LECTURE ON INTERSECTION THEORY (X)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

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Time: Thu 10:10–12:00, 2017–04–20

Place: Room 302, No.4 Science Building, Peking University

Contents

1. Albnase map

Let X be a non-singular projective variety over $\mathbb{C}$ of dimension n. In this section, we always fix a closed point $P_0 \in X$. Recall one has:

(1) $\underline{\mathrm{CH}_0(X) = \mathrm{CH}^n(X)}$:

$$\begin{tikzcd} \mathrm{CH}_0(X) \arrow[r, two heads, "\deg"] \arrow[dr, "\mathrm{cl}"'] & \mathbb{Z} \arrow[d, equal] \\ & H^{2n}(X,\mathbb{Z}) \cap H^{n,n}(X) \end{tikzcd}$$

and set $\mathrm{CH}_0(X)_0 := \ker(\deg) \subset \mathrm{CH}_0(X)$.

(2) $\underline{\mathrm{Alb}(X)}$:

$$\begin{tikzcd} (X,P_0) \arrow[r, "\mathrm{alb}"] \arrow[dr, "\forall"'] & (\mathrm{Alb}(X),0) \arrow[d, dashed, "\exists!"] \\ & (A,0) \end{tikzcd}$$

Via cohomology we can write

$$\begin{aligned} \mathrm{Alb}(X) &= H^n(X,\Omega_X^{n-1})/H^{2n-1}(X,\mathbb{Z}) \\ &= H^{n-1,n}(X)/H^{2n-1}(X,\mathbb{Z}) \end{aligned}$$

in which the quotient is taken via

$$H^{2n-1}(X,\mathbb{Z}) \xrightarrow{-\otimes_{\mathbb{Z}}\mathbb{C}} H^{2n-1}(X,\mathbb{C}) \xrightarrow{\mathrm{proj}} H^{n-1,n}(X)$$

2010 *Mathematics Subject Classification.* 42B35; 46E30; 47B38; 30H25.
Key words and phrases. Intersection theory, Chow ring.
The author is supported by ERC Advanced Grant 226257.

By Serre's duality

$$\mathrm{Alb}(X) = H^0(X, \Omega_X^1)^\vee / H_1(X, \mathbb{Z})$$

and in this case the Albanse map can be realized as

$$\mathrm{alb} : X \longrightarrow \mathrm{Alb}(X)$$
$$P \mapsto \left(\square \mapsto \int_{P_0}^{P} \square \right)$$

for any $\square \in H^0(X, \Omega_X^1)$.

Lemma 1.1. *The map* $\mathrm{alb} : X \longrightarrow \mathrm{Alb}(X)$ *is well-defined.*

Proof. For any two paths C, C' in X connecting P_0 and P, the difference

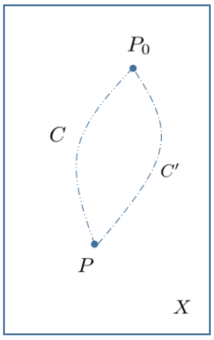


FIGURE 1. Well-definedness of alb

$$\int_C \square - \int_{C'} \square = \int_{C-C'} \square = \int_{e_{P_0}} \square = 0$$

in which the second equality involves the homotopy relation

$$[C - C'] = [e_{P_0}] \in H_1(X, \mathbb{Z})$$

as desired. Moreover, by additivity we get

$$\mathrm{alb} : Z_0(X) \to \mathrm{Alb}(X) \qquad (\dagger)$$

□

The following lemma allows us to descend the albnase map (†) to $\mathrm{CH}_0(X)_0$

$$\mathrm{alb} : \mathrm{CH}_0(X)_0 \to \mathrm{Alb}(X)$$

which will be our main object hereafter.

Lemma 1.2.

(1) *The restriction map* $\mathrm{alb} : Z_0(X)_0 \to \mathrm{Alb}(X)$ *is independent of* P_0.
(2) *For any* $\alpha \in Z_0(X)$, *we have* $\alpha \sim_{\mathrm{rat}} 0 \Rightarrow \mathrm{alb}(\alpha) = 0$.

Proof. One by one check.

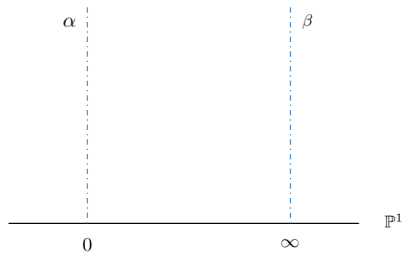
(1) For any element $\alpha = \sum n_i P_i \in Z_0(X)_0$, we have

$$\sum n_i = 0$$

Since for any closed point $P \in X$, the difference of $\mathrm{alb}(P)$ with respect to the base points P_0 and P_0' is a constant $c \in \mathrm{Alb}(X)$, then the difference of $\mathrm{alb}(\alpha)$ with respect to P_0 and P_0' is

$$\left(\sum n_i\right) \cdot c = 0$$

(2) From the deformation we obtain a morphism



$$\mathbb{P}^1 \to \mathrm{Alb}(X)$$

but any morphism from $\mathbb{P}^1$ to an abelian variety is always constant[1], we are done.

