*

   In this case $\mathrm{Aut}(\mathcal{E}) = \mathbb{G}_m$.

2. *$\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ for some line bundles $\mathcal{L}_1 \ncong \mathcal{L}_2$ with $\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) = 0$ for $i \neq j$.*

   In this case $\mathrm{Aut}(\mathcal{E}) = \mathbb{G}_m^2$.

3. *$\mathcal{E} \cong \mathcal{L} \oplus \mathcal{L}$ for some line bundle $\mathcal{L}$.*

   In this case $\mathrm{Aut}(\mathcal{E}) = \mathrm{GL}_2$.

Proof. For any $\mathcal{E} \in \mathscr{B}un_2^d(k)$ with linearly reductive automorphism group, we have two possibilities:

1. If $\mathcal{E}$ is indecomposable, then it is simple by Lemma 3.1.1. This is (1).

2. If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ for some line bundles $\mathcal{L}_1, \mathcal{L}_2$, then we have two subcases:

   (a) If $\mathcal{L}_1 \cong \mathcal{L}_2$, then $\mathrm{Aut}(\mathcal{E}) = \mathrm{GL}_2$. This is (3).

   (b) If $\mathcal{L}_1 \ncong \mathcal{L}_2$, then there are again two subcases:

– If $\mathcal{E}$ is semi-stable, then $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2)$ and $\mathcal{L}_1 \ncong \mathcal{L}_2$ implies that $\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_j) = 0$ for $i \neq j$. Hence $\mathrm{Aut}(\mathcal{E}) = \mathbb{G}_m^2$. This is (2).

– If $\mathcal{E}$ is unstable, then we may assume $\deg(\mathcal{L}_1) > \deg(\mathcal{L}_2)$ and this implies that $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = 0$. To conclude we claim that $\mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1) = 0$ and hence $\mathrm{Aut}(\mathcal{E}) = \mathbb{G}_m^2$. This is (2). Indeed, the assignment $\varphi \mapsto \mathrm{id}_{\mathcal{E}} + \varphi := \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}$ realizes $\mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)$ as a normal[1] unipotent subgroup of $\mathrm{Aut}(\mathcal{E})$, then $\mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)$ must vanish since $\mathrm{Aut}(\mathcal{E})$ is linearly reductive.

□

4.2 More on Θ-reductivity

Recall that (see Lemma 3.2.17) any open substack of $\mathscr{B}un_n^d$ supporting a Θ-testing family cannot be Θ-reductive. This fact has a simple consequence in rank 2 case.

Proposition 4.2.1. *Let $\mathscr{U} \subset \mathscr{B}un_2^d$ be a Θ-reductive open substack. Then $\mathscr{U}$ cannot contain any direct sum of line bundles of different degrees.*

Proposition 4.2.1 is proved by contradiction. If $\mathscr{U}$ contains a direct sum of line bundles of different degrees, then it will support a Θ-testing family in the sense of Lemma 3.2.17 and this preserves $\mathscr{U}$ from being Θ-reductive. The starting point is that if $\mathscr{U}$ contains a direct sum of line bundles of different degrees, then it contains a lot of such.

Lemma 4.2.2. *Let $\mathscr{U} \subset \mathscr{B}un_2^d$ be an open substack. If $\mathcal{V}_1 \oplus \mathcal{V}_2 \in \mathscr{U}(k)$ for some line bundles $\mathcal{V}_1, \mathcal{V}_2$ of degrees $d_1 < d_2$ respectively, then for any effective divisor D on C of degree $d_2 - d_1$, there exist line bundles $\mathcal{L}_1, \mathcal{L}_2$ of degrees d_1, d_2 respectively such that*

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \in \mathscr{U}(k) \text{ and } \mathcal{L}_2(-D) \oplus \mathcal{L}_1(D) \in \mathscr{U}(k).$$

Proof. Note that any effective divisor D on C of degree $d_2 - d_1$ induces an automorphism of $\mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C)$ as follows:

$$\begin{aligned} \lambda_D : \mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C) &\xrightarrow{\sim} \mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C) \\ (\mathcal{L}_1, \mathcal{L}_2) &\mapsto (\mathcal{L}_2(-D), \mathcal{L}_1(D)) \end{aligned}$$

Let $\oplus : \mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C) \to \mathscr{B}un_2^d$ be the direct sum morphism, then $\Omega := \oplus^{-1}(\mathscr{U}) \subset \mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C)$ is open and non-empty (as it contains $(\mathcal{V}_1, \mathcal{V}_2)$), therefore dense. This

[1] Since any automorphism of $\mathcal{E}$ preserves its Harder-Narasimhan filtration $0 \subset \mathcal{L}_1 \subset \mathcal{E}$, i.e., it is of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

implies that $\lambda_D^{-1}(\Omega) \cap \Omega \neq \emptyset$ and any point $(\mathcal{L}_1, \mathcal{L}_2)$ in this intersection satisfies

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \in \mathscr{U}(k) \text{ and } \mathcal{L}_2(-D) \oplus \mathcal{L}_1(D) \in \mathscr{U}(k).$$

This finishes the proof. □

Proof of Proposition 4.2.1. Suppose $\mathcal{V}_1 \oplus \mathcal{V}_2 \in \mathscr{U}(k)$ for some line bundles $\mathcal{V}_1, \mathcal{V}_2$ of degrees $d_1 < d_2$ respectively. Fix an effective divisor D on C of degree $d_2 - d_1$, then by Lemma 4.2.2 there exist line bundles $\mathcal{L}_1, \mathcal{L}_2$ of degree d_1, d_2 respectively such that

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \in \mathscr{U}(k) \text{ and } \mathcal{L}_2(-D) \oplus \mathcal{L}_1(D) \in \mathscr{U}(k).$$

Let $[\varrho] \in \mathrm{Ext}^1(\mathcal{L}_1 \oplus \mathcal{O}_D, \mathcal{L}_2(-D))$ be the extension class represented by

$$[\varrho] : 0 \to \mathcal{L}_2(-D) \xrightarrow{(0,\mathrm{incl})} \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{L}_1 \oplus \mathcal{O}_D \to 0.$$

Then by definition the quadruple

$$(\mathcal{L}_2(-D), \mathcal{L}_1(D), D, [\varrho]) \in \mathrm{Pic}^{d_1}(k) \times \mathrm{Pic}^{d_2}(k) \times \mathrm{Div}^{\mathrm{eff}}(C) \times \mathrm{Ext}^1(\mathcal{L}_1 \oplus \mathcal{O}_D, \mathcal{L}_2(-D))$$

is weird such that $\mathcal{L}_2(-D) \oplus \mathcal{L}_1(D) \in \mathscr{U}(k)$ and $[\varrho] \in \mathscr{U}(k)$. By Lemma 3.2.17 $\mathscr{U}$ cannot be Θ-reductive, we are done. □

4.3 More on S-completeness

Recall that (Proposition 3.3.17) S-completeness concerns non-separated points in $\mathscr{B}un_n^d$ (or equivalently, points in $\mathscr{B}un_n^d$ with opposite filtrations). Similar to Θ-reductivity, S-completeness also has a simple consequence in rank 2 case.

Corollary 4.3.1. *Let $\mathscr{U} \subset \mathscr{B}un_2^d$ be an S-complete open substack. If $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(k)$ are non-separated and $\mathcal{E}$ is unstable, then $\mathscr{U}$ contains a direct sum of line bundles of different degrees.*

Proof. This follows immediately from Proposition 3.3.12. □

Corollary 4.3.1 will be effective once we know when two points in $\mathscr{B}un_2^d$ are non-separated. This is characterized by the following result: they are simply the points in $\mathscr{B}un_2^d$ with the same determinant and non-zero morphisms.

Lemma 4.3.2 ([Nor78], Corollary 2 if $k = \mathbb{C}$)**.** *Two points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{B}un_2^d(k)$ are non-separated if and only if*

1. *there exists an isomorphism $\det(\mathcal{E}) \xrightarrow{\sim} \det(\mathcal{E}')$.*

2. *there exists a non-zero morphism $h : \mathcal{E} \to \mathcal{E}'$.*

Proof. Only If Part: Lemma 3.3.8. If Part: Corollary 3.3.23. $\square$

4.4 Proof of Theorem A

Using Theorem 0, we can now prove the first key step towards Theorem A. It helps to reduce our problem to a more understandable part of $\mathscr{B}un_2^d$.

Lemma 4.4.1. *If $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack that admits a good moduli space, then $\mathscr{U} \subset \mathscr{B}un_2^{d,ss} \cup \mathscr{B}un_2^{d,simple}$.*

Proof. If $\mathscr{U} \subset \mathscr{B}un_2^d$ admits a good moduli space, then $\mathscr{U}$ is locally linearly reductive and Θ-reductive by Theorem 0.

For any point $\mathcal{E} \in \mathscr{U}(\mathbb{K})$ (for some field $\mathbb{K}/k$), local linear reductivity of $\mathscr{U}$ entails that $\mathcal{E}$ specializes to a closed point $\mathcal{E}_0 \in \mathscr{U}(k)$ and $\mathrm{Aut}(\mathcal{E}_0)$ is linearly reductive. By Proposition 4.1.1, we have

1. either $\mathcal{E}_0$ is simple. Then $\mathcal{E}$ is also simple by the openness of simpleness.
2. or $\mathcal{E}_0 \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ for some line bundles $\mathcal{L}_1, \mathcal{L}_2$. But Θ-reductivity of $\mathscr{U}$ forces that $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2)$ (see Proposition 4.2.1), i.e., $\mathcal{E}_0$ is semi-stable. Then $\mathcal{E}$ is also semi-stable by the openness of semi-stability.

This shows that $\mathcal{E} \in \mathscr{B}un_2^{d,ss}(\mathbb{K}) \cup \mathscr{B}un_2^{d,simple}(\mathbb{K})$, as desired. $\square$

4.4.1 Proof of Theorem A (1)

If $(2, d) = 1$, i.e., d is odd, then $\mathscr{B}un_2^{d,ss} = \mathscr{B}un_2^{d,s} \subset \mathscr{B}un_2^{d,simple}$ and hence $\mathscr{B}un_2^{d,simple} \subset \mathscr{B}un_2^d$ is the unique maximal open substack that admits a good moduli space by Lemma 4.4.1.

4.4.2 Proof of Theorem A (2)

In this subsection, we always assume that $(2, d) \neq 1$, i.e., d is even. If $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack that admits a good moduli space, then $\mathscr{U} \subset \mathscr{B}un_2^{d,ss} \cup \mathscr{B}un_2^{d,simple}$ by Lemma 4.4.1. To conclude, it suffices to show that $\mathscr{U} \subset \mathscr{B}un_2^{d,ss}$ or $\mathscr{U} \subset \mathscr{B}un_2^{d,simple}$.

Suppose $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,ss}$ and $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,simple}$, then

1. $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,ss}$ implies that $\mathscr{U}$ contains an unstable vector bundle.

   Let $\mathcal{E} \in \mathscr{U}(k)$ be an unstable vector bundle and $\varrho : 0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$ be its Harder-Narasimhan filtration, then

$$d_1 := \deg(\mathcal{L}_1) > d/2 > \deg(\mathcal{L}_2) =: d_2$$

   and we define $a := d_1 - d/2 = d/2 - d_2 > 0$.

2. $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,simple}$ implies that $\mathscr{U}$ contains a direct sum of line bundles.

   Indeed, if $\mathscr{U}$ doesn't contain any direct sum of line bundles, then by local linear reductivity (see Proposition 4.1.1), any point in $\mathscr{U}$ specializes to a simple vector bundle in $\mathscr{U}$. By the openness of simpleness, any point in $\mathscr{U}$ is simple, i.e., $\mathscr{U} \subset \mathscr{B}un_2^{d,simple}$, a contradiction.

   Let $\mathcal{V}_1 \oplus \mathcal{V}_2 \in \mathscr{U}(k)$ be a direct sum of line bundles, then $\mathcal{V}_1 \oplus \mathcal{V}_2$ is semi-stable by Lemma 4.4.1, i.e., $\deg(\mathcal{V}_1) = \deg(\mathcal{V}_2) = d/2$. Consider the direct sum morphism

$$\oplus : \mathrm{Pic}^{d/2}(C) \times \mathrm{Pic}^{d/2}(C) \to \mathscr{B}un_2^d \text{ sending } (\mathcal{L}_1, \mathcal{L}_2) \mapsto \mathcal{L}_1 \oplus \mathcal{L}_2$$

   Then $\Omega := \oplus^{-1}(\mathscr{U}) \subset \mathrm{Pic}^{d/2}(C) \times \mathrm{Pic}^{d/2}(C)$ is an open dense (since it contains $(\mathcal{V}_1, \mathcal{V}_2)$) subset.

Building on these data, we can construct a degeneration in $\mathscr{U}$ from a strictly semi-stable vector bundle $\mathcal{G}_K$ (as an extension of line bundles of the same degree, using the open dense subset Ω defined in (2)) to an unstable vector bundle $\mathcal{G}_\kappa$ (using the unstable vector bundle $\mathcal{E}$ in (1)). Then any Jordan-Hölder filtration $\mathcal{G}_K^\bullet$ of $\mathcal{G}_K$ with $\mathrm{gr}(\mathcal{G}_K^\bullet) \in \mathscr{U}(K)$ cannot even be extended to a filtration of subbundles of $\mathcal{G}_\kappa$ (otherwise $\mathcal{G}_\kappa$ will be semi-stable as well), contradicting to Θ-reductivity of $\mathscr{U}$. Some technique employed below appears in [MS09, §2].

Let $\mathcal{P}_i \to C \times \mathrm{Pic}^{d_i}(C)$ be the universal family for $i = 1, 2$ and consider the following diagram

$$\begin{tikzcd}[column sep=small] & C \times \mathrm{Pic}^0(C) \arrow[dl, "\mathrm{id}_C \times \vartheta_1"'] \arrow[dr, "\mathrm{id}_C \times \vartheta_2"] & \\ C \times \mathrm{Pic}^{d_1}(C) & & C \times \mathrm{Pic}^{d_2}(C) \end{tikzcd}$$

where the morphism $\vartheta_i : \mathrm{Pic}^0(C) \to \mathrm{Pic}^{d_i}(C)$ is given by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{L}_i$. Let $\mathcal{Q}_i := (\mathrm{id}_C \times \vartheta_i)^* \mathcal{P}_i$ be the pull-back of the universal family to $C \times \mathrm{Pic}^0(C)$ and it satisfies

$$\mathcal{Q}_i|_{C \times \{\mathcal{L}\}} = \mathcal{L} \otimes \mathcal{L}_i \text{ for any } \mathcal{L} \in \mathrm{Pic}^0(C).$$

The families $\mathcal{Q}_1, \mathcal{Q}_2$ over $C \times \mathrm{Pic}^0(C)$ define a Cartesian diagram (see the proof of Proposition 3.2.10)

$$\begin{tikzcd} \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1) \arrow[r] \arrow[d, "\mathrm{pr}_{13}"'] \arrow[dr, phantom, "\ulcorner"] & \mathscr{E}xt(d_2, d_1) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_{13}"] & \mathscr{B}un_2^d \\ \mathrm{Pic}^0(C) \arrow[r, "(\vartheta_1, \vartheta_2)"'] & \mathrm{Pic}^{d_1}(C) \times \mathrm{Pic}^{d_2}(C) & \end{tikzcd}$$

and by definition

$$\mathrm{pr}_{13}^{-1}(\mathcal{L}) = \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)|_{\mathcal{L}} = \mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1) \text{ for any } \mathcal{L} \in \mathrm{Pic}^0(C).$$

In the above diagram we have

$$\begin{tikzcd} \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_{13}"'] & \mathscr{B}un_2^d \\ \mathrm{Pic}^0(C) \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} (\mathcal{L}, [e]) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_{13}"'] & \mathcal{E}(\mathcal{L}, [e]) \\ \mathcal{L} \end{tikzcd}$$

where $\mathcal{E}(\mathcal{L}, [e]) \in \mathscr{B}un_2^d(k)$ is the vector bundle given by $[e] \in \mathrm{pr}_{13}^{-1}(\mathcal{L})$, i.e., it fits into the short exact sequence

$$[e] : 0 \to \mathcal{L} \otimes \mathcal{L}_1 \to \mathcal{E}(\mathcal{L}, [e]) \to \mathcal{L} \otimes \mathcal{L}_2 \to 0$$

Note that $\mathrm{pr}_2((\mathcal{O}_C, [\varrho])) = \mathcal{E} \in \mathscr{U}(k)$, it follows that $\Phi := \mathrm{pr}_2^{-1}(\mathscr{U}) \subset \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)$ is open dense. Hence $\Delta := \mathrm{pr}_{13}(\Phi) \subset \mathrm{Pic}^0(C)$ is also dense since pr_{13} is a surjection between irreducible stacks. In summary we have

$$\begin{tikzcd} \Phi \arrow[r, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_{13}"'] & \mathscr{U} \\ \Delta \end{tikzcd}$$

To finish the proof, we claim that

Claim 4.4.2. There exist

$$\text{a line bundle } \mathcal{L} \in \Delta \text{ and an effective divisor } D \in \mathrm{Div}^a(C)$$

such that $(\mathcal{L}(-D) \otimes \mathcal{L}_1) \oplus (\mathcal{L}(D) \otimes \mathcal{L}_2) \in \mathscr{U}(k)$.

Indeed, take a line bundle $\mathcal{L} \in \Delta$ and an effective divisor $D \in \mathrm{Div}^a(C)$ as in Claim 4.4.2. Then by definition of Δ there exists an element $[e] \in \mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1)$ such that $(\mathcal{L}, [e]) \in \Phi$, i.e., $\mathcal{E}(\mathcal{L}, [e]) \in \mathscr{U}(k)$. From the extension

$$[e] : 0 \to \mathcal{L} \otimes \mathcal{L}_1 \to \mathcal{E}(\mathcal{L}, [e]) \to \mathcal{L} \otimes \mathcal{L}_2 \to 0$$

one obtains a twisted extension

$$[e_D] : 0 \to \mathcal{L}(-D) \otimes \mathcal{L}_1 \to \mathcal{E}(\mathcal{L}, [e]) \to (\mathcal{L} \oplus \mathcal{O}_D) \otimes \mathcal{L}_2 \to 0.$$

Then by definition the quadruple

$$(\mathcal{L}(-D)\otimes\mathcal{L}_1, \mathcal{L}(D)\otimes\mathcal{L}_2, D, [e_D]) \in \mathrm{Pic}^{d/2}(k)\times\mathrm{Pic}^{d/2}(k)\times\mathrm{Div}^a(C)\times\mathrm{Ext}^1((\mathcal{L}\oplus\mathcal{O}_D)\otimes\mathcal{L}_2, \mathcal{L}(-D)\otimes\mathcal{L}_1)$$

is weird such that $(\mathcal{L}(-D)\otimes\mathcal{L}_1)\oplus(\mathcal{L}(D)\otimes\mathcal{L}_2) \in \mathscr{U}(k)$ and $[e_D] \in \mathscr{U}(k)$. By Lemma 3.2.17 $\mathscr{U}$ cannot be Θ-reductive, a contradiction.

Proof of Claim 4.4.2. Consider the morphism

$$\begin{aligned}\lambda : \mathrm{Pic}^0(C) \times \mathrm{Div}^a(C) &\to \mathrm{Pic}^{d/2}(C) \times \mathrm{Pic}^{d/2}(C)\\ (\mathcal{L}, D) &\mapsto (\mathcal{L}_1 \otimes \mathcal{L}(-D), \mathcal{L}_2 \otimes \mathcal{L}(D))\end{aligned}$$

and it is surjective since[2]

$$((\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2^*)^{1/2}, (\mathcal{V}_1^* \otimes \mathcal{V}_2 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^*)^{1/2}) \mapsto (\mathcal{V}_1, \mathcal{V}_2).$$

for any $(\mathcal{V}_1, \mathcal{V}_2) \in \mathrm{Pic}^{d/2}(C) \times \mathrm{Pic}^{d/2}(C)$. Then $\lambda^{-1}(\Omega) \subset \mathrm{Pic}^0(C) \times \mathrm{Div}^a(C)$ is open dense and therefore

$$\lambda^{-1}(\Omega) \cap (\Delta \times \mathrm{Div}^a(C)) \neq \emptyset.$$

By definition any point $(\mathcal{L}, D)$ in this intersection satisfies the condition. $\square$

4.4.3 Proof of Theorem A (3)

Indeed, Theorem A (3) is a corollary of the following stronger lemma.

Lemma 4.4.3. *If $\mathcal{E} \in \mathscr{B}un_2^d(k)$ is unstable, then no open substack of $\mathscr{B}un_2^d$ containing $\mathcal{E}$ admits a separated good moduli space.*

The idea is essentially from [Nor78, Proposition 4]. If $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack containing an unstable vector bundle $\mathcal{E}$, then we can "flip" its Harder-Narasimhan filtration to produce another vector bundle $\mathcal{E}'$ in $\mathscr{U}$. By construction $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{U}(k)$ have opposite filtrations and hence non-separated. If $\mathscr{U}$ is S-complete, then it will contain a direct sum of line bundles of different degrees by Corollary 4.3.1. However, by Proposition 4.2.1 $\mathscr{U}$ can no longer be Θ-reductive.

Proof. Suppose $\mathscr{U} \subset \mathscr{B}un_2^d$ is an open substack containing $\mathcal{E}$ that admits a separated good moduli space. Let

$$\varrho : 0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$$

be the Harder-Narasimhan filtration of $\mathcal{E}$ and let $d_1 := \deg(\mathcal{L}_1) > \deg(\mathcal{L}_2) =: d_2$. Similar construction to §4.4.2 yields a diagram

$$\begin{array}{ccc} \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1) & \xrightarrow{\mathrm{pr}_2} & \mathscr{B}un_2^d \\ {\scriptstyle \mathrm{pr}_{13}}\downarrow & & \\ \mathrm{Pic}^0(C) & & \end{array} \quad \text{given by} \quad \begin{array}{ccc} (\mathcal{L}, [e]) & \xrightarrow{\mathrm{pr}_2} & \mathcal{E}(\mathcal{L}, [e]) \\ {\scriptstyle \mathrm{pr}_{13}}\downarrow & & \\ \mathcal{L} & & \end{array}$$

[2] The square root $(-)^{1/2}$ exists since $[2]_{\mathrm{Pic}^0(C)} : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)$ is an isogeny (see [Mil86, Theorem 8.2]).

such that

$$\mathrm{pr}_{13}^{-1}(\mathcal{L}) = \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)|_{\mathcal{L}} = \mathrm{Ext}^1(\mathcal{L}^* \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1) \text{ for any } \mathcal{L} \in \mathrm{Pic}^0(C)$$

where $\mathcal{E}(\mathcal{L}, [e]) \in \mathscr{B}un_2^d(k)$ is the vector bundle given by $[e] \in \mathrm{pr}_{13}^{-1}(\mathcal{L})$, i.e., it fits into the short exact sequence

$$[e] : 0 \to \mathcal{L} \otimes \mathcal{L}_1 \to \mathcal{E}(\mathcal{L}, [e]) \to \mathcal{L}^* \otimes \mathcal{L}_2 \to 0$$

Again, $\Phi := \mathrm{pr}_2^{-1}(\mathscr{U}) \subset \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)$ is open dense and $\Delta := \mathrm{pr}_{13}(\Phi) \subset \mathrm{Pic}^0(C)$ is also dense. In summary we have

$$\begin{tikzcd} \Phi \arrow[r, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_{13}"'] & \mathscr{U} \\ \Delta & \end{tikzcd}$$

Fix a line bundle $\beta \in \mathrm{Pic}^{d_1-d_2}(C)$ with $H^0(C, \beta) \neq 0$. Then for any $\mathcal{L} \in \mathrm{Pic}^0(C)$, by construction and choice of β there always exists a non-zero morphism

$$\mathcal{E}(\mathcal{L}, [x]) \to \mathcal{E}(\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta, [y])$$

between rank 2 vector bundles of the same determinant, for any $[x] \in \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)|_{\mathcal{L}}$ and $[y] \in \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)|_{\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta}$, i.e.,

$$[x] : 0 \to \mathcal{L} \otimes \mathcal{L}_1 \to \mathcal{E}(\mathcal{L}, [x]) \to \mathcal{L}^* \otimes \mathcal{L}_2 \to 0$$
$$[y] : 0 \to \mathcal{L}^* \otimes \mathcal{L}_2 \otimes \beta \to \mathcal{E}(\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta, [y]) \to \mathcal{L} \otimes \mathcal{L}_1 \otimes \beta^* \to 0.$$

By Corollary 4.3.2 this implies that $\mathcal{E}(\mathcal{L}, [x])$ and $\mathcal{E}(\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta, [y])$ are non-separated. Since $\mathcal{E}(\mathcal{L}, [x])$ is always unstable, if we can choose a line bundle $\mathcal{L} \in \mathrm{Pic}^0(C)$ such that both $\mathcal{E}(\mathcal{L}, [x])$ and $\mathcal{E}(\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta, [y])$ lie in $\mathscr{U}$, then $\mathscr{U}$ contains a direct sum of line bundles of different degrees by Corollary 4.3.1, a contradiction to Proposition 4.2.1. This finishes the proof.

The existence of such a line bundle is equivalent to the following claim.

Claim 4.4.4. There exists $\mathcal{L} \in \mathrm{Pic}^0(C)$ such that $\mathcal{L} \in \Delta$ and $\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta \in \Delta$.

Proof of Claim 4.4.4. This is done by codimension estimate. Let $\Sigma := \mathrm{Pic}^0(C) - \Delta$, then $\mathrm{codim}_{\mathrm{Pic}^0(C)}(\Sigma) \geq 1$ since $\Delta \subset \mathrm{Pic}^0(C)$ is dense.

Suppose otherwise that for any $\mathcal{L} \in \mathrm{Pic}^0(C)$, either $\mathcal{L} \in \Sigma$ or $\mathcal{L}^* \otimes \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta \in \Sigma$, then $\mathrm{Pic}^0(C)$ can be written as a union of subsets of strictly positive codimension

$$\mathrm{Pic}^0(C) = \Sigma \cup (\mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \beta \otimes \Sigma^*)$$

a contraction. $\square$

$\square$

Chapter 5

Higher rank case

In this chapter, we construct an open substack of $\mathscr{B}un_3^2$ that admits a separated (but non-proper) good moduli space, and is not contained in $\mathscr{B}un_3^{2,ss} = \mathscr{B}un_3^{2,s}$. This construction can be easily generalized to arbitrary higher rank.

Throughout this section we always assume that $g := g(C) > 2$.

