NORTHWESTERN UNIVERSITY

On Curve Shortening and Mean Curvature Flows

Mathematics

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ABSTRACT

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This thesis explores the behavior of geometric flows, with a focus on mean curvature flow (MCF) and curve shortening flow (CSF). We begin with an introduction to linear parabolic equations such as the heat equation, elaborate on principles including the maximum principle and the existence and uniqueness of solutions using the heat kernel. We then introduce the second fundamental form and mean curvature, and look into the evolution of hypersurfaces under MCF, explaining the smoothing and area decreasing properties of the flow.

We will study the one-dimensional case of MCF, known as the curve shortening flow, and present crucial theorems such as the Huisken's Monotonicity Theorem and the Avoidance Principle.

We then look at the formation of Type I singularities under parabolically rescaled MCF. We will show that tangent flows at such singularities are self-similar shrinkers.

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CHAPTER 1

Parabolic Partial Differential Equations and Maximum Principle

A second-order linear partial differential operator is given by

$$L(u) = \sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

where the coefficients $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are functions of the independent variables x. The operator L is said to be *elliptic* if the matrix (a_{ij}) is symmetric and there exists a constant $\lambda > 0$ such that:

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n.$$

A partial differential equation is called *parabolic* if it takes the form:

$$\frac{\partial u}{\partial t} = L(u),$$

where L is an elliptic operator.

An example of a linear parabolic PDE is the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u, \quad \text{where } \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

This equation describes the spread of heat in the medium, where $u(x,t)$ is the temperature at point $x \in \mathbb{R}^n$ at time $t \geq 0$.

By the theorem of superpositions, if c_0, c_1 are constants and u_0, u_1 are solutions to the heat equation, then $c_0u_0 + c_1u_1$ is another solution to the heat equation. This is the *linearity* of the heat equation: any linear combination of solutions is also a solution.

Throughout the section, we will consider the periodic solutions of the heat equation. I..e, solutions u that satisfy $u(x + e_i, t) = u(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ where $\{e_i\}_{i=1}^n$ is the standard basis. These solutions can be written as $u : \mathbb{T}^n \times [0, \infty) \to \mathbb{R}$ where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the *torus*.

1.0.1. Maximum Principle

The *maximum principle* is an important property of parabolic PDEs, which state that the maximum value of the solution decreases or stays constant while the minimum value of the solution increases or stays constant.

Theorem 1.1 (Maximum Principle)**.** *Let u satisfy the heat equation:*

$$\frac{\partial u}{\partial t} = \Delta u \quad on \quad \mathbb{T}^n \times [0, T)$$

Then $\max_{\mathbb{T}^n} u(x, t_2) \leq \max_{\mathbb{T}^n} u(x, t_1)$, *and* $\min_{\mathbb{T}^n} u(x, t_2) \geq \min_{\mathbb{T}^n} u(x, t_1)$, *for all* $t_2 \geq t_1$.

Proof. Define the perturbation function $u_\varepsilon(x, t) = u(x, t) - \varepsilon t$, where $\varepsilon > 0$. Suppose u_ε obtains a maximum value at some point $(x_0, t_0) \in \mathbb{T}^n \times (0, T)$. As we are at the maximum, we have:

$$\frac{\partial u_\varepsilon}{\partial t}(x_0, t_0) \geq 0, \quad \Delta u_\varepsilon(x_0, t_0) \leq 0.$$

Substituting u_ε into the heat equation, we get

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial u}{\partial t} - \varepsilon = \Delta u - \varepsilon = \Delta u_\varepsilon - \varepsilon.$$

Hence, $\Delta u_\varepsilon - \varepsilon \geq 0$. But at (x_0, t_0), this implies:

$$0 \geq \Delta u_\varepsilon(x_0, t_0) \geq \varepsilon.$$

Since $\varepsilon > 0$, this is a contradiction. Thus, u_ε cannot obtain its maximum in the interior of $\mathbb{T}^n \times (0, T)$. Hence, the maximum of u_ε must occur at $t = 0$. We get

$$\sup_{\mathbb{T}^n \times [0,T)} u_\varepsilon \leq \max_{\mathbb{T}^n} u(x, 0).$$

Taking $\varepsilon \to 0$, we now obtain

$$\sup_{\mathbb{T}^n} u(x, t) \leq \max_{\mathbb{T}^n} u(x, 0).$$

To extend this to all times $t_1 < t_2$, we shift by time. Define the function

$$v(x, \tau) := u(x, \tau + t_1), \quad \text{for } \tau \in [0, T - t_1).$$

Then v satisfies the heat equation on $\mathbb{T}^n \times [0, T - t_1)$, with $v(x, 0) = u(x, t_1)$. Following the proof above for v, we obtain

$$\sup_{\mathbb{T}^n} v(x, \tau) \leq \max_{\mathbb{T}^n} v(x, 0) = \max_{\mathbb{T}^n} u(x, t_1),$$

for all $\tau \in [0, T - t_1)$. We now see that when $\tau = t_2 - t_1$, we get

$$\max_{\mathbb{T}^n} u(x, t_2) = \sup_{\mathbb{T}^n} v(x, t_2 - t_1) \leq \max_{\mathbb{T}^n} u(x, t_1).$$

Applying the same argument for $-u$, we obtain the inequality for the minimum

$$\min_{\mathbb{T}^n} u(x, t_2) \geq \min_{\mathbb{T}^n} u(x, t_1).$$

□

The *strong maximum principle* strengthens the above result. It states that if a solution is 0 at some (x_0, t_0), then the solution must be 0 at all (x, t).

Now we introduce a maximum principle for subsolutions. This result states that subsolutions which grow at most exponentially in space cannot attain a new maximum at a later time.

