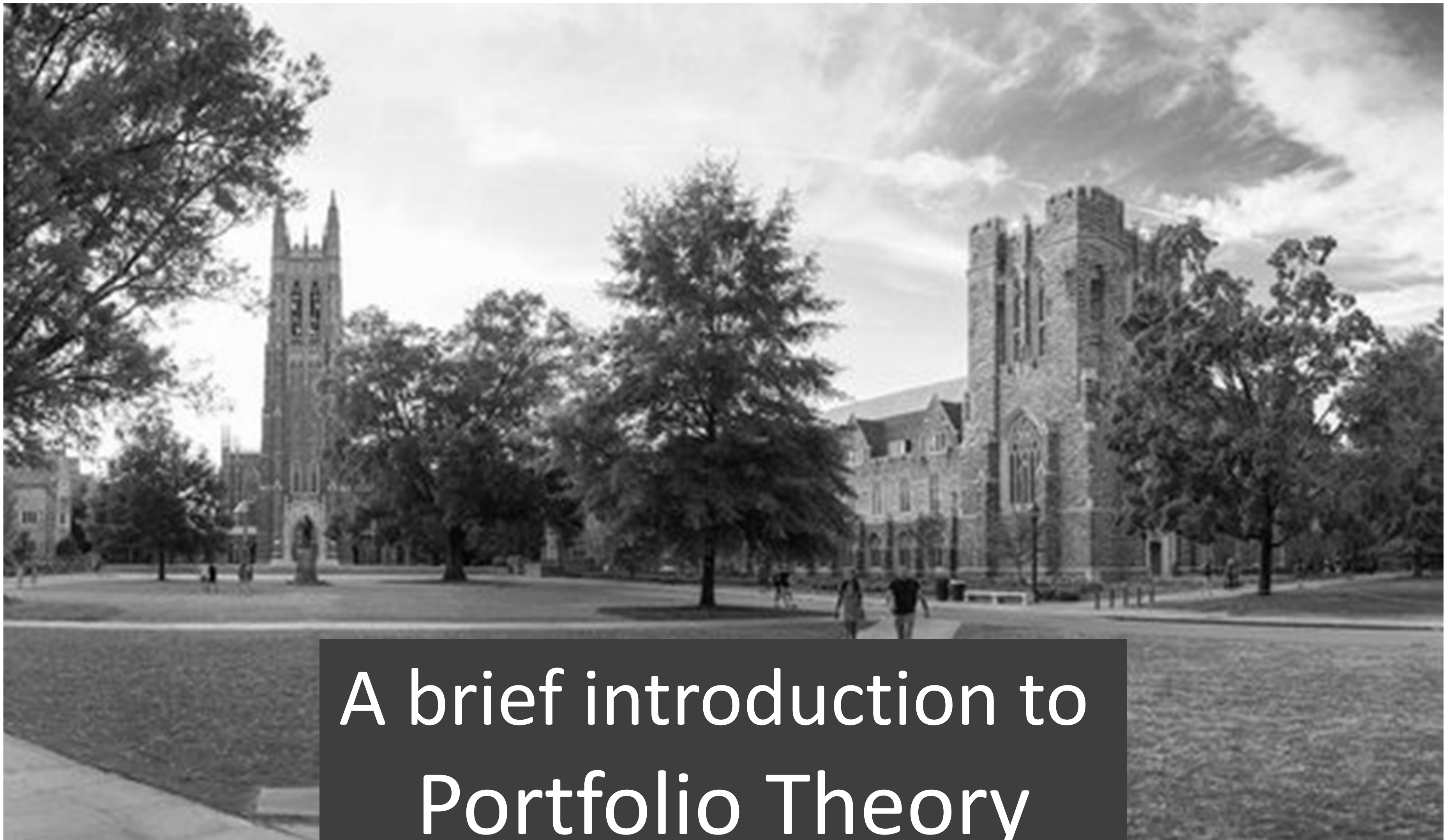
Portfolio Theory



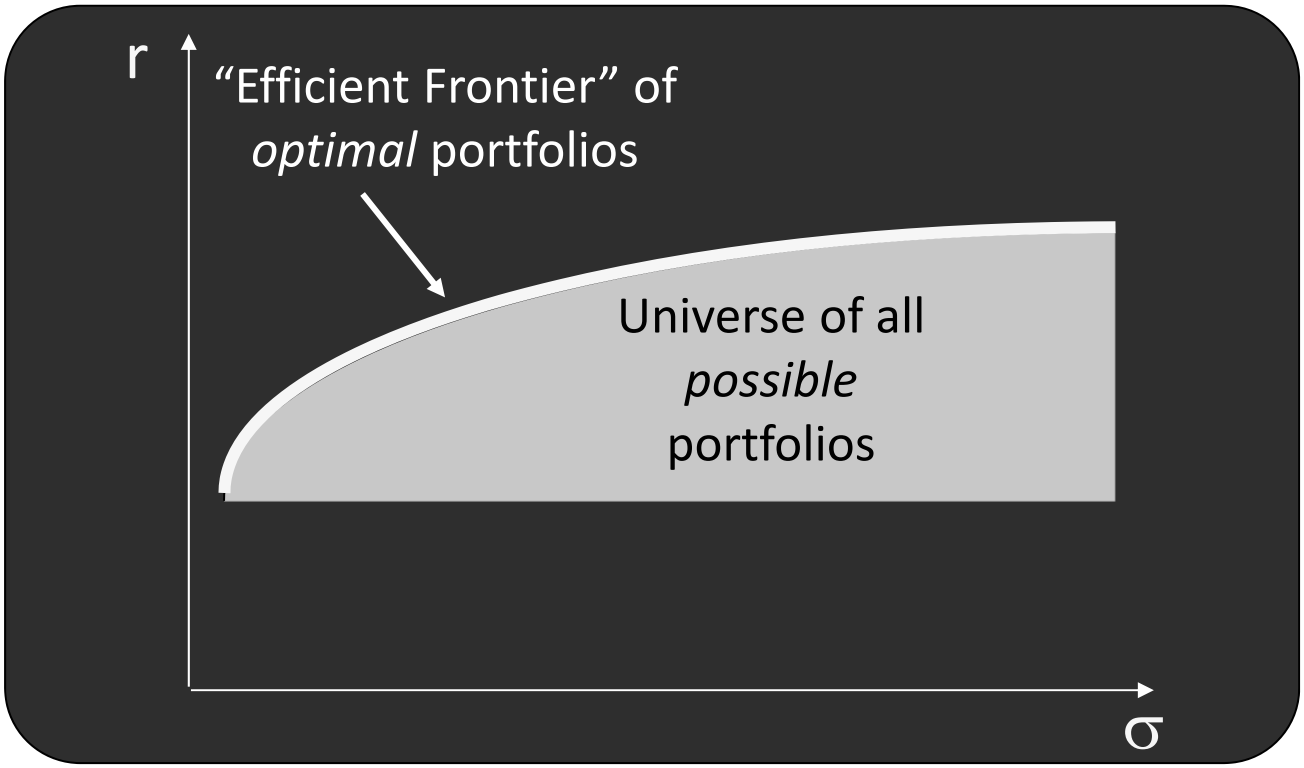
A brief introduction to

Portfolio Theory

for FinTech 510

Portfolio Theory

Portfolio Theory: a model of the tradeoff between:
- *risk* (standard deviation, σ)
- *return* (increase in value, *r*)



When you have completed these modules, you will understand:
- the tradeoff between risk and return
- the importance of diversification for portfolio optimization

Investment Return & Risk

Investors look at two key statistics when assessing financial investments:

The *rate of return* of the investment

The *riskiness* of the investment

Let’s examine each of these in detail.

Step 1: Calculating the Rate of Return

An asset's ***rate of return***: percentage change in that asset's value over a specified time period.

Rate of return is calculated as: $r_{n+1} = \dfrac{P_{n+1} - P_n}{P_n}$

Where: P_n = Price at time period *n*

P_{n+1} = Price at time period *n+1*

r_{n+1} = rate of return from period *n* to period *n+1*

All else equal, investors would like their *rate of return* to be as *high* as possible

Step 1: Calculating the Rate of Return
Example 1

Today, you purchase shares in a company at $15 / share. $P_0 = 15$

One year from now: $P_1 = 18$

The **Rate of Return r_1 =** $\dfrac{18 - 15}{15}$ = **20%**

18/15 - 15/15 =
18/15 - 1

Two years from now: $P_2 = 12$

Rate of Return r_2 = – 33%

Step 2: Calculating *Average* Rate of Return

If we would like to know the rate of return over *multiple* time periods, we simply calculate the average.