5.1 Preliminaries

Let λ be the polygon in the rank-degree plane consisting of vertices $\{(0,0),(1,1),(3,2)\}$ and $\mathscr{B}un_3^{2,\leq\lambda} \subset \mathscr{B}un_3^2$ be the open substack consisting of points whose Harder-Narasimhan polygon in the rank-degree plane don't exceed λ.

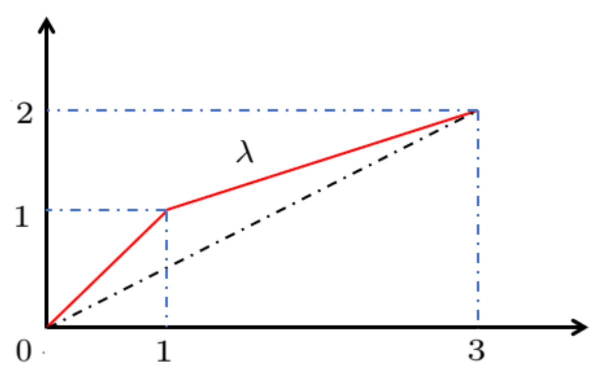

Figure 5.1: The polygon λ in the rank-degree plane

By definition the open substack $\mathscr{B}un_3^{2,\leq\lambda}$ is a disjoint union of

- the open substack $\mathscr{B}un_3^{2,ss} = \mathscr{B}un_3^{2,s}$, in which all points are indecomposable and
- the closed substack $\mathscr{B}un_3^{2,=\lambda}$.

The desired open substack $\mathscr{U} \subset \mathscr{B}un_3^2$ will be an open substack of $\mathscr{B}un_3^{2,\leq\lambda}$ and it is constructed by cutting out two closed substacks $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{B}un_3^{2,\leq\lambda}$.

5.1.1 Decomposable points in $\mathscr{B}un_3^{2,\leq\lambda}$

Our experience in rank 2 case indicates that an explicit description of decomposable points is quite useful for verifying Θ-reductivity. We start with the following easy lemma.

Lemma 5.1.1. *Any point in $\mathscr{B}un_3^{2,\leq\lambda}$ cannot admit proper subbundles of degree >1. In particular, any proper subsheaf of degree >1 of a point in $\mathscr{B}un_3^{2,\leq\lambda}$ is a subbundle.*

Proof. Any proper subsheaf of degree >1 of a point $\mathcal{E}$ in $\mathscr{B}un_3^2$ will make the Harder-Narasimhan polygon of $\mathcal{E}$ lie strictly above λ. $\square$

Lemma 5.1.2. *The intersection*

$$(\mathscr{B}un_1 \oplus \mathscr{B}un_2) \cap \mathscr{B}un_3^{2,\leq\lambda} = (\mathscr{B}un_1 \oplus \mathscr{B}un_2) \cap \mathscr{B}un_3^{2,=\lambda} = \mathscr{B}un_1^1 \oplus \mathscr{B}un_2^{1,ss}$$

exhausts all decomposable points in $\mathscr{B}un_3^{2,\leq\lambda}$.

Proof. The first equality holds since stable vector bundles are always indecomposable. For the second equality, the assertion on degree follows from Lemma 5.1.1 and on stability follows from the uniqueness of Harder-Narasimhan filtration. The last statement holds since $\mathscr{B}un_3^{2,\leq\lambda}$ doesn't contain direct sum of line bundles. $\square$

5.1.2 The (i,j)-stability

Given a pair (i,j) of non-negative integers, there is a notion of (i,j)-stability for vector bundles over C introduced in [NR75, NR78], refining the classical notion of slope-stability and thus giving a stratification of the moduli stack $\mathscr{B}un_n^{d,s} \subset \mathscr{B}un_n^d$ of stable vector bundles over C.

Definition 5.1.3. A vector bundle $\mathcal{E}$ over C is said to be (i,j)-*stable* if for any non-zero proper subbundle $0 \neq \mathcal{E}' \subsetneq \mathcal{E}$, we have

$$\frac{\deg(\mathcal{E}')+i}{\mathrm{rk}(\mathcal{E}')} < \frac{\deg(\mathcal{E})+i-j}{\mathrm{rk}(\mathcal{E})}$$

Remark 5.1.4. Let $\mathcal{E}$ be a vector bundle over C. By definition we have

1. $\mathcal{E}$ is $(0,0)$-stable if and only $\mathcal{E}$ is stable.
2. $\mathcal{E}$ is (i,j)-stable if and only if $\mathcal{E}^*$ is (j,i)-stable.
3. If $\mathcal{E}$ is (i,j)-stable, then $\mathcal{E}$ is (i',j')-stable for any $0 \leq i' \leq i$ and $0 \leq j' \leq j$. In particular, $\mathcal{E}$ is stable.

The (i,j)-stability is an open property by [NR78, Proposition 5.3] and there exists a moduli stack $\mathscr{B}un_n^d(i,j) \subset \mathscr{B}un_n^d$ of (i,j)-stable vector bundles over C. This topic is covered in [MGN17] and it gives, among many other things, a numerical criteria for the existence of (i,j)-stable vector bundles over C by codimension estimate.

Theorem 5.1.5 ([MGN17], Theorem 1.8)**.** *Let C be a smooth projective connected curve of genus $g := g(C) > 1$ over an algebraically closed field. For any pair (i,j) of non-negative integers, there exist (i,j)-stable vector bundles over C of rank n and degree d if*

$$\max\{i(n-1)+j, i+j(n-1)\} < (n-1)(g-1).$$

5.2 Construction of $\mathscr{U} \subset \mathscr{B}un_3^{2,\lambda}$

The desired open substack $\mathscr{U} \subset \mathscr{B}un_3^{2,\leq\lambda}$ is constructed by cutting out two closed substacks $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{B}un_3^{2,\leq\lambda}$.

5.2.1 Closed substack $\mathscr{A}_1 \subset \mathscr{B}un_3^{2,\leq\lambda}$

Denoted by $\mathscr{A}_{1,1} \subset \mathscr{B}un_3^{2,\leq\lambda}$ (resp., $\mathscr{A}_{1,2} \subset \mathscr{B}un_3^{2,\leq\lambda}$) the substack consisting of points $\mathcal{E} \in \mathscr{B}un_3^{2,\leq\lambda}(\mathbb{K})$ for some field $\mathbb{K}/k$ that fits into a short exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ for some $\mathcal{L} \in \mathscr{B}un_1^1(\mathbb{K})$ and $\mathcal{F} \in \mathscr{B}un_2^1(\mathbb{K})$ such that

$$\mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0 \text{ (resp., } \mathcal{F} \text{ admits a sub-line bundle of degree } \geq 0).$$

Let $\mathscr{A}_1 := \mathscr{A}_{1,1} \cup \mathscr{A}_{1,2} \subset \mathscr{B}un_3^{2,\leq\lambda}$. By definition, any point $\mathcal{E} \in \mathscr{A}_1(\mathbb{K})$ for some field $\mathbb{K}/k$ is unstable and any defining sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ of $\mathcal{E} \in \mathscr{A}_1(\mathbb{K})$ is its Harder-Narasimhan filtration (and hence unique). This means that $\mathscr{A}_1 \subset \mathscr{B}un_3^{2,=\lambda}$.

Remark 5.2.1. A few words about the substack $\mathscr{A}_1 \subset \mathscr{B}un_3^{2,\leq\lambda}$.

- Cutting the stack $\mathscr{A}_{1,1}$ is necessary for local linear reductivity. To be precise

  Lemma 5.2.2. *If $\mathscr{U} \subset \mathscr{B}un_3^{2,\leq\lambda}$ is a locally linearly reductive open substack, then $\mathscr{U} \cap \mathscr{A}_{1,1} = \emptyset$.*

  Proof. If $\mathcal{E} \in (\mathscr{U} \cap \mathscr{A}_{1,1})(\mathbb{K}) \subset \mathscr{B}un_3^{2,=\lambda}(\mathbb{K})$ for some field $\mathbb{K}/k$ and let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be its Harder-Narasimhan filtration, then we claim that any degeneration $\mathcal{E}'$ of $\mathcal{E}$ in $\mathscr{U}$ cannot have a linearly reductive automorphism group. Indeed, such $\mathcal{E}'$ also lies in $\mathscr{B}un_3^{2,=\lambda}$ and its Harder-Narasimhan filtration $0 \to \mathcal{L}' \to \mathcal{E}' \to \mathcal{F}' \to 0$ has to be the degeneration of $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$. By upper semi-continuity $\mathrm{Hom}(\mathcal{F}', \mathcal{L}') \neq 0$ since $\mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0$. But the assignment $\phi \mapsto \mathrm{id}_{\mathcal{E}'} + \phi$ realizes $\mathrm{Hom}(\mathcal{F}', \mathcal{L}')$ as a unipotent normal subgroup of $\mathrm{Aut}(\mathcal{E}')$, so $\mathrm{Aut}(\mathcal{E}')$ cannot be linearly reductive. ☐

- The stack $\mathscr{A}_{1,2} \subset \mathscr{B}un_3^{2,=\lambda}$ consists of points $\mathcal{E} \in \mathscr{B}un_3^{2,=\lambda}$ such that the quotient bundle in its Harder-Narasimhan filtration is stable but not $(1,0)$-stable.

Lemma 5.2.3. *The substack $\mathscr{A}_1 \subset \mathscr{B}un_3^{2,\leq\lambda}$ is closed.*

Proof. This is done by showing that both $\mathscr{A}_{1,1}$ and $\mathscr{A}_{1,2}$ are closed in $\mathscr{B}un_3^{2,=\lambda}$.

1. Defining condition for $\mathscr{A}_{1,1}$ is closed on $\mathscr{B}un_3^{2,=\lambda}$ by upper semi-continuity.

2. Defining condition for $\mathscr{A}_{1,2}$ is closed on $\mathscr{B}un_3^{2,=\lambda}$. Indeed, if $\mathcal{E} \in \mathscr{B}un_3^{2,=\lambda}(\mathbb{K})$ for some field $\mathbb{K}/k$ and let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be its Harder-Narasimhan filtration. Since $\mathcal{F}$ is stable, any sub-line bundle of $\mathcal{F}$ has degree ≤ 0 and condition for $\mathscr{A}_{1,2}$ requires that $\mathcal{F}$ admits a sub-line bundle of maximal possible degree. This is a closed condition by upper semi-continuity. Alternatively $\mathcal{F}$ doesn't admit a sub-line bundle of degree 0 if and only if $\mathcal{F}$ is $(1,0)$-stable, which is an open condition.

□

5.2.2 Closed substack $\mathscr{A}_2 \subset \mathscr{B}un_3^{2,\leq\lambda}$

Denoted by $\mathscr{A}_{2,1} \subset \mathscr{B}un_3^{2,\leq\lambda}$ (resp., $\mathscr{A}_{2,2} \subset \mathscr{B}un_3^{2,\leq\lambda}$) the substack consisting of points $\mathcal{E} \in \mathscr{B}un_3^{2,\leq\lambda}(\mathbb{K})$ for some field $\mathbb{K}/k$ that fits into a short exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{L} \to 0$ for some $\mathcal{F} \in \mathscr{B}un_2^1(\mathbb{K})$ and $\mathcal{L} \in \mathscr{B}un_1^1(\mathbb{K})$ such that

$$\mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0 \text{ (resp., } \mathcal{F} \text{ admits a sub-line bundle of degree } \geq 0).$$

Let $\mathscr{A}_2 := \mathscr{A}_{2,1} \cup \mathscr{A}_{2,2} \subset \mathscr{B}un_3^{2,\leq\lambda}$. Unlike $\mathscr{A}_1$, the substack $\mathscr{A}_2$ intersects $\mathscr{B}un_3^{2,s}$ non-trivially.

Lemma 5.2.4. *If $\mathcal{E} \in (\mathscr{A}_{2,1} - \mathscr{A}_{1,1})(\mathbb{K})$ for some field $\mathbb{K}/k$ and $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{L} \to 0$ is a defining sequence of $\mathcal{E} \in \mathscr{A}_{2,1}(\mathbb{K})$, then $\mathcal{E}$ is stable if and only if $\mathcal{F}$ is stable.*

Proof. Only If Part: Clear. If Part: This is proved by contradiction. If $\mathcal{E}$ is unstable and let $0 \to \mathcal{L}' \to \mathcal{E} \to \mathcal{F}' \to 0$ be its Harder-Narasimhan filtration, then $\mathrm{Hom}(\mathcal{F}', \mathcal{L}') = 0$ since $\mathcal{E} \notin \mathscr{A}_{1,1}(\mathbb{K})$. However

1. The composition $\mathcal{L}' \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{L}$ is non-zero and hence an isomorphism.

   Indeed, if it is zero, then this composition factors through an injection $\mathcal{L}' \hookrightarrow \mathcal{F}$, disrupting the stability of $\mathcal{F}$.

2. The composition $\mathcal{F} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{F}'$ is non-zero and hence an isomorphism (since they are stable of the same slope).

Now there is a contradiction since $0 = \mathrm{Hom}(\mathcal{F}', \mathcal{L}') \cong \mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0$. □

Lemma 5.2.5. *The substack $\mathscr{A}_2 \subset \mathscr{B}un_3^{2,\leq\lambda}$ is closed.*

Proof. For every DVR R over k with fraction field K and residue field κ, every family $\mathcal{E}_R \in \mathscr{B}un_3^{2,\leq\lambda}(R)$ with $\mathcal{E}_K \in \mathscr{A}_2(K)$, we need to show $\mathcal{E}_\kappa \in \mathscr{A}_2(\kappa)$.

If $0 \to \mathcal{F}_K \to \mathcal{E}_K \to \mathcal{L}_K \to 0$ is a defining sequence of $\mathcal{E}_K \in \mathscr{A}_2(K)$ and $0 \to \mathcal{F}_\kappa \to \mathcal{E}_\kappa \to \mathcal{L}_\kappa \to 0$ is its degeneration, then $\mathcal{F}_\kappa \subset \mathcal{E}_\kappa$ is a subbundle by Lemma 5.1.1. There are 2 possibilities:

1. If $\mathcal{E}_K \in \mathscr{A}_{2,1}(K)$, i.e., $\mathrm{Hom}(\mathcal{F}_K, \mathcal{L}_K) \neq 0$, then $\mathrm{Hom}(\mathcal{F}_\kappa, \mathcal{L}_\kappa) \neq 0$ by upper semi-continuity, hence $\mathcal{E}_\kappa \in \mathscr{A}_{2,1}(\kappa) \subset \mathscr{A}_2(\kappa)$.

2. If $\mathcal{E}_K \in \mathscr{A}_{2,2}(K)$, i.e., $\mathcal{F}_K$ admits a sub-line bundle of degree ≥ 0, then so does $\mathcal{F}_\kappa$, hence $\mathcal{E}_\kappa \in \mathscr{A}_{2,2}(\kappa) \subset \mathscr{A}_2(\kappa)$.

$\square$

5.2.3 Construction of $\mathscr{U} \subset \mathscr{B}un_3^{2,\leq\lambda}$

Let

$$\mathscr{U} := \mathscr{B}un_3^{2,\leq\lambda} - \mathscr{A}_1 - \mathscr{A}_2 \subset \mathscr{B}un_3^{2,\leq\lambda} \subset \mathscr{B}un_3^2.$$

This is an open substack of $\mathscr{B}un_3^2$ by Lemma 5.2.3 and Lemma 5.2.5. Our goal is to show that the open substack $\mathscr{U} \subset \mathscr{B}un_3^2$ admits a separated (but non-proper) good moduli space and $\mathscr{U} \nsubseteq \mathscr{B}un_3^{2,ss}$.

To close this subsection we describe decomposable points in $\mathscr{U}$, which is quite useful for checking Θ-reductivity. As a by-product, we show that $\mathscr{U}$ is non-empty and $\mathscr{U} \nsubseteq \mathscr{B}un_3^{2,ss}$.

Lemma 5.2.6. *Let $\mathcal{L} \in \mathscr{B}un_1^1(\mathbb{K})$ and $\mathcal{F} \in \mathscr{B}un_2^1(\mathbb{K})$ for some field $\mathbb{K}/k$. Then $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbb{K})$ if and only if*

1. $\mathrm{Hom}(\mathcal{F}, \mathcal{L}) = 0$ and

2. $\mathcal{F}$ doesn't admit a sub-line bundle of degree ≥ 0, i.e., $\mathcal{F}$ is $(1,0)$-stable.

Proof. Only If Part: This is clear from the definition of $\mathscr{U}$. If Part: We need to show $\mathcal{L} \oplus \mathcal{F} \notin (\mathscr{A}_1 \cup \mathscr{A}_2)(\mathbb{K})$. By definition $\mathcal{L} \oplus \mathcal{F} \notin \mathscr{A}_1(\mathbb{K})$. To see $\mathcal{L} \oplus \mathcal{F} \notin \mathscr{A}_2(\mathbb{K})$, if $\mathcal{L} \oplus \mathcal{F}$ fits into a short exact sequence $0 \to \mathcal{F}' \to \mathcal{L} \oplus \mathcal{F} \to \mathcal{L}' \to 0$ for some $\mathcal{F}' \in \mathscr{B}un_2^1(\mathbb{K})$ and $\mathcal{L}' \in \mathscr{B}un_1^1(\mathbb{K})$, then we consider the composition $\mathcal{L} \hookrightarrow \mathcal{L} \oplus \mathcal{F} \twoheadrightarrow \mathcal{L}'$.

- If it is non-zero, then $\mathcal{L} \cong \mathcal{L}'$ and hence $\mathcal{F}' \cong \mathcal{F}$. This shows that this short exact sequence does not realize $\mathcal{L} \oplus \mathcal{F}$ as a point in $\mathscr{A}_2$.

- If it is zero, then it factors through an injection $\mathcal{L} \hookrightarrow \mathcal{F}'$ and hence $\mathcal{F}' \cong \mathcal{L} \oplus \mathcal{L}_0$ for some degree 0 sub-line bundle $\mathcal{L}_0 \subset \mathcal{F}$, a contradiction.

$\square$

Corollary 5.2.7. *The open substack $\mathscr{U}$ is non-empty and $\mathscr{U} \nsubseteq \mathscr{B}un_3^{2,ss}$.*

Proof. For both assertions, it suffices to show that there exist $\mathcal{L} \in \mathscr{B}un_1^1(k)$ and $\mathcal{F} \in \mathscr{B}un_2^1(k)$ satisfying the conditions in Lemma 5.2.6. Recall that $\mathcal{F}$ doesn't admit a sub-line bundle of degree ≥ 0 if and only if $\mathcal{F}$ is $(1,0)$-stable, and in this case $\mathrm{Hom}(\mathcal{F}, \mathcal{L}) = 0$ for any $\mathcal{L} \in \mathscr{B}un_1^1(k)$ because any non-zero morphism $\mathcal{F} \to \mathcal{L}$ is surjective and it produces a degree 0 sub-line bundle of $\mathcal{F}$. By Theorem 5.1.5, there do exist $(1,0)$-stable vector bundles over C of rank 2 since $g > 2$. This finishes the proof. $\square$

Corollary 5.2.8. *If $\mathcal{E} \in \mathscr{B}un_3^{2,\leq\lambda}(\mathbb{K})$ for some field $\mathbb{K}/k$ fits into a short exact sequence*

$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ or $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{L} \to 0$ for some $\mathcal{L} \in \mathscr{B}un_1^1(\mathbb{K})$ and $\mathcal{F} \in \mathscr{B}un_2^1(\mathbb{K})$,

then $\mathcal{E} \in \mathscr{U}(\mathbb{K})$ if and only if $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbb{K})$.

Proof. IF PART: Since $\mathscr{U}$ is open. ONLY IF PART: If $\mathcal{L} \oplus \mathcal{F} \notin \mathscr{U}(\mathbb{K})$, then it doesn't satisfy (at least) one of the conditions in Lemma 5.2.6, but this in turn shows that $\mathcal{E} \notin \mathscr{U}(\mathbb{K})$. □

5.3 Separated good moduli space of $\mathscr{U}$

To show that $\mathscr{U}$ admits a separated (but non-proper) good moduli space, we check the conditions in Theorem 0 one-by-one in the following subsections.

5.3.1 Local linear reductivity

Lemma 5.3.1. *The open substack $\mathscr{U}$ is locally linearly reductive.*

Proof. Note that closed points in $\mathscr{U}$ are

1. either stable (indecomposable with automorphism groups $\mathbb{G}_m$),

2. or of the form $\mathcal{L} \oplus \mathcal{F}$ with $\mathcal{L} \in \mathscr{B}un_1^1(k)$ and $\mathcal{F} \in \mathscr{B}un_2^1(k)$ satisfying the two conditions in Lemma 5.2.6 (decomposable with automorphism groups $\mathbb{G}_m^2$ since there is no morphism between $\mathcal{L}$ and $\mathcal{F}$).

Indeed, if $\mathcal{E} \in \mathscr{U}(k)$ is an unstable closed point and let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be its Harder-Narasimhan filtration, then by Corollary 5.2.8 we have $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(k)$ and Rees construction provides a degeneration from $\mathcal{E}$ to $\mathcal{L} \oplus \mathcal{F}$ in $\mathscr{U}$, so the only possibility is $\mathcal{E} \cong \mathcal{L} \oplus \mathcal{F}$.

It remains to show that any point $\mathcal{E} \in \mathscr{U}(\mathbb{K})$ (for some field $\mathbb{K}/k$) degenerates to a closed point in $\mathscr{U}$. There are 2 cases:

1. If $\mathcal{E}$ is stable, then there is nothing to prove.

2. If $\mathcal{E}$ is unstable and let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be its Harder-Narasimhan filtration, then $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(k)$ by Corollary 5.2.8 and Rees construction provides a degeneration from $\mathcal{E}$ to $\mathcal{L} \oplus \mathcal{F}$ in $\mathscr{U}$.

□

5.3.2 Θ-reductivity

Lemma 5.3.2. *The open substack $\mathscr{U}$ is Θ-reductive.*

Proof. This is proved by using Proposition 3.2.6. For every DVR R over k with fraction field K and residue field κ, every family $\mathcal{E}_R \in \mathscr{U}(R)$ and every (non-trivial) filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ with $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$, by Lemma 5.1.2 we have

$$\mathrm{gr}(\mathcal{E}_K^\bullet) = \mathcal{L}_K \oplus \mathcal{F}_K \in \mathscr{U}(K) \text{ for some } \mathcal{L}_K \in \mathscr{B}un_1^1(K) \text{ and } \mathcal{F}_K \in \mathscr{B}un_2^1(K).$$

Then the filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ is of the form

$$0 \to \mathcal{L}_K \to \mathcal{E}_K \to \mathcal{F}_K \to 0 \text{ or } 0 \to \mathcal{F}_K \to \mathcal{E}_K \to \mathcal{L}_K \to 0.$$

In the first case the induced filtration $\mathcal{E}_\kappa^\bullet$ of $\mathcal{E}_\kappa$ is of the form

$$0 \to \mathcal{L}_\kappa \to \mathcal{E}_\kappa \to \mathcal{F}_\kappa \to 0$$

and we need to show $\mathrm{gr}(\mathcal{E}_\kappa^\bullet) = \mathcal{L}_\kappa \oplus \mathcal{F}_\kappa \in \mathscr{U}(\kappa)$. Note that $\mathcal{L}_\kappa \in \mathscr{B}un_1^1(k)$ and it is a subbundle of $\mathcal{E}_\kappa$ by Lemma 5.1.1, i.e., $\mathcal{F}_\kappa \in \mathscr{B}un_2^1(\kappa)$, then we are done by Corollary 5.2.8 since $\mathcal{E}_\kappa \in \mathscr{U}(\kappa)$. Similar for the second case. □

5.3.3 S-completeness

Our experience in proving S-completeness motivates us to consider all opposite filtrations of points in $\mathscr{U}$.

Lemma 5.3.3. *If $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(\mathbb{K})$ for some field $\mathbb{K}/k$ have opposite filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$ such that $\mathcal{E}$ is unstable with Harder-Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$, then one of the following holds:*

1. *The filtration $\mathcal{E}^\bullet : 0 \subset \mathcal{F} \subset \mathcal{E}$ splits and $(\mathcal{E}')^\bullet : 0 \subset \mathcal{L} \subset \mathcal{E}'$.*
2. *The filtration $\mathcal{E}^\bullet : 0 \subset \mathcal{L} \subset \mathcal{E}$ and $(\mathcal{E}')^\bullet : 0 \subset \mathcal{F} \subset \mathcal{E}'$.*

Proof. Assume that $\mathcal{E}^i \neq \mathcal{E}^j$ and $(\mathcal{E}')^i \neq (\mathcal{E}')^j$ for $i \neq j$. A priori these two filtrations

$$\mathcal{E}^\bullet : 0 = \mathcal{E}^0 \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^m = \mathcal{E} \text{ and } (\mathcal{E}')^\bullet : 0 = (\mathcal{E}')^0 \subset (\mathcal{E}')^1 \subset \cdots \subset (\mathcal{E}')^m = \mathcal{E}'$$

satisfies $\mathrm{rk}(\mathcal{E}/\mathcal{E}^{m-1}) = \mathrm{rk}((\mathcal{E}')^1) \geq 1$ and $\mathrm{rk}(\mathcal{E}'/(\mathcal{E}')^{m-1}) = \mathrm{rk}(\mathcal{E}^1) \geq 1$, hence

$$2 \geq \mathrm{rk}(\mathcal{E}^{m-1}) \geq \cdots \geq \mathrm{rk}(\mathcal{E}^1) \geq 1 \text{ and } 2 \geq \mathrm{rk}((\mathcal{E}')^{m-1}) \geq \cdots \geq \mathrm{rk}((\mathcal{E}')^1) \geq 1.$$

Applying Lemma 5.1.1 to $\mathcal{E}$ and $\mathcal{E}'$ yields that

$$\deg(\mathcal{E}^i) \leq 1 \text{ and } \deg((\mathcal{E}')^i) \leq 1 \text{ for } i = 1, \ldots, m-1.$$

However, the isomorphism $\mathcal{E}^i/\mathcal{E}^{i-1} \cong (\mathcal{E}')^{m-i+1}/(\mathcal{E}')^{m-i}$ implies that

$$\deg(\mathcal{E}^i) + \deg((\mathcal{E}')^{m-i}) = \deg(\mathcal{E}) = 2 \text{ for } i = 1, \dots, m-1,$$

i.e., $\deg(\mathcal{E}^i) = \deg((\mathcal{E}')^i) = 1$ for $i = 1, \dots, m-1$. Again by Lemma 5.1.1, we see that both filtrations $\mathcal{E}^\bullet$ and $(\mathcal{E}')^\bullet$ are filtrations of subbundles. This gives that

$$2 \geq \mathrm{rk}(\mathcal{E}^{m-1}) > \cdots > \mathrm{rk}(\mathcal{E}^1) \geq 1 \text{ and } 2 \geq \mathrm{rk}((\mathcal{E}')^{m-1}) > \cdots > \mathrm{rk}((\mathcal{E}')^1) \geq 1.$$

In particular, $3 \geq m \geq 2$. But $m = 3$ is impossible since any length 3 filtration $\mathcal{E}^\bullet : 0 \subset \mathcal{E}^1 \subset \mathcal{E}^2 \subset \mathcal{E}$ yields $\mathcal{E}^2 \oplus \mathcal{E}/\mathcal{E}^2 \in \mathscr{U}(\mathbb{K})$ by Corollary 5.2.8 and hence $\mathcal{E}^2 \in \mathscr{B}un_2^{1,ss}(\mathbb{K})$ by Lemma 5.1.2, a contradiction since $\mathcal{E}^1 \subset \mathcal{E}^2$ is a degree 1 sub-line bundle. This shows that $m = 2$, i.e.,

$$\mathcal{E}^\bullet : 0 \subset \mathcal{E}^1 \subset \mathcal{E} \text{ and } (\mathcal{E}')^\bullet : 0 \subset (\mathcal{E}')^1 \subset \mathcal{E}'.$$

There are 2 cases:

1. If $\mathrm{rk}(\mathcal{E}^1) = 1$, then $\mathcal{E}^1 \cong \mathcal{L}$ by the uniqueness of Harder-Narasimhan filtration of $\mathcal{E}$ and hence $(\mathcal{E}')^1 \cong \mathcal{E}/\mathcal{E}^1 \cong \mathcal{E}/\mathcal{L} = \mathcal{F}$. This is (2).

