Page

(1) Solve : $(D^2 + 2D + 1)y = 10e^x$

s. Auxiliary Equation is $m^2 + 2m + 1 = 0$

$m^2 + m + m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$CF = c_1 e^{-x} + c_2 e^{-x} \cdot x$

$= (c_1 + c_2 x) e^{-x}$

$P.I = \dfrac{10 \cdot e^x}{D^2 + 2D + 1}$

$= 10 \left[\dfrac{e^x}{D^2 + 2D + 1} \right] =$

$= 10 \cdot \dfrac{e^x}{(1)^2 + 2(1) + 1} = \dfrac{10 \cdot e^x}{4}$

$= \dfrac{5 \cdot e^x}{2}$

$y = (c_1 + c_2 x) e^{-x} + \dfrac{10 \cdot e^x}{4}$

Date ________
Page ________

(2) Solve : $(D^2+9)y = \cos 2x + \sin 2x$

Ans. Auxiliary form $m^2+9 = 0$

$m^2 = -9$

$m = \pm 3i$

$CF = c_1 \cos 3x + c_2 \sin 3x$

$$P.T = \frac{1}{D^2+9} \cdot (\cos 2x + \sin 2x)$$

$$= \frac{\cos 2x}{D^2+9} + \frac{\sin 2x}{D^2+9}$$

$$= \frac{\cos 2x}{-(2)^2+9} + \frac{\sin 2x}{-(2)^2+9}$$

$$= \frac{\cos 2x + \sin 2x}{5}$$

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{\cos 2x + \sin 2x}{5}$$

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Date
Page

(3) Solve: $y'' + 2y' + 3y = 2x^2$

Ans: $(D^2 + 2D + 3)y = 2x^2$

Auxiliary form: $m^2 + 2m + 3 = 0$

$$m = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \sqrt{2}\, i$$

$$CF = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$

$$P.I = \frac{2x^2}{D^2+2D+3}$$

$$= \frac{2x^2}{3\left[1 + \frac{D^2+2D}{3}\right]}$$

$$= \frac{2x^2}{3}\left[1 - \left(\frac{D^2+2D}{3}\right) + \left(\frac{D^2+2D}{3}\right)^2 + \dots\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}Dx^2 - \frac{D^2}{3}x^2 + \frac{4}{9}D^2x^2 \dots\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{4}{3}x + \frac{2}{9}\right]$$

$$y = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$

$$+ \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Date ______
Page ______

(4) Solve : $y'' + 4y = 8x^2$ [Method: Undetermined Coefficient]

Ans Auxiliary form $(D^2+4)y = 8x^2$

$m^2+4=0$. $m = \pm 2i$

$\therefore$ Solution Set $\{x^2, x, 1\}$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$y_p = k_2 x^2 + k_1 x + k_0$

$y'_p = 2k_2 x + k_1$

$y''_p = 2k_2$

$2k_2 + 4k_2 x^2 + 4k_1 x + 4k_0 = 8x^2$

$4k_2 = 8$, $4k_1 = 0$, $4k_0 + 2k_2 = 0$

$k_2 = 2$ $k_1 = 0$ $k_0 = -1$

$y_p = 2x^2 - 1$

$y = c_1 \cos 2x + c_2 \sin 2x + 2x^2 - 1.$

Date
Page

(5) Solve: $(D^2-3D+2)y = \frac{e^x}{1+e^x}$ [Method of Variation of Parameters]

