LECTURE ON INTERSECTION THEORY (I)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

Instructor: Qizheng YIN [BICMR, Peking University]

Time: Tue 13:00–14:50, 2017–02–21

Place: Room 302, No.4 Science Building, Peking University

Contents

This course will mainly cover the following three topics.

- Intersection theory.
- Cycles & Topo and Geom of complex algebraic variety
- Motives.

In this lecture, we will sketch what is contained in each topic one by one.

1. Intersection theory

Given a variety X, let V and W be two closed subvarieties of X. Our main aim is to define their *intersection* $V.W$ of the 'excepted' (which we will explain below) dimension.

1.1. Ideal case.

Suppose $\dim X = d, \dim V = i, \dim W = j$, we *expect*

- $\dim V \cap W = d - [(d-i)+(d-j)] = i+j-d$, in which case we call V and W intersect *properly*: For the reason, one recall that V is defined by $d-i$ equations and W by $d-j$ equations.
- $V \cap W$ is reduced (as scheme).

2010 *Mathematics Subject Classification.* 42B35; 46E30; 47B38; 30H25.

Key words and phrases. Intersection theory, Algebraic cycles, Birational equivalence.

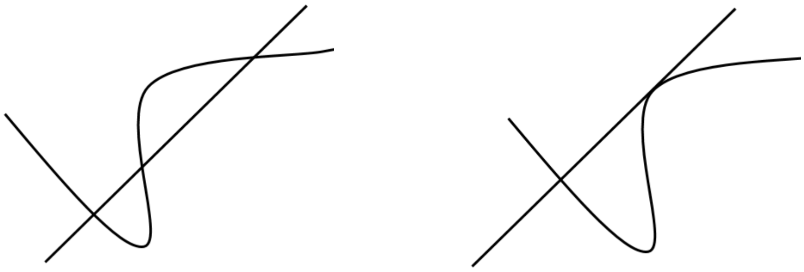
The author is supported by ERC Advanced Grant 226257.

If this is the case, we then define (as scheme/variety)

$$V.W := V \cap W$$

1.2. **Problems in general case.** In general case, we may face two problems.

(†) $V \cap W$ is non-reduced $\Rightarrow$ Use scheme or count 'multiple'.



(a) Reduced (b) Non-reduced

(‡) $\dim V \cap W > i + j - d$ (i.e., they do not intersect properly) $\Rightarrow$ Move subvariety under a suitable equivalence so that they intersect properly.

1.3. **Solution to (†): Algebraic cycles.** Define the following free abelian group

$$Z_i(X) := \mathbb{Z}\{\text{closed subvariety of } X \text{ of } \dim\ i\}$$

So each element $V \in Z_i(X)$ is of the form

$$V = \sum n_j V_j$$

where $n_j \in \mathbb{Z}$ and V_j are closed subvarieties of X of dimension i.

1.4. **Solution to (‡): Birational equivalence.** Birational equivalence is a generalization of the linear equivalence of divisors. Roughly say, it means move cycles along $\mathbb{P}^1$.

1.4.1. *Chow's moving lemma.* We introduce the i-th *Chow group* of X

$$\mathrm{CH}_i(X) := Z_i(X)/\sim$$

where $\sim$ denotes the birational equivalence. For each element $V \in Z_i(X)$, we write $[V]$ for its image in $\mathrm{CH}_i(X)$.

Lemma 1.1 (Chow). *If X is nonsingular, then for any $V \in Z_i(X)$ and $W \in Z_j(X)$, there exists $V' \in Z_i(X)$ such that*

(1) $V \sim V'$ and
(2) V' intersects properly with W.

So using Chow's moving lemma, one expect to define the intersection as

$$V.W := [V' \cap W] \in \mathrm{CH}_{i+j-d}(X) \qquad (*)$$

Problem 1.2. Unfortunately, there is no *canonical* choice of V'.

In fact, it's rather hard to prove the well-definedness of the intersection defined by $(*)$. In history, there are many selfclaimed-proof of this, some are even from very distinguished mathematicians, but finally these are all proved to be false. This problem is fixed by Futon and MacFhose alternatively in 1984, using a very different method.