2. If $\mathrm{rk}(\mathcal{E}^1) = 2$, then as before $\mathcal{E}^1 \in \mathscr{B}un_2^{1,s}(\mathbb{K})$ and the non-zero composition $\mathcal{E}^1 \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{F}$ is actually an isomorphism, yielding a splitting of the Harder-Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$. Hence $\mathcal{E} \cong \mathcal{L} \oplus \mathcal{F}$ and $(\mathcal{E}')^1 \cong \mathcal{E}/\mathcal{E}^1 \cong \mathcal{E}/\mathcal{F} = \mathcal{L}$. This is (1).

□

Corollary 5.3.4. *The open substack $\mathscr{U}$ is S-complete.*

Proof. This is proved by using Proposition 3.3.17. For any field $\mathbb{K}/k$, any points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(\mathbb{K})$ with opposite filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$, without loss of generality we may assume that $\mathcal{E}$ is unstable with Harder-Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$. By Lemma 5.3.3, such filtrations $\mathcal{E}^\bullet, (\mathcal{E}')^\bullet$ are of the forms

$$\begin{cases} \mathcal{E}^\bullet : & 0 \subset \mathcal{F} \subset \mathcal{E} \\ (\mathcal{E}')^\bullet : & 0 \subset \mathcal{L} \subset \mathcal{E}' \end{cases} \text{or} \begin{cases} \mathcal{E}^\bullet : & 0 \subset \mathcal{L} \subset \mathcal{E} \\ (\mathcal{E}')^\bullet : & 0 \subset \mathcal{F} \subset \mathcal{E}' \end{cases}$$

In either case, $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbb{K})$ by Corollary 5.2.8. This finishes the proof. □

5.3.4 Existence part of the valuative criterion for properness

In this subsection, we prove that $\mathscr{U}$ doesn't satisfy the existence part of the valuative criterion for properness.

Lemma 5.3.5. *There exists a family $\mathcal{E}_R \in \mathscr{B}un_3^{2,ss}(R)$ over some DVR R over k with fraction field K and residue field κ such that*

1. *the generic fiber $\mathcal{E}_K$ fits into a short exact sequence*

$$0 \to \mathcal{F}_K \to \mathcal{E}_K \to \mathcal{L}_K \to 0$$

for some $\mathcal{L}_K \in \mathscr{B}un_1^1(K)$ and $\mathcal{F}_K \in \mathscr{B}un_2^1(K)$ with $\mathrm{Hom}(\mathcal{F}_K, \mathcal{L}_K) = 0$ and $\mathcal{F}_K$ being $(1,0)$-stable, i.e., stable and doesn't admit a sub-line bundle of degree ≥ 0.

In particular, $\mathcal{E}_K \in \mathscr{U}(K)$.

2. *the special fiber $\mathcal{E}_\kappa$ fits into a short exact sequence*

$$0 \to \mathcal{F}_\kappa \to \mathcal{E}_\kappa \to \mathcal{L}_\kappa \to 0$$

for some $\mathcal{L}_\kappa \in \mathscr{B}un_1^1(\kappa)$ and $\mathcal{F}_\kappa \in \mathscr{B}un_2^1(\kappa)$ with $\mathrm{Hom}(\mathcal{F}_\kappa, \mathcal{L}_\kappa) = 0$ and $\mathcal{F}_\kappa$ being stable but not $(1,0)$-stable, i.e., stable and admits a sub-line bundle of degree 0.

In particular, $\mathcal{E}_\kappa \notin \mathscr{U}(\kappa)$.

Proof. Let $\mathscr{B}un_2^{1,s}(1,0) \subset \mathscr{B}un_2^{1,s}$ be the open dense substack of $(1,0)$-stable vector bundles.

- Since $\mathscr{B}un_2^{1,s}$ is irreducible, there exists a family $\mathcal{F}_R \in \mathscr{B}un_2^{1,s}(R)$ over some DVR R over k with fraction field K and residue field κ such that

$$\mathcal{F}_K \in \mathscr{B}un_2^{1,s}(1,0)(K) \text{ and } \mathcal{F}_\kappa \in (\mathscr{B}un_2^{1,s} - \mathscr{B}un_2^{1,s}(1,0))(\kappa).$$

- Choose a family $\mathcal{L}_R \in \mathscr{B}un_1^1(R)$ with $\mathrm{Hom}(\mathcal{F}_K, \mathcal{L}_K) = \mathrm{Hom}(\mathcal{F}_\kappa, \mathcal{L}_\kappa) = 0$.

  By upper semi-continuity we only need to fulfill the condition

$$\mathrm{Hom}(\mathcal{F}_\kappa, \mathcal{L}_\kappa) = 0.$$

  Since $\mathrm{Hom}(\mathcal{F}_\kappa, \mathcal{L}_\kappa) \neq 0$ if and only if $\mathcal{L}_\kappa$ is a quotient of $\mathcal{F}_\kappa$, it suffices to find a line bundle $\mathcal{L}_\kappa$ which is not a quotient of $\mathcal{F}_\kappa$. This can be done by dimension estimate. The Quot-scheme parametrizing rank 1 and degree 1 coherent quotients of $\mathcal{F}_\kappa$ has dimension ≤ 1 by Clifford's theorem, while $\dim(\mathscr{B}un_1^1) = g - 1 > 1$. So such a line bundle $\mathcal{L}_\kappa$ always exists and we can then choose any family $\mathcal{L}_R \in \mathscr{B}un_1^1(R)$ extending $\mathcal{L}_\kappa$.

- Choose an extension $[\varrho] : 0 \to \mathcal{F}_\kappa \to \mathcal{E}_\kappa \to \mathcal{L}_\kappa \to 0$ such that $\mathcal{E}_\kappa$ is stable.

  The existence of such an extension is guaranteed by [RTiB99, Proposition 1.11], which says that generic extension of stable vector bundles is again stable. But our situation is much better: we can show that such $\mathcal{E}_\kappa$ is stable if and only if the extension $[\varrho]$ does not split, using the same argument in Lemma 5.2.6.

Let $\mathcal{E}_R \in \mathscr{C}oh_3^2(R)$ be an extension family of $\mathcal{L}_R$ by $\mathcal{F}_R$ with the prescribed special fiber $[\varrho]$ (given by Proposition 3.2.11). Then $\mathcal{E}_R \in \mathscr{B}un_3^{2,ss}(R)$ since $\mathcal{E}_\kappa \in \mathscr{B}un_3^{2,ss}(\kappa)$, as asserted. $\square$

Lemma 5.3.6. *The open substack $\mathscr{U}$ doesn't satisfy the existence part of the valuative criterion for properness.*

Proof. Any family $\mathcal{E}_R \in \mathscr{B}un_3^{2,s}(R)$ in Lemma 5.3.5 defines a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "\mathcal{E}_K"] \arrow[d, hook] & \mathscr{U} \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\times" description] & \operatorname{Spec}(k) \end{tikzcd}$$

of solid arrows and we claim that there is no dotted arrow filling in.

Indeed, if there exists a family $\mathcal{E}'_R \in \mathscr{U}(R)$ such that $\mathcal{E}'_K \cong \mathcal{E}_K$, then we consider its special fiber $\mathcal{E}'_\kappa \in \mathscr{U}(\kappa)$, there are 2 cases:

1. If $\mathcal{E}'_\kappa$ is stable, then $\mathcal{E}'_R \in \mathscr{B}un_3^{2,s}(R)$ and hence $\mathcal{E}'_\kappa \cong \mathcal{E}_\kappa$ by Langton's theorem A (see [Lan75]). However by construction $\mathcal{E}_\kappa \notin \mathscr{U}(\kappa)$, a contradiction.

2. If $\mathcal{E}'_\kappa$ is unstable, then we consider the following diagram

$$\begin{tikzcd} & \mathcal{L}''_\kappa \arrow[d, hook] & \\ \mathcal{F}'_\kappa \arrow[r, hook] & \mathcal{E}'_\kappa \arrow[r, two heads] \arrow[d, two heads] & \mathcal{L}'_\kappa \\ & \mathcal{F}''_\kappa & \end{tikzcd}$$

   where the horizontal line is the induced filtration of $\mathcal{E}'_\kappa$ from that on the generic fiber

$$(\mathcal{F}'_K \hookrightarrow \mathcal{E}'_K \twoheadrightarrow \mathcal{L}'_K) \cong (\mathcal{F}_K \hookrightarrow \mathcal{E}_K \twoheadrightarrow \mathcal{L}_K)$$

   and the vertical line is the Harder-Narasimhan filtration of $\mathcal{E}'_\kappa$.

   (a) Since $\mathcal{E}'_\kappa \in \mathscr{U}(\kappa)$, both $\mathcal{F}'_\kappa$ and $\mathcal{F}''_\kappa$ don't admit sub-line bundles of degree 0 (in particular, they are both stable). Then the non-zero composition $\mathcal{F}'_\kappa \hookrightarrow \mathcal{E}'_\kappa \twoheadrightarrow \mathcal{F}''_\kappa$ is an isomorphism.

   (b) Since $\mathcal{F}'_K \cong \mathcal{F}_K$ and $\mathcal{F}'_\kappa, \mathcal{F}_\kappa \in \mathscr{B}un_2^{1,s}(\kappa)$, we have $\mathcal{F}'_\kappa \cong \mathcal{F}_\kappa$ by Langton's theorem A (see [Lan75]).

   This yields that $\mathcal{F}''_\kappa \cong \mathcal{F}_\kappa$, a contradiction since $\mathcal{F}''_\kappa$ doesn't admit a sub-line bundle of degree 0 by $\mathcal{E}'_\kappa \in \mathscr{U}(\kappa)$, while $\mathcal{F}_\kappa$ admits a sub-line bundle of degree 0 by construction.

$\square$

5.3.5 Proof of Theorem B

Theorem 5.3.7. *The open dense substack $\mathscr{U} \subset \mathscr{B}un_3^2$ admits a separated (but non-proper) good moduli space, and is not contained in $\mathscr{B}un_3^{2,ss}$.*

Proof. The open substack $\mathscr{U} \subset \mathscr{B}un_3^2$ is non-empty and not contained in $\mathscr{B}un_3^{2,ss}$ by Corollary 5.2.7. It is Θ-reductive (Lemma 5.3.2), S-complete (Corollary 5.3.4) and doesn't satisfy the existence part of valuative criterion for properness (Lemma 5.3.6). By Theorem 0, $\mathscr{U}$ admits a separated but non-proper good moduli space. $\square$

Remark 5.3.8. This construction of the open substack $\mathscr{U} \subset \mathscr{B}un_3^2$ can be easily generalized to arbitrary rank, implying that in general there are lots of open substacks of $\mathscr{B}un_n^d$ admitting separated (but non-proper) good moduli spaces.

It is also interesting to notice that under the good moduli space morphism, every unstable point in $\mathscr{U}$ is identified with a stable point. Indeed, if $\mathbb{K}/k$ is a field, $\mathcal{E} \in \mathscr{U}(\mathbb{K})$ is unstable and let $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ be its Harder-Narasimhan filtration, then by Corollary 5.2.8, $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbb{K})$ and any non-split extension $0 \to \mathcal{F} \to \mathcal{E}' \to \mathcal{L} \to 0$ is stable and satisfies $\mathcal{E}' \in \mathscr{U}(\mathbb{K})$. Then by Rees construction

$$\mathcal{L} \oplus \mathcal{F} \in \overline{\{\mathcal{E}\}} \cap \overline{\{\mathcal{E}'\}} \neq \emptyset \text{ in } \mathscr{U}(\mathbb{K})$$

and $\mathcal{E}, \mathcal{E}'$ are identified in the good moduli space of $\mathscr{U}$ by [Alp13, Theorem 4.16 (iv)].

Chapter 6

Auxiliary results

In this chapter we prove two results around Question 1.1.3, which concern about the maximality of the open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ of semi-stable vector bundles in some sense. Our failure in attacking Question 1.1.3 is lack of an effective way to check or make use of the existence part of valuative criterion for properness for open substacks of $\mathscr{B}un_n^d$.

Proposition 6.0.1. *The open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ is the unique maximal open substack that admits a quasi-projective good moduli space.*

Proof. Let $\mathscr{U} \subset \mathscr{B}un_n^d$ be an open substack that admits a quasi-projective good moduli space. Since $\mathscr{B}un_n^d$ is smooth, by [Alp13, Theorem 11.14 (ii)] there exists a line bundle $\mathcal{L}$ over $\mathscr{B}un_n^d$ such that

$$\mathscr{U} \subseteq (\mathscr{B}un_n^d)^{ss}_{\mathcal{L}}.$$

Recall that (see [Alp13]) a geometric point $\mathcal{E} \in \mathscr{B}un_n^d(k)$ is semi-stable with respect to $\mathcal{L}$ if there is a section $s \in \Gamma(\mathscr{B}un_n^d, \mathcal{L}^{\otimes m})$ for some $m > 0$ such that $s(\mathcal{E}) \neq 0$.

Note that there exists an isomorphism[1]

$$\mathrm{Pic}(\mathscr{B}un_1^d) \times \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\mathscr{B}un_n^d) \text{ sending } (\mathcal{L}_0, e) \mapsto \det{}^*\mathcal{L}_0 \otimes \mathcal{L}_{\det}^{\otimes e}$$

where $\mathcal{L}_{\det}$ is the determinant of cohomology line bundle over $\mathscr{B}un_n^d$, i.e., for any point $\mathcal{E} \in |\mathscr{B}un_n^d|$ we have

$$\mathcal{L}_{\det,\mathcal{E}} := \det(H^*(C, \mathcal{E}nd(\mathcal{E})))^*,$$

and in general for any morphism $f : T \to \mathscr{B}un_n^d$ corresponding to a vector bundle $\mathcal{E}_T$ over $C \times T$ we have

$$f^*\mathcal{L}_{\det} := \left(\det \mathbf{R}(\mathrm{pr}_T)_* \mathcal{E}nd(\mathcal{E}_T)\right)^*.$$

Since the stability condition does not change if we tensor $\mathcal{L}$ with some line bundle pulled back from $\mathscr{B}un_1^d$ via det, we may assume that $\mathcal{L} \cong \mathcal{L}_{\det}^{\otimes e}$ for some integer e.

[1] since the determinant morphism $\det : \mathscr{B}un_n^d \to \mathscr{B}un_1^d$ has a section given by $\mathcal{N} \mapsto \mathcal{N} \oplus \mathcal{O}_C^{\oplus n-1}$.

Claim 6.0.2. The integer $e > 0$.

Hence without loss of generality we may assume that $\mathcal{L} \cong \mathcal{L}_{\det}$.

Proof. Indeed, if $e = 0$, then $\mathcal{L} \cong \mathcal{O}_{\mathscr{B}un_n^d}$ and hence $(\mathscr{B}un_n^d)^{ss}_{\mathcal{L}} = \mathscr{B}un_n^d$, which doesn't admit a good moduli space. If $e < 0$, then $\mathcal{L}$ has only zero global section since $\mathcal{L}_{\det}$ is ample and hence $(\mathscr{B}un_n^d)^{ss}_{\mathcal{L}} = \emptyset$, a contradiction. $\square$

It remains to show $(\mathscr{B}un_n^d)^{ss}_{\mathcal{L}_{\det}} = \mathscr{B}un_n^{d,ss}$, i.e.,

Claim 6.0.3 ([Hei17], Lemma 1.11)**.** Let $\mathcal{E} \in \mathscr{B}un_n^d(\mathbb{K})$ for some field $\mathbb{K}/k$. Then $\mathcal{E} \in \mathscr{B}un_n^{d,ss}(\mathbb{K})$ if and only if $\mathcal{E} \in (\mathscr{B}un_n^d)^{ss}_{\mathcal{L}_{\det}}(\mathbb{K})$.

Proof. Only If Part: A result of Faltings asserts that (see [Fal93, Theorem I.2]) a vector bundle $\mathcal{E}$ over C is semi-stable if and only if there exists a vector bundle $\mathcal{F}$ over C such that $\mathcal{E} \otimes \mathcal{F}$ is cohomologically trivial, i.e., $H^0(C, \mathcal{E} \otimes \mathcal{F}) = H^1(C, \mathcal{E} \otimes \mathcal{F}) = 0$. This defines a global section $s \in \Gamma(\mathscr{B}un_n^d, \mathcal{L}_{\det})$ such that $s(\mathcal{E}) \neq 0$ (see also [HL97, Lemma 8.2.4 (1)]).

If Part: If $\mathcal{E} \in \mathscr{B}un_n^d(\mathbb{K})$ is unstable, then for any integer $m > 0$, any global section of $\mathcal{L}_{\det}^{\otimes m}$ vanishes at $\mathcal{E}$. $\square$

$\square$

Proposition 6.0.4. *Let $\mathscr{B}un_n^{d,ss} \to \mathcal{M}^{ss}(n,d)$ be the good moduli space of the open substack $\mathscr{B}un_n^{d,ss} \subset \mathscr{B}un_n^d$ of semi-stable vector bundles. If $\mathscr{U} \subset \mathscr{B}un_n^d$ is an open substack containing $\mathscr{B}un_n^{d,ss}$ that admits a separated good moduli space $\mathcal{U}$, then there is a surjection $\mathcal{M}^{ss}(n,d) \to \mathcal{U}$ such that under the good moduli space morphism $\mathscr{U} \to \mathcal{U}$, every unstable vector bundle is identified with a semi-stable one.*

Proof. If $\mathscr{U} \subset \mathscr{B}un_n^d$ is an open substack containing $\mathscr{B}un_n^{d,ss}$ that admits a separated good moduli space $\mathcal{U}$, then we have a commutative diagram

$$\begin{tikzcd}
\mathscr{B}un_n^{d,ss} \arrow[r, hook, "\text{open}"] \arrow[d, two heads, "\text{proper gms}"'] \arrow[dr, "\circ"] & \mathscr{U} \arrow[d, two heads, "\text{separated gms}"] \\
\mathcal{M}^{ss}(n,d) \arrow[r, dashed, "\exists ! \gamma"'] & \mathcal{U}
\end{tikzcd}$$

where the morphism $\gamma : \mathcal{M}^{ss}(n,d) \to \mathcal{U}$ is induced by the universal property of the good moduli space and has dense image since $\mathscr{B}un_n^{d,ss} \subset \mathscr{U}$ does. Since $\mathcal{U}$ is separated and $\mathcal{M}^{ss}(n,d)$ is proper, the morphism γ has closed image. This implies that γ is surjective. An easy diagram-chasing indicates that under the good moduli space morphism $\mathscr{U} \to \mathcal{U}$ every unstable vector bundle is identified with a semi-stable vector bundle. $\square$

Part II

Quotient stack by $\mathbb{G}_m$

Throughout this part, let

- k be a field and
- X be a separated irreducible scheme, of finite type over k with a $\mathbb{G}_m$-action.

  Denote by $\sigma : \mathbb{G}_m \times X \to X$ this action.

Chapter 7

Preliminaries

In this chapter, we present some materials on the geometry of $\mathbb{G}_m$-schemes.

7.1 Białynicki-Birula decomposition

The Białynicki-Birula decomposition provides, for any smooth projective scheme over an algebraically closed field with a $\mathbb{G}_m$-action, a decomposition into $\mathbb{G}_m$-invariant locally closed subschemes with well-behaved properties (see [BB73, Theorem 4.1]). This section is devoted to presenting a generalization of it which is sufficient for our application. First we introduce some notations.

- By [Dri15, Proposition 1.2.2] there exists a closed subscheme $X^{\mathbb{G}_m} \subset X$, of finite type over k, called the subscheme of fixed points, representing the following functor:

  $$X^{\mathbb{G}_m} := \underline{\mathrm{Hom}}_k(\mathrm{Spec}(k), X)$$

  The k-points of $X^{\mathbb{G}_m}$ are the $\mathbb{G}_m$-fixed points $x \in X(k)$.

- Let $X^{\mathbb{G}_m} = \coprod_{i=1}^r X_i$ be the decomposition of $X^{\mathbb{G}_m}$ into connected components.

  If X is geometrically normal, then each connected component X_i is irreducible. Indeed, in this case X has a $\mathbb{G}_m$-invariant affine open covering by Sumihiro's theorem (see [Sum74, Corollary 2] if $k = \bar{k}$ and [Sum75, Corollary 3.11] in general) and this enables us to apply [Fog73, Proposition 7.4] to show that $X^{\mathbb{G}_m}$ is again geometrically normal, so its connected components are irreducible (see [GW20, Exercise 3.16 (a)]).

- For any point $x \in X(\mathbb{K})$ for some field $\mathbb{K}/k$, its *orbit map* $\sigma(-,x) : \mathbb{G}_{m,\mathbb{K}} \to X$ is defined by $t \mapsto \sigma(t,x) =: t.x$. If X is proper, then $\sigma(-,x)$ has a unique extension $\overline{\sigma(-,x)} : \mathbb{P}^1_{\mathbb{K}} \to X$ and we call this morphism the *complete orbit map* of x. In this case we define[1]

$$x^- := \overline{\sigma(0,x)} \in X^{\mathbb{G}_m} \text{ and } x^+ := \overline{\sigma(\infty,x)} \in X^{\mathbb{G}_m}$$

and often regard $x^- = \lim_{t\to 0} t.x$ and $x^+ = \lim_{t\to\infty} t.x$.

- For any $\mathbb{G}_m$-invariant subset $S \subset X$, by [Dri15, Corollary 1.4.3] there exist separated schemes $S^{\pm}$, of finite type over k, representing the following functors:

$$S^{\pm} := \underline{\mathrm{Hom}}_k(\mathbb{A}^1_{\pm}, S)^{\mathbb{G}_m}$$

  where $\mathbb{A}^1_{\pm} = \mathrm{Spec}(k[\theta])$ is the affine line $\mathbb{A}^1$ with the $\mathbb{G}_m$-action such that θ has weight ± 1. Moreover, we have the following natural morphisms: a morphism $S^{\pm} \to S$ given by evaluation at 1 and a morphism $S^{\pm} \to S^{\mathbb{G}_m}$ given by evaluation at 0.

  Since X is separated, the k-points of $S^{\pm}$ are the points $x \in S(k)$ such that $x^{\pm}$ exists and lies in S.

Theorem 7.1.1 (Białynicki-Birula decomposition)**.** *Let X be a separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action.*

1. *The subset $X_i^{\pm}$ is $\mathbb{G}_m$-invariant and constructible for each i (see [Dri15, Lemma 1.4.9]).*

2. *The morphism $\pi_i^{\pm} : X_i^{\pm} \to X_i$ defined by $x \mapsto x^{\pm}$ is $\mathbb{G}_m$-equivariant and affine for each i (see [Dri15, Theorem 1.4.2]), with the closed inclusion $X_i \hookrightarrow X_i^{\pm}$ as a section.*

3. *If X has a $\mathbb{G}_m$-invariant affine open covering (e.g., X is geometrically normal), then $X_i^{\pm} \hookrightarrow X$ is a local immersion (i.e., a locally closed immersion Zariski-locally on the source) and the morphism $X_i^{\pm} \hookrightarrow X \times X_i$ defined by $x \mapsto (x, x^{\pm})$ is a locally closed immersion for each i (see [AHR20, Theorem 5.27 (3b)], also [Hes81, Theorem 4.5, page 69] and [Mil17, Proposition 13.58]).*

4. *If X is affine, then $X^{\pm} \hookrightarrow X$ is a closed embedding (see [Dri15, Proposition 1.4.11 (iv)]).*

5. *For any point $x \in X^{\mathbb{G}_m}$, the tangent map $T_x(X^{\pm}) \to T_x(X)$ corresponding to $X^{\pm} \hookrightarrow X$ induces an isomorphism $T_x(X^{\pm}) \xrightarrow{\sim} T_x(X)^{\mp}$, where $T_x(X)^{\mp} \subset T_x(X)$ is the non-positive (resp., non-negative) $\mathbb{G}_m$-weights part of $T_x(X)$. Moreover, the tangent map $T_x(X^{\mathbb{G}_m}) \to T_x(X^{\pm}) \to T_x(X^{\mathbb{G}_m})$ corresponding to $X^{\mathbb{G}_m} \hookrightarrow X^{\pm} \xrightarrow{\pi^{\pm}} X^{\mathbb{G}_m}$ is identified with the canonical maps $T_x(X)^0 \hookrightarrow T_x(X)^{\mp} \twoheadrightarrow T_x(X)^0$ (see [Dri15, Proposition 1.4.11 (vi)]).*

[1] Note that the sign conventions in this thesis are opposite to that in the literatures, e.g., [BB73]. Our $+$ is written as $-$ in *loc. cit.* and vice versa.

6. *If X is proper, then we have a decomposition of X into disjoint union of $\mathbb{G}_m$-invariant subsets*
$$X = \coprod_{i=1}^{r} X_i^+ \ (\textit{resp.}, \ X = \coprod_{i=1}^{r} X_i^-)$$
called the plus (resp., minus) Białynicki-Birula decomposition of X (see [AHR20, Theorem 5.27 (1)]).