ns: Auxiliary form $m^2-3m+2=0$

$m = 2, 1$

$y_c = c_1 e^x + c_2 e^{2x}$

$y_p = -y_1 \int \frac{y_2}{W(x)} r(x)\,dx + y_2 \int \frac{y_1}{W(x)} r(x)\,dx$

$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$

$y_p = -e^x \int \frac{dx}{1+e^x} + e^{2x} \int \frac{e^x}{e^{2x}(1+e^x)} dx$

$= e^{-x} \int \frac{e^{-x}}{1+e^{-x}} dx + e^{2x} \int \frac{dx}{e^x + e^{2x}}$

$= e^x \log|1+e^{-x}| + e^{2x} \int \frac{e^{-x}}{1+e^x} dx$

$= e^x \log|1+e^{-x}| + e^{2x} \int \frac{e^{-x} \cdot e^{-x}}{1+e^{-x}} dx$

$= e^x \log|1+e^{-x}| + e^{2x} \int \frac{e^{-x}(e^{-x}+1-1)}{e^x+1} dx$

$= e^x \log|1+e^{-x}| + e^{2x} \int e^{-x} - \frac{e^{-x}}{1+e^{-x}} dx$

$= e^x \log|1+e^{-x}| - e^x + e^{2x} \log|1+e^{-x}|$

Date
Page

(6) Solve: series solution, $y'' = y'$

Ans. $y'' = y'$

$y'' - y' = 0$ —— (1)

$\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$

$\frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

Substituting all values in Eq (1).

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \left[-\sum_{k=1}^{\infty} k a_k x^{k-1}\right] = 0$$

To obtain common power of x

$k-2 = m-1$

$k = m+1$

$$\sum_{n=1}^{\infty} (m+1)(m) a_{m+1} x^{m-1} - \sum_{n=1}^{\infty} k a_k x^{k-1} = 0$$

$\because$ m is dummy variable $\therefore$ $m = n$

$$\sum_{n=1}^{\infty} n(n+1) a_{k+1} x^{k-1} - \sum_{n=1}^{\infty} k a_k x^{k-1} = 0$$

$$k(k+1) a_{k+1} - n a_k = 0$$

$$a_{k+1} = \frac{a_k}{k(k+1)}$$ putting $k = 1, 2, 3, \ldots$

$k=1$ $\qquad$ $k=2$

$$a_2 = \frac{a_1}{2} \qquad a_3 = \frac{a_2}{6} = \frac{a_1}{12}$$

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$k=3$

$$a_4 = \frac{a_3}{12} = \frac{a_1}{144}$$

$k=4$

$$a_5 = \frac{a_4}{20} = \frac{a_1}{20 \times 144}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x + \frac{a_1}{2} x^2 + \frac{a_1}{12} x^3 + \frac{a_1}{144} x^4$$

$$+ \frac{a_1}{20 \times 144} x^5 + \cdots$$

$$y = a_0 + a_1 x + \frac{a_1 x^2}{2!} + \frac{a_1}{12} x^3 + \cdots$$

$$= a_0 + a_1 \left[x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{144} + \cdots \right]$$

Date
Page

7) By using elimination method, Solve

(A) $z = f(x^2 - y^2)$

(B) $\phi = (xyz, x+y+z) = 0$.

Ans (A)

$$z = f(x^2 - y^2)$$

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)(2x)$$

$$f'(x^2-y^2) = \frac{1}{2x}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y)$$

$$f'(x^2-y^2) = -\frac{1}{2y}\frac{\partial z}{\partial y}$$

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y}$$

$$y\frac{\partial z}{\partial x} = -x\frac{\partial z}{\partial y}$$

$$\boxed{yp + xq = 0}$$

(B) $\phi(xyz, x+y+z) = 0$

$u = xyz$

$v = x+y+z$

$$\frac{\partial \phi}{\partial x} = \phi'(u,v) = 0$$

$$= \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$0 = \phi_u \cdot yz + \phi_v$$

Date
Page

$$\frac{\partial \phi}{\partial y} : \phi(u,v)=0$$

$$0 = \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$0 = \phi_u \cdot xz + \phi_v$$

$$+\phi_v = -\phi_u \, xz$$

$$\frac{\phi_u}{\phi_v} = -\frac{1}{xz} \quad \Big| \quad \frac{\phi_u}{\phi_v} = \frac{1}{yz}$$

$$y\cancel{z} = x\cancel{z}$$

$$\boxed{y = x}$$

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Date
Page

(2) Solve: $y^2p - xyq = x(z-2y)$

Ans: $P_p + Q_q = R$ $\qquad R = xz - 2xy$

$P = y^2 \quad Q = -xy$

Lagrange's Auxiliary Eqn.

