Generating curves and swept surfaces by blended circles

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Abstract

We present an improved version of the spline curve defined by H.-J. Wenz (1996) for the interpolation of 2D curve data. This trigonometrically blended spline is curvature continuous, interpolates 3D curve data, and is generated by convex combinations of circular arcs and straight line segments. As an application we show a method for rounding off the vertices of spatial polygonal lines and generating swept surfaces. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper a new type of trigonometrically blended spline (shortly TBS) curve is presented. The basic idea of the construction stems from Wenz (1996), but a significant improvement in our method is that the generated spline curve interpolates 3D curve data with curvature continuity instead of interpolating 2D data with tangent continuity.

A practical application of the spline curve is the rounding off of polygon vertices with prescribed radii such that the resulting curve is curvature continuous, and the straight line segments and the circular arcs are exactly reproduced up to small pieces of transition segments. As another application a construction of canal shaped surfaces is shown by sweeping a cubic Bézier curve along such a TBS curve. This swept surface is tangent plane continuous, and has two types of isoparametric curves, trigonometric in one direction and polynomial in the other direction. A more generic swept surface defined by two TBS curves as leading curves and one cubic Bézier curve as generator is also shown and applied for the rounding problem of polyhedral vertices.

2. The interpolating trigonometrically blended spline curve

Let $\mathbf{p}_i$, $i = 1, \ldots, n$, be given points in three-dimensional Euclidean space. The main goal of this paper is to find a curvature continuous, i.e., GC^2 curve (Hoschek, 1992, Chapter 5) passing through the given points in the given order, which reproduces exact line segments and circular arcs if four consecutive points are lying on a line or on a circle, respectively.

The interpolating curve is constructed from $n - 1$ segments. Each segment is defined as the convex combination of two overlapping circular arcs or line segments using the neighbouring points in the data sequence. For this reason, the construction of the first and last segments requires the extension of the data sequence by the points $\mathbf{p}_0$ and $\mathbf{p}_{n+1}$. This can be done according to a properly chosen boundary condition. For the sake of simplicity, in our examples the 0th and $(n + 1)$th points are generated by the natural end conditions, what leads to zero curvature at the end points.

$$\mathbf{p}_0 := 2\mathbf{p}_1 - \mathbf{p}_2,$$
$$\mathbf{p}_{n+1} := 2\mathbf{p}_n - \mathbf{p}_{n-1}.$$

The ith segment of the required spline curve between the points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ is represented by the vector function $\mathbf{r}_i(t)$, $0 \leqslant t \leqslant 1$, $i = 1, \ldots, n - 1$, and is defined as the convex combination of two components (Fig. 1). Its *left component* $\mathbf{c}_i^L(t)$, $0 \leqslant t \leqslant 1$, is a line segment if the points $\mathbf{p}_{i-1}$, $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ are lying on a line, and is the arc between the points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ of the circle passing through $\mathbf{p}_{i-1}$, $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ otherwise. Similarly, the *right component* $\mathbf{c}_i^R(t)$, $0 \leqslant t \leqslant 1$, of the ith segment is a line segment if the points $\mathbf{p}_i$, $\mathbf{p}_{i+1}$ and $\mathbf{p}_{i+2}$ are lying on a line, and the corresponding arc of the circle passing through $\mathbf{p}_i$, $\mathbf{p}_{i+1}$ and $\mathbf{p}_{i+2}$ otherwise. The line segment $\mathbf{c}_i^L(t)$ or $\mathbf{c}_i^R(t)$ is represented by the vector function

$$(1 - t)\mathbf{p}_i + t\mathbf{p}_{i+1}, \quad t \in [0, 1], \ i = 1, \ldots, n - 1.$$

