Chapter - 3

STRAIGHT LINE

Definition

The intersection of two planes gives a straight line. Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be two planes, then the intersection of these planes gives the equations of straight line.

Note: The x–axis is formed by the intersection of xy and zx planes whose equations are $z = 0$ and $y = 0$. So equation of x–axis are $y = 0$, $z = 0$.

Similarly, the equations of the y-axis are $x = 0$, $z = 0$ and the equation of z-axis are $x = 0$, $y = 0$.

Equation of a line through (x_1, y_1, z_1) having direction ratios a, b, c

Consider a point P(x, y, z) any where on the line through $A(x_1, y_1, z_1)$. Then dr's of AP are $x-x_1$, $y - y_1$, $z - z_1$ which must be proportional to a, b, c.

$$\therefore \quad \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$, which are the equations of the line in symmetrical form.

Note: To find equations of a line one needs a point on the line and its dr's.

Equation of the line joining two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

The direction ratios of the lines joining A and B are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

Therefore using previous result, the equations of the line joining A and B, are $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$.

Example

Obtain the equation of the line passing through the point (1, 2, 1) and having dr's 2, 1, –2.

Solution: Here $(x_1, y_1, z_1) = (1, 2, 1)$ and dr's a, b, c are 2, 1, –2. Hence equations of line is $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-1}{-2}$ ...(i)

Note: For any point on the line, we let $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{-2}=r$, which gives

$x = 2r + 1$, $y = r + 2$, and $z = -2r + 1$.

So, ($x = 2r + 1$, $y = r + 2$, $z = -2r + 1$) give different points on the line (i) for different values of r.

For examples, put $r = 0$, then (1, 2, 1) is a given point on the line. For $r = 1$, (3, 3, −1) is a point on the line and similarly other.

Example

Find the point where the line $\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z+3}{4}$ meets the plane $2x+4y-z+1 = 0$.

Solution

Here, the given equation of the line is

$$\frac{x-1}{2}=\frac{y+2}{-3}=\frac{x+3}{4}=r \text{............(i) (say)}$$

And equation of the plane is

$$2x+4y-z+1 = 0 \text{..................(ii)}$$

Then any point on the line (i) is $(2r+1, -3r-2, 4r-3)$. As the line (i) meets the plane (ii),

$$\therefore \quad 2(2r+1)+4(-3r-2)-(4r-3)+1 = 0$$

$$\text{or} \quad 4r+2-12r-8-4r+3+1 = 0$$

$$\text{or} \quad -12r-2 = 0$$

$$\therefore \quad r = -1/6$$

Substituting the value for r, we get

The required point is $\left(\frac{2}{3}, \frac{-3}{2}, \frac{-11}{3}\right)$.

Example

Find the distance from (3, 4, 5) to the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meet the plane $x + y + z = 2$.

Solution:

Let M be the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meet the plane. The co-ordinate of M will be of the form $(r + 3, 2r + 4, 2r + 5)$. Since M also lies on $x + y + z = 2$. So, $r + 3 + 2r + 4 + 2r + 5 = 2$

$$\Rightarrow r = -2$$

$\therefore$ $M = (-2+3, -4+4, -4+5)$

$= (1, 0, 1)$

$\therefore$ Required distance between (3, 4, 5) and (1, 0, 1) is

$$= \sqrt{(3-1)^2 + (4-0)^2 + (5-1)^2} = 6$$

Example

Find the length of perpendicular for $A(x_1, y_1, z_1)$ upon the plane $ax + by + cz + d = 0$

Solution:

Let M be the foot of perpendicular from A on the given plane. Since AM is normal to $ax + by + cz + d = 0$ ... (i)

So, drs of AM are a, b, c. Therefore equations of AM are $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

For some r, $(ra + x_1, rb + y, rc + z_1)$ will be co-ordinates of M which also lies on equation (i).

Hence,

$$a(ra + x_1) + b(rb + y_1) + c(rc + z_1) + d = 0$$

$$r = -\frac{ax_1 + by_1 + cz_1 + d}{(a^2 + b^2 + c^2)}$$

$$\text{Length AM} = \sqrt{(ra)^2 + (rb)^2 + (rc)^2} = \pm r\sqrt{a^2 + b^2 + c^2}$$

$$= \pm\left\{-\left(\frac{ax_1 + by_1 + cz_1 + d}{(a^2 + b^2 + c^2)}\right)\right\}\sqrt{a^2 + b^2 + c^2}$$

$$= \pm\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

$\therefore$ Length of perpendicular from A(x, y, z) upon the plane $ax + by + cz + d = 0$ is

$$|AM| = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| \quad \text{.........} \ (I)$$

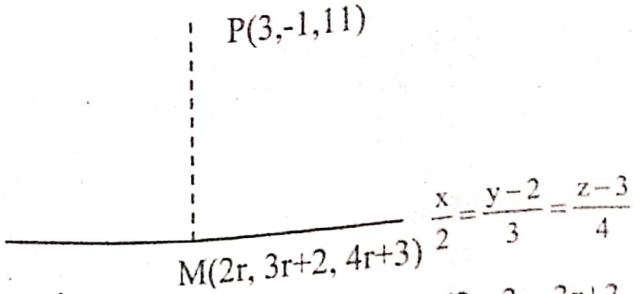
Example

Find the length of perpendicular from the point (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Also, obtain the equation of the perpendicular line.