Theorem 1.2 (Maximum principle for subsolutions). *Let u be a subsolution to the heat equation. Suppose that*

$$u(x, t) \leq a e^{a|x|^2} \quad \textit{for all} \quad (x, t) \in \mathbb{R}^n \times [0, T]$$

for some $a \leq \infty$. Then

$$\sup_{\mathbb{R}^n \times T} u \leq \sup_{\mathbb{R}^n \times 0} u$$

1.0.2. The Existence and Uniqueness of solution

We can construct solutions to the heat equation on $\mathbb{R}^n$ using the *heat kernel*. We can construct solutions to the heat equation by combining the initial function $\phi(x)$ with the heat kernel.

Theorem 1.3 (Existence and Uniqueness of Solutions to the Heat Equation)**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a bounded and continuous function representing the initial condition. Define:*

$$u(x,t) = \int_{\mathbb{R}^n} k(x,y,t)\varphi(y)\,dy,$$

where the heat kernel $k(x,y,t)$ is given by:

$$k(x,y,t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Then $u(x,t)$ satisfies the heat equation. If $v(x,t)$ is another solution with $\frac{\partial v}{\partial t} = \Delta v$, $\lim_{t\to 0^+} v(x,t) = \varphi(x)$ pointwise, and if $|v(x,t)| \leq ae^{a|x|^2}$ then $v(x,t) \equiv u(x,t)$.

Proof. Substituting $u(x,t)$ into the heat equation, we get

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^n} \frac{\partial k(x,y,t)}{\partial t}\varphi(y)\,dy.$$

Since the heat kernel satisfies $\frac{\partial k}{\partial t} = \Delta_x k$

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^n} \Delta_x k(x,y,t)\varphi(y)\,dy = \Delta_x u(x,t).$$

Hence, $u(x,t)$ satisfies $\frac{\partial u}{\partial t} = \Delta u$.

We also see that as $t \to 0^+$, the heat kernel $k(x, y, t)$ becomes sharply peaked at $y = x$, approximating the Dirac delta function $\delta(x - y)$. Hence,

$$\lim_{t\to 0^+} u(x,t) = \int_{\mathbb{R}^n} \delta(x-y)\varphi(y)\,dy = \varphi(x).$$

Now suppose $v(x, t)$ is another solution with the same initial data and satisfies the exponential growth bound

$$|v(x,t)| \leq ae^{a|x|^2}.$$

Define $w(x, t) = v(x, t) - u(x, t)$. Then by superposition,

$$\frac{\partial w}{\partial t} = \Delta w, \quad w(x,0) = 0.$$

By applying Theorem 1.2, we get

$$\sup_{\mathbb{R}^n \times [0,T)} w(x,t) \leq \sup_{\mathbb{R}^n} w(x,0) = 0.$$

Similarly, applying the same to $-w$, we get $\inf w(x, t) \geq 0$. Thus,

$$w(x,t) \equiv 0, \quad \text{so } v(x,t) = u(x,t).$$

$\square$

CHAPTER 2

Mean Curvature Flow and the Second Fundamental Form

The *mean curvature flow* (MCF) is a geometric evolution equation that describes the flow of a hypersurface in $\mathbb{R}^{n+1}$ in a way that minimizes its area. Mean curvature flow is a generalization to higher dimension of the *curve shortening flow*, which we will be discussing in the next chapter.

Under mean curvature flow, each point of a surface moves in the direction of its inward normal at rate proportional to its *mean curvature*. This allows the surface to smooth out over time.

To understand mean curvature flow, we first introduce *mean curvature* and the *second fundamental form*, which provide the geometric framework.

2.0.1. The Second Fundamental Form

A *hypersurface* in $\mathbb{R}^{n+1}$ is an n-dimensional manifold embedded in $\mathbb{R}^{n+1}$. The curvature of a hypersurface is described using the second fundamental form: a bilinear and symmetric tensor field on our manifold.

Let M^n be an n-dimensional hypersurface smoothly embedded in $\mathbb{R}^{n+1}$. Given two tangent vectors $u, v \in TM$, the second fundamental form II is defined by

$$\mathrm{II}(u, v) = (D_u V)^{\perp}.$$

where D_uV is the Euclidean covariant derivative of a vector field V and *normal projection* $(\cdot)^\perp$ is the component of D_uV that is orthogonal to M.

To show that $\mathrm{II}(u, v)$ is well-defined, we must prove that it does not depend on the choice of extension V of the tangent vector v. Suppose V and V' are two smooth extensions of v, i.e., $V(x) = V'(x) = v$ at some fixed point $x \in M$. Then their difference $W = V - V'$ is a tangent vector field on M that vanishes at x. Taking the Euclidean covariant derivative along u, we obtain

$$D_uV - D_uV' = D_uW.$$

Since W is tangent to M, its Euclidean derivative D_uW is also tangent to M at x, implying that its normal projection vanishes

$$(D_uW)^\perp = 0.$$

Thus, we conclude that

$$(D_uV)^\perp = (D_uV')^\perp,$$

proving that $\mathrm{II}(u, v)$ is independent of the choice of extension V.

Next, we verify bilinearity. Let $a, b \in \mathbb{R}$ be real numbers, and let u_1, u_2, v_1, v_2 be tangent vectors at $x \in M$. The second fundamental form satisfies

$$\mathrm{II}(au_1 + bu_2, v) = (D_{au_1+bu_2}V)^\perp.$$

Using the linearity of the Euclidean covariant derivative,

$$D_{au_1+bu_2}V = aD_{u_1}V + bD_{u_2}V.$$

Applying the normal projection,

$$\mathrm{II}(au_1 + bu_2, v) = a(D_{u_1}V)^{\perp} + b(D_{u_2}V)^{\perp} = a\mathrm{II}(u_1, v) + b\mathrm{II}(u_2, v).$$

Similarly, one can show that $\mathrm{II}(u, av_1 + bv_2) = a\mathrm{II}(u, v_1) + b\mathrm{II}(u, v_2)$, proving bilinearity in the second argument. Thus, II is a well-defined bilinear form.

The second fundamental form describes how the normal vector to the hypersurface changes as one moves along different tangent directions. It also measures the extrinsic curvature of M within $\mathbb{R}^{n+1}$.