The circular arc between the points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ is generated by rotating the radius vector $\mathbf{mp}_i$ around the center point $\mathbf{m}$ of the circle representing the left or the right component, respectively. The rotational angle is $\alpha_i = \angle(\mathbf{p}_i, \mathbf{m}, \mathbf{p}_{i+1})$. The matrix of the rotation expressed in the appropriate coordinate system attached to the plane of the circle is
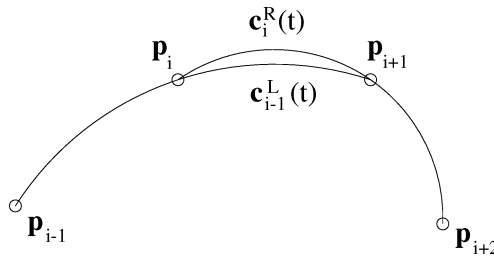

Fig. 1. The components of the ith segment.

$$\mathbf{R}_i(t) = \begin{pmatrix} \cos(t\alpha_i) & -\sin(t\alpha_i) & 0 \\ \sin(t\alpha_i) & \cos(t\alpha_i) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in [0,1], \ i = 1, \dots, n-1.$$

Consequently, the circular arc $\mathbf{c}_i^L(t)$ or $\mathbf{c}_i^R(t)$ is represented by the vector function

$$\mathbf{m} + \mathbf{R}_i(t)(\mathbf{p}_i - \mathbf{m}), \quad t \in [0,1], \ i = 1, \dots, n-1.$$

Accordingly, the left and right components of the ith segment are the overlapping parts of the two curve segments determined by the end points and one neighbouring point to the left and to the right, respectively. The ith segment of the interpolating TBS curve is defined by the convex combination of these two components.

$$\mathbf{r}_i(t) = \cos^2(t\pi/2)\mathbf{c}_i^L(t) + \sin^2(t\pi/2)\mathbf{c}_i^R(t), \quad t \in [0,1], \ i = 1, \dots, n-1. \tag{1}$$

Proposition. *The ith segment $(i = 1, \dots, n-1)$ of the constructed TBS curve between $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ is a line segment or a circular arc, if the points $\mathbf{p}_{i-1}$, $\mathbf{p}_i$, $\mathbf{p}_{i+1}$ and $\mathbf{p}_{i+2}$ are lying on a line or on a circle, respectively.*

Proof. In both cases the left component $\mathbf{c}_i^L(t)$ and the right component $\mathbf{c}_i^R(t)$, $t \in [0,1]$, of the ith segment $(i = 1, \dots, n-1)$ given by (1) are identical line segments or circular arcs, respectively. Therefore, their convex combination is the same line segment or circular arc between the points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$, $\mathbf{r}_i(t) = \mathbf{c}_i^L(t) = \mathbf{c}_i^R(t)$, $t \in [0,1]$, $i = 1, \dots, n-1$. $\square$

Theorem. *The segments of the constructed trigonometrically blended spline curve join with curvature continuity.*

Proof. We show that the segments $\mathbf{r}_i(t)$ and $\mathbf{r}_{i-1}(s)$, $t, s \in [0,1]$, $i = 1, \dots, n-1$, join with curvature continuity. Computing the derivatives of the ith segment at the starting point $t = 0$ and those of the $(i-1)$th segment at the end point $s = 1$ we obtain

$$\dot{\mathbf{r}}_i(0) = \dot{\mathbf{c}}_i^L(0), \qquad \ddot{\mathbf{r}}_i(0) = \ddot{\mathbf{c}}_i^L(0),$$

$$\mathbf{r}_{i-1}'(1) = \mathbf{c}_{i-1}'^R(1), \qquad \mathbf{r}_{i-1}''(1) = \mathbf{c}_{i-1}''^R(1),$$