Example:

Suppose we observe historic returns on a particular asset to be as follows:

r_1 = 2.2% r_2 = 2.5% r_3 = -1.0% r_4 = 1.1%

Then the average return: $r_{avg} = \boxed{\mathbf{1.2\%}}$

Note:

- For average rate of return: specify the *frequency* of the data
  - Average *annual* rates of return are different from average *monthly* returns, for example
  - Typically we report *annual* returns unless otherwise specified

Step 2: Calculating Risk

In finance, an asset's ***riskiness*** is measured by *standard deviation of returns*, and is often referred to as *volatility* (using the lower case Greek letter "sigma": σ).

Given a series of returns on an asset over time, we calculate the asset's volatility as:

$$\sigma = \left\{ \frac{1}{(n-1)} \Sigma_i \left[r - r_{avg} \right]^2 \right\}^{1/2}$$

Where: n = number of observations

r_i = the ith return observation

r_{avg} = the average return across all n observations

"Σ" (*upper* case sigma) means "summation"

All else equal, investors would like their *risk* to be as *low* as possible

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Step 2: Calculating Risk
Example

Suppose we observe historic returns on a particular asset as follows:

r_1 = 2.2% r_2 = 2.5% r_3 = -1.0% r_4 = 1.1%

(1) calculate the average return: r_{avg} = 1.2%

(2) Calculate the standard deviation of the returns

$$\sigma = \{1/3\ [(2.2\% - 1.2\%)^2 + (2.5\% - 1.2\%)^2 + (-1.0\% - 1.2\%)^2 + (1.1\% - 1.2\%)^2\}^{1/2}$$

The *risk* (or *volatility*) is σ = $\boxed{\mathbf{1.59\%}}$

Concept Check

You just got lucky and won a slightly unusual lottery. As the winner, you are invited to make a selection between one of these two options:

(1) A guaranteed payment of $1,000,000

(2) A 50% chance of getting $2,000,000, and 50% chance of getting zero

Which outcome do you prefer? (1) or (2)?

Concept Check

Why do investors want high rates of return on their investments?

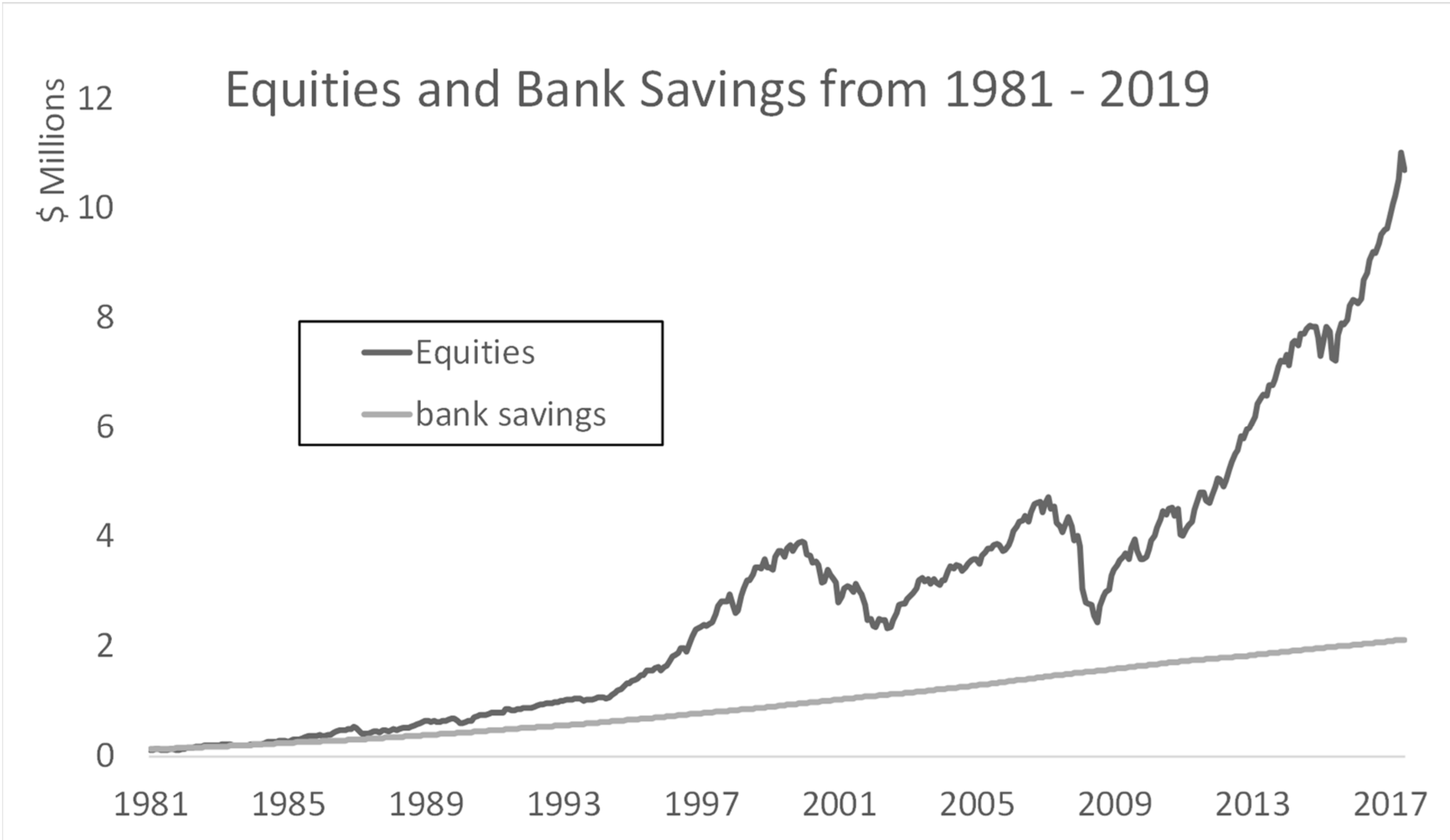
(1) The higher the rate of return, the greater the increase in the value of the investment

(2) The higher the rate of return, the more money the investor will receive, relative to the initial purchase price, when the investment is sold

(3) The higher the rate of return, the lower the risk

(A) (1) is correct
(B) (1) & (2) are correct
(C) (2) & (3) are correct
(D) (1) & (3) are correct
(E) All three of the above statements are correct

Return & Risk: Equities & Bank Savings



- Rate of return for equities is far higher than that of a savings account in a US bank (approximately 13% vs 8% on an annual basis)
- Equity *volatility* is also considerably higher on equities: 12.3% vs 1.2%

Module 2:
Portfolio Return & Risk

Correlation and Covariance

To calculate risk (volatility) for a *portfolio* (combination) of assets we also need *Covariance.*

Covariance is a measure of the extent to which two variables move *in the same direction* over time.

Formula for covariance:

Assume two assets: *a* and *b*

Returns r_{at} and r_{bt} over multiple time periods *t*

average returns $r_{a,avg}$ and $r_{b,avg}$

Covariance (σ_{ab}) between *a* and *b*: $\sigma_{ab} = 1/n \, \Sigma_t \, [\, r_{at} - r_{a,avg} \,] \, [r_{bt} - r_{b,avg}\,]$

We sometimes use Correlation ρ_{ab} instead of Covariance $\boldsymbol{\rho_{ab} = \dfrac{\sigma_{ab}}{\sigma_a \, \sigma_b}}$

Correlation has a nice feature: $-1 < \rho_{ab} < 1$

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Correlation and Covariance
Example

Two assets, *a* and *b:* $r_{a1} = 4.50\%$ $r_{a2} = 2.50\%$ $r_{a3} = 1.05\%$ $r_{a4} = -3.20\%$

$r_{b1} = 3.52\%$ $r_{b2} = 4.79\%$ $r_{b3} = 3.74\%$ $r_{b4} = 1.51\%$

(1) A*verage* returns of *a* and *b*: $r_{a,avg} = 1.21\%$ $r_{b,avg} = 3.39\%$

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(2) *Standard deviations* of *a* and *b*: $\sigma_a = 3.26\%$ $\sigma_b = 1.37\%$

(3) *Covariance* between *a* and *b:*

$$\boldsymbol{\sigma}_{\mathbf{ab}} = \tfrac{1}{4}\ \Big[(4.50\% - 1.21\%)(3.52\% - 3.39\%) + (2.5\% - 1.21\%)(4.79\% - 3.39\%) + (1.05\% - 1.21\%)(3.74\% - 3.39\%) + (-3.20\% - 1.21\%)(1.51\% - 3.39\%) \Big]$$

$$= 0.026\%$$

Correlation and Covariance
Example

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$$\boldsymbol{\sigma_{ab}} = \tfrac{1}{4}\Big[(4.50\% - 1.21\%)(3.52\% - 3.39\%) + (2.5\% - 1.21\%)(4.79\% - 3.39\%) + (1.05\% - 1.21\%)(3.74\% - 3.39\%) + (-3.20\% - 1.21\%)(1.51\% - 3.39\%)\Big]$$

$$= 0.026\%$$

Correlation between *a* and *b:* $\boldsymbol{\rho_{ab}} = \quad = 0.026\% / (3.26\% \times 1.37\%) = 58.51\%$