Solution

Let P be at (3, –1, 11) and M be the foot perpendicular on the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

For some r, M will be at (2r, 3r + 2, 4r + 3).



Here direction ratios of PM are (2r–3, 3r+3, 4r–8). As PM perpendicular to the given line so

$$2(2r-3) + 3(3r+3) + 4(4r-8) = 0$$

or $$29r = 29$$

$$r = 1.$$

Hence M is at M(2, 5, 7). The required length of perpendicular to the line is

$$PM = \sqrt{(2-3)^2 + (5+1)^2 + (7-11)^2} = \sqrt{53}.$$

Direction ratio of PM are –1, 6, –4.

So equations of PM are $\frac{x-3}{-1} = \frac{y+1}{6} = \frac{z-11}{-4}$

or $$\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$$

which is the required equations of the perpendicular the line.

Exercise 3.1

1. Find the value of k, such that the lines $\frac{x-1}{2} = \frac{y-3}{4k} = \frac{z}{2}$ and $\frac{x-2}{2k} = \frac{y-1}{3} = \frac{z-1}{4}$ are perpendicular.

2. Find the distance of the point (1,–3,5) from the plane 3x–2y+6z = 15 along a line with direction cosines proportional to (2, 1, –2).

3. Find the points in which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$ cuts the surface $11x^2 - 5y^2 + z^2 = 0$.

4. Find the point where the line joining (1, –3, 4), (9, 3, –1) cuts the plane x–y+2z = 3.

5. Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x-y+z=5$.

6. Find the two points on the line $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2}$ either side of (2, –3, –5) and at a distance 3 from it.

7. Find the distance of the point (1, –2, 3) from the plane $x-y+z=5$ measured parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

8. Find the equation to the line passing through (–1, –2, –3) and the perpendicular to each of the lines

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ and } \frac{x+2}{4} = \frac{y+3}{5} = \frac{z+4}{6}.$$

9. Show that the equation of the perpendicular from the point (1,6,3) to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is $\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$ and the foot of perpendicular is (1, 3, 5) and the length of the perpendicular is $\sqrt{13}$.

10. Find the equation of the line through (–1, 3, 2) and perpendicular to the plane $x + 2y + 2z = 3$, the length of the perpendicular and the co-ordinate of its foot.

11. Find the image of the point P(1, 3, 4) in the plane $2x - y + z + 3 = 0$.

Answers

1. $-1/2$
2. 9
3. (1, 2, 3), (2, –3, 1)
4. $(10, \frac{15}{4}, -\frac{13}{8})$
5. 13
6. (3, –1, –3), (1, – 5, –7)
7. 1.
8. $\frac{x+1}{1} = \frac{y+2}{-2} = \frac{z+3}{1}$
10. $2, \left(\frac{-5}{3}, \frac{5}{3}, \frac{2}{3}\right), \frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2}$
11. (–3, 5, 2)

Transformation of Equation of line from general form to Symmetrical form

Let $a_1x+b_1y+c_1z+d_1 = 0$.............. (i)

$a_2x+b_2y+c_2z+d_2 = 0$..............(ii)

be a equation of line in general form. Put z = 0, then the equation of line becomes,

$$a_1x+b_1y+d_1=0$$
$$a_2x+b_2y+d_2=0$$

Solving, by rule of cross multiplication, we get

$$\frac{x}{b_1d_2-b_2d_1}=\frac{y}{a_2d_1-a_1d_2}=\frac{1}{(a_1b_2-a_2b_1)}$$

$\therefore$ The point $\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1},\frac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1},0\right)$ lies on the given line.

Let l, m, n be direction cosine of the line. Since this line is obtai[ned] by the intersection of (i) and (ii), then this line is perpendicular to the nor[mals] to the planes (i) and (ii), so,

$$a_1l+b_1m+c_1n=0$$
$$a_2l+b_2m+c_2n=0$$

Solving, we get

or $$\frac{l}{(b_1c_2-b_2c_1)}=\frac{m}{(a_2c_1-a_1c_2)}=\frac{n}{(a_1b_2-a_2b_1)}=k\ (\text{say})$$

Then equation of lines in symmetrical form are

$$\frac{x-\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\frac{a_2d_1-a_1d_2}{a_1b_2-a_1c_2}}{a_2c_1-a_1c_2}=\frac{z-0}{a_1b_2-a_2b_1}.$$

Note: To find a point on the line, we may give any suitable value of one the variables x, y or z and find the corresponding values of the oth[er] two.