where *dot* denotes derivative by t and *prime* denotes derivative by s. The line or circular segments $\mathbf{c}_i^L(t)$ and $\mathbf{c}_{i-1}^R(s)$, $t, s \in [0,1]$, $i = 1, \dots, n-1$, are determined by the same points $\mathbf{p}_{i-1}$, $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$, therefore they may differ only in the parametrization. For both segments the parameter is proportional to the arc length between the corresponding end points, therefore $\mathbf{c}_{i-1}^R(s) = \mathbf{c}_i^L(\lambda t + \mu)$, where λ and μ are real numbers and $\lambda > 0$. For the derivatives we obtain

$$\frac{d}{ds}\mathbf{c}_{i-1}^R = \lambda \frac{d}{ds}\mathbf{c}_i^L, \qquad \frac{d^2}{ds^2}\mathbf{c}_{i-1}^R = \lambda^2 \frac{d^2}{ds^2}\mathbf{c}_i^L,$$

and the curvatures of the neighbouring segments at the point $\mathbf{p}_i$ are equal:

$$\frac{|\dot{\mathbf{r}}_i \times \ddot{\mathbf{r}}_i|}{|\dot{\mathbf{r}}_i|^3}\Bigg|_{t=0} = \frac{|\mathbf{r}_{i-1}' \times \mathbf{r}_{i-1}''|}{|\mathbf{r}_{i-1}'|^3}\Bigg|_{s=1}.$$

In the case, when the data points $\mathbf{p}_i$, $i = 1, \ldots, n - 1$, are equidistant, $\lambda = 1$ holds for the parametrization of the neighbouring spline segments, therefore they join with C^2 continuity. □

Remark. The circular interpolant defined by linear combination as in (Wenz, 1996)

$$\mathbf{r}_i(t) = (1 - t)\mathbf{c}_i^L(t) + t\mathbf{c}_i^R(t), \quad t \in [0, 1], \ i = 1, \ldots, n - 1,$$

leads to a GC^1 continuous spline curve.

3. Rounding off the vertices of polygonal lines

Let the points $\mathbf{q}_j$, $j = 1, \ldots, k$, determine a polygonal line in three-dimensional space. Now we round off its vertices by tangential circular arcs, the radii of which are ρ_j, $j = 2, \ldots, k - 1$. The result of this construction is a chain of straight line and circular segments, and is GC^1 continuous. In order to replace the chain with a curve which is GC^2 continuous in a neighbourhood of the connection points, we define appropriate interpolation points and generate the TBS curve described in Section 2.

Consider the line segment $\mathbf{q}_{j-1}\mathbf{q}_j$ and the tangential circular arc rounding off the vertex $\mathbf{q}_j$ with the radius ρ_j, $j = 2, \ldots, k - 1$. Let $T_{j,1}$ and $T_{j,2}$ denote the tangent point of the circle on the side $\mathbf{q}_{j-1}\mathbf{q}_j$ and on the side $\mathbf{q}_j\mathbf{q}_{j+1}$ of the given polygon, respectively. We choose an ε tolerance value, and define six interpolation points denoted by $\mathbf{p}_{j,*}$ for each consecutive line segment $T_{j-1,2}T_{j,1}$ and circular arc $T_{j,1}T_{j,2}$, $j = 2, \ldots, n - 1$, as follows: $\mathbf{p}_{j,3} = T_{j,1}$, $\mathbf{p}_{j,6} = T_{j,2}$. $\mathbf{p}_{j,1}$ and $\mathbf{p}_{j,2}$ are inner points on the segment $T_{j-1,2}T_{j,1}$ at distance ε from the end points. Similarly, $\mathbf{p}_{j,4}$ and $\mathbf{p}_{j,5}$ are inner points on the circular arc $T_{j,1}T_{j,2}$ at distance ε from the end points (Fig. 2). We set up one more interpolation point on the first and last side of the polygonal line. Finally, the sequence of the constructed interpolation
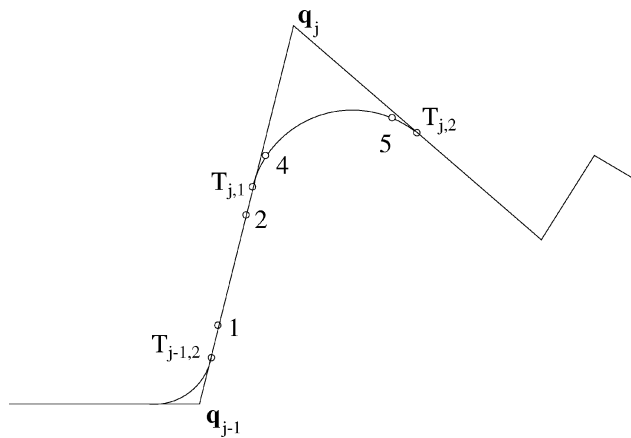

