THOMAS / FINNEY

9TH EDITION

THOMAS
FINNEY

CALCULUS

9TH EDITION

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9TH EDITION

Calculus and Analytic Geometry

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With the collaboration of
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Naval Postgraduate School

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To the Instructor

This Is a Major Revision

Throughout the 40 years that it has been in print, Thomas/Finney has been used to support a variety of teaching methods from traditional to experimental. In response to the many exciting currents in teaching calculus in the 1990s, the new edition is the most extensive revision of Thomas/Finney ever. We have built on the traditional strengths of the book—excellent exercises, sound mathematics, variety in applications—to produce a flexible text that contains all the elements needed to teach the many different kinds of courses that exist today.

A book does not make a course: The instructor and the students do. With this in mind we have added features to Thomas/Finney 9th edition to make it the most flexible calculus teaching resource yet.

- The exercises have been reorganized to facilitate assigning a subset of the material in a section.
- The grapher explorations, all accessible with any graphing calculator, many suitable for in-class and group work, have been expanded.
- New Computer Algebra System (CAS) explorations and projects that require a CAS have been included. Some of these can be done quickly while others require several hours. All are suitable for either individual or group work. You will find a list of CAS exercise topics following the Table of Contents.
- Technology Connection notes appear throughout the text suggesting experiments students might do with a grapher to supplement their understanding of a given topic. These notes are meant to encourage students to think of their grapher as a casually available tool, like a pencil.
- We revised the entire first semester and large parts of the second and third semesters to provide what we believe is a cleaner, more visual, and more accessible book.

With all these changes, we have not compromised our belief that the fundamental goal of a calculus book is to prepare students to enter the scientific community.

Students Will Find Even More Support for Creative Problem Solving

Throughout this book, we have included examples and discussions that encourage students to think visually and numerically. Almost every exercise set has easy to

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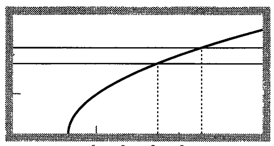


[0, 3] by [0, 3]

Keeping x between 1.75 and 2.28 will keep y between 1.8 and 2.2.

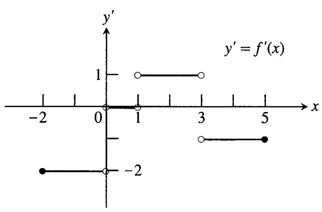
Technology *Target Values* You can experiment with target values on a graphing utility. Graph the function together with a target interval defined by horizontal lines above and below the proposed limit. Adjust the range or use zoom until the function's behavior inside the target interval is clear. Then observe what happens when you try to find an interval of x-values that will keep the function values within the target interval. (See also Exercises 7–14 and CAS Exercises 61–64.)

For example, try this for $f(x) = \sqrt{3x - 2}$ and the target interval (1.8, 2.2) on the y-axis. That is, graph $y_1 = f(x)$ and the lines $y_2 = 1.8$, $y_3 = 2.2$. Then try the target intervals (1.98, 2.02) and (1.9998, 2.0002).

32. *Recovering a function from its derivative*

a) Use the following information to graph the function f over the closed interval $[-2, 5]$.

i) The graph of f is made of closed line segments joined end to end.
ii) The graph starts at the point $(-2, 3)$.
iii) The derivative of f is the step function in Fig. 2.13.



2.13 The derivative graph for Exercise 32.

b) Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

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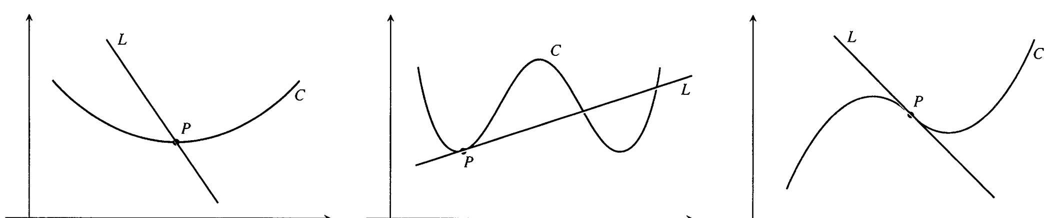
mid-level exercises that require students to generate and interpret graphs as a tool for understanding mathematical or real-world relationships. Many sections also contain a few more challenging problems to extend the range of the mathematically curious.

This edition has more than 2300 figures to appeal to the students' geometric intuition. Drawing lessons aid students with difficult 3-dimensional sketches, enhancing their ability to think in 3-space. In this edition we have increased the use of visualization internal to the discussion. The burden of exposition is shared by art in the body of the text when we feel that pictures and text together will convey ideas better than words alone.

Throughout the text, students are asked to experiment, investigate, and explain. Writing exercises are placed throughout the text. In addition, each chapter end contains a list of questions that ask students to review and summarize what they have learned. Many of these exercises make good writing assignments.



L meets C only at P but is not tangent to C.

L is tangent to C at P but meets C at several points.

L is tangent to C at P but lies on two sides of C, crossing C at P.

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1.49 Exploding myths about tangent lines.

Students Will Master Techniques

Problem Solving Strategies We believe that the students learn best when procedural techniques are laid out as clearly as possible. To this end we have revisited the summaries of the steps used to solve problems, adding some where necessary, deleting some where a thought process rather than a technique was at issue, and making each one clear and useful. As always, we are especially careful that examples in the text follow the steps outlined by the discussion.

Exercises Every exercise set has been reviewed and revised. Exercises are now *grouped by topic,* with special sections for grapher explorations. Many sections also have a set of Computer Algebra System (CAS) Explorations and Projects, a new feature for this edition. Within each group, the exercises are graded and paired. Within this framework, the exercises generally follow the order of presentation of the text.

Exercises that require a calculator or computer are identified by icons: calculator exercise, graphing utility (such as graphing calculator) exercise, and Computer Algebra System exercise.

Hidden Behavior

Sometimes graphing f' or f'' will suggest where to zoom in on a computer generated graph of f to reveal behavior hidden in the grapher's original picture.

Checklist for Graphing a Function $y = f(x)$

1. Look for symmetry.
   Is the function even? odd?
2. Is the function a shift of a known function?
3. Analyze dominant terms.
   Divide rational functions into polynomial + remainder.
4. Check for asymptotes and removable discontinuities.
   Is there a zero denominator at any point?
   What happens as $x \to \pm\infty$?
5. Compute f' and solve $f' = 0$. Identify critical points and determine intervals of rise and fall.
6. Compute f'' to determine concavity and inflection points.
7. Sketch the graph's general shape.
8. Evaluate f at special values (endpoints, critical points, intercepts).
9. Graph f, using dominant terms, general shape, and special points for guidance.

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Within the exercise sets, we have practice exercises, exercises that encourage critical thinking, more challenging exercises (in subsections marked "Applications and Theory"), and exercises that require writing in English about concepts. Writing exercises are placed both throughout the exercise sets, and in an end-of-chapter feature called "Questions to Guide Your Review."

Chapter End At the end of each chapter are three features with questions that summarize the chapter in different ways.

Questions to Guide Your Review ask students to think about concepts and verbalize their understanding without trying to calculate numeric answers. These are, as always, suitable for writing exercises.

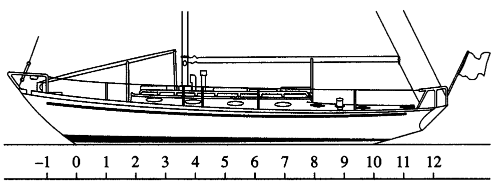
Practice Exercises provide a review of the techniques, ideas, and key applications.

Additional Exercises—Theory, Examples, Applications supply challenging applications and theoretic problems that deepen the understanding of mathematical ideas.

Applications, Technology, History—Features That Bring Calculus to Life

Applications and Examples It has been a hallmark of this book through the years that we illustrate applications of calculus with real data based on already familiar situations or situations students are likely to encounter soon. Throughout the text, we cite sources foı the data and/or articles from which the applications are drawn, helping students understand that calculus is a current, dynamic field. Most of these appli-

17. *A sailboat's displacement.* To find the volume of water displaced by a sailboat, the common practice is to partition the waterline into 10 subintervals of equal length, measure the cross section area $A(x)$ of the submerged portion of the hull at each partition point, and then use Simpson's rule to estimate the integral of $A(x)$ from one end of the waterline to the other. The table here lists the area measurements at "Stations" 0 through 10, as the partition points are called, for the cruising sloop *Pipedream,* shown here. The common subinterval length (distance between consecutive stations) is $h = 2.54$ ft (about 2′ 6 1/2″, chosen for the convenience of the builder).



a) Estimate *Pipedream*'s displacement volume to the nearest cubic foot.

Station	Submerged area (ft^2)
0	0
1	1.07
2	3.84
3	7.82
4	12.20
5	15.18
6	16.14
7	14.00
8	9.21
9	3.24
10	0

b) The figures in the table are for seawater, which weighs 64 lb/ft^3. How many pounds of water does *Pipedream* displace? (Displacement is given in pounds for small craft, and long tons [1 long ton = 2240 lb] for larger vessels.)

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cations are directed toward science and engineering, but there are many from biology and the social sciences as well.

Technology: Graphing Calculator and Computer Algebra Systems Explorations *Virtually every section* of the text contains calculator exercises that explore numerical patterns and/or graphing calculator exercises that ask students to generate and interpret graphs as a tool to understanding mathematical and real-world relationships. Many of the calculator and graphing calculator exercises are suitable for classroom demonstration or for group work by students in or out of class.

Computer Algebra System (CAS) exercises have been added to every chapter. These exercises, 160 in all, have been tested on both Mathematica and Maple. A full list of CAS exercise topics follows the Table of Contents.

As in previous editions, $\sec^{-1}x$ has been defined so that its range,$[0, \pi/2) \cup (\pi/2, \pi]$, and derivative, $1/(|x|\sqrt{x^2-1})$, agree with the results returned by Computer Algebra Systems and scientific calculators.

Notes appear throughout the text encouraging students to explore with graphers.

History Any student is enriched by seeing the human side of mathematics. As in earlier editions, we feature history boxes that describe the origins of ideas, conflicts concerning ownership of ideas, and interesting sidelights into popular topics such as fractals and chaos.

The Many Faces of This Book

Mathematics Is a Formal and Beautiful Language A good part of the beauty of the calculus lies in the fact that it is a stunning creation of the human mind. As in previous editions we have been careful to say only what is true and mathematically sound. In this edition we reviewed every definition, theorem, corollary, and proof for clarity and mathematical correctness.

Even Better Suited to Be the Reference Text in a Reform Course Whether calculus is taught by a traditional lecture or entirely in labs with individual and group learning which focuses on numeric and graphical experimentation, ideas and techniques need to be articulated clearly. This book provides the exercises for computer and grapher experiments and group learning and, in a traditional format, the summation of the lesson—the formal statement of the mathematics and the clear presentation of the technique.

Students Will Learn from This Book for Many Years to Come We provide far more material than any one instructor would want to teach. We do this intentionally. Students can continue to learn calculus from this book long after the class has ended. It provides an accessible review of the calculus a student has already studied. It is a resource for the working engineer or scientist.

Content Features of the Ninth Edition

Preliminary Material

- Lines, functions, and graphs are reviewed briefly.
- Trigonometric functions (formerly treated in an Appendix) are included.

Limits

- The limit is introduced through rates of change (Section 1.1) but defined before the derivative.
- The initial discussion of the limit is intuitive, using numeric and graphic examples of rates of change.
- The basic rules for working with limits are presented in Section 1.2.
- The limit is presented formally in Section 1.3, using input/output control systems to motivate the ε - δ definition. Covering the formal definition of the limit is optional.
- Chapter 1 concludes with the definition of the tangent line and instantaneous rate of change at a point, bringing to a close the investigation begun in Section 1.1.

Derivatives

- Chapter 2 opens with the concept and definition of the derivative as a function.
- The treatment of implicit differentiation has been revised (Section 2.6).
- The treatment of related rates has been moved earlier in the text (Section 2.7).

Applications of the Derivative

- Extrema (Section 3.1) and the Mean Value Theorem (Section 3.2) are now treated in separate sections. The first section presents the motivating problems of maximization and antidifferentiation. The second section provides motivating questions about antidifferentiation, and the Mean Value Theorem provides the answers. Testing critical points is the subject of Section 3.3.
- The graphing sections (Sections 3.4, 3.5) have been revised to emphasize the qualitative reading of graphs.
- Section 3.5 on asymptotes and dominant terms has been rewritten to present a unified approach to graphing rational functions.
- Presentation of L'Hôpital's Rule is postponed to Chapter 6 where it is applied to comparisons with exponential and logarithmic functions.
- Quadratic approximation, formerly in Chapter 3, has been included with Taylor polynomials in Chapter 8.

Integration

- As in previous editions, the indefinite integral is covered before the definite integral (Section 4.1).
- Differential equations and initial value problems are presented immediately after the indefinite integral (Section 4.2).
- Substitution is introduced for indefinite integrals in Section 4.3 and discussed again for definite integrals in Section 4.8.
- The definite integral is motivated by estimating with finite sums (Section 4.4).
- Some techniques of integration have been moved into Chapter 6 (Transcendental Functions), making Chapter 7 (Techniques of Integration) a shorter, more focused chapter.
- Integration using a Computer Algebra System (CAS) is covered in Section 7.5 along with integral tables.

Sequences and Series

- The introduction to sequences has been spread over two sections (Sections 8.1, 8.2), providing more time for this idea.
- Chapter 8 has been reorganized to allow one section per lecture. (See the Table of Contents.)
- Power series are applied to solve differential equations and initial value problems (Section 8.11).

Conic Sections

- The geometry of conic sections is treated in Section 9.1.
- Eccentricity is covered separately in Section 9.2, where it is used to classify the conics.

Differential Equations

- Differential equations and initial value problems are previewed in Section 3.2 by introducing students to the idea of determining functions from derivatives and initial values. Section 3.4 continues the preparation for differential equations by showing how to sketch the graph of a function given the formula for its first derivative and a point through which the graph must pass.
- Differential equations, initial value problems, and their applications then become the central topics of the following sections:

4.2	Differential Equations, Initial Value Problems, and Mathematical Modeling
6.5	Growth and Decay (the initial value problem $y' = ky$, $y(0) = y_0$ and its applications)
6.11	First Order Differential Equations
6.12	Euler's Numerical Method; Slope Fields

Solutions of differential equations and initial value problems appear as appropriate in exercise sets throughout the remainder of the text.

- Section 4.7 solves initial value problems using the Fundamental Theorem of Calculus, and Section 8.11 solves differential equations and initial value problems with power series.

Multivariate Calculus

- The treatments of curvature and torsion and the **TNB**-frame in Chapter 11 (Vector-Valued Functions and Motion in Space) has been simplified and unified.
- Chapter 14 (Integration in Vector Fields) has been reorganized to place all the material on line integrals before the material on surface integrals.
- There is now a section (Section 14.6) on integration over parametrized surfaces.

Supplements for the Instructor

OmniTest³ in DOS-Based Format: This easy-to-use software is developed exclusively for Addison-Wesley by ips Publishing, a leader in computerized testing and assessment. Among its features are the following.

- **DOS interface is easy to learn and operate.** The windows look-alike interface makes it easy to choose and control the items as well as the format for each test.
- **You can easily create make-up exams, customized homework assignments, and multiple test forms to prevent plagiarism.** OmniTest³ is

algorithm driven—meaning the program can automatically insert new numbers into the same equation—creating hundreds of variations of that equation. The numbers are constrained to keep answers reasonable. This allows you to create a virtually endless supply of parallel versions of the same test. This new version of OmniTest also allows you to "lock in" the values shown in the model problem, if you wish.

- **Test items are keyed by section** to the text. Within the section, you can select questions that test individual objectives from that section.
- **You can enter your own questions** by way of OmniTest^3's sophisticated editor—complete with mathematical notation.

Instructor's Solutions Manual by Maurice D. Weir (Naval Postgraduate School). This two-volume supplement contains the worked-out solutions for *all* the exercises in the text.

Answer Book contains short answers to most exercises in the text.

Supplements for the Instructor and Student

Student Study Guide by Maurice D. Weir (Naval Postgraduate School). Organized to correspond with the text, this workbook in a semiprogrammed format increases student proficiency with study tips and additional practice.

Student Solutions Manual by Maurice D. Weir (Naval Postgraduate School). This manual is designed for the student and contains carefully worked-out solutions to all of the odd-numbered exercises in the text.

Differential Equations Primer A short, supplementary manual containing approximately a chapter's-worth of material. Available should the instructor choose to cover this material within the calculus sequence.

Technology-Related Supplements

Analyzer* This program is a tool for exploring functions in calculus and many other disciplines. It can graph a function of a single variable and overlay graphs of other functions. It can differentiate, integrate, or iterate a function. It can find roots, maxima and minima, and inflection points, as well as vertical asymptotes. In addition, Analyzer* can compose functions, graph polar and parametric equations, make families of curves, and make animated sequences with changing parameters. It exploits the unique flexibility of the Macintosh wherever possible, allowing input to be either numeric (from the keyboard) or graphic (with a mouse). Analyzer* runs on Macintosh II, Plus, or better.

The Calculus Explorer Consisting of 27 programs ranging from functions to vector fields, this software enables the instructor and student to use the computer as an "electronic chalkboard." The Explorer is highly interactive and allows for manipulation of variables and equations to provide graphical visualization of mathematical relationships that are not intuitively obvious. The Explorer provides user-friendly operation through an easy-to-use menu-driven system, extensive on-line documentation, superior graphics capability, and fast operation. An accompanying manual includes sections covering each program, with appropriate examples and exercises. Available for IBM PC/compatibles.

InSight A calculus demonstration software program that enhances understanding of calculus concepts graphically. The program consists of ten simulations. Each presents an application and takes the user through the solution visually. The format is interactive. Available for IBM PC/compatibles.

Laboratories for Calculus I Using Mathematica By Margaret Höft, The University of Michigan–Dearborn. An inexpensive collection of *Mathematica* lab experiments consisting of material usually covered in the first term of the calculus sequence.

Math Explorations Series Each manual provides problems and explorations in calculus. Intended for self-paced and laboratory settings, these books are an excellent complement to the text.

Exploring Calculus with a Graphing Calculator, Second Edition, by Charlene E. Beckmann and Ted Sundstrom of Grand Valley State University.

Exploring Calculus with Mathematica, by James K. Finch and Millianne Lehmann of the University of San Francisco.

Exploring Calculus with Derive, by David C. Arney of the United States Military Academy at West Point.

Exploring Calculus with Maple, by Mark H. Holmes, Joseph G. Ecker, William E. Boyce, and William L. Seigmann of Rensselaer Polytechnic Institute.

Exploring Calculus with Analyzer*, by Richard E. Sours of Wilkes University.

Exploring Calculus with the IBM PC Version 2.0, by John B. Fraleigh and Lewis I. Pakula of the University of Rhode Island.

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Exercises

In addition, we thank the following people who reviewed the exercise sets for content and balance and contributed many of the interesting new exercises:

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Answers

We would like to thank Cynthia Hutcherson for providing answers for exercises in some of the chapters in this edition. We also appreciate the work of an outstanding team of graduate students at Stanford University, who checked every answer in the text for accuracy: Miguel Abreu, David Cardon, Tanya Kalich, Jeffrey D. Oldham, and Julie Roskies. Jeffrey D. Oldham also tested all the CAS exercises, and we thank him for his many helpful suggestions.

Other Contributors

We are particularly grateful to Maurice D. Weir, Naval Postgraduate School, who shared his teaching ideas throughout the preparation of this book. He produced the final exercise sets and wrote most of the CAS exercises for this edition. We appreciate his constant encouragement and thoughtful advice.

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To the Student

What Is Calculus?

Calculus is the mathematics of motion and change. Where there is motion or growth, where variable forces are at work producing acceleration, calculus is the right mathematics to apply. This was true in the beginnings of the subject, and it is true today.

Calculus was first invented to meet the mathematical needs of the scientists of the sixteenth and seventeenth centuries, needs that were mainly mechanical in nature. Differential calculus dealt with the problem of calculating rates of change. It enabled people to define slopes of curves, to calculate velocities and accelerations of moving bodies, to find firing angles that would give cannons their greatest range, and to predict the times when planets would be closest together or farthest apart. Integral calculus dealt with the problem of determining a function from information about its rate of change. It enabled people to calculate the future location of a body from its present position and a knowledge of the forces acting on it, to find the areas of irregular regions in the plane, to measure the lengths of curves, and to find the volumes and masses of arbitrary solids.

Today, calculus and its extensions in mathematical analysis are far reaching indeed, and the physicists, mathematicians, and astronomers who first invented the subject would surely be amazed and delighted, as we hope you will be, to see what a profusion of problems it solves and what a range of fields now use it in the mathematical models that bring understanding about the universe and the world around us. The goal of this edition is to present a modern view of calculus enhanced by the use of technology.

How to Learn Calculus

Learning calculus is not the same as learning arithmetic, algebra, and geometry. In those subjects, you learn primarily how to calculate with numbers, how to simplify algebraic expressions and calculate with variables, and how to reason about points, lines, and figures in the plane. Calculus involves those techniques and skills but develops others as well, with greater precision and at a deeper level. Calculus introduces so many new concepts and computational operations, in fact, that you will no longer be able to learn everything you need in class. You will have to learn a fair amount on your own or by working with other students. What should you do to learn?

1. Read the text. You will not be able to learn all the meanings and connections you need just by attempting the exercises. You will need to read relevant

passages in the book and work through examples step by step. Speed reading will not work here. You are reading and searching for detail in a step-by-step logical fashion. This kind of reading, required by any deep and technical content, takes attention, patience, and practice.

2. Do the homework, keeping the following principles in mind.
   a) Sketch diagrams whenever possible.
   b) Write your solutions in a connected step-by-step logical fashion, as if you were explaining to someone else.
   c) Think about why each exercise is there. Why was it assigned? How is it related to the other assigned exercises?
3. Use your calculator and computer whenever possible. Complete as many grapher and CAS (Computer Algebra System) exercises as you can, *even if they are not assigned.* Graphs provide insight and visual representations of important concepts and relationships. Numbers can reveal important patterns. A CAS gives you the freedom to explore realistic problems and examples that involve calculations that are too difficult or lengthy to do by hand.
4. Try on your own to write short descriptions of the key points each time you complete a section of the text. If you succeed, you probably understand the material. If you do not, you will know where there is a gap in your understanding.

Learning calculus is a process—it does not come all at once. Be patient, persevere, ask questions, discuss ideas and work with classmates, and seek help when you need it, right away. The rewards of learning calculus will be very satisfying, both intellectually and professionally.

G.B.T., Jr., *State College, PA*
R.L.F., *Monterey, CA*

Preliminaries

Overview This chapter reviews the main things you need to know to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, functions, and trigonometry.

1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers and the Real Line

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

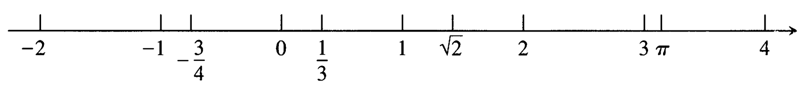
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever.

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol ℝ donotes either the real number system or, equivalently, the real line.

Properties of Real Numbers

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The algebraic properties say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

The order properties of real numbers are summarized in the following list.

The symbol $\Rightarrow$ means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign.

The completeness property of the real number system is deeper and harder to define precisely. Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many of the theorems of calculus would fail if the real number system were not complete, and the nature of the connection is important. The topic is best saved for a more advanced course, however, and we will not pursue it.

Subsets of $\mathbb{R}$

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

a) terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

b) repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}.$$

The bar indicates the block of repeating digits.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$.

Intervals

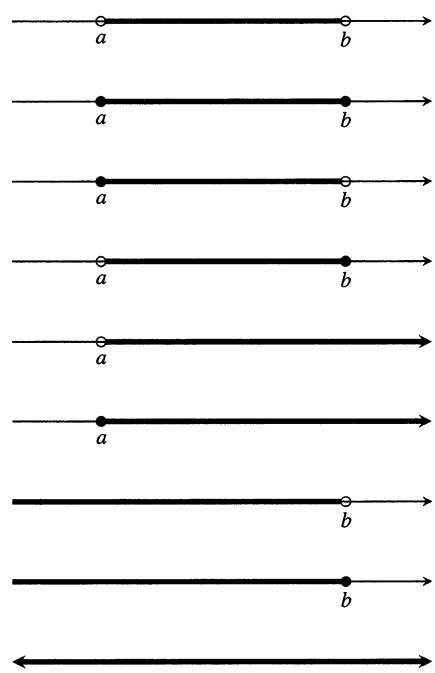
A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \le x \le 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together make up what is called the interval's **interior**.

Table 1 Types of intervals

	Notation	Set	Graph
Finite:	(a, b)	$\{x \mid a < x < b\}$	
	$[a, b]$	$\{x \mid a \le x \le b\}$	
	$[a, b)$	$\{x \mid a \le x < b\}$	
	$(a, b]$	$\{x \mid a < x \le b\}$	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	
	$[a, \infty)$	$\{x \mid x \ge a\}$	
	$(-\infty, b)$	$\{x \mid x < b\}$	
	$(-\infty, b]$	$\{x \mid x \le b\}$	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

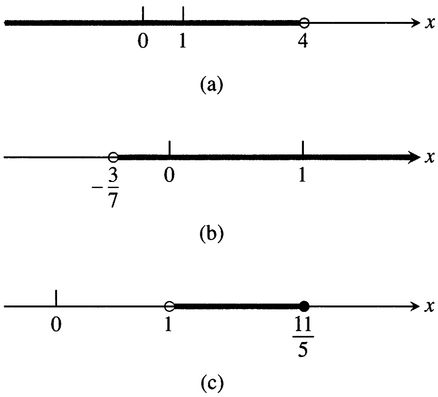


Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

EXAMPLE 1 Solve the following inequalities and graph their solution sets on the real line.

a) $2x - 1 < x + 3$ **b)** $-\dfrac{x}{3} < 2x + 1$ **c)** $\dfrac{6}{x-1} \geq 5$



1 Solutions for Example 1.

Solution

a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \qquad \text{Add 1 to both sides.}$$

$$x < 4 \qquad \text{Subtract } x \text{ from both sides.}$$

The solution set is the interval $(-\infty, 4)$ (Fig. 1a).

b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \qquad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \qquad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \qquad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \qquad \text{Divide by 7.}$$

The solution set is the interval $(-3/7, \infty)$ (Fig. 1b).

c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval $(1, 11/5]$ (Fig. 1c). ❑

Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 $|3| = 3, \quad |0| = 0, \quad |-5| = -(-5) = 5, \quad \big|-|a|\big| = |a|$ ❑

Notice that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$.

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, an alternate definition of $|x|$ is

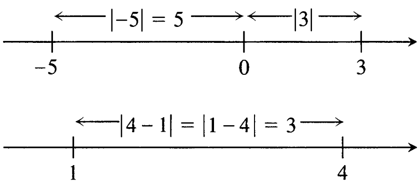
$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

Geometrically, $|x|$ represents the distance from x to the origin 0 on the real line. More generally (Fig. 2)

$$|x - y| = \text{the distance between } x \text{ and } y.$$

The absolute value has the following properties.



2 Absolute values give distances between points on the number line.

Absolute Value Properties

1. $\lvert -a \rvert = \lvert a \rvert$	A number and its negative have the same absolute value.
2. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3. $\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4. $\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	**The triangle inequality** The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$.

Notice that absolute value bars in expressions like $|-3+5|$ also work like parentheses: We do the arithmetic inside *before* taking the absolute value.

EXAMPLE 3

$$|-3+5| = |2| = 2 < |-3| + |5| = 8$$

$$|3+5| = |8| = |3| + |5|$$

$$|-3-5| = |-8| = 8 = |-3| + |-5|$$

❑

EXAMPLE 4 Solve the equation $|2x - 3| = 7$.

Solution The equation says that $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ❑

Inequalities Involving Absolute Values

The inequality $|a| < D$ says that the distance from a to 0 is less than D. Therefore, a must lie between D and $-D$.

The symbol $\Leftrightarrow$ means "if and only if," or "implies and is implied by."

Intervals and Absolute Values

If D is any positive number, then

$$|a| < D \quad \Leftrightarrow \quad -D < a < D, \tag{1}$$

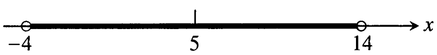
$$|a| \le D \quad \Leftrightarrow \quad -D \le a \le D. \tag{2}$$

EXAMPLE 5 Solve the inequality $|x - 5| < 9$ and graph the solution set on the real line.

Solution

$$\begin{aligned} |x - 5| &< 9 & \\ -9 < x - 5 &< 9 & \text{Eq. (1)} \\ -9 + 5 < x &< 9 + 5 & \text{Add 5 to each part to isolate } x. \\ -4 < x &< 14 & \end{aligned}$$

The solution set is the open interval $(-4, 14)$ (Fig. 3). ❑



3 The solution set of the inequality $|x - 5| < 9$ is the interval $(-4, 14)$ graphed here (Example 5).

EXAMPLE 6 Solve the inequality $\left|5 - \dfrac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5 - \frac{2}{x}\right| < 1 \quad &\Leftrightarrow \quad -1 < 5 - \frac{2}{x} < 1 && \text{Eq. (1)} \\ &\Leftrightarrow \quad -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow \quad 3 > \frac{1}{x} > 2 && \text{Multiply by } -\frac{1}{2}. \\ &\Leftrightarrow \quad \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$. ❑

EXAMPLE 7 Solve the inequality and graph the solution set:

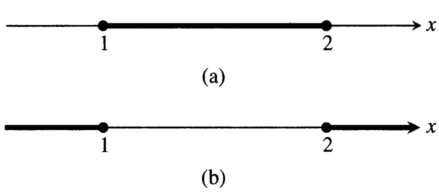
a) $|2x - 3| \le 1$ **b)** $|2x - 3| \ge 1$

Solution

a)

$$\begin{aligned} |2x - 3| &\le 1 & \\ -1 \le 2x - 3 &\le 1 & \text{Eq. (2)} \\ 2 \le 2x &\le 4 & \text{Add 3.} \\ 1 \le x &\le 2 & \text{Divide by 2.} \end{aligned}$$

The solution set is the closed interval $[1, 2]$ (Fig. 4a).



4 Graphs of the solution sets (a) $[1, 2]$ and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 7.

Union and intersection

Notice the use of the symbol $\cup$ to denote the union of intervals. A number lies in the **union** of two sets if it lies in either set. Similarly we use the symbol $\cap$ to denote intersection. A number lies in the **intersection** $I \cap J$ of two sets if it lies in *both* sets I and J. For example, $[1, 3) \cap [2, 4] = [2, 3)$.

b)
$$|2x - 3| \geq 1$$

$2x - 3 \geq 1$	or	$-(2x - 3) \geq 1$	
$2x - 3 \geq 1$	or	$2x - 3 \leq -1$	Multiply second inequality by -1.
$x - \frac{3}{2} \geq \frac{1}{2}$	or	$x - \frac{3}{2} \leq -\frac{1}{2}$	Divide by 2.
$x \geq 2$	or	$x \leq 1$	Add $\frac{3}{2}$.

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Fig. 4b). ❑

Exercises 1

Decimal Representations

1. Express $1/9$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/9$? $3/9$? $8/9$?

2. Express $1/11$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/11$? $3/11$? $9/11$?

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

a) $0 < x < 4$ **b)** $0 < x - 2 < 4$

c) $1 < \frac{x}{2} < 3$ **d)** $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$

e) $1 < \frac{6}{x} < 3$ **f)** $|x - 4| < 2$

g) $-6 < -x < 2$ **h)** $-6 < -x < -2$

4. If $-1 < y - 5 < 1$, which of the following statements about y are necessarily true, and which are not necessarily true?

a) $4 < y < 6$ **b)** $-6 < y < -4$

c) $y > 4$ **d)** $y < 6$

e) $0 < y - 4 < 2$ **f)** $2 < \frac{y}{2} < 3$

g) $\frac{1}{6} < \frac{1}{y} < \frac{1}{4}$ **h)** $|y - 5| < 1$

In Exercises 5–12, solve the inequalities and graph the solution sets.

5. $-2x > 4$ **6.** $8 - 3x \geq 5$

7. $5x - 3 \leq 7 - 3x$ **8.** $3(2 - x) > 2(3 + x)$

9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$ **10.** $\frac{6 - x}{4} < \frac{3x - 4}{2}$

11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$ **12.** $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$ **14.** $|y - 3| = 7$ **15.** $|2t + 5| = 4$

16. $|1 - t| = 1$ **17.** $|8 - 3s| = \frac{9}{2}$ **18.** $\left|\frac{s}{2} - 1\right| = 1$

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, graph each solution set on the real line.

19. $|x| < 2$ **20.** $|x| \leq 2$ **21.** $|t - 1| \leq 3$

22. $|t + 2| < 1$ **23.** $|3y - 7| < 4$ **24.** $|2y + 5| < 1$

25. $\left|\frac{z}{5} - 1\right| \leq 1$ **26.** $\left|\frac{3}{2}z - 1\right| \leq 2$ **27.** $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$

28. $\left|\frac{2}{x} - 4\right| < 3$ **29.** $|2s| \geq 4$ **30.** $|s + 3| \geq \frac{1}{2}$

31. $|1 - x| > 1$ **32.** $|2 - 3x| > 5$ **33.** $\left|\frac{r + 1}{2}\right| \geq 1$

34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Quadratic Inequalities

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and graph them. Use the result $\sqrt{a^2} = |a|$ as appropriate.

35. $x^2 < 2$ **36.** $4 \leq x^2$ **37.** $4 < x^2 < 9$

38. $\frac{1}{9} < x^2 < \frac{1}{4}$ **39.** $(x - 1)^2 < 4$ **40.** $(x + 3)^2 < 2$

41. $x^2 - x < 0$ **42.** $x^2 - x - 2 \geq 0$

Theory and Examples

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?

44. Solve the equation $|x-1| = 1-x$.

45. *A proof of the triangle inequality.* Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$
\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2 &&\\
&\le a^2+2|a||b|+b^2 && (2)\\
&\le |a|^2+2|a||b|+|b|^2 && (3)\\
&= (|a|+|b|)^2 &&\\
|a+b| &\le |a|+|b| && (4)
\end{aligned}
$$

46. Prove that $|ab| = |a||b|$ for any numbers a and b.

47. If $|x| \le 3$ and $x > -1/2$, what can you say about x?

48. Graph the inequality $|x|+|y| \le 1$.

49. GRAPHER

a) Graph the functions $f(x) = x/2$ and $g(x) = 1+(4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1+\frac{4}{x}.$$

b) Confirm your findings in (a) algebraically.

50. GRAPHER

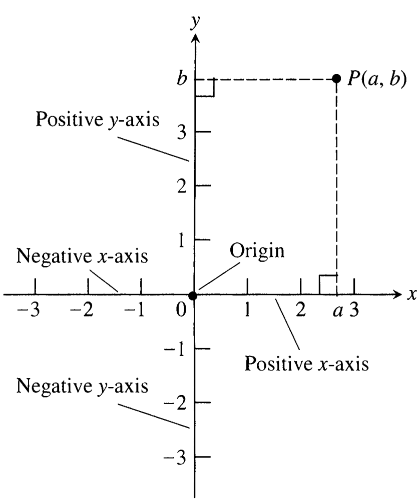
a) Graph the functions $f(x) = 3/(x-1)$ and $g(x) = 2/(x+1)$ together to identify the values of x for which

$$\frac{3}{x-1} < \frac{2}{x+1}.$$

b) Confirm your findings in (a) algebraically.

2 Coordinates, Lines, and Increments

This section reviews coordinates and lines and discusses the notion of increment.



5 Cartesian coordinates.

Cartesian Coordinates in the Plane

The positions of all points in the plane can be measured with respect to two perpendicular real lines in the plane intersecting in the 0-point of each (Fig. 5). These lines are called **coordinate axes** in the plane. On the horizontal x-axis, numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upward. The point where x and y are both 0 is the **origin** of the coordinate system, often denoted by the letter O.

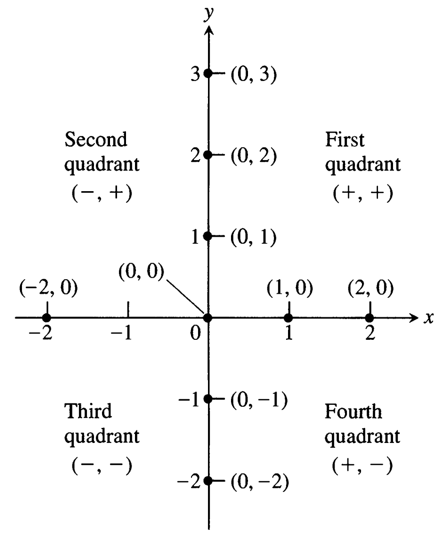
If P is any point in the plane, we can draw lines through P perpendicular to the two coordinate axes. If the lines meet the x-axis at a and the y-axis at b, then a is the ***x*-coordinate** of P, and b is the ***y*-coordinate**. The ordered pair (a, b) is the point's **coordinate pair**. The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point $(0, 0)$.

The origin divides the x-axis into the **positive *x*-axis** to the right and the **negative *x*-axis** to the left. It divides the y-axis into the **positive** and **negative *y*-axis** above and below. The axes divide the plane into four regions called **quadrants**, numbered counterclockwise as in Fig. 6.

A Word About Scales

When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If we plot time vs. thrust for a rocket motor, for example, there is no reason to place the mark that shows 1 sec on the time axis the same distance from the origin as the mark that shows 1 lb on the thrust axis.

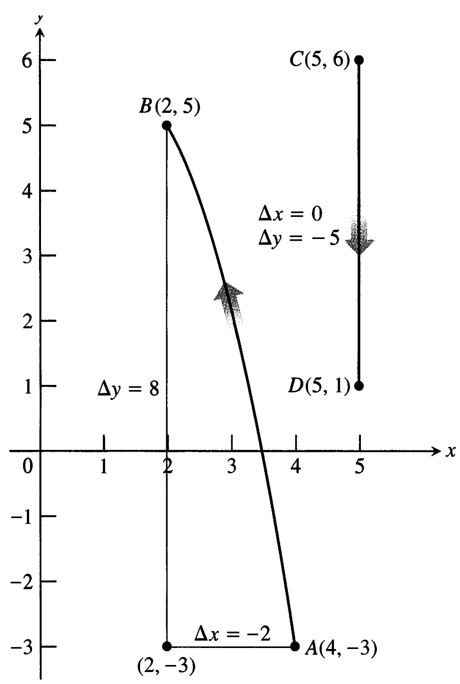
When we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry and trigonometry, we try to make the scales on the axes identical. A vertical unit



6 The points on the axes all have coordinate pairs, but we usually label them with single numbers. Notice the coordinate sign patterns in the quadrants.

of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do and angles that are supposed to be congruent will look congruent.

Computer displays and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, and there are corresponding distortions in distances, slopes, and angles. Circles may look like ellipses, rectangles may look like squares, right angles may appear to be acute or obtuse, and so on. Circumstances like these require us to take extra care in interpreting what we see. High-quality computer software usually allows you to compensate for such scale problems by adjusting the *aspect ratio* (ratio of vertical to horizontal scale). Some computer screens also allow adjustment within a narrow range. When you use a grapher, try to make the aspect ratio 1, or close to it.



7 Coordinate increments may be positive, negative, or zero.

Increments and Distance

When a particle moves from one point in the plane to another, the net changes in its coordinates are called *increments*. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

EXAMPLE 1 In going from the point $A(4, -3)$ to the point $B(2, 5)$ (Fig. 7), the increments in the x- and y-coordinates are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8. \qquad \square$$

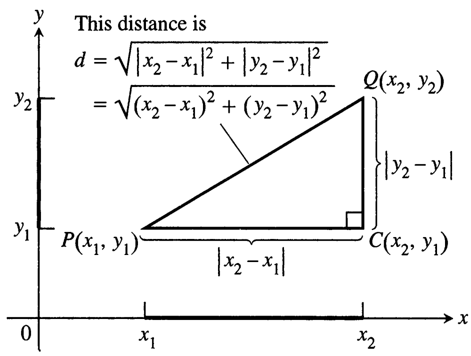
Definition

An **increment** in a variable is a net change in that variable. If x changes from x_1 to x_2, the increment in x is

$$\Delta x = x_2 - x_1.$$

EXAMPLE 2 From $C(5, 6)$ to $D(5, 1)$ (Fig. 7) the coordinate increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5. \qquad \square$$



8 To calculate the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$, apply the Pythagorean theorem to triangle *PCQ*.

The distance between points in the plane is calculated with a formula that comes from the Pythagorean theorem (Fig. 8).

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3

a) The distance between $P(-1, 2)$ and $Q(3, 4)$ is

$$\sqrt{(3-(-1))^2 + (4-2)^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}.$$

b) The distance from the origin to $P(x, y)$ is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

❑

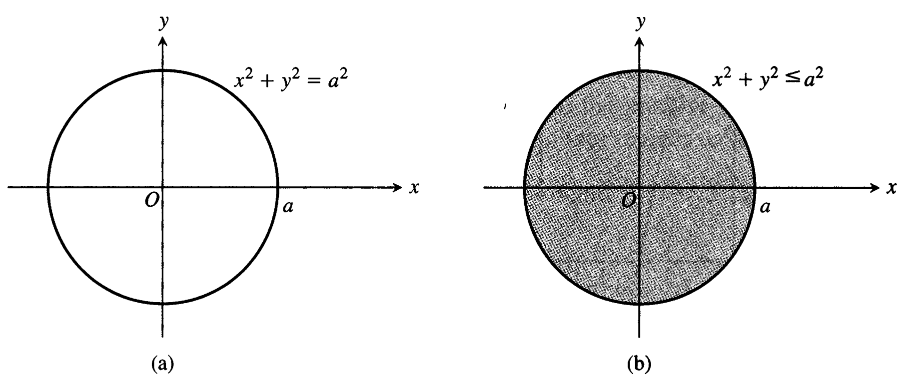
Graphs

The graph of an equation or inequality involving the variables x and y is the set of all points $P(x, y)$ whose coordinates satisfy the equation or inequality.

EXAMPLE 4 *Circles centered at the origin*

a) If $a > 0$, the equation $x^2 + y^2 = a^2$ represents all points $P(x, y)$ whose distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{a^2} = a$. These points lie on the circle of radius a centered at the origin. This circle is the graph of the equation $x^2 + y^2 = a^2$ (Fig. 9a).

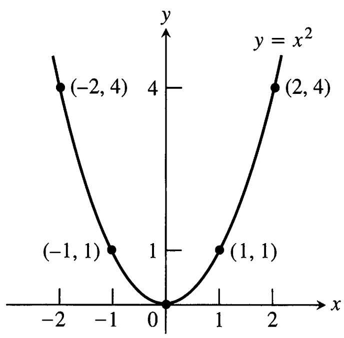
b) Points (x, y) whose coordinates satisfy the inequality $x^2 + y^2 \le a^2$ all have distance $\le a$ from the origin. The graph of the inequality is therefore the circle of radius a centered at the origin together with its interior (Fig. 9b).



9 Graphs of (a) the equation and (b) the inequality in Example 4.

❑

The circle of radius 1 unit centered at the origin is called the **unit circle**.



10 The parabola $y = x^2$.

EXAMPLE 5 Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) make up a smooth curve called a parabola (Fig. 10). ❑

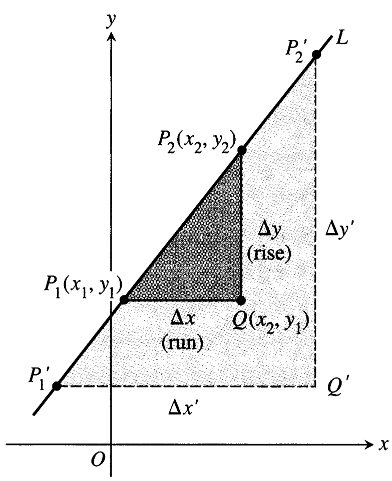
Straight Lines

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2. Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the same value for every choice of the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line (Fig. 11).



11 Triangles P_1QP_2 and $P_1'Q'P_2'$ are similar, so

$$\frac{\Delta y'}{\Delta x'} = \frac{\Delta y}{\Delta x} = m.$$

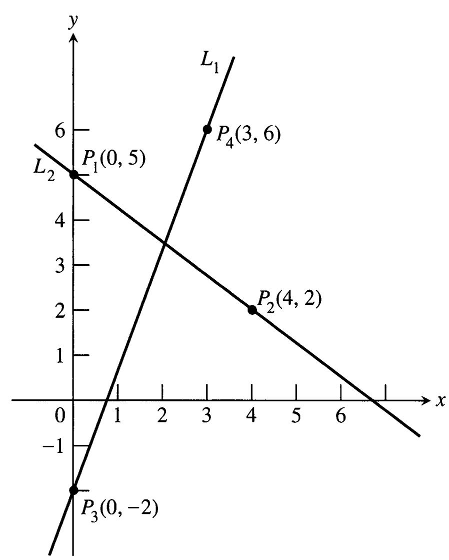
Definition

The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line P_1P_2.

The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right (Fig. 12). The greater the absolute value of the slope, the more rapid the rise or fall. The slope of a vertical line is *undefined.* Since the run Δx is zero for a vertical line, we cannot form the ratio m.



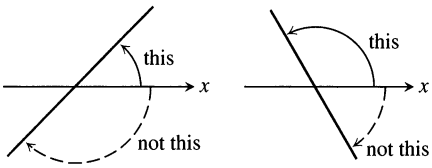
12 The slope of L_1 is

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - (-2)}{3 - 0} = \frac{8}{3}.$$

That is, y increases 8 units every time x increases 3 units. The slope of L_2 is

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 5}{4 - 0} = \frac{-3}{4}.$$

That is, y decreases 3 units every time x increases 4 units.

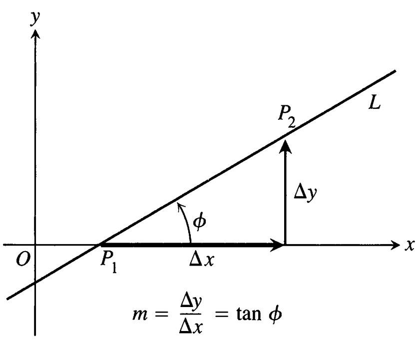


13 Angles of inclination are measured counterclockwise from the x-axis.

The direction and steepness of a line can also be measured with an angle. The **angle of inclination (inclination)** of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line (Fig. 13). The inclination of a horizontal line is 0°. The inclination of a vertical line is 90°. If ϕ (the Greek letter phi) is the inclination of a line, then $0 \le \phi < 180°$.

The relationship between the slope m of a nonvertical line and the line's inclination ϕ is shown in Fig. 14:

$$\boxed{m = \tan\phi.}$$



14 The slope of a nonvertical line is the tangent of its angle of inclination.

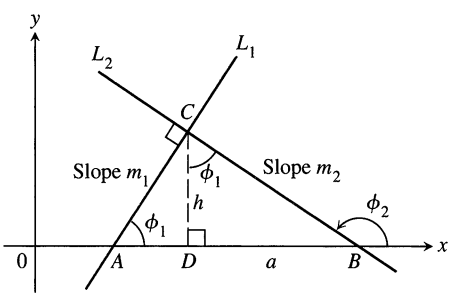
Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination. Hence, they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

The argument goes like this: In the notation of Fig. 15, $m_1 = \tan\phi_1 = a/h$, while $m_2 = \tan\phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

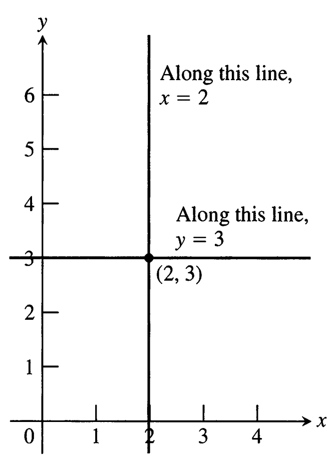


15 ΔADC is similar to ΔCDB. Hence ϕ_1 is also the upper angle in ΔCDB. From the sides of ΔCDB, we read $\tan\phi_1 = a/h$.

Equations of Lines

Straight lines have relatively simple equations. All points on the *vertical line* through the point a on the x-axis have x-coordinates equal to a. Thus, $x = a$ is an equation for the vertical line. Similarly, $y = b$ is an equation for the *horizontal line* meeting the y-axis at b.

EXAMPLE 6 The vertical and horizontal lines through the point $(2, 3)$ have equations $x = 2$ and $y = 3$, respectively (Fig. 16).



❑

16 The standard equations for the vertical and horizontal lines through (2, 3) are $x = 2$ and $y = 3$.

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

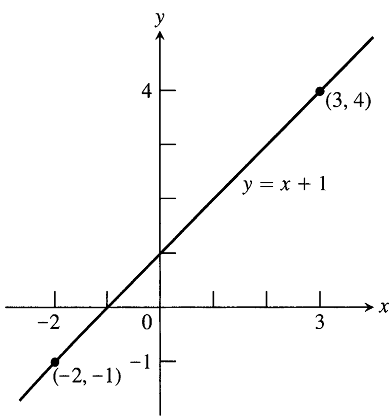
$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y = y_1 + m(x - x_1).$$

Definition

The equation

$$y = y_1 + m(x - x_1)$$

is the **point–slope equation** of the line that passes through the point (x_1, y_1) and has slope m.



16 The line in Example 8.

EXAMPLE 7 Write an equation for the line through the point (2, 3) with slope $-3/2$.

Solution We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point–slope equation and obtain

$$y = 3 - \frac{3}{2}(x - 2), \qquad \text{or} \qquad y = -\frac{3}{2}x + 6.$$ ❑

EXAMPLE 8 Write an equation for the line through $(-2, -1)$ and $(3, 4)$.

Solution The line's slope is

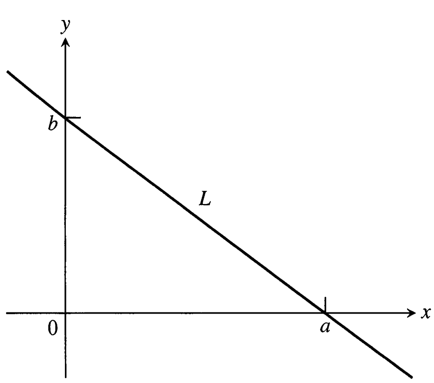
$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point–slope equation:

With $(x_1, y_1) = (-2, -1)$	**With $(x_1, y_1) = (3, 4)$**
$y = -1 + 1 \cdot (x - (-2))$	$y = 4 + 1 \cdot (x - 3)$
$y = -1 + x + 2$	$y = 4 + x - 3$
$y = x + 1$	$y = x + 1$

Same result

Either way, $y = x + 1$ is an equation for the line (Fig. 17). ❑



17 Line L has x-intercept a and y-intercept b.

The y-coordinate of the point where a nonvertical line intersects the y-axis is called the ***y*-intercept** of the line. Similarly, the ***x*-intercept** of a nonhorizontal line is the x-coordinate of the point where it crosses the x-axis (Fig. 18). A line with slope m and y-intercept b passes through the point $(0, b)$, so it has equation

$$y = b + m(x - 0), \qquad \text{or, more simply,} \qquad y = mx + b.$$

Definition

The equation

$$y = mx + b$$

is called the **slope–intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 9 The line $y = 2x - 5$ has slope 2 and y-intercept -5.