Example

Reduce the equation of line $x+y+z+1=0$, $4x+y-2z+2=0$ symmetrical form.

Solution

The equations of the line in general form are

$$x+y+z+1=0,\ 4x+y-2z+2=0$$

Putting $z=0$, we get

$$x+y+1=0,\ 4x+y+2=0$$

On solving, we get $x=-(1/3)$, $y=-(2/3)$

Thus a point on the line is $\left(\frac{-1}{3},\frac{-2}{3},0\right)$; Let l, m, n be directio[n] cosines of the line. Then

$$l.1+m.1+n.1=0 \text{ and } 4.l+m.1+n(-2)=0$$

$$\frac{l}{-2-1} = \frac{m}{4+2} = \frac{n}{1-4} = r\ (\text{say})$$

Solving,

$$l = 3r, \quad m = -6r, \; n = 3r$$

or ∴ The required equation of line in symmetrical form is

$$\frac{x+{}^1/_3}{3r} = \frac{y+{}^2/_3}{-6r} = \frac{z-0}{3r}$$

$$\therefore \quad \frac{x+{}^1/_3}{1} = \frac{y+{}^2/_3}{-2} = \frac{z}{1}.$$

Example

Find the angle between the lines $3x+2y+z-5 = 0 = x+y+2z-3$ and $2x-y-z = 0 = 7x+10y-8z$.

Solution

The given lines are

$3x+2y+z-5 = 0 = x+y+2z-3$..................(1)

and $2x-y-z = 0 = 7x+10y-8z$......................(2)

Let l_1, m_1, n_1 be direction cosines of (1). Then

$3l_1+2m_1+n_1 = 0$ and $l_1+m_1+2n_1 = 0$

or $$\frac{l_1}{4-1} = \frac{m_1}{1-6} = \frac{n_1}{3-2}$$

or $$\frac{l_1}{3} = \frac{m_1}{-5} = \frac{n_1}{1}$$

Also, let l_2, m_2, n_2 be direction cosines of (2), then

$2l_2-m_2-n_2 = 0$ and $7l_2+10m_2-8n_2 = 0$

or $$\frac{l_2}{8+10} = \frac{m_2}{-7+16} = \frac{n_2}{20+7}$$

∴ $$\frac{l_2}{2} = \frac{m_2}{1} = \frac{n_2}{3}$$

If θ be angle between (1) and (2) then

$$\cos\theta = \frac{3(2)+(-5).1+1(3)}{\sqrt{(3)^2+(-5)^2+(1)^2}\sqrt{(2)^2+(1)^2+(3)^2}}$$

$$= \frac{4}{\sqrt{9+25+1}\sqrt{4+1+9}}$$

$$\therefore \cos\theta = \frac{4}{\sqrt{35}\sqrt{14}} = \frac{4}{\sqrt{7\times5\times2\times7}} = \frac{4}{7\sqrt{10}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{4}{7\sqrt{10}}\right).$$

Exercise 3.2

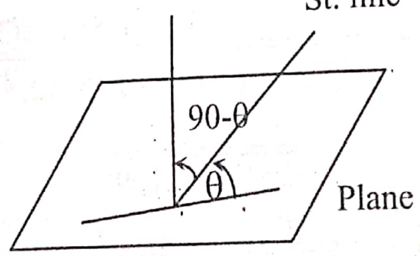
1. Change the equation $x+y+z+1=0=4x+y-2z+2$ in symmetrical form.
2. Find the equation of the plane through $(-1, 1, -1)$ and perpendicular to the line $x-2y+z=4, 4x+3y-z+4=0$.
3. Find the equation of the line through $(2, 3, 4)$ parallel to the line $x-2y+z=4, 4x+3y-z+4=0$.
4. Find the angle between the lines in which the plane $x-y+z=5$ is cut by the planes $2x+y-z=3$ and $2x+2y+3z-1=0$.
5. Prove that the lines $x=ay+b, z=cy+d$ and $x=a^1y+b^1, z=a^1y+d^1$ are perpendicular if $aa^1+cc^1+1=0$
6. Prove that the lines $x=-2y+7, z=3y+10$ and $x=5y-1, z=3y-6$ are perpendicular to each other.
7. Find the co-ordinate of the foot of the perpendicular from the origin on the straight line given by $x+2y+3z+4=0, x+y+z+1=0$.