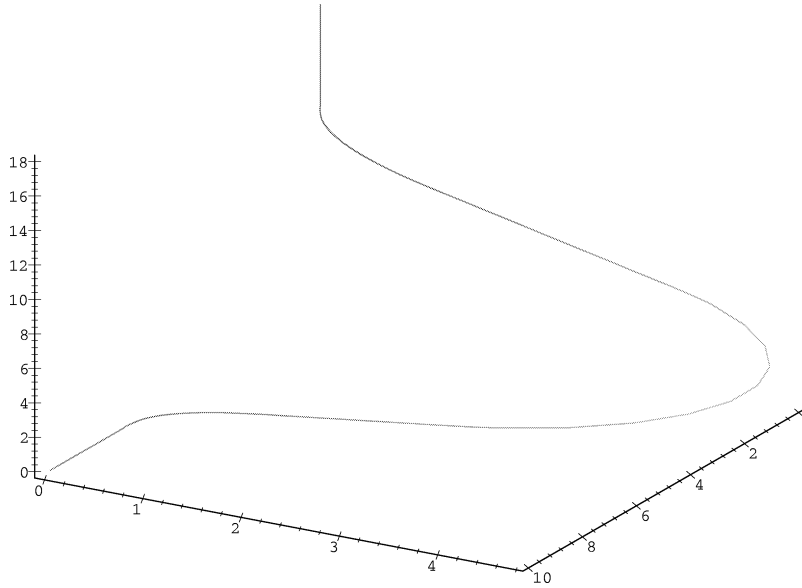Fig. 2. Interpolation points of the trigonometric spline curve.

Fig. 3. Rounded polygonal line. The radii are 2, 4 and 5, $\varepsilon = 0.5$.

points reindexed by $\mathbf{p}_i$ starts with $\mathbf{p}_1 = \mathbf{q}_1$ and ends with $\mathbf{p}_n = \mathbf{q}_k$, $n = 6(k-1) + 2$. The points $\mathbf{p}_0$ and $\mathbf{p}_{n+1}$ are determined according to the chosen boundary condition, e.g., $\mathbf{p}_0 = 2\mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{p}_{n+1} = 2\mathbf{p}_n - \mathbf{p}_{n-1}$.

The generated TBS curve interpolates the points $\mathbf{p}_i$, $i = 1, \ldots, n$, and is GC^2 continuous. The segments between the points $\mathbf{p}_{j,1}$ and $\mathbf{p}_{j,2}$ are straight line segments, the segments between $\mathbf{p}_{j,4}$ and $\mathbf{p}_{j,5}$ are circular arcs, and the segments in the neighbourhood of the tangent points $T_{j,1}$ and $T_{j,2}$ are transition segments between the tangential lines and circles, $j = 2, \ldots, k-1$.

The data of the spline curve are a polygonal line, rounding radii and tolerances.

A spatial polygonal line rounded off at vertices with prescribed radii is shown in Fig. 3.

