Brouwer's fixed point theorem

A combinatorial proof

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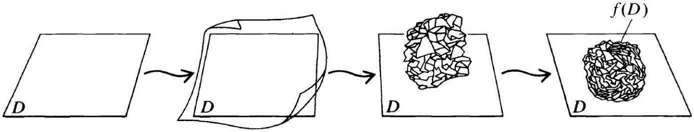
Brouwer's Theorem

Theorem

Let X be any convex compact subset of $\mathbb{R}^n$. Then $f : X \to X$ has a fixed point.



Some Definitions

Definition 1

An *n*-**dimensional simplex** is a convex linear combination of $n+1$ points in a general position. I.e., for given vertices $v_1, \ldots, v_{n+1}$, the simplex would be

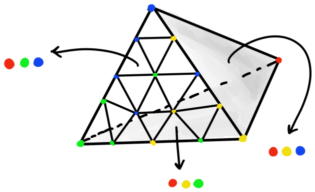
$$\Delta^n = \{\sum_{i=1}^{n+1} \alpha_i v_i : \alpha_i \geq 0,\ \sum_{i=1}^{n+1} \alpha_i = 1\}.$$

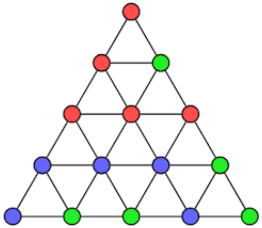
A **simplicial subdivision** of an n-dimensional simplex Δ^n is a partition of Δ^n into small simplices ("cells") such that any two cells are either di$\mathcal{S}_j$oint, or they share a full face of a certain dimension.

Some Definitions

Definition 2

A **proper coloring** of a simplicial subdivision is an assignment of $n+1$ colors to the vertices of the subdivision, so that the vertices of Δ^n receive all different colors, and points on each face of Δ^n use only the colors of the vertices defining the respective face of Δ^n.





Sperner's Lemma

Lemma

(Sperner, 1928). Every properly colored simplicial subdivision contains a cell whose vertices have all different colors.

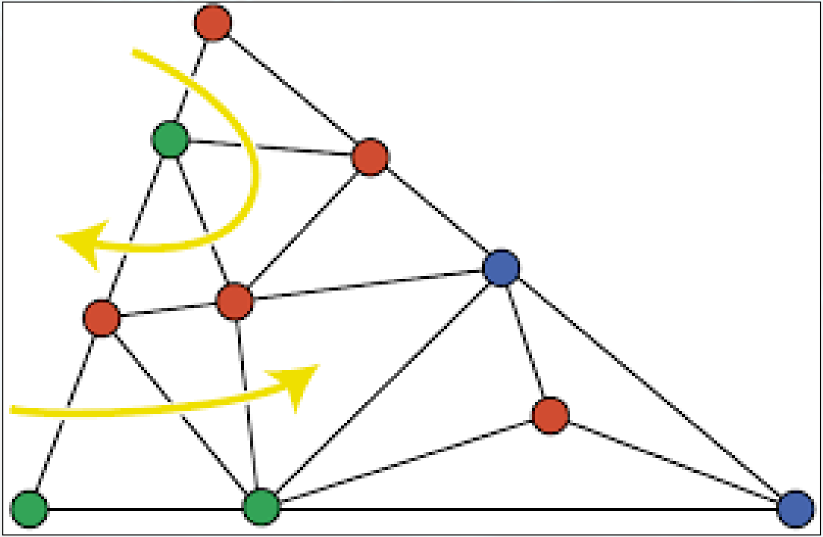
Sperner's Lemma's Proof

Let Δ^n be a properly colored n-dimensional simplex with an arbitrary subdivision. We will use induction on n to prove the lemma.

Case $n = 1$

Δ^1 is a segment such that the two endpoints have different colors. Clearly, for a subdivision $(0, 1) = x_0, x_1, \ldots, x_n = (1, 0)$ of this segment exists odd indices i such that x_i and x_{i+1} have different colors.

Sperner's Lemma's Proof



Sperner's Lemma's Proof

General Case

We have a proper coloring of a simplicial subdivision of Δ^n using $n+1$ colors. Let R denote the number of rainbow cells, using all $n+1$ colors. Let Q denote the number of simplicial cells that get all the colors except $n+1$, i.e. they are colored using $\{1, 2, \ldots, n\}$ so that exactly one of these colors is used twice and the other colors once. Also, we consider $(n-1)$-dimensional faces that use exactly the colors $\{1, 2, \ldots, n\}$. Let X denote the number of such faces on the boundary of Δ^n, and Y the number of such faces inside Δ^n. We count in two different ways.