Answers

1. $\frac{x+1/3}{1}=\frac{y+2/3}{-2}=\frac{z}{1}$
2. $x-5y-11z=5$
3. $\frac{x-2}{-1}=\frac{y-3}{5}=\frac{z-4}{11}$
4. $\cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$
7. $\left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$.

Angle between a line and a plane

Let $ax+by+cz+d=0$ be a plane and $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be a line. If θ be the angle between them, then the angle between the given line and the line normal to the given plane having direction ratios a, b and c is $90-\theta$.



Then, $\cos(90°-\theta)=\frac{al+bm+cn}{\sqrt{\Sigma a^2}\sqrt{\Sigma l^2}}$

$$\therefore \sin\theta = \frac{al + bm + cn}{\sqrt{\Sigma a^2}\sqrt{\Sigma l^2}}.$$

Note: (1) The given line is parallel to the given plane if $\theta = 0$, i.e., if $al + bm + cn = 0$.

(2) If the line is perpendicular to the plane, then it will be parallel to the normal of the plane. Then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

Condition for the line to lie on the plane

Let $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ .... (1) be a line

and $ax+by+cz+d = 0$.....................(2) be a plane.

If the line (1) lies on the plane (2) then every point on line lies in the plane. Any point on the line is $(lr+x_1, mr+y_1, nr+z_1)$ for all values of r. If it lies on the plane, then

$$a(lr+x_1) + b(mr+y_1) + c(nr+z_1) + d = 0$$

or $$(al+bm+cn)r+(ax_1+by_1+cz_1+d) = 0$$

This equation should be satisfied by every value of r and therefore it must be an identity.

$\therefore$ The coefficient r = 0 and the absolute term = 0

$\therefore$ $al+bm+cn = 0$ and $ax_1+by_1+cz_1+d = 0$

These are the required conditions for a line to lie in a plane. That is

Note: (1) The line is parallel to the plane.

(2) The point (x_1, y_1, z_1) of the line lies on the plane.

Plane containing a line

(1) Let the equation of the line be $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ ....(1)

Let $ax+by+cz+d = 0$ .... (2) be an equation of a plane containing (1), then $ax_1+by_1+cz_1+d = 0$..........(3) and $al + bm + cn = 0$...............(4)

From (3) and (2), we get, $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

which is the required equation of the plane containing the line (1), where $al+bm+cn = 0$.

(2) If the equation of line is $a_1x+b_1y+c_1z+d_1 = 0 = a_2x+b_2y+c_2z+d_2$. Then the equation of the plane containing this line is

$$a_1x+b_1y+c_1z+d_1+k\,(a_2x+b_2y+c_2z+d_2)=0$$

where k is an arbitrary constant. For different k, we get different planes containing the given line.

Example

Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $3x+y+z=7$.

Solution

The given equation of line is $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ ....(1)

and plane is $3x+y+z=7$..................(2)

If θ be the angle between them, then

$$\sin\theta=\frac{2.3+3.1+6.1}{\sqrt{2^2+3^2+6^2}\sqrt{3^2+1^2+1^2}}$$

or $$\sin\theta=\frac{15}{7\sqrt{11}}$$

$\therefore$ $$\theta=\sin^{-1}\left(\frac{15}{7\sqrt{11}}\right).$$

Example

Show that the line $2x+2y-z-6=0=2x+3y-z-8$ is parallel to the plane $y=0$ and find the co-ordinates of the points where it meets $x=0$ plane.

Solution

The given line is $2x+2y-z-6=0=2x+3y-z-8$............(1)

Let a, b, c be direction ratios of (1). Then

$2a+2b-c=0$ and $2a+3b-c=0$

or $$\frac{a}{-2+3}=\frac{b}{-2+2}=\frac{c}{6-4}$$

or $$\frac{a}{1}=\frac{b}{0}=\frac{c}{2}$$

Thus the direction ratios of the line are 1, 0, 2. The direction ratio of the line normal to $y=0$ plane are (0, 1, 0).

Here $1.0+0.1+2.0=0$, so the given line is parallel to $y=0$ plane.

Also we have to find the co-ordinates of the point where the line meets $x=0$ plane. Put $x=0$ in $2x+2y-z-6=0$ $2x+3y-z-8$, we get

$2y-z-6=0,\ 3y-z-8=0$

On solving, we get $y=2$, $z=-2$. Thus the required point is $(0,2,-2)$.

Example

Find the equation of the plane through the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ parallel to the line $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Solution

The equation of the plane through the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ .....(1)
is $A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0$....(2) such that $Al + Bm + Cn = 0$ ...... (3)

Since this plane is parallel to the line $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ .... (4)

The normal of (2) is perpendicular to the line (4). So,

$Aa + Bb + Cc = 0$ ....(5)

From (3) and (5)

$$\frac{A}{mc-bn} = \frac{B}{an-lc} = \frac{C}{lb-ma}$$

Putting the proportional values of A, B, C in equation (2) we get,

$$(mc-bn)(x-\alpha) + (an-lc)(y-\beta) + (lb-ma)(z-\gamma) = 0$$

which is the required equation of the plane.

