LECTURE ON INTERSECTION THEORY (VII)

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ABSTRACT. This is a private note taken from the course 'Topics in Algebraic Geometry'. The note taker is responsible for any inaccuracies.
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Time: Thu 10:10–12:00, 2017–04–11
Place: Room 302, No.4 Science Building, Peking University

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Recall: let $X \hookrightarrow Y$ be a closed embedding with ideal sheaf $\mathcal{I}$. Last time we see there are two related constructions.

(1) the normal bundle to X in Y is given by

$$N_X Y := \operatorname{Spec}\left(\operatorname{Sym}^\bullet \mathcal{I}/\mathcal{I}^2\right)$$

it's a vector bundle over X.

(2) the normal cone to X in Y is given by

$$C_X Y := \operatorname{Spec}\left(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

If in addition, the embedding $X \hookrightarrow Y$ is regular, then $\mathcal{I}/\mathcal{I}^2$ is locally free and the normal cone

$$C_X Y = N_X Y$$

is in fact a vector bundle.

(3) the blow-up of Y along X is given by

$$\mathfrak{Bl}_X Y := \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathcal{I}^n\right)$$

with exceptional divisor $E = \mathbb{P}(C_X Y)$.

2010 *Mathematics Subject Classification.* 42B35; 46E30; 47B38; 30H25.
Key words and phrases. Deformation, Intersection product.
The author is supported by ERC Advanced Grant 226257.

1. Deformation to the normal cone

Let $X \hookrightarrow Y$ be a closed embedding.

Goal: construct a scheme $M := M_X Y$ together with an closed embedding $X \times \mathbb{P}^1 \hookrightarrow M$ with $f : M \to \mathbb{P}^1$ flat so that

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr, "\mathrm{pr}"'] & & M \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

commutates and such that

(1) over $\mathbb{P}^1 - \{\infty\} \cong \mathbb{A}^1$
$$X \times \mathbb{A}^1 \hookrightarrow M|_{\mathbb{A}^1} \cong Y \times \mathbb{A}^1$$
is the trivial embedding induced from $X \hookrightarrow Y$.

(2) over ∞
$$X \cong X \times \{\infty\} \hookrightarrow M|_\infty = C_X Y$$
is the embedding of X into its normal cone.

Construction:

(1) Step–1: consider $X \times \{\infty\} \subset Y \times \mathbb{P}^1$ and the blow-up
$$\begin{tikzcd} \widetilde{M} := \mathfrak{Bl}_{X\times\{\infty\}} Y \times \mathbb{P}^1 \arrow[d] \arrow[dd, bend right=60, "(\dagger)"'] \\ Y \times \mathbb{P}^1 \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$
where $(\dagger)$ is flat[1]. Consider the sequence of embedding
$$X \cong X \times \{\infty\} \subset X \times \mathbb{P}^1 \subset Y \times \mathbb{P}^1$$
notice that $X \times \{\infty\}$ is a Cartier divisor in $X \times \mathbb{P}^1$, the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$ may be identified with $X \times \mathbb{P}^1$ and it can be embedded as a closed subscheme of $\widetilde{M}$.
$$\begin{tikzcd} \mathfrak{Bl}_{X\times\{\infty\}} X \times \mathbb{P}^1 \arrow[r, hook] \arrow[d, equal] & \widetilde{M} \\ X \times \mathbb{P}^1 & \end{tikzcd}$$
In addition, it's easy to see that
- over $\mathbb{P}^1 - \{\infty\} \cong \mathbb{A}^1$
$$\widetilde{M}|_{\mathbb{A}^1} \cong Y \times \mathbb{A}^1$$
as desired.
- over ∞, prior we know
$$\widetilde{M}|_\infty \supset E$$
where E is the exceptional divisor of this blow-up. Since
$$C_{X\times\{\infty\}}(Y \times \mathbb{P}^1) = C_X Y \oplus 1$$

[1] follows from the fact that (1) any blow-ups over curve is flat and (2) flatness is stable under composition.

then

$$E = \mathbb{P}(C_{X\times\{\infty\}}(Y\times\mathbb{P}^1)) = \mathbb{P}(C_XY\oplus 1)$$

and there is an canonical open embedding

$$C_XY \hookrightarrow \mathbb{P}(C_XY\oplus 1) = E \subset \widetilde{M}|_\infty$$