2.0.2. Definition of Mean Curvature

The mean curvature H of an n-dimensional hypersurface M^n embedded in $\mathbb{R}^{n+1}$ is defined as the average of the sum trace of the second fundamental form:

$$H = \frac{1}{n} trace(\mathrm{II})$$

Equivalently, the mean curvature can be written as the average of the principal curvatures $\kappa_1, \ldots, \kappa_n$

$$H = \frac{1}{n} \sum_{i=1}^{n} \kappa_i.$$

Intuitively, mean curvature measures how much a hypersurface bends on average at a given point. When $H > 0$, the surface is locally convex and bends inward. When $H < 0$, The surface is locally concave and bends outward. When $H = 0$, the surface is *minimal*, meaning it locally minimizes area, like a soap film.

2.0.3. The Mean Curvature Flow

Mean curvature flow describes the evolution of a hypersurface where each point moves in the direction of the surface normal, with rate proportional to the mean curvature. The flow is given by the PDE

$$\frac{\partial X}{\partial t} = -HN, \tag{2.1}$$

where $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is the evolving hypersurface, N is the unit normal vector to the hypersurface, and H is the mean curvature.

This equation shows how the hypersurface deforms over time, shrinking regions with high curvature. The mean curvature is deeply related to evolution of a hypersurface's area. It controls how the area of the hypersurface changes.

Lemma 5.25 in [1]. Consider a smooth family of immersions $X : M^n \times I \to \mathbb{R}^{n+1}$ evolving according to a velocity field. Let $K \subset M^n$ be a compact subset. The first variation formula for the area of a compact subset is given by

$$\frac{d}{dt} Area(K) = -\int_K \langle \frac{\partial X}{\partial t}, HN \rangle d\mu.$$

This formula states that the rate of change of the area is determined by the inner product of the velocity field with the mean curvature vector .

Under mean curvature flow, the velocity field is chosen as

$$\frac{\partial X}{\partial t} = -HN.$$

Substituting this into the change of area formula we obtain

$$\frac{d}{dt} Area(K) = -\int_K \langle -HN, HN \rangle d\mu.$$

Since the inner product simplifies to

$$\langle -HN, HN \rangle = -H^2,$$

we conclude that

$$\frac{d}{dt}\,\mathrm{Area}(K) = -\int_K H^2 d\mu \le 0.$$

This integral is always non-positive, so the area of the hypersurface is non-increasing under mean curvature flow.

2.0.4. Evolution of Shrinking Sphere

We consider an n-dimensional sphere of radius r_0 centered at the origin in $\mathbb{R}^{n+1}$, which we denote $S^n_{r_0}$.

We will assume the surface remains round as it evolves, since its curvature is the same in all directions. Hence at any time t, the surface remains a round sphere of radius $r(t)$:

$$M_t = S^n_{r(t)}.$$

We parametrize the evolving sphere M_t using the embedding:

$$X_t : S^n_1 \to \mathbb{R}^{n+1},$$

where the unit sphere S_1^n is mapped to the evolving sphere $S_{r(t)}^n$ by:

$$X_t(x) = r(t)x, \quad x \in S_1^n.$$

That is, each point x on the unit sphere is scaled by a time factor $r(t)$, to make sure that $X_t(x)$ always lies on a sphere of radius $r(t)$.

The the norm of the mean curvature vector of an n-dimensional sphere of radius $r(t)$ is

$$H(x,t) = \frac{n}{r(t)}.$$

The outward unit normal to the sphere at any point is simply

$$N(x,t) = x.$$

For the mean curvature flow

$$\frac{\partial X}{\partial t} = -HN.$$

substituting the values of H and N,

$$\frac{\partial X}{\partial t} = -\frac{n}{r(t)}x.$$

Since $X_t(x) = r(t)x$, differentiating both sides with respect to t gives

$$\frac{d}{dt}(r(t)x) = \frac{dr}{dt}(t)x.$$

Hence we get

$$\frac{dr}{dt} = -\frac{n}{r(t)}.$$

$$r(t)\, dr = -ndt.$$

Integrating both sides from $t = 0$ to some $t > 0$, with the initial condition $r(0) = r_0$, we obtain

$$\frac{1}{2}r^2(t) - \frac{1}{2}r_0^2 = -nt.$$

Rearranging,

$$r(t) = \sqrt{r_0^2 - 2nt}.$$

Thus, the evolving sphere follows the explicit solution:

$$M_t = S^n_{\sqrt{r_0^2-2nt}}.$$

From the formula for $r(t)$, we see that the radius decreases as time passes. The radius reaches zero at the time:

$$T = \frac{r_0^2}{2n}.$$

At $t = T$, the hypersurface collapses to a point. Beyond this time, the solution cannot be extended smoothly, as the curvature becomes *singular*. Hence, we see that in the beginning, the shrinkage is slow, but as the radius decreases, the mean curvature increases, accelerating the shrinkage. Eventually, the entire sphere collapses to a point at finite time T.

In particular, a hypersurface satisfying

$$H = \frac{1}{2}\langle X, N\rangle$$

is called a **shrinker**.

2.0.5. Evolution of Mean Curvature

Under mean curvature flow, the mean curvature evolves according to

$$\partial_t H = \Delta H + |\mathbf{II}|^2 H.$$

This equation shows that the mean curvature diffuses like the heat equation while also changing by the squared norm of the second fundamental form.

2.0.6. Avoidance Principle for MCF

Theorem 2.1 (Avoidance Principle)**.** *If two initially disjoint hypersurfaces evolve under mean curvature flow, they remain disjoint for all time.*

This follows from the strong maximum principle and and also shows that the mean curvature flow preserves embeddedness.

CHAPTER 3

Curve Shortening Flow

The *curve shortening flow* (CSF) is a geometric evolution equation that describes the motion of a curve in the plane such that each point moves in the direction of its normal vector scaled by its curvature. It is the one dimensional case of mean curvature flow.