Correlation and Covariance
Example

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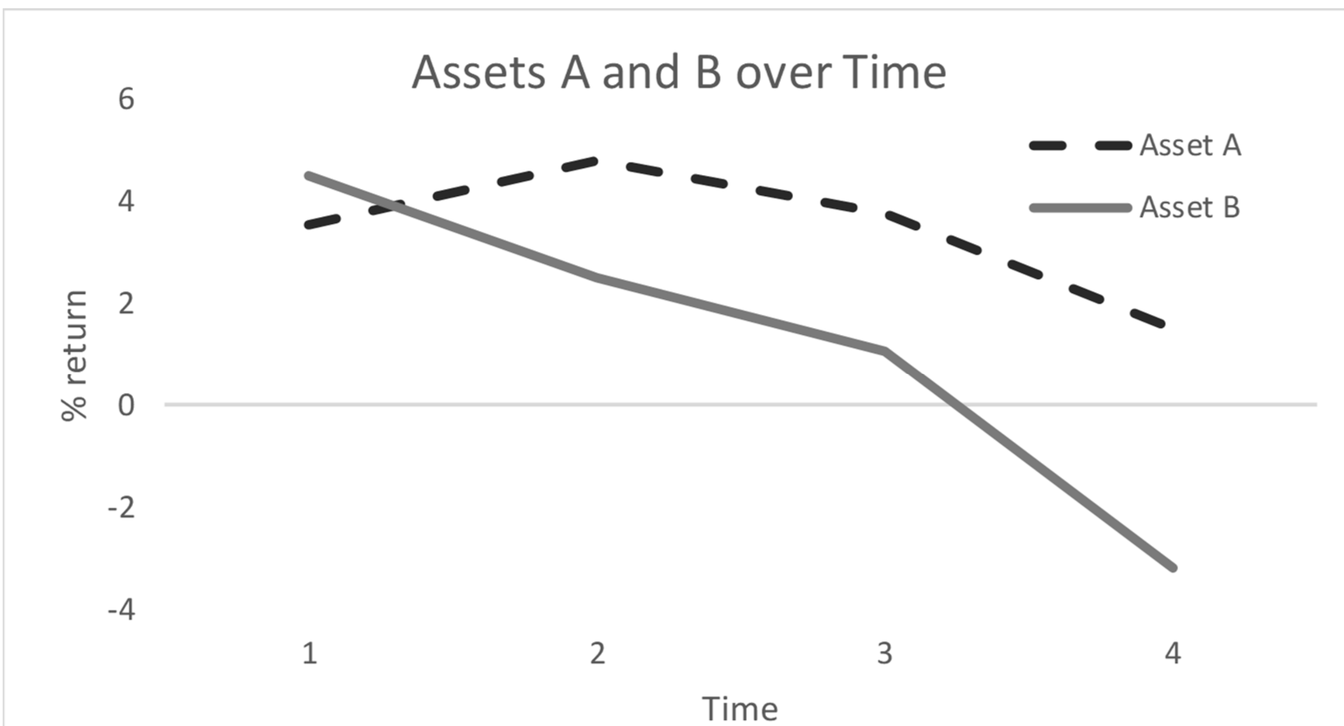
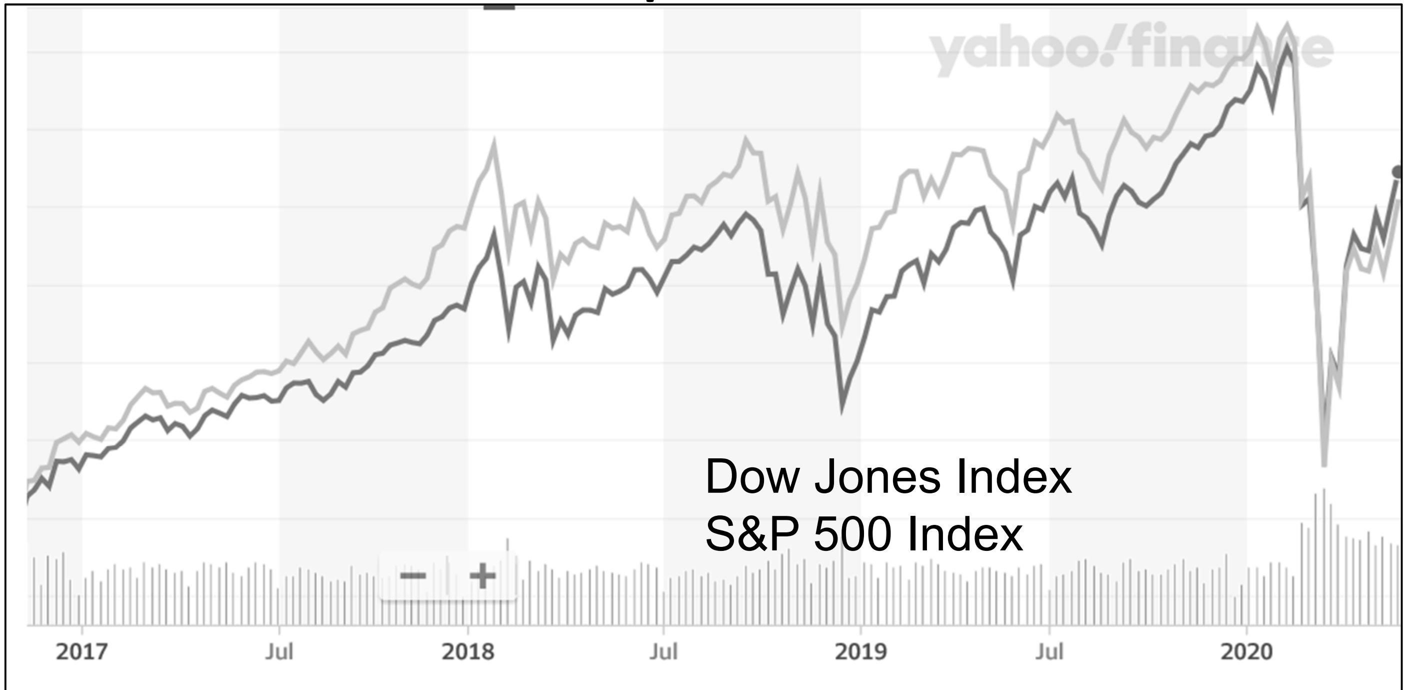


Correlation Examples



Guess the correlation between Gold & the S&P500 over this period

Statistical Relationships for Portfolios of Assets

Portfolio *return* r_P: weighted average return on the individual assets:

$r_P = \Sigma_i \, w_i \, r_i$ *with n* assets ($i = 1....n$)

Asset i represents w_i% of the overall portfolio value: $\Sigma_i w_i = 1$

Portfolio *volatility* is more complex.

$\sigma_p = (\Sigma_i \, \Sigma_j \, w_i w_j \, \sigma_i \, \sigma_j \, \rho_{ij})^{1/2}$

- As we add more assets, calculating portfolio volatility becomes cumbersome.
- We'll develop intuition for portfolio volatility in the two asset case
- To calculate volatility on a portfolio of multiple assets: use a computer!

Module 3:
Portfolio Return & Risk
The Two-Asset Case

Statistical Relationships for a Two-Asset Portfolio

	Asset *a*	Asset *b*	
Expected return on each asset	r_a	r_b	
Risk (volatility)	σ_a	σ_b	
% holding in each asset	w_a	w_b	$w_a + w_b = 1$

Statistical Relationships for a Two-Asset Portfolio

	Asset *a*	Asset *b*	
Expected return on each asset	r_a	r_b	
Risk (volatility)	σ_a	σ_b	
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Covariance between A & B: σ_{ab}

Correlation between A & B: $\rho_{ab} = \sigma_{ab} / (\sigma_a \sigma_b)$

Statistical Relationships for a Two-Asset Portfolio

	Asset *a*	Asset *b*	
Expected return on each asset	r_a	r_b	
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% holding in each asset	w_a	w_b	$w_a + w_b = 1$

Covariance between A & B: σ_{ab}

Correlation between A & B: $\rho_{ab} = \sigma_{ab} / (\sigma_a \sigma_b)$

Portfolio return r_P and volatility σ_P are then calculated as:

$$r_p = w_a r_a + w_b r_b \qquad w_a + w_b = 1$$

$$\sigma_p = [\, w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a w_b \sigma_a \sigma_b \rho_{ab} \,]^{1/2}$$