Corollary 7.1.2 ([BBS83], Lemma A.1 and A.3 if $k = \mathbb{C}$)**.** *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then there exists, after possibly renumbering, a unique fixed point component X_1 (resp., X_r), called the source (resp., the sink) of X, characterized by the property that X_1^- (resp., X_r^+) is dense in X.*

If X is geometrically normal, then X_1^- (resp., X_r^+) is an irreducible open dense subscheme of X.

Proof. Since X is proper and irreducible, the required fixed point component is singled out by looking at where its unique generic point lands, according to the minus (resp., plus) Białynicki-Birula decomposition of X given in Theorem 7.1.1 (6).

Note that the inclusion $X_1^- \hookrightarrow X$ (resp., $X_r^+ \hookrightarrow X$) can be decomposed as
$$X_1^- \xrightarrow{x \mapsto (x, x^-)} X \times X_1 \xrightarrow{\mathrm{pr}_X} X \ (\text{resp., } X_r^+ \xrightarrow{x \mapsto (x, x^+)} X \times X_r \xrightarrow{\mathrm{pr}_X} X).$$
If X is geometrically normal, then the first morphism is a locally closed immersion by Theorem 7.1.1 (3). This shows that $X_1^- \hookrightarrow X$ (resp., $X_r^+ \hookrightarrow X$) is also locally closed since the projection $\mathrm{pr}_X : X \times X_1 \to X$ (resp., $\mathrm{pr}_X : X \times X_r \to X$) is closed (since X_i is proper). Recall that a dense subset is locally closed if and only if it is open, we are done. $\square$

7.2 Chain of orbits

The notion "chain of orbits" is introduced as "equivariant chain of projective lines (of negative weight)" in [Hei17, 2.B], which turns out to be the test object for separatedness or properness of good moduli spaces (see Theorem 8.3.7 later).

Definition 7.2.1. A *chain of orbits* in X is a $\mathbb{G}_m$-equivariant morphism
$$f : C := \mathbb{P}^1 \cup_{\infty \sim 0} \cdots \cup_{\infty \sim 0} \mathbb{P}^1 \to X$$
where the $\mathbb{G}_m$-action on C induces the standard action of some negative weight $w_i < 0$ on each component $\mathbb{P}^1 = \mathrm{Proj}(k[\alpha, \beta])$ of C (with $0 = [1 : 0]$ and $\infty = [0 : 1]$), i.e., this is given by $t.\alpha = t^{w_i + d}\alpha, t.\beta = t^d \beta$ for some d.

Definition 7.2.2. Let $f : C := \mathbb{P}^1 \cup_{\infty \sim 0} \cdots \cup_{\infty \sim 0} \mathbb{P}^1 \to X$ be a chain of orbits in X.

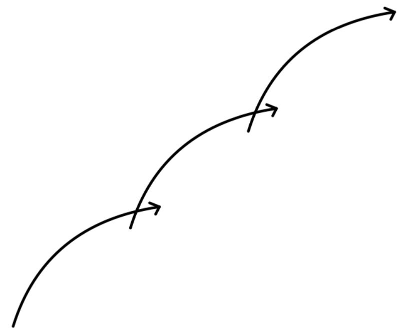

Figure 7.1: Chain of orbits

1. If X is proper (in this case Corollary 7.1.2 is applicable), then the chain f is said to be *maximal* if $f(0_{\text{first}}) \in X_1$ and $f(\infty_{\text{last}}) \in X_r$. The intuition behind this notation shall be clear after Proposition 7.3.5 later.

2. The chain f is said to be *smoothable* if there exist a DVR R with fraction field K and residue field κ, a $\mathbb{G}_m$-equivariant commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(\kappa) \arrow[d, hook] \arrow[dr, phantom, "\urcorner"] & C \arrow[l] \arrow[r, "f"] \arrow[d, hook] & X \\ \operatorname{Spec}(R) & C_R \arrow[l, "\text{flat}"] \arrow[ur, "f_R"'] & \end{tikzcd} \tag{7.2.1}$$

such that $C_K \cong \mathbb{P}^1_K$ and $f_K = \overline{\sigma(-, x_K)} : \mathbb{P}^1_K \to X$ is a complete orbit map for some point $x_K \in X$.

In particular, if f is maximal, then $x_K \in X_1^- \cap X_r^+$.

3. Let $U \subset X$ be a $\mathbb{G}_m$-invariant subset. The chain f is said to be *U-smoothable* if there exist a DVR R with fraction field K and residue field κ, a $\mathbb{G}_m$-equivariant commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(\kappa) \arrow[d, hook] \arrow[dr, phantom, "\urcorner"] & C \arrow[l] \arrow[r, "f"] \arrow[d, hook] & X \\ \operatorname{Spec}(R) & C_R \arrow[l, "\text{flat}"] \arrow[ur, "f_R"'] & \end{tikzcd}$$

such that $C_K \cong \mathbb{P}^1_K$ and $f_K = \overline{\sigma(-, x_K)} : \mathbb{P}^1_K \to X$ is a complete orbit map for some point $x_K \in U$.

In particular, if f is maximal, then $x_K \in U \cap X_1^- \cap X_r^+$.

Remark 7.2.3. The Cartesian square in (7.2.1) always exists (e.g., blowing-up $\mathbb{P}^1 \times \operatorname{Spec}(R)$ along $(\infty, \operatorname{Spec}(\kappa))$ and then iteratively blowing-up ∞ in the exceptional $\mathbb{P}^1$'s), so for smoothability of a chain f the main issue is the existence of a lifting f_R.

Proposition 7.2.4 ([BBS83], Theorem 0.1.2 and Corollary 0.2.4 if $k = \mathbb{C}$)**.** *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant dense subset. Then for any point $x \in X$, there exists a U-smoothable maximal chain of orbits in X passing through x. In other words, X is covered by U-smoothable maximal chain of orbits in X.*

Proof. There exist a DVR R with fraction field K and residue field κ, a morphism x_R : $\mathrm{Spec}(R) \to X$ such that the generic point $\mathrm{Spec}(K)$ maps to the generic point $\eta \in X$ and the closed point $\mathrm{Spec}(\kappa)$ maps to $x \in X$. Observe that $U \cap X_1^- \cap X_r^+ \subset X$ is dense by Corollary 7.1.2, it follows that $\eta \in U \cap X_1^- \cap X_r^+$.

Consider the complete orbit map $\overline{\sigma(-,\eta)} : \mathbb{P}^1 \times \mathrm{Spec}(K) \to X$ of η and the orbit map $\sigma(-, x_R) : \mathbb{G}_m \times \mathrm{Spec}(R) \to X$ of x_R. They glue together to yield a morphism

$$\begin{tikzcd} \mathbb{P}^1 \times \mathrm{Spec}(K) \cup_{\mathbb{G}_m \times \mathrm{Spec}(K)} \mathbb{G}_m \times \mathrm{Spec}(R) \arrow[r, "\overline{\sigma(-,\eta)} \cup \sigma(-,x_R)"] \arrow[d, hook, "\text{open}"'] & X \\ \mathbb{P}^1 \times \mathrm{Spec}(R) \arrow[ur, dashed] & \end{tikzcd}$$

and we regard it as a rational map $f : \mathbb{P}^1 \times \mathrm{Spec}(R) \dashrightarrow X$. The non-defined locus of f is supported at $(0, \mathrm{Spec}(\kappa))$ and $(\infty, \mathrm{Spec}(\kappa))$, it extends to a morphism $\tilde{f} : \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \to X$ after blowing-up certain ideal $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^1 \times \mathrm{Spec}(R)}$ supported at these two points since X is proper (see [GW20, Theorem 13.98]), i.e.,

$$\begin{tikzcd} \Phi \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \arrow[r, "\tilde{f}"] \arrow[d] & X \\ \mathbb{P}^1 \times \mathrm{Spec}(\kappa) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}^1 \times \mathrm{Spec}(R) \arrow[ur, dashed, "f"'] \arrow[d] & \\ \mathrm{Spec}(\kappa) \arrow[r, hook] & \mathrm{Spec}(R) \arrow[uur, bend right, "x_R"'] & \end{tikzcd}$$

and there is a natural $\mathbb{G}_m$-action on $\mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R))$ such that $\tilde{f}$ is $\mathbb{G}_m$-equivariant. To conclude, we claim that the restriction $\tilde{f}|_\Phi : \Phi \to X$ can be refined to be a chain of orbits in X, since it is already maximal, passing through x and then also U-smoothable by construction. Our proof below adapts the computations done in [Hei17, Lemma 2.1].

Indeed, it suffices to treat the affine situation that the ideal $I \subset \mathscr{O}_{\mathbb{A}^1 \times \mathrm{Spec}(R)} = R[y]$ is supported at (y, π). Then there exists an integer $n > 0$ such that $(y, \pi)^n \subset I$ and since I is $\mathbb{G}_m$-invariant it is homogeneous with respect to the grading for which y has $\mathbb{G}_m$-weight 1 and π has $\mathbb{G}_m$-weight 0. Every homogeneous generator of I (of weight $d \geq 0$) is of the form $y^d \pi^m$ for some integer $m \geq 0$ so that I is monomial.

As $(y, \pi)^n \subset I$ we may write $I = (y^a, \pi^b, y^{a_i}\pi^{b_i})_{i=1,\ldots,N}$ with $a < n, b < n, a_i + b_i < n$. This ideal becomes principal after successively blowing-up ∞ and then blowing-up ∞ in the exceptional $\mathbb{P}^1$'s: Blowing-up (y,π) we get charts with coordinates $(y,\pi) \mapsto (y'\pi,\pi)$ and $(y,\pi) \mapsto (y, y\pi')$. Since y has $\mathbb{G}_m$-weight 1 and π has $\mathbb{G}_m$-weight 0 we see that the $\mathbb{G}_m$-weights of (y',π) are $(1,0)$ and the $\mathbb{G}_m$-weights of (y,π') are $(0,-1)$.

In the first chart the proper transform of I is $(y'^a\pi^a, \pi^b, y'^{a_i}\pi^{a_i+b_i})_{i=1,\dots,N}$. This ideal is principal if $b=1$ and otherwise equals to an ideal of the form

$$\pi^c(\pi^{b-c}, \text{ mixed monomials of lower total degree}).$$

A similar computation works in the other chart. By induction this shows that the ideal will become principal after finitely many blowing ups and that in each chart the coordinates $(y^{(i)},\pi^{(i)})$ have $\mathbb{G}_m$-weights (w_i,v_i) with $w_i - v_i = 1$. $\square$

7.3 Semi-sections

The definition of semi-sections starts with an observation that there is a natural relation $<$ on the set of fixed point components of X.

Definition 7.3.1 ([BBS83], Definition, page 776)**.** Let $i,j \in \{1,\dots,r\}$.

1. We write $X_i <_d X_j$ (read: X_i is directly less than X_j) if there exists a point $x \in X$ such that $x^- \in X_i$ and $x^+ \in X_j$, or equivalently, $X_i^- \cap X_j^+ \neq \emptyset$.
2. We write $X_i < X_j$ (read: X_i is less than X_j) if there exists a sequence of points $x_1,\dots,x_n \in X$ such that $x_1^- \in X_i$, x_s^+ and x_{s+1}^- belong to the same fixed point component for $s = 1,\dots,n-1$ and $x_n^+ \in X_j$, or equivalently, $X_i <_d \cdots <_d X_j$.
3. We say X_i is *comparable* with X_j if $X_i < X_j$ or $X_j < X_i$.
4. We say there exists a *quasi-cycle* between X_i and X_j if $X_i < X_j$ and $X_j < X_i$.

Remark 7.3.2. By definition, the relation $<$ on the set of fixed point components is reflexive and transitive, but not anti-symmetric (and hence not a partial order) in general. We will shortly see in Proposition 7.3.8 that it is anti-symmetric if X is geometrically normal and projective.

Example 7.3.3 ([BBS82], Example 2.1)**.** Suppose $\mathbb{G}_m$ acts on an n-dimensional projective space $\mathbb{P}$. We can find a coordinate system on $\mathbb{P}$ such that the $\mathbb{G}_m$-action is given by

$$t.[x_0:\ldots:x_n] = [t^{\lambda_0}x_0:\ldots:t^{\lambda_n}x_n] \text{ for any } t \in \mathbb{G}_m \text{ and } [x_0:\ldots:x_n] \in \mathbb{P}$$

and $0 = \lambda_{n_0+1} = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots = \lambda_{n_2} < \cdots < \lambda_{n_{r-1}+1} = \cdots = \lambda_{n_r}$ is a sequence of integers with $n_0 := -1$ and $n_r := n$. The interval $[0,n]$ is accordingly divided into r disjoint parts

$$[0,n] = \coprod_{i=1}^{r} \mathbf{I}_i \text{ where } \mathbf{I}_i := [n_{i-1}+1, n_i].$$

The fixed locus decomposition $\mathbb{P}^{\mathbb{G}_m} = \coprod_{i=1}^r \mathbb{P}_i$ reads as

$$\mathbb{P}_i := \{[x_0 : \ldots : x_n] \in \mathbb{P} : x_j = 0 \text{ for any } j \notin \mathbf{I}_i\}$$

such that $\mathbb{P}_1$ is the source and $\mathbb{P}_r$ is the sink of $\mathbb{P}$. For any $x = [x_0 : \ldots : x_n] \in \mathbb{P}$, let $x_{\min}$ (resp., $x_{\max}$) be the first (resp., last) non-zero coordinate of x. Then

$$\begin{aligned}\mathbb{P}_i^- &= \{[x_0 : \ldots : x_n] \in \mathbb{P} : \min \in \mathbf{I}_i\}\\ \mathbb{P}_i^+ &= \{[x_0 : \ldots : x_n] \in \mathbb{P} : \max \in \mathbf{I}_i\}\end{aligned}$$

In particular, $x \in \mathbb{P}_i^- \cap \mathbb{P}_j^+$ if and only if $\min \in \mathbf{I}_i$ and $\max \in \mathbf{I}_j$. This proves that

$$\mathbb{P}_i < \mathbb{P}_j \text{ if and only if } \mathbb{P}_i <_d \mathbb{P}_j \text{ if and only if } i \leq j.$$

As a result, we have

1. $\mathbb{P}_1 < \mathbb{P}_i < \mathbb{P}_r$ for any i.

2. $\mathbb{P}_i^+ \subset \mathbb{P}_i$ (resp., $\mathbb{P}_i^- \subset \mathbb{P}_i$) if and only if $i = 1$ (resp., $i = r$).

3. $\mathbb{P}_i < \mathbb{P}_j$ and $\mathbb{P}_j < \mathbb{P}_i$ implies that $i = j$, i.e., there is no quasi-cycle on $\mathbb{P}$.

These properties of $\mathbb{G}_m$-action on $\mathbb{P}$ also hold in a slightly more general situation, see Proposition 7.3.4, 7.3.5 and 7.3.8 respectively.

Proposition 7.3.4 ([BBS83], Corollary 0.2.5 if $k = \mathbb{C}$)**.** *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then $X_1 < X_i < X_r$ for any i.*

This proposition is also proved in [BBS82, Proposition 2.3] (assuming $k = \bar{k}$), but one should be careful that for non-normal $\mathbb{G}_m$-schemes, the source and sink might coincide, see Remark 7.3.6 (1).

Proof. Choose a point $x \in X_i$ and then apply Proposition 7.2.4. □

Proposition 7.3.5 ([BBS83], Corollary A.3 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then $X_i^+ = X_i$ (resp., $X_i^- = X_i$) if and only if $i = 1$ (resp., $i = r$).*

Remark 7.3.6. This proposition gives a more intuitive characterization of the source (i.e., there is no orbit flowing in) and sink (i.e., there is no orbit flowing out). Moreover

1. The normality condition in Proposition 7.3.5 is necessary. For example we can take X to be the curve obtained from $\mathbb{P}^1$ by glueing 0 and ∞ with the standard $\mathbb{G}_m$-action. In general, we can simply identify a point of X_1 with a point of X_r and this produces many examples with $X_1 = X_r$, $X_1 \subsetneq X_1^+$ and $X_r \subsetneq X_r^-$.

2. If X is smooth, then Proposition 7.3.5 follows from a consequence of dimension computation using Theorem 7.1.1 (5).

Proof. It suffices to prove the plus case.

ONLY IF PART: If $i \neq 1$, then $X_1 < X_i$ by Proposition 7.3.4 and hence $X_i \subsetneq X_i^+$.

IF PART: By definition $X_1 \subset X_1^+$, it remains to show $X_1^+ \subset X_1$. Suppose otherwise that there exists a point $x \in X_1^+ - X_1$. Let $U \subset X$ be a $\mathbb{G}_m$-invariant affine open neighbourhood of x^+ (using X is geometrically normal). Since $\emptyset \neq (U \cap X_1)^- \subset U^-$ and

$$(U \cap X_1)^- \xrightarrow{\text{open dense}} X_1^- \xrightarrow[\text{by Cor 7.1.2}]{\text{open dense}} X \text{ has dense image,}$$

the inclusion $U^- \hookrightarrow U$ has dense image. But it is also a closed embedding by Theorem 7.1.1 (4), this forces that $U^- = U$. By Theorem 7.1.1 (5)

1. the inclusion $U^- \hookrightarrow U$ induces an isomorphism

$$T_{x^+}(X) = T_{x^+}(U) = T_{x^+}(U^-) \xrightarrow{\sim} T_{x^+}(U)^+ = T_{x^+}(X)^+,$$

   i.e., the tangent space $T_{x^+}(X)$ has only non-negative $\mathbb{G}_m$-weights part.

2. the inclusion $x \in U^+ \hookrightarrow U$ induces an isomorphism

$$0 \neq T_{x^+}(U^+) \xrightarrow{\sim} T_{x^+}(U)^- = T_{x^+}(X)^-,$$

   contributing to a strictly negative $\mathbb{G}_m$-weights part of $T_{x^+}(X)$ as $x \notin X_1$.

a contradiction. □

Corollary 7.3.7. *Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then $X_1 < X_i$ and $X_i < X_1$ (resp., $X_i < X_r$ and $X_r < X_i$) imply that $i = 1$ (resp., $i = r$), i.e., there is no quasi-cycle on X passing through X_1 (resp., X_r).*

Moreover, if X is projective, then there is even no quasi-cycle on X.

Proposition 7.3.8. *Let X be a geometrically normal, projective and irreducible scheme over a perfect field k with a $\mathbb{G}_m$-action. Then $X_i < X_j$ and $X_j < X_i$ imply that $i = j$, i.e., there is no quasi-cycle on X.*

Remark 7.3.9. A few words about Proposition 7.3.8.

1. The non-trivial IF PART in Proposition 7.3.5 follows immediately from Proposition 7.3.8 if X is projective and k is perfect. Indeed, if $X_1 \subsetneq X_1^+$ (resp., $X_r \subsetneq X_r^-$), then there exists $i \neq 1$ (resp., $i \neq r$) such that $X_i < X_1$ (resp., $X_r < X_i$). But this would produce a quasi-cycle on X since $X_1 < X_i$ (resp., $X_i < X_r$).

2. The assumption "projective" cannot be weakened to "proper". The example in [Som82, §1] gives such a counterexample and we will reproduce it in Appendix B.2.

3. The statement is true if X is a normal compact Kähler space, see [Fuj96, Theorem 1.1]. The essential point is the existence of certain "positive" class on X (here the positive Kähler class serves as such one). Our proof below is similar to it.

Proof. 1. If X is a projective space, then this is Example 7.3.3.

2. If $k = \bar{k}$, then by [Sum74, Theorem 1] (requiring $k = \bar{k}$) X can be $\mathbb{G}_m$-equivariantly embedded into a projective space (or equivalently, there exists a $\mathbb{G}_m$-equivariant ample line bundle on X). Now the assertion follows directly from that of projective space since any such X_i and X_j will be mapped to different connected components of $\mathbb{P}^{\mathbb{G}_m}$ (using the condition that they are comparable). Here we provide a more direct approach. Let $\mathscr{L}$ be a $\mathbb{G}_m$-equivariant ample line bundle over X. Since $\mathbb{G}_m$ acts trivially on X_i, it acts on $\mathscr{L}|_{X_i}$ by some character $\chi_i \in X^*(\mathbb{G}_m) \cong \mathbb{Z}$ and under this identification it corresponds to $\mathrm{wt}_{\mathbb{G}_m}(\mathscr{L}|_x)$ for any $x \in X_i$. This gives rise to a well-defined function

$$\mu_{\mathscr{L}} : \{X_1, \ldots, X_r\} \to \mathbb{Z}.$$

Claim 7.3.10. $\mu_{\mathscr{L}}(X_i) < \mu_{\mathscr{L}}(X_j)$ if $X_i < X_j$.

Hence $X_i < X_j$ and $X_i > X_j$ imply that $i = j$, we are done.

Proof. It suffices to show $\mu_{\mathscr{L}}(X_i) < \mu_{\mathscr{L}}(X_j)$ if $X_i <_d X_j$. Indeed, if $X_i <_d X_j$, i.e., there exists a complete orbit map $\overline{\sigma(-,x)} : \mathbb{P}^1 = \mathrm{Proj}(k[\alpha, \beta]) \to X$ for some $x \in X$ such that $x^- \in X_i$ and $x^+ \in X_j$, then by [Hei17, Remark 2.2]

$$0 < \deg(\mathscr{L}|_{\mathbb{P}^1}) = \frac{\mathrm{wt}_{\mathbb{G}_m}(\mathscr{L}|_{x^+}) - \mathrm{wt}_{\mathbb{G}_m}(\mathscr{L}|_{x^-})}{\mathrm{wt}_{\mathbb{G}_m}(\alpha) - \mathrm{wt}_{\mathbb{G}_m}(\beta)} = \frac{\mu_{\mathscr{L}}(X_j) - \mu_{\mathscr{L}}(X_i)}{\mathrm{wt}_{\mathbb{G}_m}(\alpha) - \mathrm{wt}_{\mathbb{G}_m}(\beta)}$$

i.e., $\mu_{\mathscr{L}}(X_j) > \mu_{\mathscr{L}}(X_i)$. □

3. If k is perfect, then passing to an algebraic closure $\bar{k}$ of k there exists a $\mathbb{G}_m$-equivariant ample line bundle $\overline{\mathscr{L}}$ on $\overline{X}$, let $k \subset k' \subset \bar{k}$ be an intermediate field such that k'/k is normal and both $\overline{X}$ and $\overline{\mathscr{L}}$ can be defined over k'. Since everything is of finite type over k, the field extension k'/k is finitely generated. By our assumption on k, the field extension k'/k is separable and hence Galois. Then the following $\mathbb{G}_m$-equivariant ample line bundle on $\overline{X}$

$$\bigotimes_{\sigma \in \mathrm{Gal}(k'/k)} \sigma^* \overline{\mathscr{L}}$$

is $\mathrm{Gal}(k'/k)$-invariant and therefore descends to a $\mathbb{G}_m$-equivariant ample line bundle on X. Then we can apply the same arguments in (2).

□

Now comes the main player in this section.

Definition 7.3.11 ([BBS83], Definition, page 776)**.** A *semi-section* of $\{1, \ldots, r\}$ is a division of $\{1, \ldots, r\}$ into three subsets (A^-, A^0, A^+), at least two of which are non-empty, satisfying one of the following equivalent conditions:

(a) $A^- \cup A^0$ is saturated with respect to $<$, i.e., if $i \in A^- \cup A^0$ and $X_j < X_i$, then $j \in A^-$.

(b) $A^0 \cup A^+$ is saturated with respect to $>$, i.e., if $i \in A^0 \cup A^+$ and $X_i < X_j$, then $j \in A^+$.

Remark 7.3.12. Let (A^-, A^0, A^+) be a semi-section of $\{1, \ldots, r\}$. By definition

1. No indices in A^0 are comparable.
2. If $X_i < X_j$ and $X_j < X_i$, then $i, j \in A^-$ or $i, j \in A^+$.
3. If X is proper, then $1 \in A^- \cup A^0$ and $r \in A^0 \cup A^+$. So by (2) the first necessary condition for the existence of semi-sections is $X_r \not< X_1$, which is guaranteed by Proposition 7.3.5 if X is further geometrically normal.

Definition 7.3.13 ([BBS82], Definition 1.3)**.** Let (A^-, A^0, A^+) be a semi-section of $\{1, \ldots, r\}$. Then

$$U(A^*) := \coprod_{\substack{i \in A^- \cup A^0 \\ j \in A^0 \cup A^+}} X_i^- \cap X_j^+ \subset X$$

is the *semi-sectional subset* defined by the semi-section (A^-, A^0, A^+), i.e.,

$$U(A^*) := \{x \in X : \text{if } x \in X_i^- \cap X_j^+, \text{ then } i \in A^- \cup A^0 \text{ and } j \in A^0 \cup A^+\}.$$

Remark 7.3.14. If X is proper, then any semi-sectional subset of X contains the dense subset $X_1^- \cap X_r^+$ by Remark 7.3.12 (3).

Example 7.3.15. Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then the following are semi-sections of $\{1, \ldots, r\}$ by Proposition 7.3.5.

1. $(A^-, A^0, A^+) = (\emptyset, \{1\}, \{2, \ldots, r\})$. In this case, $U = X_1^-$ and the corresponding quotient stack $\mathscr{U} := [U/\mathbb{G}_m] = [X_1^-/\mathbb{G}_m]$ admits X_1 as its good moduli space, which is proper.
2. $(A^-, A^0, A^+) = (\{1, \ldots, r-1\}, \{r\}, \emptyset)$. In this case, $U = X_r^+$ and the corresponding quotient stack $\mathscr{U} := [U/\mathbb{G}_m] = [X_r^+/\mathbb{G}_m]$ admits X_r as its good moduli space, which is proper.

Lemma 7.3.16. *Let X be a separated irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a semi-sectional subset defined by a semi-section (A^-, A^0, A^+) of $\{1, \ldots, r\}$. Then $i \in A^0$ if and only if $U \cap X_i \neq \emptyset$. In this case, $X_i \subset U$.*

Proof. If Part: If $U \cap X_i \neq \emptyset$, then $(X_p^- \cap X_q^+) \cap X_i \neq \emptyset$ for some $p \in A \cup A^0$ and $q \in A^0 \cup A^+$, i.e., $i = p = q \in A^0$.

Only If Part: If $i \in A^0$, then $X_i = X_i^- \cap X_i^+ \subset U$. □

Lemma 7.3.17 ([BBS82], Proposition 2.4 if $k = \bar{k}$)**.** *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then semi-sectional subsets of X are $\mathbb{G}_m$-invariant and open.*

Proof. Let $U \subset X$ be a semi-sectional subset defined by a semi-section (A^-, A^0, A^+) of $\{1, \ldots, r\}$, then U is $\mathbb{G}_m$-invariant by Theorem 7.1.1 (1). To prove the openness, note that

$$U = \coprod_{\substack{i \in A^- \cup A^0 \\ j \in A^0 \cup A^+}} (X_i^- \cap X_j^+) = \left(\coprod_{i \in A^- \cup A^0} X_i^- \right) \cap \left(\coprod_{j \in A^0 \cup A^+} X_j^+ \right)$$

and both of them are open by the following lemma. □

Lemma 7.3.18. *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $\Delta \subset \{1, \ldots, r\}$ be a subset. Then*

$$\coprod_{i \in \Delta} X_i^- \subset X \text{ is closed if and only if } \Delta \text{ is saturated with respect to } > .$$

Similar result holds for $+$ case.