3.0.1. Definition and Basic Properties

Let $X : S^1 \times [0, T) \to \mathbb{R}^2$ be a one-parameter family of smooth, immersed curves evolving over time. The *curve shortening flow* is defined by the equation

$$\frac{\partial X}{\partial t} = \kappa N,$$

where κ is the curvature of the curve, and N is the unit normal vector pointing toward the concave side.

We can also express this equation as a heat equation

$$X_t = X_{ss} \quad \text{where s is the arclength}$$

as $X_s = T$ where T is a unit tangent vector, and by the Frenet equation $T_s = -kN$. However, since the arclength derivative $\frac{d}{ds}$ depends on time, X_{ss} does not depend linearly on X. Hence, unlike the heat equation, a linear combination of solutions to curve shortening flow is not necessarily a solution to a curve shortening flow.

3.0.2. Evolution of Curvature

The curvature κ satisfies

$$\frac{\partial \kappa}{\partial t} = \kappa_{ss} + \kappa^3, \tag{3.1}$$

where κ_{ss} is the second derivative of curvature along the curve. This evolution is heat-equation like, showing that the curvature smoothes out over time.

3.0.3. Evolution of the Enclosed Area

If $A(t)$ denotes the area enclosed by a simple closed curve evolving under CSF, then:

$$\frac{dA}{dt} = -\int_{M^1} \kappa \, ds = -2\pi. \tag{3.2}$$

This shows that the enclosed area decreases monotonically over time.

3.0.4. Evolution of Length under Curve Shortening Flow

Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ be a family of closed planar curves evolving under curve shortening flow:

$$\partial_t \gamma = \kappa N.$$

Let $L(t) = \int_{\Gamma_t} ds$ be the length of the curve at time t, where $ds = |\partial_x \gamma| dx$ is the arc-length element. We now derive the evolution equation for $L(t)$.

Theorem 3.1 (Evolution of Length)**.** *Under curve shortening flow, the length $L(t)$ of the evolving closed curve satisfies:*

$$\frac{d}{dt}L(t) = -\int_{\Gamma_t} \kappa^2 ds.$$

Proof. We start from:

$$\frac{d}{dt}L(t) = \frac{d}{dt}\int_{S^1} |\partial_x \gamma| dx.$$

Differentiate under the integral:

$$\frac{d}{dt}L(t) = \int_{S^1} \left\langle \partial_x \partial_t \gamma, \frac{\partial_x \gamma}{|\partial_x \gamma|} \right\rangle dx = \int_{S^1} \langle \partial_x(\kappa N), T\rangle dx,$$

where $T = \partial_x \gamma / |\partial_x \gamma|$ is the unit tangent. Since $\langle N, T\rangle = 0$ and $\langle \partial_x N, T\rangle = -\kappa \frac{ds}{dx}$, we have:

$$\langle \partial_x(\kappa N), T\rangle = \kappa_x \langle N, T\rangle + \kappa \langle \partial_x N, T\rangle = \kappa \langle \partial_x N, T\rangle = -\kappa^2 \frac{ds}{dx}.$$

Hence:

$$\frac{d}{dt}L(t) = -\int_{S^1} \kappa^2 \frac{ds}{dx} dx = -\int_{\Gamma_t} \kappa^2 ds.$$

□

Although this shows that curve shortening flow efficiently shortens the curve by the curvature. Also note that Length$(\lambda\Gamma) = \lambda$length$(\Gamma)$, so the length evolution is not that useful when analyzing singularities, whenever λ blows up to infinity.

3.0.5. Huisken's Monotonicity Theorem

We now introduce a scale-invariant monotone quantity to resolve this issue. Let $\{\Gamma_t \subset \mathbb{R}^2\}$ be a curve shortening flow, and let $X_0 = (x_0, t_0)$ be a point in space-time.

Consider a 1-dimensional backwards heat kernel

$$\rho_{x_0}(x,t) = (4\pi(t_0 - t))^{-1/2}\, e^{-\frac{|x-x_0|^2}{4(t_0-t)}}, \quad (t < t_0),$$

The backwards heat kernel is the regular heat kernel centered at a future point (x_0, t_0) on earlier times $t < t_0$.

Theorem 3.2 (Huisken's Monotonicity Formula)**.**

$$\frac{d}{dt}\int_{\Gamma_t} \rho_{x_0}\, ds = -\int_{\Gamma_t} \left|\kappa + \frac{\langle \gamma, N\rangle}{2(t_0 - t)}\right|^2 \rho_{x_0}\, ds \quad (t < t_0).$$

Proof. Following the derivation in [5], assume first that the space-time center is at $X_0 = (0,0)$. Define

$$\rho_0(x,t) = \frac{1}{\sqrt{4\pi(-t)}} e^{-\frac{|x|^2}{4(-t)}}$$

We want to compute $\left(\frac{d}{dt} + \partial_s^2\right)\rho_0$.

Note that:

$$\partial_s \rho_0 = \nabla\rho_0 - \langle \nabla\rho_0, N\rangle N = \nabla_T \rho_0 = \langle \nabla\rho_0, T\rangle.$$

Then:

$$\partial_s^2 \rho_0 = \nabla_T(\nabla_T \rho_0) = \langle \nabla_T \nabla \rho_0, T\rangle + \langle \nabla\rho_0, \nabla_T T\rangle.$$

Since γ is parametrized by arclength, $\nabla_T T = \kappa N$,

$$\partial_s^2 \rho_0 = \langle \nabla_T \nabla \rho_0, T \rangle + \kappa \langle \nabla \rho_0, N \rangle.$$

For the time derivative along the flow

$$\frac{d}{dt}\rho_0(\gamma(x,t),t) = \partial_t \rho_0 + \langle \nabla \rho_0, \partial_t \gamma \rangle = \partial_t \rho_0 + \kappa \langle \nabla \rho_0, N \rangle.$$