Example

Find the equation of the plane through (α, β, γ) and the line $x = py+q = rz+s$.

Solution

The equation of plane through (α, β, γ) is

$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0$.......(1)

This plane contains the line $x = py+q = rz+s$.

This line can be written in the form,

$$\frac{x}{1} = \frac{y+q/p}{1/p} = \frac{z+\frac{s}{r}}{1/r} \text{ ....(2)}$$

This gives this line passes through $(0, -q/p, -s/r)$ and lies on (i) so

$$a(-\alpha) + b\left(\frac{-q}{p} - \beta\right) + c\left(\frac{-s}{r} - \gamma\right) = 0 \text{ .... (3)}$$

$$\text{and } a.1 + b.\frac{1}{p} + c.\frac{1}{r} = 0 \text{.................(4)}$$

Eliminating a, b, c from (1), (3) and (4) we get,

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ -\alpha & \frac{-q}{p}-\beta & \frac{-s}{r}-\gamma \\ 1 & \frac{1}{p} & \frac{1}{r} \end{vmatrix} = 0$$

$\therefore \begin{vmatrix} x-\alpha & yp-\beta p & zr-\gamma r \\ \alpha & q+\beta p & s+r\gamma \\ 1 & 1 & 1 \end{vmatrix} = 0$, is the required equation of plane.

Exercise 3.3

1. Find the value of k such that the line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x-3y+z=3$.
2. Find the equation of the plane parallel to the line $x-2 = \frac{y-1}{3} = \frac{z-3}{2}$ containing (0, 0, 0) and (–3, 1, 2).
3. Find the equation of a plane containing the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x+2y+z=12$.
4. Find the equation of the plane through (2,–3, 1) normal to the line joining (3, 4,–1) and (2, –1, 5).
5. Find the equations of the line through (2, –1, –1) is parallel to the plane $4x+y+z+2 = 0$ and is perpendicular to the line $2x+y=0=x-z+5$
6. Find the equation of the straight line lying in the plane $x-2y+4z-51=0$ and intersecting the straight line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-6}{7}$ at right angles.
7. Find the equation of the line through (α, β, γ) parallel to the planes $lx+my+nz=p$, $l_1x+m_1y+n_1z=p_1$.
8. Find the equations to the planes through the line $2x+3y-5z-4=0=3x-4y+5z-6$, parallel to the co-ordinate axes.
9. Find the equation of the plane through the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ parallel to y-axis.
10. Find the equation of the plane through the points (2, 2, 1), (1,–2, 3) and parallel to the joining the points (2, 1, –3), (–1, 5, –8).

Answers

1. 11
2. $2x-4y+5z=0$
3. $9x-2y-5z+4=0$
4. $x+5y-6z+19=0$
5. $2-x=y+1=\frac{z+1}{3}$
6. $\frac{x-5}{-6}=\frac{y-3}{1}=\frac{z-13}{2}$
7. $\frac{x-\alpha}{mn_1-m_1n}=\frac{y-\beta}{nl_1-n_1l}=\frac{z-\gamma}{lm_1-l_1m}$
8. $17y-25z=0,\ 17x-5z-34=0,\ 5x-y-10=0.$
9. $2x-z+1=0$
10. $12x-11y-16z+14=0$

Coplanar lines

Two lines are said to be coplanar if there exists a plane containing them.

Find the condition that the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ....(1)

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ ....(2) may lie in a plane.

Solution

The plane containing (1) is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$...............(3)

such that $al_1+bm_1+cn_1=0$.....................(4)

If (1) and (2) are coplanar, then the point (x_2, y_2, z_2) lies in the plane (3). Then

$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$................(5)

and $al_2+bm_2+cn_2=0$............................(6)

Eliminating a, b, c from then (4), (5), (6)

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

which is the required condition for lines to be coplanar. If the condition is satisfied, the equation of plane containing (1) and (2) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\text{or.} \quad \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Note: 1. If two lines are not in symmetrical form, then we change general forms to symmetrical form and use above formulae.

2. If the lines intersect each other, then they will be coplanar.

Example

Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and

$x = \frac{y-7}{-3} = \frac{z+7}{2}$ are coplanar. Also, find the equation of the plane containing them.