(2) Step–2: keep C_XY part and remove the rest in $\widetilde{M}|_\infty$. Therefore get the desired M. Similarly from the embedding sequence

$$X\times\{\infty\}\hookrightarrow Y\times\{\infty\}\hookrightarrow Y\times\mathbb{P}^1$$

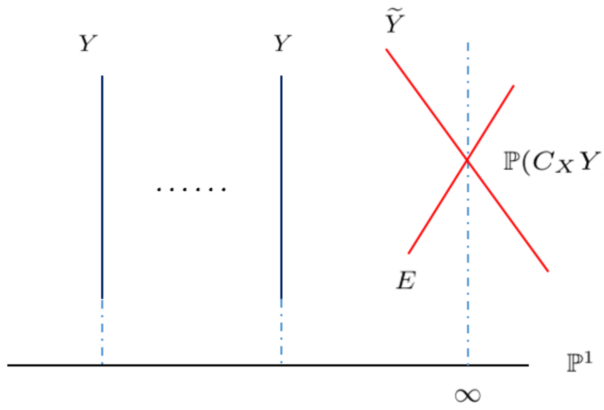
the blow-up of $Y\times\{\infty\}$ along $X\times\{\infty\}$ can be embedded as a closed subscheme of $\widetilde{M}$

$$\widetilde{Y} := \mathfrak{Bl}_{X\times\{\infty\}}Y\times\{\infty\}\subset\widetilde{M}|_\infty$$

Fact 1.1. In fact

$$\widetilde{M}|_\infty = E\bigcup_{\mathbb{P}(C_XY)}\widetilde{Y}$$

In picture, we are facing



Proof. Since we have E and $\widetilde{Y}$ globally embeds in $\widetilde{M}$, it suffices to examine their structure locally. So may assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$.

To study $\widetilde{M}$ around ∞, identify $\mathbb{P}^1 - \{0\}$ with $\mathbb{A}^1 = \mathrm{Spec}(\Bbbk[t])$ (geometrically we take the open subset around ∞ and make it into 0), then

$$Y\times\mathbb{A}^1 = \mathrm{Spec}(A[t])$$

and

$$\mathfrak{Bl}_{X\times\{0\}}Y\times\mathbb{A}^1 = \mathrm{Proj}(S^\bullet)$$

where

$$S^n = (I,t)^n = I^n\oplus I^{n-1}t\oplus\cdots\oplus It^{n-1}\oplus At^n\oplus At^{n+1}\oplus\cdots$$

The complements of $\widetilde{Y}$ is $\mathrm{Spec}(S_{(t)})$ where

$$S_{(t)} = \cdots\oplus I^nt^{-n}\oplus\cdots\oplus It^{-1}\oplus A\oplus At\oplus At^2\oplus\cdots$$

The canonical homomorphism from $A[t]$ to $S_{(t)}$ becomes an isomorphism after localization at t (i.e., after $-\otimes A[t]/tA[t]$), while

$$S_{(t)}/tS_{(t)} = \bigoplus_{n\geq 0} I^n/I^{n+1}$$

hence we get $\mathrm{Spec}(S_{(t)}/tS_{(t)}) = C_XY$. As desired. $\square$

Remark 1.2. MarPherson's description of this deformation, as a special case of his graph construction, is particularly vivid. Let $X \hookrightarrow Y$ be a closed embedding and we can take a vector bundle[2] over Y of rank r

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ Y \arrow[u, bend left=60, "s"] \arrow[u, bend right=60, "0"'] \end{tikzcd}$$

such that there exists a section s of E whose zero-scheme is X, i.e.,

$$X = \{x \in Y : s(x) = 0\}$$

For each scalar λ, the graph of λs is a line in $E \oplus 1$, this gives an embedding

$$\begin{aligned} Y \times \mathbb{A}^1 &\hookrightarrow \mathbb{P}(E \oplus 1) \times \mathbb{P}^1 \\ (y, \lambda) &\mapsto (\text{graph of } \lambda s(y), (1 : \lambda)) \end{aligned}$$

the deformation space $M_X Y$ is in fact the closure of $Y \times \mathbb{A}^1$ in this embedding. Notice that $C_X Y \subset E|_X$ and