□

Lemma 1.3. *The map* $\mathrm{alb} : \mathrm{CH}_0(X)_0 \to \mathrm{Alb}(X)$ *is surjective.*

Proof. It suffices to show $\mathrm{alb} : Z_0(X)[\text{or } Z_0(X)_0] \to \mathrm{Alb}(X)$ is surjective. One can view $X^N \subset Z_0(X)$ as a subset via

$$(P_1, \cdots, P_N) \mapsto \sum_{i=1}^{N} P_i$$

Claim: for $N \gg 0$, the restriction map

$$t : X^N \to \mathrm{Alb}(X)$$

$$(P_1, \cdots, P_N) \mapsto \sum_{i=1}^{N} \mathrm{alb}(P_i)$$

is surjective. Notice that t is proper, it suffices to check the tangent map of t is surjective at one point. This is equivalent to say that cotangent map of t is injective at one point. In fact[2]

$$t^* : H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}) \to H^0(X^N, \Omega^1_{X^N})$$

but[3]

$$H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}) = H^0(X, \Omega^1_X)$$

then over any point $(P_1, \cdots, P_N) \in X^N$ the map t^* reads as

$$t^* : H^0(X, \Omega^1_X) \to \Omega^1_{X^N,(P_1,\cdots,P_N)} = \bigoplus_{i=1}^{N} \Omega^1_{X,P_i}$$

[1]such morphism necessarily factor through $\mathrm{Alb}(\mathbb{P}^1) = \mathrm{Jac}(\mathbb{P}^1) = pt$.

[2]the cotangent bundle of an abelian variety is trivial of rank $g = \dim X$, since X is an algebraic group. Hence $\Omega^2_X = \bigwedge^2 \mathcal{O}_X$ is the trivial bundle of rank $\binom{g}{2}$. Therefore

$$\dim H^2(\mathcal{O}_X) = \dim H^0(\Omega^2_X) = \binom{g}{2}$$

where the first equality is 'Hodge symmetry' $h^{p,q} = h^{q,p}$.

[3]recall $\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z})$.

One can verify the map t^* is given by restriction. Take $N \gg 0$, we know there exists a point $(P_1, \cdots, P_N) \in X^N$ such that t^* is injective. $\square$

2. Mumford's theorem

Hereafter let $X = S$ be a surface. Our main picture is

$$\boxed{\ker[\mathrm{alb} : \mathrm{CH}_0(S) \to \mathrm{Alb}(X)] \quad v.s. \quad H^{2,0}(S)}$$

Theorem 2.1 (Mumford)**.** *If $H^{2,0}(S) \neq 0$, then $\mathrm{CH}_0(S)$ is ∞-dimensional.*

In other words, Mumford's theorem says

$$\underbrace{\mathrm{CH}_0(S) \text{ finite-dim'l}}_{\text{alg}} \Rightarrow \underbrace{H^{2,0}(S) = 0}_{\text{geom/topo}}$$

Instead of defining the ∞-*dim'l*, we give the definition of *finite-dim'l*. It has the following three characterizations.

Definition-Proposition 2.2. $\mathrm{CH}_0(S)$ is said to be *finite-dim'l* if one of the following conditions is satisfied

(1) $\ker(\mathrm{alb}) = 0$.
(2) there exists $N \gg 0$ such that the map
$$S^N \times S^N \to \mathrm{CH}_0(S)_0$$
given by
$$[(P_1, \cdots, P_N), (Q_1, \cdots, Q_N)] \mapsto \sum_{i=1}^{N} (P_i - Q_i)$$
is surjective.
(3) there exists a curve $j : C \hookrightarrow S$ on S such that
$$j_* : \mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$$
is surjective. Roughly say, all zero cycles on S are supported on C.

Remark 2.3. The equivalence of (2) $\Leftrightarrow$ (3) is Roitman's theorem.

In the following, we prove a sightly general version of Mumford's theorem.

Theorem 2.4. *If there exists $j : Y \hookrightarrow X$ such that*
$$j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$
is surjective, then
$$H^{p,0}(X) = 0 \text{ for any } p > \dim(Y)$$

Example 2.5. If $\mathrm{CH}_0(X) = \mathbb{Z}$ (for example, rational connected variety), then we can take Y to be a point and hence $\dim(Y) = 0$. In this case the Hodge diamond of X looks like

To prove Mumford's theorem, we need

- the concept of *correspondence*.
- the technique of *decomposition of the diagonal*.

