LECTURE ON INTERSECTION THEORY (IX)

ZHANG

Abstract. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.

Instructor: Qizheng YIN [BICMR, Peking University]

Time: Thu 10:10–12:00, 2017–04–18

Place: Room 302, No.4 Science Building, Peking University

Contents

Setting-up

Let X be a non-singular projective variety over $\mathbb{C}$ of dimension n, then we have the following communicative diagram

$$\begin{tikzcd} \mathrm{CH}^1(X) \arrow[r, "\cong"] & \mathrm{Pic}(X) \\ \mathrm{CH}^1(X)_{\mathrm{alg}} \arrow[u, hook] \arrow[r, "\cong"] & \mathrm{Pic}^0(X) \arrow[u, hook] \end{tikzcd}$$

such that

(1) $\mathrm{Pic}^0(X) =$ the Picard variety[1] of X, consisting of line bundles over X with zero first Chern class.

(2) $\mathrm{CH}^1(X)_{\mathrm{alg}} := Z^1(X)_{\mathrm{alg}}/Z^1(X)_{\mathrm{rat}}$ where

(2a) $Z^1(X)_{\mathrm{alg}} := Z^1(X)/\sim_{\mathrm{alg}}$.

2010 *Mathematics Subject Classification.* 42B35; 46E30; 47B38; 30H25.

Key words and phrases. Intersection theory, Chow ring.

The author is supported by ERC Advanced Grant 226257.

[1] It's an abelian variety and in the case $\Bbbk = \mathbb{C}$, we have

$$\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) = H^{0,1}(X)/H^1(X, \mathbb{Z})$$

i.e., it's isomorphic to ($\mathbb{C}^n/\Lambda$, polarization).

(2b) $\sim_{\mathrm{alg}}$ is the *algebraic equivalence* on $Z^1(X)$ defined by replacing $\mathbb{P}^1$ by any algebraic curves in the definition of $\sim_{\mathrm{rat}}$.

Notice that the object $\mathrm{CH}^1(X)_{\mathrm{alg}}$ is defined algebraically, so it's free of ground field $\Bbbk$.

therefore we have

$$\mathrm{CH}^1(X)/\mathrm{CH}^1(X)_{\mathrm{alg}} \cong \mathrm{Pic}(X)/\mathrm{Pic}^0(X) =: \mathrm{NS}(X)$$

the *Néon-Severi group* of X.

0.1. **Preliminary: GAGA.** The notation X^{an} will be used if we view X as a compact complex Kähler manifold equipped with analytic topology. In this case we have the *singular cohomology*

$$H^*(X^{\mathrm{an}}, \mathbb{Z})$$

Theorem 0.1 (GAGA)**.**

$$\text{analytic} \left\langle \begin{matrix} \text{subvarieties of } X \\ \text{vector bundles on } X \\ \text{coherent sheaves on } X \\ \vdots \end{matrix} \right\rangle \text{are algebraic}$$

Proof. For more materials on GAGA, see [Ser56]. ☐

0.2. **Exceptional sequence.** (This construction only works for $\Bbbk = \mathbb{C}$) In the analytic world, consider the exceptional sequence of sheaves on X^{an}

$$0 \longrightarrow \underbrace{\mathbb{Z}^{\mathrm{an}}}_{\text{constant sheaf}} \xrightarrow{2\pi i} (\mathcal{O}_X)^{\mathrm{an}} \xrightarrow{\exp} (\mathcal{O}_X^*)^{\mathrm{an}} \longrightarrow 0$$

One take the long exact sequence associated to it

$$\begin{gathered} H^0(X^{\mathrm{an}}, (\mathcal{O}_X)^{\mathrm{an}}) \to H^0(X^{\mathrm{an}}, (\mathcal{O}_X^*)^{\mathrm{an}}) \\ \to H^1(X^{\mathrm{an}}, \mathbb{Z}^{\mathrm{an}}) \to H^1(X^{\mathrm{an}}, (\mathcal{O}_X)^{\mathrm{an}}) \to H^1(X^{\mathrm{an}}, (\mathcal{O}_X^*)^{\mathrm{an}}) \\ \to H^2(X^{\mathrm{an}}, \mathbb{Z}^{\mathrm{an}}) \to H^2(X^{\mathrm{an}}, (\mathcal{O}_X)^{\mathrm{an}}) \end{gathered}$$