1.4.2. *Futon-MacFhose's approach: ideal.* The method of Futon-MacFhose has the following properties:

- rigorous $\Rightarrow$ can prove the well-definedness in the moving lemma.
- geometric.
- refined.

In what follows, we only give the ideal of Futon-MacFhose's method.

(1) Study (cycle theoretic) Chern classes of vector bundle. Let

$$\begin{tikzcd} E \arrow[d] \\ X \arrow[u, bend left, "0"] \end{tikzcd}$$

be a vector bundle of rank r over X, then we

(1-1) can define $0^* : \mathrm{CH}_i(E) \xrightarrow{\sim} \mathrm{CH}_{i-r}(X)$, as the pull-back of 0.

(1-2) can define intersection with zero section.

(2) Deformation to the normal cone. Let V, W be subvarieties of X, we form the following diagram

$$\begin{tikzcd} Y := V \cap W \arrow[r] \arrow[d] & W \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

Suppose V is 'good', which implies that locally

(2-1) $V \hookrightarrow X$ looks like the zero section of the normal bundle

$$\begin{tikzcd} \mathrm{N}_V(X) \arrow[d] \\ V \arrow[u, bend left, "0"] \end{tikzcd}$$

(2-2) $Y \hookrightarrow W$ looks like the zero section of the normal cone

$$\begin{tikzcd} \mathrm{C}_Y(W) \arrow[d] \\ Y \arrow[u, bend left, "0"] \end{tikzcd}$$

With these at hand, we have the following diagram

$$\begin{tikzcd} \mathrm{C}_Y(W) \arrow[r, hook] \arrow[dr] & \mathrm{N}_V(X)|_Y \\ & Y \arrow[u, "0"'] \end{tikzcd}$$

so everything reduces to the form:

$$\text{Cycle} \bigcap \text{Zero Section}$$

which we have already defined in (1), hence one can define

$$V.W := 0^*.[C_Y(W)] \in \mathrm{CH}_{i+j-d}(V \cap W)$$

Fact 1.3 (Futon-MacFhose)**.** For X nonsingular, the diagonal embedding

$$\Delta_X : X \to X \times X$$

is 'good'. For any $V \in Z_i(X)$ and $W \in Z_j(X)$, we define

$$V.W := \Delta_X.(V \times W) \in \mathrm{CH}_{i+j-d}(X)$$

where $V \times W \in Z_{i+j}(X \times X)$. Hence we get a *intersection* map

$$Z_i(X) \times Z_j(X) \dashrightarrow \mathrm{CH}_{i+j-d}(X)$$

then we verify this map respects birational equivalence, so factors through

$$\mathrm{CH}_i(X) \times \mathrm{CH}_i(X) \dashrightarrow \mathrm{CH}_{i+j-d}(X)$$

which is exactly what we want.

Remark 1.4. We usually write $\mathrm{CH}^i(X) := \mathrm{CH}_{d-i}(X)$, then

$$\mathrm{CH}^i(X) \times \mathrm{CH}^i(X) \dashrightarrow \mathrm{CH}^{i+j}(X)$$

and under this operator, we obtain the so-called *Chow ring* $(\mathrm{CH}^*(X), .)$.

2. Cycles & Topo and Geom of complex algebraic variety

2.1. **Preliminary: Singular cohomology.** Let X be a smooth complex projective algebraic variety, then we have $H^*(X, \mathbb{Z})$, the singular cohomology with coefficient $\mathbb{Z}$. It turns out that it only depends on the topology of X.

Moreover, if X is equipped with metric, then we have the *Hodge decomposition*:

$$H^i(X, \mathbb{C}) := H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \sum_{p+q=i} H^{p,q}(X)$$

where $H^{p,q}(X)$ are complex vector spaces and $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$. If we write $h^{p,q} := \dim H^{p,q}(X)$, then we have the *Hodge diamond* of X

$$\begin{array}{ccccccc}
 & & & .. & & & \\
 & & .. & & .. & & \\
 & .. & & & & .. & \\
.. & & & & & & .. \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

Table 1. Hodge diamond

Fact 2.1. The Hodge diamond is symmetric in the following way:

(1) ($\leftrightarrow$) Left and Right: follows from $\overline{H^{p,q}} = H^{q,p}$.
(2) ($\updownarrow$) Up and Down: follows from Serre Duality.

