LOGIC AND DISCRETE MATH 240

PROFESSOR PLESS

Homework 4

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Due September 30, 2014

Practice Problems:

1. Use a loop invariant to prove that the following program is correct with respect to the initial assertion that x is a positive integer and the final assertion that $ans = x^2$.

```
procedure square(x)
  i = 1
  j = 1
  while (i < x) do
    j = j + 2i + 1
    i = i + 1
  od
  return j
```

2. Use the idea of the "fast exponentiation" algorithm discussed in class to make a "faster multiplication" algorithm that returns the product of (x,y). Your program should be asymptotically faster than one that repetively adds x to itself y times. (or y to itself x times). You are not allowed to multiply in your algorithm. You are permitted to divide by two. [as an aside, for binary numbers, this can be implemented as a simple bit shift, so it isn't really cheating].

Problems to turn in:

1. Consider the game of robertNim. This game starts as a single pile of marbles. Each turn, players can take 1, 2, 3 or 5 marbles. (Note, this is not a typo, you can take, one, or two, or three, or five marbles). Whoever takes the last marble loses.

a) Make a chart and list the optimal move for player 1 for starting piles of size 1 through 11.

Solution:

Pile Size	Player 1's strategy to win
1	Loss
2	Take 1 → to put player2 in loss situation
3	Take 2→ to put player2 in loss situation
4	Take 3→ to put player2 in loss situation
5	Loss
6	Take 1 →to put player2 in loss situation
7	Take 2 → to put player2 in loss situation
8	Take 3 → to put player2 in loss situation
9	Take Loss
10	Take 1 →to put player2 in loss situation
11	Take 2 → to put player2 in loss situation
12	Take 3 →to put player2 in loss situation
13	Loss

b) Define a proposition that states when player 1 has an optimal move. This proposition might have a form similar to: $P(n)$: "player 1 can will if the number of marbles in the pile can be expressed as $3n+1$ or $3n+2$."

c) Use induction to prove your proposition correct.

Solution for (b)/(c):

(b):
From what we observe from the chart, we can conclude that Player1 will be able to force a win when the pile size is **NOT** $4n+1$, in other words when the pile size **S** is $\mathbf{4n} \lor \mathbf{4n+2} \lor \mathbf{4n+3}$ (n is an integer)

(c):

Proof. I will use **Induction** to prove that in a game of robertNIM, Player 1 will be able to force a WIN when the pile size **S** is $\mathbf{4n} \lor \mathbf{4n+2} \lor \mathbf{4n+3}$ (n is an integer , $n \geq 0$)

- **Base Case** $P(1)$: n= 0

- Pile size is 2, according to the chart, player1 can take 1 to win
- Pile size is 3, according to the chart, player1 can take 2 to win

- **Strong Inductive Hypothesis**: $k < n, \quad \forall_{0 \le j \le k} \mathrm{P(J)}$

  For any $0 \le j \le k, k < n$ Player 1 will be able to force a WIN when the pile size $\mathbf{S = 4j \vee 4j + 2 \vee 4j + 3}$

- **Goal Statement:** We want to prove If the inductive hypothesis is true, for k+1, player1 will be able to force a win when the pile size $\mathbf{S = 4(k+1) \vee 4(k+1) + 2 \vee 4(k+1) + 3}$

- **Inductive Step** Assume that the Inductive Hypothesis is True, so that: Player 1 will be able to force a WIN when the pile size $\mathbf{S = 4k \vee 4k + 2 \vee 4k + 3}$
  Inside the induction proof, I will use **proof by cases** to complete the proof

  - Case 1:

    when the current pile size $\mathbf{S} = 4(k+1) = 4k+4 \rightarrow$ Player1 take 3 $\rightarrow$ Left Pile Size $S = 4k+1$

    * **Now Player2's turn to play**, He/she will have **4 choices**
    * Player2 **Take 1**

      The left Pile size for Player1 to play will be $S = 4k$, According to our (strong)inductive hypothesis, **Player1 can force a win**

    * Player2 **Take 2**

      The left Pile size for Player1 to play will be $S = 4k-1 = 4(k-1)+3 = 4j+3$, According to our (strong)inductive hypothesis,since $(k-1) < k$, **Player1 can force a win**

    * Player2 **Take 3**

      The left Pile size for Player1 to play will be $S = 4k-2 = 4(k-1)+2 = 4j+2$, According to our (strong)inductive hypothesis,since $j = (k-2) < k$ **Player1 can force a win**

    * Player 2**Take 5**

      The left Pile size for Player1 to play will be $S = 4k-4 = 4(k-1) = 4j$, According to our (strong)inductive hypothesis,since $j = (k-1) < k$, **Player1 can force a win**

    We just proved that, for all four conditions, Player1 will have a wining strategy

  - Case 2:

    When the current pile size $\mathbf{S} = 4(k+1)+2 = 4k+6 \rightarrow$ Player1 take 5 $\rightarrow$ Left Pile size s = 4k +1

    Now it is reduced to the exact same situation as Case 1, with the exact same reasoning, we can prove that **there is a wining strategy for Player1**

