LOGIC AND DISCRETE MATH 240

PROFESSOR PLESS

Homework 3

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Practice Problems:

1. Prove for integers $n \geq 0, 3 | n^3 + 2n$
2. Prove Bernoulli's inequality:
$$(1+x)^n \geq 1 + nx$$
for all non-negative integers n, when x is a real number greater than -1.
3. Prove that $log_5 16$ is irrational.

Problems to turn in:

1. For all integers k, prove that if k^2 is a multiple of 3 then k is a multiple of 3.
Solution:

Proof. I will use **indirect proof– contrapositive** to prove the statement.
The contrapositive statement of the original statement is **for all integers k, if k is not a multiple of 3, then k^2 is not a multiple of 3**
Now the task remained is to prove the contrapositive statement

- k is not a multiple of 3 $\rightarrow$ $k = 3n + 1 \vee k = 3n + 2$ (n is an integer)
- There are only two cases. I will use the **Proof By Case** to complete the proof

- when $k = 3n + 1$

$$
\begin{aligned}
k^2 &= (3n+1)^2 \\
&= 9n^2 + 1 + 6n \\
&= 3n * (3n+2) + 1
\end{aligned}
$$

  Then obviously k^2 cannot be divided by 3 **because of the remaining 1**

- When k = 3n + 2

$$
\begin{aligned}
k^2 &= 9n^2 + 4 + 12n \\
&= 3(3n^2 + 4n + 1) + 1
\end{aligned}
$$

  So when k = 3n +2, k^2 still cannot be divided by 3 **because of the remaining 1**

- So now combining two cases, we proved that **for all integers k, if k is not a multiple of 3, then k^2 is not a multiple of 3**

- obliviously, its contrapositive statement (the original statment) is also valid
- we proved that For all integers k, prove that if k^2 is a multiple of 3 then k is a multiple of 3.

□

2. Prove that $\sqrt{3}$ is irrational using a method similar to the one we used in class (about $\sqrt{2}$). (You may use the property from the previous question, whether or not you successfully proved it).
Solution:

Proof. Before Start the proof, Let's prove this statement first. **For integer k, If k^2 is even, k is a even number; if k^2 is odd, k is also an odd number** . For the first part of the statement is proved in the class . let's prove the second half here.

Goal: For integer k, if k^2 is odd, then k is also an odd number

Proof. I will use indirect proof– contrapositive proof to prove the statement

- The contrapositive statement of the original statement is **If k is an even number, then k^2 is an even number**
- man this is trivial
- let $k = 2n$ (n is an integer)
- $k^2 = 4n^2 = 2 * 2n^2$

- So k^2 is even
- So we proved that **If k is an even number, then k2 is an even number**
- the contrapositive statement to it (which is the original statement) has to be true
- we proved that **For integer k, if k^2 is odd, then k is also an odd number**

□

Ok now combined with the first part **For integer k, If k^2 is even, k is a even number** , which has been proved in the class , we can state our Lemma

Lemma 0.1. *For integer k, If k^2 is even, k is a even number; if k^2 is odd, k is also an odd number*

Now let's back to the proof for the statement $\sqrt{3}$ is irrational.
For this proof, i will use **Proof by Contradiction**

- Assume $\sqrt{3}$ is rational.
- According to definition, there exists a, b that are relative primed. Such that $\sqrt{3} = a/b$
- $a^2/b^2 = 3 \rightarrow a^2 = 3b^2$
- There are two cases. b is even or b is odd. Let's complete the proof by **proof by cases**
- if b is even, $b = 2n$ (n is integer)

$$b = 2n \rightarrow 3b^2 = 12n^2$$
$$3b^2 = 2 * (6n^2)$$

  So $3b^2$is even $\rightarrow a^2$ is even. According to Lemma 0.1, a is even.
  Since **both a and b** are even, they will have at least a common factor 2 , Which Contradicts our original assumption that a and b are relatively primed. This ***Contradiction*** **fails** our first case for b is even. Let's move on to the next case

- When b is odd, $b = 2s + 1$ (s is an integer)

$$3b^2 = 12s^2 + 12s + 3$$
$$= 2 * 6s(s + 1) + 3$$

  Obviously, $3b^2$ is also an odd number (cannot divided by 2) . SO a^2 has to be an odd number. According to **Lemma** 0.1 a has also to be odd

- since a is odd. let $a = 2t+1$, t is an integer

$$a^2 = 4t^2 + 4t + 1 \quad (1)$$
$$a^2 = 3b^2 \quad (2)$$
$$3b^2 = 12s^2 + 12s + 3 \quad (3)$$
$$12s^2 + 12s + 3 = 4t^2 + 4t + 1 \quad \text{(from 1 ,2 ,3)}$$
$$6s^2 + 6s + 1 = 2t^2 + 2t$$
$$2 * 3s(s+1) + 1 = 2 * (t^2 + t)$$

- obviously, left side of the equations is odd while the right side of equation is a even number
- **Contradiction** arises
- **This contradiction** fails case 2 where $b = 2s+1$ (b is odd)
- Thus, after all of this, we proved both case 2 and case 1 are **FALSE** due to **contradictions**
- so The original assumption that $\sqrt{3}$ **is rational** is **FALSE**
- so the negation of the assumption has to be true
- Therefore $\sqrt{3}$ **is irrational**

□

3. Prove, by induction on n, that the following equation holds (as long as a is not exactly 1):

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$$

Solution:

Proof. I will use **Induction** to prove that for a ≠ 1, n ≥ 0 , $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$

- **Base Case** : i = 0

$$i = 0$$
$$LHS = a^i = 1 \rightarrow \sum_{i=0}^{n} a^i = 1$$
$$RHS = \frac{a^{n+1}-1}{a-1} = (a-1)/(a-1) = 1$$
$$1 = 1 \rightarrow \sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \rightarrow LHS = RHS$$

Base Case Proved

- **Inductive Hypothesis**: For any k <n, $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$
- **Goal Statement:** If the inductive hypothesis is true, for k+1, $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$, We will prove **RHS = LHS**
- **Inductive Step** Assume that the Inductive Hypothesis is True, so that: $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$

$$LHS = \sum_{i=0}^{k+1} a^i = \sum_{i=0}^{k} a^i + a^{k+1}$$
$$LHS = \frac{a^{k+1}-1}{a-1} + a^{k+1}$$
$$LHS = \frac{a^{k+1}-1+(a-1)*a^{k+1}}{a-1}$$
$$LHS = \frac{a^{k+2}-1}{a-1} = RHS$$

- We just proved that if inductive hypothesis holds, our **Goal Statement** holds (LHS = RHS)
- Since our base case is also True, According to the **principle of induction**
- The statement that for a ≠ 1, n ≥ 0 , $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ has to be True

□