This lemma can be seen as a sight generalization of [BBS83, Lemma 1.3.1].

Proof. Note that $X_i^{\pm}$ is constructible for each i by Theorem 7.1.1 (1).

1. If Part: For any $x \in \overline{\coprod_{i \in \Delta} X_i^-} = \coprod_{i \in \Delta} \overline{X_i^-} \subset X$, say $x \in \overline{X_\ell^-}$ for some $\ell \in \Delta$, we need to show

   $$x \in \coprod_{i \in \Delta} X_i^-, \text{ i.e., } x \in X_j^- \text{ for some } j \in \Delta.$$

   Passing to an irreducible component of X_ℓ^- whose closure contains x if necessary, we may assume X_ℓ^- is irreducible. Now we consider x^-. A priori

   $$x^- \in \overline{X_\ell^-} \cap X^{\mathbb{G}_m} = \overline{X_\ell^-}^{\mathbb{G}_m} \text{ since } \overline{X_\ell^-} \subset X \text{ is closed.}$$

   Let Ω be the connected component of $\overline{X_\ell^-}^{\mathbb{G}_m}$ containing x^- and $\Omega \subset X_j$ for some (uniquely determined) j, then

   $$x^- \in \Omega \subset X_j, \text{ i.e., } x \in X_j^-.$$

To conclude, we claim $j \in \Delta$. Since X_ℓ^- is dense in $\overline{X_\ell^-}$ with $\overline{X_\ell^-}$ being proper and irreducible, Corollary 7.1.2 identifies X_ℓ as the source of $\overline{X_\ell^-}$ and hence $X_\ell < X_j$ by Proposition 7.3.4. This gives $j \in \Delta$ since Δ is saturated with respect to $>$.

2. Only If Part: It suffices to show $j \in \Delta$ for any $X_j >_d X_\ell$ with $\ell \in \Delta$. In this case

$$X_\ell^- \cap X_j^+ \neq \emptyset \text{ and hence } X_j \cap \overline{X_\ell^-} \neq \emptyset.$$

Then $\coprod_{i\in\Delta} X_i^- \subset X$ being closed implies that

$$X_j \cap \coprod_{i\in\Delta} X_i^- = X_j \cap \overline{\coprod_{i\in\Delta} X_i^-} = X_j \cap \coprod_{i\in\Delta} \overline{X_i^-} = \coprod_{i\in\Delta} \left(X_j \cap \overline{X_i^-}\right) \neq \emptyset,$$

i.e., $X_j \cap X_s^- \neq \emptyset$ for some $s \in \Delta$. Thus $j = s \in \Delta$.

$\square$

The same arguments in the proof of Lemma 7.3.18 reveal the following refined structure between these strata $X_i^\pm$, although we will not use it in this thesis.

Lemma 7.3.19. *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. If $X_j <_d X_i$, then $X_i^+ \subset X_i^+ \cup X_j^+$ and $X_j^- \subset X_i^- \cup X_j^-$ are both open.*

To end this section, we show that the assignment from semi-sections of $\{1, \ldots, r\}$ to the corresponding semi-sectional subsets of X is injective.

Proposition 7.3.20. *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Then different semi-sections of $\{1, \ldots, r\}$ give rise to different semi-sectional subsets of X.*

Proof. Let $U(A^*)$ and $U(B^*)$ be the semi-sectional subsets of X defined by two different semi-sections (A^-, A^0, A^+) and (B^-, B^0, B^+) of $\{1, \ldots, r\}$, then

$$A^- \cup A^0 \neq B^- \cup B^0 \text{ or } A^0 \cup A^+ \neq B^0 \cup B^+.$$

It suffices to consider the first case. Without loss of generality we may further assume that

$$\Delta := A^- \cup A^0 - B^- \cup B^0 \neq \emptyset.$$

There are two situations:

1. If $r \in \Delta$, then $r \in A^0$ by Remark 7.3.12 (3) and hence $X_r \subset U(A^*)$ by Lemma 7.3.16. However, $X_r \nsubseteq U(B^*)$ since $r \notin B^0$ and this indicates the difference.

2. If $r \notin \Delta$, then by Lemma 7.3.4 there exist indices $i \in \Delta$ (and hence $i \notin B^- \cup B^0$, i.e., $i \in B^+$) and $j \notin \Delta$ such that $X_i^- \cap X_j^+ \neq \emptyset$, i.e., $X_i <_d X_j$. By definition of

semi-section, $i \in B^+$ implies that $j \in B^+$ and hence

$$\emptyset \neq X_i^- \cap X_j^+ \not\subset U(B^*) \text{ since } i, j \in B^+.$$

To conclude, we claim that $j \in A^+$ and hence

$$\emptyset \neq X_i^- \cap X_j^+ \subset U(A^*) \text{ since } i \in A^- \cup A^0, j \in A^+.$$

Indeed, if $j \notin A^+$, then $j \in B^+$ implies that $j \in \Delta$, a contradiction.

□

7.4 Moment measure

In this section, we give a brief introduction to the moment measure, which is the conjectural combinational date used to characterize $\mathbb{T}$-invariant open dense subsets with proper good moduli spaces (see Conjecture 1.1.8). The materials here are taken from [BB02b, §11.2].

In this section, let X be a geometrically normal and projective scheme over a field k with a $\mathbb{T} := \mathbb{G}_m^e$-action. Let $X^{\mathbb{T}} \subset X$ be the subscheme of fixed points and $X^{\mathbb{T}} = \coprod_{i=1}^r X_i$ be the decomposition of $X^{\mathbb{T}}$ into connected components. Choose a $\mathbb{T}$-equivariant ample line bundle $\mathscr{L}$ over X and it gives rise to a well-defined function

$$\mu := \mu_{\mathscr{L}} : \{X_1, \dots, X_r\} \to \mathbb{Z}^e \otimes \mathbb{R} = \mathbb{R}^e \text{ by } X_i \mapsto \mathrm{wt}_{\mathbb{T}}(\mathscr{L}|_x) \text{ for some } x \in X_i.$$

To any point $x \in X$ we attach

$$c(x) := \{i : X_i \cap \overline{\mathbb{T}.x} \neq \emptyset\}$$

and a convex subset of $\mathbb{R}^e$ defined as

$$\mu(c(x)) := \text{rel-Int}\,(\text{conv}\{\mu(X_i) : i \in c(x)\})$$

where rel-Int$(-)$ denotes the interior of the set. Then $\dim(\mathbb{T}.x) = \dim(\mu(c(x)))$.

Definition 7.4.1. Define a cell complex $\mathscr{C}_X$ in the following way:

- a *cell* of $\mathscr{C}_X$ is defined to be $c(x) \subset \{1, \dots, r\}$ for some $x \in X$.
- the *dimension* of a cell $c(x)$ is defined to be $\dim(\mu(c(x))) = \dim(\mathbb{T}.x)$.
- the *boundary* of a cell $c(x)$ is defined to be $\delta(c(x)) := \{c(y) : y \in \overline{\mathbb{T}.x}\}$.

Definition 7.4.2. A collection $\{c(x_i) : i \in I\}$ of cells is a *subdivision* of a cell $c(x)$ if

1. the collection $\{\mu(c(x_i)) : i \in I\}$ is a subdivision of the polytope $\mu(c(x))$,

2. the function μ is injective on $\{c(x_i) : i \in I\}$, i.e., if $c(x_i) \neq c(x_j)$, then $\mu(c(x_i)) \cap \mu(c(x_j)) = \emptyset$,

3. it is stable under taking boundaries, i.e., if $\mu(c(y)) \subset \mu(c(x))$ and $\mu(c(y))$ is in the boundary of $c(x_i)$ for some $i \in I$, then $c(y) = c(x_j)$ for some $j \in I$.

Definition 7.4.3. A *moment measure* on X is a non-empty collection $\mathscr{M}$ of cells of $\mathscr{C}_X$ such that if $c \in \mathscr{M}$ and $\mathscr{I}$ is a subdivision of c, then there exists a unique cell $c_i \in \mathscr{I}$ such that $c_i \in \mathscr{M}$. If $\mathscr{M}$ is a moment measure on X, then we define

$$X(\mathscr{M}) := \{x \in X : c(y) \in \mathscr{M} \text{ for some } y \in \overline{\mathbb{T}.x}\}.$$

Remark 7.4.4. A few words about Definition 7.4.3.

1. The terminology "moment measure" has been suggested by the fact that when a moment measure $\mathscr{M}$ is given, the map $\nu : \mathscr{C}_X \to \{0,1\}$ assigning to a cell the value 1 if the cell belongs to $\mathscr{M}$ and 0 otherwise, has on one hand some properties of moment maps, on the other hand, like a measure, is additive with respect to subdivisions, i.e., if $\{c(x_i) : i \in I\}$ is a subdivision of a cell $c(x)$, then

$$\nu(c(x)) = \sum_{i \in I} \nu(c(x_i)).$$

2. If $k = \mathbb{C}$, then it is proved in [BBS87b, Theorem 2.1] (see also [BBS87a, Theorem 1.3.1]) that $X(\mathscr{M}) \subset X$ is a $\mathbb{T}$-invariant open subset such that the quotient stack $[X(\mathscr{M})/\mathbb{T}]$ admits a proper good moduli space.

Example 7.4.5. If $e = 1$, then moment measures on X are in one-to-one correspondence with semi-sections of $\{1, \dots, r\}$. Indeed, for any point $x \in X$, if $x \in X_i^- \cap X_j^+$, then

$$c(x) = \{i, j\} \text{ and } \delta(c(x)) = \{\{i\}, \{i, j\}, \{j\}\}$$

and any subdivision of the cell $c(x)$ is of the form

$$\{\{i\} = \{i_1\}, \{i_1, i_2\}, \{i_2\}, \{i_2, i_3\}, \dots, \{i_{s-1}\}, \{i_{s-1}, i_s\}, \{i_s\} = \{j\}\} \text{ with } X_{i_j} <_d X_{i_{j+1}}.$$

If (A^-, A^0, A^+) is a semi-section of $\{1, \dots, r\}$, then the corresponding moment measure $\mathscr{M}$ on X contains the following cells

$$\{i\} \text{ with } i \in A^0 \text{ and } \{i, j\} \text{ with } i \in A^- - A^0 \text{ and } j \in A^+ - A^0$$

and hence the semi-sectional subset $U(A^*)$ coincides with $X(\mathscr{M})$.

Chapter 8

Geometric characterizations of existence of good moduli space

Our main goal in this chapter is to present a geometric characterization for a good moduli space of the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$, for a $\mathbb{G}_m$-invariant open subset $U \subset X$, to be separated or proper (see Theorem 8.3.7). This geometric characterization essentially comes from Theorem 0. Recall that an algebraic stack admits a proper good moduli space if and only if it is Θ-reductive, S-complete and satisfies the existence part of valuative criterion for properness. Roughly speaking, if $\mathscr{U}$ admits a good moduli space, then it is separated (resp., proper) if and only if U intersects every smoothable maximal chain of orbits in X at most one (resp., at one) orbit.

In this chapter, we will frequently use the fact without mentioning that any $\mathbb{G}_m$-torsor over a local ring (e.g., a DVR or a field) is trivial (see [SAG2, Tome 2, Exposé VIII, Corollaire 4.5]).

8.1 Consequence of Θ-reductivity

Proposition 8.1.1. *Let X be a separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset. Then the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ is Θ-reductive if and only if*

$$\text{the inclusion } (U \cap X_i)^{\pm} \subset U \text{ is a closed immersion for each } i.$$

In particular, if U contains no fixed points, then $\mathscr{U}$ is Θ-reductive.

Proof. By [AHLH18, Proposition 3.13 and Remark 3.14], the quotient stack $\mathscr{U}$ is Θ-reductive if and only if the inclusion $U^+_\lambda \subset U$ is a closed immersion, where $\lambda : \mathbb{G}_m \to \mathbb{G}_m$ is a cocharacter and

$$U^+_\lambda := \left\{ x \in U : \lim_{t \to 0} \lambda(t).x \in U \right\}.$$

Let $n_\lambda \in \mathbb{Z}$ be the integer corresponding to $\lambda \in X_*(\mathbb{G}_m) \cong \mathbb{Z}$, then we compute that

$$\lim_{t\to 0} \lambda(t).x = \lim_{t\to 0} t^{n_\lambda}.x = \begin{cases} x^- & \text{if } n_\lambda > 0 \\ x & \text{if } n_\lambda = 0 \\ x^+ & \text{if } n_\lambda < 0 \end{cases}, \text{ i.e., } U_\lambda^+ = \begin{cases} \coprod_{i=1}^r (U \cap X_i)^+ & \text{if } n_\lambda > 0 \\ U & \text{if } n_\lambda = 0 \\ \coprod_{i=1}^r (U \cap X_i)^- & \text{if } n_\lambda < 0 \end{cases}.$$

This finishes the proof. □

Corollary 8.1.2. *Let X be a proper scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ is Θ-reductive. Then $X_1 \subset U$ (resp., $X_r \subset U$) implies that $U = X_1^-$ (resp., $U = X_r^+$).*

Proof. In this case, the subset $X_1^- \subset U$ (resp., $X_r^+ \subset U$) is both dense (by Corollary 7.1.2) and closed (by Proposition 8.1.1). □

Proposition 8.1.3. *Let X be a separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. For any $\mathbb{G}_m$-invariant open subset $U \subset X - X^{\mathbb{G}_m}$, the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ admits a good moduli space.*

Proof. By [AHLH18, Theorem A], this follows from

1. $\mathscr{U}$ is locally linearly reductive.

   The stabilizer group of any point $x \in |\mathscr{U}|$, as a closed subgroup of $\mathbb{G}_m$, is either $\mathbb{G}_m$ itself (this corresponds to fixed point) or $\prod_{i=1}^n \mu_{m_i}$ (this is not connected). In both cases, they are linearly reductive.

2. $\mathscr{U}$ is Θ-reductive.

   By Proposition 8.1.1.

3. $\mathscr{U}$ has unpunctured inertia.

   For this the valuative criterion in [AHLH18, Theorem 5.2] applies since $\mathscr{U}$ is locally linearly reductive and Θ-reductive. For every DVR R with fraction field K and any morphism $x_R : \mathrm{Spec}(R) \to \mathscr{U}$, we show that any automorphism $\varphi_K \in \mathrm{Aut}_{\mathscr{U}}(x_K)$ of finite order extends to an automorphism φ_R of x_R.

   Indeed, if we choose a lift $\tilde{x}_R : \mathrm{Spec}(R) \to U$ of x_R, then $\varphi_K \in \mathrm{Aut}_{\mathscr{U}}(\xi_K)[n]$ corresponds to a K-point of $U \times_{\mathscr{U}} U = \mathbb{G}_m \times U$, say

   $$(g_K, \tilde{x}_K) : \mathrm{Spec}(K) \to \mathbb{G}_m \times U$$

   for some $g_K \in \mathbb{G}_m[n](K) \cong \mu_n(K) \cong \mu_n(R) \cong \mathbb{G}_m[n](R) \subset \mathbb{G}_m(R)$ since the group scheme μ_n is finite over k. Let $g_R \in \mathbb{G}_m(R)$ be the image of g_K, then

   $$(g_R, \tilde{x}_R) : \mathrm{Spec}(R) \to \mathbb{G}_m \times U$$

defines an element $\varphi_R \in \mathrm{Aut}_{\mathscr{U}}(x_R)$ and this extends φ_K.

$\square$

8.2 Consequence of S-completeness

The following result reveals certain structure of $\mathbb{G}_m$-invariant open subset $U \subset X$ with Θ-reductive or S-complete quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$.

Proposition 8.2.1. *Let X be a separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ is Θ-reductive or S-complete. Then there is no non-constant $\mathbb{G}_m$-equivariant morphism $\mathbb{P}^1 \to U$.*

Proof. If $\mathscr{U}$ is Θ-reductive, then the same argument in Example 2.2.4 applies.

If $\mathscr{U}$ is S-complete, then any non-constant $\mathbb{G}_m$-equivariant morphism $f : \mathbb{P}^1 \to U$ induces two morphisms

$$f_0 : \mathrm{Spec}(k[t^{-1}]_{(t^{-1})}) = \mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}) \to \mathbb{P}^1 \xrightarrow{f} U \to \mathscr{U}$$
$$f_\infty : \mathrm{Spec}(k[t]_{(t)}) = \mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,\infty}) \to \mathbb{P}^1 \xrightarrow{f} U \to \mathscr{U}$$

Identifying $k[t^{-1}]_{(t^{-1})} \xrightarrow{\sim} k[t]_{(t)}$ by mapping $t^{-1} \mapsto t$ and we denote this DVR by R and its fraction field by K. By definition $f_0|_K = f_\infty|_K$, then we obtain a morphism

$$f_0 \cup f_\infty : \mathrm{Spec}(R) \cup_{\mathrm{Spec}(K)} \mathrm{Spec}(R) = \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$$

and S-completeness of $\mathscr{U}$ implies that we can extend this morphism over 0, which is necessarily equal to both $f(0)$ and $f(\infty)$, a contradiction since f is non-constant. $\square$

Corollary 8.2.2 ([Gro84], Proposition 2.2 if $k = \mathbb{C}$)**.** *Let X be a separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ is Θ-reductive or S-complete. If $X_i \subset U$, then $X_i <_d X_j$ or $X_j <_d X_i$ for some $j \neq i$ implies that $X_j \nsubseteq U$.*

8.2.1 Geometric characterization of S-completeness

Theorem 2.4.4 indicates that S-completeness of a quotient stack by $\mathbb{G}_m$ can be checked on its dense substacks. If X is proper, then the following substack

$$\mathscr{V} := [U \cap X_1^- \cap X_r^+/\mathbb{G}_m] \subset [U/\mathbb{G}_m] = \mathscr{U}$$

is dense by Corollary 7.1.2 and it will do this job.

Proposition 8.2.3. *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset and $\mathscr{U} := [U/\mathbb{G}_m]$ be the quotient stack. The following are equivalent:*

1. *$\mathscr{U}$ is S-complete.*
2. *The image of any smoothable maximal chain of orbits in X intersects U with one of following forms:*
   (a) *$\emptyset$;*
   (b) *$\mathbb{G}_m.x$ for some $x \in X - X^{\mathbb{G}_m}$;*
   (c) *$\mathbb{G}_m.x \cup \{x^-\}$ (resp., $\mathbb{G}_m.x \cup \{x^+\}$) for some $x \in X_1^- - X_1$ (resp., $x \in X_r^+ - X_r$);*
   (d) *$\mathbb{G}_m.x_1 \cup \{x_1^+ = x_2^-\} \cup \mathbb{G}_m.x_2$ for some $x_1, x_2 \in X - X^{\mathbb{G}_m}$.*

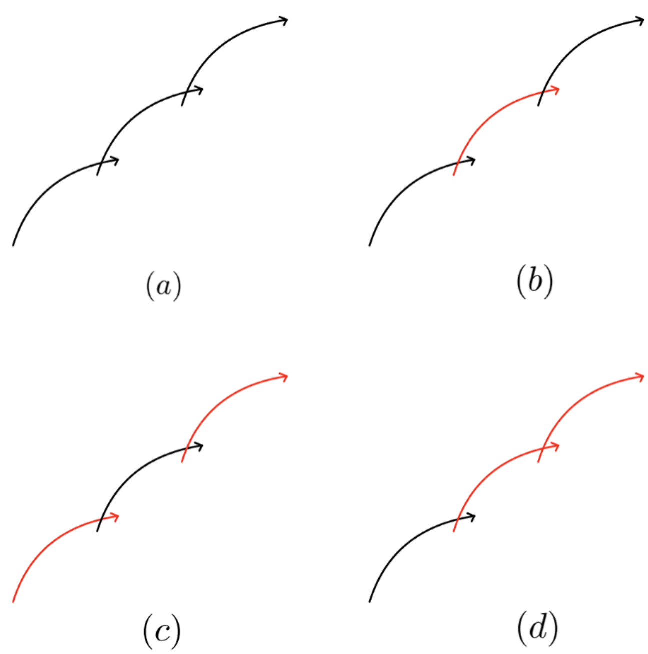

Figure 8.1: Configuration of intersection with smoothable maximal chain of orbits

Proof. Since X is proper, the $\mathbb{G}_m$-invariant subset $V := U \cap X_1^- \cap X_r^+ \subset U$ is dense by Corollary 7.1.2 and hence defines a dense substack

$$\mathscr{V} := [V/\mathbb{G}_m] \subset [U/\mathbb{G}_m] = \mathscr{U}.$$

Roughly speaking, $\mathscr{V}$ parametrizes orbits in U from the source X_1 to the sink X_r (sometimes refers to generic orbit). To check S-completeness of $\mathscr{U}$, we apply Theorem 2.4.4 to $\mathscr{V} \subset \mathscr{U}$, yielding that $\mathscr{U}$ is S-complete if and only if

(1′) For every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{U} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows such that $\mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$ factors through $\mathscr{V} \subset \mathscr{U}$ can be uniquely filled in.

To conclude, we show that (1′) ⇔ (2).

(1′) ⇒ (2). Let $f : C \to X$ be a smoothable maximal chain of orbits in X, we need to show $U \cap f(C)$ has one of the forms in (2).

By definition there exist a DVR R with fraction field K and residue field κ, a $\mathbb{G}_m$-equivariant diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa) \arrow[d, hook] & C \arrow[l] \arrow[r, "f"] \arrow[d, hook] \arrow[dl, phantom, "\lrcorner"] & X \\ \mathrm{Spec}(R) & C_R \arrow[l, "\text{flat}"] \arrow[ur, "f_R"'] & \end{tikzcd}$$

such that $C_K \cong \mathbb{P}^1_K$ and $f_K = \overline{\sigma(-, x_K)} : \mathbb{P}^1_K \to X$ is a complete orbit map for some point $x_K \in X_1^- \cap X_r^+$. If $U \cap f(C) = \emptyset$ (this could happen, e.g., if $x_K \notin U$), then this is (a) and there is nothing to prove. Hereafter we assume that $U \cap f(C) \neq \emptyset$, then it follows that

$$U \cap f_K(C_K) \neq \emptyset, \text{ i.e., } x_K \in U \cap X_1^- \cap X_r^+ = V$$

To show $U \cap f(C)$ has one of the forms in (2), it suffices to show

$$\mathbb{G}_m.x_1 \neq \mathbb{G}_m.x_2 \subset U \cap f(C) \Rightarrow x_1^+ = x_2^- \text{ or } x_1^- = x_2^+ \in U \cap f(C).$$

Since f is a chain of orbits in X, it reduces to show

$$\mathbb{G}_m.x_1 \neq \mathbb{G}_m.x_2 \subset U \cap f(C) \Rightarrow x_1^+ = x_2^- \text{ or } x_1^- = x_2^+ \in U.$$

Indeed, for any $\mathbb{G}_m.x_1 \neq \mathbb{G}_m.x_2 \subset U \cap f(C)$, there exist sections $s_i : \mathrm{Spec}(R) \to C_R$ such that the compositions

$$\Gamma_i : \mathrm{Spec}(R) \xrightarrow{s_i} C_R \xrightarrow{f_R} X \text{ satisfies } \Gamma_i|_\kappa = x_i \in U \text{ for } i = 1, 2.$$

In particular, we have

- $\mathbb{G}_m.\Gamma_1|_K = \mathbb{G}_m.\Gamma_2|_K$ since f_K is a complete orbit map.
- each Γ_i factors through U since $U \subset X$ is open.

Therefore we obtain two morphisms $\overline{\Gamma}_i : \mathrm{Spec}(R) \xrightarrow{\Gamma_i} U \to \mathscr{U}$ coinciding on $\mathrm{Spec}(K)$ such that $\mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(R) \xrightarrow{\overline{\Gamma}_i} \mathscr{U}$ factors through $\mathscr{V} \subset \mathscr{U}$, i.e., a morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$ such that $\mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$ factors through $\mathscr{V} \subset \mathscr{U}$. By $(1')$, there exists a unique extension $\Gamma : \overline{\mathrm{ST}}_R \to \mathscr{U}$. Unwinding the definition of $\Gamma(0) \in \mathscr{U}$, we have $x_1^+ = x_2^-$ or $x_1^- = x_2^+ \in U$, as desired.

$(2) \Rightarrow (1')$. For every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{ST}}_R - \{0\} \arrow[r, "u"] \arrow[d, hook] & \mathscr{U} \arrow[d] \\
\overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(k)
\end{tikzcd}$$

of solid arrows such that $\mathrm{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$ factors through $\mathscr{V} \subset \mathscr{U}$, we need to show there exists a dotted arrow filling in.

By [Hei17, Proof of Proposition 2.9], there exists a lifting u' of u and we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa) \arrow[r, hook] & \mathrm{Spec}(R) & \\
\Phi \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\llcorner"] & \mathrm{Bl}_{\mathscr{I}}(\mathrm{Bl}_z(\mathbb{P}^1 \times \mathrm{Spec}(R))) \arrow[u] \arrow[r, "\exists \tilde{u}"] \arrow[d, "\text{blow-up}"'] & X \\
& \mathrm{Bl}_z(\mathbb{P}^1 \times \mathrm{Spec}(R)) \arrow[ur, dashed, "u'"'] & \\
& \mathrm{Spec}(R[s,t]/st - \pi) \arrow[u, hook, "\text{open}"] & \\
& \mathrm{Spec}(R[s,t]/st - \pi) - \{0\} \arrow[u, hook, "\text{open}"] \arrow[r, "u'"] \arrow[d] & U \arrow[uuu] \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
& \overline{\mathrm{ST}}_R - \{0\} \arrow[r, "u"'] & \mathscr{U}
\end{tikzcd}$$

where $z := (\infty, \mathrm{Spec}(\kappa)) \in \mathbb{P}^1 \times \mathrm{Spec}(R)$ and we regard u' as a rational map $u' : \mathrm{Bl}_z(\mathbb{P}^1 \times \mathrm{Spec}(R)) \dashrightarrow X$. The non-defined locus of u' is supported at $\mathrm{Spec}(\kappa) = \mathrm{Spec}(R[s,t]/(s,t,\pi))$, it extends to a morphism $\tilde{u} : \mathrm{Bl}_{\mathscr{I}}(\mathrm{Bl}_z(\mathbb{P}^1 \times \mathrm{Spec}(R))) \to X$ after blowing-up certain ideal $\mathscr{I} \subset \mathscr{O}_{\mathrm{Bl}_z(\mathbb{P}^1 \times \mathrm{Spec}(R))}$ supported at $\mathrm{Spec}(\kappa)$ since X is proper. There is a natural $\mathbb{G}_m$-action on $\mathrm{Bl}_{\mathscr{I}}(\mathrm{Bl}_z(\mathbb{P}^1 \times \mathrm{Spec}(R)))$ such that $\tilde{u}$ is $\mathbb{G}_m$-equivariant. As in the proof of Proposition 7.2.4, the restriction $\tilde{u}|_{\Phi} : \Phi \to X$ can be refined to be a chain of orbits in X such that $U \cap \tilde{u}(\Phi) \neq \emptyset$, which is maximal (since $\operatorname{Spec}(K) \hookrightarrow \overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$ factors through $\mathscr{V} \subset \mathscr{U}$) and smoothable (by construction). Then $U \cap \tilde{u}(\Phi)$ has one of the forms given in (b), (c) or (d) by (2).