2.2. **Algebra v.s. Topo & Geom.** The first two examples are of the form Algebra $\Rightarrow$ Geometry; while the last one is of the form Geometry $\Rightarrow$ Algebra. Here $\Rightarrow$ means 'control' or 'determine'.

2.2.1. *Hodge conjecture.* Cycle class map

$$\mathrm{cl} : \mathrm{CH}^i(X) \to \mathrm{Hdg}^i(X)$$

where $\mathrm{Hdg}^i(X) := H^{2i}(X, \mathbb{Z}) \cap H^{i,i}(X)$ is the i-th *Hodge class* of X.

Conjecture 2.2 (Hodge)**.** $\mathrm{cl} : \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Hdg}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is surjective.*

2.2.2. *Mumford theorem.*

Theorem 2.3 (Mumford)**.** *For X a surface, if $\mathrm{CH}^i(X) = \mathbb{Z}$ is trivial, then*

$$h^{1,0} = h^{2,0} = 0.$$

Remark 2.4. More generally, we have $\mathrm{CH}^*(X)$ 'small' $\Rightarrow$ $H^*(X)$ 'small'.

2.2.3. *Block-Balinse conjecture.*

Conjecture 2.5 (Block)**.** *Converse of Mumford Theorem.*

More generally, we have

Conjecture 2.6 (Block-Balinse)**.** *There exists a filtration in* $\mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

$$\mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} =: F^0 \supset F^1 \supset \ldots \supset F^{i+1} = 0$$

such that F_j/F_{j+1} *is 'controlled' by* $H^{i,i-j}(X)$*.*

3. Motives

3.1. **Formation of cohomology theory and Grothendieck's dream.** Let X be a smooth projective over k. According the type of the ground field k, we have the following cohomology theories.

Table 2. Cohomology Theories

$k = \mathbb{C}$	$H^*(X, \mathbb{Z})$: singular coho
char $k = 0$	$H^*_{dR}(X, \mathbb{Z})$: (algebraic) de Rham coho
any k, $\ell \neq$ char k	$H^*_{et}(X_{\bar{k}}, \mathbb{Z}_\ell)$: ℓ-adic étale coho
char $k = p$	$H^*_{cy}(X, \mathbb{Z})$: crystallization coho

In general, let (SPV/k) denote the category of smooth projective variety over k, then we can build various cohomology theories on it, in diagram we have

$$\begin{tikzcd} & & (\mathsf{SPV}/k) \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ H_1 & H_2 & .. & H_i & .. \end{tikzcd}$$

where H_i denote various kinds of cohomology theory. Grothendieck wanted to find a category M_k, satisfying

- abelian.
- semi-simple.
- Tannakian (from representation theory)

such that the formation of cohomology theory factors through M_k, i.e., the following digram

$$\begin{tikzcd} & & (\mathsf{SPV}/k) \arrow[ddll] \arrow[ddl] \arrow[d, dotted] \arrow[ddr] \arrow[ddrr] & & \\ & & M_k \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ H_1 & H_2 & .. & H_i & .. \end{tikzcd}$$

is commutative. Clearly, (SPV/k) itself is a trivial choice of M_k.

3.2. **Grothendieck's construction.** Use cycle to define morphism in M_k, i.e.,

$$\mathrm{Hom}_{M_k}(X, Y) := Z_*(X \times Y)/\sim$$

where $\sim$ denotes some suitable equivalence, e.g., rational equivalence.

References

Institute of Mathematica, Academy of Mathematics and System Sciences, Chinese Academy of Science, Beijing 100190, China

E-mail address: zhangxucheng15@mails.ucas.cn