$$[X]^{\text{vir}} = 0^*[C_X Y] \in \text{CH}_{n-r}(X).$$

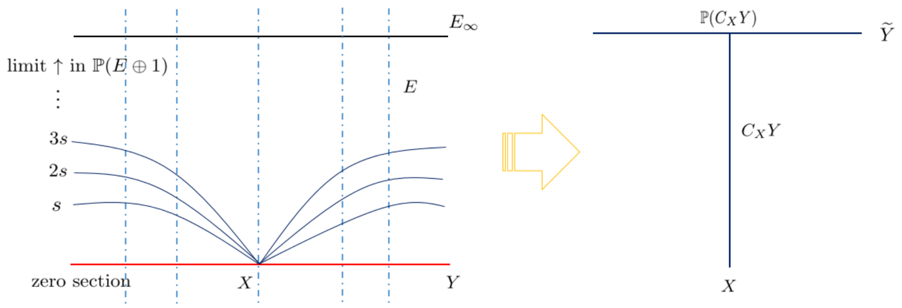


FIGURE 1. Fulton's picture

Example 1.3. Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d and $V \subset Y$ a closed subscheme of dimension k, by considering the following diagram

$$\begin{tikzcd} W := X \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

and we are led to

$$\begin{tikzcd} C_W V \arrow[rr, hook] \arrow[dr] & & N_X Y|_W \arrow[dl] \\ & W \arrow[ur, bend right, "0"'] & \end{tikzcd}$$

and therefore one can define

$$X \cdot V := 0^*[C_W V] \in \text{CH}_{k-d}(W)$$

<u>A reminding remark</u>: One can also get

$$[i^*(V) =]X \cdot V \in \text{CH}_{k-d}(X)$$

[2]if Y is quasi-projective, then such E always exists, although its rank r may be larger the the codimension of X in Y.

or

$$X \cdot V \in \mathrm{CH}_{k-d}(Y)$$

the choice of intersection theory only depends on one's own interest.

Remark 1.4. In practice, there are many different ways to define intersection theory, but $X \cdot V \in \mathrm{CH}_{k-d}(X)$ is unique if naturally require

(1) Normalization: if

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \\ X \arrow[u, bend right, "0"'] \end{tikzcd}$$

is a vector bundle over X, then for any closed subscheme $Z \subset X$ and $V := \pi^{-1}(Z) \subset Y$, we have

$$X \cdot V = [Z]$$

(2) Continuity: if

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr] & & \mathcal{Y} \supset \mathcal{V} \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

is a family of regular embedding such that $\mathcal{Y}, \mathcal{V}$ are flat over $\mathbb{P}^1$, then we have

$$X \cdot \mathcal{V}_t \in \mathrm{CH}_{k-d}(X) \text{ stay same for all } t \in \mathbb{P}^1$$

In our case,

$$\begin{array}{rcr} \mathcal{Y} = M_X Y & \text{and} & \mathcal{Y}_\infty = C_X Y = N_X Y \\ \mathcal{V} = M_W V & \text{and} & \mathcal{V}_\infty = C_W V \subset C_X Y = N_X Y \end{array}$$

In fact: (1)+(2) not only give the uniqueness of intersection theory, but also give a way to construct intersection theory, just like the Grothendieck's relation for constructing Chern class.

2. Specialization to the normal cone

Let $X \hookrightarrow Y$ be a closed embedding and $C = C_X Y$ the normal cone to X in Y. One can define the *specialization* homomorphism

$$\sigma : Z_k(Y) \to Z_k(C)$$

by the formula

$$V \mapsto \text{cycle of } C_{X \cap V} V$$

for any k-dimensional subvariety V of Y, and extending linearly to all k-cycles. As $C_{X \cap V} V$ is a scheme of pure dimension k, it has fundamental cycle.

Proposition 2.1. $\alpha \sim_{\text{rat}} 0 \Rightarrow \sigma(\alpha) \sim_{\text{rat}} 0$.