The equation

$$Ax + By = C \qquad (A \text{ and } B \text{ not both } 0)$$

is called the **general linear equation** in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).

EXAMPLE 10 Find the slope and y-intercept of the line $8x + 5y = 20$.

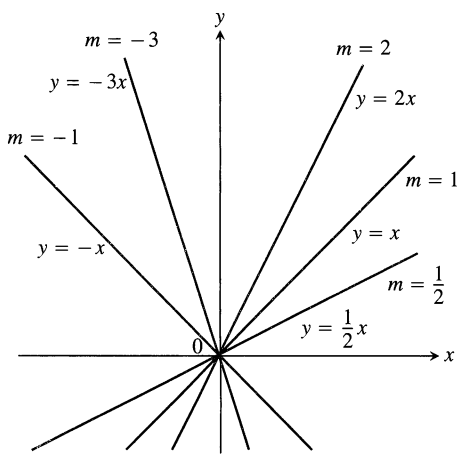
Solution Solve the equation for y to put it in slope–intercept form. Then read the slope and y-intercept from the equation:

$$\begin{aligned} 8x + 5y &= 20 \\ 5y &= -8x + 20 \\ y &= -\frac{8}{5}x + 4. \end{aligned}$$

The slope is $m = -8/5$. The y-intercept is $b = 4$.

EXAMPLE 11 *Lines through the origin*

Lines with equations of the form $y = mx$ have y-intercept 0 and so pass through the origin. Several examples are shown in Fig. 19.



19 The line $y = mx$ has slope m and passes through the origin.

Applications—The Importance of Lines and Slopes

Light travels along lines, as do bodies falling from rest in a planet's gravitational field or coasting under their own momentum (like a hockey puck gliding across the ice). We often use the equations of lines (called **linear equations**) to study such motions.

Many important quantities are related by linear equations. Once we know that a relationship between two variables is linear, we can find it from any two pairs of corresponding values just as we find the equation of a line from the coordinates of two points.

Slope is important because it gives us a way to say how steep something is (roadbeds, roofs, stairs). The notion of slope also enables us to describe how rapidly things are changing. For this reason it will play an important role in calculus.

EXAMPLE 12 *Celsius vs. Fahrenheit*

Fahrenheit temperature (F) and Celsius temperature (C) are related by a linear equation of the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus

$$32 = 0m + b, \qquad \text{and} \qquad 212 = 100m + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \qquad \text{or} \qquad C = \frac{5}{9}(F - 32).$$

❑

Exercises 2

Increments and Distance

In Exercises 1–4, a particle moves from A to B in the coordinate plane. Find the increments Δx and Δy in the particle's coordinates. Also find the distance from A to B.

1. $A(-3, 2)$, $B(-1, -2)$

2. $A(-1, -2)$, $B(-3, 2)$

3. $A(-3.2, -2)$, $B(-8.1, -2)$

4. $A(\sqrt{2}, 4)$, $B(0, 1.5)$

Describe the graphs of the equations in Exercises 5–8.

5. $x^2 + y^2 = 1$

6. $x^2 + y^2 = 2$

7. $x^2 + y^2 \leq 3$

8. $x^2 + y^2 = 0$

Slopes, Lines, and Intercepts

Plot the points in Exercises 9–12 and find the slope (if any) of the line they determine. Also find the common slope (if any) of the lines perpendicular to line AB.

9. $A(-1, 2)$, $B(-2, -1)$

10. $A(-2, 1)$, $B(2, -2)$

11. $A(2, 3)$, $B(-1, 3)$

12. $A(-2, 0)$, $B(-2, -2)$

In Exercises 13–16, find an equation for (a) the vertical line and (b) the horizontal line through the given point.

13. $(-1, 4/3)$

14. $(\sqrt{2}, -1.3)$

15. $(0, -\sqrt{2})$

16. $(-\pi, 0)$

In Exercises 17–30, write an equation for each line described.

17. Passes through $(-1, 1)$ with slope -1

18. Passes through $(2, -3)$ with slope $1/2$

19. Passes through $(3, 4)$ and $(-2, 5)$

20. Passes through $(-8, 0)$ and $(-1, 3)$

21. Has slope $-5/4$ and y-intercept 6

22. Has slope $1/2$ and y-intercept -3

23. Passes through $(-12, -9)$ and has slope 0

24. Passes through $(1/3, 4)$ and has no slope

25. Has y-intercept 4 and x-intercept -1

26. Has y-intercept -6 and x-intercept 2

27. Passes through $(5, -1)$ and is parallel to the line $2x + 5y = 15$

28. Passes through $(-\sqrt{2}, 2)$ parallel to the line $\sqrt{2}x + 5y = \sqrt{3}$

29. Passes through (4, 10) and is perpendicular to the line $6x - 3y = 5$

30. Passes through (0, 1) and is perpendicular to the line $8x - 13y = 13$

In Exercises 31–34, find the line's x- and y-intercepts and use this information to graph the line.

31. $3x + 4y = 12$

32. $x + 2y = -4$

33. $\sqrt{2}x - \sqrt{3}y = \sqrt{6}$

34. $1.5x - y = -3$

35. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Bx - Ay = C_2$ $(A \neq 0, B \neq 0)$? Give reasons for your answer.

36. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Ax + By = C_2$ $(A \neq 0, B \neq 0)$? Give reasons for your answer.

Increments and Motion

37. A particle starts at $A(-2, 3)$ and its coordinates change by increments $\Delta x = 5$, $\Delta y = -6$. Find its new position.

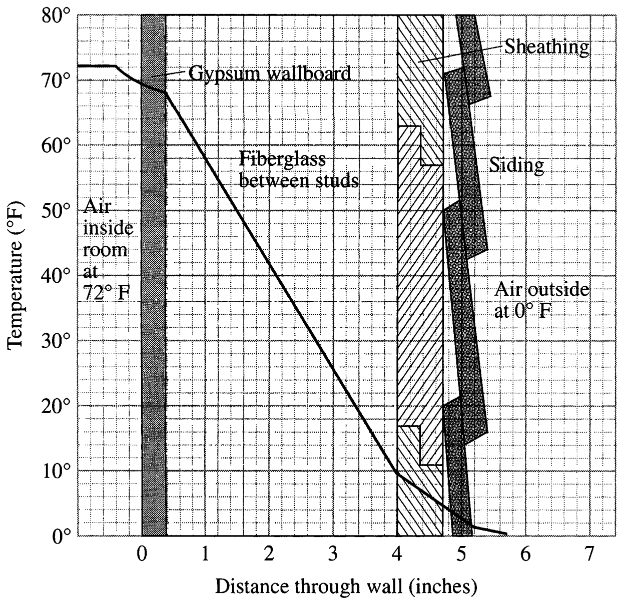
38. A particle starts at $A(6, 0)$ and its coordinates change by increments $\Delta x = -6$, $\Delta y = 0$. Find its new position.

39. The coordinates of a particle change by $\Delta x = 5$ and $\Delta y = 6$ as it moves from $A(x, y)$ to $B(3, -3)$. Find x and y.

40. A particle started at $A(1, 0)$, circled the origin once counterclockwise, and returned to $A(1, 0)$. What were the net changes in its coordinates?

Applications

41. *Insulation.* By measuring slopes in Fig. 20, estimate the temperature change in degrees per inch for (a) the gypsum wallboard; (b) the fiberglass insulation; (c) the wood sheathing. (Graphs can shift in printing, so your answers may differ slightly from those in the back of the book.)



20 The temperature changes in the wall in Exercises 41 and 42. (Source: *Differentiation,* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, Mass. [1975], p. 25.)

42. *Insulation.* According to Fig. 20, which of the materials in Exercise 41 is the best insulator? the poorest? Explain.

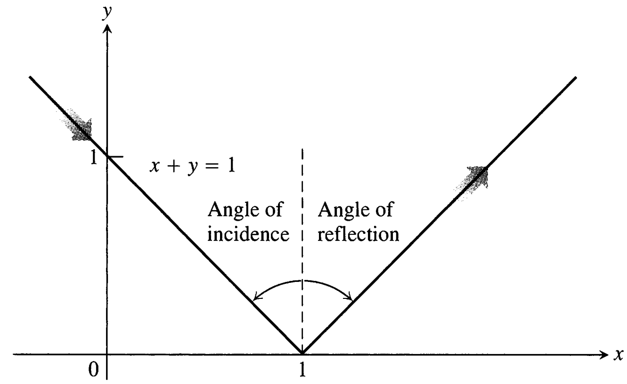
43. *Pressure under water.* The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). At the surface, the pressure is 1 atmosphere. The pressure at 100 meters is about 10.94 atmospheres. Find the pressure at 50 meters.

44. *Reflected light.* A ray of light comes in along the line $x + y = 1$ from the second quadrant and reflects off the x-axis (Fig. 21). The angle of incidence is equal to the angle of reflection. Write an equation for the line along which the departing light travels.

45. *Fahrenheit vs. Celsius.* In the FC-plane, sketch the graph of the equation

$$C = \frac{5}{9}(F - 32)$$

linking Fahrenheit and Celsius temperatures (Example 12). On the same graph sketch the line $C = F$. Is there a temperature at which a Celsius thermometer gives the same numerical reading as a Fahrenheit thermometer? If so, find it.



21 The path of the light ray in Exercise 44. Angles of incidence and reflection are measured from the perpendicular.

46. *The Mt. Washington Cog Railway.* Civil engineers calculate the slope of roadbed as the ratio of the distance it rises or falls to the distance it runs horizontally. They call this ratio the **grade** of the roadbed, usually written as a percentage. Along the coast, commercial railroad grades are usually less than 2%. In the mountains, they may go as high as 4%. Highway grades are usually less than 5%.

The steepest part of the Mt. Washington Cog Railway in New Hampshire has an exceptional 37.1% grade. Along this part of the track, the seats in the front of the car are 14 ft above those in the rear. About how far apart are the front and rear rows of seats?

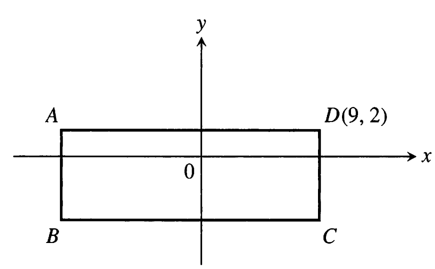
Theory and Examples

47. By calculating the lengths of its sides, show that the triangle with vertices at the points $A(1, 2)$, $B(5, 5)$, and $C(4, -2)$ is isosceles but not equilateral.

48. Show that the triangle with vertices $A(0, 0)$, $B(1, \sqrt{3})$, and $C(2, 0)$ is equilateral.

49. Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are vertices of a square, and find the fourth vertex.

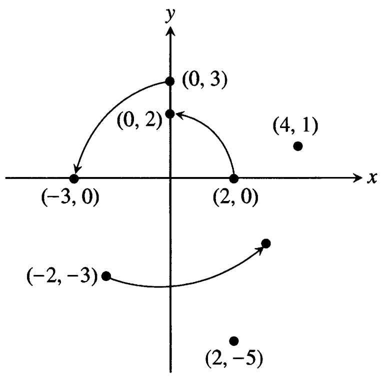
50. The rectangle shown here has sides parallel to the axes. It is three times as long as it is wide, and its perimeter is 56 units. Find the coordinates of the vertices A, B, and C.



51. Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Sketch them and find the coordinates of the fourth vertex of each.

52. A 90° rotation counterclockwise about the origin takes (2, 0) to (0, 2), and (0, 3) to (−3, 0), as shown in Fig. 22. Where does it take each of the following points?

a) $(4, 1)$ **b)** $(-2, -3)$ **c)** $(2, -5)$
d) $(x, 0)$ **e)** $(0, y)$ **f)** (x, y)
g) What point is taken to (10, 3)?



22 The points moved by the 90° rotation in Exercise 52.

53. For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

54. Find the line that passes through the point (1, 2) and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

55. Show that the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

is the midpoint of the line segment joining $P(x_1, y_1)$ to $Q(x_2, y_2)$.

56. *The distance from a point to a line.* We can find the distance from a point $P(x_0, y_0)$ to a line $L\colon Ax + By = C$ by taking the following steps (there is a somewhat faster method in Section 10.5):

1. Find an equation for the line M through P perpendicular to L.
2. Find the coordinates of the point Q in which M and L intersect.
3. Find the distance from P to Q.

Use these steps to find the distance from P to L in each of the following cases.

a) $P(2, 1)$, $L\colon y = x + 2$
b) $P(4, 6)$, $L\colon 4x + 3y = 12$
c) $P(a, b)$, $L\colon x = -1$
d) $P(x_0, y_0)$, $L\colon Ax + By = C$

3 Functions

Functions are the major tools for describing the real world in mathematical terms. This section reviews the notion of function and discusses some of the functions that arise in calculus.

Functions

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x.

The letters used for variable quantities may come from what is being described. When we study circles, we usually call the area A and the radius r. Since $A = \pi r^2$, we say that A is a function of r. The equation $A = \pi r^2$ is a *rule* that tells how to calculate a *unique* (single) output value of A for each possible input value of the radius r.

The set of all possible input values for the radius is called the **domain** of the function. The set of all output values of the area is the **range** of the function. Since circles cannot have negative radii or areas, the domain and range of the circle area function are both the interval $[0, \infty)$, consisting of all nonnegative real numbers.

The domain and range of a mathematical function can be any sets of objects; they do not have to consist of numbers. Most of the domains and ranges we will encounter in this book, however, will be sets of real numbers.

Leonhard Euler (1707–1783)

Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician who ever lived, was also an astronomer, physicist, botanist, chemist, and expert in Oriental languages. He was the first scientist to give the function concept the prominence in his work that it has in mathematics today. Euler's collected books and papers fill 70 volumes. His introductory algebra text, written originally in German (Euler was Swiss), is still read in English translation.

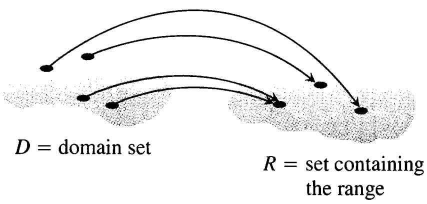
In calculus we often want to refer to a generic function without having any particular formula in mind. Euler invented a symbolic way to say "y is a function of x" by writing

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"})$$

In this notation, the symbol f represents the function. The letter x, called the **independent variable,** represents an input value from the domain of f, and y, the **dependent variable,** represents the corresponding output value $f(x)$ in the range of f. Here is the formal definition of *function.*

Definition

A **function** from a set D to a set R is a rule that assigns a *unique* element $f(x)$ in R to each element x in D.



23 A function from a set D to a set R assigns a unique element of R to each element in D.

In this definition, $D = D(f)$ (read "D of f") is the domain of the function f and R is a set *containing* the range of f. See Fig. 23.

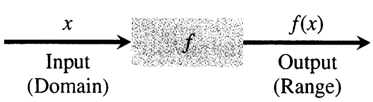
Think of a function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Fig. 24).

In this book we will usually define functions in one of two ways:

1. by giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function, or
2. by giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

Strictly speaking, we should call the function f and not $f(x)$, as the latter denotes the value of the function at the point x. However, as is common usage, we will often refer to the function as $f(x)$ in order to name the variable on which f depends.

It is sometimes convenient to use a single letter to denote both a function and its dependent variable. For instance, we might say that the area A of a circle of radius r is given by the function $A(r) = \pi r^2$.



24 A "machine" diagram for a function.

Evaluation

As we said earlier, most of the functions in this book will be **real-valued functions** of a **real variable,** functions whose domains and ranges are sets of real numbers. We evaluate such functions by substituting particular values from the domain into the function's defining rule to calculate the corresponding values in the range.

EXAMPLE 1 The volume V of a ball (solid sphere) of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3.$$

The volume of a ball of radius 3 m is

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \text{ m}^3.$$

❑

EXAMPLE 2 Suppose that the function F is defined for all real numbers t by the formula

$$F(t) = 2(t - 1) + 3.$$

Evaluate F at the input values 0, 2, $x + 2$, and $F(2)$.

Solution In each case we substitute the given input value for t into the formula for F:

$$\begin{aligned} F(0) &= 2(0 - 1) + 3 = -2 + 3 = 1 \\ F(2) &= 2(2 - 1) + 3 = 2 + 3 = 5 \\ F(x + 2) &= 2(x + 2 - 1) + 3 = 2x + 5 \\ F(F(2)) &= F(5) = 2(5 - 1) + 3 = 11. \end{aligned}$$

❑

The Domain Convention

When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly, the domain is assumed to be the largest set of x-values for which the formula gives real y-values. This is the function's so-called **natural domain.** If we want the domain to be restricted in some way, we must say so.

The domain of the function $y = x^2$ is understood to be the entire set of real numbers. The formula gives a real y-value for every real number x. If we want to restrict the domain to values of x greater than or equal to 2, we must write "$y = x^2$, $x \geq 2$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2$, $x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In symbols, the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

Most of the functions we encounter will have domains that are either intervals or unions of intervals.

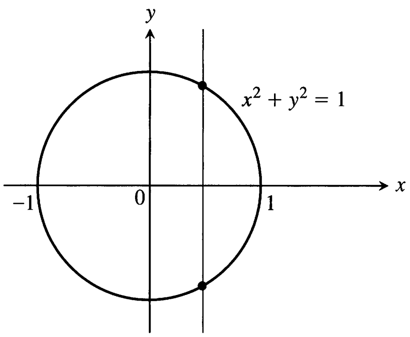
EXAMPLE 3

Function	**Domain (x)**	**Range (y)**
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = \dfrac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Beyond this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero.* The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is precisely the set of all nonzero real numbers.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).



25 This circle is not the graph of a function $y = f(x)$; it fails the vertical line test.

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all square roots of nonnegative numbers. ❑

Graphs of Functions

The **graph** of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane whose coordinates (x, y) are input–output pairs for f.

Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no *vertical line* can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Fig. 25). If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.

EXAMPLE 4 Graph the function $y = x^2$ over the interval $[-2, 2]$.

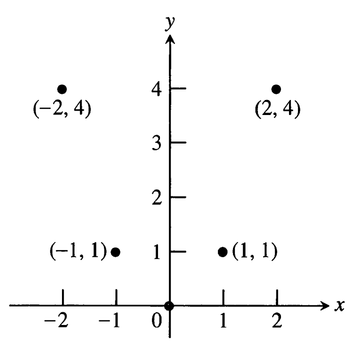
Solution

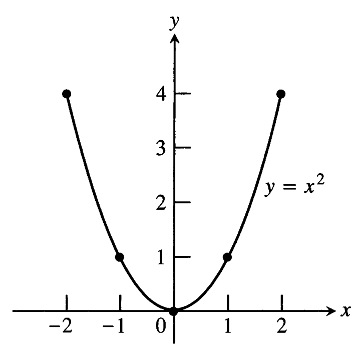
x	$y = x^2$
-2	4
-1	1
0	0
1	1
2	4

Step 1: Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

Step 2: Plot the points (x, y) whose coordinates appear in the table.

Step 3: Draw a smooth curve through the plotted points. Label the curve with its equation.

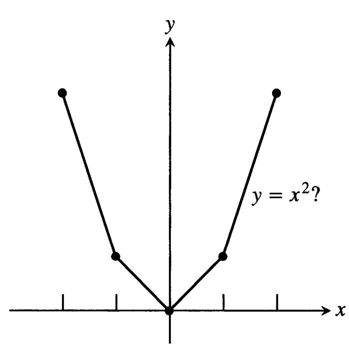


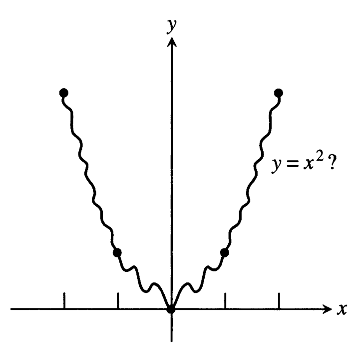


❑

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?

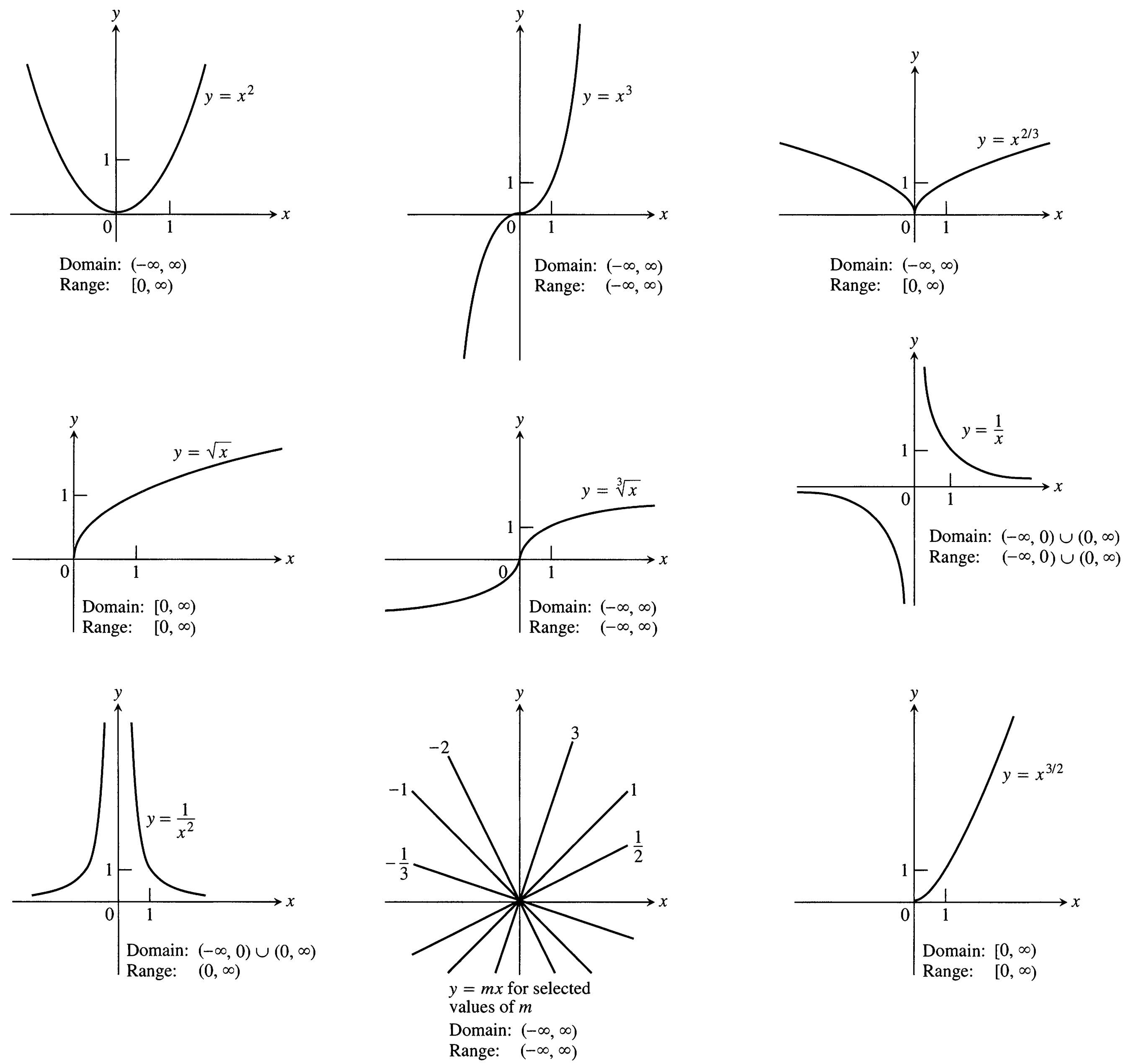
Computers and graphing calculators graph functions in much this way—by stringing together plotted points—and the same question arises.





To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? The answer lies in calculus, as we will see in Chapter 3. There we will use a marvelous mathematical tool called the *derivative* to find a curve's shape between plotted points. Meanwhile we will have to settle for plotting points and connecting them as best we can.

Figure 26 shows the graphs of several functions frequently encountered in calculus. It is a good idea to learn the shapes of these graphs so that you can recognize them or sketch them when the need arises.



26 Useful graphs.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g, we define functions $f+g$, $f-g$, and fg by the formulas

$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)g(x).$$

At any point of $D(f)\cap D(g)$ at which $g(x)\neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{(where } g(x)\neq 0\text{)}.$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 5

Function	Formula	Domain
f	$f(x)=\sqrt{x}$	$[0,\infty)$
g	$g(x)=\sqrt{1-x}$	$(-\infty,1]$
$3g$	$3g(x)=3\sqrt{1-x}$	$(-\infty,1]$
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0,1]=D(f)\cap D(g)$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0,1]$
$f\cdot g$	$(f\cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

❑

Composite Functions

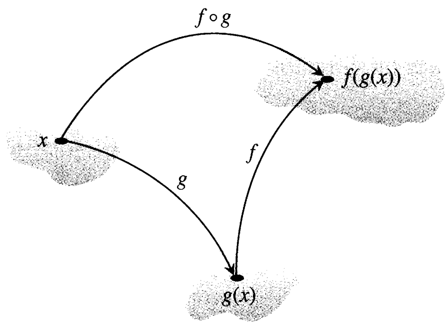
Composition is another method for combining functions.

Definition

If f and g are functions, the **composite** function $f\circ g$ ("f circle g") is defined by

$$(f\circ g)(x) = f(g(x)).$$

The domain of $f\circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.



27 The relation of $f \circ g$ to g and f.

The definition says that two functions can be composed when the range of the first lies in the domain of the second (Fig. 27). To find $(f \circ g)(x)$, we *first find* $g(x)$ and *second find* $f(g(x))$.

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 6 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

a) $(f \circ g)(x)$ **b)** $(g \circ f)(x)$ **c)** $(f \circ f)(x)$ **d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$\mathbb{R}$ or $(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, if $x \geq -1$. ❑

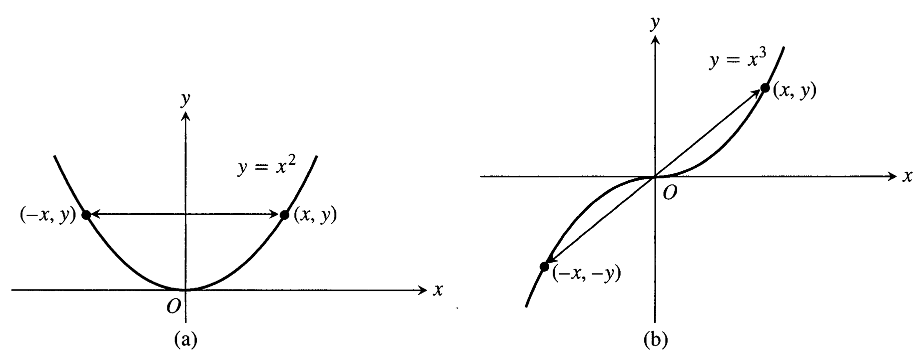
Even Functions and Odd Functions—Symmetry

A function $y = f(x)$ is **even** if $f(-x) = f(x)$ for every number x in the domain of f. Notice that this implies that both x and $-x$ must be in the domain of f. The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$.

The graph of an even function $y = f(x)$ is symmetric about the y-axis. Since $f(-x) = f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Fig. 28a). Once we know the graph on one side of the y-axis, we automatically know it on the other side.

A function $y = f(x)$ is **odd** if $f(-x) = -f(x)$ for every number x in the domain of f. Again, both x and $-x$ must lie in the domain of f. The function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$.

The graph of an odd function $y = f(x)$ is symmetric about the origin. Since $f(-x) = -f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Fig. 28b). Here again, once we know the graph of f on one side of the y-axis, we know it on both sides.



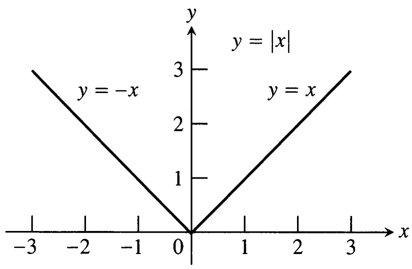
28 (a) Symmetry about the y-axis. If (x, y) is on the graph, so is $(-x, y)$. (b) Symmetry about the origin. If (x, y) is on the graph, so is $(-x, -y)$.

Piecewise Defined Functions

Sometimes a function uses different formulas on different parts of its domain. One example is the absolute value function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Fig. 29. Here are some examples.

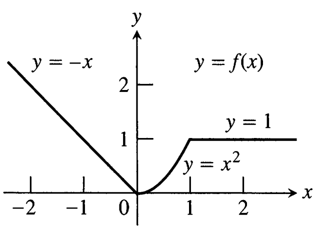


29 The absolute value function.

EXAMPLE 7 The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x (Fig. 30). ❑



30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 7).

EXAMPLE 8 *The greatest integer function*

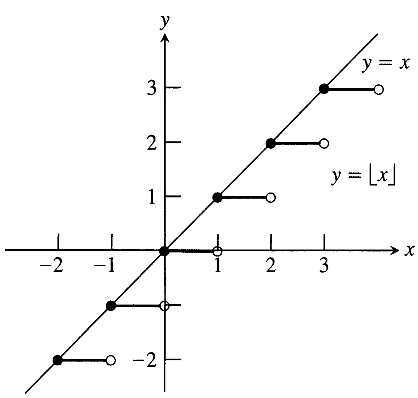
The function whose value at any number x is the *greatest integer less than or equal to x* is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$. Figure 31 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$

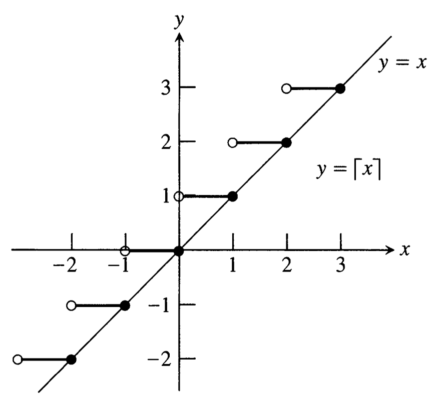
❑

EXAMPLE 9 *The least integer function*

The function whose value at any number x is the *smallest integer greater than or equal to x* is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 32 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges $1 for each hour or part of an hour.



31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x.



32 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x. ❑

Exercises 3

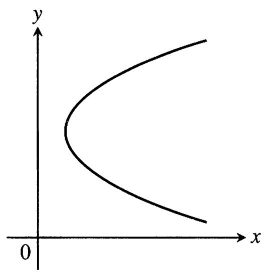
Functions

In Exercises 1–6, find the domain and range of each function.

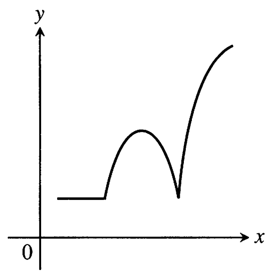
1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(t) = \dfrac{1}{\sqrt{t}}$
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$
5. $g(z) = \sqrt{4 - z^2}$
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

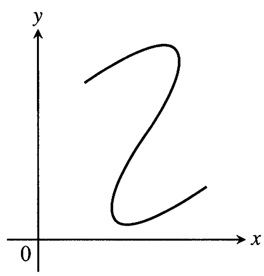
7. a)



b)



8. a)



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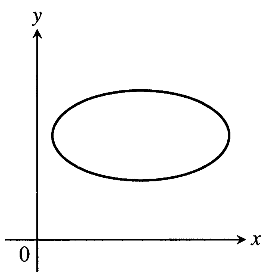


Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.
10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.
12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

Functions and Graphs

Graph the functions in Exercises 13–24. What symmetries, if any, do the graphs have? Use the graphs in Fig. 26 for guidance, as needed.

13. $y = -x^3$
14. $y = -\dfrac{1}{x^2}$
15. $y = -\dfrac{1}{x}$
16. $y = \dfrac{1}{|x|}$
17. $y = \sqrt{|x|}$
18. $y = \sqrt{-x}$
19. $y = x^3/8$
20. $y = -4\sqrt{x}$
21. $y = -x^{3/2}$
22. $y = (-x)^{3/2}$
23. $y = (-x)^{2/3}$
24. $y = -x^{2/3}$
25. Graph the following equations and explain why they are not graphs of functions of x.
    a) $|y| = x$ **b)** $y^2 = x^2$
26. Graph the following equations and explain why they are not graphs of functions of x.
    a) $|x| + |y| = 1$ **b)** $|x + y| = 1$

Even and Odd Functions

In Exercises 27–38, say whether the function is even, odd, or neither.

27. $f(x) = 3$
28. $f(x) = x^{-5}$
29. $f(x) = x^2 + 1$
30. $f(x) = x^2 + x$
31. $g(x) = x^3 + x$
32. $g(x) = x^4 + 3x^2 - 1$
33. $g(x) = \dfrac{1}{x^2 - 1}$
34. $g(x) = \dfrac{x}{x^2 - 1}$
35. $h(t) = \dfrac{1}{t - 1}$
36. $h(t) = |t^3|$
37. $h(t) = 2t + 1$
38. $h(t) = 2|t| + 1$

Sums, Differences, Products, and Quotients

In Exercises 39 and 40, find the domains and ranges of $f, g, f + g$, and $f \cdot g$.

39. $f(x) = x, \quad g(x) = \sqrt{x - 1}$
40. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 41 and 42, find the domains and ranges of $f, g, f/g$, and g/f.

41. $f(x) = 2, \quad g(x) = x^2 + 1$
42. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

43. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a) $f(g(0))$ **b)** $g(f(0))$
c) $f(g(x))$ **d)** $g(f(x))$
e) $f(f(-5))$ **f)** $g(g(2))$
g) $f(f(x))$ **h)** $g(g(x))$

44. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a) $f(g(1/2))$ **b)** $g(f(1/2))$
c) $f(g(x))$ **d)** $g(f(x))$
e) $f(f(2))$ **f)** $g(g(2))$
g) $f(f(x))$ **h)** $g(g(x))$

45. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a) $u(v(f(x)))$ **b)** $u(f(v(x)))$
c) $v(u(f(x)))$ **d)** $v(f(u(x)))$
e) $f(u(v(x)))$ **f)** $f(v(u(x)))$

46. If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

a) $h(g(f(x)))$ **b)** $h(f(g(x)))$
c) $g(h(f(x)))$ **d)** $g(f(h(x)))$
e) $f(g(h(x)))$ **f)** $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 47 and 48 as a composite involving one or more of $f, g, h,$ and j.

47. **a)** $y = \sqrt{x} - 3$ **b)** $y = 2\sqrt{x}$
c) $y = x^{1/4}$ **d)** $y = 4x$
e) $y = \sqrt{(x-3)^3}$ **f)** $y = (2x - 6)^3$

48. **a)** $y = 2x - 3$ **b)** $y = x^{3/2}$
c) $y = x^9$ **d)** $y = x - 6$
e) $y = 2\sqrt{x - 3}$ **f)** $y = \sqrt{x^3 - 3}$

49. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a)	$x - 7$	$\sqrt{x}$	
b)	$x + 2$	$3x$	
c)		$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d)	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	
e)		$1 + \dfrac{1}{x}$	x
f)	$\dfrac{1}{x}$		x

50. *A magic trick.* You may have heard of a magic trick that goes like this: Take any number. Add 5. Double the result. Subtract 6. Divide by 2. Subtract 2. Now tell me your answer, and I'll tell you what you started with.

Pick a number and try it.

You can see what is going on if you let x be your original number and follow the steps to make a formula $f(x)$ for the number you end up with.

Piecewise Defined Functions

Graph the functions in Exercises 51–54.

51. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

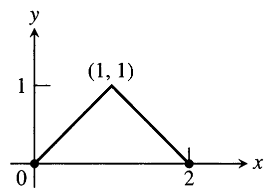
52. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

53. $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

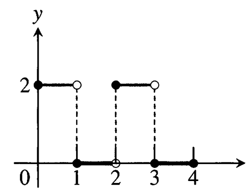
54. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

55. Find a formula for each function graphed.

a)

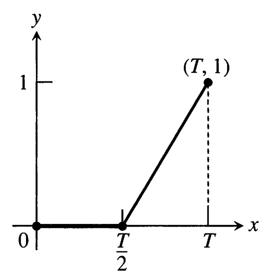


b)



56. Find a formula for each function graphed.

a)



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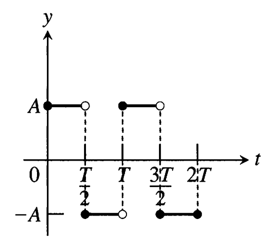


The Greatest and Least Integer Functions

57. For what values of x is (a) $\lfloor x \rfloor = 0$? (b) $\lceil x \rceil = 0$?

58. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

59. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

60. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0 \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Even and Odd Functions

61. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

a) fg b) f/g c) g/f
d) $f^2 = ff$ e) $g^2 = gg$ f) $f \circ g$
g) $g \circ f$ h) $f \circ f$ i) $g \circ g$

62. Can a function be both even and odd? Give reasons for your answer.

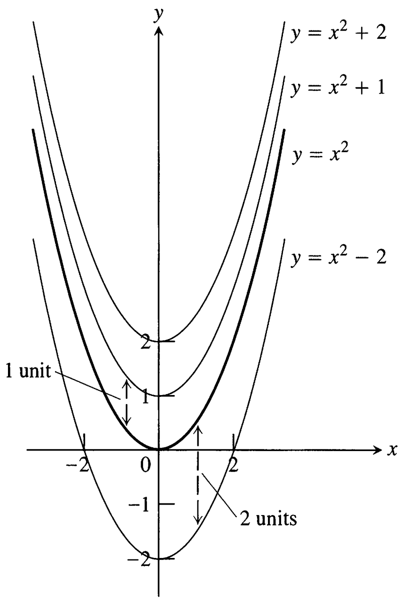
Grapher

63. (*Continuation of Example 5.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

64. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

4 Shifting Graphs

This section shows how to change an equation to shift its graph up or down or to the right or left. Knowing about this can help us spot familiar graphs in new locations. It can also help us graph unfamiliar equations more quickly. We practice mostly with circles and parabolas (because they make useful examples in calculus), but the methods apply to other curves as well. We will revisit parabolas and circles in Chapter 9.



33 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f.

How to Shift a Graph

To shift the graph of a function $y = f(x)$ straight up, we add a positive constant to the right-hand side of the formula $y = f(x)$.

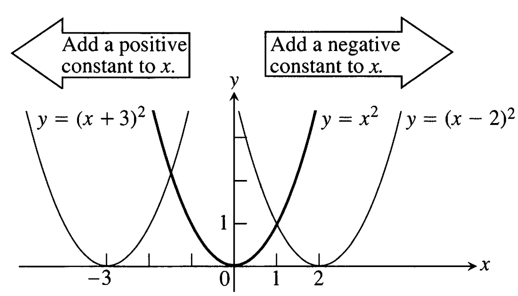
EXAMPLE 1 Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Fig. 33). ❑

To shift the graph of a function $y = f(x)$ straight down, we add a negative constant to the right-hand side of the formula $y = f(x)$.

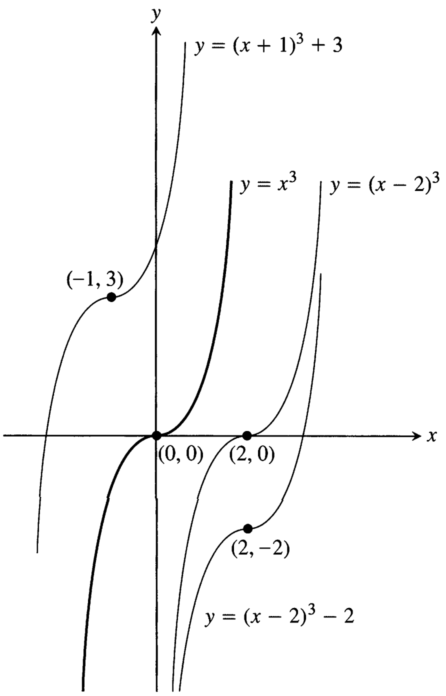
EXAMPLE 2 Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Fig. 33). ❑

To shift the graph of $y = f(x)$ to the left, we add a positive constant to x.

EXAMPLE 3 Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Fig. 34). ❑



34 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x.



35 The graph of $y = x^3$ shifted to three new positions in the xy-plane.

To shift the graph of $y = f(x)$ to the right, we add a negative constant to x.

EXAMPLE 4 Adding -2 to x in $y = x^2$ to get $y = (x-2)^2$ shifts the graph 2 units to the right (Fig. 34). ❑

Shift Formulas

VERTICAL SHIFTS

$y - k = f(x)$ or $y = f(x) + k$	Shifts the graph *up* k units if $k > 0$ Shifts it *down* $\lvert k\rvert$ units if $k < 0$

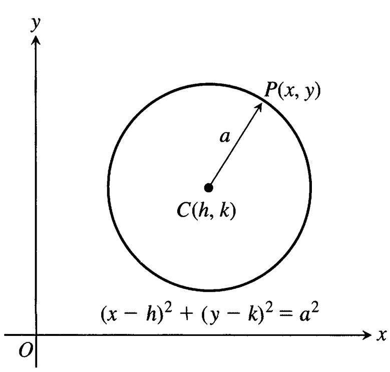
HORIZONTAL SHIFTS

$y = f(x-h)$	Shifts the graph *right* h units *if* $h > 0$ Shifts it *left* $\lvert h\rvert$ units if $h < 0$

EXAMPLE 5 The graph of $y = (x-2)^3 - 2$ is the graph of $y = x^3$ shifted 2 units to the right and 2 units down. The graph of $y = (x+1)^3 + 3$ is the graph of $y = x^3$ shifted 1 unit to the left and 3 units up (Fig. 35). ❑

Equations for Circles

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant (Fig. 36). The fixed point is the **center** of the circle; the constant distance is the **radius**. We saw in Section 2, Example 4, that the circle of radius a centered at the origin has equation $x^2 + y^2 = a^2$. If we shift the circle to place its center at the point (h, k), its equation becomes $(x-h)^2 + (y-k)^2 = a^2$.



36 A circle of radius a in the xy-plane, with center at (h, k).

The Standard Equation for the Circle of Radius *a* Centered at the Point (*h*, *k*)

$$(x-h)^2 + (y-k)^2 = a^2 \qquad \textbf{(1)}$$

EXAMPLE 6 If the circle $x^2 + y^2 = 25$ is shifted 2 units to the left and 3 units up, its new equation is $(x+2)^2 + (y-3)^2 = 25$. As Eq. (1) says it should be, this is the equation of the circle of radius 5 centered at $(h, k) = (-2, 3)$. ❑

EXAMPLE 7 The standard equation for the circle of radius 2 centered at (3, 4) is

$$(x-3)^2 + (y-4)^2 = (2)^2$$

or

$$(x-3)^2 + (y-4)^2 = 4.$$

There is no need to square out the x- and y-terms in this equation. In fact, it is better not to do so. The present form reveals the circle's center and radius. ❑

EXAMPLE 8 Find the center and radius of the circle

$$(x-1)^2+(y+5)^2=3.$$

Solution Comparing

$$(x-h)^2+(y-k)^2=a^2$$

with

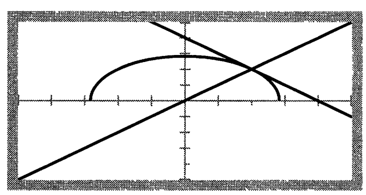
$$(x-1)^2+(y+5)^2=3$$

shows that $h=1, k=-5$, and $a=\sqrt{3}$. The center is the point $(h,k)=(1,-5)$; the radius is $a=\sqrt{3}$. ❑

Technology *Square Windows* We use the term "square window" when the units or scalings on both axes are the same. In a square window graphs are true in shape. They are distorted in a nonsquare window.

The term square window does not refer to the shape of the graphic display. Graphing calculators usually have rectangular displays. The displays of Computer Algebra Systems are usually square. When a graph is displayed, the x-unit may differ from the y-unit in order to fit the graph in the display, resulting in a distorted picture. The graphing window can be made square by shrinking or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window "square." If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

On your graphing utility, compare the perpendicular lines $y_1=x$ and $y_2=-x+4$ in a square window and a nonsquare one such as $[-10, 10]$ by $[10, 10]$. Graph the semicircle $y=\sqrt{8-x^2}$ in the same windows.

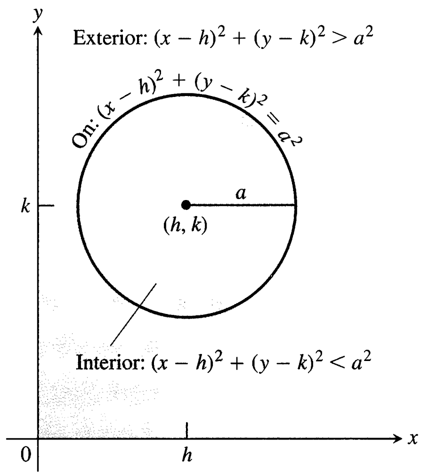


Two perpendicular lines and a semicircle graphed distorted by a rectangular window.

If an equation for a circle is not in standard form, we can find the circle's center and radius by first converting the equation to standard form. The algebraic technique for doing so is *completing the square* (see inside front cover).

EXAMPLE 9 Find the center and radius of the circle

$$x^2+y^2+4x-6y-3=0.$$



37 The interior and exterior of the circle $(x - h)^2 + (y - k)^2 = a^2$.

Solution We convert the equation to standard form by completing the squares in x and y:

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

Start with the given equation.

$$(x^2 + 4x \quad) + (y^2 - 6y \quad) = 3$$

Gather terms. Move the constant to the right-hand side.

$$\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) + \left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right) = 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{-6}{2}\right)^2$$

Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

Write each quadratic as a squared linear expression.

With the equation now in standard form, we read off the center's coordinates and the radius: $(h, k) = (-2, 3)$ and $a = 4$. ❑

Interior and Exterior

The points that lie inside the circle $(x - h)^2 + (y - k)^2 = a^2$ are the points less than a units from (h, k). They satisfy the inequality

$$(x - h)^2 + (y - k)^2 < a^2.$$

They make up the region we call the **interior** of the circle (Fig. 37).

The circle's **exterior** consists of the points that lie more than a units from (h, k). These points satisfy the inequality

$$(x - h)^2 + (y - k)^2 > a^2.$$

EXAMPLE 10

Inequality	Region
$x^2 + y^2 < 1$	Interior of the unit circle
$x^2 + y^2 \leq 1$	Unit circle plus its interior
$x^2 + y^2 > 1$	Exterior of the unit circle
$x^2 + y^2 \geq 1$	Unit circle plus its exterior

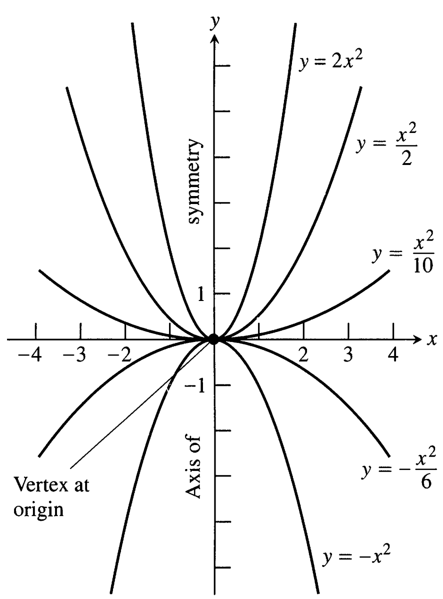
❑

Parabolic Graphs

The graph of an equation like $y = 3x^2$ or $y = -5x^2$ that has the form

$$y = ax^2$$

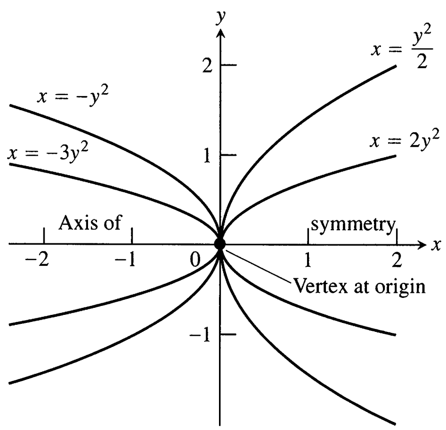
is a **parabola** whose **axis** (axis of symmetry) is the y-axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a > 0$ and downward if $a < 0$. The larger the value of $|a|$, the narrower the parabola (Fig. 38).



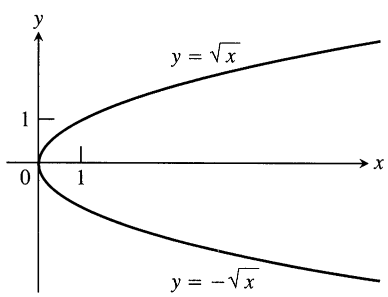
38 Besides determining the direction in which the parabola $y = ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

If we interchange x and y in the formula $y = ax^2$, we obtain the equation

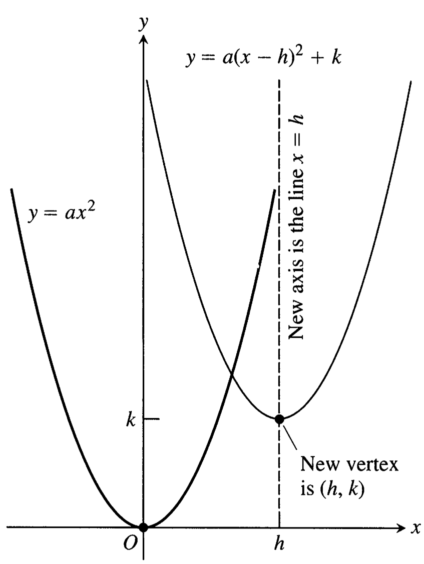
$$x = ay^2.$$



39 The parabola $x = ay^2$ is symmetric about the x-axis. It opens to the right if $a > 0$ and to the left if $a < 0$.



40 The graphs of the functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ join at the origin to make the graph of the equation $x = y^2$ (Example 11).



41 The parabola $y = ax^2$, $a > 0$, shifted h units to the right and k units up.

With x and y now reversed, the graph is a parabola whose axis is the x-axis and whose vertex lies at the origin (Fig. 39).

EXAMPLE 11 The formula $x = y^2$ gives x as a function of y but does *not* give y as a function of x. If we solve for y, we find that $y = \pm\sqrt{x}$. For each positive value of x we get *two* values of y instead of the required single value.

When taken separately, the formulas $y = \sqrt{x}$ and $y = -\sqrt{x}$ do define functions of x. Each formula gives exactly one value of y for each possible value of x. The graph of $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$. The graph of $y = -\sqrt{x}$ is the lower half (Fig. 40). ❑

The Quadratic Equation $y = ax^2 + bx + c, \quad a \neq 0$

To shift the parabola $y = ax^2$ horizontally, we rewrite the equation as

$$y = a(x - h)^2.$$

To shift it vertically as well, we change the equation to

$$y - k = a(x - h)^2. \tag{2}$$

The combined shifts place the vertex at the point (h, k) and the axis along the line $x = h$ (Fig. 41).

