Optimization approach for efficient frontier

1 Problem formulation

For the classic efficient frontier problem, we have a universe of assets, each with an expected return and standard deviation, and covariance between them. We wish to create a portfolio that minimizes the variance for a specified portfolio return.

This can be formulated as follows:

$$\begin{aligned}
\min \quad & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} && (1)\\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 && (2)\\
& \sum_{i=1}^{n} x_i r_i = r_p && (3)\\
& x_i \geq 0 \quad \forall i && (4)
\end{aligned}$$

where x_i is the weight of the ith asset in the portfolio of n assets, σ_{ij} is the covariance between the ith and jth asset, r_i is the return of the ith asset, and r_p is the portfolio return.

Eq. 1 specifies to minimize one half the variance (which is equivalent to minimizing the variance and mathematically more convenient), and Eqs. 2–4 require the weights sum to 1, the portfolio return is a specified value, and there are no short sales.

It is convenient to write these in matrix form:

$$\begin{aligned}
\min \quad & \frac{1}{2}x^T \sigma x && (5)\\
\text{s.t.} \quad & 1^T x = 1 && (6)\\
& r^T x = r_p && (7)\\
& x \geq 0 && (8)
\end{aligned}$$

where x and r are vectors of the weights and average returns respectively, and σ is the covariance matrix.

In standard form, we define the objective function $f(x)$, equality constraints $g(x) = 0$, and inequality constraints $h(x) \leq 0$.

$$\begin{aligned}
f(x) &\equiv \frac{1}{2}x^T \sigma x && (9)\\
g(x) &\equiv Ax - b = 0 && (10)\\
h(x) &\equiv -x \leq 0 && (11)
\end{aligned}$$

Here, $Ax - b$ are Eqs. 6–7 in matrix form.

2 Method of Lagrange multipliers

We form the Lagrangian

$$\mathcal{L} = \frac{1}{2}x^T \sigma x + \lambda^T(Ax - b) + \mu^T(-x) \tag{12}$$

where λ is the vector of Lagrange multipliers associated with the equality constraints (of length 2), and μ is the vector of Lagrange multipliers associated with the inequality constraints (of length n).

While the objective function has a global minimum, which we seek, the Lagrangian is a saddle shape, and the optimal solution is a saddle point of the Lagrangian. We need to simultaneously minimize with respect to x and maximize with respect to λ, μ.

3 Unrestricted case

To solve this problem, consider first the case where we do not have the inequality constraint, *i.e.*, we allow short selling.

The gradients with respect to x and λ are, respectively:

$$\nabla_x \mathcal{L} = \sigma x + A^T \lambda \tag{13}$$

$$\nabla_\lambda \mathcal{L} = Ax - b \tag{14}$$

The solution is optimal when these gradients are 0, so we can write these as a matrix equation:

$$\begin{bmatrix} \sigma & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \tag{15}$$

For the case where short selling is allowed, these equations can be solved directly, as we have $n+2$ equations and $n+2$ unknowns. However, introducing the nonnegativity constraint on the weights increases the number of unknowns by n without n additional equality constraints. Therefore, we will give two suggestions for solving the unrestricted case (short selling allowed).

1. Directly solve matrix equation 15 with a linear algebra package.

2. Numerically solve Eqs. 13–14, by performing gradient descent in the x directions and gradient ascent in the λ directions. This is known as the primal-dual gradient method.

Strategy 2) can be described as

Initialize x and step sizes η_1,η_2
for $t = 0, 1, ...$ **do**

$$\begin{aligned} x_{t+1} &= x_t - \eta_1 \nabla_x \mathcal{L}(x_t, \lambda_t) \\ &= x_t - \eta_1(\sigma x_t + A^T \lambda_t) \\ \lambda_{t+1} &= \lambda_t + \eta_2 \nabla_\lambda \mathcal{L}(x_t, \lambda_t) \\ &= \lambda_t + \eta_2(Ax_t - b) \end{aligned}$$

end for

and we end when the gradient or change in x is sufficiently small.

4 Restricted case

When when include the constraint in Eq. 11, we can use the the Karush-Kuhn-Tucker (KKT) conditions, an extension of the method of Lagrange multipliers to find the equations that will optimize the problem.

$$\sigma x + A^T \lambda - 1^T \mu = 0 \tag{16}$$
$$Ax = b \tag{17}$$
$$-x \leq 0 \tag{18}$$
$$\mu \geq 0 \tag{19}$$
$$-\mu^T x = 0 \tag{20}$$

However, these are difficult to solve exactly, as we now have $2n + 2$ unknowns and only $n + 2$ linear equations. Therefore, we suggest adding a projection step to the primal-dual gradient method, so that the algorithm becomes:

Initialize x and step sizes η_1,η_2
for $t = 0, 1, ...$ **do**
  $x_{t+1} = \big(x_t - \eta_1(\sigma x_t + A^T \lambda_t)\big)_+$
  $\lambda_{t+1} = \lambda_t + \eta_2(Ax_t - b)$
end for

The easiest way to do this projection is: after stepping x, set any $x < 0$ equal to 0;