Statistical Relationships for a Two-Asset Portfolio: *Calculation Check*

		Asset A	Asset B
Expected return	r	5%	8%
Risk (volatility)	σ	9%	15%
Correlation between A & B	ρ	50%	

r_P = __________

σ_P = __________

Please calculate the return and volatility for the portfolio of A & B when w_a = 40%

$$r_p = w_a r_a + w_b r_b \qquad w_a + w_b = 1$$

$$\sigma_p = [\, w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \sigma_a \sigma_b \rho_{ab} \,]^{1/2}$$

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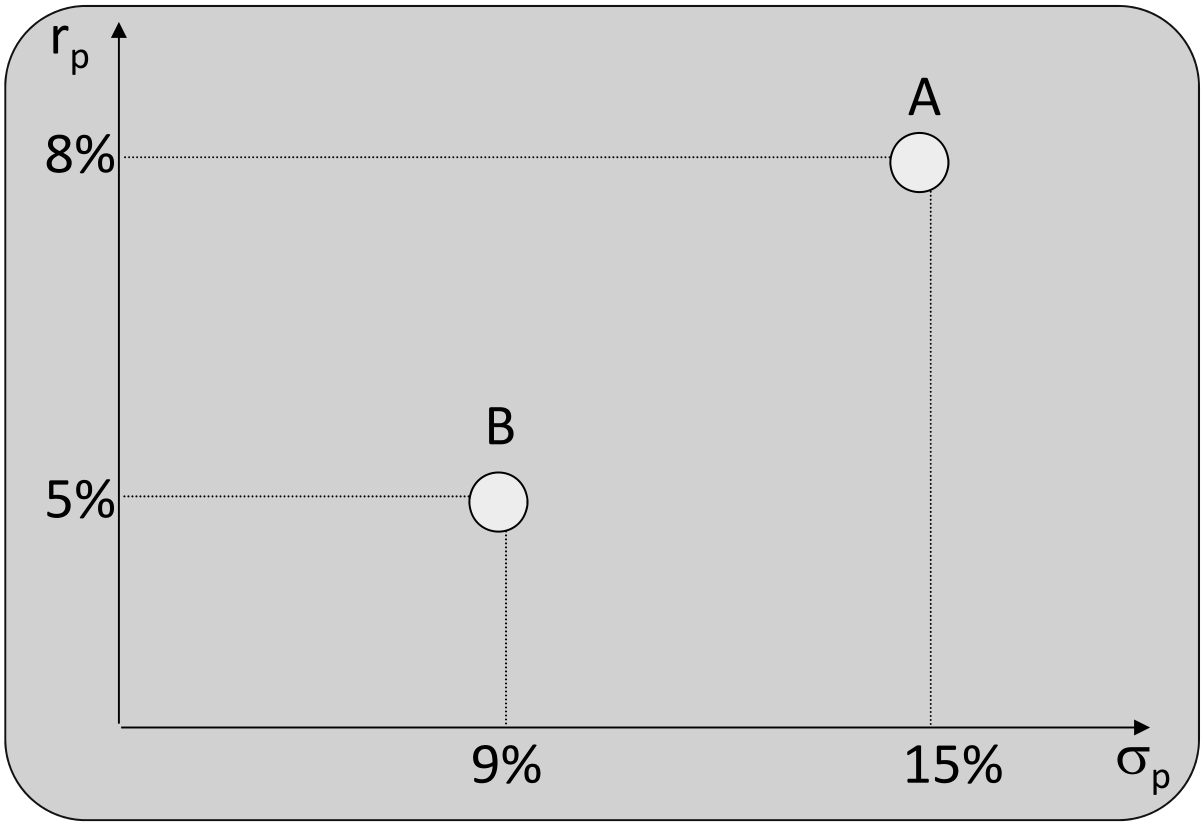
Return & Risk for a Two-Asset Portfolio

- Graphical demonstration of range of possible portfolios of two assets
  - Asset A has higher return and higher risk than Asset B
  - What range of portfolios can we create from these two assets?



$r_p = w_a r_a + w_b r_b \qquad w_a + w_b = 1$

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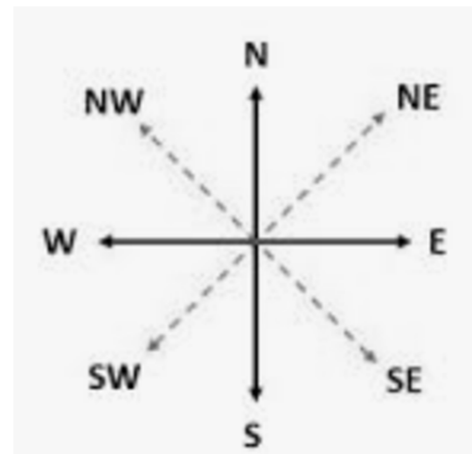

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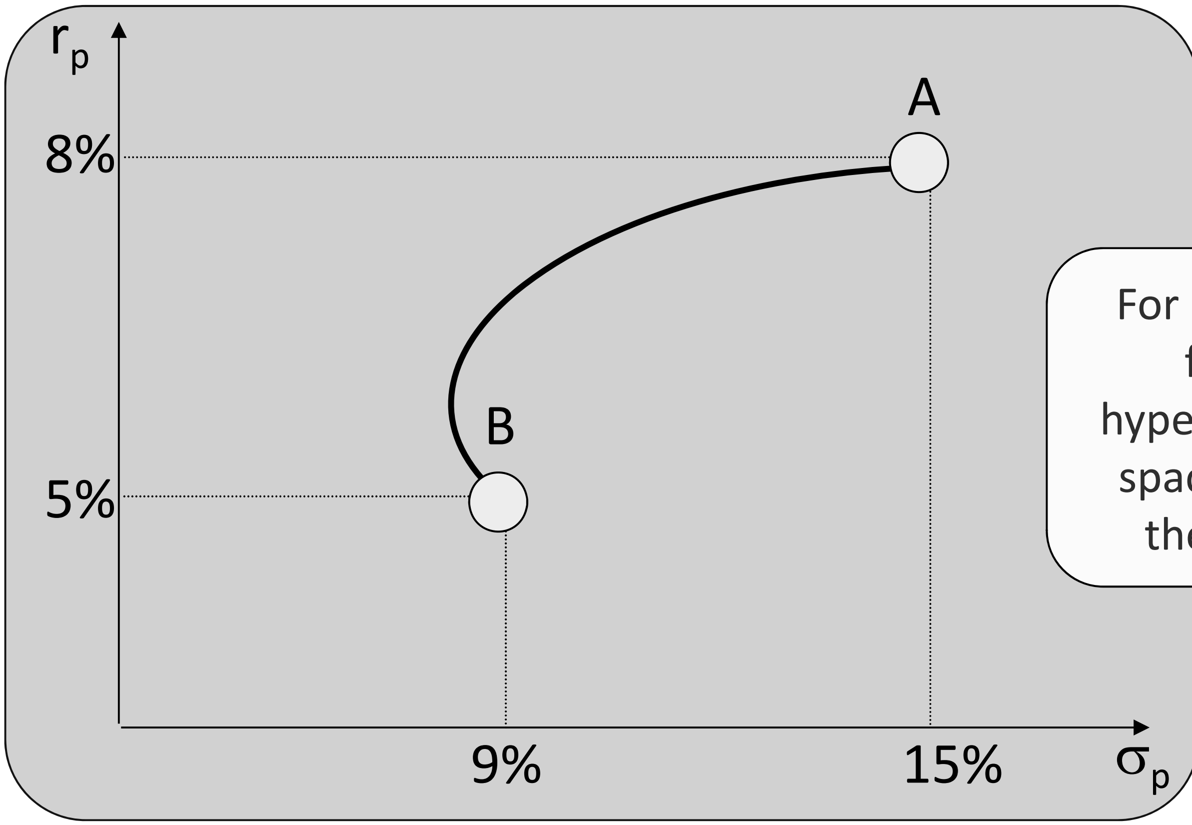
Return & Risk for a Two-Asset Portfolio

- Graphical demonstration of range of possible portfolios of two assets
  - The white curve demonstrates the range of all possible portfolios
  - Portfolio risk-return outcome driven by selection of w_a (and hence w_b)



