LOGIC AND DISCRETE MATH 240

PROFESSOR PLESS

Homework 5

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Problems to Turn In:

1. Define the set S as follows:

   a) $1 \in S$

   b) $n \in S \longrightarrow 5n \in S$

   c) $n \in S \longrightarrow n^2 \in S$

   d) no other elements are in S.

   Given these definitions:

   a) What are the 5 smallest elements of this set?

   Solution:

Rank	Value
1	1
2	5
3	25
4	125
5	625

   b) Prove that every element of set S can be written as $4k+1$ for some integer k.

   Solution:

Proof. I will employ **proof by induction** to carry out the proof here
Define **P(n)** as **if any element x is generated by k number of rules (for constructing set S) $\longrightarrow$ x = 4k + 1**

If we can prove the correctness of P(n), we have proved that any element in set S can be written as $4k+1$ for some integer k since all elements in S are constructed through rules listed above (a,b,c) and there is no other element in set s (d).

i. **base case–** P(1):if element x is generated by 1 rule $\longrightarrow x = 4k+1$

   - The only number that use only one rule to construct is gonna be 1,the rule invoked is $1 \in S$ (rule 0.1)
   - $1 = 4k+1$ when k = 0
   - so base case is proved

ii. **Strong Inductive Hypothesis (IH):**for any k<n, Assume $\forall_{0<j<k} P(j) \equiv P(1) \wedge P(2) \wedge P(3) \ldots \wedge P(k)$
   if x is generated by j number of rules, $1 \leq j \leq k, k < n \rightarrow x = 4k+1$ (k is an integer)

iii. **Goal Statement** P(k+1): if any x is generated via k+1 rules, then $x = 4k+1$ (k is an integer)

iv. **inductive step:** Assume $\forall_{1 \leq j \leq k} P(j)$, we try to prove p(k+1)

   - imagine a number A generated by k+1 number of rules, there are two cases for the very last rule applied to construct A (i will use proof by cases to complete the proof)
   - case 1 – $A = 5t$
     - $A = 5t$, where t is an element in Set S. apparently, t is generated via y number of rules where y < k
     - therefore, $t = 4k_1 + 1$ (k_1 is an integer), according to IH
     -
       $$\begin{aligned} A &= 5t \\ &= 20k_1 + 5 \\ &= 20k_1 + 4 + 1 \\ &= 4(5k_1 + 1) + 1 \end{aligned}$$
     - replace $5k_1 + 1$ with another integer k_2, we get $A = 4k_2 + 1$
     - so in this case, we proved that A can be expressed as $4k+1$
   - case 2 – $A = s^2$

- $A = s^2$, where s is an element in Set s, apparently, s is generated via y' number of rules where y' < k
- therefore $s = 4k'_1 + 1$ (k'_1 is an integer), according to IH
-

$$
\begin{aligned}
A &= s^2 \\
&= 16k_1'^2 + 1 + 8k'_1 \\
&= 4(4k_1'^2 + 2k'_1) + 1
\end{aligned}
$$

- replace $(4k_1'^2 + 2k'_1)$ by k'_2
- so in this case we proved that, A can be expressed as $4k + 1$

- for both case 1 and case 2, we proved that A can be expressed as $4k + 1$. That is to say, we proved that for any element x constructed via k+1 rules, when IH holds, x can be expressed as $4k + 1$
- so we proved that $\forall_{1<j\leq k} P(j) \longrightarrow P(k+1)$

v. **Conclusion:**
we have proved

- Base case - P(1) is true
- $\forall_{1<j\leq k} P(j) \longrightarrow P(k+1)$

So we can conclude P(n): **if any element x is generated by k number of rules (for constructing set S) $\longrightarrow$ x = 4k + 1**
Since P(n) is true, we have proved that **any element in set S can be written as** $4k + 1$ **for some integer** k since all elements in S are constructed through rules listed above (a,b,c) and there is no other element in set s (d).

□

2. Define G_i recursively as follows:

$$G_0 = 0 \tag{0.1}$$

$$G_1 = 1 \tag{0.2}$$

$$G_n = G_{n-1} + G_{n-2} \qquad \text{, for } n > 1 \tag{0.3}$$

Prove, for $n > 0$, $\sum_{i=0}^{n} G_i = G_{n+2} - 1$.
Solution:

Proof. I will employ **prove by induction** here to carry out the proof

a) **Base case–P(1):** $\sum_{i=0}^{n} G_i = G_{n+2} - 1$. n = 1

$$\begin{aligned}
LHS &= \sum_{i=0}^{1} G_i \\
&= G_0 + G_1 \\
&= 0 + 1 && \text{(by rule 0.1 and 0.2)} \\
&= 1 \\
RHS &= G_3 - 1 \\
&= G_2 + G_1 - 1 && \text{(by rule 0.3)} \\
&= G_1 + G_0 + G_1 - 1 && \text{(by rule 0.3)} \\
&= 0 + 1 + 1 - 1 && \text{(by rule 0.1 and 0.2)} \\
&= 1 \\
LHS &= RHS
\end{aligned}$$

Therefore Base case is proved

b) **Inductive Hypothesis–P(k):**
for any $k < n$, $\sum_{i=0}^{k} G_i = G_{k+2} - 1$

c) **Goal Statement–P(k+1):**
for k+1, $\sum_{i=0}^{k} +1G_i = G_{k+3} - 1$

d) **Inductive Step:**
We try to prove when **P(k) (IH)** is true P(k+1): $\sum_{i=0}^{k} +1G_i = G_{k+3} - 1$ is TRUE

$$\begin{aligned}
LHS &= \sum_{i=0}^{k+1} G_i \\
&= \sum_{i=0}^{k} G_i + G_{k+1} \\
&= G_{k+2} - 1 + G_{k+1} && \text{(according to IH)} \\
&= G_{k+2} + G_{k+1} - 1 \\
&= G_{k+3} - 1 && \text{(by rule 0.3)} \\
&= RHS
\end{aligned}$$

So we have proved that $P(k) \to P(k+1)$

e) **Conclusion**
We have proved that

- Base case– $P(1)$ is True
- for any k < n, $P(k) \to P(k+1)$
- we can then conclude P(n): for $n > 0$, $\boldsymbol{\sum_{i=0}^{n} G_i = G_{n+2} - 1}$.

□