Solution

The lines are

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} \quad \text{.........................(1)}$$

and $$x = \frac{y-7}{-3} = \frac{z+7}{2} \quad \text{.........................(2)}$$

If (1) and (2) are coplanar, then

$$\begin{vmatrix} 0+1 & 7-3 & -7+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} 1 & 4 & -5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or $1(4+3) - 4(-6-1) - 5(9-2) = 0$

or $7+28-35 = 0$

or $0 = 0$. Thus the given lines are coplanar.

The equation of plane containing the lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or $(x+1)(4+3) - (y-3)(-6-1) + (z+2)(9-2) = 0$

or $7(x+1) + 7(y-3) + 7(z+2) = 0$

or $x+y+z+1-3+2 = 0$

$\therefore x+y+z = 0.$

Example

Show that the lines $x+y+z-3=0=2x+3y+4z-5$ and $4x-y+5z-7=0=2x-5y-z-3$ are coplanar. Find the plane in which they lie.

Solution

The equations of lines are

$$\left.\begin{array}{l} x+y+z-3=0 \\ 2x+3y+4z-5=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\left.\begin{array}{l} 4x-y+5z-7=0 \\ 2x-5y-z-3=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let a, b, c be direction ratios of the line(1), then

$a+b+c=0$ and $2a+3b+4c=0$

or $\quad \dfrac{a}{4-3}=\dfrac{b}{2-4}=\dfrac{c}{3-2} \Rightarrow \dfrac{a}{1}=\dfrac{b}{-2}=\dfrac{c}{1}$

Thus, we get direction ratios of line (1) as $1, -2, 1$.

$a=1, b=-2, c=1.$

To find a point on the line (1) put $z=0$, then

$x+y=3$ and $\quad 2x+3y=5 \quad \therefore x=4, y=-1.$

$\therefore \quad (4, -1, 0)$ is a point on (1)

Thus the equation of line through $(4, -1, 0)$ having direction ratios $1, -2, 1$ are $\dfrac{x-4}{1}=\dfrac{y+1}{-2}=\dfrac{z-0}{1}$ ....(3)

Any plane containing the second line is

$(4x-y+5z-7)+k(2x-5y-z-3)=0$

or $\quad (4+2k)x+(-1-5k)y+(5-k)z-7-3k=0$..........(4)

If (1) is parallel to (4), then,

$(4+2k)1+(-1-5k)(-2)+(5-k)1=0$

$\Rightarrow \quad k=-1.$

Putting $k=-1$ in (4), we get

$2x+4y+6z-4=0$

i.e. $x+2y+3z-2=0$...(5)

which is the equation of plane containing (2) and parallel to (1). The point $(4,-1,0)$ on (1), satisfies (5), because

$4+2(-1)+3.0-2=0$

or $\quad 4-2+0-2=0$

$\therefore \quad 0=0$. Hence the line (1) also lies on (5)

Thus (1) and (2) are coplanar and the required equation of plane containing (1) and (2) is

$x+2y+3z-2=0.$

Exercise 3.4

1. Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Also obtain the equation of the plane containing them.

2. Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = y-4 = \frac{z-5}{3}$ are coplanar. Find their common point and the equation of the plane in which they lie.

3. Show that the lines $x-1 = 2y-4 = 3z$ and $3x-5 = 4y-9 = 3z$ meet in a point and the equation of the plane in which they lie is
$$3x-8y+3z+13 = 0$$

4. Show that the lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{1-z}{2}$ and $3x-2y+z+5 = 0 = 2x+3y+4z-4$ are coplanar. Find their point of intersection and the plane in which they lie. .

5. Prove that the lines $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3}$ and $x+2y+3z-8 = 0 = 2x+3y+4z-11$ are coplanar and find the point of intersection and the equation of the plane containing them.

6. Find the equation of the plane containing
$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{3} \text{ and } \frac{x-2}{3} = \frac{y-1}{4} = \frac{z-4}{5}.$$

7. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x-3y+1 = 0 = 5x-3z+2$ are coplanar. Also find their point of çontact.

Answers

1. $x-2y+z = 0$
2. $(1, 3, 2)$, $17x-47y-24z+172 = 0$
4. $(2, 4, -3)$, $45x-17y+25z+53 = 0$
5. $(0, 1, 2)$, $4x+y-2z+3 = 0$
6. $2x+y-2z+3 = 0$
7. $(-1, -1, -1)$