4. Construction of swept surfaces

First we generate a swept surface with the TBS curve constructed in Section 3 as leading curve and a cubic Bézier curve as generator. Instead of sweeping with the generator along the leading curve, we move its control polygon, and generate the Bézier curve (Rogers, 1990, Chapter 5) at any point of the spline. During the sweeping the control polygon should move together with the Frenet frame of the curve. As the Frenet frame of a straight line segment does not exist, we define a local coordinate system at each of its end points, and use them as a Frenet frame. Then we construct the control polygon of the Bézier curve at the inner points of the spline segments as follows.

Let $\mathbf{V}_k$ ($k = 1, \ldots, 4$) denote the control vertices of a given Bézier curve in the starting position $\mathbf{p}_1$ which is the first end point of the leading curve. At this point we define an

orthogonal local coordinate system. Its first axis lies on the straight line $\mathbf{p}_1\mathbf{p}_2$. The second axis is orthogonal to this line in the plane of the first rounding circle and points to the side of the center point of this circle. The third axis forms a right oriented system with the first two axes. The coordinates of the control points $\mathbf{V}_k$ $(k = 1, \ldots, 4)$ referring to this local coordinate system are kept fixed while placing the control frame into the Frenet frame at the interpolation points $\mathbf{p}_i$ $(i = 1, \ldots, n)$. In this way we obtain the control points $\mathbf{V}_{i,k}$ from the points $\mathbf{V}_k$ $(k = 1, \ldots, 4)$. At an interpolation point which is an end point of a straight line segment ($\mathbf{p}_{j,1}$ or $\mathbf{p}_{j,2}$) we set up a local coordinate system with parallel axes to the axes of the Frenet frame computed at the neighbouring interpolation point at ε distance ($T_{j-1,2}$ or $T_{j,1}$, respectively in Fig. 2) from this end point. Here ε is the tolerance used in the construction of the curve. Consider the segment $\mathbf{r}_i(t)$, $i = 1, \ldots, n - 1$, of the TBS curve between the interpolation points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ and the vertices of the control polygons $\mathbf{V}_{i,k}$ and $\mathbf{V}_{i+1,k}$, $k = 1, \ldots, 4$, respectively.

We define the control vertices of the Bézier curve at the inner point of the curve segment as follows

$$\mathbf{V}_{i,k}(t) = \cos^2(t\pi/2)\mathbf{V}_{i,k} + \sin^2(t\pi/2)\mathbf{V}_{i+1,k} + \mathbf{r}_i(t),$$
$$t \in [0, 1], \quad k = 1, \ldots, 4, \ i = 1, \ldots, n - 1. \tag{2}$$

Generating the Bézier curve from the control points $\mathbf{V}_{i,k}(t)$, we obtain the equation of the ith patch of the swept surface written in matrix form

$$\mathbf{R}_i(t, u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{B}^{(3)} \begin{bmatrix} \mathbf{V}_{i,1}(t) & \mathbf{V}_{i,2}(t) & \mathbf{V}_{i,3}(t) & \mathbf{V}_{i,4}(t) \end{bmatrix}^{\mathrm{T}},$$
$$t, u \in [0, 1], \ i = 1, \ldots, n - 1, \tag{3}$$

where the matrix

$$\mathbf{B}^{(3)} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is the coefficient matrix of the cubic Bernstein polynomials.

A canal shaped surface generated from a planar cubic Bézier curve sweeping along the TBS curve in Fig. 3 is shown in Fig. 4. For each patch eight Bézier curves are drawn, each with six points. Their control frames having the vertices written in (2) are situated close to the curves, therefore are not shown.