Normally there would be no point in multiplying out the right-hand side of Eq. (2). In this case, however, we can learn something from doing so because the resulting equation, when rearranged, takes the form

$$y = ax^2 + bx + c. \tag{3}$$

This tells us that the graph of every equation of the form $y = ax^2 + bx + c, a \neq 0$, is the graph of $y = ax^2$ shifted somewhere else. Why? Because the steps that take us from Eq. (2) to Eq. (3) can be reversed to take us from (3) back to (2). The curve $y = ax^2 + bx + c$ has the same shape and orientation as the curve $y = ax^2$.

The axis of the parabola $y = ax^2 + bx + c$ turns out to be the line $x = -b/(2a)$. The y-intercept, $y = c$, is obtained by setting $x = 0$.

The Graph of $y = ax^2 + bx + c, \quad a \neq 0$

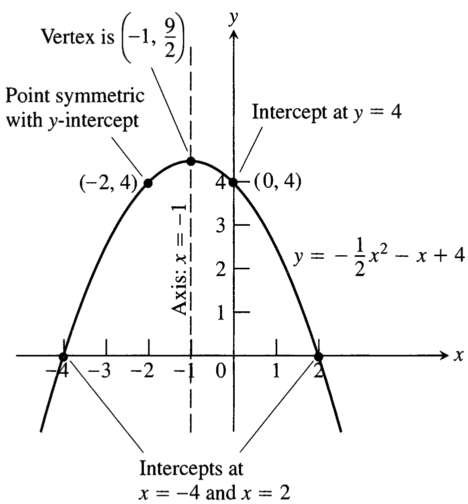
The graph of the equation $y = ax^2 + bx + c, a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The axis is the line

$$x = -\frac{b}{2a}. \tag{4}$$

The vertex of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

EXAMPLE 12 *Graphing a parabola*

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.



42 The parabola in Example 12.

Solution We take the following steps.

Step 1: *Compare the equation with $y = ax^2 + bx + c$ to identify a, b, and c.*

$$a = -\frac{1}{2}, \qquad b = -1, \qquad c = 4$$

Step 2: *Find the direction of opening.* Down, because $a < 0$.

Step 3: *Find the axis and vertex.* The axis is the line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1, \qquad \text{Eq. (4)}$$

so the x-coordinate of the vertex is -1. The y-coordinate is

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

Step 4: *Find the x-intercepts (if any).*

$$-\frac{1}{2}x^2 - x + 4 = 0 \qquad \text{Set } y = 0 \text{ in the parabola's equation.}$$

$$x^2 + 2x - 8 = 0 \qquad \text{Solve as usual.}$$

$$(x - 2)(x + 4) = 0$$

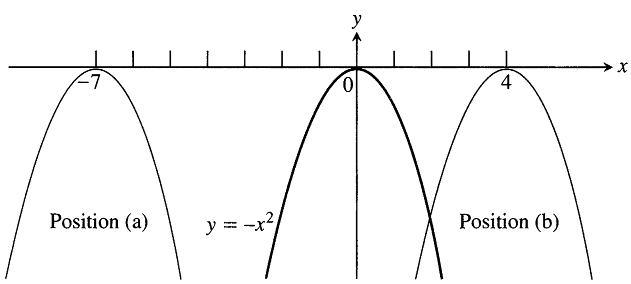
$$x = 2, \qquad x = -4$$

Step 5: *Sketch the graph.* We plot points, sketch the axis (lightly), and use what we know about symmetry and the direction of opening to complete the graph (Fig. 42). ❑

Exercises 4

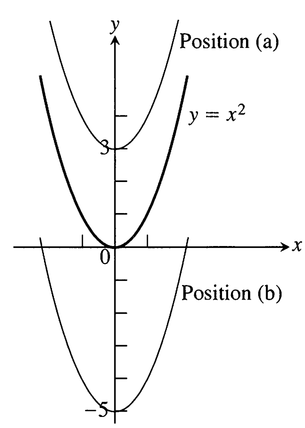
Shifting Graphs

1. Figure 43 shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



43 The parabolas in Exercise 1.

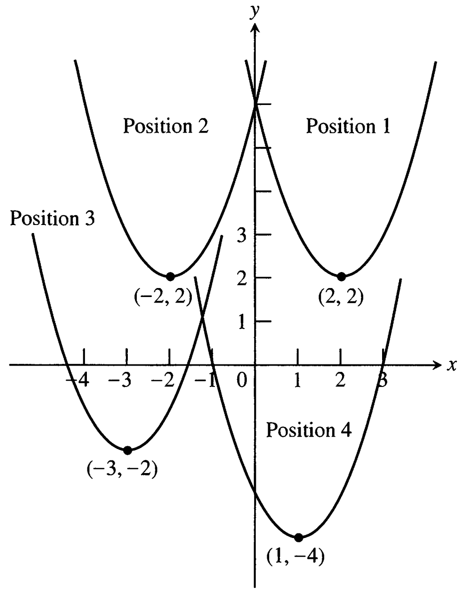
2. Figure 44 shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



44 The parabolas in Exercise 2.

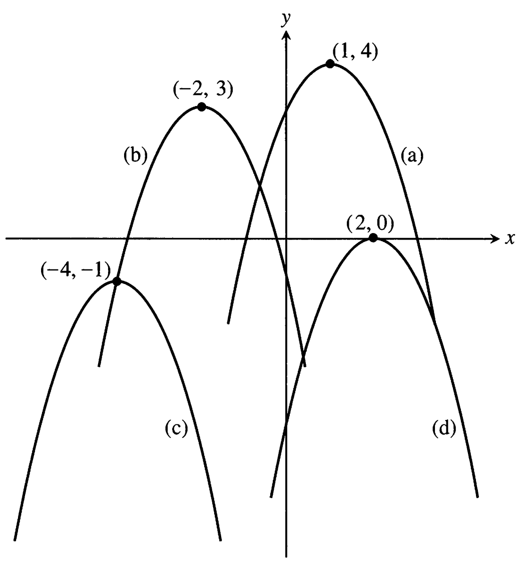
3. Match the equations listed in (a)–(d) to the graphs in Fig. 45.

a) $y = (x-1)^2 - 4$ **b)** $y = (x-2)^2 + 2$
c) $y = (x+2)^2 + 2$ **d)** $y = (x+3)^2 - 2$



45 The parabolas in Exercise 3.

4. Figure 46 shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



46 The parabolas in Exercise 4.

Exercises 5–16 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation. Use the graphs in Fig. 26 for reference as needed.

5. $x^2 + y^2 = 49$ Down 3, left 2

6. $x^2 + y^2 = 25$ Up 3, left 4

7. $y = x^3$ Left 1, down 1

8. $y = x^{2/3}$ Right 1, down 1

9. $y = \sqrt{x}$ Left 0.81

10. $y = -\sqrt{x}$ Right 3

11. $y = 2x - 7$ Up 7

12. $y = \dfrac{1}{2}(x+1) + 5$ Down 5, right 1

13. $x = y^2$ Left 1

14. $x = -3y^2$ Up 2, right 3

15. $y = 1/x$ Up 1, right 1

16. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 17–36. Use the graphs in Fig. 26 for reference as needed.

17. $y = \sqrt{x+4}$

18. $y = \sqrt{9-x}$

19. $y = |x-2|$

20. $y = |1-x| - 1$

21. $y = 1 + \sqrt{x-1}$

22. $y = 1 - \sqrt{x}$

23. $y = (x+1)^{2/3}$

24. $y = (x-8)^{2/3}$

25. $y = 1 - x^{2/3}$

26. $y + 4 = x^{2/3}$

27. $y = \sqrt[3]{x-1} - 1$

28. $y = (x+2)^{3/2} + 1$

29. $y = \dfrac{1}{x-2}$

30. $y = \dfrac{1}{x} - 2$

31. $y = \dfrac{1}{x} + 2$

32. $y = \dfrac{1}{x+2}$

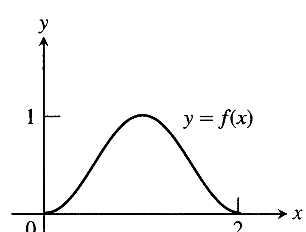
33. $y = \dfrac{1}{(x-1)^2}$

34. $y = \dfrac{1}{x^2} - 1$

35. $y = \dfrac{1}{x^2} + 1$

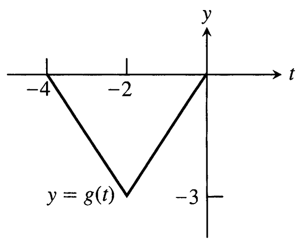
36. $y = \dfrac{1}{(x+1)^2}$

37. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $f(x) + 2$ **b)** $f(x) - 1$
c) $2f(x)$ **d)** $-f(x)$
e) $f(x+2)$ **f)** $f(x-1)$
g) $f(-x)$ **h)** $-f(x+1) + 1$

38. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $g(-t)$ **b)** $-g(t)$
c) $g(t) + 3$ **d)** $1 - g(t)$
e) $g(-t + 2)$ **f)** $g(t - 2)$
g) $g(1 - t)$ **h)** $-g(t - 4)$

Circles

In Exercises 39–44, find an equation for the circle with the given center $C(h, k)$ and radius a. Then sketch the circle in the xy-plane. Include the circle's center in your sketch. Also, label the circle's x- and y-intercepts, if any, with their coordinate pairs.

39. $C(0, 2), \quad a = 2$ **40.** $C(-3, 0), \quad a = 3$

41. $C(-1, 5), \quad a = \sqrt{10}$ **42.** $C(1, 1), \quad a = \sqrt{2}$

43. $C(-\sqrt{3}, -2), \quad a = 2$ **44.** $C(3, 1/2), \quad a = 5$

Graph the circles whose equations are given in Exercises 45–50. Label each circle's center and intercepts (if any) with their coordinate pairs.

45. $x^2 + y^2 + 4x - 4y + 4 = 0$

46. $x^2 + y^2 - 8x + 4y + 16 = 0$

47. $x^2 + y^2 - 3y - 4 = 0$

48. $x^2 + y^2 - 4x - (9/4) = 0$

49. $x^2 + y^2 - 4x + 4y = 0$

50. $x^2 + y^2 + 2x = 3$

Parabolas

Graph the parabolas in Exercises 51–58. Label the vertex, axis, and intercepts in each case.

51. $y = x^2 - 2x - 3$ **52.** $y = x^2 + 4x + 3$

53. $y = -x^2 + 4x$ **54.** $y = -x^2 + 4x - 5$

55. $y = -x^2 - 6x - 5$ **56.** $y = 2x^2 - x + 3$

57. $y = \frac{1}{2}x^2 + x + 4$ **58.** $y = -\frac{1}{4}x^2 + 2x + 4$

59. Graph the parabola $y = x - x^2$. Then find the domain and range of $f(x) = \sqrt{x - x^2}$.

60. Graph the parabola $y = 3 - 2x - x^2$. Then find the domain and range of $g(x) = \sqrt{3 - 2x - x^2}$.

Inequalities

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 61–68.

61. $x^2 + y^2 > 7$

62. $x^2 + y^2 < 5$

63. $(x - 1)^2 + y^2 \le 4$

64. $x^2 + (y - 2)^2 \ge 4$

65. $x^2 + y^2 > 1, \quad x^2 + y^2 < 4$

66. $x^2 + y^2 \le 4, \quad (x + 2)^2 + y^2 \le 4$

67. $x^2 + y^2 + 6y < 0, \quad y > -3$

68. $x^2 + y^2 - 4x + 2y > 4, \quad x > 2$

69. Write an inequality that describes the points that lie inside the circle with center $(-2, 1)$ and radius $\sqrt{6}$.

70. Write an inequality that describes the points that lie outside the circle with center $(-4, 2)$ and radius 4.

71. Write a pair of inequalities that describe the points that lie inside or on the circle with center (0, 0) and radius $\sqrt{2}$, and on or to the right of the vertical line through (1, 0).

72. Write a pair of inequalities that describe the points that lie outside the circle with center (0, 0) and radius 2, and inside the circle that has center (1, 3) and passes through the origin.

Shifting Lines

73. The line $y = mx$, which passes through the origin, is shifted vertically and horizontally to pass through the point (x_0, y_0). Find an equation for the new line. (This equation is called the line's *point–slope equation.*)

74. The line $y = mx$ is shifted vertically to pass through the point $(0, b)$. What is the new line's equation?

Intersecting Lines, Circles, and Parabolas

In Exercises 75–82, graph the two equations and find the points in which the graphs intersect.

75. $y = 2x, \quad x^2 + y^2 = 1$

76. $x + y = 1, \quad (x - 1)^2 + y^2 = 1$

77. $y - x = 1, \quad y = x^2$

78. $x + y = 0, \quad y = -(x - 1)^2$

79. $y = -x^2, \quad y = 2x^2 - 1$

80. $y = \frac{1}{4}x^2, \quad y = (x - 1)^2$

81. $x^2 + y^2 = 1, \quad (x - 1)^2 + y^2 = 1$

82. $x^2 + y^2 = 1, \quad x^2 + y = 1$

CAS Explorations and Projects

In Exercises 83–86, you will explore graphically what happens to the graph of $y = f(ax)$ as you change the value of the constant a. Use a CAS or computer grapher to perform the following steps.

a) Plot the function $y = f(x)$ together with the function $y = f(ax)$ for $a = 2, 3$, and 10 over the specified interval. Describe what happens to the graph as a increases through positive values.

b) Plot the function $y = f(x)$ and $y = f(ax)$ for the negative values $a = -2, -3$. What happens to the graph in this situation?

c) Plot the function $y = f(x)$ and $y = f(ax)$ for the fractional values $a = 1/2, 1/3, 1/4$. Describe what happens to the graph when $|a| < 1$.

83. $f(x) = \dfrac{5x}{x^2+4}, \quad [-10, 10]$

84. $f(x) = \dfrac{2x(x-1)}{x^2+1}, \quad [-3, 2]$

85. $f(x) = \dfrac{x+1}{2x^2+1}, \quad [-2, 2]$

86. $f(x) = \dfrac{x^4 - 4x^3 + 10}{x^2+4}, \quad [-1, 4]$

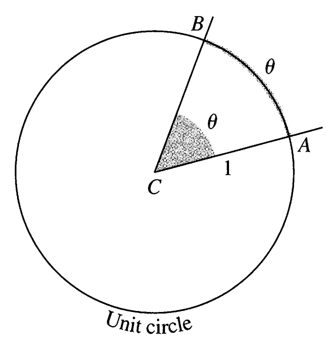
5 Trigonometric Functions

This section reviews radian measure, trigonometric functions, periodicity, and basic trigonometric identities.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations (Section 2.4).

Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle *ACB* is the length of the arc *AB*.

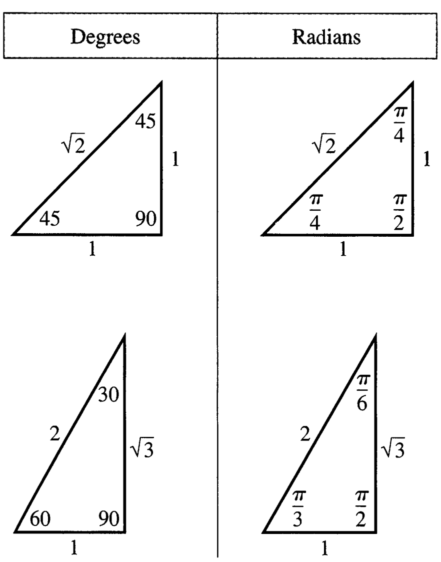
The **radian measure** θ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by the following equation.

$$\boxed{\pi \text{ radians} = 180^\circ}$$

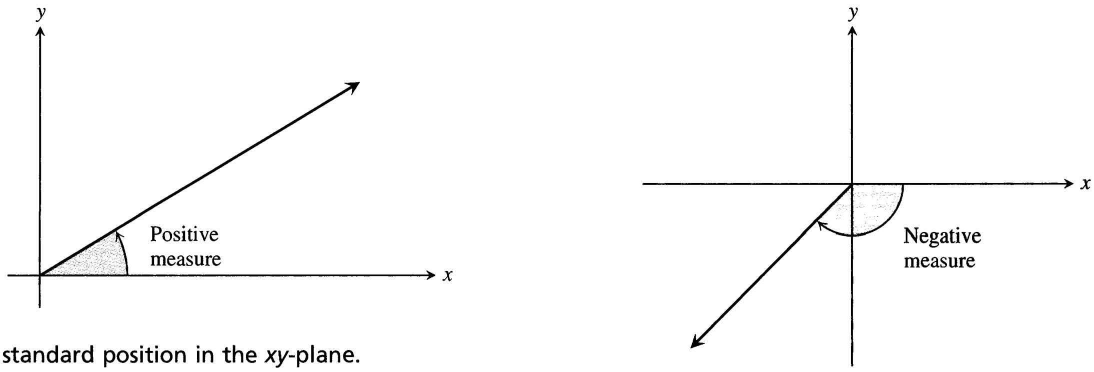
EXAMPLE 1 *Conversions (Fig. 48)*

Convert 45° to radians: $\quad 45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\quad \dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30^\circ$ ❑



48 The angles of two common triangles, in degrees and radians.



49 Angles in standard position in the *xy*-plane.

Conversion formulas

1 degree $= \dfrac{\pi}{180}$ (≈ 0.02) radians

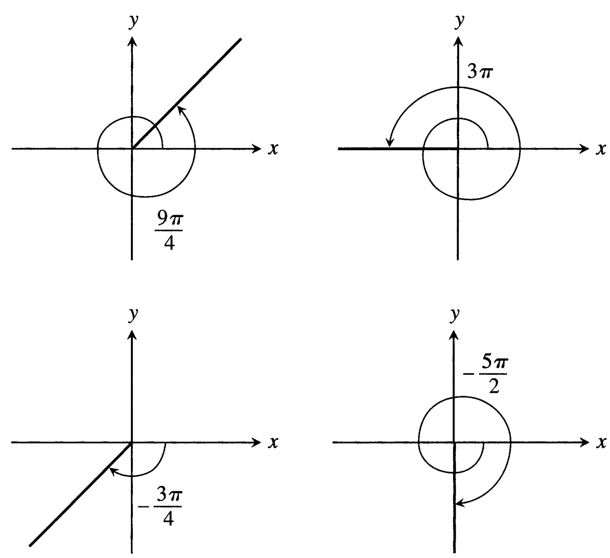
Degrees to radians: multiply by $\dfrac{\pi}{180}$

1 radian $= \dfrac{180}{\pi}$ (≈ 57) degrees

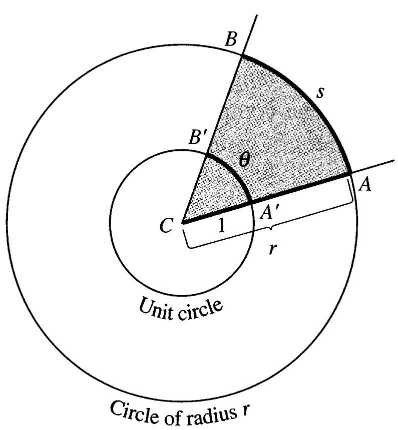
Radians to degrees: multiply by $\dfrac{180}{\pi}$

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig. 49). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

When angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Fig. 50).

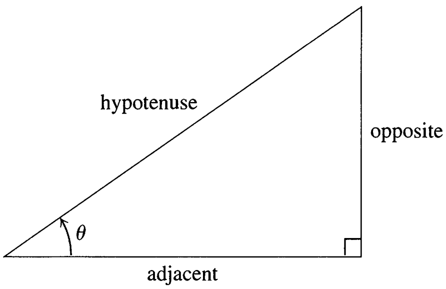


50 Nonzero radian measures can be positive or negative.



51 The radian measure of angle *ACB* is the length θ of arc $A'B'$ on the unit circle centered at *C*. The value of θ can be found from any other circle as *s*/*r*.

There is a useful relationship between the length s of an arc AB on a circle of radius r and the radian measure θ of the angle the arc subtends at the circle's center C (Fig. 51). If we draw a unit circle with the same center C, the arc $A'B'$ cut by the angle will have length θ, by the definition of radian measure. From the similarity of the circular sectors ACB and $A'CB'$, we then have $s/r = \theta/1$.

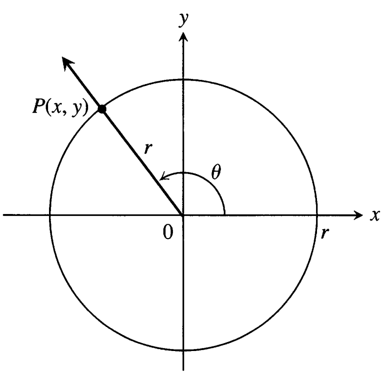


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

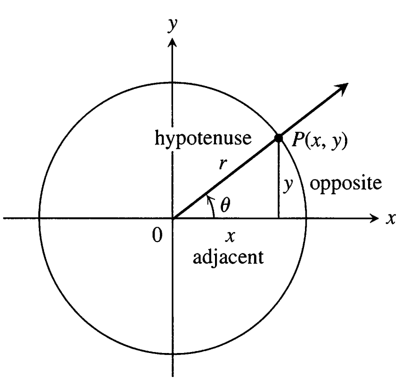
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

52 Trigonometric ratios of an acute angle.



53 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.



54 The new and old definitions agree for acute angles.

Radian Measure and Arc Length

$$\frac{s}{r} = \theta, \qquad \text{or} \qquad s = r\theta$$

Notice that these equalities hold precisely because we are measuring the angle in radians.

Angle Convention: Use Radians

From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

EXAMPLE 2 Consider a circle of radius 8. (a) Find the central angle subtended by an arc of length 2π on the circle. (b) Find the length of an arc subtending a central angle of $3\pi/4$.

Solution

a) $\theta = \dfrac{s}{r} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$ **b)** $s = r\theta = 8\left(\dfrac{3\pi}{4}\right) = 6\pi$ ❑

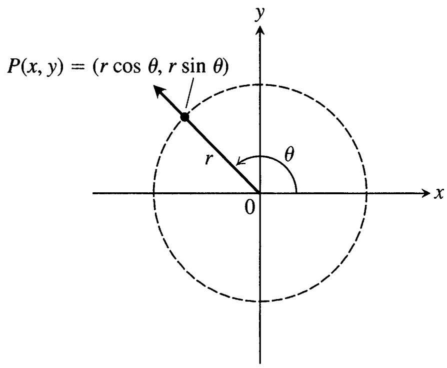
The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Fig. 52). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Fig. 53).

Sine:	$\sin\theta = \dfrac{y}{r}$	Cosecant:	$\csc\theta = \dfrac{r}{y}$
Cosine:	$\cos\theta = \dfrac{x}{r}$	Secant:	$\sec\theta = \dfrac{r}{x}$
Tangent:	$\tan\theta = \dfrac{y}{x}$	Cotangent:	$\cot\theta = \dfrac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute (Fig. 54).

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means they are



55 The Cartesian coordinates of a point in the plane expressed in terms of r and θ.

not defined if θ is $\pm\pi/2,\ \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0,\ \pm\pi,\ \pm 2\pi, \ldots$.

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

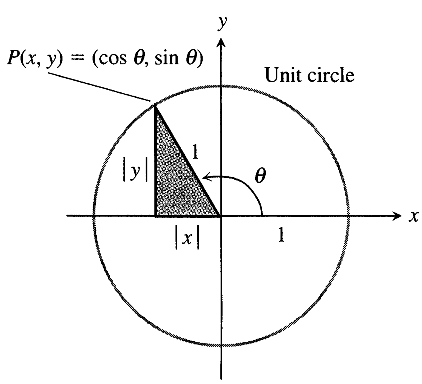
The coordinates of any point $P(x, y)$ in the plane can now be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Fig. 55). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

Values of Trigonometric Functions

If the circle in Fig. 53 has radius $r = 1$, the equations defining $\sin\theta$ and $\cos\theta$ become

$$\cos\theta = x, \qquad \sin\theta = y.$$



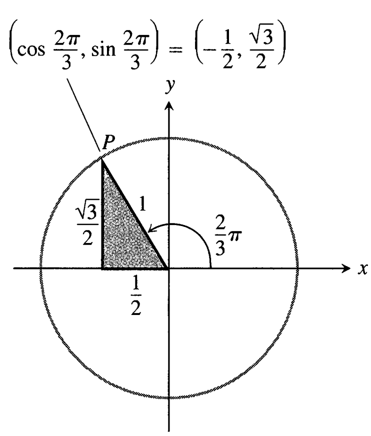
56 The acute reference triangle for an angle θ.

We can then calculate the values of the cosine and sine directly from the coordinates of P, if we happen to know them, or indirectly from the acute reference triangle made by dropping a perpendicular from P to the x-axis (Fig. 56). We read the magnitudes of x and y from the triangle's sides. The signs of x and y are determined by the quadrant in which the triangle lies.

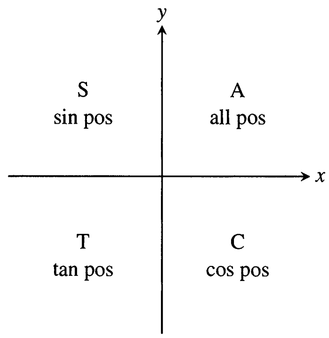
EXAMPLE 3 Find the sine and cosine of $2\pi/3$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 57).



57 The triangle for calculating the sine and cosine of $2\pi/3$ radians (Example 3).



58 The CAST rule.

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

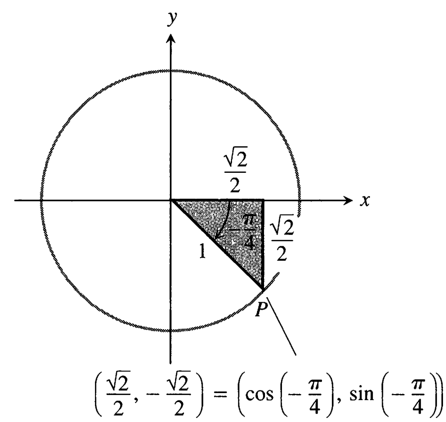
$$\cos\frac{2\pi}{3} = x\text{-coordinate of } P = -\frac{1}{2}$$
$$\sin\frac{2\pi}{3} = y\text{-coordinate of } P = \frac{\sqrt{3}}{2}. \qquad \square$$

A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule (Fig. 58).

EXAMPLE 4 Find the sine and cosine of $-\pi/4$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 59).



59 The triangle for calculating the sine and cosine of $-\pi/4$ radians (Example 4).

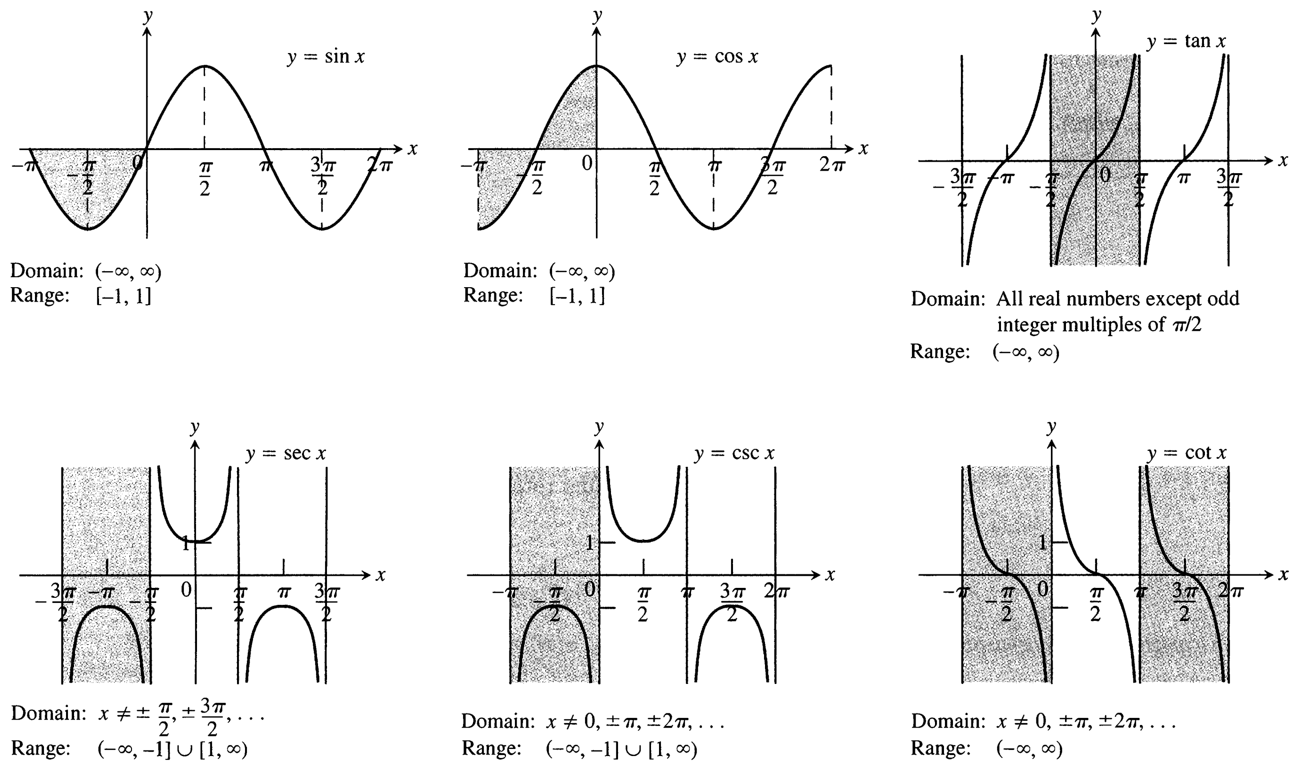
Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

$$\cos\left(-\frac{\pi}{4}\right) = x\text{-coordinate of } P = \frac{\sqrt{2}}{2},$$
$$\sin\left(-\frac{\pi}{4}\right) = y\text{-coordinate of} P = -\frac{\sqrt{2}}{2}. \qquad \square$$

Calculations similar to those in Examples 3 and 4 allow us to fill in Table 2.

Table 2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	-180	-135	-90	-45	0	30	45	60	90	135	180
θ (radians)	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π
$\sin\theta$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{2}/2$	0
$\cos\theta$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-\sqrt{2}/2$	-1
$\tan\theta$	0	1		-1	0	$\sqrt{3}/3$	1	$\sqrt{3}$		-1	0



60 The graphs of the six basic trigonometric functions as functions of radian measure. Each function's periodicity shows clearly in its graph.

Graphs

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Fig. 60.

Periodicity

When an angle of measure x and an angle of measure $x + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric values. For example, $\cos(x + 2\pi) = \cos x$. Functions like the trigonometric functions whose values repeat at regular intervals are called periodic.

Definition

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for all x. The smallest such value of p is the **period** of f.

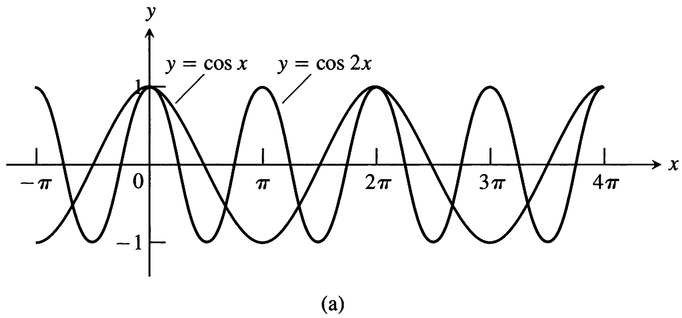
Periods of trigonometric functions

Period π:
$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π:
$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

As we can see in Fig. 60, the tangent and cotangent functions have period $p = \pi$. The other four functions have period 2π.

Figure 61 shows graphs of $y = \cos 2x$ and $y = \cos(x/2)$ plotted against the graph of $y = \cos x$. Multiplying x by a number greater than 1 speeds up a trigonometric function (increases the frequency) and shortens its period. Multiplying x by a positive number less than 1 slows a trigonometric function down and lengthens its period.



61 (a) Shorter period: $\cos 2x$. (b) Longer period: $\cos(x/2)$

The importance of periodic functions stems from the fact that much of the behavior we study in science is periodic. Brain waves and heartbeats are periodic, as are household voltage and electric current. The electromagnetic field that heats food in a microwave oven is periodic, as are cash flows in seasonal businesses and the behavior of rotational machinery. The seasons are periodic—so is the weather. The phases of the moon are periodic, as are the motions of the planets. There is strong evidence that the ice ages are periodic, with a period of 90,000–100,000 years.

If so many things are periodic, why limit our discussion to trigonometric functions? The answer lies in a surprising and beautiful theorem from advanced calculus that says that every periodic function we want to use in mathematical modeling can be written as an algebraic combination of sines and cosines. Thus, once we learn the calculus of sines and cosines, we will know everything we need to know to model the mathematical behavior of periodic phenomena.

Even vs. Odd

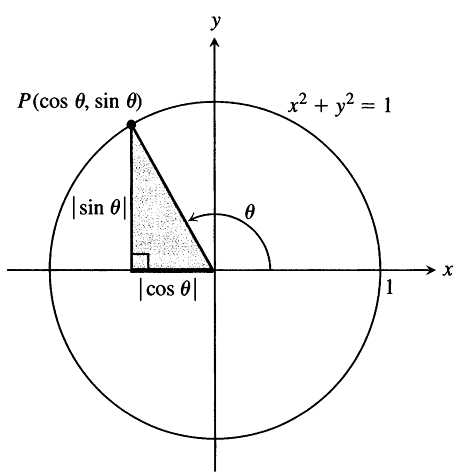
The symmetries in the graphs in Fig. 60 reveal that the cosine and secant functions are even and the other four functions are odd:

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Identities

Applying the Pythagorean theorem to the reference right triangle we obtain by dropping a perpendicular from the point $P(\cos\theta, \sin\theta)$ on the unit circle to the x-axis (Fig. 62) gives

$$\boxed{\cos^2\theta + \sin^2\theta = 1. \qquad (2)}$$



62 The reference triangle for a general angle θ.

This equation, true for all values of θ, is probably the most frequently used identity in trigonometry.

Dividing Eq. (2) in turn by $\cos^2\theta$ and $\sin^2\theta$ gives the identities

$$1+\tan^2\theta = \sec^2\theta,$$
$$1+\cot^2\theta = \csc^2\theta.$$

You may recall the following identities from an earlier course.

All the trigonometric identities you will need in this book derive from Eqs. (2) and (3).

Angle Sum Formulas

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{3}$$

These formulas hold for all angles A and B. There are similar formulas for $\cos(A-B)$ and $\sin(A-B)$ (Exercises 35 and 36).

Substituting θ for both A and B in the angle sum formulas gives two more useful identities:

Instead of memorizing Eqs. (3) you might find it helpful to remember Eqs. (4), and then recall where they came from.

Double-angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{4}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

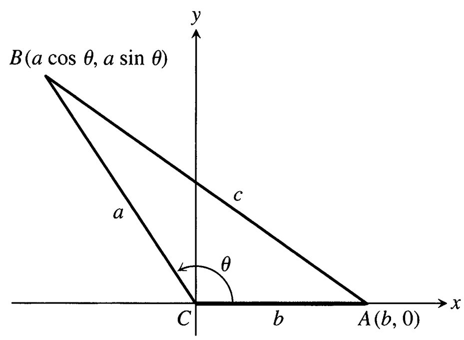
We add the two equations to get $2\cos^2\theta = 1+\cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1-\cos 2\theta$.

Additional Double-angle Formulas

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{5}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{6}$$

When θ is replaced by $\theta/2$ in Eqs. (5) and (6), the resulting formulas are called **half-angle** formulas. Some books refer to Eqs. (5) and (6) by this name as well.



63 The square of the distance between A and B gives the law of cosines.

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (7)$$

This equation is called the **law of cosines.**

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Fig. 63. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta,\ a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

Combining these equalities gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Exercises 5

Radians, Degrees, and Circular Arcs

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) $110°$?
2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.
3. CALCULATOR You want to make an $80°$ angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. CALCULATOR If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the table of function values shown on the following page. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \dfrac{3}{5}, \quad x \text{ in } \left[\dfrac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \text{ in } \left[0, \dfrac{\pi}{2}\right]$

9. $\cos x = \dfrac{1}{3}, \quad x \text{ in } \left[-\dfrac{\pi}{2}, 0\right]$

10. $\cos x = -\dfrac{5}{13}, \quad x \text{ in } \left[\dfrac{\pi}{2}, \pi\right]$

11. $\tan x = \dfrac{1}{2}, \quad x \text{ in } \left[\pi, \dfrac{3\pi}{2}\right]$

12. $\sin x = -\dfrac{1}{2}, \quad x \text{ in } \left[\pi, \dfrac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$

14. $\sin(x/2)$

15. $\cos \pi x$

16. $\cos \dfrac{\pi x}{2}$

17. $-\sin \dfrac{\pi x}{3}$

18. $-\cos 2\pi x$

19. $\cos\left(x - \dfrac{\pi}{2}\right)$

20. $\sin\left(x + \dfrac{\pi}{2}\right)$

21. $\sin\left(x - \dfrac{\pi}{4}\right) + 1$

22. $\cos\left(x + \dfrac{\pi}{4}\right) - 1$

Graph the functions in Exercises 23–26 in the ts-plane (t-axis horizontal, s-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot\ 2t$

24. $s = -\tan\ \pi t$

25. $s = \sec\left(\dfrac{\pi t}{2}\right)$

26. $s = \csc\left(\dfrac{t}{2}\right)$

27. GRAPHER

a) Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.

b) Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

28. GRAPHER Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Additional Trigonometric Identities

Use the angle sum formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the angle sum formulas? Do the results agree with something you already know?

Using the Angle Sum Formulas

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\dfrac{3\pi}{2} - x\right)$

42. $\cos\left(\dfrac{3\pi}{2} + x\right)$

43. Evaluate $\sin \dfrac{7\pi}{12}$ as $\sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right)$.

44. Evaluate $\cos \dfrac{11\pi}{12}$ as $\cos\left(\dfrac{\pi}{4} + \dfrac{2\pi}{3}\right)$.

45. Evaluate $\cos \dfrac{\pi}{12}$.

46. Evaluate $\sin \dfrac{5\pi}{12}$.

Using the Double-angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \dfrac{\pi}{8}$

48. $\cos^2 \dfrac{\pi}{12}$

49. $\sin^2 \dfrac{\pi}{12}$

50. $\sin^2 \dfrac{\pi}{8}$

Theory and Examples

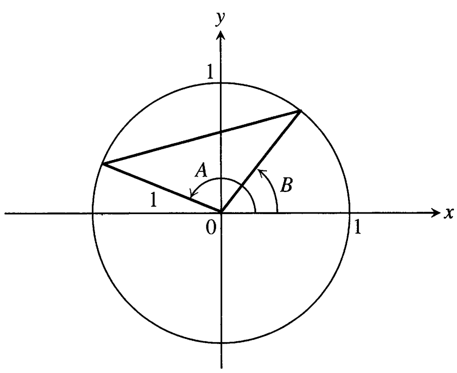
51. *The tangent sum formula.* The standard formula for the tangent of the sum of two angles is

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

52. *(Continuation of Exercise 51.)* Derive a formula for $\tan(A - B)$.

53. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



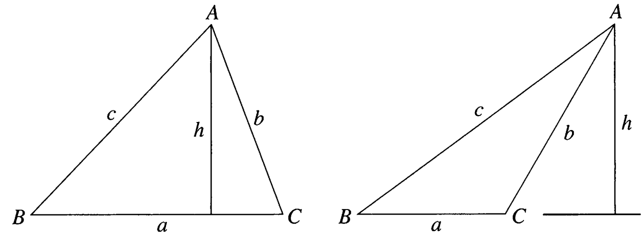
54. When applied to a figure similar to the one in Exercise 53, the law of cosines leads directly to the formula for $\cos(A + B)$. What is that figure and how does the derivation go?

55. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

56. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

57. *The law of sines.* The **law of sines** says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.

58. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 55). Find the sine of angle B using the law of sines.

59. CALCULATOR A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

60. *The approximation $\sin x \approx x$.* It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.7, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a) With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b) With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

c) *A quick radian mode check.* Is your calculator in radian mode? Evaluate $\sin x$ at a value of x near the origin, say $x = 0.1$. If $\sin x \approx x$, the calculator is in radian mode; if not, it isn't. Try it.

General Sine Curves

Figure 64 on the following page shows the graph of a **general sine function** of the form

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

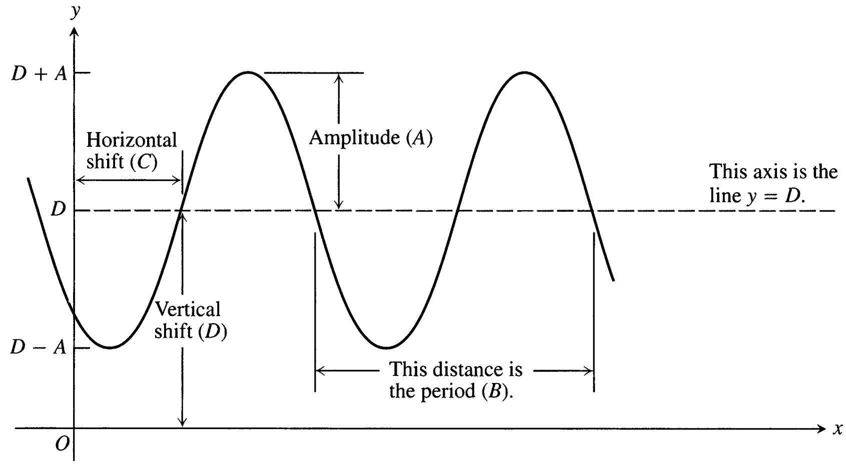
where $|A|$ is the *amplitude,* $|B|$ is the *period,* C is the *horizontal shift,* and D is the *vertical shift.* Identify A, B, C, and D for the sine functions in Exercises 61–64 and sketch their graphs.

61. $y = 2\sin(x + \pi) - 1$

62. $y = \dfrac{1}{2}\sin(\pi x - \pi) + \dfrac{1}{2}$

63. $y = -\dfrac{2}{\pi}\sin\left(\dfrac{\pi}{-2}t\right) + \dfrac{1}{\pi}$

64. $y = \dfrac{L}{2\pi}\sin\dfrac{2\pi t}{L}, \quad L > 0$



64 The general sine curve

$$y = A\sin[(2\pi/B)(x - C)] + D,$$

shown for *A*, *B*, *C*, and *D* positive.

The Trans-Alaska Pipeline

The builders of the Trans-Alaska Pipeline used insulated pads to keep the heat from the hot oil in the pipeline from melting the permanently frozen soil beneath. To design the pads, it was necessary to take into account the variation in air temperature throughout the year. Figure 65 shows how we can use a general sine function, defined in the introduction to Exercises 61–64, to represent temperature data. The data points in the figure are plots of the mean air temperature for Fairbanks, Alaska, based on records of the National Weather Service from 1941 to 1970. The sine function used to fit the data is

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25,$$

where f is temperature in degrees Fahrenheit and x is the number of the day counting from the beginning of the year. The fit is remarkably good.

65. *Temperature in Fairbanks, Alaska.* Find the (a) amplitude, (b) period, (c) horizontal shift, and (d) vertical shift of the general sine function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

66. *Temperature in Fairbanks, Alaska.* Use the equation in Exercise 65 to approximate the answers to the following questions about the temperature in Fairbanks, Alaska, shown in Fig. 65. Assume that the year has 365 days.

a) What are the highest and lowest mean daily temperatures shown?

b) What is the average of the highest and lowest mean daily temperatures shown? Why is this average the vertical shift of the function?

CAS Explorations and Projects

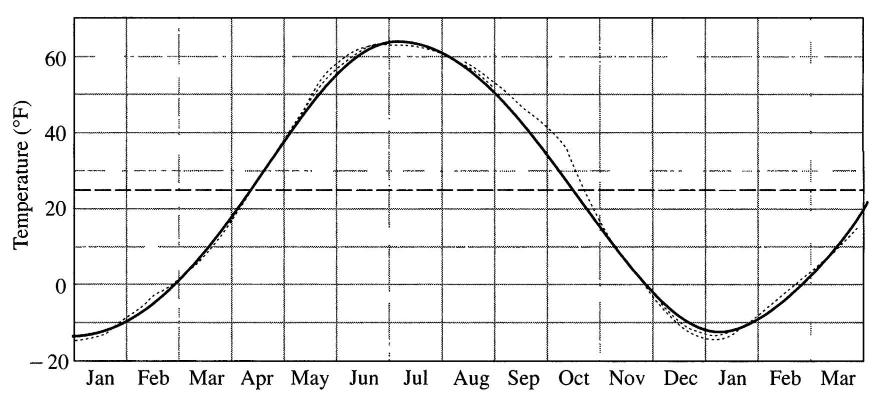
In Exercises 67–70, you will explore graphically the general sine function

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

67. *The period B.* Set the constants $A = 3$, $C = D = 0$.

a) Plot $f(x)$ for the values $B = 1,\ 3,\ 2\pi,\ 5\pi$ over the interval



65 Normal mean air temperature at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

(Source: "Is the Curve of Temperature Variation a Sine Curve?" by B. M. Lando and C. A. Lando, *The Mathematics Teacher,* 7:6, Fig. 2, p. 535 [September 1977].)

$-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b) What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

68. *The horizontal shift C.* Set the constants $A = 3$, $B = 6$, $D = 0$.

a) Plot $f(x)$ for the values $C = 0$, 1, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b) What happens to the graph for negative values of C?

c) What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

69. *The vertical shift D.* Set the constants $A = 3$, $B = 6$, $C = 0$.

a) Plot $f(x)$ for the values $D = 0$, 1, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b) What happens to the graph for negative values of D?

70. *The amplitude A.* Set the constants $B = 6$, $C = D = 0$.

a) Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1$, 5, and 9.

b) What happens to the graph for negative values of A?

PRELIMINARIES — QUESTIONS TO GUIDE YOUR REVIEW

1. What are the order properties of the real numbers? How are they used in solving inequalities?
2. What is a number's absolute value? Give examples. How are $|-a|$, $|ab|$, $|a/b|$, and $|a+b|$ related to $|a|$ and $|b|$?
3. How are absolute values used to describe intervals or unions of intervals? Give examples.
4. How do you find the distance between two points in the coordinate plane?
5. How can you write an equation for a line if you know the coordinates of two points on the line? the line's slope and the coordinates of one point on the line? the line's slope and y-intercept? Give examples.
6. What are the standard equations for lines perpendicular to the coordinate axes?
7. How are the slopes of mutually perpendicular lines related? What about parallel lines? Give examples.
8. When a line is not vertical, what is the relation between its slope and its angle of inclination?
9. What is a function? Give examples. How do you graph a real-valued function of a real variable?
10. Name some typical algebraic and trigonometric functions and draw their graphs.
11. What is an even function? an odd function? What geometric properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd. What, if anything, can you say about sums, products, quotients, and composites involving even and odd functions?
12. If f and g are real-valued functions, how are the domains of $f+g$, $f-g$, fg, and f/g related to the domains of f and g? Give examples.
13. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
14. How do you change the equation $y = f(x)$ to shift its graph up or down? to the left or right? Give examples.
15. Describe the steps you would take to graph the circle $x^2 + y^2 + 4x - 6y + 12 = 0$.
16. If a, b, and c are constants and $a \ne 0$, what can you say about the graph of the equation $y = ax^2 + bx + c$? In particular, how would you go about sketching the curve $y = 2x^2 + 4x$?
17. What inequality describes the points in the coordinate plane that lie inside the circle of radius a centered at the point (h, k)? that lie inside or on the circle? that lie outside the circle? that lie outside or on the circle?
18. What is radian measure? How do you convert from radians to degrees? degrees to radians?
19. Graph the six basic trigonometric functions. What symmetries do the graphs have?
20. How can you sometimes find the values of trigonometric functions from triangles? Give examples.
21. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
22. Starting with the identity $\cos^2\theta + \sin^2\theta = 1$ and the formulas for $\cos(A+B)$ and $\sin(A+B)$, show how a variety of other trigonometric identities may be derived.

PRELIMINARIES PRACTICE EXERCISES

Geometry

1. A particle in the plane moved from $A(-2, 5)$ to the y-axis in such a way that Δy equaled $3\,\Delta x$. What were the particle's new coordinates?

2. a) Plot the points $A(8, 1)$, $B(2, 10)$, $C(-4, 6)$, $D(2, -3)$, and $E(14/3, 6)$.
   b) Find the slopes of the lines *AB, BC, CD, DA, CE,* and *BD.*
   c) Do any four of the five points *A, B, C, D,* and *E* form a parallelogram?
   d) Are any three of the five points collinear? How do you know?
   e) Which of the lines determined by the five points pass through the origin?

3. Do the points $A(6, 4)$, $B(4, -3)$, and $C(-2, 3)$ form an isosceles triangle? a right triangle? How do you know?

4. Find the coordinates of the point on the line $y = 3x + 1$ that is equidistant from $(0, 0)$ and $(-3, 4)$.

Functions and Graphs

5. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.

6. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.

7. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.

8. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of lift-off. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

Composition with absolute values. In Exercises 9–14, graph f_1 and f_2 together. Then describe how applying the absolute value function before applying f_1 affects the graph.

	$f_1(x)$	$f_2(x) = f_1(\|x\|)$
9.	x	$\|x\|$
10.	x^3	$\|x\|^3$
11.	x^2	$\|x\|^2$
12.	$\frac{1}{x}$	$\frac{1}{\|x\|}$
13.	$\sqrt{x}$	$\sqrt{\|x\|}$
14.	$\sin x$	$\sin \|x\|$

Composition with absolute values. In Exercises 15–20, graph g_1 and g_2 together. Then describe how taking absolute values after applying g_1 affects the graph.

	$g_1(x)$	$g_2(x) = \|g_1(x)\|$
15.	x^3	$\|x^3\|$
16.	$\sqrt{x}$	$\|\sqrt{x}\|$
17.	$\frac{1}{x}$	$\left\|\frac{1}{x}\right\|$
18.	$4 - x^2$	$\|4 - x^2\|$
19.	$x^2 + x$	$\|x^2 + x\|$
20.	$\sin x$	$\|\sin x\|$

Trigonometry

In Exercises 21–24, sketch the graph of the given function. What is the period of the function?

21. $y = \cos 2x$

22. $y = \sin \dfrac{x}{2}$

23. $y = \sin \pi x$

24. $y = \cos \dfrac{\pi x}{2}$

25. Sketch the graph $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

26. Sketch the graph $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 27–30, *ABC* is a right triangle with the right angle at *C.* The sides opposite angles *A, B,* and *C* are *a, b,* and *c,* respectively.

27. a) Find a and b if $c = 2$, $B = \pi/3$.
    b) Find a and c if $b = 2$, $B = \pi/3$.

28. a) Express a in terms of A and c.
    b) Express a in terms of A and b.

29. a) Express a in terms of B and b.
    b) Express c in terms of A and a.