Therefore,

$$\left(\frac{d}{dt} + \partial_s^2 \right) \rho_0 = \partial_t \rho_0 + \langle \nabla_T \nabla \rho_0, T \rangle + 2\kappa \langle \nabla \rho_0, N \rangle.$$

Note we see that for ρ_0,

$$\nabla \rho_0 = -\frac{x}{2t}\rho_0, \quad \Rightarrow \quad \langle \nabla \rho_0, N \rangle = -\frac{\langle x, N \rangle}{2t}\rho_0.$$

Using this in our formula, we get

$$\left(\frac{d}{dt} + \partial_s^2 \right) \rho_0 = -\left| \kappa + \frac{\langle x, N \rangle}{2t} \right|^2 \rho_0 + \kappa^2 \rho_0.$$

Therefore,

$$\left(\frac{d}{dt} + \partial_s^2 - \kappa^2 \right) \rho_0 = -\left| \kappa + \frac{\langle x, N \rangle}{2t} \right|^2 \rho_0.$$

Using the evolution of arc length element $\frac{d}{dt}ds = -\kappa^2 ds$, we differentiate the integral:

$$\frac{d}{dt}\int_{\Gamma_t} \rho_0 \, ds = \int_{\Gamma_t} \left(\frac{\partial \rho_0}{\partial t} - \kappa^2 \rho_0 \right) ds.$$

Substituting from above:

$$= \int_{\Gamma_t} \left(-\partial_s^2 \rho_0 - \left| \kappa + \frac{\langle x, N \rangle}{2t} \right|^2 \rho_0 \right) ds.$$

Now, since Γ_t is a closed curve,

$$\int_{\Gamma_t} \partial_s^2 \rho_0 \, ds = 0.$$

Hence, we conclude

$$\frac{d}{dt} \int_{\Gamma_t} \rho_0 \, ds = - \int_{\Gamma_t} \left| \kappa + \frac{\langle x, N \rangle}{2t} \right|^2 \rho_0 \, ds.$$

This completes the proof in the case $X_0 = (0, 0)$.

By parabolic rescaling invariance below, the formula holds for all (x_0, t_0). $\square$

3.0.6. Invariance under Parabolic Rescaling

Let $\{\Gamma_t\}_{t<T} \subset \mathbb{R}^2$ be a family of curves in plane evolving under curve shortening flow. Fix a point in space-time $(x_0, T) \in \mathbb{R}^2 \times \mathbb{R}$. Define the backwards heat kernel centered at (x_0, T) by

$$\rho_{X_0}(x, t) = \frac{1}{\sqrt{4\pi(T - t)}} \exp\left(- \frac{|x - x_0|^2}{4(T - t)} \right), \quad t < T.$$

Huisken's monotonicity formula states

$$\frac{d}{dt} \int_{\Gamma_t} \rho_{X_0}(x, t) \, ds = - \int_{\Gamma_t} \left(\kappa + \frac{\langle \gamma - x_0, N \rangle}{2(T - t)} \right)^2 \rho_{X_0}(x, t) \, ds.$$

Now define the *parabolically rescaled flow* Γ_t^λ centered at (x_0, T) by

$$\Gamma_t^\lambda := \lambda \cdot \left(\Gamma_{T + \lambda^{-2} t} - x_0 \right), \quad t \in [-\lambda^2 T, 0).$$

We now show that the monotonicity formula is invariant under the parabolic rescaling. Let $\gamma(x,t) \in \Gamma_t$ be a parametrization of the flow. Under parabolic rescaling, define

$$\gamma^\lambda(x,t) := \lambda \cdot \left(\gamma(x, T + \lambda^{-2}t) - x_0\right).$$

We then see that the arclength becomes

$$ds^\lambda = \left|\partial_x \gamma^\lambda(x,t)\right| dx = \left|\lambda \cdot \partial_x \gamma(x, T+\lambda^{-2}t)\right| dx = \lambda \cdot \left|\partial_x \gamma(x, T+\lambda^{-2}t)\right| dx = \lambda \cdot ds$$

And the curvature becomes

$$\kappa^\lambda = \frac{dT}{ds^\lambda} = \frac{dT}{\lambda\, ds} = \frac{1}{\lambda}\frac{dT}{ds} = \frac{1}{\lambda}\kappa$$

By change of variables, let $y = \lambda x + x_0$

$$|y - x_0|^2 = \lambda^2 |x|^2, \quad T - (T + \lambda^{-2}t) = -\lambda^{-2}t.$$

Therefore, the heat kernel is

$$\rho_{X_0}(y, T+\lambda^{-2}t) = \frac{1}{\sqrt{4\pi(-\lambda^{-2}t)}} \exp\left(-\frac{\lambda^2|x|^2}{4(-\lambda^{-2}t)}\right) = \lambda \cdot \frac{1}{\sqrt{4\pi(-t)}} \exp\left(-\frac{|x|^2}{4(-t)}\right) = \lambda \cdot \rho_0(x,t),$$

where $\rho_0(x,t)$ denotes the heat kernel centered at the origin in space-time.

Using the change of variables above, we compute the rescaled quantity:

$$\int_{\Gamma_{T+\lambda^{-2}t}} \rho_{X_0}(x,t)\, ds = \int_{\Gamma_t^\lambda} \rho_0(x,t)\, ds^\lambda.$$

Differentiating both sides and applying Huisken's monotonicity formula in the rescaled coordinates centered at the origin, we obtain

$$\frac{d}{dt}\int_{\Gamma_t^\lambda} \rho_0(x,t)\, ds^\lambda = -\int_{\Gamma_t^\lambda} \left(\kappa^\lambda + \frac{\langle \gamma^\lambda, N^\lambda\rangle}{2(-t)}\right)^2 \rho_0(x,t)\, ds^\lambda.$$

This has the same form as the original formula, now centered at the origin in rescaled coordinates. Thus, Huisken's monotonicity formula is invariant under the parabolic rescaling.