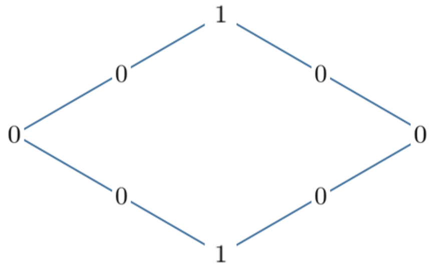


Figure 2. Hodge diamond

2.1. **Correspondence.**

Definition 2.6. Let X, Y be non-singular projective varieties over $\mathbb{C}$, then the *correspondence* from X to Y is defined as

$$\mathrm{Corr}(X,Y) := \mathrm{CH}^*(X\times Y)$$

Some properties of $\mathrm{Corr}(X,Y)$ are summarized as follows.

(1) For any $\Gamma\in\mathrm{Corr}(X,Y)$ and $\Gamma'\in\mathrm{Corr}(Y,Z)$, we can define their composition as follows

$$\Gamma'\circ\Gamma := (p_{X,Z})_*[p^*_{X,Y}(\Gamma)\cdot p^*_{Y,Z}(\Gamma')]\in\mathrm{Corr}(X,Z)$$

where

$$\begin{tikzcd} & X\times Y\times Z \arrow[dl, "p_{X,Y}"'] \arrow[d, "p_{X,Z}"] \arrow[dr, "p_{Y,Z}"] & \\ X\times Y & X\times Z & Y\times Z \end{tikzcd}$$

And this composition is associated:

$$\Gamma''\circ\Gamma'\circ\Gamma = \Gamma''\circ(\Gamma'\circ\Gamma) = (\Gamma''\circ\Gamma')\circ\Gamma$$

(2) For any $\Gamma\in\mathrm{Corr}(X,Y)$, it induces a homomorphism of groups

$$\begin{aligned}\Gamma_*: \mathrm{CH}^*(X)&\to\mathrm{CH}^*(Y)\\ \alpha&\mapsto (p_Y)_*[p_X^*(\alpha)\cdot\Gamma]\end{aligned}$$

where

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

And this homomorphism satisfies

$$(\Gamma'\circ\Gamma)_* = \Gamma'_*\circ\Gamma_*$$

Similarly one can define

$$\Gamma_*: H^*(X,\mathbb{Z})\to H^*(Y,\mathbb{Z})$$

(3) $\mathrm{Corr}(X,Y)$ is graded:

$$\mathrm{Corr}(X,Y) = \bigoplus_r \mathrm{Corr}^r(X,Y)$$

where

$$\mathrm{Corr}^r(X,Y) = \mathrm{CH}^{\dim(X)+r}(X\times Y)$$

And in this level, the homomorphism Γ_* respect the graded structure on both sides.

$$\Gamma_* : \mathrm{CH}^k(X) \to \mathrm{CH}^{k+r}(Y)$$
$$\Gamma_* : H^k(X,\mathbb{Z}) \to H^{k+2r}(Y,\mathbb{Z})$$
$$\Gamma_* : H^{p,q}(X) \to H^{p+r,q+r}(Y)$$

(4) Any morphism $f : Y \to X$ induces an element
$$\Gamma_f^t \in \mathrm{Corr}^0(X,Y)$$
where Γ_f^t is the transport of graph of f. It turns out
$$(\Gamma_f^t)_* = f^*$$
In this case, the identity $\mathrm{id}_X : X \to X$ corresponds to
$$[\Delta_X] \in \mathrm{Corr}^0(X,X) = \mathrm{CH}^{\dim(X)}(X,X)$$

2.2. **Decomposition of the diagonal.**

Theorem 2.7 (Bloch–Srinivas). *Let X be a non-singular projective variety over $\mathbb{C}$ of dimension n. If there exists $j : Y \hookrightarrow X$ with*
$$j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$
is surjective, then there exists $N \in \mathbb{N}$ such that
$$N \cdot [\Delta_X] = \Gamma_1 + \Gamma_2 \in \mathrm{Corr}^0(X,X)$$
with $\Gamma_i \in \mathrm{Corr}^0(X,X)$ and

(1) *Γ_1 is supported on $Y \times X$.*
(2) *Γ_2 is supported on $X \times D$ where D is a divisor on X.*