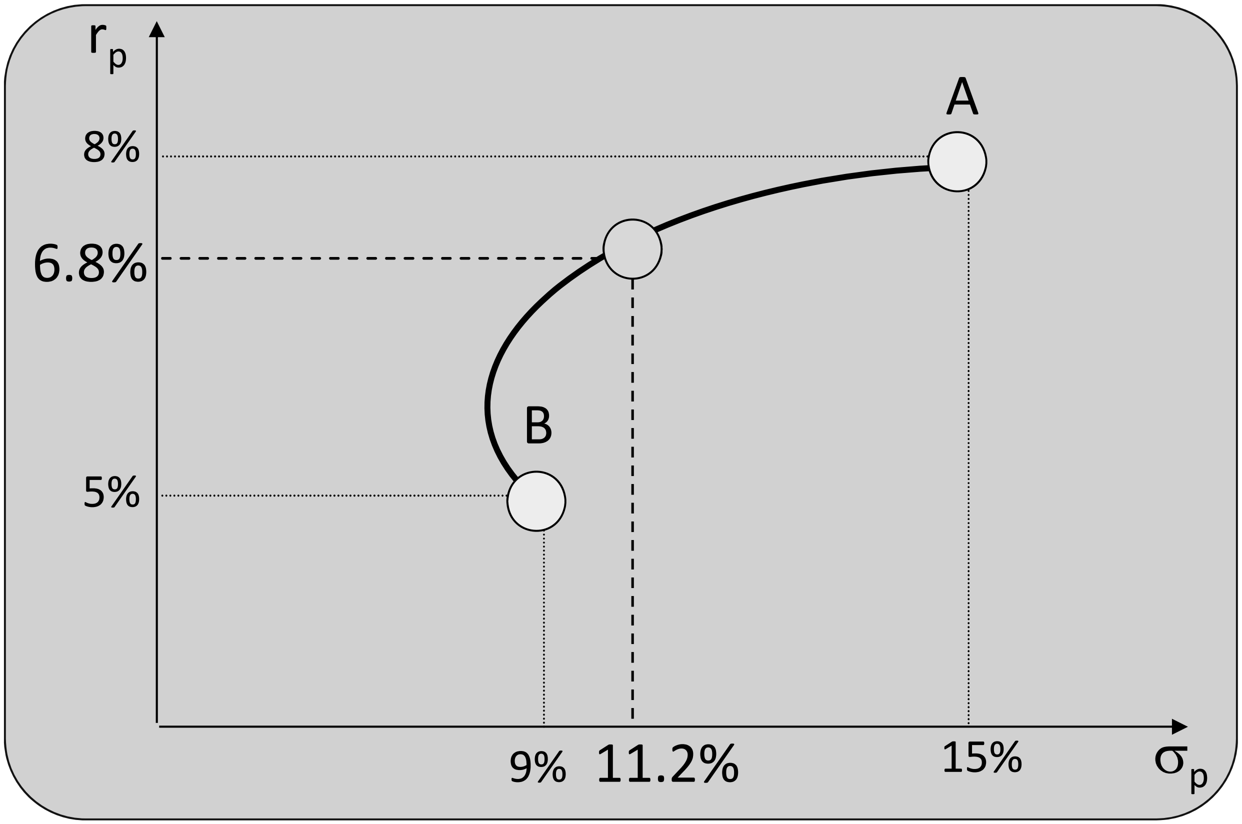
For partial proof that these formulas generate a hyperbola in return-volatility space, see the Appendix at the back of the handout

$$r_p = w_a r_a + w_b r_b \qquad w_a + w_b = 1$$

$$\sigma_p = [\, w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a w_b \, \sigma_a \sigma_b \rho_{ab} \,]^{1/2}$$

Return & Risk for a Two-Asset Portfolio
Example

If ρ = 0.5 & w_a = 0.4 as before: r_p = 6.8%, σ_p = 11.2%



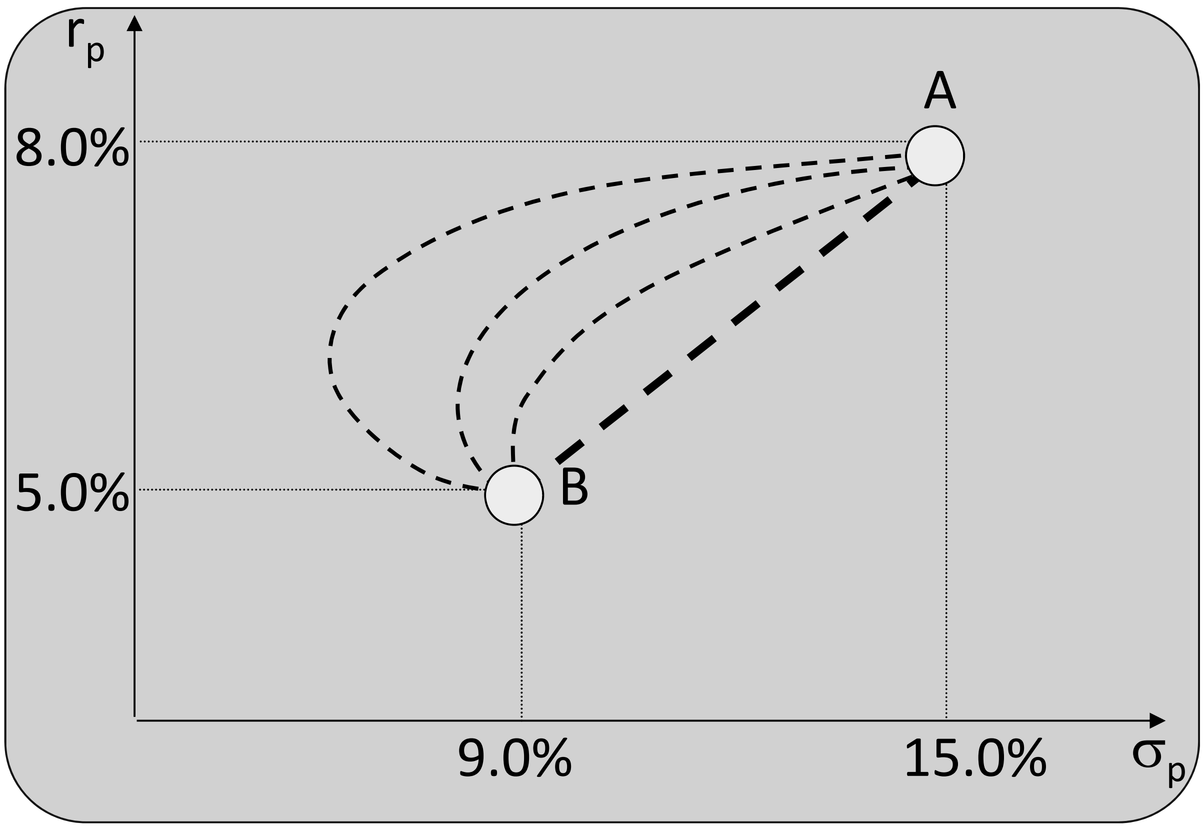
$r_p = w_a r_a + w_b r_b$ $\qquad w_a + w_b = 1$

$\sigma_p = [\, w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a w_b \sigma_a \sigma_b \rho_{ab} \,]^{1/2}$

Return & Risk for a Two-Asset Portfolio
Concept Check

(1) Which statistical variable drives curvature?

(i) r_a	(iv) σ_a
(ii) r_b	(v) σ_b
(iii) w_a	(vi) ρ_{ab}



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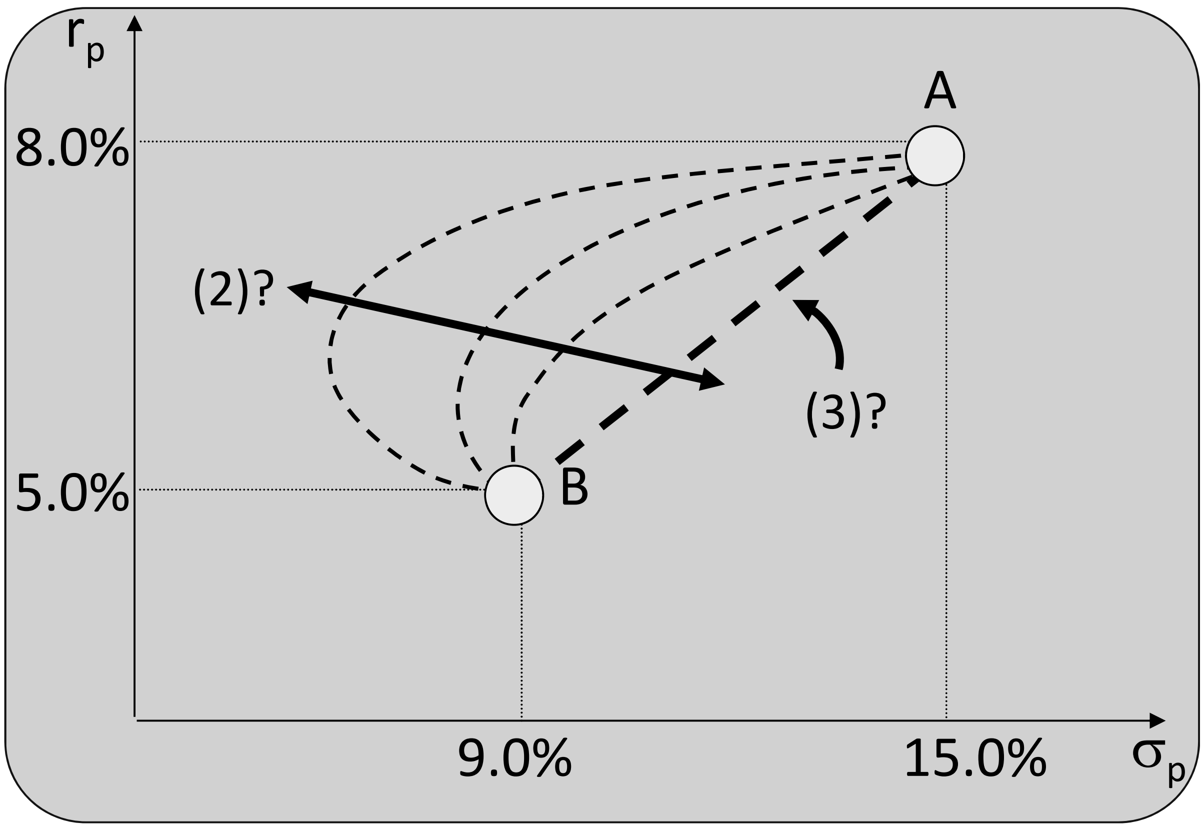

(2) As the variable *decreases*, which way does the curve stretch?