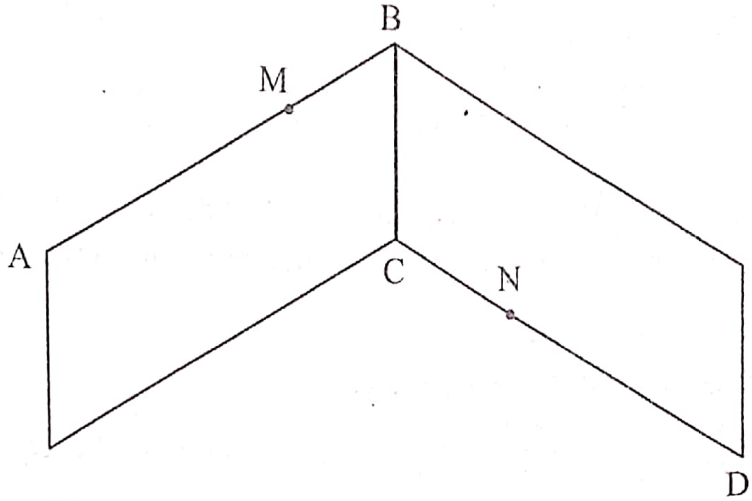
The shortest distance

When two straight lines neither intersect nor parallel, are said to be skew lines. The perpendicular distance between two skew lines is said to be the shortest distance and is denoted by S.D.

Find the shortest distance and equation of shortest distance between two given lines

Let the equation of skew lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \;\ldots\ldots\ldots\ldots(1) \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \;\ldots\ldots\ldots\ldots(2)$$



Let M and N be at (x_1, y_1, z_1) and (x_2, y_2, z_2) of the points of (1) and (2) respectively. Let BC be the shortest distance between AB and CD. Let a, b, c be the direction ratios of the line of shortest distance. Since BC is perpendicular to AB and CD, so

Then, $al_1+bm_1+cn_1 = 0$ and $al_2+bm_2+cn_2 = 0$

$$\therefore \quad \frac{a}{m_1n_2-m_2n_1}=\frac{b}{l_2n_1-l_1n_2}=\frac{c}{l_1m_2-l_2m_1}.$$

l, m, n are dc's of S.D.

Thus, the direction cosines of S.D. are

$$\frac{m_1n_2-m_2n_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}, \frac{n_1l_2-l_1n_2}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}, \frac{l_1m_2-l_2m_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

i.e. l, m, n (say)

Here, $\quad BC =$ Projection of MN on BC

$\quad\quad = (x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)n$

where l, m, n are the direction cosines of BC.

$\therefore$ The length of shortest distance

$$=\frac{(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-l_1n_2)+(z_2-z_1)(l_1m_2-l_2m_1)}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

$$=\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & m_2 \end{vmatrix} \div \sqrt{\Sigma\,(m_1n_2-m_2n_1)^2}$$

The equation of the plane containing line AB and the line of shortest distance BC is.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \text{.............}(3)$$

And the equation of plane containing CD and the line of shortest distance BC is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \text{ ....}(4)$$

The equations (3) and (4) give the equations of the line of shortest distance. In (3) and (4) one can replace l, m, n by direction ratios of shortest distance.

Example

Find the shortest distance between

$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$. Find also the equations of shortest distance.

Solution

We have equation of skew lines are

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \text{ ....}(1)$$

and $$\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2} \text{ ....}(2)$$

Let a, b, c be direction ratios of the line of shortest distance between (1) and (2), then

$$2a-3b+c = 0, \; 3a-5b+2c = 0$$

or $$\frac{a}{-6+5} = \frac{b}{3-4} = \frac{c}{-10+9} \qquad \therefore \frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

Then direction cosines of the line of shortest distance are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

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Then, length of shortest distance

$$= (2-0)\frac{1}{\sqrt{3}} + (1-0)\frac{1}{\sqrt{3}} + (-2-0)\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Hence the required length of shortest distance $= \frac{1}{\sqrt{3}}$.

And equation of the line of shortest distance are

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-2 & z+3 \\ 1 & -3 & 2 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}$$

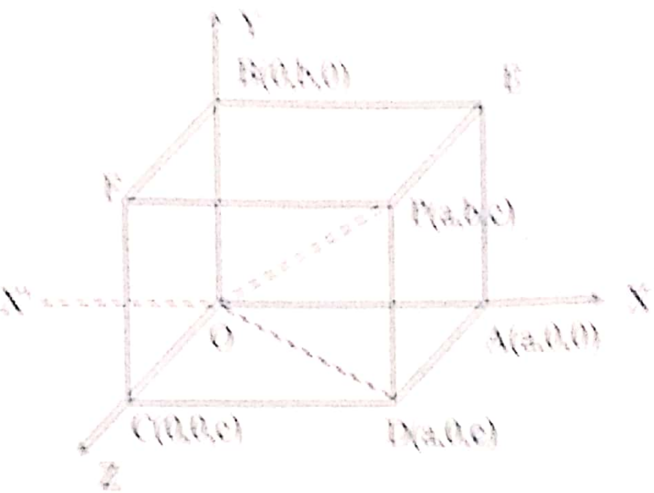
$4x+y-5z = 0 = 7x+y-8z-31.$

Example

Prove that the shortest distance between the diagonal of a rectangular parallelopiped and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}$, $\frac{ca}{\sqrt{c^2+a^2}}$, $\frac{ab}{\sqrt{a^2+b^2}}$ where a, b, c are the lengths of the edges.