Proposition. *The path* $\mathbf{V}_{i,k}(t)$, *$t \in [0, 1]$, $i = 1, \ldots, n - 1$, of the control point $\mathbf{V}_k$, $k = 1, \ldots, 4$, defined by (2) is a GC^1 continuous curve.*

Proof. Consider the control polygons at the inner points of the neighbouring curve segments $\mathbf{r}_{i-1}(s)$ and $\mathbf{r}_i(t)$, $s, t \in [0, 1]$ of the TBS curve, $i = 1, \ldots, n - 1$,

$$\mathbf{V}_{i-1,k}(s) = \cos^2(s\pi/2)\mathbf{V}_{i-1,k} + \sin^2(s\pi/2)\mathbf{V}_{i,k} + \mathbf{r}_{i-1}(s),$$
$$s \in [0, 1], \ k = 1, \ldots, 4,$$

and

$$\mathbf{V}_{i,k}(t) = \cos^2(t\pi/2)\mathbf{V}_{i,k} + \sin^2(t\pi/2)\mathbf{V}_{i+1,k} + \mathbf{r}_i(t), \quad t \in [0, 1], \ k = 1, \ldots, 4,$$
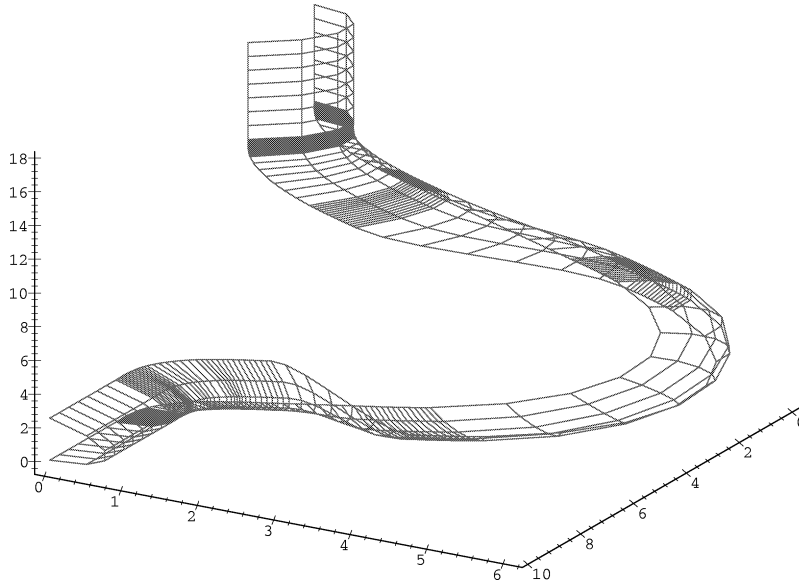
Fig. 4. Swept surface with one leading curve.

where the points $\mathbf{V}_{i,k}$ are placed into the Frenet frame at the interpolation point $\mathbf{p}_i$, $i = 1, \ldots, n - 1$. We compute the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{V}_{i-1,k}(s) = -\frac{\pi}{2}\sin(s\pi)\mathbf{V}_{i-1,k} + \frac{\pi}{2}\sin(s\pi)\mathbf{V}_{i,k} + \mathbf{r}'_{i-1}(s), \quad s \in [0, 1],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{i,k}(t) = -\frac{\pi}{2}\sin(t\pi)\mathbf{V}_{i,k} + \frac{\pi}{2}\sin(t\pi)\mathbf{V}_{i+1,k} + \dot{\mathbf{r}}_i(t), \quad t \in [0, 1].$$

The values at the joining point of the curve segments $\mathbf{r}_{i-1}(s)$ and $\mathbf{r}_i(t)$, $i = 1, \ldots, n - 1$, are

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{V}_{i-1,k}(1) = \mathbf{r}'_{i-1}(1) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{i,k}(0) = \dot{\mathbf{r}}_i(0).$$

These vectors are parallel and equally oriented according to the theorem in Section 2. $\quad \square$

Corollary. *The t-isoparametric lines of the swept surface defined in* (3) *are GC^1 continuous. Consequently, the surface patches join with tangent plane continuity.*

5. Examples

The next swept surfaces are generated by two leading curves and one generator. The leading curves are TBS curves defined by polygonal lines, rounding radii and tolerance values as in Section 3. The first leading curve passes through the interpolation points $\mathbf{p}_i$, and the second one passes through the points $\mathbf{q}_i$, $i = 1, \ldots, n$. Let us denote the curve segments by $\mathbf{r}_{1i}(t)$ and $\mathbf{r}_{2i}(t)$, respectively, $t \in [0, 1]$, $i = 1, \ldots, n - 1$. The sweeping