- Case 3:

  When the current pile size $\mathbf{S} = 4(k+1)+3 = 4k+7 \rightarrow$ Player1 take 2 $\rightarrow$ left pile size $S = 4(k+1)+1$

  * **Now Player2's turn to play**,facing pile size $= 4k+5$, He/she will have **4 choices**
  * Player2 **Take 1**

    The left Pile size for Player1 to play will be $S = 4k+4 \rightarrow$ This is Case 1 for Player1 where i have proved that there exits a wining strategy
  * Player2 **Take 2**

    The left Pile size for Player1 to play will be $S = 4k+3$, According to our (strong)inductive hypothesis **Player1 can force a win**
  * Player2 **Take 3**

    The left Pile size for Player1 to play will be $S = 4k+2$, According to our (strong)inductive hypothesis **Player1 can force a win**
  * Player2 **Take 5**

    The left Pile size for Player1 to play will be $S = 4k$, According to our (strong)inductive hypothesis **Player1 can force a win**

  We just proved that, for all four conditions, Player1 will have a wining strategy

- Combining the results, we have proved that, for all three cases there will be an wining strategy for Player 1
- That is to say we proved $\forall_{0 \le j \le k}$ P(k) $\rightarrow$ P(k+1) :

  If our strong inductive hypothesis that for any $0 \le j \le k, k < n$ Player 1 will be able to force a WIN when the pile size $\mathbf{S = 4j \vee 4j+2 \vee 4j+3}$ holds
  our goal statement that for k+1, player1 will be able to force a win when the pile size $\mathbf{S = 4(k+1) \vee 4(k+1)+2 \vee 4(k+1)+3}$ has to be TRUE
- Since our base case is also True, According to the **principle of induction**
- P(n): Player1 will be able to force a win when the pile size **S** is $\mathbf{4n \vee 4n+2 \vee 4n+3}$ (n is an integer) IS True

□

2. Consider the following very small (one line!) Hoare triples. For each, say if the triples are valid. Argue for their correctness, including the proof that the final assertion is true and that the program terminates. If the Hoare triple is not correct, state why. You may assume that x is an integer.

a) $\{x > 0\}\ x := x + 1\ \{x > 0\}$
Solution:
It is a Valid Triple

Proof. **The correctness of the triple**

- P–Pre condition for the algorithm, $P: \ x > 0$
- Q–Post condition for algorithm, $Q: \ x > 0$
- p': P' is P with every instance of x replaced with x+1 ≡ $x = x + 1$
- Q': Q' is Q with every instance of x replaced with x+1 ≡ $x + 1 > 0$
- $x > 0 \to x + 1 > 0$: $\longrightarrow P' \to Q'$
- i have shown that, any instance of x, if satisfies pre-condition, after urning code, it will satisfy the post-condition
- Therefore the triple is correct

□

b) $\{x > 0\}\ x := x - 1\ \{x > 0\}$
Solution:
It is NOT a Valid triple

Proof. The triple is not correct
Counter Example, suppose x = 1

- Precondition P is satisefied when x = 1
- $x' = x - 1 = 0$
- $0 \ngtr 1$
- Thus the Post Condition Q is not satisfied → THe triple is not correct

□

c) $\{x < 0\}$
while $x \neq 0$ do
$x := x + 1$
od
$\{x = 0\}$

Solution:
THe triple is Valid

Proof. Prove the correctness using loop invariant through 3 Parts
The **Loop Invariant (LI)** is $\mathbf{x < 0}$

- Part1: $\{P\}\mathrm{S}_0\{LI\}$ ≡ to prove LI is true initially
  - $P: x < 0$, S0 does nothing (no initial instantiations)

- $P : x < 0 \rightarrow x < 1$ (LI)

  So LI is initially True (before enter the loop)

- Part2: $\{LI\} \wedge R \;\; S1\{LI\} \equiv$ LI holds each iteration of the loop
  - $x < 1 \wedge x \neq 0$
  - $x' = x - 1$
  - After the loop
  - $x' < x < 1$(by LI) $\longrightarrow$

  So LI holds after each iteration.

- Part3: $\{LI\} \wedge \neg R \rightarrow \{Q\}$
  - LI: $x < 1$
  - $\neg R \equiv x = 0$
  - Therefore $\{LI\} \wedge \neg R \rightarrow x = 0 \equiv \{Q\}$

  Post-condition can be implied from LI and the negation of loop condition

- Proof for termination
  Loop condition: $R : x \neq 0$ Obviously the loop will terminates since x is increasing by 1 each iteration. Eventually x will hit 0 and terminates the loop

- Combine the proof of partial correctness with proof of termination, we conclude that the triple is VALID

□

d) $\{true\}$
while $x \neq 0$
$x := x + 1$
od
$\{x = 0\}$
Solution:

It's an invalid triple

Proof. **Counterexample**:

- when initial input x = 1, it satisfy the pre-condition
- after it enters the loop, x will increment by 1 and will never hit x=0 $\longrightarrow$ the loop goes on forever, NEVER terminates
- Also, the post-condition condition will never be satisfied

So The triple is Invalid □