Choosing a section $\sigma : \operatorname{Spec}(R[s,t]/st-\pi) \to \mathrm{Bl}_{\mathscr{I}}(\mathrm{Bl}_z(\mathbb{P}^1 \times \operatorname{Spec}(R)))$ such that the composition

$$\zeta : \operatorname{Spec}(R[s,t]/st-\pi) \xrightarrow{\sigma} \mathrm{Bl}_{\mathscr{I}}(\mathrm{Bl}_z(\mathbb{P}^1 \times \operatorname{Spec}(R))) \xrightarrow{\tilde{u}} X$$

maps the unique closed point $\operatorname{Spec}(\kappa)$ to a closed point of $U \cap \tilde{u}(\Phi)$, then ζ factors through U since $U \subset X$ is open. Since ζ is $\mathbb{G}_m$-equivariant, it descends to a morphism $\overline{\mathrm{ST}}_R \to \mathscr{U}$ extending u by construction. The uniqueness also follows since there is a unique closed point in $U \cap \tilde{u}(\Phi)$ by (2).

□

Corollary 8.2.4 ([BBS83], Theorem 1.4 if $k = \mathbb{C}$ and $U \subset X - X^{\mathbb{G}_m}$)**.** *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ is S-complete. If $Z := X - U$ is the closed complement, then the following are equivalent:*

1. *Z is disconnected.*
2. *Z has two connected components.*

In either case, X_1 and X_r are contained in the different connected components of Z.

Proof. It suffices to prove (1) ⇒ (2). For any point $x \in Z$, by Proposition 7.2.4 there exists a maximal smoothable chain of orbits $f : C \to X$ in X passing through x, then x is in the same connected component of Z as X_1 or X_r by Proposition 8.2.3. This shows that Z has at most two connected components. □

8.3 Consequence of existence part of valuative criterion

Lemma 8.3.1 ([Gro84], Proposition 2.1 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ satisfies the existence part of valuative criterion for properness. If $U \cap X_i \neq \emptyset$, then $X_i \subset U$.*

Proof. We need to show $U \cap X_i = X_i$. Since $\emptyset \neq U \cap X_i \subset X_i$ is open and X_i is irreducible (using that X is geometrically normal), it suffices to show $U \cap X_i \subset X_i$ is closed, i.e., for every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "u_K"] \arrow[d, hook] & U \cap X_i \arrow[d, hook] \\ \operatorname{Spec}(R) \arrow[r, "u_R"'] \arrow[ur, dashed, "\exists"] & X_i \end{tikzcd}$$

of solid arrows can be filled in. Starting from u_K we consider the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "u_K"] \arrow[d, hook] & U \cap X_i \arrow[r, hook] & U \arrow[r, two heads] & \mathscr{U} \arrow[d] \\
\mathrm{Spec}(R) \arrow[rrr] \arrow[rrru, dashed] & & & \mathrm{Spec}(k)
\end{tikzcd}$$

and the dotted arrow exists since $\mathscr{U}$ satisfies the existence part of the valuative criterion for properness. Choosing a section[1] $s_R : \mathrm{Spec}(R) \to \mathbb{G}_m \times \mathrm{Spec}(R)$ and composing in the following Cartesian diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Spec}(R) \arrow[r] \arrow[d] & U \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
\mathrm{Spec}(R) \arrow[r] \arrow[u, bend left, "s_R"] \arrow[ru, "u'_R"'] & \mathscr{U}
\end{tikzcd}$$

gives rise to a morphism $u'_R : \mathrm{Spec}(R) \xrightarrow{s_R} \mathbb{G}_m \times \mathrm{Spec}(R) \to U$ with $u'_K = \eta_K.u_K = u_K$ for some $\eta_K \in \mathbb{G}_m(K)$. The following two morphisms

$$u_R : \mathrm{Spec}(R) \to X_i \hookrightarrow X \text{ and } u'_R : \mathrm{Spec}(R) \to U \hookrightarrow X$$

coincide on $\mathrm{Spec}(K)$, thus must be equal since X is separated. This shows that u_R factors through $U \cap X_i \subset X_i$, as desired. $\square$

Corollary 8.3.2. *Let X be a geometrically normal, separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ satisfies the existence part of valuative criterion for properness. Let $Z := X - U$ be the closed complement. If $Z \cap X_i \neq \emptyset$, then $X_i \subset Z$.*

Lemma 8.3.3 ([BBS83], Lemma 1.1.1 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ satisfies the existence part of valuative criterion for properness. If $E \subset X_i^- \cap X_j^+$ is a $\mathbb{G}_m$-invariant irreducible subset such that $U \cap E \neq \emptyset$, then $E \subset U$.*

Remark 8.3.4. A few words about Lemma 8.3.3.

1. Lemma 8.3.3 indicates that the irreducible components of $X_i^- \cap X_j^+$ are the building blocks for the $\mathbb{G}_m$-invariant open subsets $U \subset X$ with proper good moduli spaces. This is not so surprising by itself but we would like to point out the unexpected simplicity of Theorem C: the $\mathbb{G}_m$-invariant open subsets with proper good moduli spaces are built out of the subsets $X_i^- \cap X_j^+$ (and not their irreducible components).

[1] Indeed, all possible choices of such section correspond to elements of $\mathbb{G}_m(R)$.

2. There exist examples that $X_i^- \cap X_j^+$ has unfavourable properties, e.g., disconnected and singular, see [Som82, §2]. Such examples are reproduced in Appendix B.1.

Proof. If $U \cap X_i \neq \emptyset$ or $U \cap X_j \neq \emptyset$, then $X_i \subset U$ or $X_j \subset U$ by Lemma 8.3.1. This implies that $X_i^- \subset U$ or $X_j^+ \subset U$ since $U \subset X$ is open. In particular $E \subset X_i^- \cap X_j^+ \subset U$, we are done. Hereafter we assume that $U \cap X_i = \emptyset = U \cap X_j$.

We need to prove $U \cap E = E$. Since $\emptyset \neq U \cap E \subset E$ is open and E is irreducible, it suffices to show $U \cap E \subset E$ is closed, i.e., for every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "u_K"] \arrow[d, hook] & U \cap E \arrow[d, hook] \\
\mathrm{Spec}(R) \arrow[r, "u_R"'] \arrow[ur, dashed, "\exists"] & E
\end{tikzcd} \tag{8.3.1}$$

of solid arrows can be filled in.

1. Starting from u_K we consider the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "u_K"] \arrow[d, hook] & U \cap E \arrow[r, hook] & U \arrow[r] & \mathscr{U} \arrow[d] \\
\mathrm{Spec}(R) \arrow[rrr] \arrow[urrr, dashed] & & & \mathrm{Spec}(k)
\end{tikzcd}$$

   and the dotted arrow exists since $\mathscr{U}$ satisfies the existence part of the valuative criterion for properness. Choosing a section $s_R : \mathrm{Spec}(R) \to \mathbb{G}_m \times \mathrm{Spec}(R)$ and composing in the following Cartesian diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Spec}(R) \arrow[r] \arrow[d] & U \arrow[d, "\mathbb{G}_m\text{-tor}"] \\
\mathrm{Spec}(R) \arrow[r] \arrow[u, bend left, "s_R"] \arrow[ur, "u'_R"'] & \mathscr{U}
\end{tikzcd}$$

   gives rise to a morphism $u'_R : \mathrm{Spec}(R) \xrightarrow{s_R} \mathbb{G}_m \times \mathrm{Spec}(R) \to U$ with $u'_K = \eta_K.u_K$ for some $\eta_K \in \mathbb{G}_m(K) \subset \mathbb{P}^1(K)$. Let $\eta_R \in \mathbb{P}^1(R)$ be the unique element extending $\eta_K \in \mathbb{P}^1(K)$.

2. The complete orbit map of $u_R \in E \subset X_i^- \cap X_j^+$ has image in

$$\overline{\sigma(-, u_R)} : \mathbb{P}^1 \times \mathrm{Spec}(R) \to X_i \cup E \cup X_j.$$

Combing (1) and (2) yields a morphism

$$u''_R : \mathrm{Spec}(R) \xrightarrow{(\eta_R, \mathrm{id})} \mathbb{P}^1 \times \mathrm{Spec}(R) \xrightarrow{\overline{\sigma(-,u_R)}} X_i \cup E \cup X_j \hookrightarrow X.$$

By definition u''_R and u'_R coincide on $\mathrm{Spec}(K)$, then $u''_R = u'_R$ since X is separated. This implies that u''_R factors through $U \cap (X_i \cup E \cup X_j) = U \cap E \subset X$ (using that $U \cap X_i = \emptyset = U \cap X_j$). In particular

$$u''_\kappa = (\eta_R . u_R)_\kappa = \eta_\kappa . u_\kappa \in U \cap E.$$

There are two situations:

1. If $\eta_\kappa \in \mathbb{G}_m(\kappa)$, then $u_\kappa \in U \cap E$ since $U \cap E$ is $\mathbb{G}_m$-invariant. Coming back to (8.3.1), this means that u_R factors through $U \cap E$, as desired.

2. If $\eta_\kappa \in \mathbb{P}^1(\kappa) - \mathbb{G}_m(\kappa)$, then either $U \cap X_i \neq \emptyset$ or $U \cap X_j \neq \emptyset$, a contradiction.

□

Corollary 8.3.5 ([BBS83], Lemma 1.1.1 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open dense subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ satisfies the existence part of valuative criterion for properness. Then $X_1^- \cap X_r^+ \subset U$.*

Proof. The subset $X_1^- \cap X_r^+ \subset X$ is irreducible and dense by Corollary 7.1.2. Then $U \cap (X_1^- \cap X_r^+) \neq \emptyset$ since they are both dense in X and Lemma 8.3.3 applies. □

8.3.1 Geometric characterization of the existence part of valuative criterion

Theorem 2.5.1 indicates that the existence part of valuative criterion of a quotient stack by $\mathbb{G}_m$ can be checked on to its dense substacks. If X is proper, then the same dense substack $\mathscr{V} := [U \cap X_1^- \cap X_r^+/\mathbb{G}_m] \subset [U/\mathbb{G}_m] = \mathscr{U}$ as before will do this job.

Proposition 8.3.6 ([BBS83], Lemma 1.2 if $k = \mathbb{C}$)**.** *Let X be a proper irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset and $\mathscr{U} := [U/\mathbb{G}_m]$ be the quotient stack. The following are equivalent:*

1. *$\mathscr{U}$ satisfies the existence part of valuative criterion for properness.*

2. *The image of any U-smoothable maximal chain of orbits in X intersects U non-trivially.*

Proof. As in the proof of Proposition 8.2.3, we again consider the dense substack

$$\mathscr{V} := [U \cap X_1^- \cap X_r^+/\mathbb{G}_m] \subset [U/\mathbb{G}_m] = \mathscr{U}.$$

To check the existence part of valuative criterion for properness of $\mathscr{U}$, we apply Theorem 2.5.1 to $\mathscr{V} \subset \mathscr{U}$, yielding that $\mathscr{U}$ satisfies the existence part of valuative criterion for properness if and only if

(1′) For every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & \mathscr{V} \arrow[r, hook] & \mathscr{U} \arrow[d] \\ \mathrm{Spec}(R) \arrow[rr] \arrow[urr, dashed] & & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows, up to a finite extension of the fraction field K, can be filled in.

To conclude, we show that (1′) $\Leftrightarrow$ (2).

(1′) $\Rightarrow$ (2). Let $f : C \to X$ be a U-smoothable maximal chain of orbits in X, we need to show

$$U \cap f(C) \neq \emptyset.$$

By definition there exists a DVR R with fraction field K and residue field κ, a $\mathbb{G}_m$-equivariant diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa) \arrow[d, hook] & C \arrow[l] \arrow[r, "f"] \arrow[d, hook] \arrow[dl, phantom, "\urcorner"] & X \\ \mathrm{Spec}(R) & C_R \arrow[l, "\text{flat}"'] \arrow[ur, "f_R"'] & \end{tikzcd}$$

such that $C_K \cong \mathbb{P}^1_K$ and $f_K = \overline{\sigma(-, x_K)} : \mathbb{P}^1_K \to X$ is a complete orbit map for some point $x_K \in U \cap X_1^- \cap X_r^+ =: V$. Starting from x_K we consider the following diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "x_K"] \arrow[d, hook] & V \arrow[r, two heads] & \mathscr{V} \arrow[r, hook] & \mathscr{U} \arrow[d] \\ \mathrm{Spec}(R) \arrow[rrr] \arrow[urrr, dashed, "x'_R"'] & & & \mathrm{Spec}(k) \end{tikzcd}$$

and the dotted arrow exists by (1′). Choosing a section $s_R : \mathrm{Spec}(R) \to \mathbb{G}_m \times \mathrm{Spec}(R)$ and composing in the following Cartesian diagram

$$\begin{tikzcd} \mathbb{G}_m \times \mathrm{Spec}(R) \arrow[r] \arrow[d] & U \arrow[d, "\mathbb{G}_m\text{-tor}"] \\ \mathrm{Spec}(R) \arrow[r, "x'_R"'] \arrow[u, bend left, "s_R"] \arrow[ur, "x''_R"'] & \mathscr{U} \end{tikzcd}$$

gives rise to a morphism $x''_R : \mathrm{Spec}(R) \xrightarrow{s_R} \mathbb{G}_m \times \mathrm{Spec}(R) \to U$ with $x''_K = \eta_K.x_K$ for some $\eta_K \in \mathbb{G}_m(K)$. Consider the following commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Spec}(R) & \mathrm{Spec}(\kappa) \arrow[l, hook'] \\
C_R \arrow[r, "f_R"] \arrow[urr, bend left] & X & \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \arrow[l, "\exists u"'] \arrow[u] \arrow[d, "\text{blow-up}"] \arrow[ur, phantom, "\lrcorner"] & \Phi \arrow[l, hook'] \arrow[u] \\
 & U \arrow[u, hook, "\text{open}"] & \mathbb{P}^1 \times \mathrm{Spec}(R) \arrow[ul, dashed] & \\
C_K \cong \mathbb{P}^1 \times \mathrm{Spec}(K) \arrow[uu, hook, "\text{open}"] & \mathbb{G}_m \times \mathrm{Spec}(K) \arrow[l, hook', "\text{open}"] \arrow[r, hook, "\text{open}"'] \arrow[u, "\sigma(-{,}x_K)"] & \mathbb{G}_m \times \mathrm{Spec}(R) \arrow[u, hook, "\text{open}"'] \arrow[ul, "\sigma(-{,}x''_R)"']
\end{tikzcd}$$

where we regard $\sigma(-, x''_R)$ as a rational map $\sigma(-, x''_R) : \mathbb{P}^1 \times \mathrm{Spec}(R) \dashrightarrow X$. The non-defined locus of $\sigma(-, x''_R)$ is supported at $(0, \mathrm{Spec}(\kappa))$ and $(\infty, \mathrm{Spec}(\kappa))$, it extends to a morphism $u : \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \to X$ after blowing-up certain ideal $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^1 \times \mathrm{Spec}(R)}$ supported at these two points since X is proper. Then

$$\mathrm{Im}(u) = \overline{\mathrm{Im}(\mathbb{G}_m \times \mathrm{Spec}(K))} = \mathrm{Im}(f_R) \text{ in } X$$

because both are closed and contain the image of $\mathbb{G}_m \times \mathrm{Spec}(K)$ as a dense subset. In particular this implies that $u(\Phi) = f(C)$ and hence

$$\emptyset \neq U \cap u(\Phi) = U \cap f(C).$$

$(2) \Rightarrow (1')$. For every DVR R with fraction field K, any commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d, hook] & \mathscr{V} \arrow[r, hook] & \mathscr{U} \arrow[d] \\
\mathrm{Spec}(R) \arrow[rr] \arrow[urr, dashed] & & \mathrm{Spec}(k)
\end{tikzcd}$$

of solid arrows, we need to show there exists a dotted arrow filling in.

Choosing a section $s_K : \mathrm{Spec}(K) \to \mathbb{G}_m \times \mathrm{Spec}(K)$ and composing in the following Cartesian diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Spec}(K) \arrow[r] \arrow[d, two heads] & V \arrow[d, two heads] \\
\mathrm{Spec}(K) \arrow[r] \arrow[u, bend left, "s_K"] \arrow[ur, "x_K"] & \mathscr{V}
\end{tikzcd}$$

gives rise to a morphism $x_K : \mathrm{Spec}(K) \xrightarrow{s_K} \mathbb{G}_m \times \mathrm{Spec}(K) \to V$. Starting from x_K we consider the following commutative digram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "x_K"] \arrow[d, hook] & V \arrow[r, hook] & X \arrow[d] \\
\mathrm{Spec}(R) \arrow[rr] \arrow[urr, dashed, "x_R"'] & & \mathrm{Spec}(k)
\end{tikzcd}$$

and the dotted arrow exists since X is proper. Consider the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(R) & \mathrm{Spec}(\kappa) \arrow[l, hook'] \\
X & \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \arrow[l, "\exists u"'] \arrow[u] \arrow[d, "\text{blow-up}"] \arrow[ur, phantom, "\lrcorner"] & \Phi \arrow[l, hook'] \arrow[u] \\
V \arrow[u, hook, "\text{open}"] & \mathbb{P}^1 \times \mathrm{Spec}(R) \arrow[ul, dashed] & \\
\mathbb{G}_m \times \mathrm{Spec}(K) \arrow[u, "\sigma(-,x_K)"] \arrow[r, hook, "\text{open}"'] & \mathbb{G}_m \times \mathrm{Spec}(R) \arrow[u, hook, "\text{open}"'] \arrow[uul, "\sigma(-,x_R)"] &
\end{tikzcd}$$

where we regard $\sigma(-, x_R)$ as a rational map $\sigma(-, x_R) : \mathbb{P}^1 \times \mathrm{Spec}(R) \dashrightarrow X$. The non-defined locus of $\sigma(-, x_R)$ is supported at $(0, \mathrm{Spec}(\kappa))$ and $(\infty, \mathrm{Spec}(\kappa))$, it extends to a morphism $u : \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \to X$ after blowing-up certain ideal $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^1 \times \mathrm{Spec}(R)}$ supported at these two points since X is proper. As in the proof of Proposition 7.2.4, the restriction $u|_\Phi : \Phi \to X$ can be refined to be a chain of orbits in X, which is maximal (since $x_K \in V$) and U-smoothable (by construction). Then $U \cap u(\Phi) \neq \emptyset$ by (2).

Choosing a section $s : \mathrm{Spec}(R) \to \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R))$ such that the composition

$$\zeta : \mathrm{Spec}(R) \xrightarrow{s} \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^1 \times \mathrm{Spec}(R)) \xrightarrow{u} X$$

maps the unique closed point $\mathrm{Spec}(\kappa)$ to $U \cap u(\Phi)$, then ζ factors through U since $U \subset X$ is open and the composition $\mathrm{Spec}(R) \xrightarrow{\zeta} U \to \mathscr{U}$ extends $\mathrm{Spec}(K) \to \mathscr{V}$.

□

The conclusion so far is summarized as follows.

Theorem 8.3.7. *Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset and $\mathscr{U} := [U/\mathbb{G}_m]$ be the quotient stack. Then $\mathscr{U}$ is S-complete if and only if the image of any smoothable maximal chain of orbits in X intersects U with one of the following forms:*

1. *$\emptyset$;*

2. $\mathbb{G}_m.x$ *for some* $x \in X - X^{\mathbb{G}_m}$*;*

3. $\mathbb{G}_m.x \cup \{x^-\}$ *(resp.* $\mathbb{G}_m.x \cup \{x^+\}$*) for some* $x \in X_1^- - X_1$ *(resp.* $x \in X_r^+ - X_r$*);*

4. $\mathbb{G}_m.x_1 \cup \{x_1^+ = x_2^-\} \cup \mathbb{G}_m.x_2$ *for some* $x_1, x_2 \in X - X^{\mathbb{G}_m}$.

Moreover, if $\mathscr{U}$ *is S-complete, then it satisfies the existence part of valuative criterion for properness if and only if the case (1) doesn't appear.*

Proof. The only extra input is Corollary 8.3.5, asserting that in this case every smoothable maximal chain of orbits in X is actually U-smoothable. □

8.4 A topological characterization of properness of good moduli space

Our goal in this section is to prove the following topological characterization of properness of a good moduli space whenever it exists.

Proposition 8.4.1 ([BBS83], Theorem 1.4 if $k = \mathbb{C}$)**.** *Let* X *be a geometrically normal, proper and geometrically irreducible scheme over a field* k *with a* $\mathbb{G}_m$*-action. Let* $U \subset X - X^{\mathbb{G}_m}$ *be a* $\mathbb{G}_m$*-invariant open dense subset such that the quotient stack* $\mathscr{U} := [U/\mathbb{G}_m]$ *admits a separated good moduli space* $U/\mathbb{G}_m$*. Then the following are equivalent:*

1. *The separated good moduli space* $U/\mathbb{G}_m$ *is proper.*

2. *The closed complement* $Z := X - U$ *has two connected components.*

Remark 8.4.2. In either case of Proposition 8.4.1, the source X_1 and the sink X_r of X are contained in the different connected components of Z by Corollary 8.2.4.

Let $k_{\text{prime}} \subset k$ be its prime field. Since everything is of finite type over k, there exists an intermediate field $k_{\text{prime}} \subset k' \subset k$ such that the field extension k'/k_{prime} is finitely generated and everything is already defined over k'. Thus we may assume, without loss of generality, that k is finitely generated over its prime field, i.e., $k/\mathbb{Q}$ (resp., $k/\mathbb{F}_p$) is finitely generated if $\text{char}(k) = 0$ (resp., $\text{char}(k) = p > 0$).

First we deal with the relatively easy case that X is smooth and then explain how to modify the arguments to work in general.

8.4.1 Proof of Proposition 8.4.1: smooth case

Our proof in the smooth case is motivated by [BBS85, Theorem 1.3].

- Case $\mathrm{char}(k) > 0$ **or** $k = \mathbb{C}$**.**

Hereafter, all cohomology groups are either

1. étale cohomology groups with coefficient $\mathbb{Q}_\ell$ if $\mathrm{char}(k) > 0$, where ℓ is a prime invertible in k, or

2. singular cohomology groups with coefficient $\mathbb{Q}$ if $k = \mathbb{C}$.

Denote by $\mathbb{K}$ the coefficients in both cases. Since Z has two connected components if and only if $H^0(Z, \mathbb{K}) = \mathbb{K}^2$, it is equivalent to show

$$U/\mathbb{G}_m \text{ is proper if and only if } H^0(Z, \mathbb{K}) = \mathbb{K}^2.$$

Let $\jmath : U \hookrightarrow X$ (resp., $\imath : Z \hookrightarrow X$) be the open (resp., closed) embedding. Applying the cohomology functor $H_c^*(X, -)$ to the exact triangle $\mathbf{R}\jmath_!\mathbb{K} \to \mathbb{K} \to \imath_*\mathbb{K} \xrightarrow{+1}$ yields a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, phantom, "U \text{ irred \& non-proper}"] & H_c^0(U, \mathbb{K}) \arrow[r] \arrow[d, equal] & H^0(X, \mathbb{K}) \arrow[r] \arrow[d, equal, "X \text{ irred}"] & H^0(Z, \mathbb{K}) \arrow[r] \arrow[d, equal] & H_c^1(U, \mathbb{K}) \arrow[r, "\jmath_*"] \arrow[d, equal, "(*)"] & H^1(X, \mathbb{K}) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{K} \arrow[r] & H^0(Z, \mathbb{K}) \arrow[r] & H_c^0(U/\mathbb{G}_m, \mathbb{K}) \arrow[r, "0", "(**)"'] & H^1(X, \mathbb{K}) \arrow[r] & \cdots
\end{tikzcd}$$

and we claim that it equals to the second row, which will conclude the proof.

The only non-trivial parts are $(*)$ and $(**)$. Roughly speaking, $(*)$ is a Leray spectral sequence computation (see Claim 8.4.3) and $(**)$ is done via comparing the weights on both sides (see Claim 8.4.4).

Claim 8.4.3. The composition $\lambda : U \xrightarrow{f} \mathscr{U} \xrightarrow{g} U/\mathbb{G}_m$, where $f : U \to \mathscr{U}$ is the $\mathbb{G}_m$-torsor and $g : \mathscr{U} \to U/\mathbb{G}_m$ is the good moduli space, induces an isomorphism

$$H_c^0(U/\mathbb{G}_m, \mathbb{K}) \cong H_c^1(U, \mathbb{K}).$$

Proof. The Leray spectral sequence for λ reads

$$E_2^{p,q} := H_c^p(U/\mathbb{G}_m, \mathbf{R}^q\lambda_!\mathbb{K}) \Rightarrow H_c^{p+q}(U, \mathbb{K}).$$

Since $\mathbf{R}\lambda_! = \mathbf{R}(g \circ f)_! = \mathbf{R}g_! \circ \mathbf{R}f_!$, we will compute $\mathbf{R}\lambda_!$ step-by-step.