Proof. Let $U := X \setminus Y$, then by assumption $\mathrm{CH}_0(U) = 0$. Notice that X, Y are defined over a finitely generated[4] field $K/\mathbb{Q}$. Let $\eta \in X_{K(X)}$ be the generic point, then
$$[\eta_U] \in \mathrm{CH}_0(U_{K(X)})$$
Choose an embedding $K(X)/K \hookrightarrow \mathbb{C}/K$, then
$$0 = [\eta_U] \in \mathrm{CH}_0(U_{\mathbb{C}})$$
so there exists $N \in \mathbb{N}$ such that [5]
$$N \cdot [\eta_U] = 0 \in \mathrm{CH}_0(U_{K(X)})$$
hence
$$\begin{aligned} N \cdot [\eta_U] &= 0 \in \mathrm{CH}_0(U_{\mathbb{C}(X)}) \\ &= \delta \in \mathrm{CH}_0(X_{\mathbb{C}(X)}) = \mathrm{CH}^n(X_{\mathbb{C}(X)}) \\ &\in \mathrm{CH}_0(Y_{\mathbb{C}(X)}) \end{aligned}$$

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[4]involve all the coefficients of defining equations of X and Y: they are finitely many.

[5] the formula $[\eta_U] = 0 \in \mathrm{CH}_0(U_{\mathbb{C}})$ implies that there exists an immediate field $L/K(X) < \infty$ such that $0 = [\eta_U] \in \mathrm{CH}_0(U_L)$. Consider the following fiber product diagram

$$\begin{array}{ccc} X_L & \xrightarrow{f} & X_{K(X)} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(K(X)) \end{array}$$

Since $f^*([\eta_U]) = 0$, then $f_* f^*([\eta_U]) = 0$. But on the other hand
$$f_* f^*([\eta_U]) = N \cdot [\eta_U]$$
for some $N \in \mathbb{N}$ since $L/K(X) < \infty$. This confirm the existence of such N.

where the second equality follows from exceptional sequence. Take closure

$$N \cdot [\Delta_X] - \Gamma_1 \in \ker[\mathrm{CH}^n(X \times X) \to \mathrm{CH}^n(X_{\mathbb{C}(X)})]$$

where $\Gamma_1 \in \mathrm{CH}^n(Y \times X)$.

Fact 2.8. $\mathrm{CH}^n(X_{\mathbb{C}(X)}) = \lim_{\to D} \mathrm{CH}^n(X \times (X \setminus D))$ where D runs through all divisors of X.

Then there exists a divisor D on X such that $N \cdot [\Delta_X] - \Gamma_1$ is supported on $X \times D$. Hence

$$N \cdot [\Delta_X] = \Gamma_1 + \Gamma_2$$

satisfying the required conditions. □

2.3. **Proof of Mumford's theorem.** Here we prove the general version of Mumford's theorem, i.e., Theorem 2.4.

Proof. Suppose there exists $j : Y \hookrightarrow X$ such that

$$j_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$

is surjective. By Theorem 2.7, we have the decomposition of diagonal

$$N \cdot [\Delta_X] = \Gamma_1 + \Gamma_2$$

Consider

$$(N \cdot [\Delta_X])_* = (\Gamma_1)_* + (\Gamma_2)_* \text{ on } H^*(X, \mathbb{Z})$$

(LHS) $(N \cdot [\Delta_X])_* = N \cdot \mathrm{id}_{H^*(X,\mathbb{Z})}$.
(RHS) 1^{st} term: $(\Gamma_1)_*$ factors through $H^*(\widetilde{Y}, \mathbb{Z})$ where

$$\begin{array}{c} \widetilde{Y} \times X \\ \downarrow \\ Y \times X \\ \downarrow \\ X \times X \end{array}$$

is a resolution of singularities for Y. Since $\dim(\widetilde{Y}) = \dim(Y)$, then

$$(\Gamma_1)_*(H^{p,0}(X)) = 0 \text{ for any } p > \dim(Y)$$

(RHS) 2^{nd} term: $(\Gamma_2)_*$ factors through $H^*(\widetilde{D}, \mathbb{Z})$ where

$$\begin{array}{c} X \times \widetilde{D} \\ \downarrow \\ X \times D \\ \downarrow \\ X \times X \end{array}$$

is a resolution of singularities for D. But we know

$$H^{p,q}(\widetilde{D}) \to H^{p+1,q+1}(X)$$

then

$$(\Gamma_2)_*(H^{p,0}(X)) = 0 \text{ for any } p > 0$$

Hence (RHS) implies that

$$N \cdot \mathrm{id}_{H^*(X,\mathbb{Z})}(H^{p,0}(X)) = 0 \text{ for any } p > \dim(Y)$$

i.e.,

$$H^{p,0}(X) = 0 \text{ for } \text{ for any } p > \dim(Y)$$

$\square$

References

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