Bézier curve is given by its four control points in the starting position. We assume that the first control point $\mathbf{V}_1$ of the sweeping Bézier curve is moving along the leading curve $\mathbf{r}_1(t)$ and the fourth one, $\mathbf{V}_4$, is moving along the second leading curve $\mathbf{r}_2(t)$. Then we determine the position of the control point $\mathbf{V}_2$ referring to the Frenet frame of $\mathbf{r}_1(t)$ at the interpolation point $\mathbf{p}_i$ and similarly, the control point $\mathbf{V}_3$ with the help of Frenet frame of $\mathbf{r}_2(t)$ at the interpolation point $\mathbf{q}_i$, $i = 1, \ldots, n$. The shape of the control frames $\mathbf{V}_{i,k}$, $(i = 1, \ldots, n; k = 1, \ldots, 4)$ may, of course, distorted for each i in this construction.

The definition of the moving control polygon is the following

$$\mathbf{V}_{i,1}(t) = \mathbf{r}_{1i}(t),$$
$$\mathbf{V}_{i,2}(t) = \cos^2(t\pi/2)\mathbf{V}_{i,2} + \sin^2(t\pi/2)\mathbf{V}_{i+1,2} + \mathbf{r}_{1i}(t),$$
$$\mathbf{V}_{i,3}(t) = \cos^2(t\pi/2)\mathbf{V}_{i,3} + \sin^2(t\pi/2)\mathbf{V}_{i+1,3} + \mathbf{r}_{2i}(t),$$
$$\mathbf{V}_{i,4}(t) = \mathbf{r}_{2i}(t), \quad t \in [0, 1], \ i = 1, \ldots, n-1.$$

The equation of the swept surface $\mathbf{R}_i(t, u)$, $t, u \in [0, 1]$, $i = 1, \ldots, n-1$, is then obtained by generating the Bézier curve with the parameter u from the control points $\mathbf{V}_{i,k}(t)$, $k = 1, \ldots, 4$ as in (3).

The following examples show swept surfaces generated with two leading curves and with one cubic Bézier curve as generator. The leading curves are constructed as rounded polygons represented by interpolating TBS curves.

The surface shown in Figs. 5 and 6 has its two leading curves in parallel planes, both are constructed from rectangles by rounding off the vertices with circular arcs of equal radii.
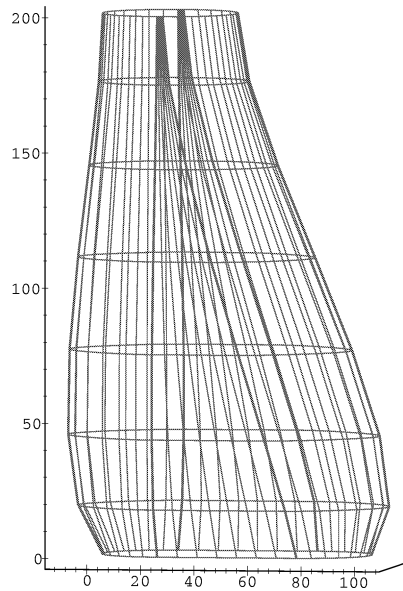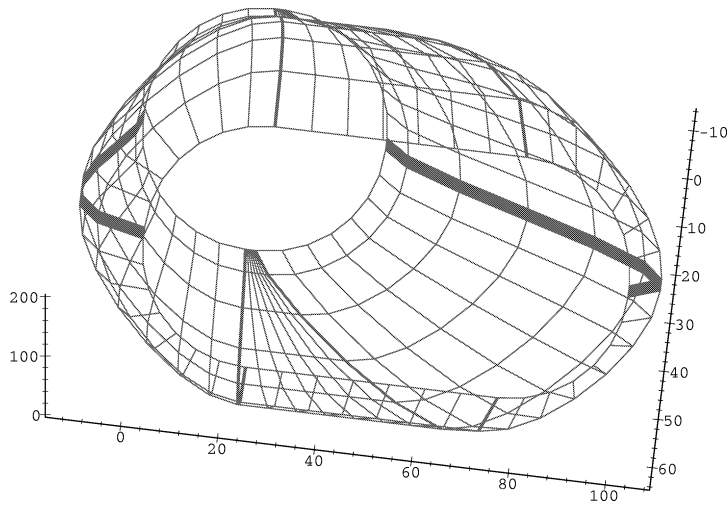