Sperner's Lemma's Proof

General Case (Continued)

Each cell of type R contributes exactly one face colored $\{1, 2, \ldots, n\}$. Each cell of type Q contributes exactly two faces colored $\{1, 2, \ldots, n\}$. However, inside faces appear in two cells while boundary faces appear in one cell. Hence, we get the equation $2Q + R = X + 2Y$.

On the boundary, the only $(n-1)$-dimensional faces colored $\{1, 2, \ldots, n\}$ can be on the face $F \subset \Delta^n$ whose vertices are colored $\{1, 2, \ldots, n\}$. Here, we use the inductive hypothesis for F, which forms a properly colored $(n-1)$-dimensional subdivision. By the hypothesis, F contains an odd number of rainbow $(n-1)$-dimensional cells, i.e. X is odd. We conclude that R is odd as well.

Brouwer's Theorem's Proof

Lemma

If $Y \subset \mathbb{R}^n$ is homeomorphic to $X \subset \mathbb{R}^n$ and X has the fixed point property, then so does Y.

Brouwer's Theorem's Proof

Proof

Consider the homeomorphism $g : X \to Y$, and let $f : Y \to Y$ be continuous. Now consider $h : X \to X$, where $h = g^{-1}fg$. Then since h is the composition of continuous functions, it is also continuous, so h has a fixed point x_0 in X. Now let $y_0 = g(x_0)$. We claim that y_0 is a fixed point under f in Y. To see this,$f^{-1}(y_0) = f^{-1}g(x_0) = f^{-1}gh(x_0) = g(x_0) = y_0$,so of course, $f(y_0) = y_0$. Since f was an arbitrary continuous mapping to begin with, we have shown that any function $f : Y \to Y$ has a fixed point, so Y has the fixed point property.

Brouwer's Theorem Proof

By the previous lemma, it is sufficient to work with a simplex instead of a compact convex subset of $\mathbb{R}^n$ (which is equivalent by a homeomorphism).
Suppose the vertices of $S = \Delta^n$ are $v_1 = (1, 0, \dots, 0)$, $v_2 = (0, 1, \dots, 0)$, ..., $v_{n+1} = (0, 0, \dots, 1)$.
Let $f : S \to S$ be a continuous map and assume that it has no fixed point.

Brouwer's Theorem Proof

We construct a sequence of subdivisions of S that we denote by $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$ Each S_j is a subdivision of $\mathcal{S}_{j-1}$, so that the size of each cell in $\mathcal{S}_j$ tends to zero as $j \to \infty$.

Now we define a coloring of $\mathcal{S}_j$. For each vertex x $\in$S$_j$, we assign a color $c(x) \in [n+1]$ such that $(f(x))_{c(x)} < x_{c(x)}$. To see that this is possible, note that for each point $x \in \mathcal{S}, \sum x_i = 1$, and also $\sum f(x)_i = 1$. Unless $f(x) = x$, there are coordinates such that $(f(x))_i < x_i$ and also $(f(x))_{i'} > x_{i'}$. In case there are multiple coordinates such that $(f(x))_{i'} < x_{i'}$, we pick the smallest i.

Brouwer's Theorem Proof

Let us check that this is a proper coloring in the sense of Sperner's lemma. For vertices of S, $v_i = (0, \dots, 1, \dots, 0)$, we have $\mathrm{c(x) = i}$ because i is the only coordinate where $(f(x))_i < x_i$ is possible. Similarly, for vertices on a certain faces of S, e.g. $x = conv\{v_i : i \in A\}$, the only coordinates where $(f(x))_i < x_i$ is possible are those where $i \in A$, and hence $c(x) \in A$.

Brouwer's Theorem Proof

Sperner's lemma implies that there is a rainbow cell with vertices $x^{(j,1)}, \ldots, x^{(j,n+1)} \in \mathcal{S}_j$. In other words, $(f(x^{(j,i)}))_i < x_i^{(j,i)}$ for each $i \in [n+1]$. Since this is true for each $\mathcal{S}_j$, we get a sequence of points $\{x^{(j,1)}\}$ inside a compact set S which has a convergent subsequence. Let us throw away all the elements outside of this subsequence - we can assume that $\{x^{(j,1)}\}$ itself is convergent.

Brouwer's Theorem Proof

Since the size of the cells in $\mathcal{S}_j$ tends to zero, the limits $\lim_{j \to \infty} x^{(j,i)}$ are the same in fact for all $i \in [n+1]$- let's call this common limit point $x^* = \lim_{j \to \infty} x^{(j,i)}$. We assumed that there is no fixed point, therefore $f(x^*) \neq x^*$. This means that $(f(x^*))_i > x^* i$ for some coordinate i. But we know that $(f(x^{(j,i)}))_i < x_i^{(j,i)}$ for all j and $\lim_{j \to \infty} x^{(j,i)} = x^*$, which implies $(f(x^*))_i \leq x_i^*$ by continuity. This contradicts the assumption that there is no fixed point.

References

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