30. a) Express $\sin A$ in terms of a and c.
    b) Express $\sin A$ in terms of b and c.

31. CALCULATOR Two guy wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal, and wire CT makes an angle of 50° with the horizontal, how high is the pole?

32. CALCULATOR Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to

be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

33. Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

34. Express $\cos 3x$ in terms of $\sin x$ and $\cos x$.

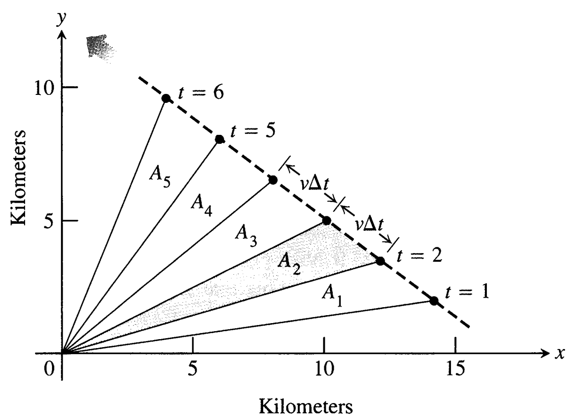
35. a) GRAPHER Graph the function $f(x) = \sin x + \cos(x/2)$.
b) What appears to be the period of this function?
c) Confirm your finding in (b) algebraically.

36. a) GRAPHER Graph $f(x) = \sin(1/x)$.
b) What are the domain and range of f?
c) Is f periodic? Give reasons for your answer.

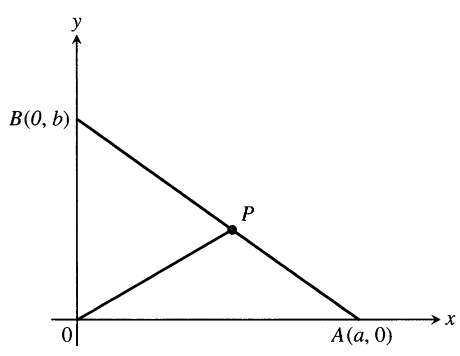
PRELIMINARIES — ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Geometry

1. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 11.5), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



2. a) Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure $(a, b > 0)$.



b) When is OP perpendicular to AB?

Functions and Graphs

3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ *is* a straight line. Give reasons for your answer.

5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

7. Graph the equation $|x| + |y| = 1 + x$.

8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}$, $A = \pi/3$, $B = \pi/4$.

10. Find $\sin B$ if $a = 4$, $b = 3$, $A = \pi/4$.

11. Find $\cos A$ if $a = 2$, $b = 2$, $c = 3$.

12. Find c if $a = 2$, $b = 3$, $C = \pi/4$.

13. Find $\sin B$ if $a = 2$, $b = 3$, $c = 4$.

14. Find $\sin C$ if $a = 2$, $b = 4$, $c = 5$.

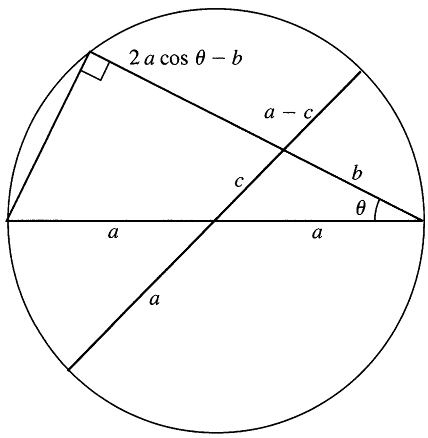
Derivations and Proofs

15. Prove the following identities.

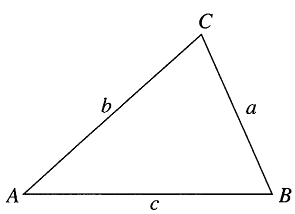
a) $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

b) $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine,* Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab\sin C = (1/2)bc\sin A = (1/2)ca\sin B$.



* 18. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter of the triangle.*

19. *Properties of inequalities.* If a and b are real numbers, we say that a **is less than** b and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

If a, b, and c are real numbers, then:

1. $a < b \implies a + c < b + c$
2. $a < b \implies a - c < b - c$
3. $a < b$ and $c > 0 \implies ac < bc$
4. $a < b$ and $c < 0 \implies bc < ac$
   (Special case: $a < b \implies -b < -a$)
5. $a > 0 \implies \dfrac{1}{a} > 0$
6. $0 < a < b \implies \dfrac{1}{b} < \dfrac{1}{a}$
7. $a < b < 0 \implies \dfrac{1}{b} < \dfrac{1}{a}$

20. *Properties of absolute values.* Prove the following properties of absolute values of real numbers.

a) $|-a| = |a|$

b) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

21. Prove that the following inequalities hold for any real numbers a and b.

a) $|a| < |b|$ if and only if $a^2 < b^2$
b) $|a - b| \geq ||a| - |b||$

22. *Generalizing the triangle inequality.* Prove by mathematical induction that the following inequalities hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

a) $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$
b) $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a) *Even-odd decompositions.* Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:
$$f(x) = E(x) + O(x),$$
where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b) *Uniqueness.* Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

a) a changes while b and c remain fixed?
b) b changes (a and c fixed, $a \neq 0$)?
c) c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x + b)^3 + c$ as

a) a changes while b and c remain fixed?
b) b changes (a and c fixed, $a \neq 0$)?
c) c changes (a and b fixed, $a \neq 0$)?

27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

*Asterisk denotes more challenging problem.

CHAPTER 1

Limits and Continuity

OVERVIEW The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry.

In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function. The derivative, which we investigate thoroughly in Chapter 2, quantifies the way a function's values change.

1.1 Rates of Change and Limits

In this section we introduce two rates of change, speed and population growth. This leads to the main idea of the section, the idea of limit.

Speed

A moving body's **average speed** over any particular time interval is the amount of distance covered during the interval divided by the length of the interval.

Free fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time elapsed. At least, that is what happens when the body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of their fall through air, before their velocities build up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall.*

EXAMPLE 1 A rock falls from the top of a 150-ft cliff. What is its average speed (a) during the first 2 sec of fall? (b) during the 1-sec interval between second 1 and second 2?

Solution Physical experiments show that a solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2 \text{ ft}$$

during the first t sec. The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt.

a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\frac{\text{ft}}{\text{sec}}$$

b) From second 1 to second 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\frac{\text{ft}}{\text{sec}}$$ ❑

Table 1.1 Average speeds over short time intervals

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(t_0+h)^2 - 16{t_0}^2}{h}$

Length of time interval h	**Average speed over interval of length h starting at $t_0 = 1$**	**Average speed over interval of length h starting at $t_0 = 2$**
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

EXAMPLE 2 Find the speed of the rock at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0+h)^2 - 16{t_0}^2}{h}.$$

We cannot use this formula to calculate the "instantaneous" speed at t_0 by substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, we see a pattern (Table 1.1).

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Similarly, the rock's speed at $t_0 = 2$ sec would appear to be 64 ft/sec. ❑

Average Rates of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in value of y, $\Delta y = f(x_2) - f(x_1)$, by the length of the interval $\Delta x = x_2 - x_1 = h$ over which the change occurred.

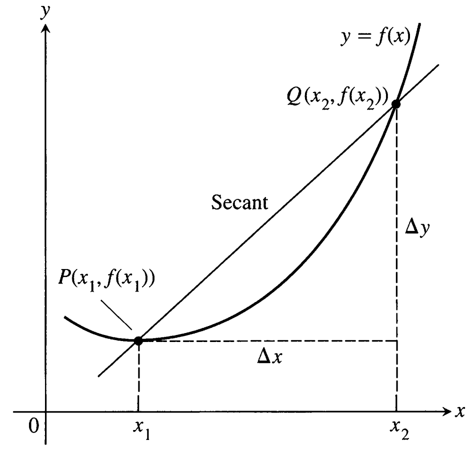
Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

Geometrically, an average rate of change is a secant slope.

Notice that the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Fig. 1.1). In geometry, a line joining two points of a curve is called a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.



1.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions.

EXAMPLE 3 *The average growth rate of a laboratory population*

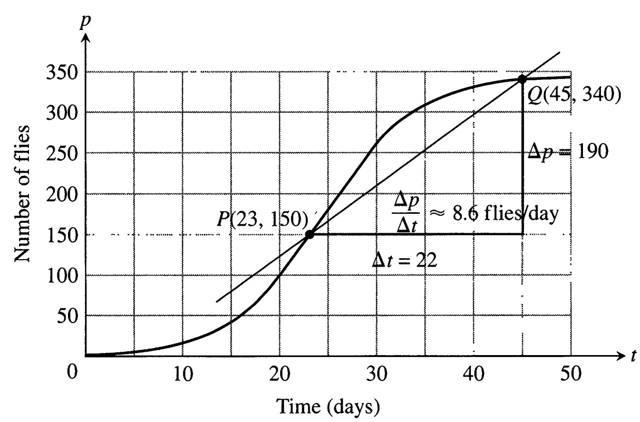
Figure 1.2 shows how a population of fruit flies *(Drosophila)* grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve. Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

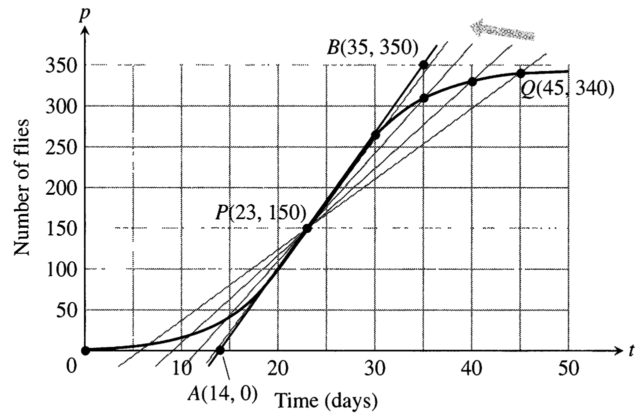
This average is the slope of the secant through the points P and Q on the graph in Fig. 1.2. ❑

The average rate of change from day 23 to day 45 calculated in Example 3 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.



1.2 Growth of a fruit fly population in a controlled experiment. (Source: *Elements of Mathematical Biology* by A. J. Lotka, 1956, Dover, New York, p. 69.)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\dfrac{340-150}{45-23} \approx 8.6$
(40, 330)	$\dfrac{330-150}{40-23} \approx 10.6$
(35, 310)	$\dfrac{310-150}{35-23} \approx 13.3$
(30, 265)	$\dfrac{265-150}{30-23} \approx 16.4$



1.3 The positions and slopes of four secants through the point P on the fruit fly graph.

EXAMPLE 4 How fast was the number of flies in the population of Example 3 growing on day 23 itself?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Fig. 1.3).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red line in the figure, a line that goes through P in the same direction that the curve goes through P. We will see that this line is called the *tangent* to the curve at P. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350-0}{35-14} = 16.7 \text{ flies/day (approximately).}$$

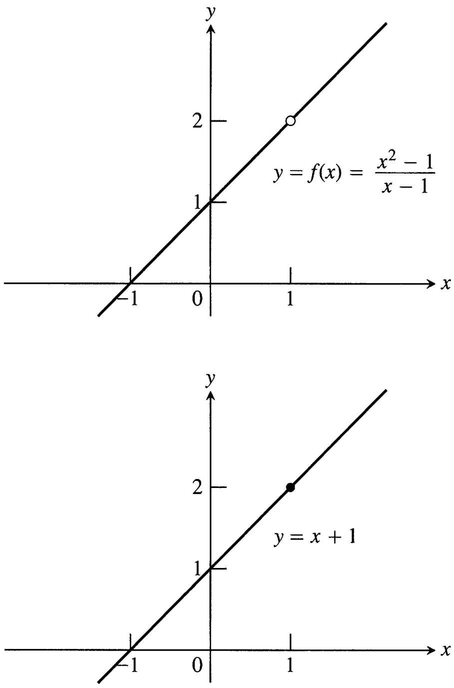
On day 23 the population was increasing at a rate of about 16.7 flies/day. ❑

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ and the rate at which the population in Example 4 was changing on day $t = 23$ are called *instantaneous rates of change.* As the examples suggest, we find instantaneous rates as limiting values of average rates. In Example 4, we also pictured the tangent line to the population curve on day 23 as a limiting position of secant lines. Instantaneous rates and tangent lines, intimately connected, appear in many other contexts. To talk about the two constructively, and to understand the connection further, we need to investigate the process by which we determine limiting values, or *limits,* as we will soon call them.

Limits of Function Values

Before we give a definition of limit, let us look at another example.

EXAMPLE 5 How does the function $f(x) = \dfrac{x^2-1}{x-1}$ behave near $x = 1$?



1.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined.

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$ we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for} \quad x \neq 1.$$

The graph of f is thus the line $y = x + 1$ with one point removed, namely the point $(1, 2)$. This removed point is shown as a "hole" in Fig. 1.4. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing *x close enough* to 1 (Table 1.2).

We say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, $f(x)$ approaches the *limit* 2 as x approaches 1. We write this as

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

❑

Table 1.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2.

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

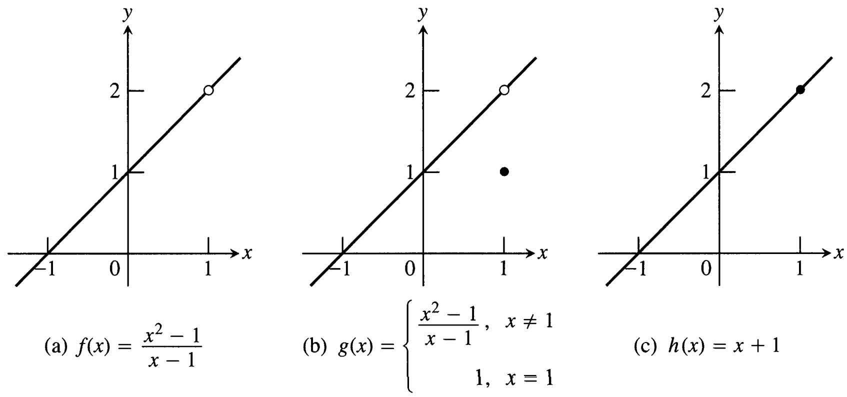
Definition

Informal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L.$$

This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch.* To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years.* The definition is clear enough, however, to enable us to recognize and evaluate limits of specific functions. We will need the more precise definition of Section 1.3, however, when we set out to prove theorems about limits.



1.5 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2.$

EXAMPLE 6 The existence of a limit as $x \to x_0$ does not depend on how the function may be defined at x_0. The function f in Fig 1.5 has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$. For h we have $\lim_{x \to 1} h(x) = h(1)$. This kind of equality of limit and function value is special, and we will return to it in Section 1.5. ❑

Sometimes $\lim_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$. This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(x_0)$ is defined. (We will say more about this in Sections 1.2 and 1.5.)

EXAMPLE 7

a) $\lim_{x \to 2} (4) = 4$

b) $\lim_{x \to -13} (4) = 4$

c) $\lim_{x \to 3} x = 3$

d) $\lim_{x \to 2} (5x - 3) = 10 - 3 = 7$

e) $\lim_{x \to -2} \dfrac{3x + 4}{x + 5} = \dfrac{-6 + 4}{-2 + 5} = -\dfrac{2}{3}$ ❑

EXAMPLE 8

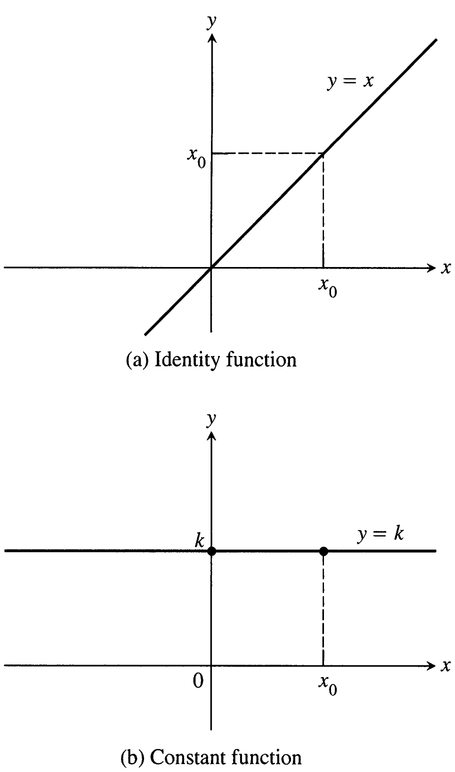
a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Fig. 1.6a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

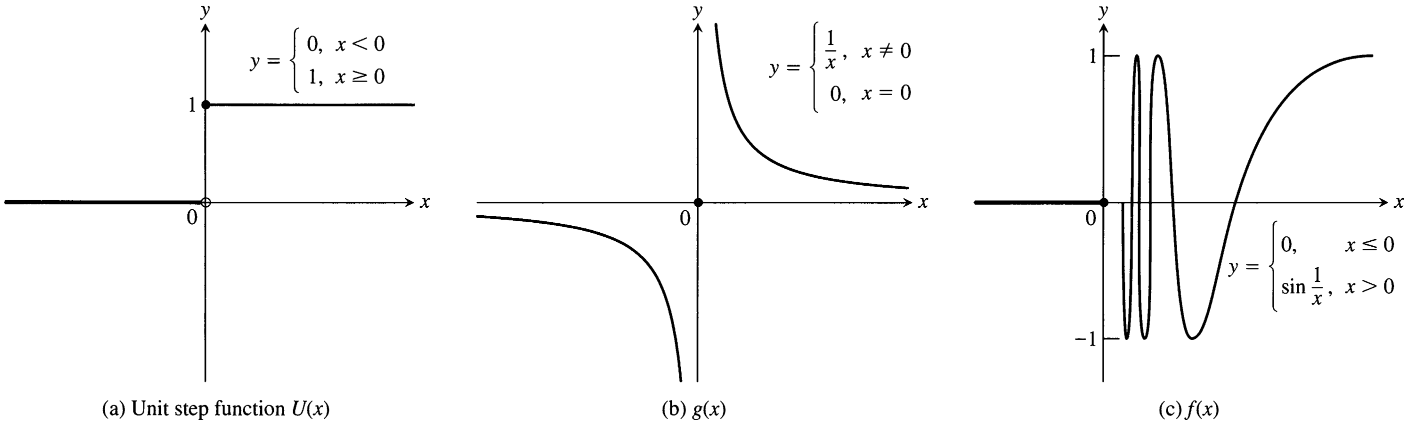
b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Fig. 1.6b),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$ ❑

Some ways that limits can fail to exist are illustrated in Fig. 1.7 and described in the next example.



1.6 The functions in Example 8.



1.7 The functions in Example 9.

EXAMPLE 9 *A function may fail to have a limit at a point in its domain.*

Discuss the behavior of the following functions as $x \to 0$.

a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

b) $g(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \dfrac{1}{x}, & x > 0 \end{cases}$

Solution

a) It jumps: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Fig. 1.7a).

b) It grows too large: $g(x)$ has no limits as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* real number (Fig. 1.7b).

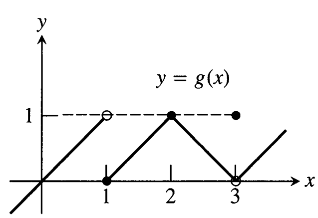
c) It oscillates too much: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$ (Fig. 1.7c). ❑

Exercises 1.1

Limits from Graphs

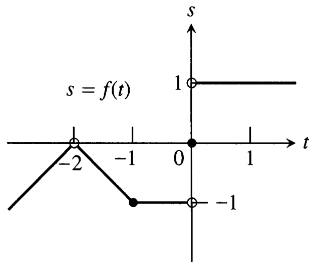
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a) $\lim_{x \to 1} g(x)$ **b)** $\lim_{x \to 2} g(x)$ **c)** $\lim_{x \to 3} g(x)$

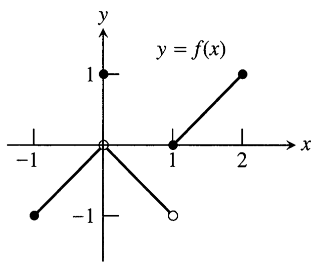


2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a) $\lim_{t\to -2} f(t)$ b) $\lim_{t\to -1} f(t)$ c) $\lim_{t\to 0} f(t)$

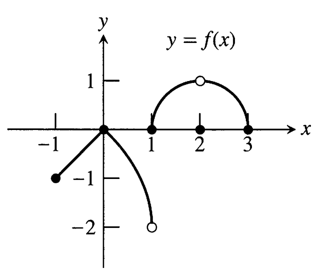


3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



   a) $\lim_{x\to 0} f(x)$ exists b) $\lim_{x\to 0} f(x) = 0$
   c) $\lim_{x\to 0} f(x) = 1$ d) $\lim_{x\to 1} f(x) = 1$
   e) $\lim_{x\to 1} f(x) = 0$
   f) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$

4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



   a) $\lim_{x\to 2} f(x)$ does not exist
   b) $\lim_{x\to 2} f(x) = 2$
   c) $\lim_{x\to 1} f(x)$ does not exist
   d) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$
   e) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(1, 3)$

Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to 0} \dfrac{x}{|x|}$ 6. $\lim_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = x_0$. Can anything be said about the existence of $\lim_{x\to x_0} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculator/Grapher Exercises—Estimating Limits

11. Let $f(x) = (x^2 - 9)/(x + 3)$.
   a) CALCULATOR Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x\to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?
   b) GRAPHER Support your conclusions in (a) by graphing f near $x_0 = -3$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to -3$.
   c) Find $\lim_{x\to -3} f(x)$ algebraically.

12. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.
   a) CALCULATOR Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x\to\sqrt{2}} g(x)$.
   b) GRAPHER Support your conclusion in (a) by graphing g near $x_0 = \sqrt{2}$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to \sqrt{2}$.
   c) Find $\lim_{x\to\sqrt{2}} g(x)$ algebraically.

13. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.
   a) CALCULATOR Make a table of the values of G at $x = -5.9, -5.99, -5.999, \ldots$. Then estimate $\lim_{x\to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \ldots$ instead?
   b) GRAPHER Support your conclusions in (a) by graphing G and using ZOOM and TRACE to estimate y-values on the graph as $x \to -6$.
   c) Find $\lim_{x\to -6} G(x)$ algebraically.

14. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.
   a) CALCULATOR Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x\to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b) GRAPHER Support your conclusions in (a) by graphing h near $x_0 = 3$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to 3$.

c) Find $\lim_{x\to 3} h(x)$ algebraically.

15. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a) CALCULATOR Make tables of the values of f at values of x that approach $x_0 = -1$ from above and below. Then estimate $\lim_{x\to -1} f(x)$.

b) GRAPHER Support your conclusion in (a) by graphing f near $x_0 = -1$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to -1$.

c) Find $\lim_{x\to -1} f(x)$ algebraically.

16. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a) CALCULATOR Make tables of values of F at values of x that approach $x_0 = -2$ from above and below. Then estimate $\lim_{x\to -2} F(x)$.

b) GRAPHER Support your conclusion in (a) by graphing F near $x_0 = -2$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to -2$.

c) Find $\lim_{x\to -2} F(x)$ algebraically.

17. Let $g(\theta) = (\sin\theta)/\theta$.

a) CALCULATOR Make tables of values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta\to 0} g(\theta)$.

b) GRAPHER Support your conclusion in (a) by graphing g near $\theta_0 = 0$.

18. Let $G(t) = (1 - \cos t)/t^2$.

a) CALCULATOR Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t\to 0} G(t)$.

b) GRAPHER Support your conclusion in (a) by graphing G near $t_0 = 0$.

19. Let $f(x) = x^{1/(1-x)}$.

a) CALCULATOR Make tables of values of f at values of x that approach $x_0 = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b) GRAPHER Support your conclusions in (a) by graphing f near $x_0 = 1$.

20. Let $f(x) = (3^x - 1)/x$.

a) CALCULATOR Make tables of values of f at values of x that approach $x_0 = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b) GRAPHER Support your conclusions in (a) by graphing f near $x_0 = 0$.

Limits by Substitution

In Exercises 21–28, find the limits by substitution. *Support your answers with a grapher or calculator if available.*

21. $\lim_{x\to 2} 2x$

22. $\lim_{x\to 0} 2x$

23. $\lim_{x\to 1/3} (3x - 1)$

24. $\lim_{x\to 1} \dfrac{-1}{(3x-1)}$

25. $\lim_{x\to -1} 3x(2x - 1)$

26. $\lim_{x\to -1} \dfrac{3x^2}{2x-1}$

27. $\lim_{x\to \pi/2} x\sin x$

28. $\lim_{x\to \pi} \dfrac{\cos x}{1-\pi}$

Average Rates of Change

In Exercises 29–34, find the average rate of change of the function over the given interval or intervals.

29. $f(x) = x^3 + 1$;
(a) $[2, 3]$, (b) $[-1, 1]$

30. $g(x) = x^2$;
(a) $[-1, 1]$, (b) $[-2, 0]$

31. $h(t) = \cot t$;
(a) $[\pi/4, 3\pi/4]$, (b) $[\pi/6, \pi/2]$

32. $g(t) = 2 + \cos t$;
(a) $[0, \pi]$, (b) $[-\pi, \pi]$

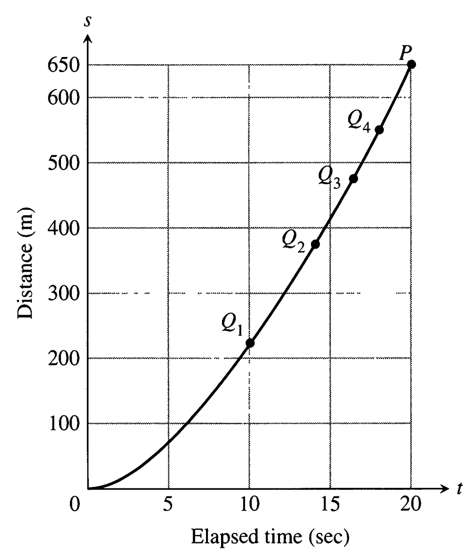
33. $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$

34. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

35. Figure 1.8 shows the time-to-distance graph for a 1994 Ford Mustang Cobra accelerating from a standstill.

a) Estimate the slopes of secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What are the appropriate units for these slopes?

b) Then estimate the Cobra's speed at time $t = 20$ sec.

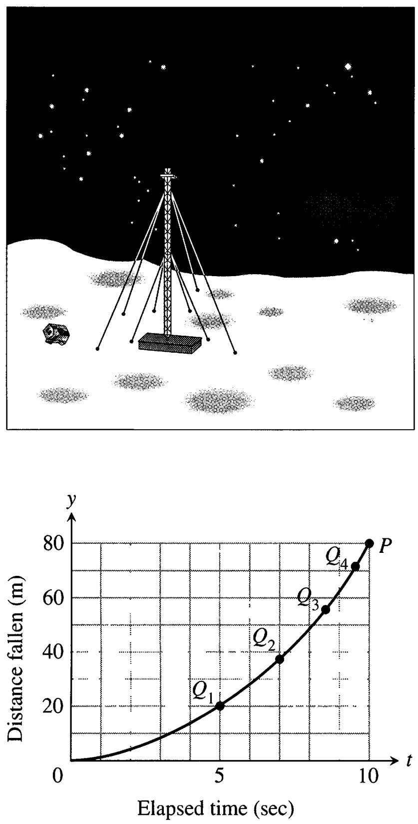


1.8 The time-to-distance graph for Exercise 35.

36. Figure 1.9 shows the plot of distance fallen (m) vs. time for a wrench that fell from the top platform of a communications mast on the moon to the station roof 80 m below.

a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Fig. 1.3.

b) About how fast was the wrench going when it hit the roof?



1.9 The time-to-distance graph for Exercise 36.

37. CALCULATOR The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in $1000s
1990	6
1991	27
1992	62
1993	111
1994	174

a) Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

b) What is the average rate of increase of the profits between 1992 and 1994?

c) Use your graph to estimate the rate at which the profits were changing in 1992.

38. CALCULATOR Make a table of values for the function $F(x) = (x+2)/(x-2)$ at the points $x = 2, x = 11/10, x = 101/100, x = 1001/1000, x = 10001/10000$, and $x = 1$.

a) Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

b) Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

39. CALCULATOR Let $g(x) = \sqrt{x}$ for $x \geq 0$.

a) Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$, and $[1, 1+h]$.

b) Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1+h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001.

c) What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?

d) Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1+h]$.

40. CALCULATOR Let $f(t) = 1/t$ for $t \neq 0$.

a) Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

b) Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001.

c) What does your table indicate is the rate of change of f with respect to t at $t = 2$?

d) Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T. You will have to do some algebra before you can substitute $T = 2$.

CAS Explorations and Projects

In Exercises 41–46, use a CAS to perform the following steps:

a) Plot the function near the point x_0 being approached.

b) From your plot guess the value of the limit.

c) Evaluate the limit symbolically. How close was your guess?

41. $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x - 2}$

42. $\lim\limits_{x \to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x+1)^2}$

43. $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1+x} - 1}{x}$

44. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{\sqrt{x^2+7} - 4}$

45. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x \sin x}$

46. $\lim\limits_{x \to 0} \dfrac{2x^2}{3 - 3\cos x}$

1.2

Rules for Finding Limits

This section presents theorems for calculating limits. The first three let us build on the results of Example 8 in the preceding section to find limits of polynomials, rational functions, and powers. The fourth prepares for calculations later in the text.

Limits of Powers and Algebraic Combinations

Theorem 1
Properties of Limits

The following rules hold if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ (L and M real numbers).

1. *Sum Rule:* $\lim_{x\to c} [f(x) + g(x)] = L + M$
2. *Difference Rule:* $\lim_{x\to c} [f(x) - g(x)] = L - M$
3. *Product Rule:* $\lim_{x\to c} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to c} kf(x) = kL$ (any number k)
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If m and n are integers, then
$$\lim_{x\to c} [f(x)]^{m/n} = L^{m/n},$$
provided $L^{m/n}$ is a real number.

In words, the formulas in Theorem 1 say:

1. The limit of the sum of two functions is the sum of their limits.
2. The limit of the difference of two functions is the difference of their limits.
3. The limit of the product of two functions is the product of their limits.
4. The limit of a constant times a function is that constant times the limit of the function.
5. The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. The limit of any rational power of a function is that power of the limit of the function, provided the latter is a real number.

We will prove the Sum Rule in Section 1.3. Rules 2–5 are proved in Appendix 2. Rule 6 is proved in more advanced texts.

EXAMPLE 1 Find $\displaystyle\lim_{x\to c} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$.

Solution Starting with the limits $\lim_{x\to c} x = c$ and $\lim_{x\to c} k = k$ from Section 1.1, Example 8, and combining them using various parts of Theorem 1, we obtain:

a) $\lim_{x\to c} x^2 = \left(\lim_{x\to c} x\right)\left(\lim_{x\to c} x\right) = c \cdot c = c^2$ — Product or Power

b) $\lim_{x\to c} (x^2+5) = \lim_{x\to c} x^2 + \lim_{x\to c} 5 = c^2 + 5$ — Sum and (a)

c) $\lim_{x\to c} 4x^2 = 4 \lim_{x\to c} x^2 = 4c^2$ — Constant Multiple and (a)

d) $\lim_{x\to c} (4x^2-3) = \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 = 4c^2 - 3$ — Difference and (c)

e) $\lim_{x\to c} x^3 = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c} x\right) = c^2 \cdot c = c^3$ — Product and (a), or Power

f) $\lim_{x\to c} (x^3+4x-3) = \lim_{x\to c} x^3 + \lim_{x\to c} (4x^2-3)$ — Sum

$= c^3 + 4c^2 - 3$ — (d) and (e)

g) $\lim_{x\to c} \dfrac{x^3+4x^2-3}{x^2+5} = \dfrac{\lim_{x\to c} (x^3+4x^2-3)}{\lim_{x\to c} (x^2+5)}$ — Quotient

$= \dfrac{c^3+4c^2-3}{c^2+5}$ — (f) and (b) ❑

EXAMPLE 2 Find $\lim_{x\to -2} \sqrt{4x^2-3}$.

Solution

$$\lim_{x\to -2} \sqrt{4x^2-3} = \sqrt{4(-2)^2-3}$$ Example 1(d) and Power Rule with $n = 1/2$

$$= \sqrt{16-3}$$
$$= \sqrt{13}$$ ❑

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero,* substitute c for x in the formula for the function.

Theorem 2
Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Theorem 3
Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 3

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}=\frac{0}{6}=0.$$

This is the limit in Example 1 with $c=-1$, now done in one step. ❑

Identifying common factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c)=0$, then $(x-c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x=c$, then $(x-c)$ is a common factor.

Eliminating Zero Denominators Algebraically

Theorem 3 applies only when the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator will sometimes reduce the fraction to one whose denominator is no longer zero at c. When this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 4 *Canceling a common factor*

Evaluate $\displaystyle\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$.

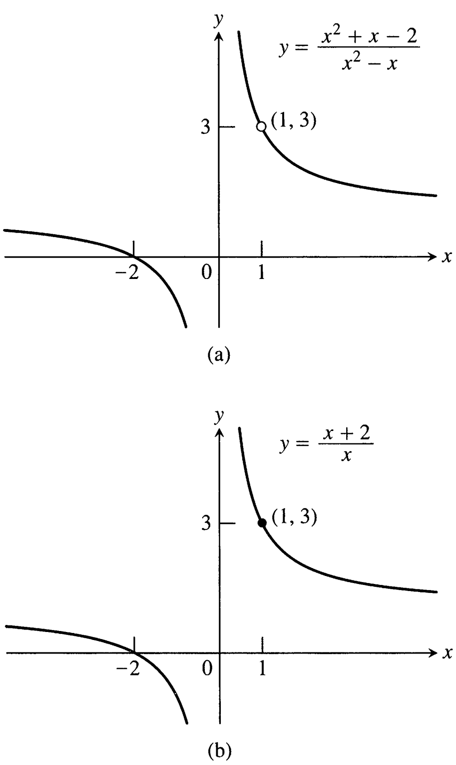
Solution We cannot just substitute $x=1$, because it makes the denominator zero. However, we can factor the numerator and denominator and cancel the common factor to obtain

$$\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x},\quad \text{if } x\neq 1.$$

Thus

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3.$$

See Fig. 1.10. ❑



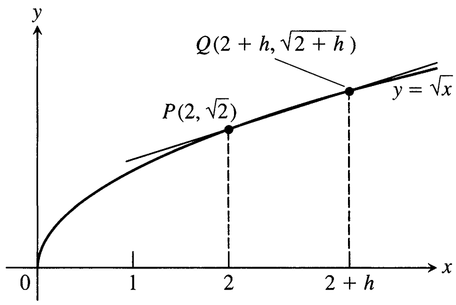
1.10 The graph of $f(x)=(x^2+x-2)/(x^2-x)$ in (a) is the same as the graph of $g(x)=(x+2)/x$ in (b) except at $x=1$, where f is undefined. The functions have the same limit as $x\to 1$.

EXAMPLE 5 *Creating and canceling a common factor*

Find $\displaystyle\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}$.

Solution We cannot find the limit by substituting $h=0$, and the numerator and denominator do not have obvious factors. However, we can create a common factor in the numerator by multiplying it (and the denominator) by the so-called *conjugate expression* $\sqrt{2+h}+\sqrt{2}$, obtained by changing the sign between the square roots:

$$\begin{aligned}\frac{\sqrt{2+h}-\sqrt{2}}{h}&=\frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}\\&=\frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}\\&=\frac{h}{h(\sqrt{2+h}+\sqrt{2})}\qquad\text{We have created a common factor of } h\ldots\\&=\frac{1}{\sqrt{2+h}+\sqrt{2}}\qquad\ldots\text{which we cancel.}\end{aligned}$$



1.11 The limit of the slope of secant PQ as $Q \to P$ along the curve is $1/(2\sqrt{2})$ (Example 5).

Therefore,

$$\lim_{h \to 0} \frac{\sqrt{2+h}-\sqrt{2}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}$$

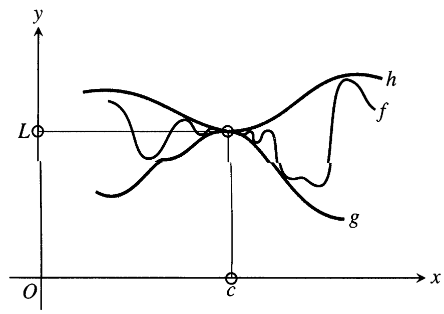
$$= \frac{1}{\sqrt{2+0}+\sqrt{2}} \qquad \text{The denominator is no longer 0 at } h = 0\text{, so we can substitute.}$$

$$= \frac{1}{2\sqrt{2}}.$$

Notice that the fraction $(\sqrt{2+h}-\sqrt{2})/h$ is the slope of the secant through the point $P(2, \sqrt{2})$ and the point $Q(2+h, \sqrt{2+h})$ nearby on the curve $y = \sqrt{x}$. Figure 1.11 shows the secant for $h > 0$. Our calculation shows that the limiting value of this slope as $Q \to P$ along the curve from either side is $1/(2\sqrt{2})$. ❑

The Sandwich Theorem

The following theorem will enable us to calculate a variety of limits in subsequent chapters. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Fig. 1.12). You will find a proof in Appendix 2.



1.12 The graph of f is sandwiched between the graphs of g and h.

Theorem 4
The Sandwich Theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE 6 Given that

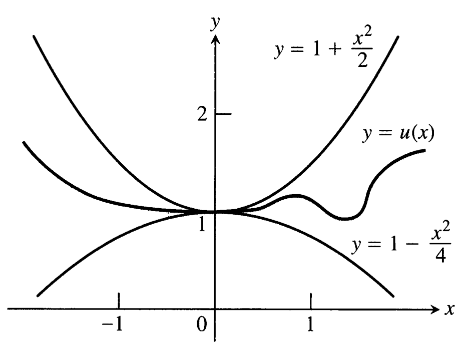
$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \text{ for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$.

Solution Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Fig. 1.13). ❑



1.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$.

EXAMPLE 7 Show that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

Solution Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches c, $\lim_{x \to c} f(x) = 0$ by the Sandwich Theorem. ❑

Exercises 1.2

Limit Calculations

Find the limits in Exercises 1–16.

1. $\lim_{x\to -7} (2x+5)$

2. $\lim_{x\to 12} (10-3x)$

3. $\lim_{x\to 2} (-x^2+5x-2)$

4. $\lim_{x\to -2} (x^3-2x^2+4x+8)$

5. $\lim_{t\to 6} 8(t-5)(t-7)$

6. $\lim_{s\to 2/3} 3s(2s-1)$

7. $\lim_{x\to 2} \dfrac{x+3}{x+6}$

8. $\lim_{x\to 5} \dfrac{4}{x-7}$

9. $\lim_{y\to -5} \dfrac{y^2}{5-y}$

10. $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

11. $\lim_{x\to -1} 3(2x-1)^2$

12. $\lim_{x\to -4} (x+3)^{1984}$

13. $\lim_{y\to -3} (5-y)^{4/3}$

14. $\lim_{z\to 0} (2z-8)^{1/3}$

15. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$

16. $\lim_{h\to 0} \dfrac{5}{\sqrt{5h+4}+2}$

Find the limits in Exercises 17–30.

17. $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$

18. $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$

19. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$

20. $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

21. $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$

22. $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$

23. $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$

24. $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$

25. $\lim_{u\to 1} \dfrac{u^4-1}{u^3-1}$

26. $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$

27. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

28. $\lim_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$

29. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

30. $\lim_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Using Limit Rules

31. Suppose $\lim_{x\to 0} f(x)=1$ and $\lim_{x\to 0} g(x)=-5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x)-g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \qquad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{2/3}} \qquad \text{(b)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \qquad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

32. Let $\lim_{x\to 1} h(x)=5$, $\lim_{x\to 1} p(x)=1$, and $\lim_{x\to 1} r(x)=2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4-r(x)))} \qquad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)} \qquad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \qquad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

33. Suppose $\lim_{x\to c} f(x)=5$ and $\lim_{x\to c} g(x)=-2$. Find

a) $\lim_{x\to c} f(x)g(x)$ **b)** $\lim_{x\to c} 2f(x)g(x)$

c) $\lim_{x\to c} (f(x)+3g(x))$ **d)** $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

34. Suppose $\lim_{x\to 4} f(x)=0$ and $\lim_{x\to 4} g(x)=-3$. Find

a) $\lim_{x\to 4} (g(x)+3)$ **b)** $\lim_{x\to 4} xf(x)$

c) $\lim_{x\to 4} (g(x))^2$ **d)** $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$

35. Suppose $\lim_{x\to b} f(x)=7$ and $\lim_{x\to b} g(x)=-3$. Find

a) $\lim_{x\to b} (f(x)+g(x))$ **b)** $\lim_{x\to b} f(x)\cdot g(x)$

c) $\lim_{x\to b} 4g(x)$ **d)** $\lim_{x\to b} f(x)/g(x)$

36. Suppose that $\lim_{x\to -2} p(x)=4$, $\lim_{x\to -2} r(x)=0$, and $\lim_{x\to -2} s(x)=-3$. Find

a) $\lim_{x\to -2} (p(x)+r(x)+s(x))$

b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$

c) $\lim_{x\to -2} (-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 37–42, evaluate this limit for the given value of x and function f.

37. $f(x) = x^2, \quad x = 1$

38. $f(x) = x^2, \quad x = -2$

39. $f(x) = 3x - 4, \quad x = 2$

40. $f(x) = 1/x, \quad x = -2$

41. $f(x) = \sqrt{x}, \quad x = 7$

42. $f(x) = \sqrt{3x+1}, \quad x = 0$

Using the Sandwich Theorem

43. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

44. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

45. a) It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{x \sin x}{2 - 2\cos x}?$$

Give reasons for your answer.

b) GRAPHER Graph

$$y = 1 - (x^2/6),\ y = (x \sin x)/(2 - 2\cos x), \text{ and } y = 1$$

together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

46. a) Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 8.10.) What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

b) GRAPHER Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Theory and Examples

47. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

48. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

49. If $\displaystyle\lim_{x\to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\displaystyle\lim_{x\to 4} f(x)$.

50. If $\displaystyle\lim_{x\to -2} \frac{f(x)}{x^2} = 1$, find (a) $\displaystyle\lim_{x\to -2} f(x)$ and (b) $\displaystyle\lim_{x\to -2} \frac{f(x)}{x}$.

51. a) If $\displaystyle\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\displaystyle\lim_{x\to 2} f(x)$.

b) If $\displaystyle\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\displaystyle\lim_{x\to 2} f(x)$.

52. If $\displaystyle\lim_{x\to 0} \frac{f(x)}{x^2} = 1$, find (a) $\displaystyle\lim_{x\to 0} f(x)$ and (b) $\displaystyle\lim_{x\to 0} \frac{f(x)}{x}$.

53. a) GRAPHER Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b) Confirm your estimate in (a) with a proof.

54. a) GRAPHER Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b) Confirm your estimate in (a) with a proof.

1.3

Target Values and Formal Definitions of Limits

In this section we give a formal definition of the limit introduced in the previous two sections. We replace vague phrases like "gets arbitrarily close" in the informal definition with specific conditions that can be applied to any particular example. To do this we first examine how to control the input of a function to ensure that the output is kept within preset bounds.

Keeping Outputs near Target Values

We sometimes need to know what input values x will result in output values of the function $y = f(x)$ near a particular target value. How near depends on the context.

A gas station attendant, asked for \$5.00 worth of gas, will try to pump a volume of gas worth \$5.00 to the nearest cent. An automobile mechanic grinding a 3.385-in. cylinder will not let the bore exceed this value by more than 0.002 in. A pharmacist making ointments will measure ingredients to the nearest milligram.

EXAMPLE 1 *Controlling a linear function*

How close to $x_0 = 4$ must we hold the input x to be sure that the output $y = 2x - 1$ lies within 2 units of $y_0 = 7$?

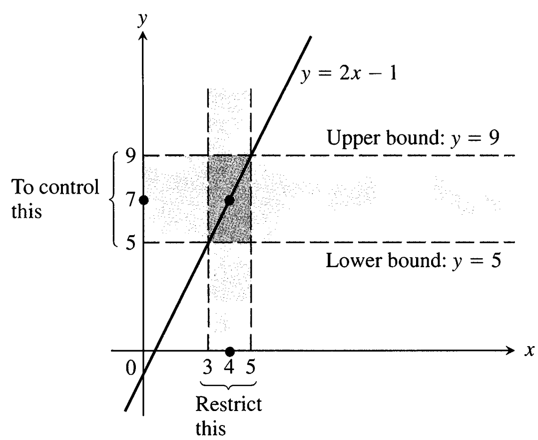
Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: What values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 \\ -1 < x - 4 &< 1. \end{aligned}$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Fig. 1.14).

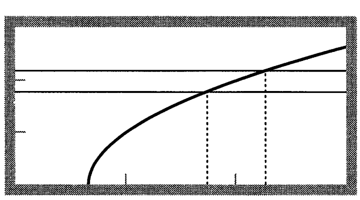


1.14 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$.

❑

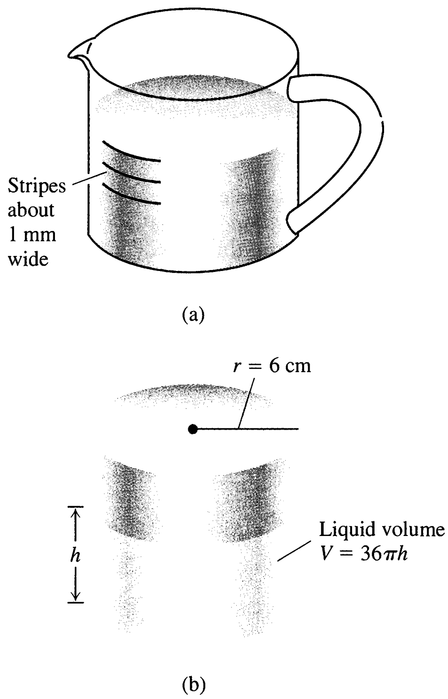
Technology *Target Values* You can experiment with target values on a graphing utility. Graph the function together with a target interval defined by horizontal lines above and below the proposed limit. Adjust the range or use zoom until the function's behavior inside the target interval is clear. Then observe what happens when you try to find an interval of x-values that will keep the function values within the target interval. (See also Exercises 7–14 and CAS Exercises 61–64.)

For example, try this for $f(x) = \sqrt{3x - 2}$ and the target interval (1.8, 2.2) on the y-axis. That is, graph $y_1 = f(x)$ and the lines $y_2 = 1.8$, $y_3 = 2.2$. Then try the target intervals (1.98, 2.02) and (1.9998, 2.0002).



[0, 3] by [0, 3]

Keeping x between 1.75 and 2.28 will keep y between 1.8 and 2.2.



1.15 A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm (Example 2).

EXAMPLE 2 *Why the stripes on a 1-liter kitchen measuring cup are about a millimeter wide*

The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (Fig. 1.15). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm^3) with an error of no more than 1% (10 cm^3)?

Solution We want to know in what interval to hold values of h to make V satisfy the inequality

$$|V - 1000| = |36\pi h - 1000| \leq 10.$$

To find out, we solve the inequality:

$$|36\pi h - 1000| \leq 10$$

$$-10 \leq 36\pi h - 1000 \leq 10$$

$$990 \leq 36\pi h \leq 1010$$

$$\frac{990}{36\pi} \leq h \leq \frac{1010}{36\pi}$$

$$\underset{\substack{\text{rounded up,}\\ \text{to be safe}}}{8.8} \leq h \leq \underset{\substack{\text{rounded down,}\\ \text{to be safe}}}{8.9}$$

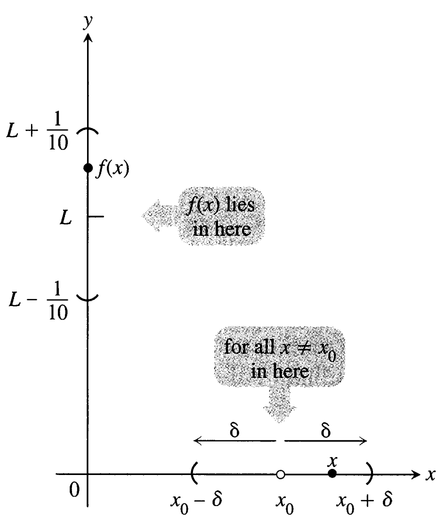
The interval in which we should hold h is about $8.9 - 8.8 = 0.1$ cm wide (1 mm). With stripes 1 mm wide, we can expect to measure a liter of water with an accuracy of 1%, which is more than enough accuracy for cooking. ❑

The Precise Definition of Limit

In a target-value problem, we determine how close to hold a variable x to a particular value x_0 to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a target value L. To show that the limit of $f(x)$ as $x \to x_0$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error,* no matter how small, by holding x close enough to x_0.

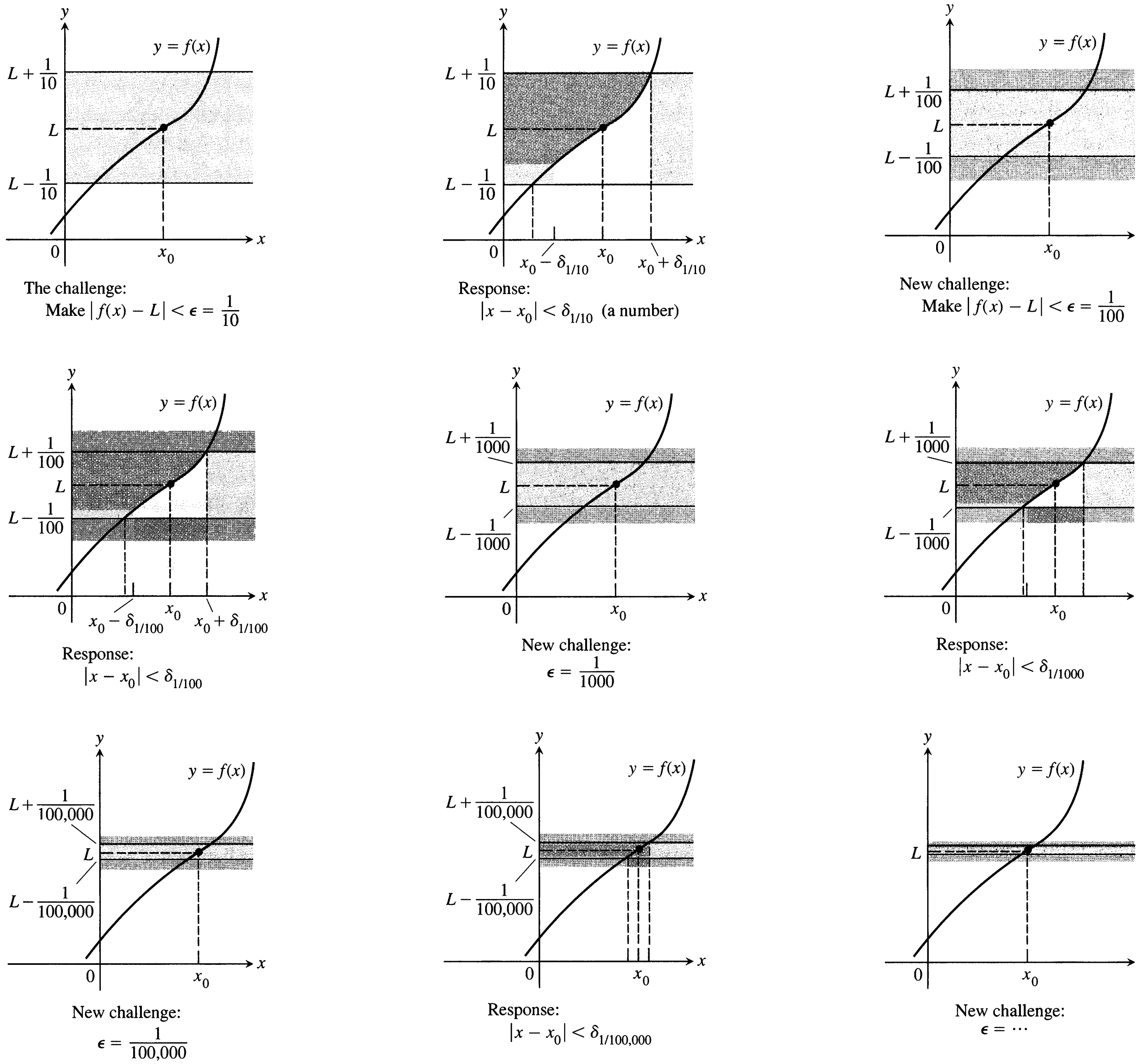
Suppose we are watching the values of a function $f(x)$ as x approaches x_0 (without taking on the value of x_0 itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit of L as soon as x stays within some distance δ of x_0 (Fig. 1.16). But that in itself is not enough, because as x continues on its course toward x_0, what is to prevent $f(x)$ from jittering about within the interval from $L - 1/10$ to $L + 1/10$ without tending toward L?