Applying Serre's GAGA to drop the superscript $\square^{\mathrm{an}}$ everywhere

$$\begin{gathered} H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*) \\ \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \\ \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \end{gathered}$$

Notice that the first line is surjective thus reduced to

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

this leads to another exact sequence

$$\begin{tikzcd}
 & H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d, two heads] & H^1(X,\mathcal{O}_X^*) \arrow[r, "\mathrm{cl}/c_1"] \arrow[d, equal] & H^2(X,\mathbb{Z}) & \\
0 \arrow[r] & H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{O}_X^*) \arrow[r] \arrow[d, equal] & \ker[H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)] \arrow[r] \arrow[u, hook] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d, equal] & \mathrm{Pic}(X) \arrow[r] \arrow[d, equal] & \mathrm{NS}(X) \arrow[r] \arrow[d, equal] & 0 \\
 & Z^1(X)_{\mathrm{alg}}/Z^1(X)_{\mathrm{rat}} \arrow[d, equal] & Z^1(X)/Z^1(X)_{\mathrm{rat}} \arrow[d, equal] & Z^1(X)/Z^1(X)_{\mathrm{alg}} \arrow[d, equal] & \\
 & \mathrm{CH}^1(X)_{\mathrm{alg}} & \mathrm{CH}^1(X) \arrow[uuuur, dashed] & \mathrm{CH}^1(X)/\mathrm{CH}^1(X)_{\mathrm{alg}} &
\end{tikzcd}$$

Hereafter, we will frequently use this diagram and denote it by ♣.

1. Cyclic class map

1.1. **Definition via Poincáre duality.** Under a series of identifications in the diagram ♣, we obtain the *cohomology class of divisors*

$$\mathrm{cl} : \mathrm{CH}^1(X) \to H^2(X,\mathbb{Z})$$

or equivalently *first Chern class*

$$c_1 : \mathrm{CH}^1(X) \to H^2(X,\mathbb{Z})$$

The cohomology class of divisors provides us a toy version/picture of a general concept: the cycle class map, which maps *every* cycle class to cohomology. The quickest definition may be as follows.

Definition-Proposition 1.1. The *cycle class map*

$$\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X,\mathbb{Z})$$

is defined as follow: for any closed subvariety $V \subset X$ of codimension k, let

$$f : \widetilde{V} \to V \hookrightarrow X$$

be a resolution of singularies for V with $\widetilde{V}$ non-singular, then $\mathrm{cl}(V) \in H^{2k}(X,\mathbb{Z})$ is defined by the composition

$$\begin{aligned} H^0(\widetilde{V},\mathbb{Z}) \cong H_{2k}(\widetilde{V},\mathbb{Z}) &\xrightarrow{f_*} H_{2n-2k}(X,\mathbb{Z}) \cong H^{2k}(X,\mathbb{Z}) \\ \widetilde{V} &\longmapsto \mathrm{cl}(\widetilde{V}) \end{aligned}$$

such that

(1) $H^0(\widetilde{V},\mathbb{Z}) = H_{2k}(\widetilde{V},\mathbb{Z}) \cong \mathbb{Z}[\widetilde{V}]$ is generated by the fundamental class $[\widetilde{V}]$.
(2) two isomorphisms are Poincáre duality.

Proof. It's sufficient to show that $\mathrm{cl}(V)$ is independent of the choice of resolution of singularities $\widetilde{V}$. Indeed, if V_1, V_2 are both resolutions of V, then we can find a third resolution V_3 of V dominating both V_1, V_2, i.e., there exists a commutative diagram[2]

$$\begin{tikzcd} & V_3 \arrow[dl] \arrow[dr] \arrow[dd] & \\ V_1 \arrow[dr] & & V_2 \arrow[dl] \\ & V & \end{tikzcd}$$

then the well-definedness follows. So we get

$$\mathrm{cl}: Z^k(X) \to H^{2k}(X, \mathbb{Z})$$

Lemma 1.2. $\alpha \sim_{\mathrm{rat}} 0 \Rightarrow \mathrm{cl}(\alpha) = 0$.