Hence σ passes through the rational equivalence, defining *specialization homomorphism*

$$\sigma : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(C)$$

Proof. Let $M = M_X Y$ be the deformation space and consider

$$C \xhookrightarrow{i} M \xhookleftarrow[j]{} Y \times \mathbb{A}^1$$

so we have the localization sequence

$$\begin{tikzcd}
\mathrm{CH}_{k+1}(C) \arrow[r, "i_*"] & \mathrm{CH}_{k+1}(M) \arrow[r, "j^*"] \arrow[r, "\Downarrow"'] \arrow[d, "i^*"'] & \mathrm{CH}_{k+1}(Y \times \mathbb{A}^1) \arrow[r] \arrow[dl, dashed, "\tau"] & 0 \\
& \mathrm{CH}_k(C) & \mathrm{CH}_k(Y) \arrow[l, dashed, "\sigma"] \arrow[u, "\mathrm{pr}^*", "\sim"'] &
\end{tikzcd}$$

where

(1) i^* is the Gysin pull-back of divisors, since C is a Cartier divisor of M.
(2) pr^* is isomorphism.

If we want to construct σ, it then suffices to construct τ. <u>Claim:</u> $i^* i_* = 0$, then by universal property of cokernel, there is an unique map from $\mathrm{CH}_{k+1}(Y \times \mathbb{A}^1)$ to $\mathrm{CH}_k(C)$, which is our τ. In fact

$$i^* i_* = c_1(N_C M) \cap -$$

whereas $N_C M$ is trivial[3]. Define

$$\sigma := \tau \circ \mathrm{pr}^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(C)$$

To prove this proposition, it's remaining to show

$$\sigma([V]) = [C_{X \cap V} V]$$

In fact, $\mathrm{pr}^*([V]) = [V \times \mathbb{A}^1]$. The subvariety $M_{X \cap V} V$ is a closed subvariety of M which restricts to $\mathbb{A}^1$ is $V \times \mathbb{A}^1$, i.e.,

$$M_{X \cap V} V|_{\mathbb{A}^1} = V \times \mathbb{A}^1$$

then

$$j^*([M_{X \cap V} V]) = \mathrm{pr}^*([V])$$

and now

$$\sigma([V]) = \tau \circ \mathrm{pr}^*([V]) = \tau \circ j^*([M_{X \cap V} V]) = i^*([M_{X \cap V} V])$$

Notice that

$$M_{X \cap V} V|_\infty = C_{X \cap V} V$$

i.e., $i^*([M_{X \cap V} V]) = [C_{X \cap V} V]$, which completes the proof. □

3. Consequence: Intersection theory

Let $i : X \hookrightarrow Y$ is a closed regular embedding of codimension d so the normal bundle $N := N_X Y = C_X Y$. Define the *Gysin pull-back*

$$i^* : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k-d}(X)$$

to be the composition

$$\mathrm{CH}_k(Y) \xrightarrow{\sigma} \mathrm{CH}_k(N) \xrightarrow{0^*} \mathrm{CH}_{k-d}(X)$$

(1) If $d = 1$ (resp. i is the zero section of a vector bundle), this Gysin pull-back agrees with that defined before.
(2) If Y is pure of dimension n, then $i^*([Y]) = [X]$.
(3) For all $\alpha \in \mathrm{CH}_*(X), i^* i_*(\alpha) = c_d(N) \cap \alpha$.

[3] let $\widetilde{M}$ be the blow-up as before, then $E = \mathbb{P}(C_X Y \oplus 1), N_E \widetilde{M} = \mathcal{O}_{C_X Y \oplus 1}(-1)$. Consider the support yields the triviality.

(4) If X is smooth of dimension n, the diagonal embedding

$$\Delta_X : X \hookrightarrow X \times X$$

is regular of codimension n, therefore

$$\mathrm{CH}_k(X) \times \mathrm{CH}_\ell(X) \xrightarrow{\times} \mathrm{CH}_{k+\ell}(X \times X) \xrightarrow{\Delta_X^*} \mathrm{CH}_{k+\ell-n}(X)$$

defines an intersection product on $\mathrm{CH}_*(X)$

$$\mathrm{CH}_k(X) \times \mathrm{CH}_\ell(X) \xrightarrow{\cdot} \mathrm{CH}_{k+\ell-n}(X)$$

or

$$\mathrm{CH}^k(X) \times \mathrm{CH}^\ell(X) \xrightarrow{\cdot} \mathrm{CH}^{k+\ell}(X)$$

References

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