Fig. 5. Swept surface with two leading curves.

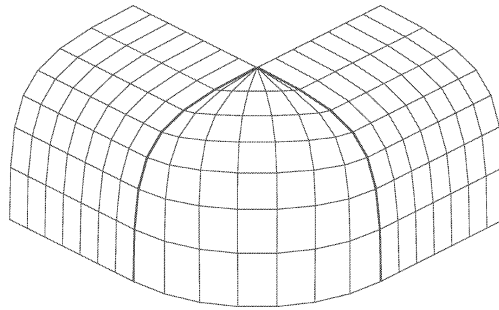Fig. 6. The surface in Fig. 5 in different projection.

Fig. 7. Rounded convex corner.

The next two examples show how the swept surface can be applied in praxis, for example in the solution of the suit case corner problem.

The surface in Fig. 7 is constructed along the edges of a rectangular prism in such a way that the two leading curves are lying on neighbouring faces joining along the specified edges. The rounding radius of the upper leading curve is ε and that of the lower one is a given ρ. The first and last edges of the Bézier control polygon are lying in the horizontal and vertical faces of the prism, respectively. This ensures the tangential fitting of the generated swept surface and the polyhedral faces. The value of ε is defined as the prescribed tolerance value. The sweeping Bézier curve can approximate a circular arc within a given tolerance as shown in (Goldapp, 1991), or can be constructed with given geometric properties as in (Juhász, 1998).

The surface in Fig. 8 is an example for rounding off a concave vertex. The leading curves of the swept surface are constructed similarly as in the example in Fig. 7, but the sweeping Bézier curve is constructed tangentially to the inside of the prism.
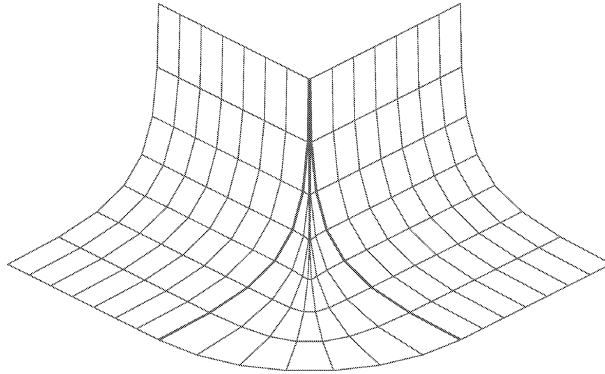
Fig. 8. Rounded concave corner.

This rounding effect is similar to that of the blending surface definition with two rail (leading) curves in (Filip, 1989).

The computations and the figures have been made by the symbolical algebraic program package Maple V R3.

6. Conclusions

An interpolating trigonometrically blended spline (TBS) curve has been defined from circular arcs and line segments, which is curvature continuous and gives a solution for rounding off polygonal vertices. The new result is the specific convex combination of the interpolating components, which reproduces circular arcs and straight line segments if four consecutive points are lying on a circle or on a line, respectively. A swept surface of mixed type, trigonometric in one parameter and polynomial in the other one, has been constructed with the help of the TBS curve, and applied for rounding off polyhedral vertices with tangent plane continuity.

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