Proof. Reduce to the case of principal divisor. And in that case, such verification coincides with the exceptional sequence. □

Hence cl passes through the rational equivalence and we obtain

$$\mathrm{cl}: \mathrm{CH}^k(X) \to H^{2k}(X, \mathbb{Z})$$

as desired. □

1.2. **Properties.** In last section we finally get

$$\mathrm{cl}: \underbrace{\mathrm{CH}^k(X)}_{\mathrm{alg}} \to \underbrace{H^{2k}(X, \mathbb{Z})}_{\mathrm{topo}}$$

and this will be our main object hereafter.

Proposition 1.3. *Cycle class map satisfies the following functorial properties.*

(1) (Respect/Compatible with the ring structure on both sides) *For any* $\alpha \in \mathrm{CH}^k(X)$ *and* $\beta \in \mathrm{CH}^\ell(X)$*, we have*

$$\mathrm{cl}(\alpha \cdot \beta) = \mathrm{cl}(\alpha) \cup \mathrm{cl}(\beta)$$

In picture,

$$\begin{tikzcd} \mathrm{CH}^k(X) \times \mathrm{CH}^\ell(X) \arrow[r, "\cdot"] \arrow[d, "\mathrm{cl}\times\mathrm{cl}"'] & \mathrm{CH}^{k+\ell}(X) \arrow[d, "\mathrm{cl}"] \\ H^{2k}(X) \times H^{2\ell}(X) \arrow[r, "\cup"] & H^{2(k+\ell)}(X, \mathbb{Z}) \end{tikzcd}$$

(2) (Pull-back) *For any morphism* $f : X \to Y$ *and* $\alpha \in \mathrm{CH}^k(Y)$*, we have*

$$\mathrm{cl}_X(f^*(\alpha)) = f^*(\mathrm{cl}_Y(\alpha))$$

[2] Indeed, we can construct V_3 as follows: form the fiber product

$$\begin{tikzcd} V_1 \times_V V_2 \arrow[r] \arrow[d] & V_2 \arrow[d] \\ V_1 \arrow[r] & V \end{tikzcd}$$

which may be singular and then take V_3 to be the resolution of singularities for $V_1 \times_V V_2$.

In picture,

$$\begin{tikzcd}
\mathrm{CH}^k(Y) \arrow[r, "\mathrm{cl}_Y"] \arrow[d, "f^*"'] & H^{2k}(Y,\mathbb{Z}) \arrow[d, "f^*"] \\
\mathrm{CH}^{k+\dim(Y)}(X) \arrow[r, "\mathrm{cl}_X"] & H^{2k+2\dim(Y)}(X,\mathbb{Z})
\end{tikzcd}$$

Proof. Use earlier deformation to the normal cone or moving lemma. Note

$$\mathrm{cl} : \mathrm{CH}^1(X) \to \ker(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X))$$

□

2. Hodge theory: A glance

2.1. Hodge decomposition.

Theorem 2.1 (Hodge).

(1) *There exists a decomposition of ring*

$$(\dagger) \quad H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^*(X,\mathbb{C}) = \bigoplus_{k=0}^{2n} H^k(X,\mathbb{C})$$

$$= \bigoplus_{k=0}^{2n} \left[\bigoplus_{\substack{p+q=k \\ p,q\geq 0}} H^{p,q}(X) \right]$$

such that

$$\overline{H^{p,q}(X)} = H^{q,p}(X)$$

(2) $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$, *where* Ω_X^p *is the sheaf of holomorphic/differential* p*-forms on* X.

Sketch of the proof.