(3) Under what circumstances would the curve from A to B be linear (so that investors' portfolio options are limited to the straight line between A & B)?

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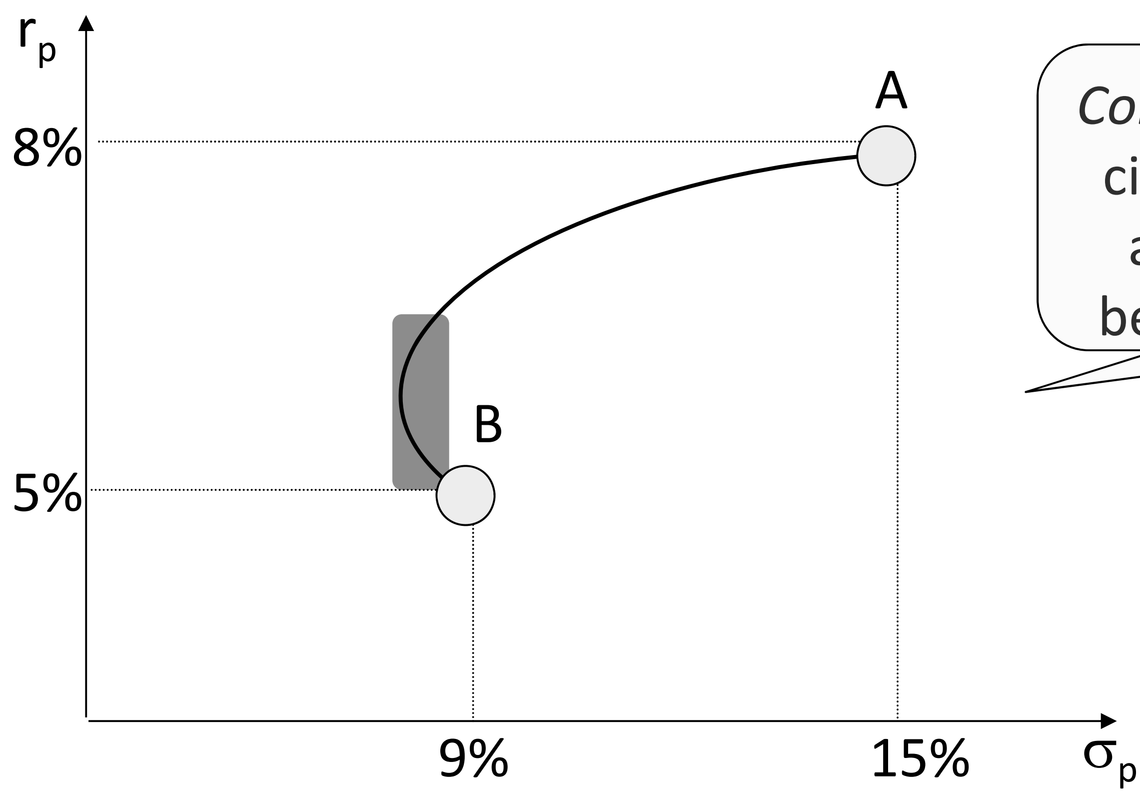
Return & Risk for a Two-Asset Portfolio

- When $\rho_{ab} < 1$, portfolio possibilities exist that have ***lower* volatility *and higher* return**, than Asset B alone
  - Shaded region of the curve

- This is the **benefit of diversification**: combinations of assets *almost invariably* offer better risk-return profiles than individual assets



Concept Check: under what circumstances will a two-asset portfolio have **no** benefit of diversification?

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Module 4:
Return & Risk for Portfolios of Multiple Assets

Return & Risk for a Multi-Asset Portfolio

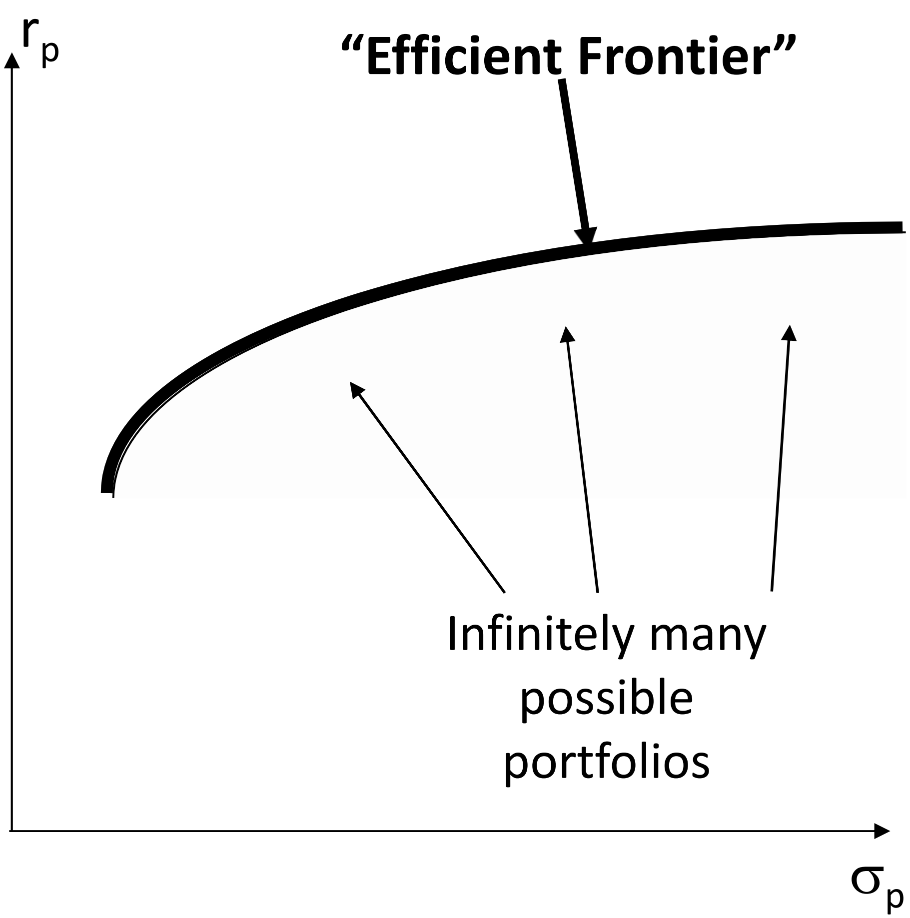
Expected Return and Standard Deviation formulas for portfolios with many assets:

$$r_p = \Sigma w_i r_i$$

$$\sigma_p = (\Sigma_i \Sigma_j w_i w_j \sigma_i \sigma_j \rho_{ij})^{1/2}$$

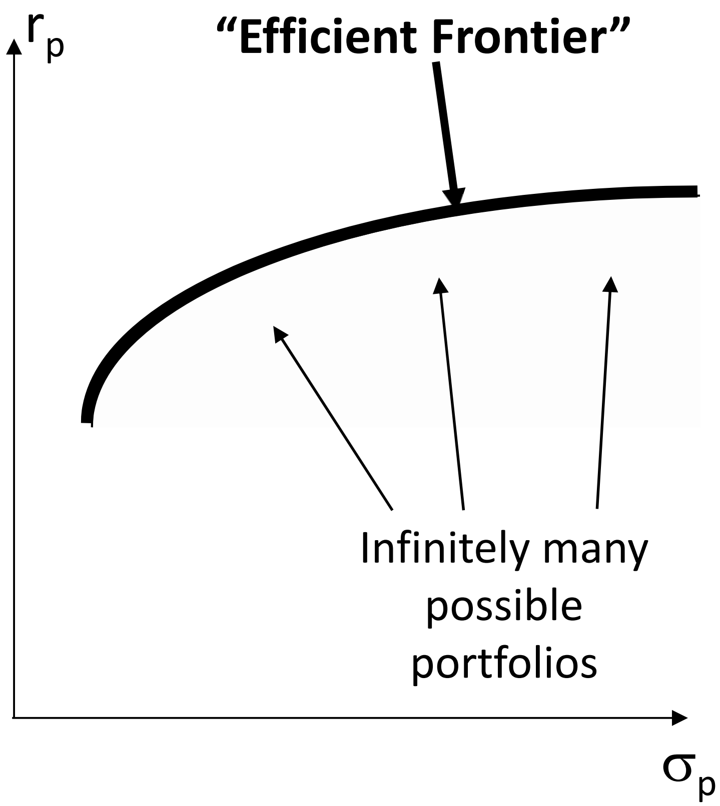
- *Portfolio return:*
  - Linear weighted average return of the assets in the portfolio
  - Does not depend on correlations between pairs of assets
- *Portfolio volatility*:
  - Non-linear: increasingly complex to calculate as more assets are added (let a computer do the work!)
  - *Does* depend on correlations between pairs of assets
  - Adding more assets incrementally increases the potential for lower portfolio volatility for any given rate of return