$$k = \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz - 2xy}$$

$$-xy\,dx = y^2\,dy$$

$$xy\,dx + y^2\,dy = 0 \quad \text{———— (1)}$$

$M = xy \qquad N = y^2$

$$\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 0$$

$$I.F = \frac{1}{Mx + Ny} = \frac{1}{x^2y + y^3}$$

$$\frac{xy}{x^2y + y^3}\,dx + \frac{y^2}{x^2y + y^3}\,dy = 0.$$

$$\underbrace{\frac{x}{x^2 + y^2}}\,dx + \underbrace{\frac{y}{x^2 + y^2}}\,dy = 0.$$

Date ______ Page ______

$M = \dfrac{x}{x^2+y^2}$ $N = \dfrac{y}{x^2+y^2}$

$\left(\dfrac{\partial M}{\partial y}\right) = \dfrac{-x(2y)}{(x^2+y^2)^2}$ $\left(\dfrac{\partial N}{\partial x}\right) = \dfrac{-y(2x)}{(x^2+y^2)^2}$

$\therefore$ Equation is Exact

$C = \int M\,dx + \int N\,dy$

exclude ~~terms with x~~

$= \dfrac{1}{2}\int \dfrac{2x}{x^2+y^2}\,dx + \int \dfrac{y}{x^2+y^2}\,dy$

$\boxed{C_1 = \log|x^2+y^2|} + 0$

$\dfrac{dx}{y^2} = \dfrac{dz}{xz-2xy}$

$xz - 2xy\,dx = y^2\,dz$

$xz - 2xy\,dx - y^2\,dz = 0.$ $y = \text{constant}$

Date ________

Page ________

(9) Solve : $x^2p^2 + y^2q^2 = z^2$

$F = x^2p^2 + y^2q^2 - z^2 = 0$

$F_p = 2px^2 \qquad F_q = 2qy^2 \qquad F_z = -2z$

$F_x = 2xp^2 \qquad F_y = 2yq^2$

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x + pF_z}$$

$$= \frac{-dq}{F_y + qF_z}$$

$$\frac{dx}{2px^2} = \frac{dy}{2qy^2} = \frac{dz}{2p^2x^2 + 2q^2y^2}$$

$$= \frac{-dp}{2xp^2 - 2zp} = \frac{-dq}{2yq^2 - 2zq}$$

Take

$$\frac{pdx + xdp}{2p^2x^2 - 2p^2x^2 + 2zpx} = \frac{qdy + ydq}{2q^2y^2 - 2y^2q^2 + 2zqy}$$

$$\frac{d(xp)}{2zpx} = \frac{d(yq)}{2zqy}$$

$$\frac{d(xp)}{xp} = \frac{d(yq)}{yq}$$

Date ______
Page ______

$$\int \frac{d(xp)}{xp} = \int \frac{d(yq)}{yq}$$

$\log xp = \log yq + \log a$ $\quad a = $ constant

$$xp = yq\,a$$

$$x^2p^2 + y^2q^2 = z^2$$

$$y^2q^2a^2 + y^2q^2 = z^2$$

$$q^2 = \frac{z^2}{y^2(a^2+1)} \qquad q = \frac{z}{y\sqrt{a^2+1}}$$

$$\therefore p = \frac{az}{x\sqrt{a^2+1}}$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{az}{x\sqrt{a^2+1}}\,dx + \frac{z}{y\sqrt{a^2+1}}\,dy$$

$$\sqrt{a^2+1}\int \frac{dz}{z} = a\int \frac{dx}{x} + \int \frac{dy}{y}$$

$$\boxed{\sqrt{a^2+1}\cdot \log z = a\log x + \log y + b}$$

Date ______
Page ______

(10) Using Separation of Variables

(A) $u_x = 2u_t$; $u(0,t) = 2e^{6t}$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t}$$