(1) In $(\dagger)$, the first equality follows from universal coefficient theorem for cohomology; the second equality follows from the decomposition of $H^*(X,\mathbb{Z})$. So it's remaining to show the third one, i.e.,

$$H^k(X,\mathbb{C}) = \bigoplus_{\substack{p+q=k \\ p,q\geq 0}} H^{p,q}(X)$$

such that

$$\overline{H^{p,q}(X)} = H^{q,p}(X)$$

Fact 2.2.

$$H^k(X,\mathbb{C}) \cong H^k_{\mathrm{dR}}(X,\mathbb{C}) \cong \mathscr{H}^k(X)$$

(a) $H^k_{\mathrm{dR}}(X,\mathbb{C}) = k$-th de Rham cohomology of X.

(b)

$$\mathscr{H}^k(X) := \{k\text{-forms } \omega \text{ on } X : \Delta_d(\omega) = 0\}$$

the set of harmonic k-forms on X.

Then all the remaining statements follow from the well-known decomposition of $\mathscr{H}^k(X)$

$$\mathscr{H}^k(X) = \bigoplus_{\substack{p+q=k \\ p,q\geq 0}} \mathscr{H}^{p,q}(X)$$

where

$$\mathscr{H}^{p,q}(X) = \{(p,q)\text{-forms } \omega \text{ on } X : \Delta_d(\omega) = 0\}$$

is set of harmonic[3] (p,q)-forms[4] on X. Hence one obtain

$$H^k(X,\mathbb{C}) = \bigoplus_{\substack{p+q=k\\p,q\geq 0}} H^{p,q}(X)$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

(2) Recall that

$$H^q(X,\Omega_X^p) \cong \{(p,q)\text{-forms } \omega \text{ on } X : \Delta_{\bar\partial}(\omega) = 0\}$$

Since X is Kälher, then

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$$

and therefore

$$\begin{aligned} H^q(X,\Omega_X^p) &\cong \{(p,q)\text{-forms } \omega \text{ on } X : \Delta_d(\omega) = 0\} \\ &= \mathscr{H}^{p,q}(X) \cong H^{p,q}(X) \end{aligned}$$

□

Definition 2.3 (Hodge diamond)**.** Define the *Hodge numbers* as

$$h^{p,q} := \dim H^{p,q}(X)$$

and then we can arrange these numbers into *Hodge diamond*. To illustrate what it looks like, we present the case when X is a surface

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Remark 2.4. Hodge diamond is symmetric in the following ways:

(1) $\updownarrow$ by Serre's duality. Indeed, for any holomorphic vector bundle E over a smooth compact complex manifold V of dimension n

$$H^k(V,E) = H^{n-k}(V, E^\vee \otimes \Omega_X^n)^\vee$$

Applied to our case, we get

$$H^q(X,\Omega_X^p) = H^{n-q}(X,\Omega_X^{n-p})^\vee \cong H^{n-q}(X,\Omega_X^{n-p})$$

i.e.,

$$H^{p,q}(X) = H^{n-p,n-q}(X) \Rightarrow h^{p,q} = h^{n-p,n-q}$$

(2) $\leftrightarrow$ by Hodge decomposition.

2.2. **Hodge conjecture.**

[3]since X is compact Kälher.

[4]in local coordinates, such ω can be written as

$$\omega = \sum_{|I|=p,|J|=q} f_{IJ}\, dz^I \wedge d\bar z^J$$

consisting of exactly p holomorphic and q anti-holomorphic components.

2.2.1. *Refined cycle class map.* Since $H^k(X, \mathcal{O}_X) = H^{0,k}(X)$, then the diagram ♣ reads as

$$\begin{tikzcd}
0 \arrow[r] & H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{O}_X^*) \arrow[r] \arrow[d, equal] & \ker[H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^{0,1}(X)/H^1(X,\mathbb{Z}) \arrow[r] & \mathrm{CH}^1(X) \arrow[r] & \ker[H^2(X,\mathbb{Z}) \to H^{0,2}(X)] \arrow[r] & 0
\end{tikzcd}$$

where the map $H^2(X, \mathbb{Z}) \to H^{0,2}(X)$ is given by the composition

$$H^2(X,\mathbb{Z}) \xrightarrow{-\otimes_{\mathbb{Z}}\mathbb{C}} H^2(X,\mathbb{C}) = \bigoplus_{\substack{p+q=2\\ p,q\geq 0}} H^{p,q}(X) \xrightarrow{\text{proj}} H^{0,2}(X)$$

