LOGIC AND DISCRETE MATH 240

PROFESSOR PLESS

Homework 6

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Practice Problems:

1. Given two arbitrary sets A,B, prove that (A intersection B) union (A intersection B-complement) equals A. You may *not* use a venn diagram. You may define characteristic functions for A,B so that A = {x | PA(x}), and B = {x | PB(x)}.

2. Let f(x) = 2x + 1 and g(x) = (x-3)2. For each definition of Domain and Co-Domain defined below, answer (i) is the function injective? (ii) is the function surjective? (iii) Is the function a bijection? (iv) If the function if bijective, give its inverse

   (a) the function f, when the domain and co-domain are the reals.
   (b) the function f, when the domain in the integers and co-domain is the reals.
   (c) the function (f o g) when the domain and co-domain are the reals. [Recall, (f o g)(x) = f(g(x))]

3. List the elements of the following sets:

   a) $\{1,2,3\} \times \{a,b\}$

   b) $Powerset(\{1,2,3\}) \bigcap Powerset(\{1,2,4\})$

4. Prove the cardinality of all finite length binary strings (strings over the alphabet $\{0,1\}$) is the same as the cardinality of all finite length strings over the alphabet $\{0,1,2,3\}$

5. Sets that are the same size as the integers are called "countable", which is short for countably infinite. Which of the following sets are countable? Give an intuitive answer that describes a mapping between these sets and the integers, or argue in a few sentences why this isn't possible.

   (example) The set of even integers. ANSWER: $f(x) = \frac{x}{2}$ is a bijection between even integers and integers.

   (example) The set of finite length binary strings. ANSWER: there is a bijection between the binary strings and the positive integers they represent.

   (a) $P(\mathbb{N})$ (the powerset of natural numbers).

   (b) The set of all possible functions $f : \mathbb{N} \longrightarrow \{0, 1\}$.

   (c) The set of all possible functions $f : \{0, 1\} \longrightarrow \mathbb{N}$.

6. Let A, B, and C be any sets that each have at least 1 element. Also, let f, g, h be functions, where: $g : A \longrightarrow B$, $h : A \longrightarrow C$, and $f : A \longrightarrow B \times C$, where f is defined by g, h so that:

$$f(x) = (g(x), h(x))$$

   Prove or give a counterexample for each of the following statements. (In this case, a counterexample must include a definition of sets A,B,C and a definition of the functions g,h that make the statement false.).

   a) If g or h is one-to-one, then f is one-to-one

   b) If neither g or h is one-to-one, then f is not one-to-one

Problems to turn in:

1. Let A be the set $\{2, 3, 4\}$, and B be the set $\{1, 2, 3, 4, 5\}$. Let P(A) be the powerset of A, and P(B) be the powerset of B.

   a) How many elements does the set $A \times B$ have?

   b) How many elements does $A \times P(A)$ have?

   c) list two different elements of $A \times P(A)$

   d) How many elements does $P(A) \times P(B)$ have?

   Solution:
   a):
   $|A| = 3$, $|B| = 5$
   $|A \times B| = |A| * |B| = 15$
   Therefore there are 15 elements in A x B
   b):
   $|P(A)| = 2^3 = 8$
   $|A \times P(A)| = |A| * |P(A)| = 3 * 8 = 24$

c):
list two distinctive elements of $|A \times P(A)|$
First: $(2, \{2, 3, 4\})$
Second: $(3, \{2, 3, 4\})$
d):
$|P(A)| = 2^3 = 8$
$|P(B)| = 2^5 = 32$
$|P(A) \times P(B)| = |P(A)| * |P(B)| = 8 * 32 = 256$

2. Let A, B, and C be any sets that each have at least 1 element. Also, let f, g, h be functions, where: $g : A \longrightarrow B$, $h : A \longrightarrow C$, and $f : A \longrightarrow B \times C$, where f is defined by g, h so that:

$$f(x) = (g(x), h(x))$$

Prove or give a counterexample for each of the following statements. (In this case, a counterexample must include a definition of sets A,B,C and a definition of the functions g,h that make the statement false.).

   a) If f is onto, then both g and h are onto.
   b) if g and h are both onto, then f is onto.

solution:
a):
The claim that If f is onto, then both g and h are onto is **TRUE**

Proof. We are gonna use DIRECT proof here

   a) If f is onto. Then for all (b, c) in BxC, $\exists a$ in A, such that $f(a) = (b, c)$
   b) Suppose any (b_1, c_1) in BxC, then there must exist a in A, such that $f(a_1) = (b_1, c_1)$ [this is essentially a universal universal instantiation of (a)]
   c) $f(x) = (g(x), h(x)) \longrightarrow f(a_1) = (g(a_1), h(b_1)) = (b1, c1)$. Therefore $g(a_1) = b_1$ and $h(a_1) = c_1$
   d) since (b_1, c_1) is any element in B x C, then b_1 is any element in B and c_1 is any element in C
   e) combine b),c),d), we have shown if $f : A \longrightarrow B$ is onto, then for any element $b_1 \in B$, there exists $a_1 \in A$, such that $g(a_1) = b_1$. Therefore $g : A \longrightarrow B$ is onto
   f) Similarly, from b),c),d), if $f : A \longrightarrow B$ is onto, then for any element $c_1 \in c$, there exists $a_1 \in A$, such that $h(a_1) = c_1$. Therefore $h : A \longrightarrow C$ is onto

We have proved that If f is onto, then both $g : A \longrightarrow B$ and $h : A \longrightarrow C$ are onto
Therefore claim (a) is TRUE □

b):
The claim that if g and h are both onto, then f is onto is **FALSE**

Proof. we are gonna prove the false of the claim by a **counterexample**

a) suppose $A = \{1,2\}, g(x) = x, h(x) = x$. both g and h are onto

b) so $B = \{1,2\}, C = \{1,2\} \longrightarrow C = AxB = \{(1,1),(1,2),(2,1),(2,2)\}$

c) $f : A \longrightarrow B \times C, f(x) = (g(x), h(x))$

- $f(1) = (1,1)$
- $f(2) = (2,2)$

d) from c) and b), we can see that element $(1,2)$ and $(2,1)$ in B x C are not get mapped from A for $f : A \longrightarrow B \times C$

e) Therefore $f : A \longrightarrow B \times C$ is not onto

so our counterexample has shown that when $g : A \longrightarrow B$ and $h : A \longrightarrow C$, there exists at least one case f is not onto
Therefore Claim (b) is **FALSE** □