We can be told that the error can be no more than 1/100 or 1/1000 or 1/100,000. Each time, we find a new δ-interval about x_0 so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ jitters away from L at the last minute.

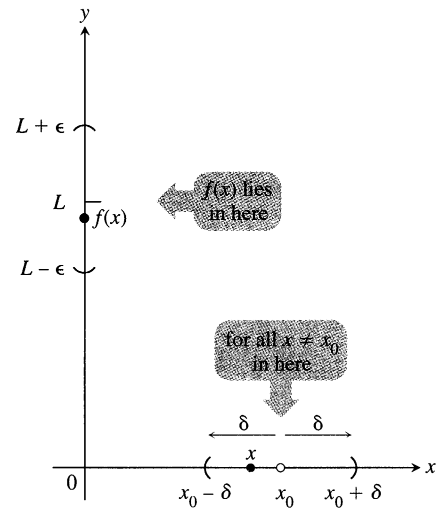


1.16 A preliminary stage in the development of the definition of limit.

The following figures illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ϵ-challenges to prove that the limit does not exist or, more precisely, that there is room for doubt, and the scholar answers every challenge with a δ-interval around x_0.



How do we stop this seemingly endless series of challenges and responses? By proving that for every error tolerance ϵ that the challenger can produce, we can find, calculate, or conjure a matching distance δ that keeps x "close enough" to x_0 to keep $f(x)$ within that tolerance of L (Fig. 1.17 on the following page).



1.17 The relation of δ and ϵ in the definition of limit.

Here, at last, is a mathematical way to say that the closer x gets to x_0, the closer $y = f(x)$ gets to L.

Definition
A Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

The Weierstrass definition

The concepts of limit and continuity (and, indeed, real number and function) did not enter mathematics overnight with the great discoveries of Sir Isaac Newton (1642–1727) and Baron Gottfried Wilhelm Leibniz (1646–1716). Mathematicians had an imperfect understanding of these fundamental ideas even as late as the last century. Definitions of the limit given by French mathematician Augustin-Louis Cauchy (1789–1857) and others referred to variables "approaching indefinitely" a fixed value and frequently made use of "infinitesimals," quantities that become infinitely small but not zero. The now accepted ϵ-δ definition of limit was formulated by German mathematician Karl Weierstrass (1815–1897) in the middle of the nineteenth century as part of his attempt to put mathematical analysis on a sound logical foundation.

To return to the idea of target values, suppose you are machining a generator shaft to a close tolerance. You may try for diameter L, but since nothing is perfect, you must be satisfied with a diameter $f(x)$ somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate your control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, you may have to adjust δ. That is, the value of δ, how tight your control setting must be, depends on the value of ϵ, the error tolerance.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. The following examples show how the definition can be used to verify limit statements for specific functions. (The first two examples correspond to parts of Examples 7 and 8 in Section 1.1.) However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.

EXAMPLE 3 Show that $\lim_{x\to 1} (5x - 3) = 2$.

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$ we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, if

$$0 < |x - 1| < \delta,$$

then $f(x)$ is within distance ϵ of $L = 2$, that is

$$|f(x) - 2| < \epsilon.$$

We find δ by working backwards from the ϵ-inequality:

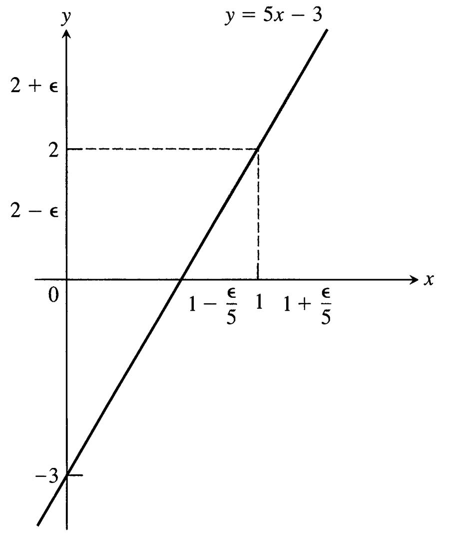
$$\begin{aligned} |(5x-3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5 \end{aligned}$$

Thus we can take $\delta = \epsilon/5$ (Fig. 1.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x-3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon.$$

This proves that $\lim_{x\to 1} (5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ❑



1.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 3).

EXAMPLE 4 *Two important limits*

Verify: (a) $\lim\limits_{x\to x_0} x = x_0$ (b) $\lim\limits_{x\to x_0} k = k$ (k constant).

Solution

a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

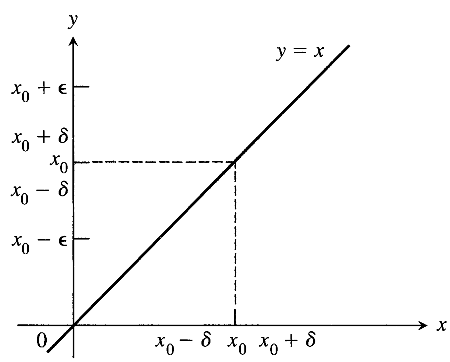
$$0 < |x - x_0| < \delta \qquad \text{implies} \qquad |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number (Fig. 1.19). This proves that $\lim_{x\to x_0} x = x_0$.

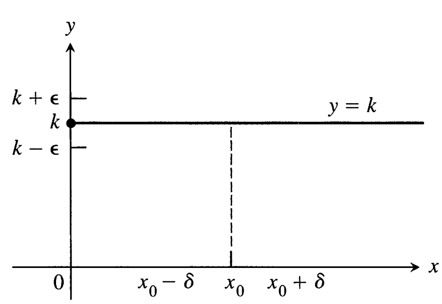
b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \text{implies} \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Fig. 1.20). This proves that $\lim_{x\to x_0} k = k$. ❑



1.19 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 4a).



1.20 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 4b).

Finding Deltas Algebraically for Given Epsilons

In Examples 3 and 4, the interval of values about x_0 for which $|f(x) - L|$ was less than ϵ was symmetric about x_0 and we could take δ to be half the length of the interval. When such symmetry is absent, as it usually is, we can take δ to be the distance from x_0 to the interval's nearer endpoint.

EXAMPLE 5 For the limit $\lim_{x\to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

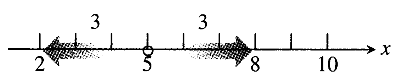
$$0 < |x - 5| < \delta \implies |\sqrt{x-1} - 2| < 1.$$

Solution We organize the search into two steps. First we solve the inequality $|\sqrt{x-1}-2| < 1$ to find an interval (a, b) about $x_0 = 5$ on which the inequality holds for all $x \neq x_0$. Then we find a value of $\delta > 0$ that places the interval $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) inside the interval (a, b).

Step 1: *Solve the inequality* $|\sqrt{x-1}-2| < 1$ *to find an interval about* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$|\sqrt{x-1}-2| < 1$$
$$-1 < \sqrt{x-1} - 2 < 1$$
$$1 < \sqrt{x-1} < 3$$
$$1 < x - 1 < 9$$
$$2 < x < 10$$

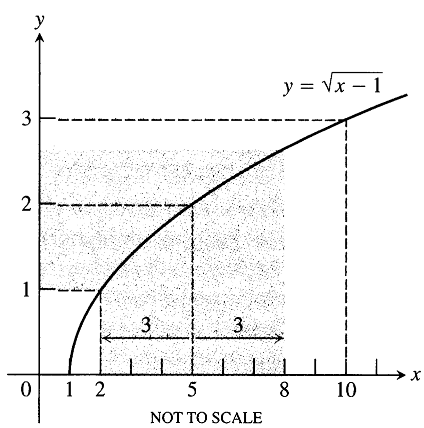
The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.



1.21 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

Step 2: *Find a value of* $\delta > 0$ *that places the centered interval* $5 - \delta < x < 5 + \delta$ *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Fig. 1.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x-1}-2| < 1$ (Fig. 1.22):

$$0 < |x - 5| < 3 \quad \Longrightarrow \quad |\sqrt{x-1}-2| < 1. \qquad \square$$



1.22 The function and intervals in Example 5.

How to Find a δ for a Given f, L, x_0, and $\epsilon > 0$ Algebraically

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

Step 1 *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) about x_0 on which the inequality holds for all $x \neq x_0$.

Step 2 *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

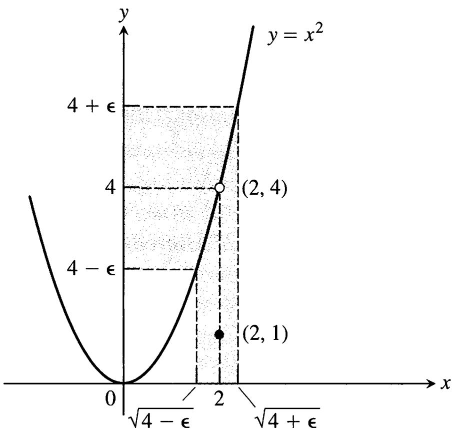
EXAMPLE 6 Prove that $\lim_{x\to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \quad \Longrightarrow \quad |f(x) - 4| < \epsilon.$$

Step 1: *Solve the inequality* $|f(x) - 4| < \epsilon$ *to find an open interval about* $x_0 = 2$ *on which the inequality holds for all* $x \neq x_0$.



1.23 The function in Example 6.

For $x \neq x_0 = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$|x^2 - 4| < \epsilon$$
$$-\epsilon < x^2 - 4 < \epsilon$$
$$4 - \epsilon < x^2 < 4 + \epsilon$$
$$\sqrt{4-\epsilon} < |x| < \sqrt{4+\epsilon} \qquad \text{Assumes } \epsilon < 4\text{; see below.}$$
$$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}. \qquad \text{An open interval about } x_0 = 2 \text{ that solves the inequality}$$

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$ (Fig. 1.23).

Step 2: *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$. In other words, take $\delta = \min\ \{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x,

$$0 < |x - 2| < \delta \qquad \Longrightarrow \qquad |f(x) - 4| < \epsilon.$$

This completes the proof.

Why was it all right to assume $\epsilon < 4$? Because, in finding a δ such that for all x, $0 < |x - 2| < \delta$ implied $|f(x) - 4| < \epsilon < 4$, we found a δ that would work for any larger ϵ as well.

Finally, notice the freedom we gained in letting $\delta = \min\ \{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$. We did not have to spend time deciding which, if either, number was the smaller of the two. We just let δ represent the smaller and went on to finish the argument. ❑

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather we appeal to general theorems about limits, in particular the theorems of Section 1.2. The definition is used to prove these theorems. As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 7 *Proving the rule for the limit of a sum*

Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, prove that

$$\lim_{x \to c} (f(x) + g(x)) = L + M.$$

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that for all x

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) + g(x) - (L + M)| < \epsilon.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M|. \end{aligned} \qquad \begin{aligned} &\text{Triangle Inequality:} \\ &|a + b| \le |a| + |b| \end{aligned}$$

Since $\lim_{x \to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x

$$0 < |x - c| < \delta_1 \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon/2.$$

Similarly, since $\lim_{x \to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \qquad \Longrightarrow \qquad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \epsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \epsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x \to c} (f(x) + g(x)) = L + M$. ❑

Exercises 1.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Longrightarrow a < x < b$.

1. $a = 1, \quad b = 7, \quad x_0 = 5$
2. $a = 1, \quad b = 7, \quad x_0 = 2$
3. $a = -7/2, \quad b = -1/2, \quad x_0 = -3$
4. $a = -7/2, \quad b = -1/2, \quad x_0 = -3/2$
5. $a = 4/9, \quad b = 4/7, \quad x_0 = 1/2$
6. $a = 2.7591, \quad b = 3.2391, \quad x_0 = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon.$$

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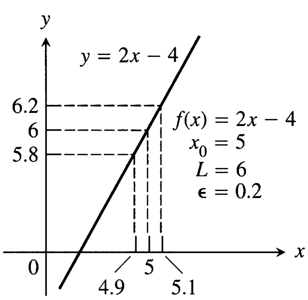

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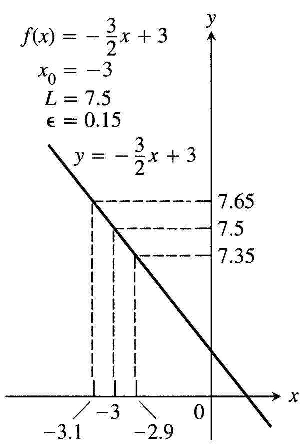

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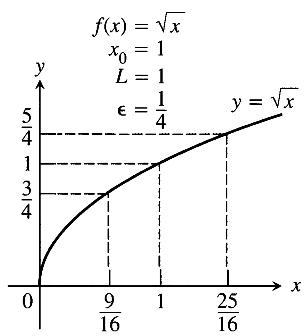

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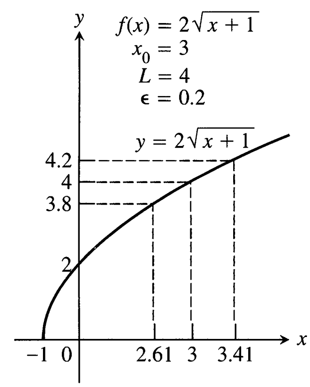

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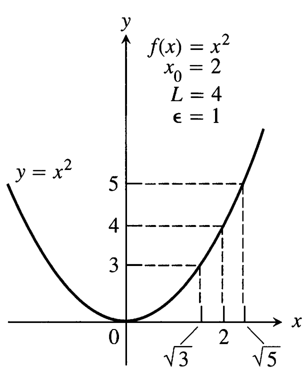

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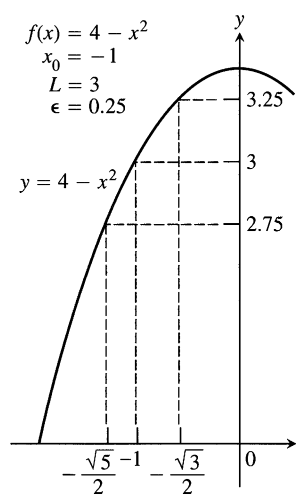

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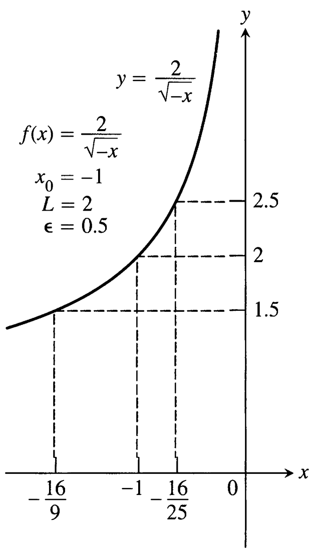

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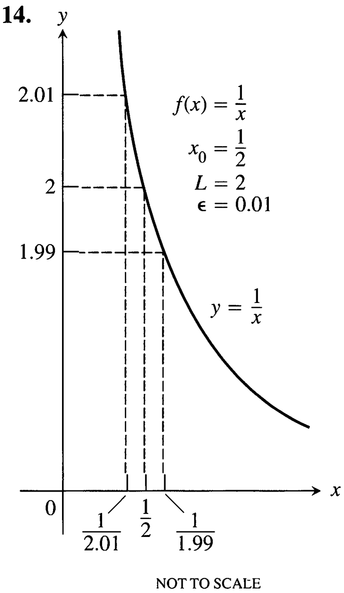


Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, x_0, and $\epsilon > 0$. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$

16. $f(x) = 2x - 2, \quad L = -6, \quad x_0 = -2, \quad \epsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad x_0 = 0, \quad \epsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad x_0 = 1/4, \quad \epsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad x_0 = 10, \quad \epsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad x_0 = 23, \quad \epsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad x_0 = 4, \quad \epsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \epsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad x_0 = -2, \quad \epsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad x_0 = -1, \quad \epsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad x_0 = 4, \quad \epsilon = 1$

26. $f(x) = 120/x, \quad L = 5, \quad x_0 = 24, \quad \epsilon = 1$

27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad x_0 = 2, \quad \epsilon = 0.03$

28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad x_0 = 3, \quad \epsilon = c > 0$

29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b, \quad x_0 = 1/2,$
$\epsilon = c > 0$

30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad x_0 = 1, \quad \epsilon = 0.05$

More on Formal Limits

Each of Exercises 31–36 gives a function $f(x)$, a point x_0, and a positive number ϵ. Find $L = \lim_{x \to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

31. $f(x) = 3 - 2x, \quad x_0 = 3, \quad \epsilon = 0.02$

32. $f(x) = -3x - 2, \quad x_0 = -1, \quad \epsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad x_0 = 2, \quad \epsilon = 0.05$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad x_0 = -5, \quad \epsilon = 0.05$

35. $f(x) = \sqrt{1 - 5x}, \quad x_0 = -3, \quad \epsilon = 0.5$

36. $f(x) = 4/x, \quad x_0 = 2, \quad \epsilon = 0.4$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x \to 4} (9 - x) = 5$

38. $\lim_{x \to 3} (3x - 7) = 2$

39. $\lim_{x \to 9} \sqrt{x - 5} = 2$

40. $\lim_{x \to 0} \sqrt{4 - x} = 2$

41. $\lim_{x \to 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

42. $\lim_{x \to -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$

43. $\lim_{x \to 1} \dfrac{1}{x} = 1$

44. $\lim_{x \to \sqrt{3}} \dfrac{1}{x^2} = \dfrac{1}{3}$

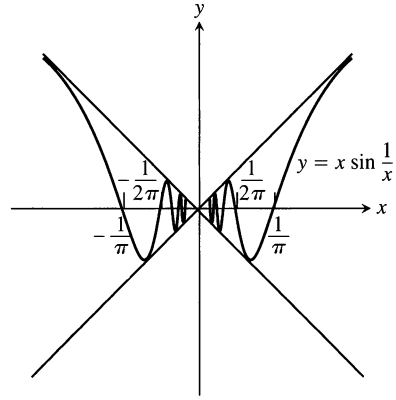
45. $\lim_{x \to -3} \dfrac{x^2 - 9}{x + 3} = -6$

46. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$

47. $\lim_{x \to 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$

48. $\lim_{x \to 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$

49. $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$



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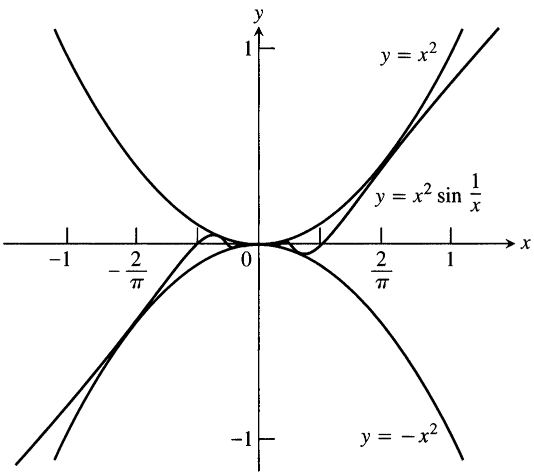
50. $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$



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Theory and Examples

51. Define what it means to say that $\lim_{x \to 2} f(x) = 5$.

52. Define what it means to say that $\lim_{x \to 0} g(x) = k$.

53. *A wrong statement about limits.* Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches x_0 if $f(x)$ gets closer to L as x approaches x_0.

Explain why the function in your example does not have the given value of L as a limit as $x \to x_0$.

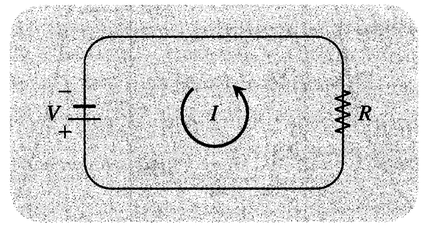
54. *Another wrong statement about limits.* Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to x_0$.

55. *Grinding engine cylinders.* Before contracting to grind engine cylinders to a cross-section area of 9 in^2, you need to know how much deviation from the ideal cylinder diameter of $x_0 = 3.385$ in. you can allow and still have the area come within 0.01 in^2 of the required 9 in^2. To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 9| \le 0.01$. What interval do you find?

56. *Manufacturing electrical resistors.* Ohm's law for electrical circuits like the one shown in Fig. 1.24 states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the target value $I_0 = 5$?



1.24 The circuit in Exercise 56.

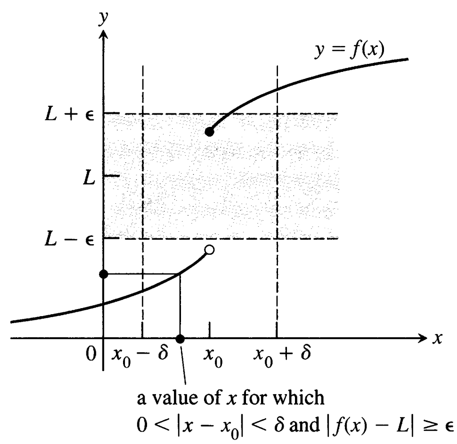
When Is a Number *L Not* the Limit of *f(x)* as $x \to x_0$?

We can prove that $\lim_{x \to x_0} f(x) \ne L$ by providing an $\epsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

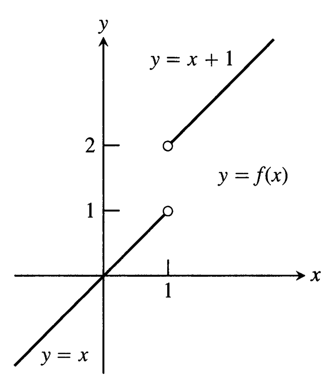
$$\text{For all } x, \quad 0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

We accomplish this for our candidate ϵ by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - x_0| < \delta \quad \text{and} \quad |f(x) - L| \ge \epsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



a) Let $\epsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

For all x, $0 < |x - 1| < \delta \implies |f(x) - 2| < 1/2$.

That is, for each $\delta > 0$ show that there is a value of x such that

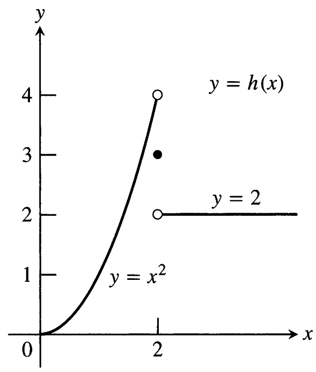
$$0 < |x - 1| < \delta \quad \text{and} \quad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \to 1} f(x) \neq 2$.

b) Show that $\lim_{x \to 1} f(x) \neq 1$.

c) Show that $\lim_{x \to 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a) $\lim_{x \to 2} h(x) \neq 4$

b) $\lim_{x \to 2} h(x) \neq 3$

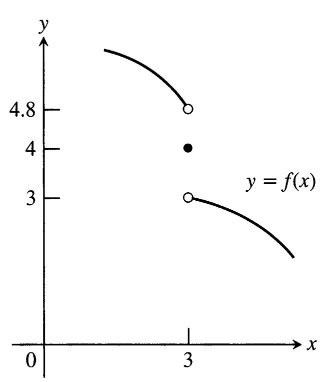
c) $\lim_{x \to 2} h(x) \neq 2$

59. For the function graphed here, show that

a) $\lim_{x \to 3} f(x) \neq 4$

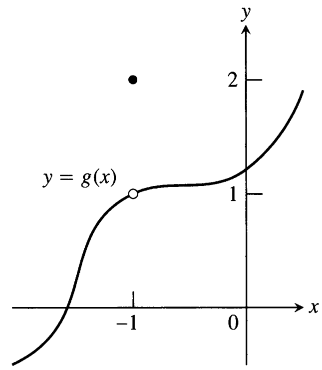
b) $\lim_{x \to 3} f(x) \neq 4.8$

c) $\lim_{x \to 3} f(x) \neq 3$



60. a) For the function graphed here, show that $\lim_{x \to -1} g(x) \neq 2$.

b) Does $\lim_{x \to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



CAS Explorations and Projects

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a) Plot the function $y = f(x)$ near the point x_0 being approached.

b) Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c) Using the value $\epsilon = 0.2$, graph the banding lines $y_1 = L - \epsilon$ and $y_2 = L + \epsilon$ together with the function f near x_0.

d) From your graph in part (c), estimate a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - x_0| < \delta$. For your viewing window use $x_0 - 2\delta \leq x \leq x_0 + 2\delta$ and $L - 2\epsilon \leq y \leq L + 2\epsilon$. If any function values lie outside the interval $[L - \epsilon, L + \epsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e) Repeat parts (c) and (d) successively for $\epsilon = 0.1, 0.05$, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}, \quad x_0 = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad x_0 = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}, \quad x_0 = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}, \quad x_0 = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}, \quad x_0 = 1$

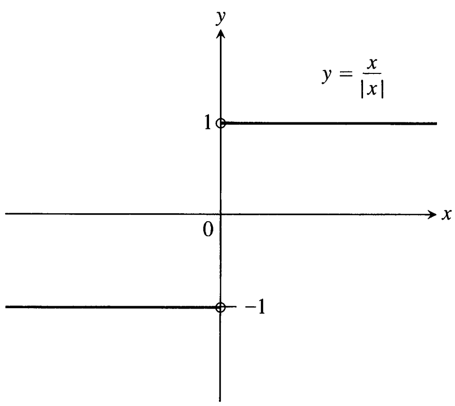
66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad x_0 = 1$

1.4

Extensions of the Limit Concept

In this section we extend the concept of limit to

1. *one-sided limits,* which are limits as x approaches a from the left-hand side or the right-hand side only,
2. *infinite limits,* which are not really limits at all, but provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large, positive or negative.



1.25 Different right-hand and left-hand limits at the origin.

One-Sided Limits

To have a limit L as x approaches a, a function f must be defined on *both sides* of a, and its values $f(x)$ must approach L as x approaches a from either side. Because of this, ordinary limits are sometimes called **two-sided** limits.

It is possible for a function to approach a limiting value as x approaches a from only one side, either from the right or from the left. In this case we say that f has a **one-sided** (either right-hand or left-hand) limit at a. The function $f(x) = x/|x|$ graphed in Fig. 1.25 has limit 1 as x approaches zero from the right, and limit -1 as x approaches zero from the left.

Definition

Informal Definition of Right-hand and Left-hand Limits

Let $f(x)$ be defined on an interval (a, b) where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has **right-hand limit** L at a, and we write

$$\lim_{x \to a^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (c, a) where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within the interval (c, a), then we say that f has **left-hand limit** M at a, and we write

$$\lim_{x \to a^-} f(x) = M.$$

For the function $f(x) = x/|x|$ in Fig. 1.25, we have

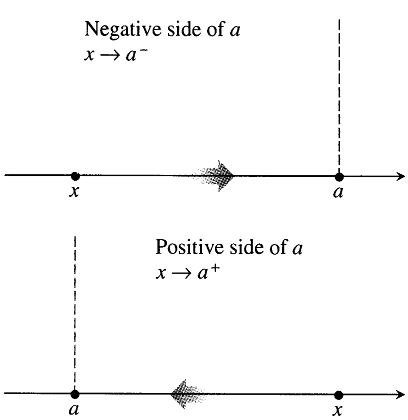
$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$

The "+" and "−"

The significance of the signs in the notation for one-sided limits is this:

$\boldsymbol{x \longrightarrow a^-}$ means x approaches a from the negative side of a, through values less than a.

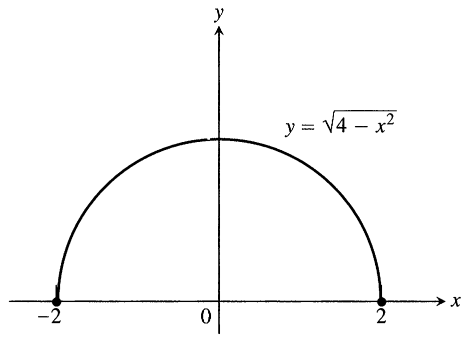
$\boldsymbol{x \longrightarrow a^+}$ means x approaches a from the positive side of a, through values greater than a.



A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Fig. 1.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$



1.26 $\lim_{x\to 2^-} \sqrt{4-x^2} = 0$, $\lim_{x\to -2^+} \sqrt{4-x^2} = 0$.

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. ❑

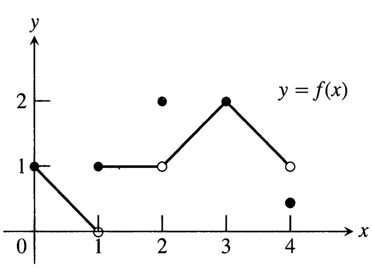
One-sided limits have all the limit properties listed in Theorem 1, Section 1.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem.

The connection between one-sided and two-sided limits is stated in the following theorem (proved at the end of this section).

Theorem 5
One-sided vs. Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



1.27 Graph of the function in Example 2.

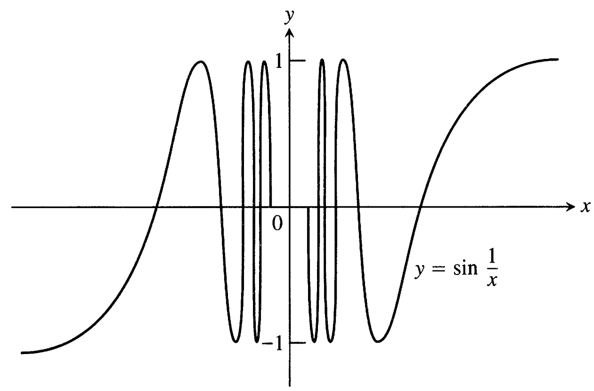
EXAMPLE 2 All of the following statements about the function graphed in Figure 1.27 are true.

At $x = 0$:	$\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. (The function is not defined to the left of $x = 0$.)
At $x = 1$:	$\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$, $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1} f(x)$ does not exist. (The right- and left-hand limits are not equal.)
At $x = 2$:	$\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.
At $x = 3$:	$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$
At $x = 4$:	$\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$, $\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. (The function is not defined to the right of $x = 4$.)

At every other point a in $[0, 4]$, $f(x)$ has limit $f(a)$. ❑

In the examples so far in this section, the functions that failed to have a limit at some point at least had one existing one-sided limit there. The function in the following example has neither a left-hand limit nor a right-hand limit at $x = 0$ even though it is defined everywhere except at $x = 0$.

EXAMPLE 3 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Fig. 1.28).



1.28 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 3).

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ❑

Infinite Limits

Let us look closely at the function $f(x) = 1/x$ that drives the sine in Example 3. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Fig. 1.29). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

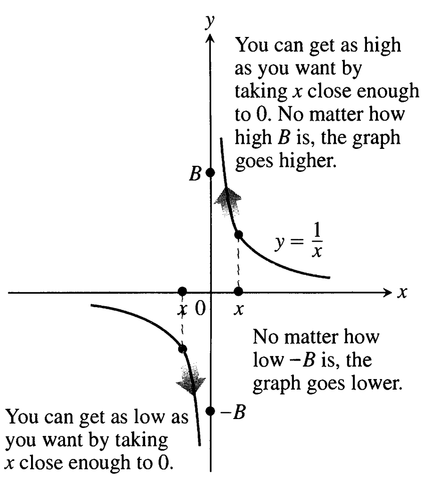
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

In writing this, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Fig. 1.29.) We write

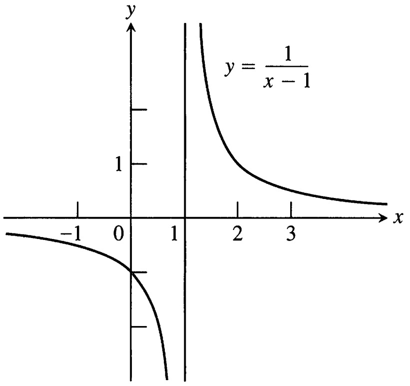
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative.*



1.29 One-sided infinite limits:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$



1.30 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right.

EXAMPLE 4 *One-sided infinite limits*

Find $\lim_{x\to 1^+} \dfrac{1}{x-1}$ and $\lim_{x\to 1^-} \dfrac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Fig. 1.30). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $1/(x-1) \to -\infty$. ❑

EXAMPLE 5 *Two-sided infinite limits*

Discuss the behavior of

a) $f(x) = \dfrac{1}{x^2}$ near $x = 0$,

b) $g(x) = \dfrac{1}{(x+3)^2}$ near $x = -3$.

Solution

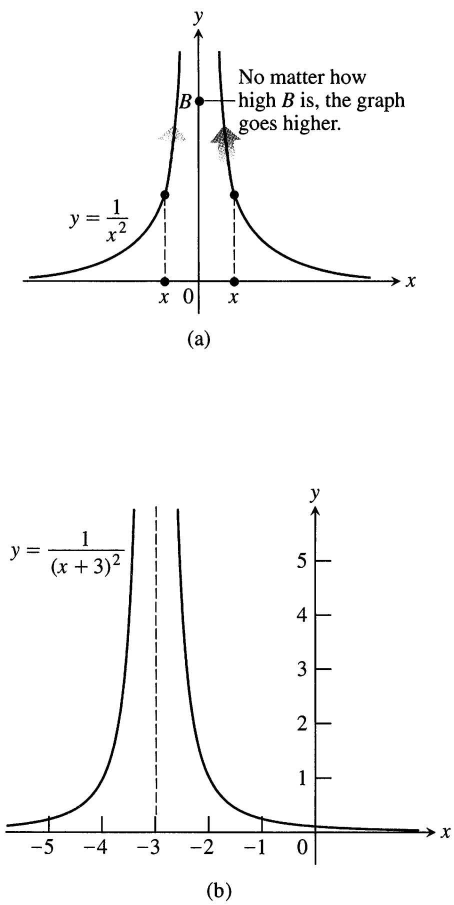
a) As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Fig. 1.31a):

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

b) The graph of $g(x) = 1/(x+3)^2$ is the graph of $f(x) = 1/x^2$ shifted 3 units to the left (Fig. 1.31b). Therefore, g behaves near -3 exactly the way f behaves near 0.

$$\lim_{x\to -3} g(x) = \lim_{x\to -3} \frac{1}{(x+3)^2} = \infty.$$ ❑

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}(1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}(1/x^2) = \infty$.



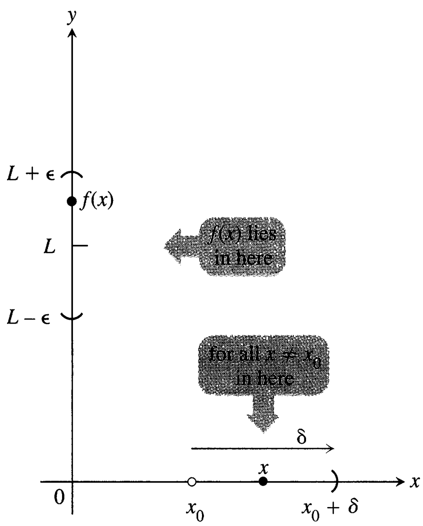
1.31 The graphs of the functions in Example 5.

EXAMPLE 6 *Rational functions can behave in various ways near zeros of their denominators.*

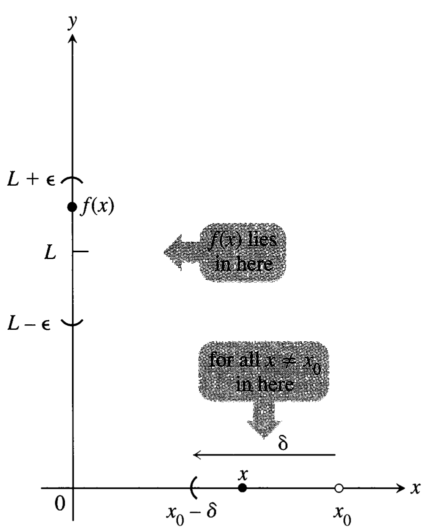
a) $\displaystyle \lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

b) $\displaystyle \lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

c) $\displaystyle \lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ The values are negative for $x > 2$, x near 2.



1.32 Diagram for the definition of right-hand limit.



1.33 Diagram for the definition of left-hand limit.

d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ — The values are positive for $x < 2$, x near 2.

e) $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. — See (c) and (d).

f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero in the denominator. ❑

Precise Definitions of One-sided Limits

The formal definition of two-sided limit in Section 1.3 is readily modified for one-sided limits.

Definitions

Right-hand Limit

We say that $f(x)$ has right-hand limit L at x_0, and write

$$\lim_{x\to x_0^+} f(x) = L \qquad \text{(See Fig. 1.32)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{1}$$

Left-hand Limit

We say that f has left-hand limit L at x_0, and write

$$\lim_{x\to x_0^-} f(x) = L \qquad \text{(See Fig. 1.33)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{2}$$

The Relation Between One- and Two-sided Limits

If we subtract x_0 from the δ-inequalities in implications (1) and (2), we can see the logical relation between the one-sided limits just defined and the two-sided limit defined in Section 1.3. For right-hand limits, subtracting x_0 gives

$$0 < x - x_0 < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon; \tag{3}$$

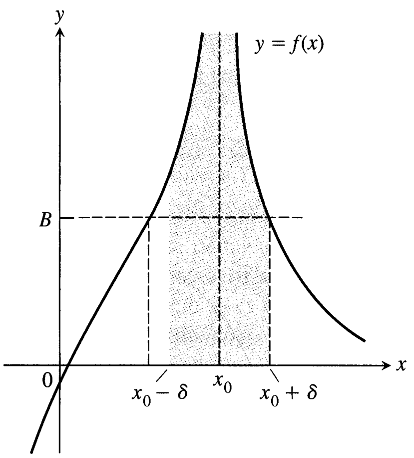
for left-hand limits we get

$$-\delta < x - x_0 < 0 \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{4}$$

Together, (3) and (4) say the same thing as

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon, \tag{5}$$

the implication required for two-sided limit. Thus, f has limit L at x_0 if and only if f has right-hand limit L and left-hand limit L at x_0.



1.34 $\lim_{x \to x_0} f(x) = \infty$.

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to x_0, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from the origin. Except for this change, the language is identical with what we have seen before. Figures 1.34 and 1.35 accompany these definitions.

Definitions
Infinite Limits

1. We say that $f(x)$ approaches infinity as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

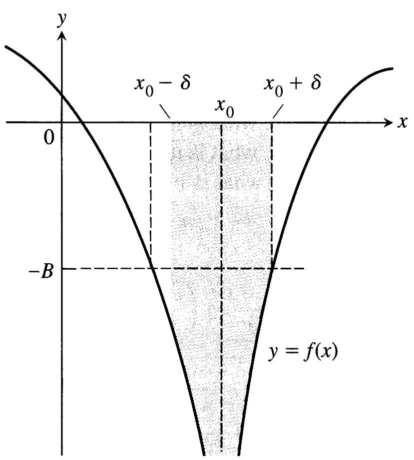
$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that $f(x)$ approaches minus infinity as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$



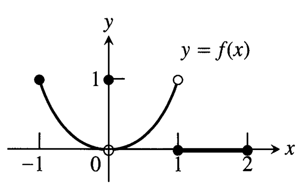
1.35 $\lim_{x \to x_0} f(x) = -\infty$.

The precise definitions of one-sided infinite limits at x_0 are similar and are stated in the exercises.

Exercises 1.4

Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x \to -1^+} f(x) = 1$

b) $\lim_{x \to 0^-} f(x) = 0$

c) $\lim_{x \to 0^-} f(x) = 1$

d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

e) $\lim_{x \to 0} f(x)$ exists

f) $\lim_{x \to 0} f(x) = 0$

g) $\lim_{x \to 0} f(x) = 1$

h) $\lim_{x \to 1} f(x) = 1$

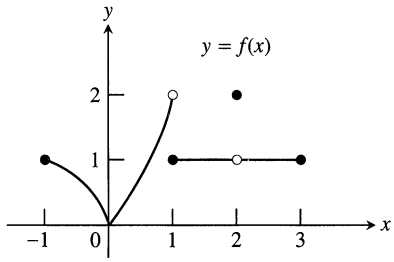
i) $\lim_{x \to 1} f(x) = 0$

j) $\lim_{x \to 2^-} f(x) = 2$

k) $\lim_{x \to -1^-} f(x)$ does not exist.

l) $\lim_{x \to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x \to -1^+} f(x) = 1$ **b)** $\lim_{x \to 2} f(x)$ does not exist.

c) $\lim_{x \to 2} f(x) = 2$ **d)** $\lim_{x \to 1^-} f(x) = 2$

e) $\lim_{x \to 1^+} f(x) = 1$ **f)** $\lim_{x \to 1} f(x)$ does not exist.

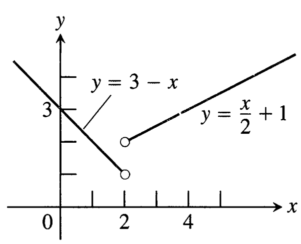
g) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

h) $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i) $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j) $\lim_{x \to -1^-} f(x) = 0$ **k)** $\lim_{x \to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$



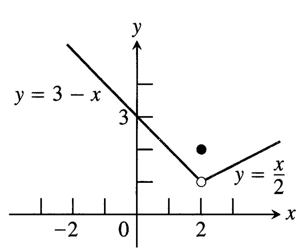
a) Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.

b) Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

c) Find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

d) Does $\lim_{x \to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



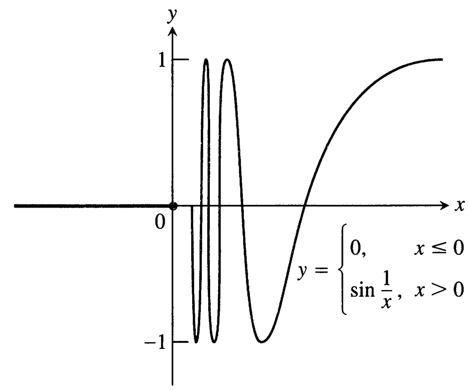
a) Find $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, and $f(2)$.

b) Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

c) Find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$.

d) Does $\lim_{x \to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

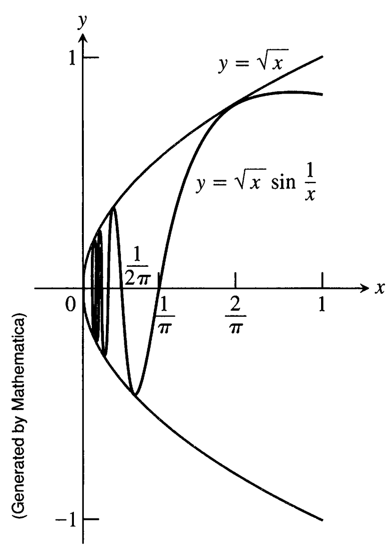


a) Does $\lim_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x} \sin(1/x)$.



a) Does $\lim_{x \to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a) Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

c) Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a) Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

c) Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a) What are the domain and range of f?

b) At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c) At what points does only the left-hand limit exist?

d) At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } x = 2 \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < -1, \text{ or } x > 1 \end{cases}$

Finding Limits Algebraically

Find the limits in Exercises 11–20.

11. $\lim_{x \to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

12. $\lim_{x \to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

13. $\lim_{x \to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

14. $\lim_{x \to 1^-} \left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right)$

15. $\lim_{h \to 0^+} \dfrac{\sqrt{h^2+4h+5} - \sqrt{5}}{h}$

16. $\lim_{h \to 0^-} \dfrac{\sqrt{6} - \sqrt{5h^2+11h+6}}{h}$

17. a) $\lim_{x \to -2^+} (x+3)\dfrac{|x+2|}{x+2}$ **b)** $\lim_{x \to -2^-} (x+3)\dfrac{|x+2|}{x+2}$

18. a) $\lim_{x \to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$ **b)** $\lim_{x \to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

19. a) $\lim_{\theta \to 3^+} \dfrac{\lfloor \theta \rfloor}{\theta}$ **b)** $\lim_{\theta \to 3^-} \dfrac{\lfloor \theta \rfloor}{\theta}$

20. a) $\lim_{t \to 4^+} (t - \lfloor t \rfloor)$ **b)** $\lim_{t \to 4^-} (t - \lfloor t \rfloor)$

Infinite Limits

Find the limits in Exercises 21–32.

21. $\lim_{x \to 0^+} \dfrac{1}{3x}$

22. $\lim_{x \to 0^-} \dfrac{5}{2x}$

23. $\lim_{x \to 2^-} \dfrac{3}{x-2}$

24. $\lim_{x \to 3^+} \dfrac{1}{x-3}$

25. $\lim_{x \to -8^+} \dfrac{2x}{x+8}$

26. $\lim_{x \to -5^-} \dfrac{3x}{2x+10}$

27. $\lim_{x \to 7} \dfrac{4}{(x-7)^2}$

28. $\lim_{x \to 0} \dfrac{-1}{x^2(x+1)}$

29. a) $\lim_{x \to 0^+} \dfrac{2}{3x^{1/3}}$ **b)** $\lim_{x \to 0^-} \dfrac{2}{3x^{1/3}}$

30. a) $\lim_{x \to 0^+} \dfrac{2}{x^{1/5}}$ **b)** $\lim_{x \to 0^-} \dfrac{2}{x^{1/5}}$

31. $\lim_{x \to 0} \dfrac{4}{x^{2/5}}$

32. $\lim_{x \to 0} \dfrac{1}{x^{2/3}}$

Find the limits in Exercises 33–36.

33. $\lim_{x \to (\pi/2)^-} \tan x$

34. $\lim_{x \to (-\pi/2)^+} \sec x$

35. $\lim_{\theta \to 0^-} (1 + \csc \theta)$

36. $\lim_{\theta \to 0} (2 - \cot \theta)$

Additional Calculations

Find the limits in Exercises 37–42.

37. $\lim \dfrac{1}{x^2-4}$ as

a) $x \to 2^+$ **b)** $x \to 2^-$

c) $x \to -2^+$ **d)** $x \to -2^-$

38. $\lim \dfrac{x}{x^2-1}$ as

a) $x \to 1^+$ **b)** $x \to 1^-$

c) $x \to -1^+$ **d)** $x \to -1^-$

39. $\lim \left(\dfrac{x^2}{2} - \dfrac{1}{x}\right)$ as

a) $x \to 0^+$ **b)** $x \to 0^-$

c) $x \to \sqrt[3]{2}$ **d)** $x \to -1$

40. $\lim \dfrac{x^2-1}{2x+4}$ as

a) $x \to -2^+$ **b)** $x \to -2^-$

c) $x \to 1^+$ **d)** $x \to 0^-$

41. $\lim \dfrac{x^2-3x+2}{x^3-2x^2}$ as

a) $x \to 0^+$ **b)** $x \to 2^+$

c) $x \to 2^-$ **d)** $x \to 2$

e) What, if anything, can be said about the limit as $x \to 0$?

42. $\lim \dfrac{x^2-3x+2}{x^3-4x}$ as

a) $x \to 2^+$ **b)** $x \to -2^+$

c) $x \to 0^-$ **d)** $x \to 1^+$

e) What, if anything, can be said about the limit as $x \to 0$?

Find the limits in Exercises 43–46.

43. $\lim \left(2 - \dfrac{3}{t^{1/3}}\right)$ as

a) $t \to 0^+$ **b)** $t \to 0^-$

44. $\lim \left(\dfrac{1}{t^{3/5}} + 7\right)$ as

a) $t \to 0^+$
b) $t \to 0^-$

45. $\lim \left(\dfrac{1}{x^{2/3}} + \dfrac{2}{(x-1)^{2/3}}\right)$ as

a) $x \to 0^+$
b) $x \to 0^-$
c) $x \to 1^+$
d) $x \to 1^-$

46. $\lim \left(\dfrac{1}{x^{1/3}} - \dfrac{1}{(x-1)^{4/3}}\right)$ as

a) $x \to 0^+$
b) $x \to 0^-$
c) $x \to 1^+$
d) $x \to 1^-$

Theory and Examples

47. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-sided Limits

51. Given $\epsilon > 0$, find an interval $I = (5, 5+\delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x-5} < \epsilon$. What limit is being verified and what is its value?

52. Given $\epsilon > 0$, find an interval $I = (4-\delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4-x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|} = -1$

54. $\displaystyle\lim_{x\to 2^+} \frac{x-2}{|x-2|} = 1$

55. Find (a) $\lim_{x\to 400^+} \lfloor x \rfloor$ and (b) $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. (c) Based on your conclusions in (a) and (b), can anything be said about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answers.

56. Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find (a) $\lim_{x\to 0^+} f(x)$ and (b) $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. (c) Based on your conclusions in (a) and (b), can anything be said about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

The Formal Definition of Infinite Limit

Use formal definitions to prove the limit statements in Exercises 57–60.

57. $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$

58. $\displaystyle\lim_{x\to 0} \frac{-1}{x^2} = -\infty$

59. $\displaystyle\lim_{x\to 3} \frac{-2}{(x-3)^2} = -\infty$

60. $\displaystyle\lim_{x\to -5} \frac{1}{(x+5)^2} = \infty$

Formal Definitions of Infinite One-sided Limits

61. Here is the definition of **infinite right-hand limit.**

> We say that $f(x)$ approaches infinity as x approaches x_0 from the right, and write
>
> $$\lim_{x\to x_0^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that for all x
>
> $$x_0 < x < x_0 + \delta \quad \Rightarrow \quad f(x) > B.$$

Modify the definition to cover the following cases.

a) $\displaystyle\lim_{x\to x_0^-} f(x) = \infty$

b) $\displaystyle\lim_{x\to x_0^+} f(x) = -\infty$

c) $\displaystyle\lim_{x\to x_0^-} f(x) = -\infty$

Use the formal definitions from Exercise 61 to prove the limit statements in Exercises 62–67.