1. Firstly we have an quasi-isomorphism

$$\mathbf{R}f_!\mathbb{K} \simeq \mathbb{K}[-1] \oplus \mathbb{K}[-2]. \tag{8.4.1}$$

Consider the following diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[d, "f"'] & {[U \times \mathbb{A}^1/\mathbb{G}_m]} \arrow[dl, "\bar{f}"'] \\ \mathscr{U} \arrow[ur, bend right, "s"'] & \end{tikzcd}$$

where $\bar{f} : [U \times \mathbb{A}^1/\mathbb{G}_m] \to \mathscr{U}$ is the line bundle associated to the $\mathbb{G}_m$-torsor $f : U \to \mathscr{U}$ and $s : \mathscr{U} \to [U \times \mathbb{A}^1/\mathbb{G}_m]$ is its zero section. Applying $\mathbf{R}\bar{f}_!$ to the exact triangle $\mathbf{R}j_!\mathbb{K} \to \mathbb{K} \to s_*\mathbb{K} \xrightarrow{+1}$ yields

$$\mathbf{R}f_!\mathbb{K} \to \mathbf{R}\bar{f}_!\mathbb{K} \to \mathbb{K} \xrightarrow{+1}$$

To conclude we claim that there is a quasi-isomorphism $\mathbf{R}\bar{f}_!\mathbb{K} \simeq \mathbb{K}[-2]$ and therefore $\mathbf{R}f_!\mathbb{K}$ is quasi-isomorphic to the (-1)-shifted mapping cone of $\mathbf{R}\bar{f}_!\mathbb{K} \to \mathbb{K}$. Indeed, since the fibres of $\bar{f}$ are $\mathbb{A}^1$, which are $\mathbb{K}$-acyclic, we have an quasi-isomorphism $\mathbf{R}\bar{f}_*\mathbb{K} \simeq \mathbb{K}$ and Poincaré duality then tells us that $\mathbf{R}\bar{f}_! \cong \mathbf{R}\bar{f}_*[-2]$ since $\bar{f}$ is smooth of relative dimension 1.

2. Secondly we have an quasi-isomorphism

$$\mathbf{R}g_!\mathbb{K} \simeq \mathbb{K}. \tag{8.4.2}$$

Indeed, since U has no fixed points, for any $\bar{x} \in U/\mathbb{G}_m$, the fiber $g^{-1}(\bar{x}) = [\mathbb{G}_m.x/\mathbb{G}_m] = \mathrm{B}I_x$ where $x \in \lambda^{-1}(\bar{x})$ is any point in the fiber of $\bar{x}$ and $I_x \subset \mathbb{G}_m$ is its stabilizer group, which is finite. Therefore

$$(\mathbf{R}^i g_!\mathbb{K})_{\bar{x}} = H^i_c(\mathrm{B}I_x, \mathbb{K}) = H^{-i}(\mathrm{B}I_x, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

This shows $\mathbf{R}^i g_!\mathbb{K} = 0$ for $i \neq 0$. As for $i = 0$, there exists a morphism $g_!\mathbb{K} \hookrightarrow g_*\mathbb{K} \cong \mathbb{K}$ inducing isomorphisms on stalks, witnessing $g_!\mathbb{K} \cong \mathbb{K}$. Thus the natural morphism $\mathbf{R}g_!\mathbb{K} \to g_*\mathbb{K} \cong \mathbb{K}$ is a quasi-isomorphism.

Altogether, this shows

$$\mathbf{R}\lambda_!\mathbb{K} = \mathbf{R}g_! \circ \mathbf{R}f_!\mathbb{K} \simeq \mathbb{K}[-1] \oplus \mathbb{K}[-2]$$

and hence $H^1_c(U, \mathbb{K}) = H^0_c(U/\mathbb{G}_m, \mathbf{R}^1\lambda_!\mathbb{K}) = H^0_c(U/\mathbb{G}_m, \mathbb{K})$. We are done. $\square$

Claim 8.4.4. $\jmath_* : H^1_c(U, \mathbb{K}) \to H^1(X, \mathbb{K})$ is a zero map.

Proof. The morphism $\jmath_* : H^1_c(U, \mathbb{K}) \to H^1(X, \mathbb{K})$ preserves the weights on both sides and this claim is proved by comparing them.

1. $H^1_c(U, \mathbb{K}) = H^0_c(U/\mathbb{G}_m, \mathbb{K})$ is pure of weight 0.

   This follows from Deligne's Weil II, see [Del80, Corollaire 3.3.3] (resp., Deligne's Hodge III, see [Del74, Théorème 8.2.4]) if $\mathrm{char}(k) > 0$ (resp., $k = \mathbb{C}$).

2. $H^1(X, \mathbb{K})$ is pure of weight 1. This seems to be the only place where we use the smoothness of X.

   This follows from Deligne's Weil II, see [Del80, Corollaire 3.3.6] (resp., Deligne's Hodge II, see [Del71, Corollaire 3.2.15]) if $\mathrm{char}(k) > 0$ (resp., $k = \mathbb{C}$).

Then the morphism $\jmath_*$ must vanish. $\square$

- Case $\mathrm{char}(k) = 0$**.**

Since we assume that $k/\mathbb{Q}$ is finitely generated, there exists an embedding $k \hookrightarrow \mathbb{C}$ and we fix such one. The following diagram finishes the proof in this general case, where the bar denotes the base-change to $\mathbb{C}$.

$$\begin{tikzcd}
\overline{U}/\mathbb{G}_m \text{ is proper} \arrow[r, equal, "\text{uniqueness of}", "\text{gms}"'] \arrow[d, Leftrightarrow, "k=\mathbb{C}"'] & \overline{U/\mathbb{G}_m} \text{ is proper} \arrow[r, Leftrightarrow] & U/\mathbb{G}_m \text{ is proper} \arrow[d, dashed, leftrightarrow] \\
\overline{Z} \text{ has two connected components} \arrow[rr, Leftrightarrow, "\text{Corollary 8.2.4}"'] & & Z \text{ has two connected components}
\end{tikzcd}$$

Here is the argument for the bottom equivalence. If Z has two connected components, then $\overline{Z}$ has at least two connected components since base-change can only increase the number of connected components. But $\overline{Z}$ cannot have more by Corollary 8.2.4. Conversely, if $\overline{Z}$ has two connected components, then Z has at most two connected components. By Corollary 8.2.4 the two connected components of $\overline{Z}$ are indexed by $\overline{X}_1$ and $\overline{X}_r$ respectively, which cannot be identified by descent (see Proposition 7.3.5), i.e., both of them are invariant under the action of the absolute Galois group $G_k := \mathrm{Gal}(\bar{k}/k)$. This shows that Z has two connected components.

8.4.2 Proof of Proposition 8.4.1: normal case

As mentioned in the proof of Claim 8.4.4, the only place where the smoothness of X comes into a play is the purity of its cohomology groups. To retain this kind of purity in the normal case a natural idea is to replace the constant sheaf $\mathbb{K}$ by the intersection complex IC with respect to the middle perversity p (and we will drop the superscript p) and coefficient $\mathbb{K}$. This replacement leads to two issues to be proved: (1) the resulting intersection cohomology groups coincide with the usual cohomology groups in degree 0 and (2) an analogy of Claim 8.4.3 holds.

The first issue is settled in the following lemma, which will be used frequently in the computation of hypercohomology spectral sequence. This is also the major extra input in normal case, as a price of replacing sheaves by complexes.

Lemma 8.4.5. *Let V be a normal, separated and irreducible scheme, of finite type and dimension n over k. Then we have*

$$\mathscr{H}^{-n}(\mathrm{IC}_V) = \mathbb{K}.$$

In particular, there exists an exact triangle $\mathbb{K}[n] \to \mathrm{IC}_V \to \tau_{\geq 1}\mathrm{IC}_X \xrightarrow{+1}$ *and*

$$\mathrm{IH}^0_c(V) = H^0_c(V, \mathbb{K}).$$

Proof. Let $j : V_{sm} \hookrightarrow V$ be the open embedding of the smooth locus, then the strong support condition reads

$$\mathscr{H}^{<-n}(\mathrm{IC}_V) = 0 \text{ and } \mathscr{H}^{-n}(\mathrm{IC}_V) = j_*\mathbb{K}.$$

Since V is normal, the complement of V_{sm} in V has codimension at least two, it follows that $j_*\mathbb{K} = \mathbb{K}$. In particular, the standard exact triangle

$$\tau_{\leq 0}(\mathrm{IC}_V[-n]) \to \mathrm{IC}_V[-n] \to \tau_{\geq 1}(\mathrm{IC}_V[-n]) \xrightarrow{+1}$$

now becomes $\mathbb{K} \to \mathrm{IC}_V[-n] \to (\tau_{\geq -n+1}\mathrm{IC}_V)[-n] \xrightarrow{+1}$, i.e.,

$$\mathbb{K}[n] \to \mathrm{IC}_V \to \tau_{\geq -n+1}\mathrm{IC}_V \xrightarrow{+1}$$

and

$$\mathrm{IH}^0_c(V) := \mathbb{H}^{-n}_c(V, \mathrm{IC}_V) = H^0_c(V, \mathscr{H}^{-n}(\mathrm{IC}_V)) = H^0_c(V, \mathbb{K})$$

where the first equality follows from the hypercohomology spectral sequence

$$E_2^{p,q} := H^p_c(V, \mathscr{H}^q(\mathrm{IC}_V)) \Rightarrow \mathbb{H}^{p+q}_c(V, \mathrm{IC}_V)$$

and the fact $H^p_c(V, -) \neq 0$ only for $p \geq 0$ and $\mathscr{H}^q(\mathrm{IC}_V) \neq 0$ only for $q \geq -n$. $\square$

As before, applying the hypercohomology functor $\mathbb{H}_c^{*-n}(X,-)$ to the exact triangle $\jmath_!\jmath^*\mathrm{IC}_X \to \mathrm{IC}_X \to \imath_*\imath^*\mathrm{IC}_X \xrightarrow{+1}$ yields a long exact sequence

$$\begin{tikzcd}
\mathrm{IH}_c^0(U) \arrow[r] \arrow[d, equal, "\text{Lemma 8.4.5}"'] & \mathrm{IH}^0(X) \arrow[r] \arrow[d, equal, "\text{Lemma 8.4.5}"'] & \mathbb{H}^{-n}(Z,\imath^*\mathrm{IC}_X) \arrow[r] \arrow[d, equal, "\text{Lemma 8.4.5}"', "\text{hypercohomology s.s}"] & \mathrm{IH}_c^1(U) \arrow[r, "\jmath_*"] \arrow[d, equal] & \mathrm{IH}^1(X) \arrow[d, equal] \\
H_c^0(U,\mathbb{K}) \arrow[r] \arrow[d, equal, "U \text{ irred \& non-proper}"'] & H^0(X,\mathbb{K}) \arrow[r] \arrow[d, equal, "X \text{ irred}"'] & H^0(Z,\mathbb{K}) \arrow[r] \arrow[d, equal] & \mathrm{IH}_c^1(U) \arrow[r] \arrow[d, equal, "(*)"'] & \mathrm{IH}^1(X) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{K} \arrow[r] & H^0(Z,\mathbb{K}) \arrow[r] & H_c^0(U/\mathbb{G}_m,\mathbb{K}) \arrow[r, "0", "(**)"'] & \mathrm{IH}^1(X)
\end{tikzcd}$$

and we claim that it equals to the third row. Note that now we get $(**)$ for free by the nature of our replacement: $\mathrm{IH}_c^1(X)$ is pure of weight 1, using Gabber's purity theorem, see [BBD82, Corollaire 5.3.2 and Théorème 5.4.1] (resp., [Max19, Corollary 11.3.5]) if $\mathrm{char}(k) > 0$ (resp., $k = \mathbb{C}$). Then it remains to prove $(*)$.

Claim 8.4.6. The composition $\lambda : U \xrightarrow{f} \mathscr{U} \xrightarrow{g} U/\mathbb{G}_m$, where $f : U \to \mathscr{U}$ is the $\mathbb{G}_m$-torsor and $g : \mathscr{U} \to U/\mathbb{G}_m$ is the good moduli space, induces an isomorphism

$$\mathrm{IH}_c^1(U) \cong H_c^0(U/\mathbb{G}_m,\mathbb{K}).$$

Proof. The proof essentially uses the results from smooth case. We keep the notation in smooth case. The Leray spectral sequence for λ reads

$$E_2^{p,q} := H_c^p(U/\mathbb{G}_m, \mathbf{R}^q\lambda_!\mathrm{IC}_U) \Rightarrow \mathbb{H}_c^{p+q}(U,\mathrm{IC}_U)$$

1. Firstly we have an quasi-isomorphism

$$\mathbf{R}f_!\mathrm{IC}_U \simeq \mathrm{IC}_{\mathscr{U}} \oplus \mathrm{IC}_{\mathscr{U}}[-1]$$

   Since f is a $\mathbb{G}_m$-torsor, in particular smooth, we could apply [LO09, Lemma 6.1] (or [GM83, 5.4.2 Theorem]) to obtain that

$$f^!\mathrm{IC}_{\mathscr{U}} \cong \mathrm{IC}_U[-1] \tag{8.4.3}$$

   By projection formula we have

$$\begin{tikzcd}
\mathbf{R}f_!(f^!\mathrm{IC}_{\mathscr{U}} \otimes \mathbb{K}) \arrow[r, equal, "\text{proj}", "\text{formula}"'] \arrow[d, "(8.4.3)"', "\cong"] & \mathrm{IC}_{\mathscr{U}} \otimes \mathbf{R}f_!\mathbb{K} \arrow[d, "\simeq"', "(8.4.1)"] \\
\mathbf{R}f_!\mathrm{IC}_U[-1] \arrow[r, "\simeq"] & \mathrm{IC}_{\mathscr{U}} \otimes (\mathbb{K} \oplus \mathbb{K}[-1])
\end{tikzcd}$$

2. Secondly we have

$$\mathbf{R}^i g_! \mathrm{IC}_{\mathscr{U}} = \begin{cases} 0 & \text{if } i < 1-n \\ \mathbb{K} & \text{if } i = 1-n \end{cases}$$

Indeed, for any $\bar{x} \in U/\mathbb{G}_m$

$$\begin{aligned}(\mathbf{R}^i g_! \mathrm{IC}_{\mathscr{U}})_{\bar{x}} &= \mathbb{H}^i_c(\mathrm{B}I_x, \iota_x^* \mathrm{IC}_{\mathscr{U}}) = H^0_c(\mathrm{B}I_x, \mathscr{H}^i(\iota_x^* \mathrm{IC}_{\mathscr{U}})) = H^0(\mathrm{B}I_x, \iota_x^* \mathscr{H}^i(\mathrm{IC}_{\mathscr{U}})) \\ &= \begin{cases} 0 & \text{if } i < 1-n \\ \mathbb{K} & \text{if } i = 1-n \end{cases}\end{aligned}$$

where $\iota_x : \mathrm{B}I_x = [\mathbb{G}_m.x/\mathbb{G}_m] \hookrightarrow [U/\mathbb{G}_m] = \mathscr{U}$ is the inclusion. This shows $\mathbf{R}^i g_! \mathrm{IC}_{\mathscr{U}} = 0$ for $i < 1-n$. As for $i = 1-n$, since there exists a morphism $\mathbb{K}[n-1] \to \mathrm{IC}_{\mathscr{U}}$ (see the proof of Lemma 8.4.5), using (8.4.2) we obtain a morphism $\mathbb{K} \to \mathbf{R}^{1-n} g_! \mathrm{IC}_{\mathscr{U}}$ inducing isomorphisms on stalks, witnessing $\mathbf{R}^{1-n} g_! \mathrm{IC}_{\mathscr{U}} \cong \mathbb{K}$.

Altogether, this shows

$$\mathbf{R}\lambda_! \mathrm{IC}_U = \mathbf{R}g_! \circ \mathbf{R}f_! \mathrm{IC}_U \simeq \mathbf{R}g_! \mathrm{IC}_{\mathscr{U}} \oplus \mathbf{R}g_! \mathrm{IC}_{\mathscr{U}}[-1]$$

and hence $\mathbf{R}^q \lambda_! \mathrm{IC}_U = \mathbf{R}^q g_! \mathrm{IC}_{\mathscr{U}} \oplus \mathbf{R}^{q-1} g_! \mathrm{IC}_{\mathscr{U}}$, i.e.,

$$\mathrm{IH}^1_c(U) = \mathbb{H}^{1-n}_c(U, \mathrm{IC}_U) = H^0_c(U/\mathbb{G}_m, \mathbf{R}^{1-n} g_! \mathrm{IC}_{\mathscr{U}}) = H^0_c(U/\mathbb{G}_m, \mathbb{K}).$$

□

Remark 8.4.7. A generalization of Proposition 8.4.1 (see Proposition 10.2.1) turns out to be the key point in the proof of the “Surjectivity” part of Theorem C.

Chapter 9

Proof of Theorem C: Injectivity

In this chapter, we establish an assignment

$$\left\{\begin{matrix}\text{Semi-sections of}\\ \{1,\ldots,r\}.\end{matrix}\right\} \to \left\{\begin{matrix}\mathbb{G}_m\text{-invariant open dense subsets } U \subset X\\ \text{such that the quotient stacks } [U/\mathbb{G}_m] \text{ admit}\\ \text{proper good moduli spaces.}\end{matrix}\right\}$$
$$(A^-, A^0, A^+) \mapsto U(A^*)$$

Once this is done, we already know that it is injective by Proposition 7.3.20. This gives the "Injectivity" part of Theorem C.

Theorem 9.0.1 ([BBS83], Theorem 1.3 plus [Gro84], Theorem 2.8 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Suppose the fixed point locus $X^{\mathbb{G}_m}$ has r connected components. If (A^-, A^0, A^+) is a semi-section of $\{1,\ldots,r\}$, then the semi-sectional subset $U := U(A^*) \subset X$ is a $\mathbb{G}_m$-invariant open dense subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ admits a proper good moduli space.*

Proof. Semi-sectional subsets are $\mathbb{G}_m$-invariant and open dense by Lemma 7.3.17. To prove that the quotient stack $\mathscr{U}$ admits a proper good moduli space we apply Theorem 0:

1. $\mathscr{U}$ is Θ-reductive.

   By Proposition 8.1.1 and duality, it suffices to prove

   $$\text{the inclusion } (U \cap X_i)^- \subset U \text{ is a closed immersion for each } i.$$

   If $U \cap X_i = \emptyset$, then there is nothing to prove. If $U \cap X_i \neq \emptyset$, then Lemma 7.3.16 implies that $X_i \subset U$ and $i \in A^0$. Thus it suffices to show the inclusion $X_i^- \subset U$ is a closed immersion for each $i \in A^0$, i.e., $\overline{X_i^-}^U = X_i^-$. Indeed, for any point $x \in \overline{X_i^-}^U = \overline{X_i^-} \cap U$, the proof of Lemma 7.3.18 gives that $x \in X_j^-$ for some $X_i < X_j$. However, we have $j \in A^- \cup A^0$ since $x \in U$, then $i \in A^0$ tells us that $i = j$.

2. $\mathscr{U}$ is S-complete.

   Let $f : C \to X$ be a smoothable maximal chain of orbits in X. To show $\mathscr{U}$ is S-complete, it is equivalent to show $U \cap f(C)$ has one of the forms in Proposition 8.2.3. This follows immediately if we could prove that

$$\mathbb{G}_m.x_1 \neq \mathbb{G}_m.x_2 \subset U \cap f(C) \Rightarrow x_1^+ = x_2^- \text{ or } x_1^- = x_2^+ \in U \cap f(C).$$

   Indeed, we may assume that x_1 appears earlier than x_2 along the direction of f, i.e., if $x_1^+ \in X_i$ and $x_2^- \in X_j$, then $i = j$ or $X_i < X_j$.

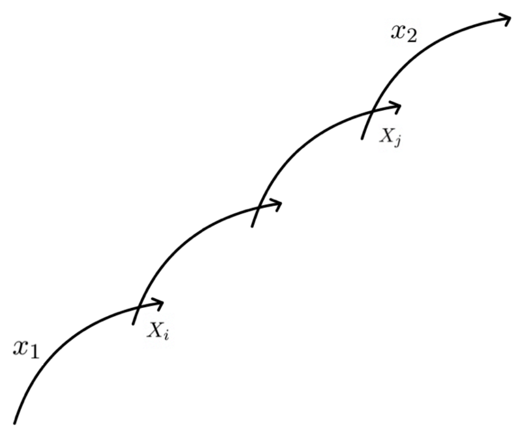

Figure 9.1: The relative configuration of x_1 and x_2

   Since f is a chain of orbits, it suffices to show $i = j$. Note that $x_1, x_2 \in U$, we have $i \in A^0 \cup A^+$ and $j \in A^- \cup A^0$. To avoid the contradiction with $X_i < X_j$, the only possibility is $i = j \in A^0$.

3. $\mathscr{U}$ satisfies the existence part of valuative criterion for properness.

   Let $f : C \to X$ be a smoothable maximal chain of orbits in X. To show $\mathscr{U}$ satisfies the existence part of valuative criterion for properness, it is equivalent to show $U \cap f(C) \neq \emptyset$ by Proposition 8.3.6. Assume otherwise, then

$$\{1, r\} \subset \{i : X_i \cap f(C) \neq \emptyset\} \subset A^- \text{ or } A^+$$

   Indeed, the first inclusion follows from the maximality of f; for the second inclusion, suppose otherwise, then we could find a point $x \in f(C)$ such that $x \in X_i^- \cap X_j^+$ for some $i \in A^- \cup A^0$ and $j \in A^0 \cup A^+$, then $x \in U$ by definition, i.e., $x \in U \cap f(C) \neq \emptyset$, a contradiction. But $\{1, r\} \subset A^-$ or A^+ implies that (using Proposition 7.3.4)

$$A^0 \cup A^+ = \emptyset \text{ or } A^- \cup A^0 = \emptyset$$

   again a contradiction.

□

Chapter 10

Proof of Theorem C: Surjectivity

In this chapter, we prove that the assignment defined in §9

$$\left\{\begin{matrix}\text{Semi-sections of}\\ \{1,\dots,r\}.\end{matrix}\right\} \to \left\{\begin{matrix}\mathbb{G}_m\text{-invariant open dense subsets } U \subset X\\ \text{such that the quotient stacks } [U/\mathbb{G}_m] \text{ admit}\\ \text{proper good moduli spaces.}\end{matrix}\right\}$$
$$(A^-, A^0, A^+) \mapsto U(A^*)$$

is surjective. This is the "Surjectivity" part of Theorem C.

Theorem 10.0.1 ([BBS83], Theorem 1.4 plus [Gro84], Theorem 2.11 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, proper and geometrically irreducible scheme over a field k with a $\mathbb{G}_m$-action. Suppose the fixed point locus $X^{\mathbb{G}_m}$ has r connected components. If $U \subset X$ is a $\mathbb{G}_m$-invariant open dense subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ admits a proper good moduli space, then $U = U(A^*)$ for some semi-section (A^-, A^0, A^+) of $\{1,\dots,r\}$.*

Remark 10.0.2. By Lemma 8.3.1 and Corollary 8.1.2, all such subsets $U \subset X$ fall into two cases:

A. the subset U intersects the source X_1 or the sink X_r of X.

B. the subset U doesn't intersect the source X_1 and the sink X_r of X.

and therefore the proof of Theorem 10.0.1 is divided into two cases. It will be also clear and surprising (see Proposition 10.2.1) that the two cases are identified topologically: whether the closed complement $Z := X - U$ is connected or not.

10.1 Proof of Theorem 10.0.1: Case A

Proposition 10.1.1 ([Gro84], Proposition 2.4 if $k = \mathbb{C}$)**.** *Let X be a geometrically normal, proper and irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open dense subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ admits a proper good moduli space. If U intersects the source X_1 or the sink X_r of X, then $U = X_1^-$ or $U = X_r^+$.*

Proof. Since $\mathscr{U}$ satisfies the existence part of valuative criterion for properness, $U \cap X_1 \neq \emptyset$ (resp., $U \cap X_r \neq \emptyset$) implies that $X_1 \subset U$ (resp., $X_r \subset U$) by Lemma 8.3.1. Then $\mathscr{U}$ being Θ-reductive implies that $U = X_1^-$ (resp., $U = X_r^+$) by Corollary 8.1.2, which is semi-sectional by Example 7.3.15. □

10.2 Proof of Theorem 10.0.1: Case B

The starting point in Case B is to read off the semi-section (A^-, A^0, A^+) from U. It is a little bit surprising that this can be achieved via the topological characterization in Proposition 8.4.1.

Proposition 10.2.1 ([BBS83], Theorem 1.4 if $k = \mathbb{C}$ and $U \subset X - X^{\mathbb{G}_m}$)**.** *Let X be a geometrically normal, proper and geometrically irreducible scheme over a field k with a $\mathbb{G}_m$-action. Suppose the fixed point locus $X^{\mathbb{G}_m}$ has r connected components. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open dense subset such that*

i. the subset U doesn't intersect the source X_1 and the sink X_r of X.

ii. the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ admits a separated good moduli space $U/\mathbb{G}_m$.

Let $Z := X - U$ be the closed complement. Then the following are equivalent:

1. *the separated good moduli space $U/\mathbb{G}_m$ is proper.*
2. *the closed complement Z is disconnected.*
3. *the closed complement Z has two connected components.*
4. *the subset $U = U(A^*)$ for some semi-section (A^-, A^0, A^+) of $\{1, \dots, r\}$.*

Remark 10.2.2. Condition (i) in Proposition 10.2.1 is necessary. Indeed, if $U \cap X_1 \neq \emptyset$ (resp., $U \cap X_r \neq \emptyset$), then $U = X_1^-$ (resp., $U = X_r^+$) by Proposition 10.1.1. In each case the closed complement Z is connected by Proposition 7.2.4 since every point in Z can be connected (in Z) to X_r (resp., X_1).

Proof. Our proof here is similar to [Gro84, Theorem 2.11] if $k = \mathbb{C}$. The proof road is

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] & (2) \arrow[d, Leftrightarrow, "\text{Corollary 8.2.4}"] \\ (4) \arrow[u, Rightarrow, "\text{Theorem 9.0.1}"] & (3) \arrow[l, Rightarrow] \end{tikzcd}$$

(1) $\Rightarrow$ (2). Suppose $U/\mathbb{G}_m$ is proper. Let $\Delta := \{i : X_i \subset U\}$ (by assumption $1, r \notin \Delta$) and define

$$U^\circ := U - \coprod_{i \in \Delta} X_i^- \subset U.$$

Then $U^\circ \subset X - X^{\mathbb{G}_m}$ and it is $\mathbb{G}_m$-invariant. Since $\mathscr{U}$ is Θ-reductive, we see

$$\coprod_{i\in\Delta} X_i^- = \coprod_{i\in\Delta} (U \cap X_i)^- \subset U \text{ is closed by Proposition 8.1.1}$$

and hence $U^\circ \subset U$ is open (and further open in X). We claim that

Claim 10.2.3 ([Gro84], Lemma 2.9 if $k = \mathbb{C}$)**.** The quotient stack $\mathscr{U}^\circ := [U^\circ/\mathbb{G}_m]$ admits a proper good moduli space.