For any element in $H^2(X, \mathbb{Z})$, it lies in the kernel iff it has no $H^{0,2}(X)$-part, and hence no $H^{2,0}(X)$-part, then

$$\ker[H^2(X,\mathbb{Z}) \to H^{0,2}(X)] = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

so we get a 'refined' cycle class map[5]

$$\mathrm{cl} : \mathrm{CH}^1(X) \to H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

More general we have

Lemma 2.5. *For any $k \in \mathbb{Z}$, we have*

$$\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X,\mathbb{Z}) \cap H^{k,k}(X) =: \mathrm{Hdg}^k(X)$$

Proof. For any closed subvariety $V \subset X$ of codimension k, i.e., $\dim(V) = n - k$. Consider $\mathrm{cl}(V)$, due to the dimension obstruction, $\widetilde{V}$ cannot support (p, q)-forms with $p > n - k$ or $q > n - k$. Then $\mathrm{cl}(V)$ is represented by a (k, k)-form on X. □

2.2.2. *Hodge conjecture.* The celebrated Hodge conjecture says

Conjecture 2.6 (Hodge)**.** *The map*

$$\mathrm{cl} : \mathrm{CH}^k(X)_{\mathbb{Q}} \to \mathrm{Hdg}^k(X)_{\mathbb{Q}}$$

is surjective, where $\square_{\mathbb{Q}} := \square \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 2.7. By far, we know Hodge conjecture is

(1) trivial for $k = 0, n$: just unwinding the fundamental class.
(2) true for $k = 1$, even with $\mathbb{Z}$-coefficient: use exceptional sequence. In fact

    Fact 2.8 (Lefschetz theorem on $(1, 1)$-classes)**.** Any element of

    $$H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

    is the cohomology class of a divisor on X. In particular, the Hodge conjecture is true for $k = 1$.

[5]in fact, we have

$$\mathrm{cl}(\mathrm{CH}^1(X)) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

2.2.3. *Hard Lefschetz.* For any ample line bundle L on X, it determines a cohomology class of the Kähler $(1,1)$-form on X

$$[\omega] \in H^{1,1}(X) \subset H^2(X, \mathbb{Z})$$

Fact 2.9. The cup product with ω sends

(1) closed forms (i.e. forms killed by d) to closed forms.
(2) exact forms (i.e. forms in the image of d) to exact forms.

Thus it induces an operator, called the *Lefschetz operator*

$$L : H^k(X, \mathbb{Z}) \xrightarrow{\cup\omega} H^{k+2}(X, \mathbb{Z})$$

which preserves the Hodge decomposition

$$L : H^{p,q}(X) \xrightarrow{\cup\omega} H^{p+1,q+1}(X)$$

Theorem 2.10 (Hard Lefschetz)**.** *For any $k \leq n$ we have an isomorphism*

$$L^{n-k} : H^k(X, \mathbb{Q}) \xrightarrow{\sim} H^{2n-k}(X, \mathbb{Q})$$

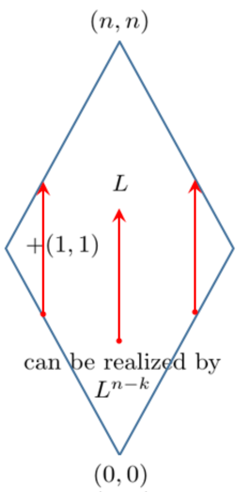

Figure 1. Hard Lefschetz theorem on Hodge diamond

By Hard Lefschetz theorem, we know Hodge conjecture is

(1) true for $k = n - 1$.
(2) true for $n \leq 3$: Indeed, if Hodge conjecture is true for $k \leq \lfloor n/2 \rfloor$, then it's also true for $n - k$.

Remark 2.11. By far, we know Hodge conjecture holds in the cases:

- for $n \leq 3$ and any k.
- for any n and $k = 0, 1, n-1, n$.
- if true for $k \leq \lfloor n/2 \rfloor$, then also true for $n - k$.