3.0.7. Characterization of Shrinkers using Huisken's Monotonicity

We see that the Huisken's monotonicity formula is always non-positive and strictly decreasing unless the integrand vanishes. In particular,

$$\frac{d}{dt}\int_{\Gamma_t} \rho_{X_0}(x,t)\, ds = 0,$$

if and only if

$$\kappa + \frac{\langle \gamma - x_0, N\rangle}{2(t_0 - t)} = 0$$

which imply

$$\kappa N = -\frac{(\gamma - x_0)^\perp}{2(t_0 - t)}.$$

This is the condition for **self-similarly shrinking solutions**, or shrinkers, as curve is evolving by shrinking toward the spacetime point (x_0, t_0), with each point flowing inward along its normal direction. Hence, the equality case in Huisken's monotonicity formula characterizes shrinkers.

3.0.8. The Avoidance Principle

The avoidance principle is an important result in curve shortening flow. If there are two curves that are initially disjoint evolving under the curve shortening flow, the curves remain disjoint for all times.

Theorem 3.3 (Avoidance Principle)**.** *Let $X_i : M_i^1 \times [0,T) \to \mathbb{R}^2$ for $i = 1, 2$ be solutions to the curve shortening flow. If $X_1(M_1^1, 0) \cap X_2(M_2^1, 0) = \emptyset$, then $X_1(M_1^1, t) \cap X_2(M_2^1, t) = \emptyset$ for all $t \in [0,T)$.*

Proof. Following the proof in [1], we show a stronger result. The minimum distance between two curves evolving under the curve shortening flow is non-decreasing in time. We will use the maximum principle and analyze the evolution of the distance function between the curves.

Define the distance function $d : M_1^1 \times M_2^1 \times [0,T) \to \mathbb{R}$ as

$$d(x, y, t) = \|X_2(y, t) - X_1(x, t)\|,$$

where $x \in M_1^1$ and $y \in M_2^1$. At the initial time $t = 0$, the minimum distance between the two curves is

$$d_0 = \inf_{(x,y) \in M_1^1 \times M_2^1} d(x, y, 0) > 0.$$

Since M_1^1 and M_2^1 are compact, by extreme value theorem such infimum exists, and d_0 is positive as the curves are initially disjoint.

To prove that the minimum distance does not decrease, we proceed with contradiction. Suppose that $d(x, y, t)$ decreases at some time t. We will show this results in a contradiction by looking at the points where $d(x, y, t)$ is minimum.

Let $t_0 \in (0, T)$ be the first time such that the minimum distance $d(x, y, t)$ decreases below d_0. Then, there exists a pair of points $(x_0, y_0) \in M_1^1 \times M_2^1$ such that

$$d(x_0, y_0, t_0) = \inf_{(x,y) \in M_1^1 \times M_2^1} d(x, y, t_0).$$

At this point (x_0, y_0, t_0), the spatial derivatives of d is zero, and the second derivative matrix is nonnegative definite by the definition of being minimum.

At the point of minimum distance, let

$$w = \frac{X_2(y_0, t_0) - X_1(x_0, t_0)}{\|X_2(y_0, t_0) - X_1(x_0, t_0)\|}$$

be the unit vector pointing from $X_1(x_0, t_0)$ to $X_2(y_0, t_0)$. The tangent vectors to the curves at x_0 and y_0 are denoted by T_1 and T_2, respectively, while the normal vectors are N_1 and N_2. At the point of minimum distance, we have:

$$w \cdot T_1 = 0, \quad w \cdot T_2 = 0,$$

indicating that w is orthogonal to both tangent vectors. By choosing appropriate parametrizations of M_1^1 and M_2^1, we can make sure that the normal vectors N_1 and N_2 align with w.

The second derivatives of d with respect to x and y at the point (x_0, y_0, t_0) are given by

$$\frac{\partial^2 d}{\partial x^2} = \frac{\langle T_1 - \langle w, T_1 \rangle w, T_1 \rangle}{|X_1 - X_2|} - \langle w, \kappa_1 N_1 \rangle$$

$$\frac{\partial^2 d}{\partial y^2} = \frac{\langle T_2 - \langle w, T_2 \rangle w, T_2 \rangle}{|X_1 - X_2|} + \langle w, \kappa_2 N_2 \rangle$$

$$\frac{\partial^2 d}{\partial x \partial y} = -\frac{\langle T_1 - \langle w, T_1 \rangle w, T_2 \rangle}{|X_1 - X_2|}.$$

where κ_1, κ_2 are the curvatures at x_0, y_0, respectively. However, we see that w is orthogonal to T_1, T_2 and alligns with N, so we have

$$\frac{\partial^2 d}{\partial x^2} = \frac{1}{d} - \kappa_1, \quad \frac{\partial^2 d}{\partial y^2} = \frac{1}{d} + \kappa_2,$$

$$\frac{\partial^2 d}{\partial x \partial y} = -\frac{1}{d}.$$

From the non-negativity of the Hessian matrix, we have

$$\kappa_2 - \kappa_1 \geq 0.$$

Taking the derivative with respect to time and applying the chain rule,

$$\frac{d}{dt} d(x, y, t) = \frac{1}{|X_2 - X_1|} \left\langle X_2 - X_1, \frac{d}{dt}(X_2 - X_1) \right\rangle.$$

Since the curves evolve under the curve shortening flow, their velocity at each point is

$$\frac{\partial X_1}{\partial t} = \kappa_1 N_1, \quad \frac{\partial X_2}{\partial t} = \kappa_2 N_2.$$

Thus, the time derivative of the vector $X_2 - X_1$ is

$$\frac{d}{dt}(X_2 - X_1) = \kappa_2 N_2 - \kappa_1 N_1.$$

We project this equation onto the unit vector w, which points from X_1 to X_2 and is given by:

$$w = \frac{X_2 - X_1}{|X_2 - X_1|}.$$

Taking the inner product with w, we get

$$\frac{d}{dt} d(x, y, t) = \langle w, \kappa_2 N_2 - \kappa_1 N_1 \rangle.$$

Hence the time derivative of $d(x, y, t)$ is

$$\frac{\partial d}{\partial t} = -\kappa_1 (w \cdot N_1) + \kappa_2 (w \cdot N_2).$$

Since w alligns with N_1 and N_2, we get

$$\frac{\partial d}{\partial t} = \kappa_2 - \kappa_1.$$

From the earlier inequality $\kappa_2 - \kappa_1 \geq 0$, we conclude

$$\frac{\partial d}{\partial t} \geq 0.$$

This contradicts the assumption that $d(x, y, t)$ decreases at t_0.