$$x't = 2xt' = k$$

$x't = k$ $2xt' = k$ $\div$ by xt

$$\frac{x'}{x} = k \qquad \frac{2t'}{t} = k$$

$$\log x = kx + c_1 \qquad 2\log t = kt + c_2$$

$$2e^{6t} = c_1 c_2 \cdot e^{kx} \cdot e^{t\left(\frac{k-1}{4}\right)}$$

$$c_1 \cdot c_2 = 2 \qquad \frac{k-1}{4} = 6$$

$$\boxed{k = 25}$$

$$u = 2e^{25x} e^{6t}$$

Date ______
Page ______

1) Find Fourier Series

$$f(x) = \begin{cases} x^2 ; & 0 < x < 2 \\ 0 ; & 2 < x < 4 \end{cases}$$

→ period $p = 2l = 4 = \boxed{l = 2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right]$$

$$a_0 = \frac{1}{2} \int_0^4 \cancel{x^2} f(x)\, dx$$

$$= \frac{1}{2} \left[\int_0^2 x^2\, dx \right]$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{1}{2} \left[\frac{8}{3} \right] = \frac{4 \times \cancel{2}}{3 \times \cancel{2}} = 4/3$$

$$a_n = \frac{1}{2} \left[\int_0^4 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \right]$$

$$= \frac{1}{2} \left[\int_0^{2} x^2 \cos\left(\frac{n\pi x}{2}\right) dx \right]$$

Date ______ Page ______

$$= \frac{1}{2}\left[+ \frac{x^2 \sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{2}{\frac{n\pi}{2}} \int x \sin\frac{n\pi x}{2}\, dx \right]$$

$$= \frac{1}{2}\left[\frac{2x^2}{n\pi} \sin\left(\frac{\pi n x}{2}\right) + \frac{4}{n\pi}\left[\frac{x \cos\frac{n\pi x}{2}}{\frac{n\pi x}{2}} + \left(\frac{2}{n\pi}\right)^2 \left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^4\right.\right.$$

$$= \frac{1}{2}\left(\frac{2x^2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{2\times 4x \cos\left(\frac{n\pi x}{2}\right)}{(n\pi)^2} + \frac{16}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\frac{8}{n\pi}(0) + \frac{32}{n^2\pi^2}\right] = \boxed{\frac{16}{n^2\pi^2} = a_n}$$

$$b_n = \frac{1}{2}\left[\int_0^4 x^2 \sin\left(\frac{n\pi x}{2}\right) dx \right]$$

even odd

$$= \frac{1}{2}\left[\frac{-x^2 \cdot 2\cos\frac{n\pi x}{2}}{n\pi} + \frac{4\times 2}{(n\pi)^2}\left[\int x \cancel{\sin} \cos\left(\frac{n\pi x}{2}\right) dx\right)\right.$$

Date______
Page______

$$= \frac{1}{2}\left[-\frac{2x^2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n\pi}\left[\frac{2x}{n\pi}\sin\left(\frac{n x\pi}{2}\right) + \frac{4}{(n\pi)^2}\cos\left(\frac{n x\pi}{2}\right)\right]\right]_0^4$$

$$= \frac{1}{2}\left[\frac{-2x^2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{8x}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right) + \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi x}{2}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\left(\frac{-32}{n\pi} + \cancel{\frac{16}{n^3\pi^3}}\right) - \left(\cancel{\frac{16}{n^3\pi^3}}\right)\right]$$

$$= \boxed{\frac{-16}{n\pi} = b_n}$$

Half Range of Sine Series

$f(x) = x^2 , \; 0 < x < 4$

$l = 4$

$$b_n = \frac{1}{2} \int_0^4 \left[x^2 \sin\left(\frac{n\pi x}{4}\right) dx \right]$$

$$= \frac{1}{2} \left[\frac{-x^2 \cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}} + \frac{2\times 4}{n\pi} \left[\int x \frac{\cos \frac{n\pi x}{4}\, dx}{\frac{n\pi}{4}} \right] \right.$$

$$= \frac{1}{2} \left[\frac{-4x^2}{n\pi} \cos\left(\frac{n\pi x}{4}\right) + \frac{8}{n\pi} \left[\frac{+x \sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi x}{4}\right)} \right. \right.$$