62. $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = \infty$

63. $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$

64. $\displaystyle\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$

65. $\displaystyle\lim_{x\to 2^+} \frac{1}{x-2} = \infty$

66. $\displaystyle\lim_{x\to 1^+} \frac{1}{1-x^2} = -\infty$

67. $\displaystyle\lim_{x\to 1^-} \frac{1}{1-x^2} = \infty$

1.5

Continuity

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure. In doing so, we are assuming that we are working with a continuous function, a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between.

So many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as quite a surprise when the physicists of the 1920s discovered that the vibrating atoms in a hydrogen molecule can oscillate only at discrete energy levels, that light comes in particles, and that, when heated, atoms emit light at discrete frequencies and not in continuous spectra. As a result of these and other discoveries, and because of the heavy use of discrete functions in computer science and statistics, the issue of continuity has become one of practical as well as theoretical importance.

In this section, we define continuity, show how to tell whether a function is continuous at a given point, and examine the intermediate value property of continuous functions.

Continuity at a Point

In practice, most functions of a real variable have domains that are intervals or unions of separate intervals, and it is natural to restrict our study of continuity to functions with these domains. This leaves us with only three kinds of points to consider: **interior points** (points that lie in an open interval in the domain), **left endpoints,** and **right endpoints.**

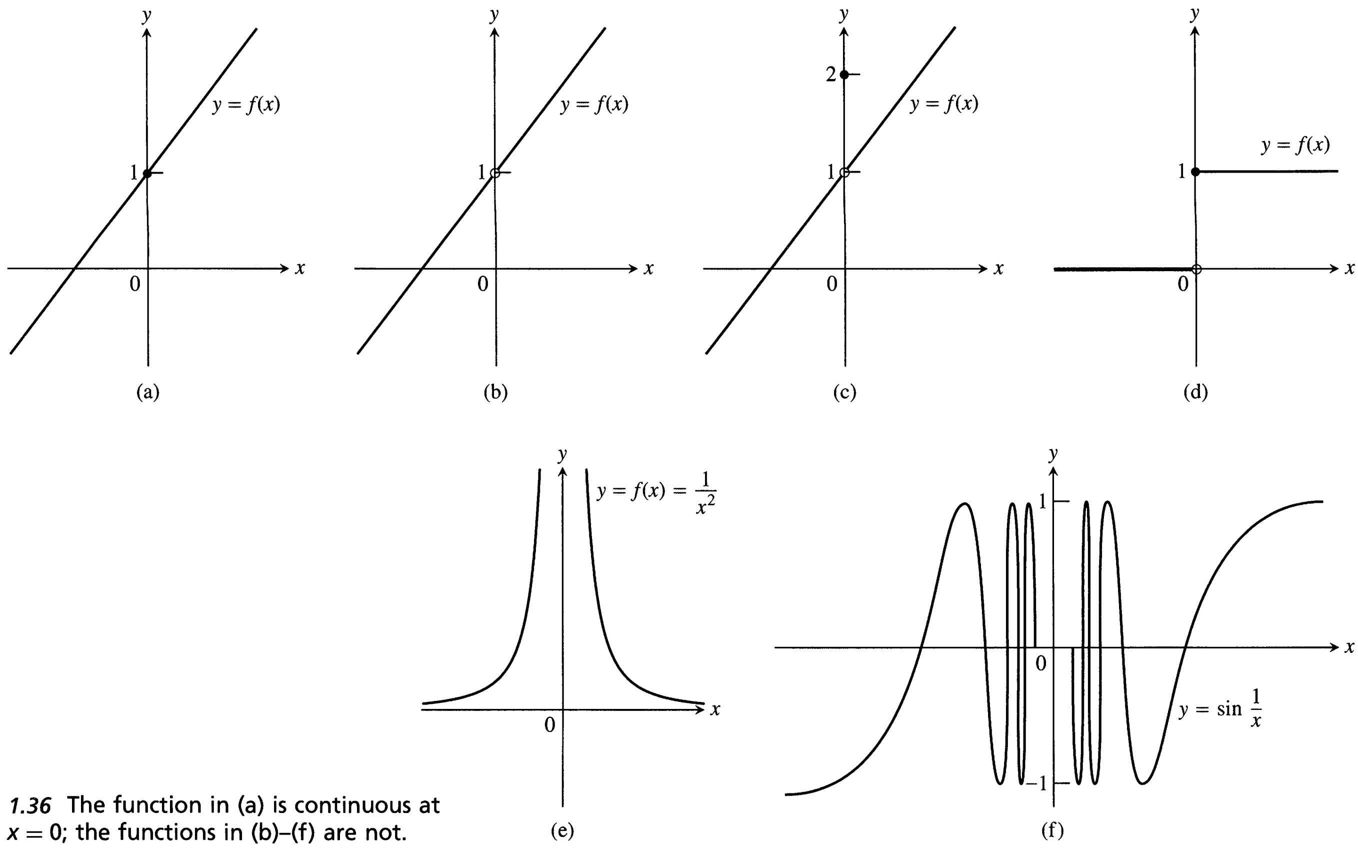
Definition

A function f is **continuous at an interior point** $x = c$ of its domain if

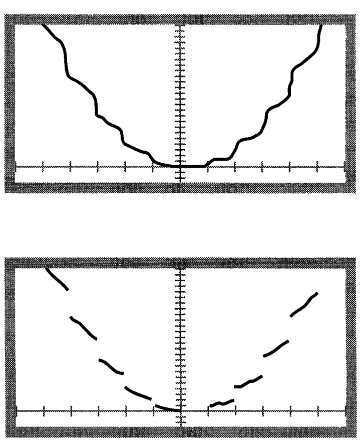
$$\lim_{x\to c} f(x) = f(c).$$

In Fig. 1.36 on the following page, the first function is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in parts (d)–(f) of Fig. 1.36 are more serious: $\lim_{x\to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** Jumps and infinite discontinuities are the ones most frequently encountered, but there are others. The function in (f) is discontinuous at the origin because it oscillates too much to have a limit as $x \to 0$.



1.36 The function in (a) is continuous at $x = 0$; the functions in (b)–(f) are not.

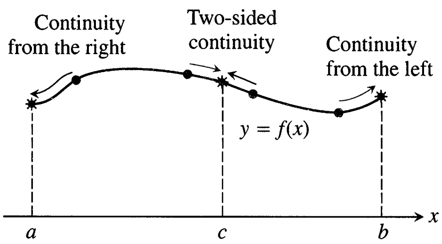


a) $y_1 = x$*int x incorrectly graphed in connected mode.
b) $y_1 = x$*int x correctly graphed in dot mode.

Technology *Deceptive Pictures* A graphing utility (calculator or Computer Algebra System—CAS*) plots a graph much as you do when plotting by hand: by plotting points, or *pixels*, and then connecting them in succession. The resulting picture may be misleading when points on opposite sides of a point of discontinuity in the graph are incorrectly connected. To avoid incorrect connections some systems allow you to use a "dot mode," which plots only the points. Dot mode, however, may not reveal enough information to portray the true behavior of the graph. Try the following four functions on your graphing device. If you can, plot them in both "connected" and "dot" modes.

$y_1 = x^*\text{int } x$	at $x = 2$	jump discontinuity
$y_2 = \sin \dfrac{1}{x}$	at $x = 0$	oscillating discontinuity
$y_3 = \dfrac{1}{x - 2}$	at $x = 2$	infinite discontinuity
$y_4 = \dfrac{x^2 - 2}{x - \sqrt{2}}$	at $x = 2$	removable discontinuity

*Rhymes with class.



1.37 Continuity at points *a*, *b*, and *c*.

Continuity at endpoints is defined by taking one-sided limits.

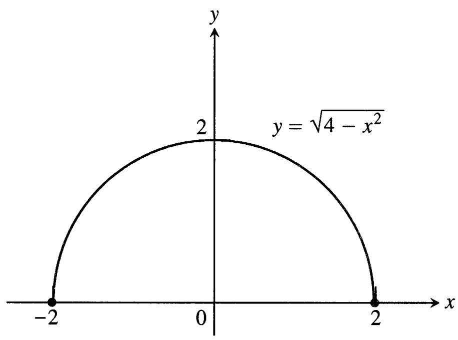
Definition

A function f is **continuous at a left endpoint** $x = a$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

and **continuous at a right endpoint** $x = b$ of its domain if

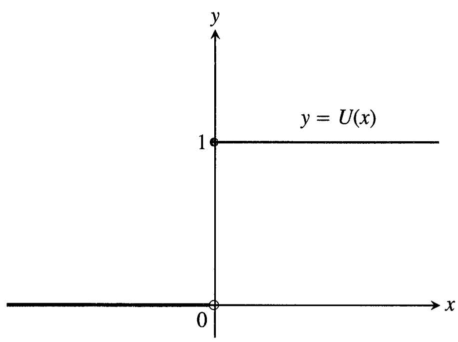
$$\lim_{x \to b^-} f(x) = f(b).$$



1.38 Continuous at every domain point.

In general, a function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Fig. 1.37).

EXAMPLE 1 The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Fig. 1.38). This includes $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ❑



1.39 Right-continuous at the origin.

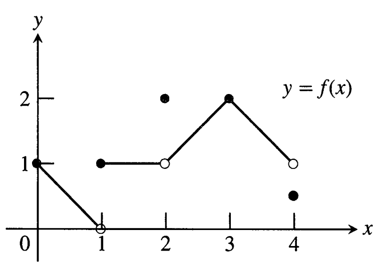
EXAMPLE 2 The unit step function $U(x)$, graphed in Fig. 1.39, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. ❑

We summarize continuity at a point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f)
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$)
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value)



1.40 This function, defined on the closed interval [0, 4], is discontinuous at $x = 1$, 2, and 4. It is continuous at all other points of its domain.

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 3 Consider the function $y = f(x)$ in Fig. 1.40, whose domain is the closed interval [0, 4]. Discuss the continuity of f at $x = 0, 1, 2, 3$, and 4.

Solution The continuity test gives the following results:

a) f is continuous at $x = 0$ because

i) $f(0)$ exists $(f(0) = 1)$,
ii) $\lim_{x\to 0^+} f(x) = 1$ (the right-hand limit exists at this left endpoint),
iii) $\lim_{x\to 0^+} f(x) = f(0)$ (the limit equals the function value).

b) f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist. Part 2 of the test fails: f has different right- and left-hand limits at the interior point $x = 1$. However, f *is* right-continuous at $x = 1$ because

i) $f(1)$ exists $(f(1) = 1)$,
ii) $\lim_{x\to 1^+} f(x) = 1$ (the right-hand limit exists at $x = 1$),
iii) $\lim_{x\to 1^+} f(x) = f(1)$ (the right-hand limit equals the function value).

c) f is discontinuous at $x = 2$ because $\lim_{x\to 2} f(x) \neq f(2)$. Part 3 of the test fails.

d) f is continuous at $x = 3$ because

i) $f(3)$ exists $(f(3) = 2)$,
ii) $\lim_{x\to 3} f(x) = 2$ (the limit exists at $x = 2$),
iii) $\lim_{x\to 3} f(x) = f(3)$ (the limit equals the function value).

e) f is discontinuous at the right endpoint $x = 4$ because $\lim_{x\to 4^-} f(x) \neq f(4)$. The right-endpoint version of Part 3 of the test fails. ❑

Rules of Continuity

It follows from Theorem 1 in Section 1.2 that if two functions are continuous at a point, then various algebraic combinations of those functions are continuous at that point.

Theorem 6
Continuity of Algebraic Combinations

If functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $f + g$ and $f - g$
2. fg
3. kf, where k is any number
4. f/g (provided $g(c) \neq 0$)
5. $(f(x))^{m/n}$ (provided $f(x))^{m/n}$ is defined on an interval containing c, and m and n are integers)

As a consequence, polynomials and rational functions are continuous at every point where they are defined.

Theorem 7
Continuity of Polynomials and Rational Functions

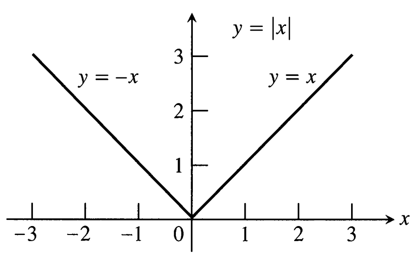
Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

EXAMPLE 4 The functions $f(x) = x^4 + 20$ and $g(x) = 5x(x-2)$ are continuous at every value of x. The function

$$r(x) = \frac{f(x)}{g(x)} = \frac{x^4 + 20}{5x(x-2)}$$

is continuous at every value of x except $x = 0$ and $x = 2$, where the denominator is 0. ❑

EXAMPLE 5 *Continuity of f(x) = |x|*



1.41 The sharp corner does not prevent the function from being continuous at the origin (Example 5).

The function $f(x) = |x|$ is continuous at every value of x (Fig. 1.41). If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x\to 0} |x| = 0 = |0|$. ❑

EXAMPLE 6 *Continuity of trigonometric functions*

We will show in Chapter 2 that the functions $\sin x$ and $\cos x$ are continuous at every value of x. Accordingly, the quotients

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

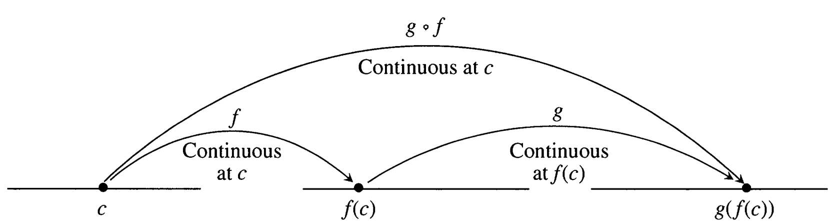
$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

are continuous at every point where they are defined. ❑

Theorem 8 tells us that continuity is preserved under the operation of composition.

Theorem 8
Continuity of Composites

If f is continuous at c, and g is continuous at $f(c)$, then $g \circ f$ is continuous at c (see Fig. 1.42).



1.42 The continuity of composites.

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied. For an outline of the proof of Theorem 8, see Exercise 6 in Appendix 2.

EXAMPLE 7 The following functions are continuous everywhere on their respective domains.

a) $y = \sqrt{x}$ — Theorems 6 and 7 (rational power of a polynomial)

b) $y = \sqrt{x^2 - 2x - 5}$ — Theorems 6 and 7, or (a) plus Theorems 7 and 8 (power of a polynomial or composition with the square root)

c) $y = \dfrac{x\cos(x^{2/3})}{1 + x^4}$ — Theorems 6, 7, and 8 (power, composite, product, polynomial, and quotient)

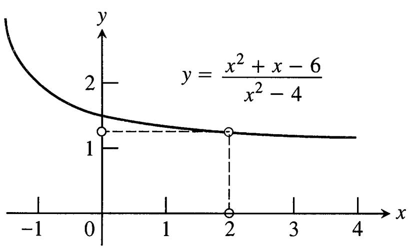
d) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ — Theorems 7 and 8 (composite of absolute value and a rational function) ❑

Continuous Extension to a Point

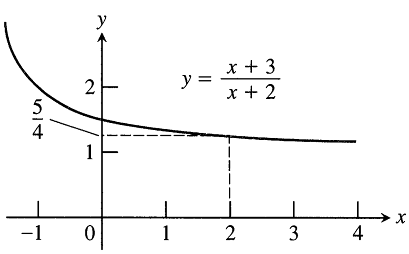
As we saw in Section 1.2, a rational function may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension** of f to $x = c$. For rational functions f, continuous extensions are usually found by canceling common factors.



(a)



(b)

1.43 (a) The graph of

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

and (b) the graph of its continuous extension

$$F(x) = \frac{x+3}{x+2} = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - 4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$$

(Example 8).

EXAMPLE 8 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$, but is also continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x\to 2} f(x) = \frac{5}{4}.$$

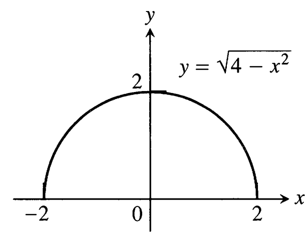
The graph of f is shown in Fig. 1.43. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. ❑

Continuity on Intervals

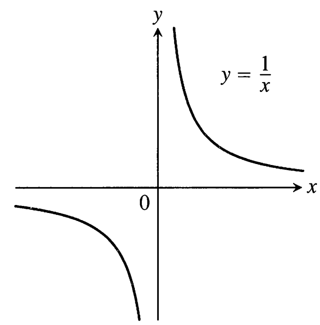
A function is called **continuous** if it is continuous everywhere in its domain. A function that is not continuous throughout its entire domain may still be continuous when restricted to particular intervals within the domain.

A function f is said to be **continuous on an interval** I in its domain if $\lim_{x\to c} f(x) = f(c)$ at every interior point c and if the appropriate one-sided limits equal the function values at any endpoints I may contain. A function continuous on an interval I is automatically continuous on any interval contained in I. Polynomials are continuous on every interval, and rational functions are continuous on every interval on which they are defined.

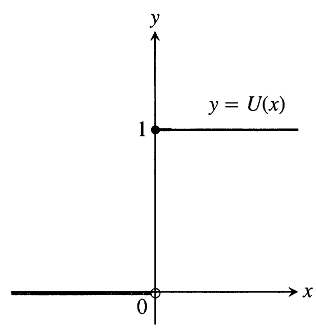
EXAMPLE 9 *Functions continuous on intervals*



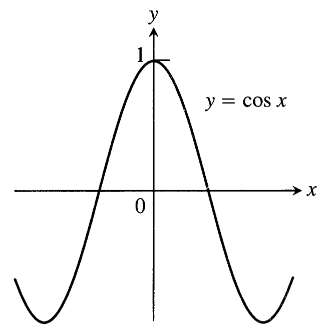
(a) Continuous on $[-2, 2]$



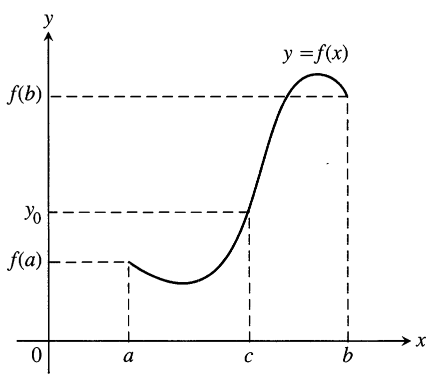
(b) Continuous on $(-\infty, 0)$ and $(0, \infty)$



(c) Continuous on $(-\infty, 0)$ and $[0, \infty)$



(d) Continuous on $(-\infty, \infty)$ ❑

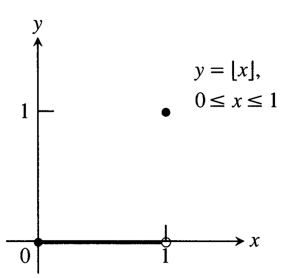


1.44 The function f, being continuous on $[a, b]$, takes on every value between $f(a)$ and $f(b)$.

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the intermediate value property. A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

Theorem 9
The Intermediate Value Theorem

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$ (Fig. 1.44).



1.45 The function $f(x) = \lfloor x \rfloor, 0 \le x \le 1$, does not take on any value between $f(0) = 0$ and $f(1) = 1$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system and can be found in more advanced texts.

The continuity of f on I is essential to the theorem. If f is discontinuous at even one point of I, the theorem's conclusion may fail, as it does for the function graphed in Fig. 1.45.

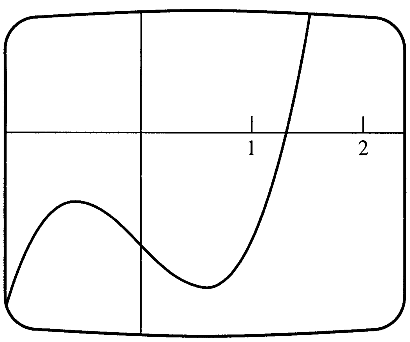
A Consequence for Graphing: Connectivity Theorem 9 is the reason the graph of a function continuous on an interval I cannot have any breaks. It will be **connected,** a single, unbroken curve, like the graph of $\sin x$. It will not have jumps like the graph of the greatest integer function $\lfloor x \rfloor$ or separate branches like the graph of $1/x$.

The Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** or **zero** of the function f. The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign must contain a zero of the function.

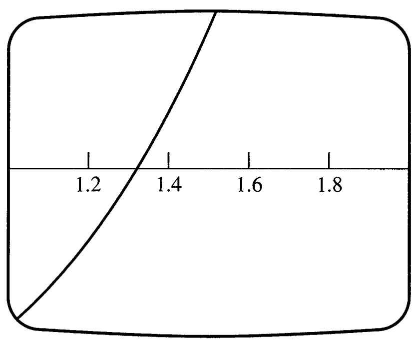
This observation is the basis of the way we solve equations of the form $f(x) = 0$ with a graphing calculator or computer grapher (when f is continuous). The solutions are the x-intercepts of the graph of f. We graph the function $y = f(x)$ over a large interval to see roughly where its zeros are. Then we zoom in on the intersection points one at a time to estimate their coordinates. Figure 1.46 shows a typical sequence of steps in a graphical solution of the equation $x^3 - x - 1 = 0$.

Graphical procedures for solving equations and finding zeros of functions, while instructive, are relatively slow. We usually get faster results from numerical methods, as you will see in Section 3.8.

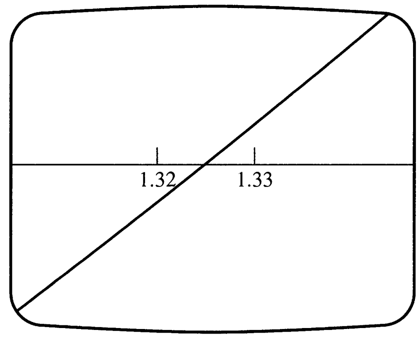
1.46 A graphical solution of the equation $x^3 - x - 1 = 0$. We graph the function $f(x) = x^3 - x - 1$ and, with successive screen enlargements, estimate the coordinates of the point where the graph crosses the x-axis.



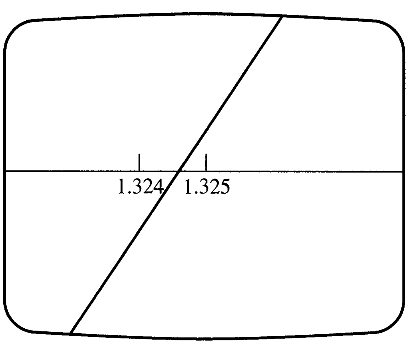
First we make a graph with a relatively large scale. It reveals a root (zero) between $x = 1$ and $x = 2$.



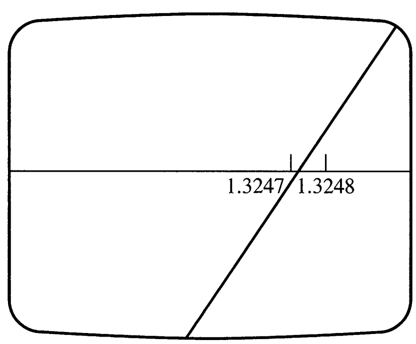
We change the viewing window to $1 \le x \le 2, -1 \le y \le 1$. We now see that the root lies between 1.3 and 1.4.



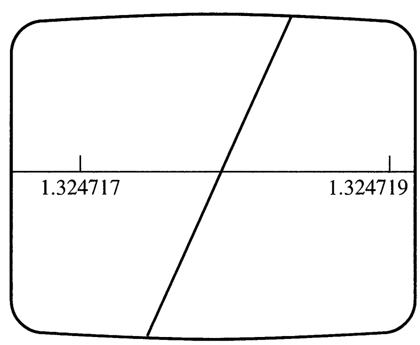
We change the window to $1.3 \le x \le 1.35, -0.1 \le y \le 0.1$. The root lies between 1.32 and 1.33.



We change the window to $1.32 \le x \le 1.33, -0.01 \le y \le 0.01$. The root lies between 1.324 and 1.325.



We change the window to $1.324 \le x \le 1.325, -0.001 \le y \le 0.001$. The root lies between 1.3247 and 1.3248.



After two more enlargements, we arrive at a screen that shows the root to be approximately 1.324718.

EXAMPLE 10 Is any real number exactly 1 less than its cube?

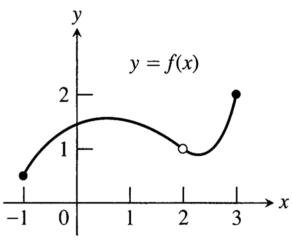
Solution This is the question that gave rise to the equation we just solved. Any such number must satisfy the equation $x = x^3 - 1$ or $x^3 - x - 1 = 0$. Hence, we are looking for a zero of $f(x) = x^3 - x - 1$. By trial, we find that $f(1) = -1$ and $f(2) = 5$ and conclude from Theorem 9 that there is at least one number in $[1, 2]$ where f is zero. So, yes, there is a number that is 1 less than its cube, and we just estimated its value graphically to be about 1.3247 18. ❑

Exercises 1.5

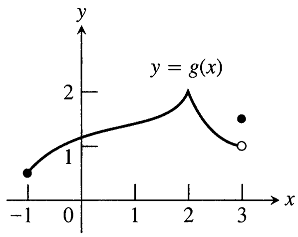
Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

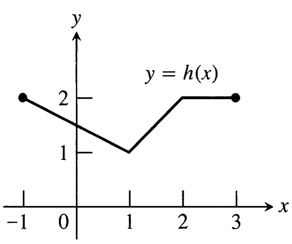
1.



2.



3.



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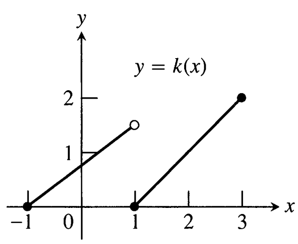
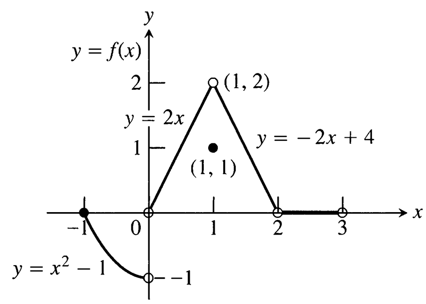


Exercises 5–10 are about the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in Fig. 1.47.



1.47 The graph for Exercises 5–10.

5. a) Does $f(-1)$ exist?
b) Does $\lim_{x \to -1^+} f(x)$ exist?
c) Does $\lim_{x \to -1^+} f(x) = f(-1)$?
d) Is f continuous at $x = -1$?

6. a) Does $f(1)$ exist?
b) Does $\lim_{x \to 1} f(x)$ exist?
c) Does $\lim_{x \to 1} f(x) = f(1)$?
d) Is f continuous at $x = 1$?

7. a) Is f defined at $x = 2$? (Look at the definition of f.)
b) Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in the following exercises fail to be continuous? At which points, if any, are the discontinuities removable? not removable? Give reasons for your answers.

11. Exercise 1, Section 1.4 **12.** Exercise 2, Section 1.4

At what points are the functions in Exercises 13–28 continuous?

13. $y = \dfrac{1}{x-2} - 3x$ **14.** $y = \dfrac{1}{(x+2)^2} + 4$

15. $y = \dfrac{x+1}{x^2 - 4x + 3}$ **16.** $y = \dfrac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$ **18.** $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$ **20.** $y = \dfrac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

Limits of Composite Functions

Find the limits in Exercises 29–34.

29. $\lim_{x \to \pi} \sin(x - \sin x)$

30. $\lim_{t \to 0} \sin\left(\dfrac{\pi}{2} \cos(\tan t)\right)$

31. $\lim_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

32. $\lim_{x \to 0} \tan\left(\dfrac{\pi}{4} \cos(\sin x^{1/3})\right)$

33. $\lim_{t \to 0} \cos\left(\dfrac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$

34. $\lim_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$

Continuous Extensions

35. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

36. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

37. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

38. Define $g(4)$ in a way that extends $g(x) = (x^2 - 16)/(x^2 - 3x - 4)$ to be continuous at $x = 4$.

39. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

40. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

Grapher Explorations—Continuous Extension

In Exercises 41–44, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use TRACE and ZOOM to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

41. $f(x) = \dfrac{10^x - 1}{x}$

42. $f(x) = \dfrac{10^{|x|} - 1}{x}$

43. $f(x) = \dfrac{\sin x}{|x|}$

44. $f(x) = (1 + 2x)^{1/x}$

Theory and Examples

45. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

46. Explain why the equation $\cos x = x$ has at least one solution.

47. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

48. Show that the function $F(x) = (x - a)^2(x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

49. If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals (a) π; (b) $-\sqrt{3}$; (c) 5,000,000.

50. Explain why the following five statements ask for the same information.

a) Find the roots of $f(x) = x^3 - 3x - 1$.
b) Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.
c) Find all the values of x for which $x^3 - 3x = 1$.
d) Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.
e) Solve the equation $x^3 - 3x - 1 = 0$.

51. Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

52. Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

* **53.** * *A function discontinuous at every point.*

a) Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b) Is f right-continuous or left-continuous at any point?

54. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

55. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

---

*Asterisk denotes a challenging problem.

56. Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 8? Give reasons for your answer.

57. Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

58. Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

59. *A fixed point theorem.* Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

60. *The sign-preserving property of continuous functions.* Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta,\ c + \delta)$ about c where f has the same sign as $f(c)$. Notice how remarkable this conclusion is. Although f is defined throughout (a, b), it is not required to be continuous at any point except c. That and the condition $f(c) \neq 0$ are enough to make f different from zero (positive or negative) throughout an entire interval.

61. Explain how Theorem 6 follows from Theorem 1 in Section 1.2.

62. Explain how Theorem 7 follows from Theorems 2 and 3 in Section 1.2.

Solving Equations Graphically

Use a graphing calculator or computer grapher to solve the equations in Exercises 63–70.

63. $x^3 - 3x - 1 = 0$

64. $2x^3 - 2x^2 - 2x + 1 = 0$

65. $x(x - 1)^2 = 1$ (one root)

66. $x^x = 2$

67. $\sqrt{x} + \sqrt{1 + x} = 4$

68. $x^3 - 15x + 1 = 0$ (three roots)

69. $\cos x = x$ (one root). Make sure you are using radian mode.

70. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

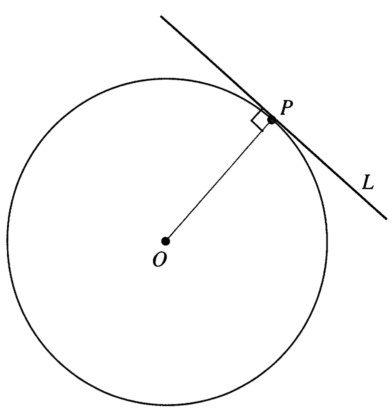
1.6 Tangent Lines

This section continues the discussion of secants and tangents begun in Section 1.1. We calculate limits of secant slopes to find tangents to curves.

What *Is* a Tangent to a Curve?

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Fig. 1.48). Such a line just *touches* the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P? Generalizing from the geometry of the circle, we might say that it means one of the following.

1. L passes through P perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.



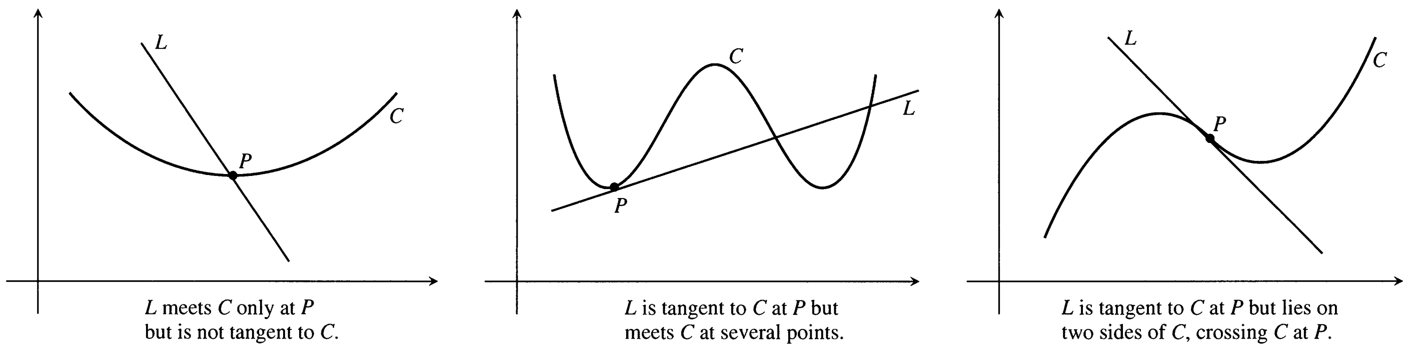
1.48 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

While these statements are valid if C is a circle, none of them work consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency (Fig. 1.49 on the following page).

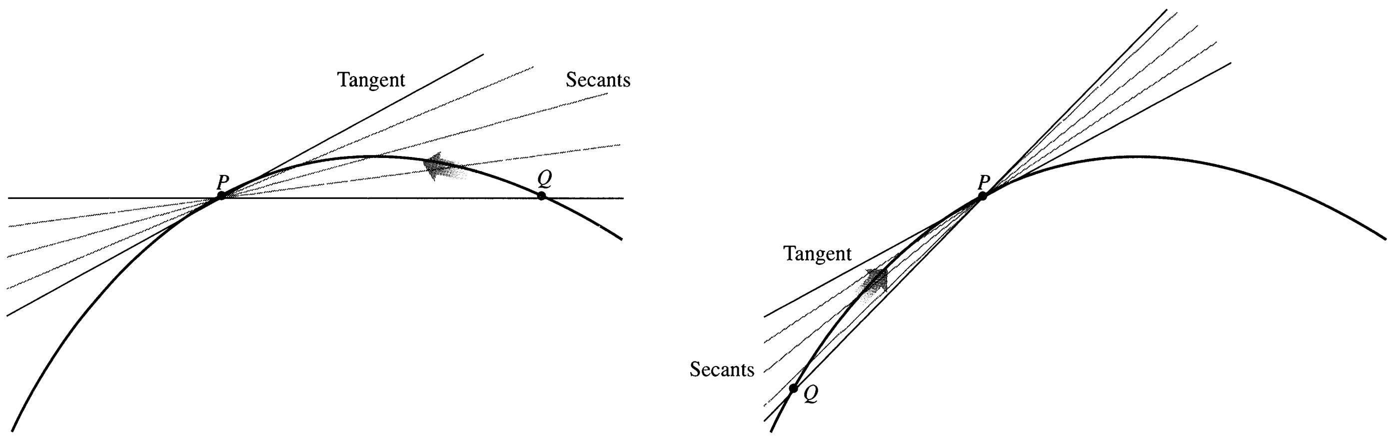
To define tangency for general curves, we need a dynamic approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Fig. 1.50 on the following page). It goes like this:

1. We start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

This is what we were doing in the fruit fly example in Section 1.1.



1.49 Exploding myths about tangent lines.



1.50 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

How do you find a tangent to a curve?

This was the dominant mathematical question of the early seventeenth century and it is hard to overestimate how badly the scientists of the day wanted to know the answer. In optics, the tangent determined the angle at which a ray of light entered a curved lens. In mechanics, the tangent determined the direction of a body's motion at every point along its path. In geometry, the tangents to two curves at a point of intersection determined the angle at which the curves intersected. Descartes went so far as to say that the problem of finding a tangent to a curve was "the most useful and most general problem not only that I know but even that I have any desire to know."

EXAMPLE 1 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

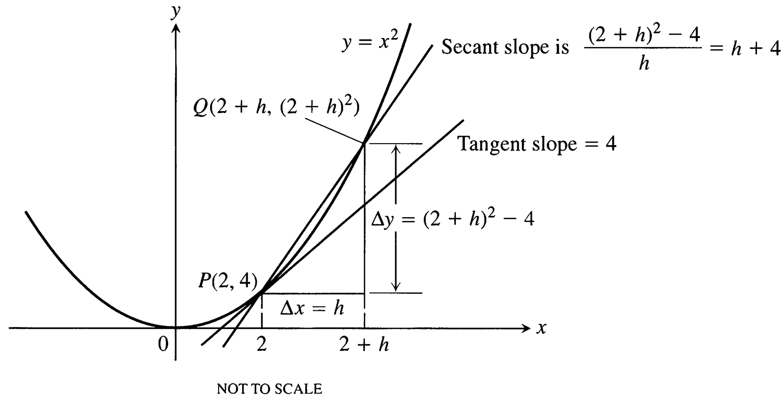
Solution We begin with a secant line through $P(2, 4)$ and $Q(2+h, (2+h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned}\text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4.\end{aligned}$$

If $h > 0$, Q lies above and to the right of P, as in Fig. 1.51. If $h < 0$, Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope approaches 4:

$$\lim_{h \to 0} (h + 4) = 4.$$

We take 4 to be the parabola's slope at P.



1.51 Diagram for finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

The tangent to the parabola at P is the line through P with slope 4:

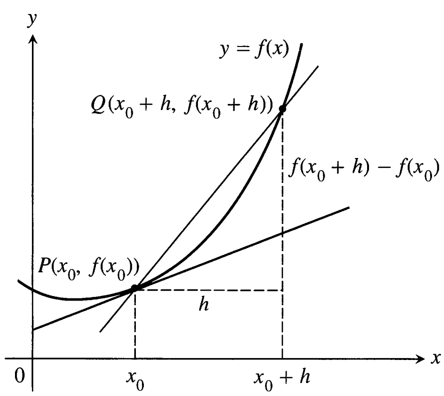
$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$

$$y = 4x - 4.$$

❑

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$ we use the same dynamic procedure. We calculate the slope of the secant through P and a point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Fig. 1.52). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.



1.52 The tangent slope is

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Definitions

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Whenever we make a new definition it is a good idea to try it on familiar objects to be sure it gives the results we want in familiar cases. The next example shows that the new definition of slope agrees with the old definition when we apply it to nonvertical lines.

How to Find the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as $y = y_0 + m(x - x_0)$.

EXAMPLE 2 *Testing the definition*

Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution We let $f(x) = mx + b$ and organize the work into three steps.

Step 1: *Find $f(x_0)$ and $f(x_0 + h)$.*

$$f(x_0) = mx_0 + b$$

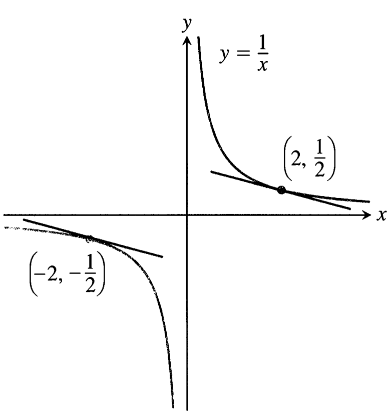
$$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$$

Pierre de Fermat (1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the seventeenth century's major contributions to calculus.

Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His famous "last theorem" (that $a^n + b^n = c^n$ has no positive integer solutions for a, b, and c if n is an integer greater than 2) is known only from a note he jotted in the margin of a book. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

Besides the work in physics and number theory for which he is best known, Fermat found the areas under curves as limits of sums of rectangle areas (as we do today) and developed a method for finding the centroids of shapes bounded by curves in the plane. The standard formula for the first derivative of a polynomial function, the formulas for calculating arc length and for finding the area of a surface of revolution, and the second derivative test for extreme values of functions can all be found in his papers. We will see what these are as the text continues.



1.53 The two tangent lines to $y = 1/x$ having slope $-1/4$.

Step 2: *Find the slope* $\lim_{h\to 0} (f(x_0+h) - f(x_0))/h$.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h} = m$$

Step 3: *Find the tangent line using the point–slope equation.* The tangent line at the point $(x, mx_0 + b)$ is

$$y = (mx_0 + b) + m(x - x_0)$$
$$y = mx_0 + b + mx - mx_0$$
$$y = mx + b.$$

❑

EXAMPLE 3

a) Find the slope of the curve $y = 1/x$ at $x = a$.
b) Where does the slope equal $-1/4$?
c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\frac{a - (a+h)}{a(a+h)}$$
$$= \lim_{h\to 0} \frac{-h}{ha(a+h)}$$
$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" at the beginning of each line until the stage where we could evaluate the limit by substituting $h = 0$.

b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

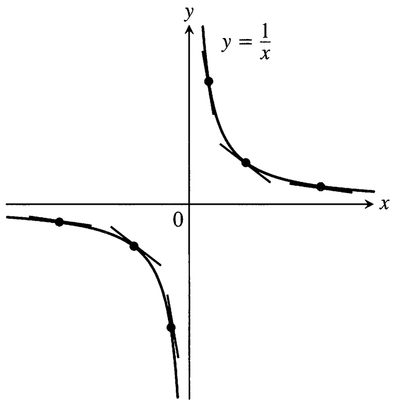
This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Fig. 1.53).

c) Notice that the slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Fig. 1.54). We see this again as $a \to 0^-$. As a moves away from the origin, the slope approaches 0^- and the tangent levels off. ❑

Rates of Change

The expression

$$\frac{f(x_0+h) - f(x_0)}{h}$$



1.54 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

is called the **difference quotient of f at x_0**. If the difference quotient has a limit as h approaches zero, that limit is called the **derivative of f at x_0**. If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, as we did in Section 1.1, the derivative gives the function's rate of change with respect to x at the point $x = x_0$. The derivative is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 2.

All of these refer to the same thing.

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

EXAMPLE 4 *Instantaneous speed (Continuation of Section 1.1, Examples 1 and 2)*

In Examples 1 and 2 in Section 1.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t seconds, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. Exactly what *was* the rock's speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds was

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}.$$

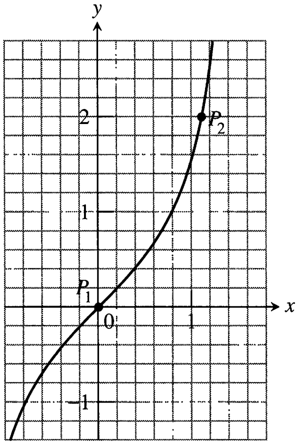
Our original estimate of 32 ft/sec was right. ❑

Exercises 1.6

Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y-units per x-unit) at the points P_1 and P_2. Graphs can shift during a press run, so your estimates may be somewhat different from those in the back of the book.

1.



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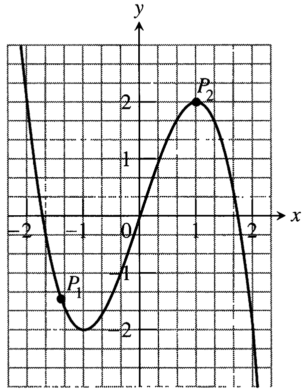

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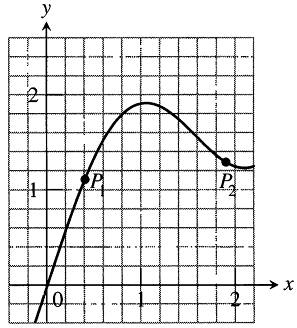

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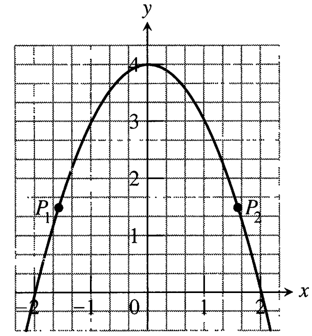


In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

5. $y = 4 - x^2, \quad (-1, 3)$

6. $y = (x-1)^2 + 1, \quad (1, 1)$

7. $y = 2\sqrt{x}, \quad (1, 2)$

8. $y = \dfrac{1}{x^2}, \quad (-1, 1)$

9. $y = x^3, \quad (-2, -8)$

10. $y = \dfrac{1}{x^3}, \quad \left(-2, -\dfrac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1, \quad (2, 5)$

12. $f(x) = x - 2x^2, \quad (1, -1)$

13. $g(x) = \dfrac{x}{x-2}, \quad (3, 3)$

14. $g(x) = \dfrac{8}{x^2}, \quad (2, 2)$

15. $h(t) = t^3, \quad (2, 8)$

16. $h(t) = t^3 + 3t, \quad (1, 4)$

17. $f(x) = \sqrt{x}, \quad (4, 2)$

18. $f(x) = \sqrt{x+1}, \quad (8, 3)$

In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x^2, \quad x = -1$

20. $y = 1 - x^2, \quad x = 2$

21. $y = \dfrac{1}{x-1}, \quad x = 3$

22. $y = \dfrac{x-1}{x+1}, \quad x = 0$

Tangent Lines with Specified Slopes

At what points do the graphs of the functions in Exercises 23 and 24 have horizontal tangents?

23. $f(x) = x^2 + 4x - 1$

24. $g(x) = x^3 - 3x$

25. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x-1)$.

26. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

27. An object is dropped from the top of a 100-m-high tower. Its height aboveground after t seconds is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

28. At t seconds after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

29. What is the rate of change of the area of a circle ($A = \pi r^2$) with respect to its radius when the radius is $r = 3$?

30. What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

Testing for Tangents

31. Does the graph of

$$f(x) = \begin{cases} x^2 \sin\ (1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

32. Does the graph of

$$g(x) = \begin{cases} x \sin\ (1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

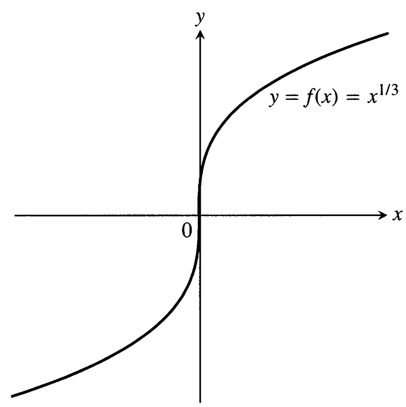
have a tangent at the origin? Give reasons for your answer.

Vertical Tangents

We say that the curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if $\lim_{h \to 0}\ (f(x_0 + h) - f(x_0))/h = \infty$ or $-\infty$.

Vertical tangent at $x = 0$:

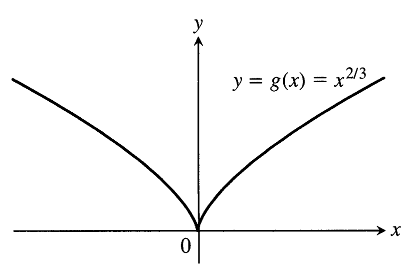
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h}$$
$$= \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty$$



No vertical tangent at $x = 0$:

$$\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{h^{2/3} - 0}{h}$$
$$= \lim_{h \to 0} \frac{1}{h^{1/3}}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



33. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

34. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent at the point (0, 1)? Give reasons for your answer.

Grapher Explorations—Vertical Tangents

a) Graph the curves in Exercises 35–44. Where do the graphs appear to have vertical tangents?
b) Confirm your findings in (a) with limit calculations.

35. $y = x^{2/5}$
36. $y = x^{4/5}$
37. $y = x^{1/5}$
38. $y = x^{3/5}$
39. $y = 4x^{2/5} - 2x$
40. $y = x^{5/3} - 5x^{2/3}$
41. $y = x^{2/3} - (x - 1)^{1/3}$
42. $y = x^{1/3} + (x - 1)^{1/3}$

43. $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

44. $y = \sqrt{|4 - x|}$

CAS Explorations and Projects

Use a CAS to perform the following steps for the functions in Exercises 45–48.

a) Plot $y = f(x)$ over the interval $x_0 - \frac{1}{2} \leq x \leq x_0 + 3$.
b) Define the difference quotient q at x_0 as a function of the general step size h.
c) Find the limit of q as $h \to 0$.
d) Define the secant lines $y = f(x_0) + q^*(x - x_0)$ for $h = 3, 2,$ and 1. Graph them together with f and the tangent line over the interval in part (a).

45. $f(x) = x^3 + 2x, \quad x_0 = 0$

46. $f(x) = x + \dfrac{5}{x}, \quad x_0 = 1$

47. $f(x) = x + \sin\ (2x), \quad x_0 = \pi/2$

48. $f(x) = \cos\ x + 4\sin\ (2x), \quad x_0 = \pi$

Chapter 1 Questions to Guide Your Review

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
3. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain, and give examples.
4. What theorems are available for calculating limits? Give examples of how the theorems are used.
5. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
6. How is the problem of controlling the input x of a function f so that the output $y = f(x)$ will be within a certain specified tolerance ϵ of a target value $y_0 = f(x_0)$ related to the problem of proving that f has limit y_0 as $x \to x_0$?
7. What exactly does $\lim_{x \to x_0} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given $f, L, x_0,$ and $\epsilon > 0$ in the formal definition of limit.
8. Give formal definitions of the following statements.
   a) $\lim_{x \to 2^-} f(x) = 5$
   b) $\lim_{x \to 2^+} f(x) = 5$
   c) $\lim_{x \to 2} f(x) = \infty$
   d) $\lim_{x \to 2} f(x) = -\infty$
9. What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? at an endpoint?
10. How can looking at the graph of a function help you tell where the function is continuous?
11. What does it mean for a function to be right-continuous at a point? left-continuous? How are continuity and one-sided continuity related?
12. What can be said about the continuity of polynomials? of rational functions? of trigonometric functions? of rational powers and algebraic combinations of functions? of composites of functions? of absolute values of functions?
13. Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.
14. What does it mean for a function to be continuous on an interval?
15. What does it mean for a function to be continuous? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.

16. What property must a function f that is continuous on an interval $[a, b]$ have? Show by examples that f need not have this property if it is discontinuous at some point of the interval.

17. It is often said that a function is continuous if you can draw its graph without having to lift your pen from the paper. Why is that?

18. What does continuity have to do with solving equations?

19. When is a line tangent to a curve C at a point P?

20. What is the significance of the formula

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}?$$

CHAPTER 1 PRACTICE EXERCISES

Limit Calculations and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

Then discuss, in complete detail, limits, one-sided limits, continuity, and one-sided continuity of f at each of the points $x = -1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \le -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to c} f(x) = -7$ and $\lim_{x\to c} g(x) = 0$. Find the limit as $x \to c$ of the following functions.

a) $3f(x)$ **b)** $(f(x))^2$

c) $f(x) \cdot g(x)$ **d)** $\dfrac{f(x)}{g(x) - 7}$

e) $\cos(g(x))$ **f)** $|f(x)|$

4. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to 0} f(x) = 1/2$ and $\lim_{x\to 0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.

a) $-g(x)$ **b)** $g(x) \cdot f(x)$

c) $f(x) + g(x)$ **d)** $1/f(x)$

e) $x + f(x)$ **f)** $\dfrac{f(x) \cdot \cos x}{x - 1}$

In Exercises 5 and 6, find the value that $\lim_{x\to 0} g(x)$ must have if the given limit statements hold.

5. $\displaystyle\lim_{x\to 0} \left(\frac{4 - g(x)}{x}\right) = 1$ **6.** $\displaystyle\lim_{x\to -4} \left(x \lim_{x\to 0} g(x)\right) = 2$

In Exercises 7–10, find the limit of $g(x)$ as x approaches the indicated value.

7. $\displaystyle\lim_{x\to 0^+} (4g(x))^{1/3} = 2$ **8.** $\displaystyle\lim_{x\to\sqrt{5}} \frac{1}{x + g(x)} = 2$

9. $\displaystyle\lim_{x\to 1} \frac{3x^2+1}{g(x)} = \infty$ **10.** $\displaystyle\lim_{x\to -2} \frac{5 - x^2}{\sqrt{g(x)}} = 0$

In Exercises 11–18, find the limit or explain why it does not exist.