Since $d(x, y, t)$ cannot decrease, the minimum distance between the two curves remains non-decreasing in time. Therefore, if $X_1(M_1^1, 0) \cap X_2(M_2^1, 0) = \emptyset$, the two curves remain disjoint for all $t \in [0, T)$. $\square$

If a smooth closed curve evolves under the curve shortening flow, it can only exist for a finite amount of time before collapsing to a point. This behavior occurs because the curve is always contained within a shrinking circle that encloses it. Initially, the curve lies inside a ball of radius r_0. As time progresses, the radius of this enclosing circle decreases according to the formula $r(t) = \sqrt{r_0^2 - 2t}$, where $r(t)$ is the radius at time t. Eventually, at the time $T = r_0^2/2$, the radius of the circle becomes zero, and the curve collapses entirely. This result follows from the avoidance principle, which ensures that the curve remains inside the enclosing circle for all times $t > 0$. Thus, the curve shortening flow demonstrates that any closed, compact curve will shrink to a single point within a finite time.

The *Gage-Hamilton Theorem* is a crucial result in the curve shortening flow. It states that if an initially convex embedded curve evolves under curve shortening flow, it remains convex for all time and asymptotically becomes circular before collapsing to a point.

Theorem 3.4 (Gage-Hamilton Theorem)**.** *If M is a convex curve embedded in the plane R^2, the curve shortening flow shrinks M to a point. The curve remains convex and becomes circular as it shrinks, meaning the ratio of scribed and circumscribed radius approaches 1, ratio of maximum and minimum curvature approaches 1, and the higher order derivative of curvature converge to 0 uniformly.*

CHAPTER 4

Tangent Flows are Shrinkers

In this section, we prove the result that any tangent flow arising from a Type I singularity of mean curvature flow is a *shrinker*. This means that the blowup limit under parabolic rescalings satisfies the self-similar shrinking equation.

4.0.1. Parabolic Rescaling

Let $L_t \subset \mathbb{R}^{n+1}$ be a family of smooth hypersurfaces evolving under mean curvature flow:

$$\partial_t X = -HN,$$

where H is the mean curvature and N the inward pointing unit normal vector. Suppose that the flow develops a singularity at point $(0,0) \in \mathbb{R}^{n+1} \times \mathbb{R}$. Without loss of generality, we assume the singularity occurs at time $t = 0$ and location $X_0 = 0$, i.e.,

$$\limsup_{t \nearrow 0} \sup_{x \in L_t} |A(x,t)| = \infty.$$

The assumption that the singularity occurs at $(0,0)$ is without loss of generality. If the singularity happens at (x_0, t_0), we apply spacetime translation to define a new flow $\widetilde{L}_t := L_{t+t_0} - x_0$, which centers the singularity at $(0,0)$.

We assume the singularity is of *Type I*, which means the second fundamental form is controlled as

$$|A(x,t)| \le \frac{C}{\sqrt{-t}} \quad \text{for all } t \in [-T,0),$$

for some $T > 0$ and constant $C > 0$.

To analyze the behavior near the singularity, we consider the parabolically rescaled flows

$$L_s^k := \lambda_k L_{\lambda_k^{-2}s}, \quad s < 0, \quad \lambda_k \to \infty.$$

This is the same type of rescaling used in Huisken's monotonicity formula, where the flow is centered at a point (x_0, t_0) and the rescaled flow is given by

$$\Gamma_t^\lambda = \lambda \left(\Gamma_{t_0+\lambda^{-2}t} - x_0 \right).$$

In our setting, the center is $(0,0)$, so the translation term disappears and get

$$L_s^k = \lambda_k L_{\lambda_k^{-2}s}.$$

The surfaces L_s^k still evolve by mean curvature flow as we are just rescaling in time and space. The Type I assumption implies that the second fundamental forms of the rescaled flows satisfy

$$|A^k(x,s)| \le \frac{C}{\sqrt{-s}} \quad \text{for all } s \in [-2,-1/4].$$

This follows from the rescaling formula:

$$|A^k(x,s)| = \lambda_k^{-1}|A(x',t')|, \quad \text{where } x' = \lambda_k^{-1}x,\ t' = \lambda_k^{-2}s.$$

Using $|A(x',t')| \leq \frac{C}{\sqrt{-t'}} = \frac{C}{\sqrt{-\lambda_k^{-2}s}} = \frac{C\lambda_k}{\sqrt{-s}}$, we deduce:

$$|A^k(x,s)| \leq \frac{C}{\sqrt{-s}}.$$

This uniform bound in k over a time interval $s \in [-2,-1/4]$ is important for showing parabolic regularity.

4.0.2. Parabolic Regularity

We will now show that the higher derivatives of the second fundamental form is uniformly bounded.