$$- \frac{4}{n\pi} \left[\int \sin\left(\frac{n\pi x}{4}\right) dx \right]$$

$$= \frac{1}{2} \left[\frac{-4x^2}{n\pi} \cos\left(\frac{n\pi x}{4}\right) + \frac{8}{n\pi} \left[\frac{4x}{n\pi} \sin\left(\frac{n\pi x}{4}\right) + \frac{4\times 4}{(n\pi)^2} \right. \right.$$

$$\left. \cos\left(\frac{n\pi x}{4}\right) \right]_0^4$$

$$= \frac{1}{2} \left[\frac{-4x^2}{n\pi} \cos\left(\frac{n\pi x}{4}\right) + \frac{32x}{n^2\pi^2} \sin\left(\frac{n\pi x}{4}\right) \right.$$

Date ______
Page ______

Fourier transform

$F(x) = e^{|x|}$

$F(-x) = e^{|-x|} = e^{|x|} = f(x)$

$\therefore$ function is Even $\quad -\pi < x < \pi$

$\therefore b_n = 0$

if $F(x) = e^{x}$; $\quad 0 < x < \pi \leftarrow$ Even

$= e^{-x}$; $\quad -\pi < x < 0$

$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{1}{\pi} \int_0^{\pi} e^{x}\, dx$

$= \frac{1}{2 \times \pi} \left[e^{x} \right]_0^{\pi} = \frac{1}{2 \times \pi} \left[e^{\pi} - 1 \right]$

$a_n = \frac{1}{\pi} \int_0^{\pi} e^{x} \cos n\pi \, dx$

$= \frac{1}{\pi} \left| \frac{e^{x}}{n^2+1} \left(\cos nx + n \sin nx \right) \right|_0^{\pi}$

$$(x^2+y^2+3)\,dx - 2xy\,dy = 0.$$

$$M = x^2+y^2+3 \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \left(2y+2y\right)$$

$$= -\frac{2}{x}$$

$$IF = e^{\int -\frac{2}{x}dx} = x^{-2} = 1/x^2$$

$$\frac{1}{x^2}\left(x^2+y^2+3\right)dx - \frac{1}{x^2}\,2xy\,dy = 0$$

$$\left(1 + \left(\frac{y}{x}\right)^2 + \frac{3}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N_1}{\partial x} = \frac{2y}{x^2}$$

$$\int M\,dx + \int N\,(\text{Excluded terms } x)\,dy$$

$$\boxed{x - \frac{y^2+3}{x} = c}$$

Date
Page

$(x^2y^2+2)y\,dx + (2-x^2y^2)x\,dy = 0$

$M = x^2y^3 + 2y$

$N = 2x - x^3y^2$

$IF = \dfrac{1}{Mx - Ny} = \dfrac{1}{x^3y^3 + \cancel{2xy} - \cancel{2yx} + x^3y^3}$

$= \dfrac{1}{2x^3y^3}$

$(x^2y^2+2)y\left(\dfrac{1}{2x^3y^3}\right)dx + (2-x^2y^2)\dfrac{x}{2x^3y^3}\,dy = 0$

$\dfrac{1}{2}\left(\dfrac{1}{x} + \dfrac{2}{x^3y^3}\right)dx + \dfrac{1}{2}\left[\dfrac{2}{x^3y^3} - \dfrac{1}{y}\right]dy = 0$

$M_1 = \dfrac{1}{2}\left[\dfrac{1}{x} + \dfrac{2}{x^3y^2}\right]$ $\quad N_1 = \dfrac{1}{2}\left[\dfrac{2}{x^2y^3} - \dfrac{1}{y}\right]$

$\dfrac{\partial M_1}{\partial y} = -\dfrac{2}{x^3y^3}$ $\quad \dfrac{\partial N_1}{\partial x} = -\dfrac{2}{x^3y^3}$

$= \int M\,dx + \int N\,(\text{Excluding terms } x)\,dy$

$= \dfrac{1}{2}\left[\log x - \dfrac{1}{x^2y^2}\right] - \dfrac{1}{2}\log y$

$\boxed{C = \log\left(\dfrac{x}{y}\right) - \dfrac{1}{x^2y^2}}$