Solution

We place the rectangular parallelopiped in such a way that one corners coincide with origin and three of its sides lies on the axes. From figure A (a, 0, 0), B (0, b, 0), C(0, 0, c), P(a, b, c), O(0, 0, 0).



From figure OP is a diagonal and AD is the edge which does not meet OP.

Equation of OP is

$$\frac{x-0}{a-0} = \frac{y-0}{b-0} = \frac{z-0}{c-0} \qquad \text{or} \qquad \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

And equation of AD is $\frac{x-a}{a-a} = \frac{y-0}{0-0} = \frac{z-0}{0-c}$

i.e., $\frac{x-a}{0} = \frac{y}{0} = \frac{z}{-c}$

Let A, B, C be direction ratios of shortest distance between OP and AD, then

$$Aa+Bb+Cc = 0$$
$$A0+B0+C(-c) = 0$$

By cross multiplication,

$$\therefore \frac{A}{-bc} = \frac{B}{ac} = \frac{C}{0}$$

$\Rightarrow$ Direction ratios of shortest distance are $-bc, ac, 0$.

$\therefore$ The direction cosines of shortest distance are $\left(\frac{-b}{\sqrt{b^2+a^2}}, \frac{a}{\sqrt{a^2+b^2}}, 0\right)$

Length of shortest distance = Projection of OD on S.D.

$$= (a-0)\frac{-b}{\sqrt{b^2+a^2}} + (0-0)\frac{a}{\sqrt{a^2+b^2}} + (0-0).0 = \frac{-ab}{\sqrt{a^2+b^2}}$$

Taking positive sign, the required shortest distance between the diagonal and the edge not meeting it which is parallel to z-axis is $\frac{ab}{\sqrt{a^2+b^2}}$.

Similarly, the shortest distances to the edge parallel to x and y-axes are $\frac{bc}{\sqrt{b^2+c^2}}$ and $\frac{ac}{\sqrt{a^2+c^2}}$ respectively.

Example

Find the shortest distance between axis of z and the line $ax + by + cz + d = 0, \quad a'x + b'y + c'z + d' = 0$.

Solution

We know, the equations of z-axis,

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1} \quad \text{.........} (1)$$

and given line

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \quad \text{.........} (2)$$

Any plane containing line (2) is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0 \quad \text{......} (3)$$

Let it is parallel to the line (1), then

$$(a + ka')\,0 + (b + kb')0 + (c + kc')1 = 0$$

$$\Rightarrow \quad k = -\frac{c}{c'}$$

Putting the value of k in equation (3), we get

$$ax + by + cz + d - \frac{c}{c'}(a'x + b'y + c'z + d') = 0$$

$\Rightarrow$ $(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0$ ...... (4)

Here equation (4) is the plane through equation (2) and parallel to equations of line (1).

Thus the required shortest distance = the perpendicular distance from any point of (1) to the plane (4)

= perpendicular distance from (0, 0, 0) to the plane (4)

$$\therefore \quad S.D. = \pm \frac{0+0+dc'-d'c}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}$$

$$S.D. = \pm \frac{dc'-d'c}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}.$$

Exercise 3.5

1. Find the magnitude and equation of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

2. Show that the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is $3\sqrt{30}$. Find out the equation of the line of shortest distance.

3. Find the shortest distance between the lines $x = y+4 = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y}{2} = z$.

4. Find the shortest distance between the lines $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5}$. Find also the equation of shortest distance.

5. Find the magnitude and equation of the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{1-y}{5} = \frac{z+2}{2}$.

6. Find the length and equations of the shortest distance between the lines $x-y+z = 0 = 2x-3y+4z$ & $x+y+2z-3 = 0 = 2x+3y+3z-4$.

7. Show that the shortest distance between the y-axis and the line $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ is $x=0$ $z=0$

$$\pm \frac{bd'-b'd}{\sqrt{(ba'-b'a)^2+(bc'-b'c)^2}}$$

Answers

1. $14,\ 117x+4y-41z-490 = 0 = 9x-4y-z-14$
2. $4x-5y-17z+79 = 0 = 22x-5y+19z-83$
3. $9/\sqrt{10}$
4. $14, \dfrac{x-5}{2} = \dfrac{y-7}{3} = \dfrac{z-3}{6}$.
5. $\dfrac{1}{\sqrt{3}}$, $4x+y-5z = 0 = 7x+y-8z-31$.
6. $13/\sqrt{66}$, $3x-y-z = 0 = x+2y+z-1$.