Theorem 4.1 (Parabolic Regularity)**.** *Let M^n be a compact manifold and $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ a solution to mean curvature flow. Suppose $T < \infty$ and $C_0 := \sup_{M\times[0,T)} |\boldsymbol{I\!I}| < \infty$. Then for each $m \in \mathbb{N}$, there exists a constant $C_m < \infty$, depending only on n, m, T, C_0, and $\sup_{M\times\{0\}} |\nabla^\ell \boldsymbol{I\!I}|$ for $\ell = 1, \ldots, m$, such that:*

$$\sup_{M\times[0,T)} |\nabla^m \boldsymbol{I\!I}| \leq C_m.$$

Applying this to the rescaled flows L^k_s, we get uniform bounds

$$|\nabla^m A^k(x,s)| \leq C_m \quad \text{for all } s \in [-1,-1/2],\ m \in \mathbb{N}.$$

4.0.3. Local Graph Representation

Fix $s = -1$. By the uniform curvature bounds, each point $p \in L^k_{-1}$ lies in a neighborhood where L^k_{-1} can be written as the graph of a smooth function over its tangent space. In other words, there exists $r > 0$ such that

$$L_{-1}^k \cap B_r(p) = \{(x, u^k(x)) : x \in B_r \subset \mathbb{R}^n\}.$$

Here, $u^k(x)$ denotes the local graph representation of the hypersurface L_{-1}^k near point p.

Since L_{-1}^k is compact, it can be covered by finitely many such coordinate patches $B_r(p_i)$, $i = 1, \ldots, N$. Let u_i^k denote the graph functions in each patch.

From the parabolic regularity bounds, we know that if the second fundamental form $|A^k(x,s)|$ is uniformly bounded on a time interval, then all its higher covariant derivatives $\nabla^m A^k(x,s)$ are also uniformly bounded for every $m \in \mathbb{N}$. Applying this to the rescaled flow L_s^k, which satisfies the Type I curvature condition on the larger interval $s \in [-2, -1/4]$, we obtain uniform bounds on all derivatives of curvature over the slightly smaller interval $s \in [-1, -1/2]$. This implies that in each local coordinate patch, the associated graph functions u_i^k are uniformly bounded in C^m for every $m \in \mathbb{N}$. Therefore, for each fixed patch i, we may apply the Arzela-Ascoli theorem to extract a subsequence $u_i^{k_j} \to u_i^\infty$ converging in C^∞ on compact subsets.

Now, we apply a further diagonal argument to obtain smooth convergence across all patches simultaneously. That is, for each index $i \in \mathbb{N}$, let $\{k_j^{(i)}\}$ be a subsequence such that $u_i^{k_j^{(i)}} \to u_i^\infty$ in C^∞ on patch i. Then define a diagonal subsequence $\{k_j\}$ where we take the jth entry of the jth subsequence.

$$k_j := k_j^{(j)}.$$

This ensures that for each fixed patch i, there exists a large enough j such that $k_j \geq k_i^{(i)}$, and thus $u_i^{k_j} \to u_i^\infty$ in C^∞. Then, the sequence $\{L_s^{k_j}\}$ converges smoothly on all patches simultaneously.

As the charts cover the compact surface L_{-1}^k, the convergence of the graphs implies:

$$L_{-1}^{k_j} \to L_{-1}^\infty \quad \text{in } C_{\text{loc}}^\infty.$$

Repeating the argument for $s = -1/2$, the same diagonalization yields smooth convergence. That is, $L_s^{k_j} \to L_s^\infty$ smoothly on every compact subset of spacetime $\mathbb{R}^{n+1} \times (-\infty, 0)$. Moreover, the limit L_s^∞ still satisfies the mean curvature flow.

4.0.4. Application of Huisken's Monotonicity Formula

To identify the limiting flow as a *shrinker*, we now apply Huisken's monotonicity formula:

$$\frac{d}{dt} \int_{L_t} \rho_{0,0}(x,t)\, d\mu = -\int_{L_t} \left| H + \frac{x^\perp}{2t} \right|^2 \rho_{0,0}(x,t)\, d\mu,$$

where $\rho_{0,0}(x,t) = (4\pi|t|)^{-n/2} e^{-\frac{|x|^2}{4|t|}}$ is the backward heat kernel centered at $(0,0)$.

Because L_t evolves by MCF, this quantity is monotone decreasing in t. We first define

$$\Phi_k(s) := \int_{L_s^k} \rho_{0,0}(x,s)\, d\mu.$$

Then for any $s_1 < s_2 < 0$,

$$\Phi_k(s_1) \geq \Phi_k(s_2).$$

By smooth convergence $L_s^k \to L_s^\infty$, we obtain

$$\lim_{k\to\infty} \Phi_k(s) = \int_{L_s^\infty} \rho_{0,0}(x,s)\, d\mu.$$

The difference between time -1 and -1/2 is

$$\Phi_k(-1) - \Phi_k(-1/2) = \int_{L_{-1}^k} \rho_{0,0} - \int_{L_{-1/2}^k} \rho_{0,0}.$$

This corresponds to:

$$\int_{L_{-\lambda_k^{-2}}} \rho_{0,0}(x, -\lambda_k^{-2})\, d\mu - \int_{L_{-\frac{1}{2}\lambda_k^{-2}}} \rho_{0,0}(x, -\frac{1}{2}\lambda_k^{-2})\, d\mu.$$

Since this quantity is nonnegative by monotonicity and the sum over k of these differences is bounded above by $\int_{L_{-\lambda_1^{-2}}} \rho_{0,0}$, the series

$$\sum_{k=1}^{\infty} (\Phi_k(-1) - \Phi_k(-1/2))$$

is a sum of nonnegative terms bounded above. Hence, we conclude

$$\lim_{k\to\infty} (\Phi_k(-1) - \Phi_k(-1/2)) = 0.$$

This implies

$$\int_{L_{-1}^\infty} \rho_{0,0} = \int_{L_{-1/2}^\infty} \rho_{0,0}.$$

But from Huisken's formula,

$$\int_{-1}^{-1/2} \int_{L_s^\infty} \left| H + \frac{x^\perp}{2s} \right|^2 \rho_{0,0}\, ds = 0,$$

which implies:

$$H = -\frac{x^{\perp}}{2s} \quad \text{on } L_s^{\infty} \text{ for } s \in [-1, -1/2].$$

Since the above argument can be repeated for any closed interval in $(-\infty, 0)$, it follows that L_s^{∞} satisfies the self-shrinker equation for all $s < 0$.

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