11. $\displaystyle\lim \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$ (a) as $x \to 0$, (b) as $x \to 2$

12. $\displaystyle\lim \frac{x^2 + x}{x^5 + 2x^4 + x^3}$ (a) as $x \to 0$, (b) as $x \to -1$

13. $\displaystyle\lim_{x\to 1} \frac{1 - \sqrt{x}}{1 - x}$ **14.** $\displaystyle\lim_{x\to a} \frac{x^2 - a^2}{x^4 - a^4}$

15. $\displaystyle\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$ **16.** $\displaystyle\lim_{x\to 0} \frac{(x+h)^2 - x^2}{h}$

17. $\displaystyle\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$ **18.** $\displaystyle\lim_{x\to 0} \frac{(2+x)^3 - 8}{x}$

19. On what intervals are the following functions continuous?

a) $f(x) = x^{1/3}$ **b)** $g(x) = x^{3/4}$

c) $h(x) = x^{-2/3}$ **d)** $k(x) = x^{-1/6}$

20. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or -1? Give reasons for your answers. (Graph the function—you will find the graph interesting.)

Grapher Explorations—Continuous Extensions

In Exercises 21–24, graph the function to see whether it appears to have a continuous extension to the given point a. If it does, use TRACE and ZOOM to find a good candidate for the extended function's value at a. If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

21. $f(x) = \dfrac{x-1}{x-\sqrt[4]{x}}, \quad a = 1$

22. $g(\theta) = \dfrac{5\cos\theta}{4\theta - 2\pi}, \quad a = \pi/2$

23. $h(t) = (1+|t|)^{1/t}, \quad a = 0$

24. $k(x) = \dfrac{x}{1-2^{|x|}}, \quad a = 0$

Grapher Explorations—Roots

25. Let $f(x) = x^3 - x - 1$.

a) Show that f must have a zero between -1 and 2.

b) Solve the equation $f(x) = 0$ graphically with an error of at most 10^{-8}.

c) It can be shown that the exact value of the solution in (b) is

$$\left(\frac{1}{2}+\frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2}-\frac{\sqrt{69}}{18}\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value determined in (b).

26. Let $f(x) = x^3 - 2x + 2$.

a) Show that f must have a zero between -2 and 0.

b) Solve the equation $f(x) = 0$ graphically with an error of at most 10^{-4}.

c) It can be shown that the exact value of the solution in (b) is

$$\left(\sqrt{\frac{19}{27}}-1\right)^{1/3} - \left(\sqrt{\frac{19}{27}}+1\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value determined in (b).

CHAPTER 1 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. a) If $\lim_{x\to c} f(x) = 5$, must $f(c) = 5$?

b) If $f(c) = 5$, must $\lim_{x\to c} f(x) = 5$?

Give reasons for your answers.

2. Can $\lim_{x\to c} (f(x)/g(x))$ exist even if $\lim_{x\to c} f(c) = 0$ and $\lim_{x\to c} g(x) = 0$? Give reasons for your answer.

3. *Assigning a value to 0^0.* The rules of exponents tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0, we would get conflicting results. The first rule would say $0^0 = 1$ while the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here are two examples that might help you to decide. (See Exercise 4 for another example.)

a) CALCULATOR Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Write down the value you get each time. What pattern do you see?

b) GRAPHER Graph the function $y = x^x$ (as $y = x\hat{\ }x$) for $0 \le x \le 1$. Even though the function is not defined for $x \le 0$, the graph will approach the y-axis from the right. Toward what y-value does it seem to be headed? Zoom in to estimate the value more closely. What do you think it is?

4. *A reason you might want 0^0 to be something other than 0 or 1.* As the number x increases through positive values, the numbers $1/x$ and $1/(\ln\ x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln\ x)}$$

as x increases? Here are two ways to find out.

a) CALCULATOR Evaluate f for $x = 10, 100, 1000$, and so on, as far as your calculator can reasonably go. What pattern do you see?

b) GRAPHER Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Use TRACE to read y-values along the graph. What do you find? Chapter 6 will explain what is going on.

5. *Lorentz contraction.* In relativity theory the length of an object, say a rocket, appears, to an observer, to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the rocket's length will appear to be

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}.$$

The Lorentz contraction formula.

Here, $c \approx 3 \times 10^8$ m/sec is the speed of light in a vacuum. What happens to L as v increases? Find $\lim_{v\to c^-} L$. Why was the left-hand limit needed?

6. *Roots of a quadratic equation that is almost linear.* The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1+a}}{a}, \quad r_-(a) = \frac{-1 - \sqrt{1+a}}{a}.$$

a) What happens to $r_+(a)$ as $a \to 0$? as $a \to -1^+$?
b) What happens to $r_-(a)$ as $a \to 0$? as $a \to -1^+$?
c) GRAPHER Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.
d) GRAPHER For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05.

7. If $\lim_{x\to 0^+} f(x) = A$ and $\lim_{x\to 0^-} f(x) = B$, find

a) $\lim_{x\to 0^+} f(x^3 - x)$
b) $\lim_{x\to 0^-} f(x^3 - x)$
c) $\lim_{x\to 0^+} f(x^2 - x^4)$
d) $\lim_{x\to 0^-} f(x^2 - x^4)$

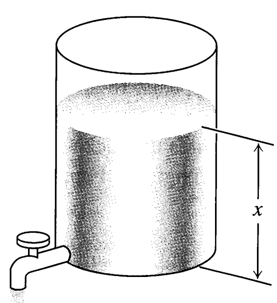
8. Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} (f(x) + g(x))$ does not exist.
b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} (f(x) + g(x))$ does not exist.
c) If f is continuous at a, then so is $|f|$.
d) If $|f|$ is continuous at a, then so is f.

9. Show that the equation $x + 2\cos x = 0$ has at least one solution.

10. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

11. *Controlling the flow from a draining tank.* Torricelli's law says that if you drain a tank like the one in the figure below, the rate y at which water runs out is a constant times the square root of the water's depth x. The constant depends on the size of the exit valve. Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by pouring more water into the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate (a) within 0.2 ft^3/min of the rate $y_0 = 1$ ft^3/min? (b) within 0.1 ft^3/min of the rate $y_0 = 1$ ft^3/min?



Exit rate y ft^3/min

12. *Thermal expansion in precise equipment.* As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the temperature in the shop where it is made and the laboratory where it is used must not be allowed to vary. A typical aluminum bar that is 10 cm wide at 70°F will be

$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t. Suppose you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 70$°F must you maintain the temperature to ensure that this tolerance is not exceeded?

13. *Antipodal points.* Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on the earth's equator where the temperatures are the same? Explain.

14. *Uniqueness of limits.* Show that a function cannot have two different limits at the same point. That is, if $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} f(x) = L_2$, then $L_1 = L_2$.

In Exercises 15–18, use the formal definition of limit to prove that the function is continuous at x_0.

15. $f(x) = x^2 - 7, \quad x_0 = 1$

16. $g(x) = 1/(2x), \quad x_0 = 1/4$

17. $h(x) = \sqrt{2x - 3}, \quad x_0 = 2$

18. $F(x) = \sqrt{9 - x}, \quad x_0 = 5$

In Exercises 19 and 20, use the formal definition of limit to prove that the function has a continuous extension to the given value of x.

19. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad x = -1$

20. $g(x) = \dfrac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

21. *Max {a, b} and min {a, b}.*

a) Show that the expression

$$\max\ \{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, max (a, b) gives the larger of the two numbers a and b.

b) Find a similar expression for min $\{a, b\}$, the smaller of a and b.

***22.** *A function continuous at only one point.* Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

a) Show that f is continuous at $x = 0$.
b) Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

***23.** *Bounded functions.* A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D. We call N, when it exists, an **upper bound** for f on D and say that f is bounded from above by N. In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D. We call M, when it exists, a **lower bound** for f on D and say that f is bounded from below by M. We say that f is **bounded** on D if it is bounded from both above and below.

a) Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D.

b) Suppose that f is bounded from above by N. Show that if $\lim_{x \to x_0} f(x) = L$, then $L \leq N$.

c) Suppose that f is bounded from below by M. Show that if $\lim_{x \to x_0} f(x) = L$, then $L \geq M$.

***24.** *The Dirichlet ruler function.* If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms.* For example, 6/4 written in lowest terms is 3/2.) Let $f(x)$ be defined for all x in the interval [0, 1] by

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

a) Show that f is discontinuous at every rational number in [0, 1].

b) Show that f is continuous at every irrational number in [0, 1]. (*Hint:* If ϵ is a given positive number, show that there are only finitely many rational numbers r in [0, 1] such that $f(r) \geq \epsilon$.)

c) Sketch the graph of f. Why do you think f is called the "ruler function"?

CHAPTER 2

Derivatives

OVERVIEW In Chapter 1 we defined the slope of a curve at a point as the limit of secant slopes. This limit, called a derivative, measures the rate at which a function changes and is one of the most important ideas in calculus. Derivatives are used widely in science, economics, medicine, and computer science to calculate velocity and acceleration, to explain the behavior of machinery, to estimate the drop in water levels as water is pumped out of a tank, and to predict the consequences of making errors in measurements. Finding derivatives by evaluating limits can be lengthy and difficult. In this chapter we develop techniques to make calculating derivatives easier.

2.1 The Derivative of a Function

At the end of Chapter 1, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We called this limit, when it existed, the derivative of f at x_0. In this section, we investigate the derivative as a *function* derived from f by considering the limit at each point of f's domain.

Definition

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f **has a derivative (is differentiable)** at x.

Notation

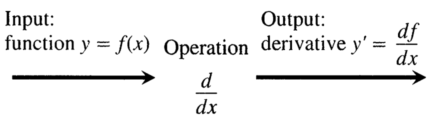
There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief but does not name the independent variable
$\dfrac{dy}{dx}$	"dy dx"	Names the variables and uses d for derivative
$\dfrac{df}{dx}$	"df dx"	Emphasizes the function's name
$\dfrac{d}{dx}f(x)$	"ddx of $f(x)$"	Emphasizes the idea that differentiation is an operation performed on f (Fig. 2.1)
$D_x f$	"dx of f"	A common operator notation
$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives

We also read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x."

Why all these notations?

The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. Each has its own strengths and weaknesses.



2.1 Flow diagram for the operation of taking a derivative with respect to x.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation.** Examples 2 and 3 of Section 1.6 illustrate the process for the functions $y = mx + b$ and $y = 1/x$. Example 2 shows that

$$\frac{d}{dx}(mx + b) = m.$$

In Example 3, we see that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Here are two more examples.

Steps for Calculating $f'(x)$ from the Definition of Derivative

1. Write expressions for $f(x)$ and $f(x + h)$.
2. Expand and simplify the difference quotient
$$\frac{f(x+h) - f(x)}{h}.$$
3. Using the simplified quotient, find $f'(x)$ by evaluating the limit
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

EXAMPLE 1

a) Differentiate $f(x) = \dfrac{x}{x - 1}$.

b) Where does the curve $y = f(x)$ have slope -1?

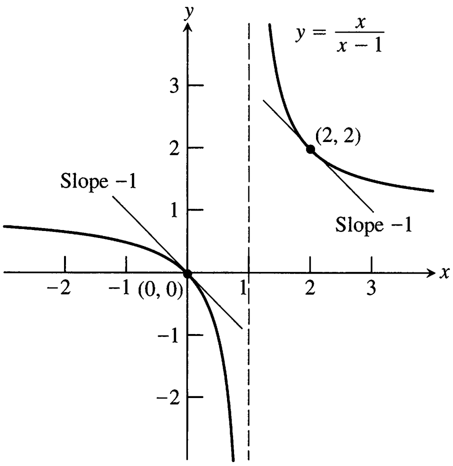
Solution

a) We take the three steps listed in the margin.

Step 1: Here we have $f(x) = \dfrac{x}{x-1}$

and

$$f(x + h) = \frac{(x+h)}{(x+h) - 1}, \text{ so}$$



2.2 $y' = -1$ at $x = 0$ and $x = 2$.

$$\textbf{Step 2:}\quad \frac{f(x+h)-f(x)}{h} = \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \frac{1}{h}\cdot\frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$$

$$= \frac{1}{h}\cdot\frac{-h}{(x+h-1)(x-1)}, \text{ and}$$

$$\textbf{Step 3:}\quad f'(x) = \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

b) The slope of $y = f(x)$ will be -1 provided

$$-\frac{1}{(x-1)^2} = -1.$$

This equation is equivalent to $(x-1)^2 = 1$, so $x = 2$ or $x = 0$ (Fig. 2.2). ❑

EXAMPLE 2

a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.
b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

You will often need to know the derivative of $\sqrt{x}$ for $x > 0$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Try to remember it.

a) ***Step 1:*** $f(x) = \sqrt{x}$ and $f(x+h) = \sqrt{x+h}$

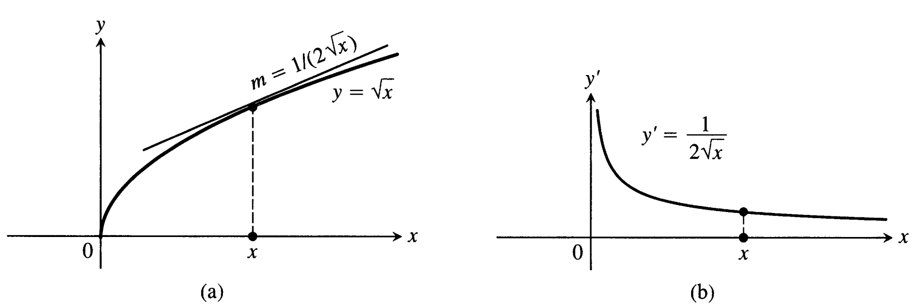
$$\textbf{Step 2:}\quad \frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \qquad \text{Multiply by } \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

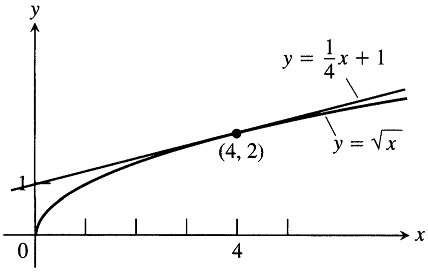
$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$\textbf{Step 3:}\quad f'(x) = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

See Fig. 2.3.



2.3 The graphs of (a) $y = \sqrt{x}$ and (b) $y' = 1/(2\sqrt{x})$, $x > 0$ (Example 2). The function is defined at $x = 0$, but its derivative is not.



2.4 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating *dy/dx* at $x = 4$ (Example 2).

b) The slope of the curve at $x = 4$ is

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Fig. 2.4).

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$

❑

The symbol for evaluation

In addition to

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

the value of the derivative of $y = f(x)$ with respect to x at $x = a$ can be denoted in the following ways:

$$y'\,|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

Here the symbol $|_{x=a}$, called an **evaluation symbol,** tells us to evaluate the expression to its left at $x = a$.

Graphing f' from Estimated Values

When we measure the values of a function $y = f(x)$ in the laboratory or in the field (pressure vs. temperature, say, or population vs. time) we usually connect the data points with lines or curves to picture the graph of f. We can often make a reasonable plot of f' by estimating slopes on this graph. The following examples show how this is done and what can be learned from the process.

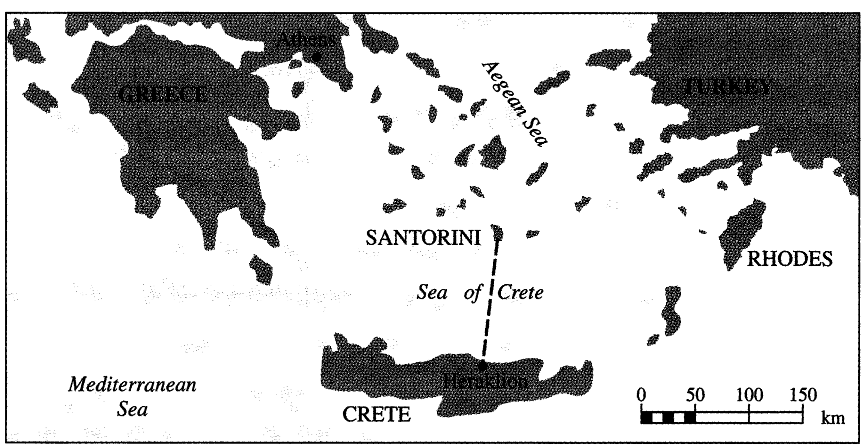
EXAMPLE 3 *Medicine*

On April 23, 1988, the human-powered airplane *Daedalus* flew a record-breaking 119 km from Crete to the island of Santorini in the Aegean Sea, southeast of mainland Greece. During the 6-h endurance tests before the flight, researchers monitored the prospective pilots' blood-sugar concentrations. The concentration graph for one of the athlete-pilots is shown in Fig. 2.5(a), where the concentration in milligrams/deciliter is plotted against time in hours.

The graph is made of line segments connecting data points. The constant slope of each segment gives an estimate of the derivative of the concentration between measurements. We calculated the slope of each segment from the coordinate grid and plotted the derivative as a step function in Fig. 2.5(b). To make the plot for the first hour, for instance, we observed that the concentration increased from about 79 mg/dL to 93 mg/dL. The net increase was $\Delta y = 93 - 79 = 14$ mg/dL. Dividing this by $\Delta t = 1$ h gave the rate of change as

$$\frac{\Delta y}{\Delta t} = \frac{14}{1} = 14 \text{ mg/dL per h.}$$

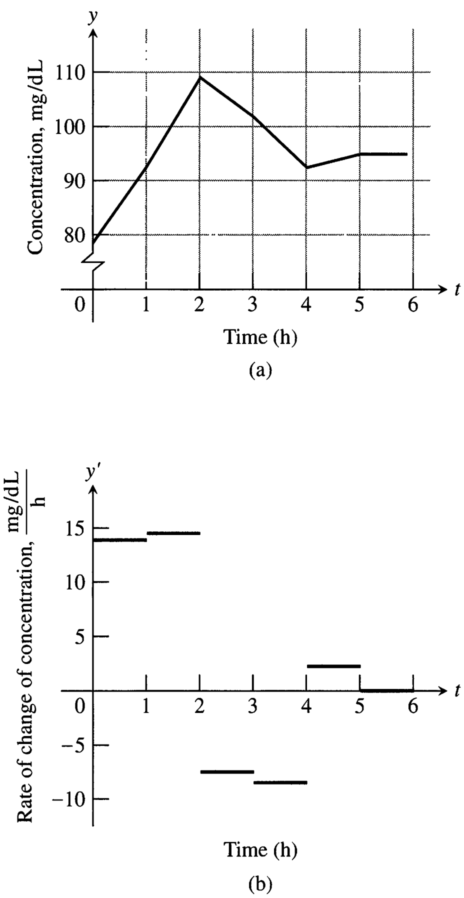
❑



Daedalus's flight path on April 23, 1988.

Notice that we can make no estimate of the concentration's rate of change at times $t = 1, 2, \ldots, 5$, where the graph we have drawn for the concentration has a corner and no slope. The derivative step function is not defined at these times.

When we have so many data that the graph we get by connecting the data points resembles a smooth curve, we may wish to plot the derivative as a smooth curve. The next example shows how this is done.



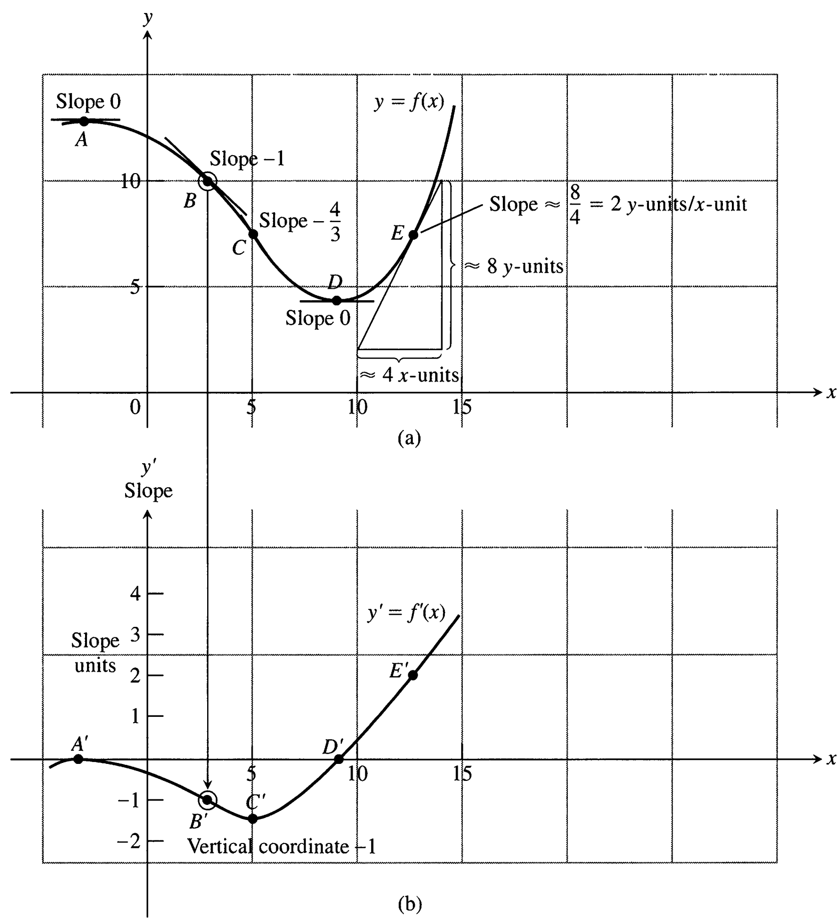
2.5 (a) The sugar concentration in the blood of a *Daedalus* pilot during a 6-h preflight endurance test. (b) The derivative of the pilot's blood-sugar concentration shows how rapidly the concentration rose and fell during various portions of the test. (Source: *The Daedalus Project: Physiological Problems and Solutions* by Ethan R. Nadel and Steven R. Bussolari, *American Scientist,* Vol. 76, No. 4, July–August 1988, p. 358.)

EXAMPLE 4 Graph the derivative of the function $y = f(x)$ in Fig. 2.6(a).

Solution We draw a pair of axes, marking the horizontal axis in x-units and the vertical axis in y'-units (Fig. 2.6b). Next we sketch tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $y' = f'(x)$ at these points. We plot the corresponding (x, y') pairs and connect them with a smooth curve.

From the graph of $y' = f'(x)$ we see at a glance

1. where f's rate of change is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing. ❑



2.6 We made the graph of $y' = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B, and so on. The graph of $y' = f'(x)$ is a visual record of how the slope of f changes with x.

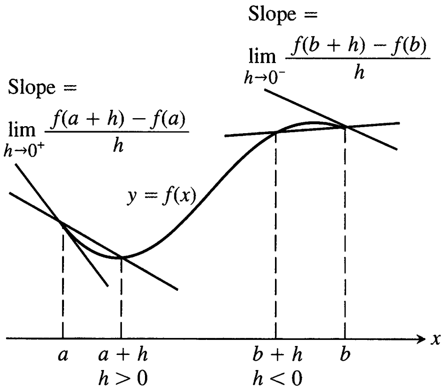
Differentiable on an Interval; One-sided Derivatives

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Fig. 2.7).



2.7 Derivatives at endpoints are one-sided limits.

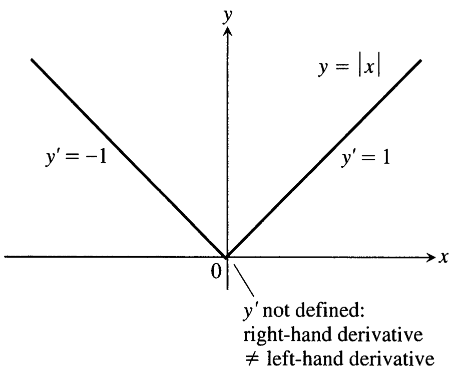
Right-hand and left-hand derivatives may be defined at any point of a function's domain. The usual relation between one-sided and two-sided limits holds for these derivatives. Because of Theorem 5, Section 1.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

EXAMPLE 5 The function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$. To the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1$$



2.8 Not differentiable at the origin.

(Fig. 2.8). There can be no derivative at the origin because the one-sided derivatives differ there:

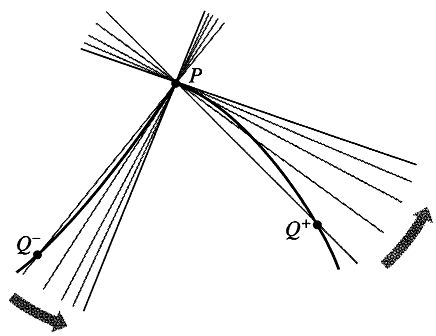
$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0 \\
&= \lim_{h\to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0 \\
&= \lim_{h\to 0^-} -1 = -1.
\end{aligned}$$

❑

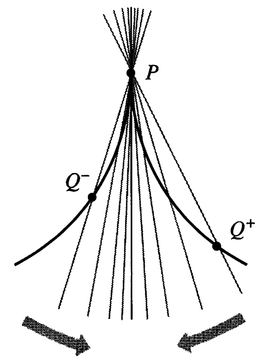
When Does a Function *Not* Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. A function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

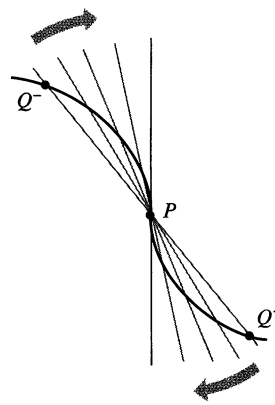
1. a *corner,* where the one-sided derivatives differ



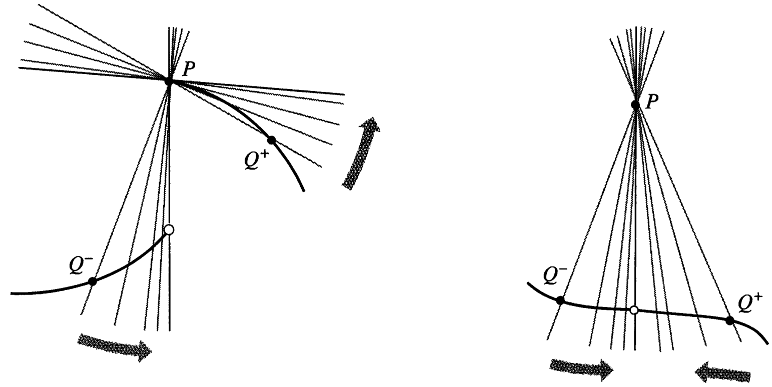
2. a *cusp,* where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



3. a *vertical tangent,* where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)

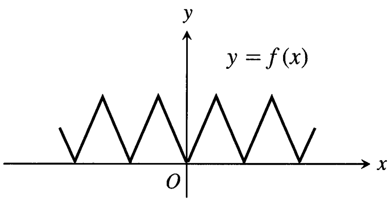


4. a *discontinuity.*



How rough can the graph of a continuous function be?

The absolute value function fails to be differentiable at a single point. Using a similar idea, we can use a sawtooth graph to define a continuous function that fails to have a derivative at infinitely many points.



But can a continuous function fail to have a derivative at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass (1815–1897) found in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a graph that is too bumpy in the limit to have a tangent anywhere.

Continuous curves that fail to have a tangent anywhere play a useful role in chaos theory, in part because there is no way to assign a finite length to such a curve. We will see what length has to do with derivatives when we get to Section 5.5.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem 1

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + (f(c+h) - f(c)) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 1.2,

$$\begin{aligned} \lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

❑

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

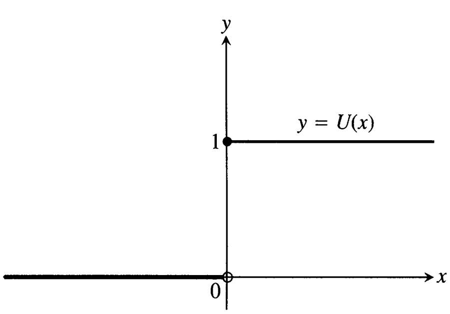
Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw in Example 5.

The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.

Theorem 2

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.



2.9 The unit step function does not have the intermediate value property and cannot be the derivative of a function on the real line.

Theorem 2 (which we will not prove) says that a function cannot *be* a derivative on an interval unless it has the intermediate value property there (Fig. 2.9). The question of when a function is a derivative is one of the central questions in all calculus, and Newton's and Leibniz's answer to this question revolutionized the world of mathematics. We will see what their answer was when we reach Chapter 4.

Exercises 2.1

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2;\quad f'(-3), f'(0), f'(1)$

2. $F(x) = (x-1)^2 + 1;\quad F'(-1), F'(0), F'(2)$

3. $g(t) = \dfrac{1}{t^2};\quad g'(-1), g'(2), g'(\sqrt{3})$

4. $k(z) = \dfrac{1-z}{2z};\quad k'(-1), k'(1), k'(\sqrt{2})$

5. $p(\theta) = \sqrt{3\theta};\quad p'(1), p'(3), p'(2/3)$

6. $r(s) = \sqrt{2s+1};\quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$

8. $\dfrac{dr}{ds}$ if $r = \dfrac{s^3}{2} + 1$

9. $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t+1}$

10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$

11. $\dfrac{dp}{dq}$ if $p = \dfrac{1}{\sqrt{q+1}}$

12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{3w-2}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \dfrac{9}{x},\quad x = -3$

14. $k(x) = \dfrac{1}{2+x},\quad x = 2$

15. $s = t^3 - t^2,\quad t = -1$

16. $y = (x+1)^3,\quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \dfrac{8}{\sqrt{x-2}},\quad (x, y) = (6, 4)$

18. $w = g(z) = 1 + \sqrt{4-z},\quad (z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$

20. $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$

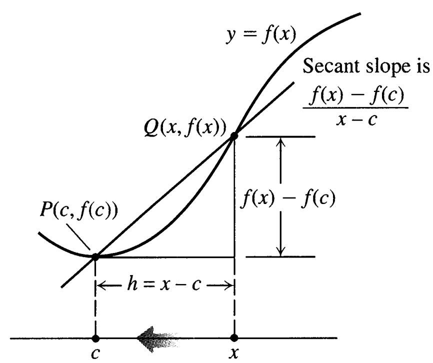
21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4-\theta}}$

22. $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

An Alternative Formula for Calculating Derivatives

The formula for the secant slope whose limit leads to the derivative depends on how the points involved are labeled. In the notation of Fig. 2.10, the secant slope is $(f(x) - f(c))/(x-c)$ and the slope of the curve at P is

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$



Derivative of f at c is

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

2.10 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

The use of this formula simplifies some derivative calculations. Use it in Exercises 23–26 to find the derivative of the function at the given value of c.

23. $f(x) = \dfrac{1}{x+2},\quad c = -1$

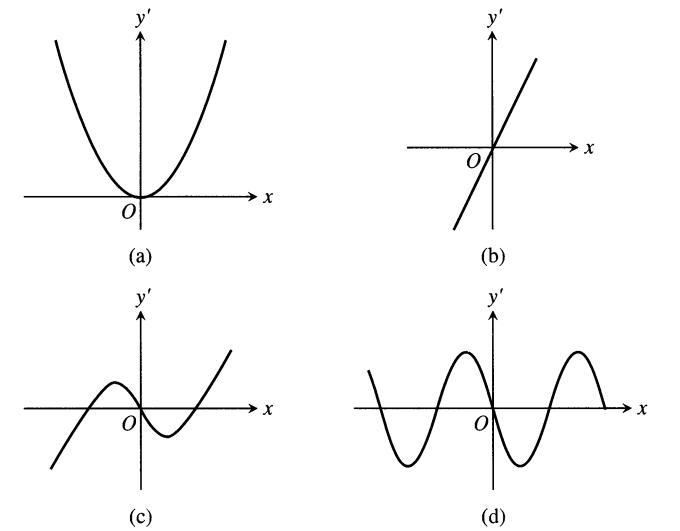
24. $f(x) = \dfrac{1}{(x-1)^2},\quad c = 2$

25. $g(t) = \dfrac{t}{t-1},\quad c = 3$

26. $k(s) = 1 + \sqrt{s},\quad c = 9$

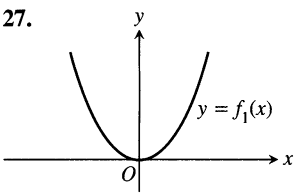
Graphs

Match the functions graphed in Exercises 27–30 with the derivatives graphed in Fig. 2.11.



2.11 The derivative graphs for Exercises 27–30.

27.



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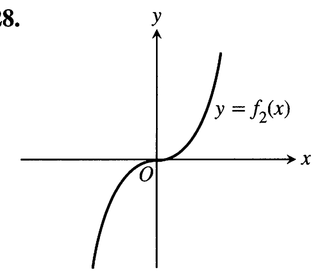

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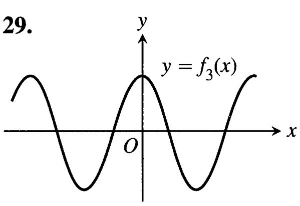
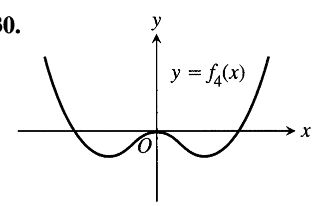
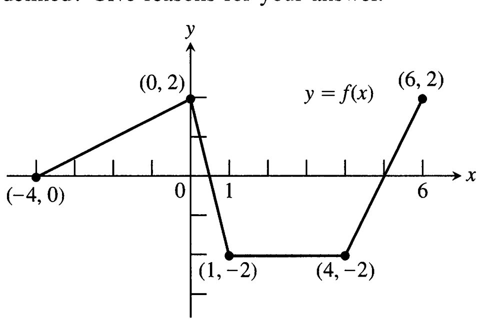


30.



31. a) The graph in Fig. 2.12 is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.



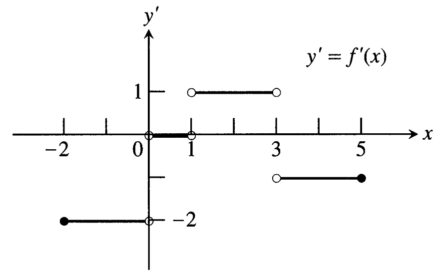
2.12 The graph for Exercise 31.

b) Graph the derivative of f. Call the vertical axis the y'-axis. The graph should show a step function.

32. *Recovering a function from its derivative*

a) Use the following information to graph the function f over the closed interval $[-2, 5]$.

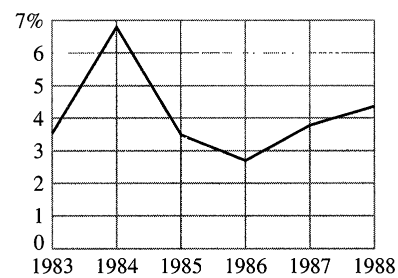
i) The graph of f is made of closed line segments joined end to end.
ii) The graph starts at the point $(-2, 3)$.
iii) The derivative of f is the step function in Fig. 2.13.



2.13 The derivative graph for Exercise 32.

b) Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

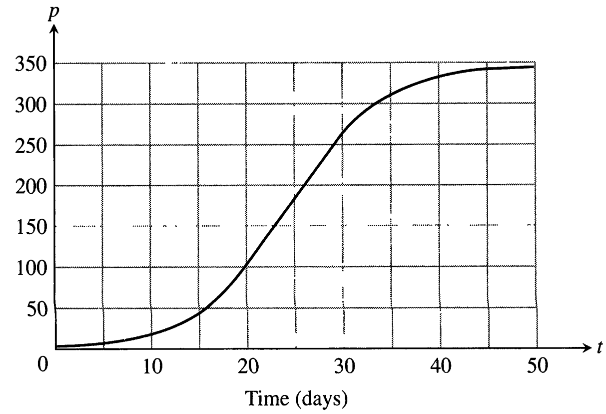
33. *Growth in the economy.* The graph in Fig. 2.14 shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 1983–1988. Graph dy/dt (where defined). (Source: *Statistical Abstracts of the United States,* 110th Edition, U.S. Department of Commerce, p. 427.)



2.14 The graph for Exercise 33.

34. *Fruit flies. (Continuation of Example 3, Section 1.1.)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

a) Use the graphical technique of Example 4 to graph the derivative of the fruit fly population introduced in Section 1.1. The graph of the population is reproduced here as Fig. 2.15. What units should be used on the horizontal and vertical axes for the derivative's graph?

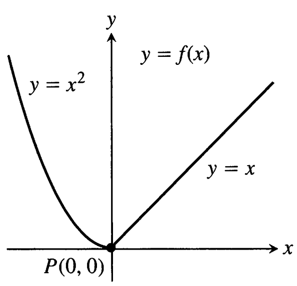


2.15 The graph for Exercise 34.

b) During what days does the population seem to be increasing fastest? slowest?

Compare the right-hand and left-hand derivatives to show that the functions in Exercises 35–38 are not differentiable at the point *P*.

35.



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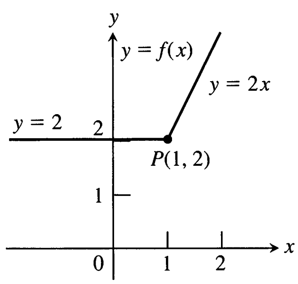

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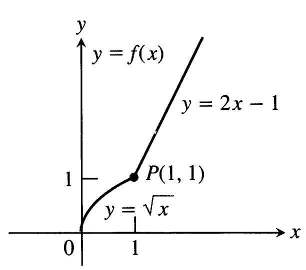

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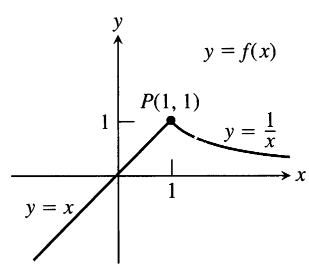
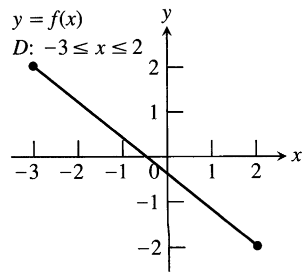


Each figure in Exercises 39–44 shows the graph of a function over a closed interval *D*. At what domain points does the function appear to be

a) differentiable?
b) continuous but not differentiable?
c) neither continuous nor differentiable?

Give reasons for your answers.

39.



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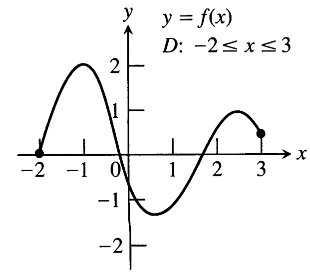

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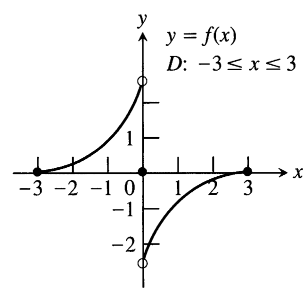

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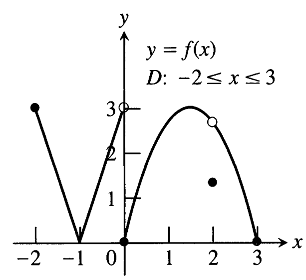

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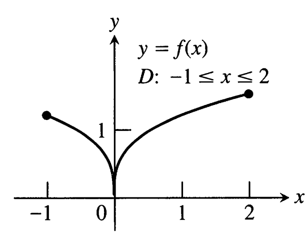

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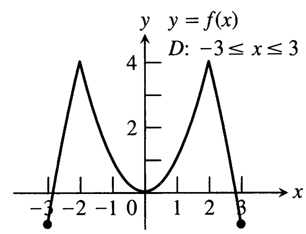


Theory and Examples

In Exercises 45–48,

a) Find the derivative $y' = f'(x)$ of the given function $y = f(x)$.
b) Graph $y = f(x)$ and $y' = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.
c) For what values of x, if any, is y' positive? zero? negative?
d) Over what intervals of x-values, if any, does the function $y = f(x)$ increase as x increases? decrease as x increases? How is this related to what you found in (c)? (We will say more about this relationship in Chapter 3.)

45. $y = -x^2$

46. $y = -1/x$

47. $y = x^3/3$

48. $y = x^4/4$

49. Does the curve $y = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

50. Does the curve $y = 2\sqrt{x}$ have any horizontal tangents? If so, where? Give reasons for your answer.

51. Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is -1? If so, find an equation for the line and the point of tangency. If not, why not?

52. Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

53. Does any function differentiable on $(-\infty, \infty)$ have $y = \lfloor x \rfloor$ as its derivative? Give reasons for your answer.

54. Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

55. Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

56. Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

57. Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and that $g(0) = h(0) = 0$. Can $\lim_{t \to 0}(g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

58. **a)** Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b) Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

Grapher Explorations

59. Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

60. Graph $y = 3x^2$ in a window that has $-2 \le x \le 2, 0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

61. *Weierstrass's nowhere differentiable continuous function.* The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty}(2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2 \pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3 \pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

CAS Explorations and Projects

Use a CAS to perform the following steps for the functions in Exercises 62–67.

a) Plot $y = f(x)$ to see that function's global behavior.

b) Define the difference quotient q at a general point x, with general stepsize h.

c) Take the limit as $h \to 0$. What formula does this give?

d) Substitute the value $x = x_0$ and plot the function together with its tangent line at that point.

e) Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?

f) Graph the formula obtained in part (c). What does it mean when its values are negative? zero? positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

62. $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

63. $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

64. $f(x) = \dfrac{4x}{x^2 + 1}, \quad x_0 = 2$

65. $f(x) = \dfrac{x - 1}{3x^2 + 1}, \quad x_0 = -1$

66. $f(x) = \sin 2x, \quad x_0 = \pi/2$

67. $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

2.2 Differentiation Rules

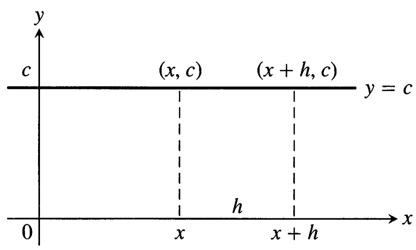
This section shows how to differentiate functions without having to apply the definition each time.

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

> **Rule 1 Derivative of a Constant**
>
> If c is constant, then $\dfrac{d}{dx}c = 0$.

EXAMPLE 1 $\dfrac{d}{dx}(8) = 0, \quad \dfrac{d}{dx}\left(-\dfrac{1}{2}\right) = 0, \quad \dfrac{d}{dx}\left(\sqrt{3}\right) = 0$ ❑



2.16 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Fig. 2.16). At every value of x, we find that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$ ❑

The next rule tells how to differentiate x^n if n is a positive integer.

> **Rule 2 Power Rule for Positive Integers**
>
> If n is a positive integer, then
>
> $$\frac{d}{dx}x^n = nx^{n-1}.$$

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 2

f	x	x^2	x^3	x^4	$\cdots$
f'	1	$2x$	$3x^2$	$4x^3$	$\cdots$

❑

Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Since n is a positive integer, we can use the fact that

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

to simplify the difference quotient for f. Taking $x+h=a$ and $x=b$, we have $a-b=h$. Thus

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h}\\ &= \frac{(h)[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}]}{h}\\ &= \underbrace{(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}}_{n \text{ terms, each with limit } x^{n-1} \text{ as } h\to 0}.\end{aligned}$$

Hence

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = nx^{n-1}.$$ ❑

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Rule 3 The Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

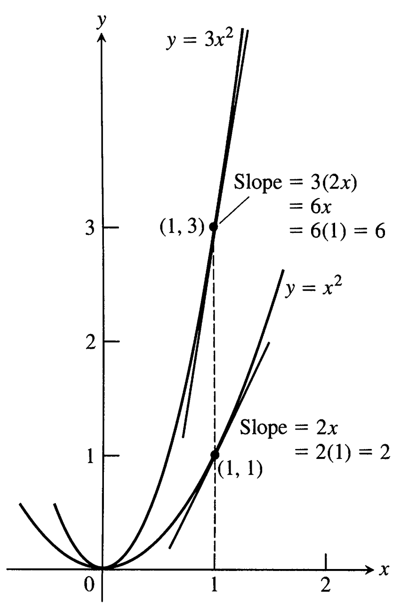
In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cn\,x^{n-1}.$$

EXAMPLE 3 The derivative formula

$$\frac{d}{dx}(3x^2) = 3\cdot 2x = 6x$$

says that if we rescale the graph of $y=x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Fig. 2.17). ❑



2.17 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

EXAMPLE 4 *A useful special case*

The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with $c=-1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1\cdot\frac{d}{dx}(u) = -\frac{du}{dx}.$$ ❑

Proof of Rule 3

$$\begin{aligned}\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} && \text{Derivative definition with } f(x)=cu(x)\\ &= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} && \text{Limit property}\\ &= c\frac{du}{dx} && u \text{ is differentiable.}\end{aligned}$$ ❑

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Denoting functions by *u* and *v*

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find it using these same letters in some other way. To guard against this, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

Rule 4 The Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Combining the Sum Rule with the Constant Multiple Rule gives the equivalent **Difference Rule,** which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

EXAMPLE 5

a) $y = x^4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$

$$= 4x^3 + 12$$

b) $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

❑

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in Example 5.

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x)+v(x)] = \lim_{h \to 0} \frac{[u(x+h)+v(x+h)] - [u(x)+v(x)]}{h}$$

$$= \lim_{h \to 0} \left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right]$$

$$= \lim_{h \to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h \to 0} \frac{v(x+h)-v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.$$

❑

Proof by mathematical induction

Many formulas can be shown to hold for every positive integer n greater than or equal to some lowest integer n_0 by applying an axiom called the *mathematical induction principle.* A proof using this axiom is called a *proof by mathematical induction* or a *proof by induction.* The steps in proving a formula by induction are

1. Check that it holds for $n = n_0$.
2. Prove that if it holds for any positive integer $n = k \geq n_0$, then it holds for $n = k + 1$.

Once these steps are completed, the axiom says, we know that the formula holds for all $n \geq n_0$. For more mathematical induction, see Appendix 1.

Proof of the Sum Rule for Sums of More Than Two Functions We prove the statement

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$$

by mathematical induction. The statement is true for $n = 2$, as was just proved. This is step 1 of the induction proof.

Step 2 is to show that if the statement is true for any positive integer $n = k$, where $k \geq n_0 = 2$, then it is also true for $n = k + 1$. So suppose that

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_k) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}. \tag{1}$$

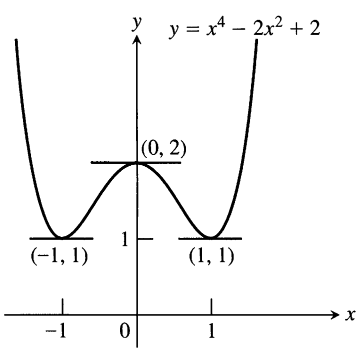
Then

$$\frac{d}{dx}\,(\underbrace{u_1 + u_2 + \cdots + u_k}_{\substack{\text{Call the function}\\ \text{defined by this sum } u.}} + \underbrace{u_{k+1}}_{\substack{\text{Call this}\\ \text{function } v.}})$$

$$= \frac{d}{dx}(u_1 + u_2 + \cdots + u_k) + \frac{du_{k+1}}{dx} \qquad \text{Rule 4 for } \frac{d}{dx}(u+v)$$

$$= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx}. \qquad \text{Eq. (1)}$$

With these steps verified, the mathematical induction principle now guarantees the Sum Rule for every integer $n \geq 2$. ❑



2.18 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

EXAMPLE 6 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we

1. Calculate dy/dx: $\quad \dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$

2. Solve the equation $\dfrac{dy}{dx} = 0$ for x:
$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1 \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and -1. The corresponding points on the curve are (0, 2), (1, 1) and $(-1, 1)$. See Fig. 2.18. ❑

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Rule 5 The Product Rule

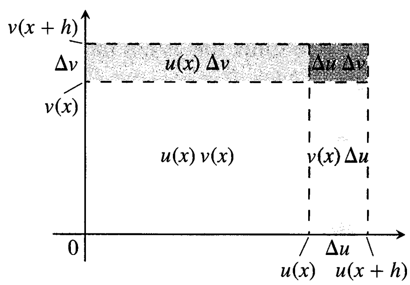
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation,* $(uv)' = uv' + vu'$.

Picturing the product rule

If $u(x)$ and $v(x)$ are positive and increase when x increases, and if $h > 0$,



the total shaded area in the picture is

$$u(x+h)v(x+h) - u(x)v(x) =$$
$$u(x+h)\Delta v + v(x+h)\Delta u - \Delta u \Delta v.$$

Dividing both sides of this equation by h gives

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$
$$= u(x+h)\frac{\Delta v}{h} + v(x+h)\frac{\Delta u}{h} - \Delta u\frac{\Delta v}{h}.$$

As $h \to 0^+$, $\Delta u \cdot \dfrac{\Delta v}{h} \to 0 \cdot \dfrac{dv}{dx} = 0$, leaving

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of Rule 5

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right] \\
&= \lim_{h\to 0} u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}.
\end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ❑

EXAMPLE 7 Find the derivative of $y = (x^2+1)(x^3+3)$.

Solution From the Product Rule with $u = x^2+1$ and $v = x^3+3$, we find

$$\begin{aligned}
\frac{d}{dx}[(x^2+1)(x^3+3)] &= (x^2+1)(3x^2) + (x^3+3)(2x) \\
&= 3x^4 + 3x^2 + 2x^4 + 6x \\
&= 5x^4 + 3x^2 + 6x.
\end{aligned}$$ ❑

Example 7 can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check:

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$
$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation.

There are times, however, when the Product Rule *must* be used. In the following example, we have only numerical values to work with.

EXAMPLE 8 Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \qquad u'(2) = -4, \qquad v(2) = 1, \qquad \text{and} \qquad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

we have

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) = 6 - 4 = 2. \end{aligned}$$

❑

Quotients

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives. What happens instead is this:

Rule 6 The Quotient Rule

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof of Rule 6

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} \\ &= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h\to 0} \frac{v(x)\dfrac{u(x+h) - u(x)}{h} - u(x)\dfrac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limit in the numerator and denominator now gives the Quotient Rule.

❑

EXAMPLE 9 Find the derivative of $y = \dfrac{t^2 - 1}{t^2 + 1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}.$$

❑

The Power Rule for Negative Integers

The Power Rule for negative integers is the same as the rule for positive integers.