And Hodge conjecture fails in the cases:

- with $\mathbb{Z}$-coefficient: see Atiyah, Hirzebruch [AH62], Kollár, Talum.
- for Kälher manifold, see Voisin [Voi02].

3. From Chow ring to geometry

Fix $P \in X$ a closed points.

3.1. **Albnase variety.** The Albanese variety is a generalization of the Jacobian variety of a curve. But in general, it's not necessarily unique.

Definition 3.1 (Albnase variety)**.** The *Albnase variety* of (X,P), usually denoted by $\mathrm{Alb}(X)$, is an abelian variety equipped with a morphism

$$\mathrm{alb}:(X,P)\to(\mathrm{Alb}(X),0)$$

such that any morphism $(X,P)\to(A,0)$ into an abelian variety uniquely factors through alb. In picture,

$$\begin{tikzcd}
(X,P) \arrow[rr, "\mathrm{alb}"] \arrow[dr] & & (\mathrm{Alb}(X),0) \arrow[dl, dashed, "\exists\,!"] \\
& (A,0) &
\end{tikzcd}$$

Remark 3.2. The Albanese variety $\mathrm{Alb}(X)$ has an explicit description.

(1) via Picard variety $\mathrm{Pic}^0(X)$:

$$\mathrm{Alb}(X):=(\mathrm{Pic}^0(X))^\vee$$

For algebraic curves, the Abel–Jacobi theorem implies that the Albanese and Picard varieties are isomorphic.

(2) Over $\mathbb{C}$, we have

$$\begin{aligned}\mathrm{Alb}(X)&=H^n(X,\Omega_X^{n-1})/H^{n-1}(X,\mathbb{Z})\\&=H^{n-1,n}(X)/H^{n-1}(X,\mathbb{Z})\end{aligned}$$

hence for compact Kähler manifold X, the dimension of its Albanese variety is the Hodge number $h^{1,0}$.

Remark 3.3. Recall the degree map

$$\deg:\mathrm{CH}_0(X)\longrightarrow\mathbb{Z}$$

and set $\mathrm{CH}_0(X)_0:=\ker(\deg)$. Since the ground field $\Bbbk$ is algebraically closed, the Albanese map

$$\mathrm{alb}:(X,P)\to(\mathrm{Alb}(X),0)$$

factors through a group homomorphism (also called the *Albanese map*)

$$\mathrm{alb}:\mathrm{CH}_0(X)_0\to\mathrm{Alb}(X)$$

since $\mathrm{CH}_0(X)_0$ is generated by the elements of the form $[*-P]$.

3.2. **Toy example: surface.** To illustrate how these objects are used to detect the geometry of X, we see what is happening when X be a surface. Consider its Hodge diamond

$$\begin{array}{ccccc}
 & & h^{2,2} & & \\
 & h^{2,1} & & h^{1,2} & \\
h^{2,0} & & h^{1,1} & & h^{0,2}\\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}$$

And we already know

TABLE 1. Algebra ⇒ Geom/Topo

Geom/Topo	Algebra
$h^{0,0}$	by the fundamental class $[X]$
$h^{1,0} = h^{0,1}$	by $\mathrm{Pic}^0(S)$
$h^{2,0} = h^{0,2}$	by ker(alb) (will be confirmed by Bloch)
$h^{1,1}$	by Hard Lefschetz theorem
$h^{2,1} = h^{1,2}$	by symmetry
$h^{2,2}$	by points of X

References

[AH62] M. F. Atiyah and F. Hirzebruch. Analytic cycles on complex manifolds. *Topology*, 1:25–45, 1962. 8

[Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Annales de l'Institut Fourier*, 6:1–42, 1956. 2

[Voi02] Claire Voisin. A counterexample to the Hodge conjecture extended to Kähler varieties. *Int. Math. Res. Not.*, (20):1057–1075, 2002. 8

INSTITUTE OF MATHEMATICA, ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, CHINESE ACADEMY OF SCIENCE, BEIJING 100190, CHINA

E-mail address: zhangxucheng15@mails.ucas.cn