Return & Risk for a Multi-Asset Portfolio: The *Efficient Frontier*



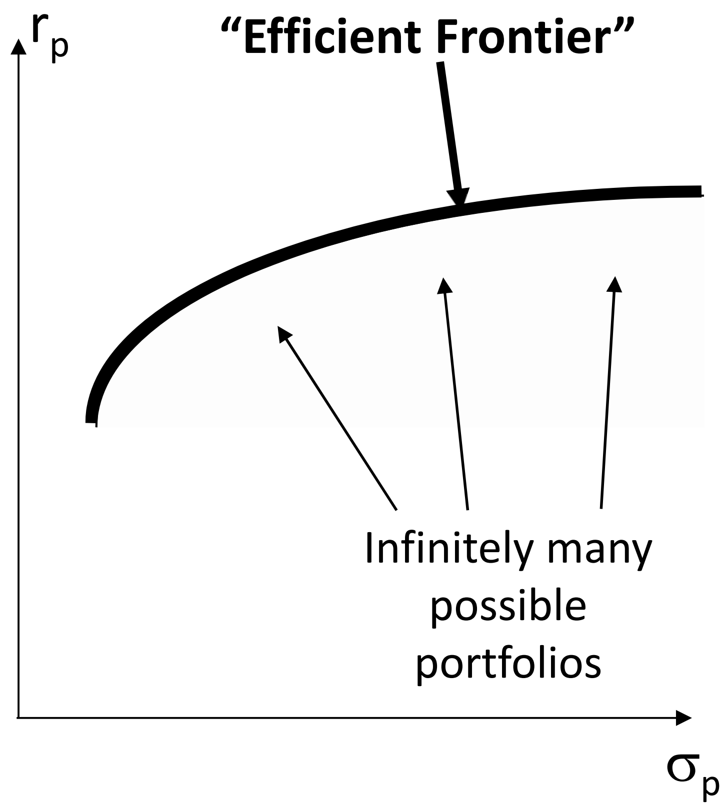
- Infinite numbers of possible portfolios in the shaded area
    - We can *estimate* risk & return characteristics of each, and hence "build" the frontier (how?)
- All possible portfolios will lie on or below a hyperbola (mathematically)
    - The "Efficient Frontier"
- Portfolios *on* the frontier have *best possible* risk-return characteristics
    - Called "optimal" portfolios
- Investors should choose portfolios that lie *on,* rather than *below*, this Frontier
    - Why?
    - How do we know which portfolios are truly optimal?!
    - Lots of caveats…

Return & Risk for a Multi-Asset Portfolio: The *Efficient Frontier: Caveats*



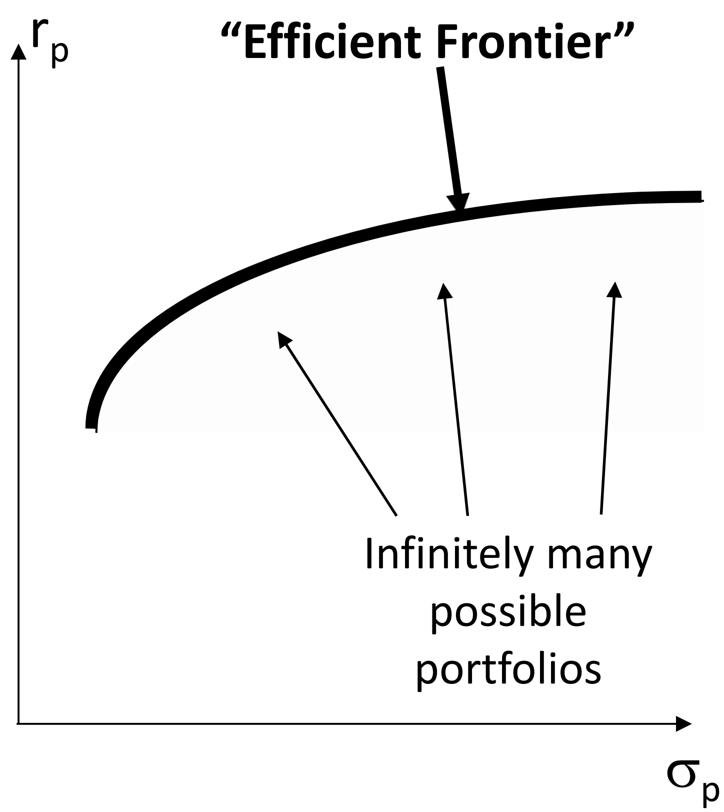
- Impossible to estimate all possible global portfolios
    - Infinitely many combinations

Return & Risk for a Multi-Asset Portfolio: The *Efficient Frontier: Caveats*



- Impossible to estimate all possible global portfolios
    - Infinitely many combinations
- Impossible to know the future return, volatility and pairwise correlations of all global assets
    - Historical data is a poor estimate at best
    - With incorrect assumptions, the efficient frontier will be in the "wrong" place

Return & Risk for a Multi-Asset Portfolio: The *Efficient Frontier: Caveats*



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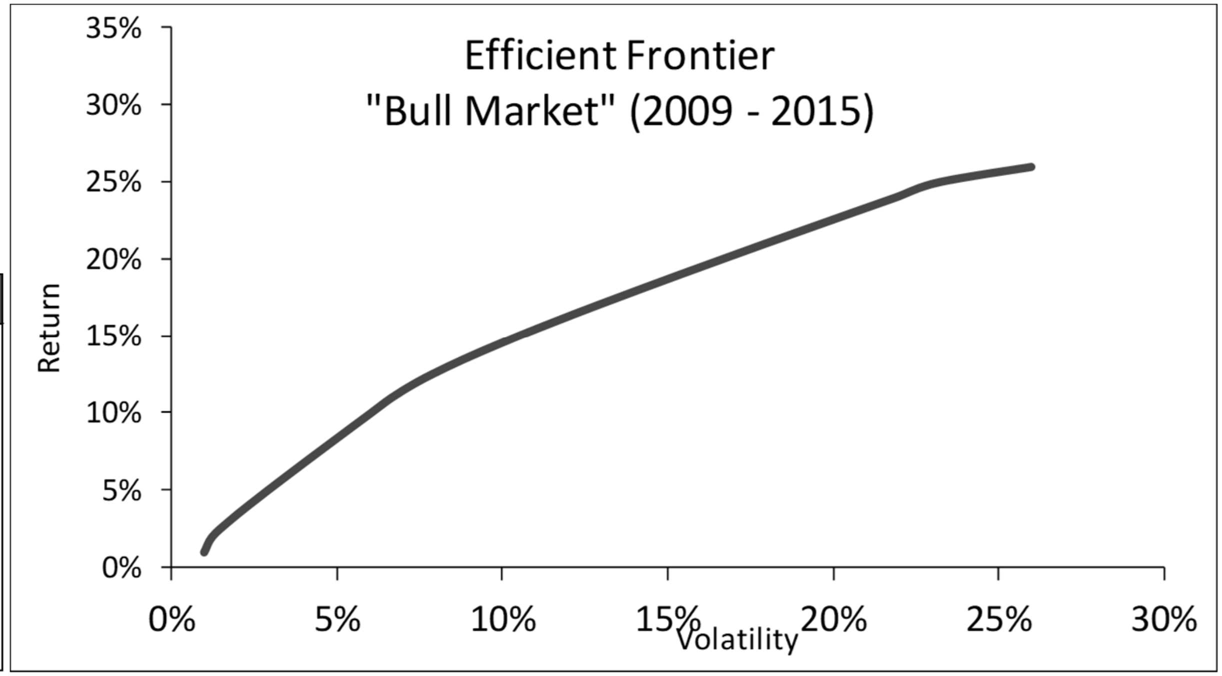
- Impossible to know the future return, volatility and pairwise correlations of all global assets
  - Historical data is a poor estimate at best
  - With incorrect assumptions, the efficient frontier will be in the "wrong" place
- Why do we care about the frontier?
  - Diversification is important!
  - Experienced financial managers *estimate* the frontier