Proof. Since $U^\circ \subset X - X^{\mathbb{G}_m}$, the quotient stack $\mathscr{U}^\circ$ admits a good moduli space by Proposition 8.1.3. To show this good moduli space is proper, we use Theorem 8.3.7. For any smoothable maximal chain of orbits $f : C \to X$ in X, a priori we have

$$f(C) \cap U^\circ \subset f(C) \cap U.$$

Since $\mathscr{U}$ admits a proper good moduli space, there are only two configurations for the intersection $f(C) \cap U$ by Theorem 8.3.7 (since U doesn't intersect X_1 and X_r):

(b) If $f(C) \cap U = \mathbb{G}_m.x$ for some $x \in X - X^{\mathbb{G}_m}$, then $x \notin X_i$ for any $i \in \Delta$ and hence $f(C) \cap U^\circ = \mathbb{G}_m.x$.

(d) If $f(C) \cap U = \mathbb{G}_m.x_1 \cup \{x_1^+ = x_2^-\} \cup \mathbb{G}_m.x_2$ for some $x_1, x_2 \in X - X^{\mathbb{G}_m}$, then $x_1 \notin X_i$ for any $i \in \Delta$ and $x_2 \in X_j$ for some $j \in \Delta$, hence $f(C) \cap U^\circ = \mathbb{G}_m.x_1$.

Then the good moduli space of $\mathscr{U}^\circ$ is proper by Theorem 8.3.7. □

Applying Proposition 8.4.1 to U° yields that the closed complement $X - U^\circ$ has two connected components, one containing X_1 and the other containing X_r. Since

$$Z = X - U = (X - U^\circ) - \coprod_{i\in\Delta} X_i^-.$$

and $1, r \notin \Delta$, the closed complement Z is still disconnected since X_1 and X_r are again in the different connected components of Z.

(3) $\Rightarrow$ (4). Suppose the closed complement $Z := X - U$ has two connected components. By Lemma 8.2.4, let Ω_1 and Ω_2 be the connected component of Z containing X_1 and X_r respectively, then we can read off (A^-, A^0, A^+) from U by setting

$$\begin{aligned} A^- &:= \{i : X_i \subset \Omega_1\} = \{i : X_i \cap \Omega_1 \neq \emptyset\}, \\ A^0 &:= \{i : X_i \subset U\} = \{i : X_i \cap U \neq \emptyset\}, \\ A^+ &:= \{i : X_i \subset \Omega_2\} = \{i : X_i \cap \Omega_2 \neq \emptyset\}. \end{aligned}$$

where the second equalities follow from Corollary 8.3.2 and Lemma 8.3.1.

Claim 10.2.4. (A^-, A^0, A^+) is a semi-section of $\{1, \ldots, r\}$.

Proof. Clearly (A^-, A^0, A^+) forms a division of $\{1, \ldots, r\}$ and $1 \in A^-, r \in A^+$. It remains to check that

$$\text{If } i \in A^- \cup A^0 \text{ and } X_j < X_i, \text{ then } j \in A^-.$$

By induction it suffices to consider the case $X_j <_d X_i$, i.e., there exists a point $x \in X_j^- \cap X_i^+$. Pick a smoothable maximal chain of orbits $f : C \to X$ in X passing through x (exists by Proposition 7.2.4). Notice that $f(C)$ cannot intersect U earlier than X_i (using the configurations in Theorem 8.3.7), since otherwise X_i would be in the same connected component as X_r, i.e., $i \in A^+$, a contradiction. This implies that $X_j \cap Z \neq \emptyset$ and X_j is in the same connected component as X_1 (using the configurations in Theorem 8.3.7), i.e., $j \in A^-$. $\square$

Claim 10.2.5. The subset

$$U = \coprod_{\substack{i \in A^- \cup A^0 \\ j \in A^0 \cup A^+}} (X_i^- \cap X_j^+).$$

This concludes the proof (3) $\Rightarrow$ (4).

Proof. This is done by two-sided inclusions.

$\subset$. For any point $x \in U$, say $x \in X_i^- \cap X_j^+$, i.e., $X_i <_d X_j$, we need to show

$$i \in A^- \cup A^0 \text{ and } j \in A^0 \cup A^+.$$

Indeed, pick a smoothable maximal chain of orbits $f : C \to X$ in X passing through x (exists by Proposition 7.2.4). If $j \in A^-$ (resp., $i \in A^+$), then by Theorem 8.3.7 we have

$$\mathbb{G}_m.x \cup x^+ \subset f(C) \cap U \text{ (resp., } x^- \cup \mathbb{G}_m.x \subset f(C) \cap U)$$

i.e., $X_j \cap U \neq \emptyset$ (resp., $X_i \cap U \neq \emptyset$), this shows that $j \in A^0$ (resp., $i \in A^0$) by Lemma 8.3.1, a contradiction.

$\supset$. For any point $x \in X_i^- \cap X_j^+$ such that $i \in A^- \cup A^0$ and $j \in A^0 \cup A^+$, we need to show $x \in U$. Indeed, if $x \in \Omega_1$ (resp., $x \in \Omega_2$), then its complete orbit map

$$\overline{\sigma(-, x)} : \mathbb{P}^1 \to X$$

has image in Ω_1 (resp., Ω_2) since $\Omega_1 \subset X$ (resp., $\Omega_2 \subset X$) is closed, then

$$\overline{\sigma(\infty, x)} = x^+ \in X_j \cap \Omega_1 \text{ (resp., } \overline{\sigma(0, x)} = x^- \in X_i \cap \Omega_2),$$

i.e., $j \in A^-$ (resp., $i \in A^+$). This proves that $x \in X - \Omega_1 - \Omega_2 = X - Z = U$.

$\square$

$\square$

Remark 10.2.6. To prove Proposition 10.2.1 (i.e., Theorem 10.0.1 in Case B), it suffices to prove that Z is disconnected. Surprisingly, the proof of this fact takes advantage of certain global arguments (e.g., Proposition 8.4.1), while we still expect some local arguments could apply. We have tried many ways to attack this but it seems that the cohomological arguments given in Proposition 8.4.1 is the only working way for the moment.

Chapter 11

Application

The following proposition is presented only to compare with our result on the moduli stack of rank 2 vector bundles over a curve, where only semi-stable objects can admit open neighbourhoods with separated good moduli spaces (see Lemma 4.4.3).

Proposition 11.0.1 ([BBS83], Theorem 1.6 if $k = \mathbb{C}$ and $U \subset X - X^{\mathbb{G}_m}$)**.** *Let X be a geometrically normal, proper and geometrically irreducible scheme over a field k with a $\mathbb{G}_m$-action. Let $\{U_\omega : \omega \in \Omega\}$ be the set of $\mathbb{G}_m$-invariant open subsets of X such that the quotient stack $\mathscr{U}_\omega := [U_\omega/\mathbb{G}_m]$ admits a proper good moduli space. Then*

$$\bigcup_{\omega\in\Omega} U_\omega = \bigcup_{X_i \not> X_j \text{ or } X_j \not> X_i} X_i^- \cap X_j^+.$$

In particular, if k is perfect and X is projective, then

$$\bigcup_{\omega\in\Omega} U_\omega = X,$$

i.e., X is covered by such $\mathbb{G}_m$-invariant open subsets.

Proof. Each U_ω is semi-sectional by Theorem C. By Remark 7.3.12 (2), it follows that

$$\bigcup_{\omega\in\Omega} U_\omega \subset \bigcup_{X_i \not> X_j \text{ or } X_j \not> X_i} X_i^- \cap X_j^+.$$

To see the inverse inclusion, it is enough to show that for any $X_i^- \cap X_j^+ \neq \emptyset$ with $X_i \not> X_j$ or $X_j \not> X_i$, i.e., $X_i <_d X_j$ and $X_i \not> X_j$, there exists a semi-sectional subset $U \subset X$ containing $X_i^- \cap X_j^+$. There are two cases:

1. If $i \neq j$. Define

$$A_0^- := \{\ell : X_\ell < X_i\} \text{ and } A_0^+ := \{\ell : X_\ell > X_j\},$$

and they are non-empty since $i \in A_0^-$ and $j \in A_0^+$. For any $p \in A_0^-$ and $q \in A_0^+$, it follows that

$$X_q > X_j >_d X_i > X_p \text{ and } X_p \not> X_q \text{ (otherwise } X_i > X_j).$$

Let (A^-, A^+) be a maximal pair of subsets of $\{1, \ldots, r\}$ satisfying:

- $A_0^- \subset A^-$ and $A_0^+ \subset A^+$.
- $X_p \not> X_q$ for any $p \in A^-$ and $q \in A^+$.

then we claim that $(A^-, \emptyset, A^+)$ is a semi-section of $\{1, \ldots, r\}$. To prove this, it remains to show $A^- \cup A^+ = \{1, \ldots, r\}$. Indeed, for any $\ell \in \{1, \ldots, r\} - A^- - A^+$, the maximality of the pair (A^-, A^+) implies that there exist $p \in A^-$ and $q \in A^+$ such that $X_p > X_\ell > X_q$, a contradiction.

Let $U \subset X$ be the semi-sectional subset defined by $(A^-, \emptyset, A^+)$, then $X_i^- \cap X_j^+ \subset U$.

2. If $i = j$. Define

$$A_0^- := \{\ell : X_\ell < X_i\}, A_0^0 := \{i\} \text{ and } A_0^+ := \{\ell : X_\ell > X_i\},$$

and they are non-empty since $1 \in A_0^-$ and $r \in A_0^+$. For any $p \in A_0^- \cup A_0^0$ and $q \in A_0^0 \cup A_0^+$, it follows that

$$X_q > X_i > X_p \text{ and } X_p \not> X_q \text{ (otherwise } X_i > X_j).$$

Let (A^-, A^0, A^+) be a maximal triple of subsets of $\{1, \ldots, r\}$ satisfying:

- $A_0^- \subset A^-, A_0^0 \subset A^0$ and $A_0^+ \subset A^+$.
- $X_p \not> X_q$ for any $p \in A^- \cup A^0$ and $q \in A^0 \cup A^+$.

then we claim that (A^-, A^0, A^+) is a semi-section of $\{1, \ldots, r\}$ as before.

Let $U \subset X$ be the semi-sectional subset defined by (A^-, A^0, A^+), then $X_i^- \cap X_i^+ \subset U$.

In particular, if k is perfect and X is projective, the condition $X_i > X_j$ and $X_j > X_i$ imply that $i = j$ by Proposition 7.3.8, hence

$$\bigcup_{X_i \not> X_j \text{ or } X_j \not> X_i} X_i^- \cap X_j^+ = X$$

and we are done. $\square$

Appendix A

Deformations of affine chains

The results in the appendix will NOT be used anywhere in the thesis.

In this appendix, we prove a deformation result for affine chains of two orbits in X (see Proposition A.0.2). The starting point is the following étale-local structure around fixed points of X.

Lemma A.0.1 ([BR85], Lemma 8.3 and [Mil17], Lemma 13.36)**.** *Let X be a geometrically normal and separated scheme, of finite type and dimension c over a field k with a $\mathbb{G}_m$-action. If $x \in X^{\mathbb{G}_m}$ is a closed point of X, then there exist*

1. *a $\mathbb{G}_m$-invariant affine open neighbourhood U of x.*

2. *a $\mathbb{G}_m$-equivariant étale morphism $\lambda : U \to T_x(X)$ such that*

$$\lambda(x) = 0 \text{ and } \mathrm{d}\lambda_x = \mathrm{id} : T_x(X) \to T_x(X).$$

Proof. Since X is geometrically normal, there exists a $\mathbb{G}_m$-invariant affine open neighbourhood of x by Sumihiro's theorem. Then we reduce to the affine case

$$X = \mathrm{Spec}(A) \text{ for some } \mathbb{Z}\text{-graded } k\text{-algebra } A = \oplus A_d.$$

Let $m_x \subset A$ be the maximal ideal corresponding to the closed point $x \in X$, then we have a $\mathbb{G}_m$-equivariant commutative diagram

$$\begin{tikzcd} \mathbb{G}_m \times \mathrm{Spec}(A/m_x) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(A/m_x) \arrow[d, hook] \\ \mathbb{G}_m \times \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

corresponding to the $\mathbb{Z}$-graded commutative diagram

$$\begin{tikzcd}
k[t,t^{-1}] \otimes m_x & m_x \arrow[l] \arrow[d, hook] \\
k[t,t^{-1}] \otimes A \arrow[u, hookleftarrow] \arrow[d, two heads] & A \arrow[l] \arrow[d, two heads] \\
k[t,t^{-1}] \otimes A/m_x & A/m_x \arrow[l]
\end{tikzcd}$$

i.e., m_x is also $\mathbb{Z}$-graded. Consider the $\mathbb{G}_m$-equivariant quotient $m_x \twoheadrightarrow m_x/m_x^2$. Since $\mathbb{G}_m$ is linearly reductive, this quotient has a section, i.e., a $\mathbb{G}_m$-invariant k-subspace $E \subset m_x$ such that $m_x = E \oplus m_x^2$. Since $x \in X$ is normal, there exists (exactly c) homogeneous elements $f_1, \dots, f_c \in m_x$ such that their images span m_x/m_x^2 as a k-vector space, i.e.,

$$m_x/m_x^2 = (\bar{f}_1, \dots, \bar{f}_c) \cong E$$

For any $a \in X$, let $\lambda(a) \in E^*$ be defined by $\lambda(a)(\bar{f}_i) := \bar{f}_i(a)$ for each i. This gives rise to a $\mathbb{G}_m$-equivariant morphism

$$\lambda : X \to E^* = (m_x/m_x^2)^* \cong T_x(X).$$

Clearly, $\lambda(x) = 0$ and the induced morphism on tangent spaces is given by

$$(\bar{f}_1, \dots, \bar{f}_c)^* \xrightarrow{\text{id}} ((f_1, \dots, f_c)/(f_1, \dots, f_c)^2)^*$$

In particular, λ is étale at x. Since étaleness is an open condition on the source, up to shrinking we may assume that λ is étale everywhere. Finally we apply Sumihiro's theorem to make X affine again. $\square$

Proposition A.0.2. *Let X be a geometrically normal and separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. If $f : C^\circ := (\mathbb{P}^1 - 0) \cup_{\infty \sim 0} (\mathbb{P}^1 - \infty) \to X$ is an open subchain of two orbits in X such that the unique fixed point in the image of f is a closed point of X, then f is smoothable.*

Proof. Let $x \in X^{\mathbb{G}_m}$ be the unique fixed point in the image of f. By Lemma A.0.1, there exists a $\mathbb{G}_m$-invariant affine open neighbourhood U of x and a $\mathbb{G}_m$-equivariant étale morphism $\lambda : U \to T_x(X)$. Then f factors through U.

Locally around the nodal point $\infty \sim 0$ of C°, we could write C° as $\mathrm{Spec}(k[\alpha, \beta]/\alpha\beta)$ such that α, β have $\mathbb{G}_m$-weights $1, -1$ respectively. Let $R := k[[\pi]]$ be a DVR over k and

$$C_R^\circ := \mathrm{Spec}(R[\alpha, \beta]/\alpha\beta - \pi)$$

then we have a $\mathbb{G}_m$-equivariant commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & C^\circ \arrow[l] \arrow[r, "f"] \arrow[d, hook] & U \\
\operatorname{Spec}(R) & C^\circ_R \arrow[l, "\text{flat}"'] \arrow[ur, dashed, "f_R"'] &
\end{tikzcd}$$

To construct a $\mathbb{G}_m$-equivariant morphism f_R lifting f, we complete this diagram by introducing the morphism λ

$$\begin{tikzcd}
C^\circ \arrow[r, "f"] \arrow[d, hook] \arrow[dr, "\zeta"] & U \arrow[d, "\lambda"] \\
C^\circ_R \arrow[r, dashed, "\zeta_R"'] & T_x(X)
\end{tikzcd}$$

and it suffices to construct a $\mathbb{G}_m$-equivariant morphism ζ_R lifting ζ, since $C^\circ \hookrightarrow C^\circ_R$ is an infinitesimal thickening of affine schemes and λ is étale. To proceed, we consider the corresponding $\mathbb{G}_m$-equivariant morphism of k-algebras

$$\begin{tikzcd}
k[u_1, \dots, u_c] \arrow[r, "\zeta"] \arrow[dr, dashed, "\zeta_R"'] & k[\alpha, \beta]/\alpha\beta \\
 & R[\alpha, \beta]/\alpha\beta - \pi \arrow[u, two heads, "\pi=0"']
\end{tikzcd}$$

and we can define ζ_R as follows (viewing $k[\alpha, \beta]/\alpha\beta$ as a subring of $R[\alpha, \beta]/\alpha\beta - \pi$ via the canonical inclusion):

$$\zeta_R(u_i) := \begin{cases} \zeta(u_i) & \text{if } \zeta(u_i) \neq 0 \\ \pi\alpha^{\mathrm{wt}_{\mathbb{G}_m}(u_i)} & \text{if } \zeta(u_i) = 0 \text{ and } \mathrm{wt}_{\mathbb{G}_m}(u_i) \geq 0 \\ \pi\beta^{-\mathrm{wt}_{\mathbb{G}_m}(u_i)} & \text{if } \zeta(u_i) = 0 \text{ and } \mathrm{wt}_{\mathbb{G}_m}(u_i) < 0 \end{cases} .$$

This finishes the proof. $\square$

Lemma A.0.3. *Let X be a geometrically normal and separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} = [U/\mathbb{G}_m]$ is S-complete. If $x_1, x_2 \in U$ such that $x_1^+ = x_2^-$ is a closed point of X, then $x_1^+ = x_2^- \in U$.*

Proof. Let $f : C^\circ := (\mathbb{P}^1 - 0) \cup_{\infty \sim 0} (\mathbb{P}^1 - \infty) \to X$ be the open subchain of two orbits in X corresponding to x_1 and x_2. By Proposition A.0.2 there exists a DVR R with fraction field

K and a $\mathbb{G}_m$-equivariant commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & C^\circ \arrow[l] \arrow[r, "f"] \arrow[d, hook] & X \\
\mathrm{Spec}(R) & C_R^\circ \arrow[l, "\text{flat}"'] \arrow[ur, "f_R"'] &
\end{tikzcd}$$

such that $C_K \cong \mathbb{G}_{m,K}$ and $f_K = \sigma(-, x_K) : \mathbb{G}_{m,K} \to X$ is an orbit map for some $x_K \in X$. Choose two sections $s_1, s_2 : \mathrm{Spec}(R) \to C_R^\circ$ such that the compositions

$$\Gamma_i : \mathrm{Spec}(R) \xrightarrow{s_i} C_R^\circ \xrightarrow{f_R} X \text{ satisfies } (\Gamma_i)_k = x_i \in U \text{ for } i = 1, 2.$$

Note that $\mathbb{G}_m.(\Gamma_1)_K = \mathbb{G}_m.(\Gamma_2)_K$ (since f_K is an orbit map) and Γ_i factors through U (since $U \subset X$ is open), the two morphisms $\overline{\Gamma}_i : \mathrm{Spec}(R) \xrightarrow{\Gamma_i} U \to \mathscr{U}$ coinciding on $\mathrm{Spec}(K)$, i.e., define a morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{U}$. Since $\mathscr{U}$ is S-complete, there exists a unique extension $\Gamma : \overline{\mathrm{ST}}_R \to \mathscr{U}$. Unwinding the definition of $\Gamma(0) \in \mathscr{U}$, it follows that $x_1^+ = x_2^- \in U$. $\square$

Corollary A.0.4 ([BBS83], Lemma 3.1.1 if $k = \mathbb{C}$ and $U \subset X - X^{\mathbb{G}_m}$)**.** *Let X be a geometrically normal and separated scheme, of finite type over a field k with a $\mathbb{G}_m$-action. Let $U \subset X$ be a $\mathbb{G}_m$-invariant open subset such that the quotient stack $\mathscr{U} := [U/\mathbb{G}_m]$ is S-complete. Then for any $i = 1, \ldots, r$, one of the following occurs:*

1. *$U \cap X_i \neq \emptyset$.*
2. *$U \cap X_i = \emptyset$ and either $U \cap X_i^+ = \emptyset$ or $U \cap X_i^- = \emptyset$.*

Proof. It is equivalent to prove that $U \cap X_i^+ \neq \emptyset \neq U \cap X_i^-$ implies $U \cap X_i = \emptyset$. Since the morphism $\pi_i^\pm : X_i^\pm \to X_i$ is surjective and $U \cap X_i^\pm \subset X_i^\pm$ is dense, we have

$$\pi_i^+(U \cap X_i^+) \cap \pi_i^-(U \cap X_i^-) \neq \emptyset,$$

i.e., there exists two points $x_1, x_2 \in U$ such that $x_1^+ = x_2^- \in X_i$. However $x_1^+ = x_2^- \in U$ by Lemma A.0.3, then $U \cap X_i \neq \emptyset$. $\square$

Appendix B

Examples of $\mathbb{G}_m$-action

This appendix is devoted to presenting several examples of $\mathbb{G}_m$-action. All materials are reproduced from [Som82] and no originality is claimed here.

B.1 Example of $\mathbb{G}_m$-action with prescribed $X_i^- \cap X_j^+$

In this section we give an example of a smooth projective scheme X with a $\mathbb{G}_m$-action such that $X_i^- \cap X_j^+$ is disconnected and singular. This is reproduced from [Som82, §2].

Example B.1.1. Starting with a connected scheme Z together with two closed subschemes $Z_1, Z_2 \subset Z$, we construct a scheme X with a $\mathbb{G}_m$-action such that

$$X_1^- \cap X_2^+ \cong \mathbb{G}_m \times (Z_1 \cap Z_2).$$

Consider the diagonal $\mathbb{G}_m$-action on $\mathbb{P}^1 \times Z$ which is standard on $\mathbb{P}^1$ and trivial on Z. Then $(\mathbb{P}^1 \times Z)^{\mathbb{G}_m} = Z_0 \sqcup Z_\infty$ such that $Z_0 \cong Z$ is the source and $Z_\infty \cong Z$ is the sink of $\mathbb{P}^1 \times Z$ respectively. Let

$$\pi : X := \mathrm{Bl}_{(\{0\}\times Z_1)\cup(\{\infty\}\times Z_2)}(\mathbb{P}^1 \times Z) \to \mathbb{P}^1 \times Z$$

be the blowing-up of $\mathbb{P}^1 \times Z$ at $\{0\} \times Z_1$ and $\{\infty\} \times Z_2$, then X admits a canonical $\mathbb{G}_m$-action such that π is $\mathbb{G}_m$-equivariant. Apart from the source and sink (which come from those of $\mathbb{P}^1 \times Z$ respectively), X has another two fixed point components X_1, X_2 such that $X_1 \cong Z_1$ and $X_2 \cong Z_2$ respectively. It is not difficult to see that

$$X_1^- \cap X_2^+ \cong \mathbb{G}_m \times (Z_1 \cap Z_2).$$

Remark B.1.2. This example shows that the subsets $X_i^- \cap X_j^+$ of a $\mathbb{G}_m$-action can be rather unpleasant (e.g., singular and disconnected), even if X is smooth and projective (e.g., if Z is smooth projective and both $Z_1, Z_2 \subset Z$ are smooth).

B.2 Example of $\mathbb{G}_m$-action with (quasi-)cycles

In this section we give an example of a smooth proper (but not projective) scheme X with a $\mathbb{G}_m$-action such that it has (quasi-)cycles. This is reproduced from [Som82, §1] and can be seen as a variation on Hironaka's famous example of proper non-projective varieties (see [Hir60] and also [Har77, Appendix B, Example 3.4.1]).

Example B.2.1. Let $\mathbb{G}_m$ act on $\mathbb{P}^3$ such that $t.[x_0 : x_1 : x_2 : x_3] := [x_0 : tx_1 : tx_2 : tx_3]$ and consider the following decomposition

$$\{0\} \in \mathbb{A}^3 \xhookrightarrow{x_0 \neq 0} \mathbb{P}^3 \xhookleftarrow{x_0 = 0} \mathbb{P}^2$$

Then the induced $\mathbb{G}_m$-action on $\mathbb{A}^3 \subset \mathbb{P}^3$ has weights $(1, 1, 1)$ and $\mathbb{P}^2 \subset \mathbb{P}^3$ is a fixed point component. Moreover $(\mathbb{P}^3)^{\mathbb{G}_m} = \{0\} \sqcup \mathbb{P}^2$ such that $\{0\}$ is the source and $\mathbb{P}^2$ is the sink of $\mathbb{P}^3$ respectively.

Choose two smooth curves $C_1, C_2 \subset \mathbb{P}^2$ which meet transversally at two points p_1, p_2 and nowhere else. Let

$$\pi : X := \mathrm{Bl}_{\tilde{C}_2}(\mathrm{Bl}_{C_1}(\mathbb{P}^3 - \{p_1\})) \bigcup_{\mathrm{Bl}_{C_1 \cup C_2 - \{p_1, p_2\}}(\mathbb{P}^3 - \{p_1, p_2\})} \mathrm{Bl}_{\tilde{C}_1}(\mathrm{Bl}_{C_2}(\mathbb{P}^3 - \{p_2\}) \to \mathbb{P}^3$$

be the glueing of two blowing-ups of $\mathbb{P}^3$, i.e.,

1. Over $\mathbb{P}^3 - \{p_1\}$, first blowing-up the curve C_1, then blowing-up the strict transform of the curve C_2.
2. Over $\mathbb{P}^3 - \{p_2\}$, first blowing-up the curve C_2, then blowing-up the strict transform of the curve C_1.
3. Over $\mathbb{P}^3 - \{p_1, p_2\}$, it doesn't matter in which order we blowing-up the curves C_1 and C_2, so we can glue our two blowing-ups along the inverse image of $\mathbb{P}^3 - \{p_1, p_2\}$.

Then X is a smooth proper (but not projective) scheme and admits a canonical $\mathbb{G}_m$-action such that π is $\mathbb{G}_m$-equivariant. Apart from the source and sink (which come from those of $\mathbb{P}^3$ respectively), X has another two fixed point components $\tilde{C}_1, \tilde{C}_2$ such that $\tilde{C}_1 \cong C_1$ and $\tilde{C}_2 \cong C_2$ respectively. In fact

$$X^{\mathbb{G}_m} = \{0\} \sqcup \tilde{C}_1 \sqcup \tilde{C}_2 \sqcup \tilde{\mathbb{P}^2}$$

such that $\{0\}$ is the source and $\tilde{\mathbb{P}^2}$ is the sink of X respectively. By construction there exist $\mathbb{G}_m$-orbits from $\tilde{C}_1$ to $\tilde{C}_2$ and vice versa, yielding a (quasi-)cycle on X.

Remark B.2.2. For an illustrative procedure of this construction we highly recommend the beautiful figures in [Thi09, Figure 3].

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Index