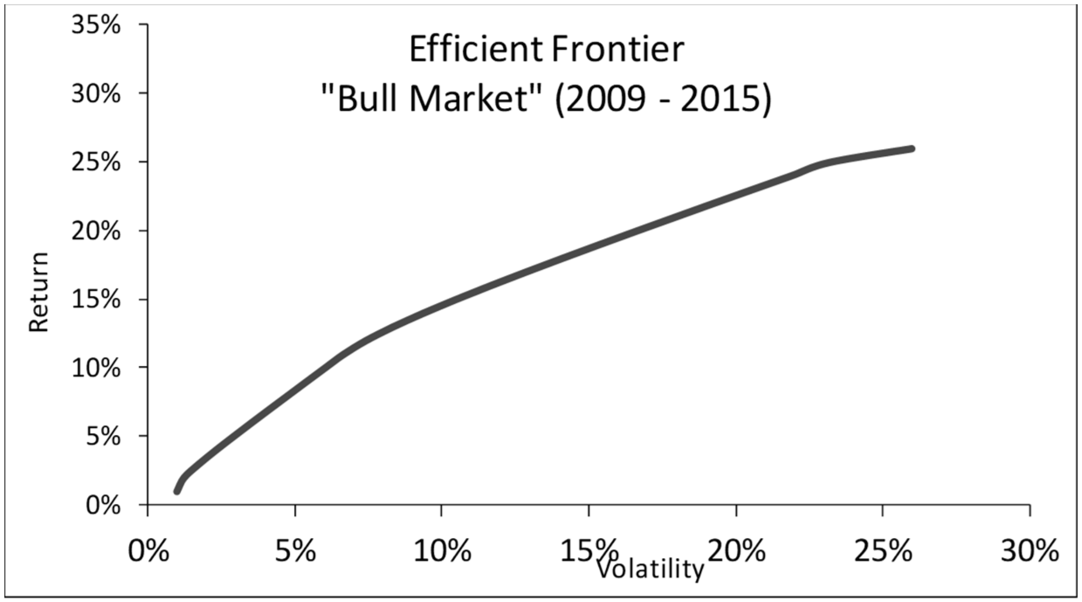
Return & Risk for a Multi-Asset Portfolio: The *Efficient Frontier: Samples*

	Avg rtn	volatility
Intl Equity	26.51%	31.27%
Commodities	-0.64%	20.89%
Corporate bonds	7.91%	5.93%
7-10yr UST	4.07%	6.92%
Inflation-sensitive	3.91%	5.75%
Real Estate	25.05%	25.30%
1-3yr UST	0.93%	0.99%

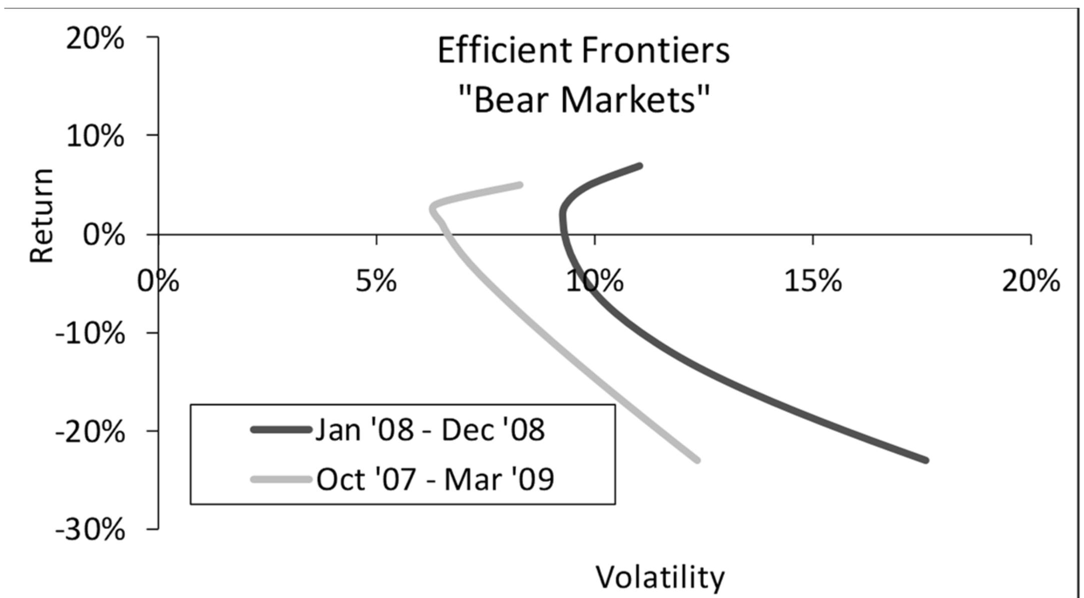


Correlation Matrix	Intl Equity	Commod	Corporate bonds	7-10yr UST	Inflation-sensitive	Real Estate	1-3yr UST
Intl Equity	100%	32%	-2%	-30%	-19%	51%	-16%
Commodities	32%	100%	-2%	-27%	-2%	43%	-15%
Corporate bonds	-2%	-2%	100%	72%	67%	11%	52%
7-10yr UST	-30%	-27%	72%	100%	79%	-27%	72%
Inflation-sensitive	-19%	-2%	67%	79%	100%	-9%	55%
Real Estate	51%	43%	11%	-27%	-9%	100%	-21%
1-3yr UST	-16%	-15%	52%	72%	55%	-21%	100%

Return & Risk for a Multi-Asset Portfolio: The *Efficient Frontier: Samples*



- From 2009 – 2015
  - Most diversified portfolios did well



- In bear markets (e.g., credit crisis)
  - Almost all asset classes and portfolios did badly!

Glossary

Bull Market: A market in which assets are going up in value.
Bear Market: A market in which assets are going down in value.

Equities (or shares): fractional ownership stake in a publicly traded company
Bonds: publicly traded debt (may be the debt of governments, corporations, municipalities etc.)

***S&P 500 Index ("S&P 500"):** A value-weighted index comprised of shares of the 500 largest US publicly traded companies, where "value" = share price x number of shares (called "Market Capitalization" or MCAP). Because it is value-weighted, it is heavily influenced by swings in the largest companies (by MCAP) in the index

***Dow Jones Index (DJIA or "DOW"):** A price-weighted index comprised of shares of the 30 largest US publicly traded companies. Because it is price-weighted (unlike the S&P 500), changes in the price of relatively larger stocks in the index do not have as much impact on the overall level of the Dow than they do on the level of the S&P

Risk Averse Investor: someone who, given the choice between two investments with equal expected returns but different volatilities, would always choose the investment with the lower volatility
Portfolio: A combination of different assets held together as an investment vehicle
Optimal Portfolio: In the context of portfolio theory, an optimal portfolio is one that offers the best possible rate of return for any given level of volatility.
Efficient Frontier: In risk-return space, a hypothetical continuous curve of "optimal" portfolios

Investopedia (www.Investopedia.com): a good online source of definitions for financial terms

** The S&P 500 can be viewed as an individual asset, because it can be purchased on a stand-alone basis (via an ETF: see Investopedia.com for a definition of ETFs). However, the S&P can also be viewed as a well diversified portfolio of US equities (as can the DOW and other US Indexes).*

Appendix
Why is the Efficient Frontier a Hyperbola?
Partial Explanation: Two-Asset Case

$$r_p = w_a r_a + (1 - w_a) r_b \quad (1)$$

$$\sigma_p^2 = w_a^2 \sigma_a^2 + (1 - w_a)^2 \sigma_b^2 + 2 w_a (1 - w_a) \sigma_{ab} \quad (2)$$

where r_a, r_b, σ_a, σ_b, and σ_{ab} are known constants which we will replace with a, b, c, d, and e to simplify

hence from (1): $w_a = \frac{(r_p - b)}{(a - b)}$

Substitute into (2): $\sigma_p^2 = \frac{(r_p - b)}{(a - b)} c^2 + \left(1 - \frac{(r_p - b)}{(a - b)}\right)^2 d^2 + 2e \frac{(r_p - b)}{(a - b)} \left(1 - \frac{(r_p - b)}{(a - b)}\right)$

From here, some tedious algebraic manipulation yields the formula for a hyperbola:

$$\frac{(x - h)^2}{p^2} - \frac{(y - j)^2}{q^2} = 1 \quad \textit{where } x = r_p \ \ \& \ \ y = \sigma_p$$

Even more tedious algebra can show that this generalizes to the multi-asset case