> **Rule 7 Power Rule for Negative Integers**
>
> If n is a negative integer and $x \neq 0$, then
>
> $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 The proof uses the Quotient Rule in a clever way. If n is a negative integer, then $n = -m$ where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$ and

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2} \qquad \text{Quotient Rule with } u = 1 \text{ and } v = x^m$$

$$= \frac{0 - mx^{m-1}}{x^{2m}} \qquad \text{Since } m > 0,\ \frac{d}{dx}(x^m) = mx^{m-1}$$

$$= -mx^{-m-1}$$

$$= nx^{n-1}. \qquad \text{Since } -m = n$$

❑

EXAMPLE 10

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

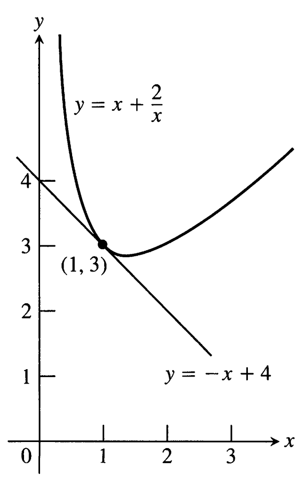
$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

❑

EXAMPLE 11 Find an equation for the tangent to the curve

$$y = x + \frac{2}{x}$$

at the point (1, 3) (Fig. 2.19).



2.19 The tangent to the curve $y = x + (2/x)$ at (1, 3). The curve has a third-quadrant portion not shown here. We will see how to graph functions like this in Chapter 3.

Solution The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1.$$

The line through (1, 3) with slope $m = -1$ is

$$\begin{aligned} y - 3 &= (-1)(x - 1) && \text{Point-slope equation} \\ y &= -x + 1 + 3 \\ y &= -x + 4. \end{aligned}$$

❑

Choosing Which Rules to Use

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 12 Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned} \frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}. \end{aligned}$$

❑

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is the **first (first order) derivative** of y with respect to x. This derivative may itself be a differentiable function of x; if so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

is called the **second (second order) derivative** of y with respect to x.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third (third order) derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)}$$

denoting the ***n*th (*n*th order) derivative** of y with respect to x, for any positive integer n.

Notice that

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

does not mean multiplication. It means "the derivative of the derivative."

How to read the symbols for derivatives

y'	"y prime"	y''	"y double prime"
$\dfrac{d^2y}{dx^2}$	"d squared y dx squared"		
y'''	"y triple prime"		
$y^{(n)}$	"y super n"		
$\dfrac{d^ny}{dx^n}$	"d to the n of y by dx to the n"		

EXAMPLE 13 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ❑

Exercises 2.2

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = x^2 + x + 8$

3. $s = 5t^3 - 3t^5$

4. $w = 3z^7 - 7z^3 + 21z^2$

5. $y = \dfrac{4x^3}{3} - x$

6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$

7. $w = 3z^{-2} - \dfrac{1}{z}$

8. $s = -2t^{-1} + \dfrac{4}{t^2}$

9. $y = 6x^2 - 10x - 5x^{-2}$

10. $y = 4 - 2x - x^{-3}$

11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$

12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$

14. $y = (x - 1)(x^2 + x + 1)$

15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$

16. $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \dfrac{2x + 5}{3x - 2}$

18. $z = \dfrac{2x + 1}{x^2 - 1}$

19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$

20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$

22. $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$

24. $u = \dfrac{5x + 1}{2\sqrt{x}}$

25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$

26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$

28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29 and 30.

29. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

30. $y = \dfrac{x^5}{120}$

Find the first and second derivatives of the functions in Exercises 31–38.

31. $y = \dfrac{x^3 + 7}{x}$

32. $s = \dfrac{t^2 + 5t - 1}{t^2}$

33. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

34. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$

35. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$

36. $w = (z + 1)(z - 1)(z^2 + 1)$

37. $p = \left(\dfrac{q^2 + 3}{12q}\right)\left(\dfrac{q^4 - 1}{q^3}\right)$

38. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

Using Numerical Values

39. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c)** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

40. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c)** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

Slopes and Tangents

41. a) Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

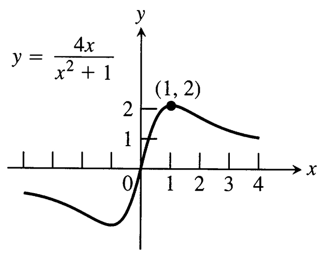
b) What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c) Find equations for the tangents to the curve at the points where the slope of the curve is 8.

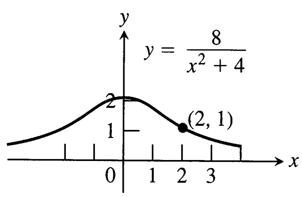
42. a) Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

b) What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

43. Find the tangents to *Newton's Serpentine* (graphed here) at the origin and the point (1, 2).



44. Find the tangent to the *Witch of Agnesi* (graphed here) at the point (2, 1). There is a nice story about the name of this curve in the marginal note on Agnesi in Section 9.4.



45. The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $y = x$ at the origin. Find a, b, and c.

46. The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point (1, 0). Find a, b, and c.

47. a) Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

b) GRAPHER Graph the curve and tangent line together. The tangent intersects the curve at another point. Use ZOOM and TRACE to estimate the point's coordinates.

c) GRAPHER Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (SOLVER key).

48. a) Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

b) GRAPHER Graph the curve and tangent together. The tangent intersects the curve at another point. Use ZOOM and TRACE to estimate the point's coordinates.

c) GRAPHER Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (SOLVER key).

Physical Applications

49. *Pressure and volume.* If the gas in a closed container is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV.

50. *The body's reaction to medicine.* The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine. In Section 3.6, we will see how to find the amount of medicine to which the body is most sensitive. (Source: *Some Mathematical Models in Biology,* Revised Edition, R. M Thrall, J. A. Mortimer, K. R. Rebman, R. F. Baum, eds., December 1967, PB-202 364, p. 221; distributed by NTIS, U.S. Department of Commerce.)

Theory and Examples

51. Suppose that the function v in the Product Rule has a constant value c. What does the Product Rule then say? What does this say about the Constant Multiple Rule?

52. *The Reciprocal Rule*

a) The **Reciprocal Rule** says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Quotient Rule.

b) Show that the Reciprocal Rule and the Product Rule together imply the Quotient Rule.

53. *Another proof of the Power Rule for positive integers.* Use the algebra formula

$$x^n - c^n = (x - c)(x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1})$$

together with the derivative formula

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

from Exercises 2.1 to show that $(d/dx)(x^n) = nx^{n-1}$.

54. *Generalizing the Product Rule.* The Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for the derivative of the product uv of two differentiable functions of x.

a) What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x?

b) What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of x?

c) What is the formula for the derivative of a product $u_1u_2u_3 \cdots u_n$ of a finite number n of differentiable functions of x?

55. *Rational Powers*

a) Find $\frac{d}{dx}(x^{3/2})$ by writing $x^{3/2}$ as $x \cdot x^{1/2}$ and using the Product Rule. Express your answer as a rational number times a rational power of x. Work parts (b) and (c) by a similar method.

b) Find $\frac{d}{dx}(x^{5/2})$.

c) Find $\frac{d}{dx}(x^{7/2})$.

d) What patterns do you see in your answers to (a), (b), and (c)? Rational powers are one of the topics in Section 2.6.

2.3 Rates of Change

In this section we examine some applications in which derivatives are used to represent and interpret the rates at which things change in the world around us. It is natural to think of change in terms of dependence on time, such as the position, velocity, and acceleration of a moving object, but there is no need to be so restrictive. Change with respect to variables other than time can be treated in the same way. For example, a physician may want to know how small changes in dosage can affect the body's response to a drug. An economist may want to study how investment changes with respect to variations in interest rates. These questions can all be expressed in terms of the rate of change of a function with respect to a variable.

Average and Instantaneous Rates of Change

We start by recalling the concept of average rate of change of a function over an interval, introduced in Section 1.1. The derivative of the function is the limit of this average rate as the length of the interval goes to zero.

Definitions

The **average rate of change** of a function $f(x)$ with respect to x over the interval from x_0 to $x_0 + h$ is

$$\text{Average rate of change} = \frac{f(x_0 + h) - f(x_0)}{h}.$$

The **(instantaneous) rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change,* we mean *instantaneous rate of change.*

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

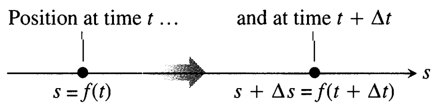
$$A = \frac{\pi}{4} D^2.$$

How fast is the area changing with respect to the diameter when the diameter is 10 m?

Solution The (instantaneous) rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing at rate $(\pi/2)10 = 5\pi\ \mathrm{m}^2/\mathrm{m}$. This means that a small change ΔD m in the diameter would result in a change of about $5\pi\,\Delta D$ m^2 in the area of the circle. ❑



2.20 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

Motion Along a Line—Displacement, Velocity, Speed, and Acceleration

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Fig. 2.20) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

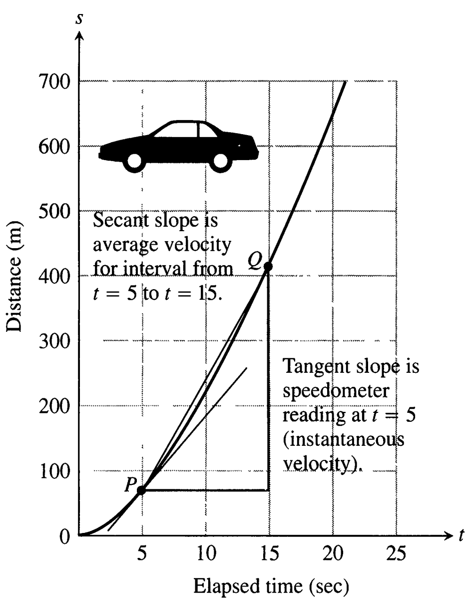
$$v_{\mathrm{av}} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

Definition

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

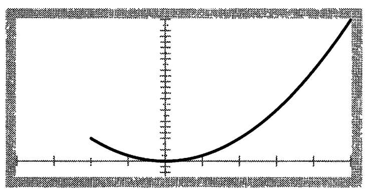


2.21 The time-to-distance data for Example 2.

EXAMPLE 2 Figure 2.21 shows a distance–time graph of a 1994 Ford Mustang Cobra. The slope of the secant PQ is the average velocity for the 10-sec interval from $t = 5$ to $t = 15$ sec, in this case 35.5 m/sec or 128 km/h. The slope of the tangent at P is the speedometer reading at $t = 5$ sec, about 20 m/sec or 72 km/h. The car's top speed is 220 km/h (about 137 mph). (Source: *Car & Driver,* April 1994.) ❑

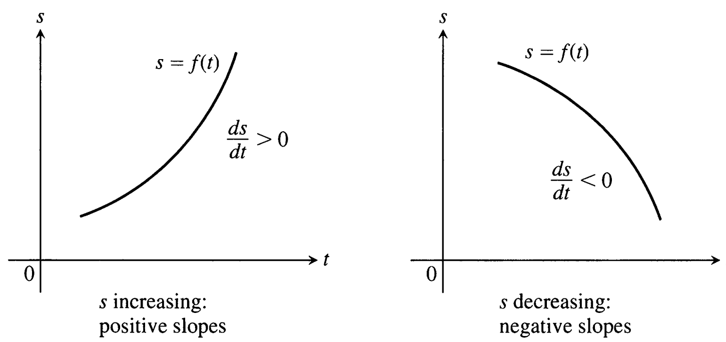
Technology *Parametric Functions* To graph curves $y = f(x)$, where y is a function of x, your graphing utility should be set in *function mode*. Not all curves can be represented in that mode, so most graphing utilities have a *parametric mode* as well. In this mode you plot the points $(x(t), y(t))$ whose coordinates are functions of the varying "time" parameter t. Thus you can think of the curve as the path of a moving particle as it changes its (x, y) position over time (see Section 9.4). A curve $y = f(x)$ can be graphed in parametric mode using the equations $x = t$, $y = f(t)$. Set your graphing utility to parametric mode and try the following equations.

Relation	Parametrization
$y = x^2$ (y a function of x)	$x(t) = t, \quad y(t) = t^2, \quad -\infty < t < \infty$
$x^2 + y^2 = 4$ (y not a function of x)	$x(t) = 2\cos t, \quad y(t) = 2\sin t,$ $0 \le t \le 2\pi$



The parabola $x(t) = t$, $y(t) = t^2$, for $t \ge -2$

Besides telling us how fast the object is moving, the velocity also tells us in what direction it is moving. When the object is moving forward (s increasing) the velocity is positive; when the body is moving backward (s decreasing) the velocity is negative (Fig. 2.22).



2.22 $v = ds/dt$ is positive when s increases and negative when s decreases.

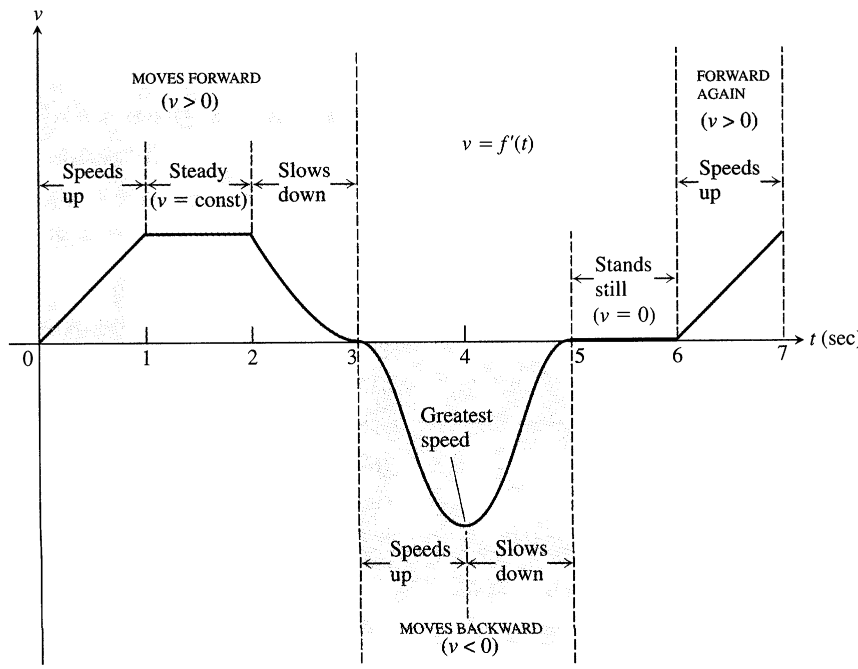
If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows speed, which is the absolute value of velocity. Speed measures the rate of forward progress regardless of direction.

Definition

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 3 Figure 2.23 shows the velocity $v = f'(t)$ of a particle moving on a coordinate line. The particle moves forward for the first 3 seconds, moves backward for the next 2 seconds, stands still for a second, and moves forward again. Notice that the particle achieves its greatest speed at time $t = 4$, while moving backward.



2.23 The velocity graph for Example 3.

❑

The rate at which a body's velocity changes is called the body's acceleration. The acceleration measures how quickly the body picks up or loses speed.

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

We can illustrate all this with free fall. As we mentioned at the beginning of Chapter 1, near the surface of the earth all bodies fall with the same constant acceleration. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls **free fall.** The mathematical description of this type of motion captured the imagination of many great scientists, including Aristotle, Galileo, and Newton. Experimental and theoretical investigations revealed that the distance a body released from rest falls in time t is proportional to the square of the amount of time it has fallen. We express this by saying that

$$s = \frac{1}{2}gt^2,$$

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, but it closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before air resistance starts to slow them down.

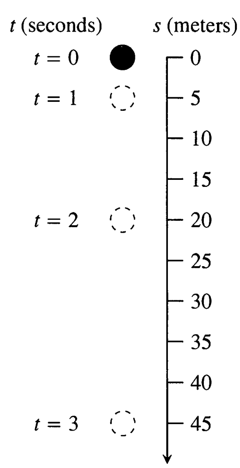
The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s. With t in seconds (the usual unit), we have the following values:

Free-Fall Equations (Earth)

English units:	$g = 32\dfrac{\text{ft}}{\text{sec}^2}$,	$s = \dfrac{1}{2}(32)t^2 = 16t^2$	(s in feet)
Metric units:	$g = 9.8\dfrac{\text{m}}{\text{sec}^2}$,	$s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$	(s in meters)

The abbreviation ft/sec^2 is read "feet per second squared" or "feet per second per second," and m/sec^2 is read "meters per second squared."

This description allows us to answer many questions concerning the position and velocity of a falling object.



2.24 A ball bearing falling from rest (Example 4).

EXAMPLE 4 Figure 2.24 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

a) How many meters does the ball fall in the first 2 sec?
b) What is its velocity, speed, and acceleration then?

Solution

a) The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6 \text{ m}.$$

b) At any time t, *velocity* is the derivative of displacement:

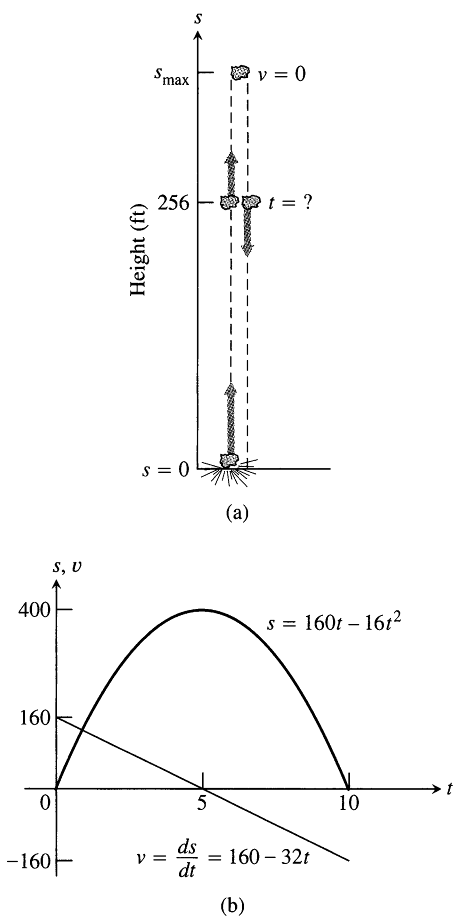
$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 2$, the velocity is

$$v(2) = 19.6 \text{ m/sec}$$

in the downward (increasing s) direction. The *speed* at $t = 2$ is

$$\text{speed} = |v(2)| = 19.6 \text{ m/sec}.$$



2.25 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: it is a plot of height vs. time. The slope is the rock's velocity.

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/sec}^2.$$

At $t = 2$, the acceleration is 9.8 m/sec². ❑

EXAMPLE 5 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Fig. 2.25a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

a) How high does the rock go?
b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? on the way down?
c) What is the acceleration of the rock at any time t during its flight (after the blast)?
d) When does the rock hit the ground again?

Solution

a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. Therefore, to find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0, \qquad \text{or} \qquad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Fig. 2.25(b).

b) To find the rock's velocity at 256 ft on the way up and again on the way down, we find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation we write

$$16t^2 - 160t + 256 = 0$$
$$16(t^2 - 10t + 16) = 0$$
$$(t - 2)(t - 8) = 0$$
$$t = 2 \text{ sec}, \quad t = 8 \text{ sec}.$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec},$$
$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec.

c) At any time during its flight following the explosion, the rock's acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. When the rock is rising, it is slowing down; when it is falling, it is speeding up.

d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$ the blast occurred and the rock was thrown upward. It returned to the ground 10 seconds later. ❑

Technology *Simulation of Motion on a Vertical Line* The parametric equations

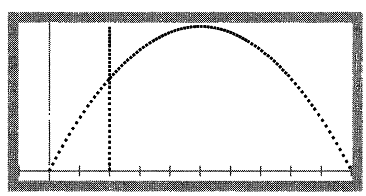
$$x(t) = c, \qquad y(t) = f(t)$$

will illuminate pixels along the vertical line $x = c$. If $f(t)$ denotes the height of a moving body at time t, graphing $(x(t), y(t)) = (c, f(t))$ will simulate the actual motion. Try it for the rock in Example 5 with $x(t) = 2$, say, and $y(t) = 160t - 16t^2$, in dot mode with tStep $= 0.1$. Why does the spacing of the dots vary? Why does the grapher seem to stop after it reaches the top? (Try the plots for $0 \le t \le 5$ and $5 \le t \le 10$ separately.)

For a second experiment, plot the parametric equations

$$x(t) = t, \qquad y(t) = 160t - 16t^2$$

together with the vertical line simulation of the motion, again in dot mode. Use what you know about the behavior of the rock from the calculations of Example 5 to select a window size that will display all the interesting behavior.



$\begin{cases} x(t) = 2 \\ y(t) = 160t - 16t^2 \end{cases}$

and

$\begin{cases} x(t) = t \\ y(t) = 160t - 16t^2 \end{cases}$

in dot mode

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of the sensitivity to change at x.

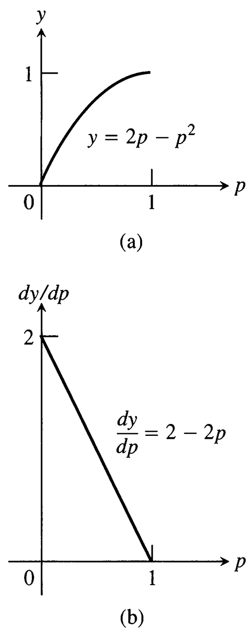
Why peas wrinkle

British geneticists have recently discovered that the wrinkling trait comes from an extra piece of DNA that prevents the gene that directs starch synthesis from functioning properly. With the plant's starch conversion impaired, sucrose and water build up in the young seeds. As the seeds mature, they lose much of this water, and the shrinkage leaves them wrinkled.

EXAMPLE 6 *Sensitivity to change*

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization. His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the population at large is

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$



2.26 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas. (b) The graph of dy/dp.

The graph of y versus p in Fig. 2.26(a) suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this is borne out by the derivative graph in Fig. 2.26(b), which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

We will say more about sensitivity in Section 3.7. ❑

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals.*

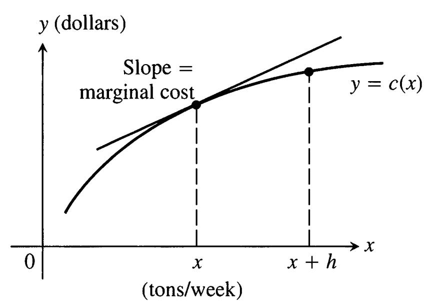
In a manufacturing operation, the *cost of production* $c(x)$ is a function of x, the number of units produced. The *marginal cost of production* is the rate of change of cost (c) with respect to level of production (x), so it is dc/dx.

For example, let $c(x)$ represent the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ units, and the cost difference, divided by h, is the average increase in cost per ton per week:

$$\frac{c(x+h) - c(x)}{h} = \begin{array}{l}\text{average increase in cost/ton/wk}\\ \text{to produce the next } h \text{ tons of steel}\end{array}$$

The limit of this ratio as $h \to 0$ is the *marginal cost* of producing more steel when the current production level is x tons (Fig. 2.27):

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production.}$$

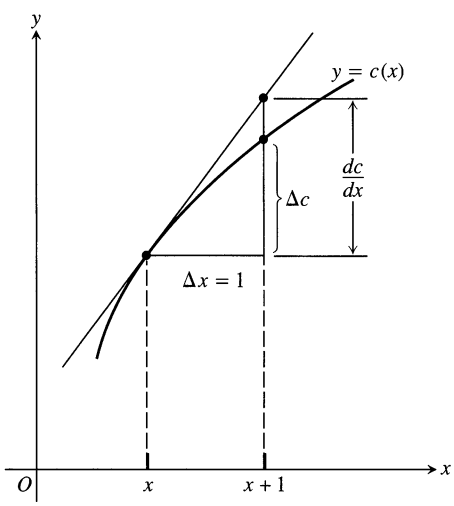


2.27 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x+h) - c(x)$.

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximately the value of dc/dx at x. To see why this is an acceptable approximation, observe that if the slope of c does not change quickly near x, then the difference quotient will be close to its limit, the derivative dc/dx, even if $\Delta x = 1$ (Fig. 2.28). In practice, the approximation works best for large values of x.



2.28 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

EXAMPLE 7 *Marginal cost*

Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. ❑

Choosing functions to illustrate economics

In case you are wondering why economists use polynomials of low degree to illustrate complicated phenomena like cost and revenue, here is the rationale: While formulas for real phenomena are rarely available in any given instance, the theory of economics can still provide valuable guidance. The functions about which theory speaks can often be illustrated with low degree polynomials on relevant intervals. Cubic polynomials provide a good balance between being easy to work with and being complicated enough to illustrate important points.

EXAMPLE 8 *Marginal tax rate*

To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1,000, you can expect to have to pay an extra \$280 in income taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I, the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 out of every extra dollar you earn in taxes. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ❑

EXAMPLE 9 *Marginal revenue*

If

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x thousand candy bars, $5 \le x \le 20$, the marginal revenue when x thousand are sold is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

As with marginal cost, the marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 thousand candy bars a week, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 thousand bars a week. ❑

Exercises 2.3

Motion Along a Coordinate Line

Exercises 1–6 give the position $s = f(t)$ of a body moving on a coordinate line for $a \le t \le b$, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.

b) Find the body's speed and acceleration at the endpoints of the interval.

c) When during the interval does the body change direction (if ever)?

1. $s = 0.8t^2$, $0 \le t \le 10$ (free fall on the moon)

2. $s = 1.86t^2$, $0 \le t \le 0.5$ (free fall on Mars)

3. $s = -t^3 + 3t^2 - 3t$, $0 \le t \le 3$

4. $s = (t^4/4) - t^3 + t^2$, $0 \le t \le 2$

5. $s = \dfrac{25}{t^2} - \dfrac{5}{t}, \quad 1 \le t \le 5$

6. $s = \dfrac{25}{t+5}, \quad -4 \le t \le 0$

7. At time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t$ m. (a) Find the body's acceleration each time the velocity is zero. (b) Find the body's speed each time the acceleration is zero. (c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

8. At time $t \ge 0$, the velocity of a body moving along the s-axis is $v = t^2 - 4t + 3$. (a) Find the body's acceleration each time the velocity is zero. (b) When is the body moving forward? moving backward? (c) When is the body's velocity increasing? decreasing?

Free-Fall Applications

9. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars, $s = 11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec (about 100 km/h) on each planet?

10. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

    a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
    b) How long does it take the rock to reach its highest point?
    c) How high does the rock go?
    d) How long does it take the rock to reach half its maximum height?
    e) How long is the rock aloft?

11. On Earth, in the absence of air, the rock in Exercise 10 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds.

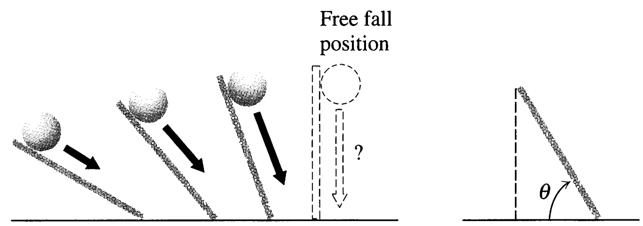
    a) Find the rock's velocity and acceleration at time t.
    b) How long would it take the rock to reach its highest point?
    c) How high would the rock go?
    d) How long would it take the rock to reach half its maximum height?
    e) How long would the rock be aloft?

12. Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. Because the acceleration of gravity at the planet's surface was g_s m/sec^2, the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ meters t seconds later. The ball bearing reached its maximum height 20 sec after being launched. What was the value of g_s?

13. A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t seconds. How long will the bullet be aloft in each case? How high would the bullet go?

14. *(Continuation of Exercise 13.)* On Jupiter, in the absence of air, the bullet's height would be $s = 832t - 37.53t^2$ feet after t seconds. On Mars it would be $s = 832t - 6.1t^2$ feet after t seconds. How high would the bullet go in each case?

15. *Galileo's free-fall formula.* Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely (part a of the accompanying figure). He found that, for any given angle of the plank, the ball's velocity t seconds into the motion was a constant multiple of t. That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.



(a) (b)

In modern notation (part b of the figure), with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ, the ball's velocity t seconds into the roll was

$$v = 9.8(\sin\theta)t \text{ m/sec.}$$

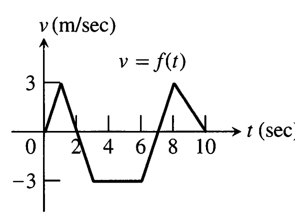
a) What is the equation for the ball's velocity during free fall?
b) Building on your work in (a), what constant acceleration does a freely falling body experience near the surface of the earth?

16. *Free fall from the tower of Pisa.* Had Galileo dropped a cannonball from the tower of Pisa, 179 ft above the ground, the ball's height aboveground t seconds into the fall would have been $s = 179 - 16t^2$.

    a) What would have been the ball's velocity, speed, and acceleration at time t?
    b) About how long would it have taken the ball to hit the ground?
    c) What would have been the ball's velocity at the moment of impact?

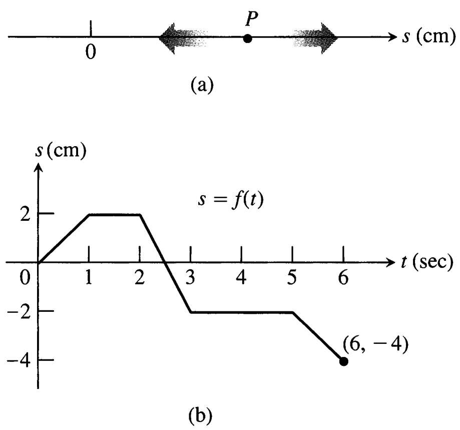
Conclusions About Motion from Graphs

17. The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



a) When does the body reverse direction?
b) When (approximately) is the body moving at a constant speed?
c) Graph the body's speed for $0 \le t \le 10$.
d) Graph the acceleration, where defined.

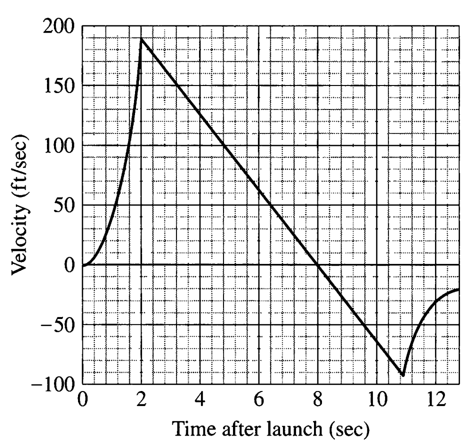
18. A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



a) When is P moving to the left? moving to the right? standing still?
b) Graph the particle's velocity and speed (where defined).

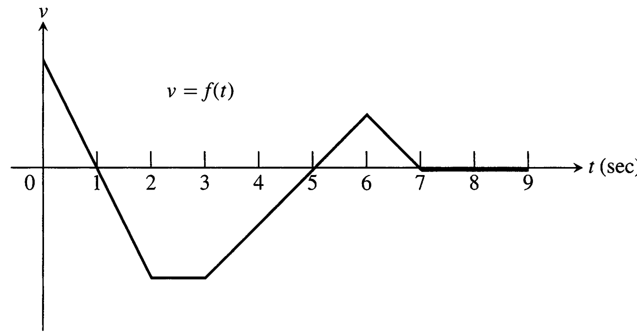
19. When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.



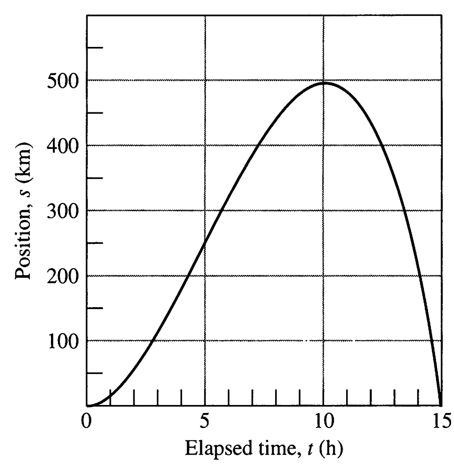
a) How fast was the rocket climbing when the engine stopped?
b) For how many seconds did the engine burn?
c) When did the rocket reach its highest point? What was its velocity then?
d) When did the parachute pop out? How fast was the rocket falling then?
e) How long did the rocket fall before the parachute opened?
f) When was the rocket's acceleration greatest?
g) When was the acceleration constant? What was its value then (to the nearest integer)?

20. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.



a) When does the particle move forward? move backward? speed up? slow down?
b) When is the particle's acceleration positive? negative? zero?
c) When does the particle move at its greatest speed?
d) When does the particle stand still for more than an instant?

21. The graph here shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$.



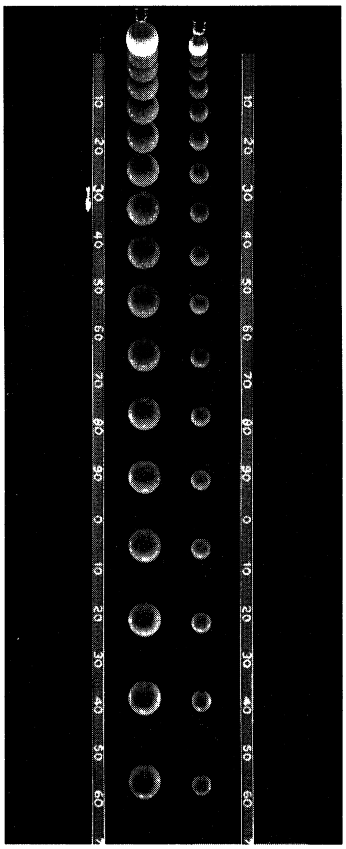
a) Use the technique described in Section 2.1, Example 4, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.
b) Suppose $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2 and compare your graphs with those in (a).

22. The multiflash photograph in Fig. 2.29 on the following page shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.

a) How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

b) How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

c) About how fast was the light flashing (flashes per second)?



2.29 Two balls falling from rest (Exercise 22).

23. The graphs in Fig. 2.30 show the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.

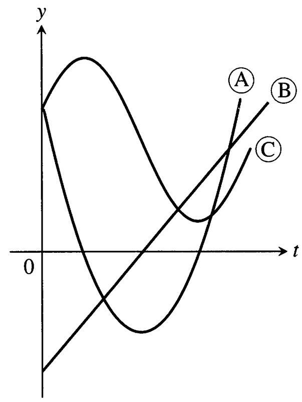
24. The graphs in Fig. 2.31 show the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along the coordinate line as functions of time t. Which graph is which? Give reasons for your answers.

Economics

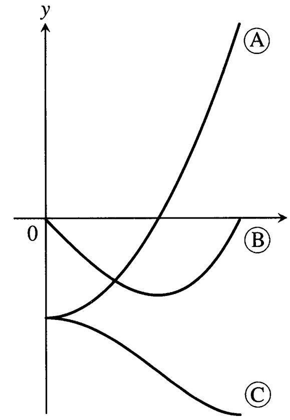
25. *Marginal cost.* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a) Find the average cost per machine of producing the first 100 washing machines.

b) Find the marginal cost when 100 washing machines are produced.



2.30 The graphs for Exercise 23.



2.31 The graphs for Exercise 24.

c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

26. *Marginal revenue.* Suppose the revenue from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right)$$

dollars.

a) Find the marginal revenue when x desks are produced.

b) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 5 desks a week to 6 desks a week.

c) Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

Additional Applications

27. When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at (a) $t = 0$; (b) $t = 5$; and (c) $t = 10$ hours.

28. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

29. It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

a) Find the rate dy/dt (m/h) at which the tank is draining at time t.

b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

c) GRAPHER Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

30. The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

a) At what rate does the volume change with respect to the radius when $r = 2$ ft?

b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

31. Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. If the aircraft will become airborne when its speed reaches 200 km/hr, how long will it take to become airborne, and what distance will it travel in that time?

32. *Volcanic lava fountains.* Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour?

(*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0 t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Grapher Explorations

Exercises 33–36 give the position function $s = f(t)$ of a body moving along the s-axis as a function of time t. Graph f together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the body's behavior in relation to the signs and values v and a. Include in your commentary such topics as the following.

a) When is the body momentarily at rest?

b) When does it move to the left (down) or to the right (up)?

c) When does it change direction?

d) When does it speed up and slow down?

e) When is it moving fastest (highest speed)? slowest?

f) When is it farthest from the axis origin?

33. $s = 200t - 16t^2, \quad 0 \le t \le 12.5$ (A heavy object fired straight up from the earth's surface at 200 ft/sec)

34. $s = t^2 - 3t + 2, \quad 0 \le t \le 5$

35. $s = t^3 - 6t^2 + 7t, \quad 0 \le t \le 4$

36. $s = 4 - 7t + 6t^2 - t^3, \quad 0 \le t \le 4$

2.4 Derivatives of Trigonometric Functions

Trigonometric functions are important because so many of the phenomena we want information about are periodic (electromagnetic fields, heart rhythms, tides, weather). A surprising and beautiful theorem from advanced calculus says that every periodic function we are likely to use in mathematical modeling can be written as an algebraic combination of sines and cosines, so the derivatives of sines and cosines play a key role in describing important changes. This section shows how to differentiate the six basic trigonometric functions.

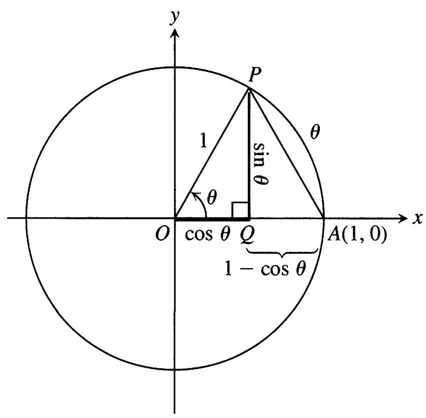
Some Special Limits

Our first step is to establish some inequalities and limits. It is assumed throughout that angles are measured in radians.

Theorem 3

If θ is measured in radians, then

$$-|\theta| < \sin\theta < |\theta| \qquad \text{and} \qquad -|\theta| < 1 - \cos\theta < |\theta|.$$



2.32 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 < \theta^2$.

Proof To establish these inequalities, we picture θ as an angle in standard position (Fig. 2.32). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 < \theta^2. \tag{1}$$

The terms on the left side of Eq. (1) are both positive, so each is smaller than their sum and hence is less than θ^2:

$$\sin^2\theta < \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 < \theta^2.$$

By taking square roots, we can see that this is equivalent to saying that

$$|\sin\theta| < |\theta| \qquad \text{and} \qquad |1 - \cos\theta| < |\theta|$$

or

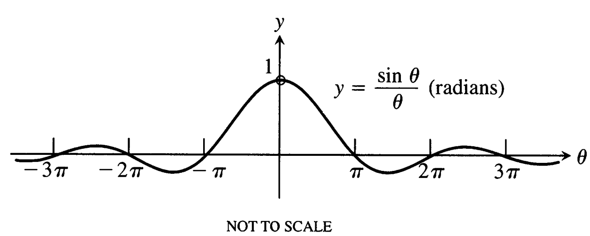
$$-|\theta| < \sin\theta < |\theta| \qquad \text{and} \qquad -|\theta| < 1 - \cos\theta < |\theta|. \qquad \square$$

EXAMPLE 1 Show that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$. That is,

$$\lim_{\theta\to 0} \sin\theta = 0 \qquad \text{and} \qquad \lim_{\theta\to 0} \cos\theta = 1.$$

Solution As $\theta \to 0$, both $|\theta|$ and $-|\theta|$ approach 0. The values of the limits therefore follow immediately from Theorem 3 and the Sandwich Theorem. $\square$

The function $f(\theta) = (\sin\theta)/\theta$ graphed in Fig. 2.33 appears to have a removable discontinuity at $\theta = 0$. As the figure suggests, $\lim_{\theta\to 0} f(\theta) = 1$.



2.33 The graph of $f(\theta) = (\sin\theta)/\theta$.

Theorem 4

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{2}$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with values of θ that are positive and less than $\pi/2$ (Fig. 2.34). Notice that

$$\text{Area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT.$$

Equation (3) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

We can express these areas in terms of θ as follows:

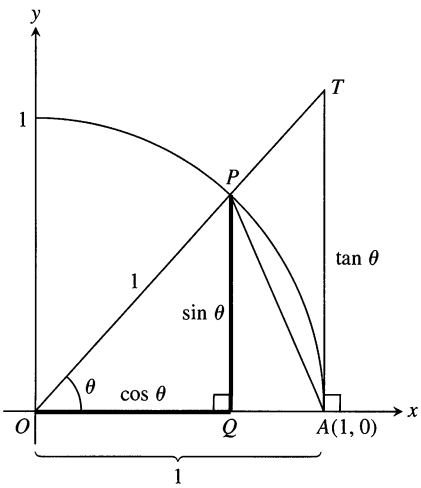
$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{3}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta,$$

so

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$



2.34 The figure for the proof of Theorem 4. *TA*/*OA* = $\tan\theta$, but *OA* = 1, so *TA* = $\tan\theta$.

This last inequality will go the same way if we divide all three terms by the positive number $(1/2)\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

We next take reciprocals, which reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

Finally, observe that $\sin\theta$ and θ are both *odd functions.* Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function,* with a graph symmetric about the y-axis (see Fig. 2.33). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 5 of Section 1.4. ❑

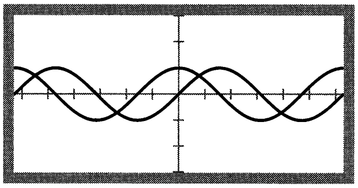
Theorem 4 can be combined with limit rules and known trigonometric identities to yield other trigonometric limits.

EXAMPLE 2 Show that $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$

Solution Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = h/2. \\ &= -(1)(0) = 0.\end{aligned}$$

❑



$y_1 = \sin x, -2\pi \le x \le 2\pi$
$y_2 = d(y_1)/dx, -2\pi \le x \le 2\pi$

Technology *Conjectures Based on Grapher Images* What you see in the window of a graphing utility can suggest conjectures, sometimes rather strongly. Graph the functions

$$y_1 = \sin x$$
$$y_2 = d(y_1)/dx \qquad \text{(This is computed by a built-in differentiation utility.)}$$

Does the graph of y_2 look familiar? What function do you think it is? Test your conjecture by adding the function's graph to the screen.

The Derivative of the Sine

To calculate the derivative of $y = \sin x$, we combine the limits in Example 2 and Theorem 4 with the addition formula

$$\sin(x+h) = \sin x \cos h + \cos x \sin h. \tag{4}$$

We have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition}\\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Eq. (4)}\\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}\\
&= \lim_{h\to 0}\left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right)\\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 2 and Theorem 4}\\
&= \cos x.
\end{aligned}$$

In short, the derivative of the sine is the cosine.

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

EXAMPLE 3

a) $y = x^2 - \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule}\\
&= 2x - \cos x
\end{aligned}$$

b) $y = x^2 \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule}\\
&= x^2 \cos x + 2x \sin x
\end{aligned}$$

Radian measure in calculus

In case you are wondering why calculus uses radian measure when the rest of the world seems to use degrees, the answer lies in the argument that the derivative of the sine is the cosine. The derivative of $\sin x$ is $\cos x$ *only* if x is measured in radians. The argument requires that when h is a small increment in x,

$$\lim_{h\to 0} (\sin h)/h = 1.$$

This is true only for radian measure, as we saw during the proof of Theorem 4. You will see what the degree-mode derivatives of the sine and cosine are if you do Exercise 76.

c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x\cos x - \sin x}{x^2}$$

❑

The Derivative of the Cosine

With the help of the addition formula,

$$\cos(x+h) = \cos x \cos h - \sin x \sin h, \tag{5}$$

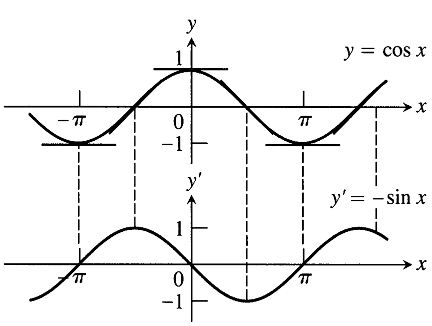
we have

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition}\\
&= \lim_{h\to 0} \frac{(\cos x\cos h - \sin x \sin h) - \cos x}{h} && \text{Eq. (5)}\\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}\\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h}\\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h}\\
&= \cos x \cdot 0 - \sin x \cdot 1 && \text{Example 2 and Theorem 4}\\
&= -\sin x.
\end{aligned}$$

In short, the derivative of the cosine is the negative of the sine.

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Figure 2.35 shows another way to visualize this result.



2.35 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

EXAMPLE 4

a) $y = 5x + \cos x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) && \text{Sum Rule}\\
&= 5 - \sin x
\end{aligned}$$

b) $y = \sin x \cos x$

$$\begin{aligned}
\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule}\\
&= \sin x(-\sin x) + \cos x(\cos x)\\
&= \cos^2 x - \sin^2 x
\end{aligned}$$

c)

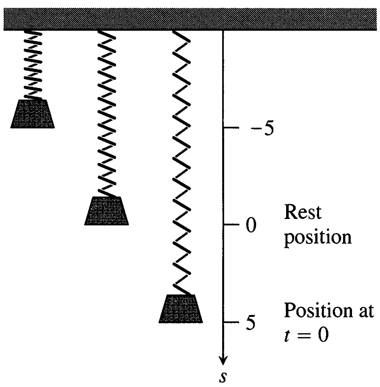
$$y = \frac{\cos x}{1 - \sin x}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 - \sin x)\dfrac{d}{dx}(\cos x) - \cos x \dfrac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\
&= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\
&= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\
&= \frac{1}{1 - \sin x}
\end{aligned}$$

❑

Simple Harmonic Motion

The motion of a body bobbing up and down on the end of a spring is an example of *simple harmonic motion.* The next example describes a case in which there are no opposing forces like friction or buoyancy to slow the motion down.



2.36 The body in Example 5.

EXAMPLE 5 A body hanging from a spring (Fig. 2.36) is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5\cos t.$$

What are its velocity and acceleration at time t?

Solution We have

Position: $s = 5\cos t$

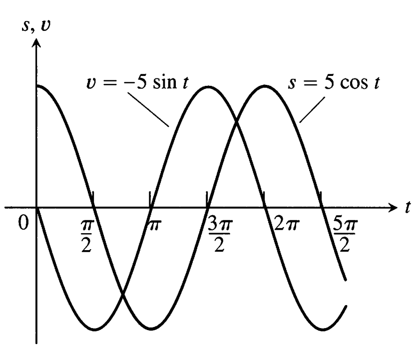
Velocity: $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(5\cos t) = 5\dfrac{d}{dt}(\cos t) = -5\sin t$

Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(-5\sin t) = -5\dfrac{d}{dt}(\sin t) = -5\cos t.$

Here is what we can learn from these equations:

1. As time passes, the body moves up and down between $s = 5$ and $s = -5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π, the period of $\cos t$.
2. The function $\sin t$ attains its greatest magnitude (1) when $\cos t = 0$, as the graphs of the sine and cosine show (Fig. 2.37). Hence, the body's speed, $|v| = 5|\sin t|$, is greatest every time $\cos t = 0$, i.e., every time the body passes its rest position.

   The body's speed is zero when $\sin t = 0$. This occurs at the endpoints of the interval of motion, when $\cos t = \pm 1$.
3. The acceleration, $a = -5\cos t$, is zero only at the rest position, where the cosine is zero. When the body is anywhere else, the spring is either pulling on it or pushing on it. The acceleration is greatest in magnitude at the points farthest from the origin, where $\cos t = \pm 1$. ❑



2.37 The graphs of the position and velocity of the body in Example 5.

Jerk

A sudden change in acceleration is called a "jerk." When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt. Jerk is what spills your soft drink. The derivative responsible for jerk is d^3s/dt^3.

Definition

Jerk is the derivative of acceleration. If a body's position at time t is $s = f(t)$, the body's jerk at time t is

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Recent tests have shown that motion sickness comes from accelerations whose changes in magnitude or direction take us by surprise. Keeping an eye on the road helps us to see the changes coming. A driver is less likely to become sick than a passenger reading in the backseat.

EXAMPLE 6

a) The jerk of the constant acceleration of gravity ($g = 32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

We don't experience motion sickness if we are just sitting around.

b) The jerk of the simple harmonic motion in Example 5 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t)$$
$$= 5\sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the origin, where the acceleration changes direction and sign. ❑

The Derivatives of the Other Basic Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas.

Notice the minus signs in the derivative formulas for the cofunctions.

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad (6) \qquad\qquad \frac{d}{dx}(\sec x) = \sec x \tan x \qquad (7)$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad (8) \qquad\qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad (9)$$

To show how a typical calculation goes, we derive Eq. (6). The other derivations are left to Exercises 67 and 68.

EXAMPLE 7 Find dy/dx if $y = \tan x$.

Solution

$$\begin{aligned}\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule}\\ &= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

❑

EXAMPLE 8 Find y'' if $y = \sec x$.

Solution

$$\begin{aligned} y &= \sec x\\ y' &= \sec x \tan x && \text{Eq. (7)}\\ y'' &= \frac{d}{dx}(\sec x \tan x)\\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Product Rule}\\ &= \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x)\\ &= \sec^3 x + \sec x \tan^2 x\end{aligned}$$

❑

EXAMPLE 9

a) $\displaystyle \frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \csc^2 x$

b) $\displaystyle \frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\csc x) = 2\frac{d}{dx}(\csc x)$

$$= 2(-\csc x \cot x) = -2\csc x \cot x$$

❑

Continuity of Trigonometric Functions

Since the six basic trigonometric functions are differentiable throughout their domains they are also continuous throughout their domains by Theorem 1, Section 2.1. This means that $\sin x$ and $\cos x$ are continuous for all x, that $\sec x$ and $\tan x$ are continuous except when x is a nonzero integer multiple of $\pi/2$, and that $\csc x$ and $\cot x$ are continuous except when x is an integer multiple of π. For each function, $\lim_{x\to c} f(x) = f(c)$ whenever $f(c)$ is defined. As a result, we can calculate the limits of many algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 10

$$\lim_{x\to 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$ ❑

Other Limits Calculated with Theorem 4

The equation $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ holds no matter how θ may be expressed:

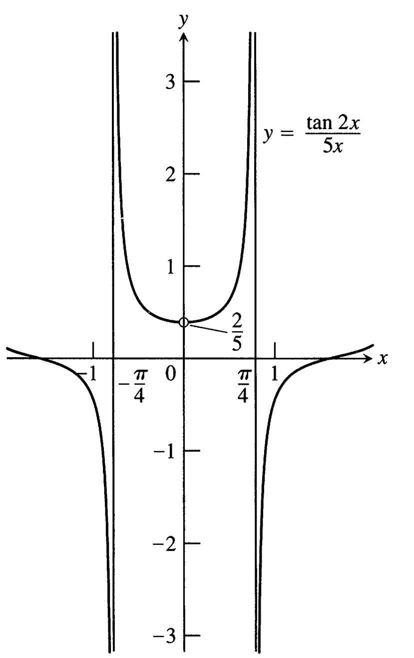
$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \theta = x; \qquad \lim_{x\to 0} \frac{\sin 7x}{7x} = 1, \quad \theta = 7x;$$

As $x \to 0$, $\theta \to 0$ As $x \to 0$, $\theta \to 0$

$$\lim_{x\to 0} \frac{\sin(2/3)x}{(2/3)x} = 1, \quad \theta = (2/3)x$$

As $x \to 0$, $\theta \to 0$

Knowing this helps us calculate related limits involving angles in radian measure.



2.38 The graph of $y = (\tan 2x)/5x$ steps across the y-axis at $y = 2/5$ (Example 11).

EXAMPLE 11

a) $$\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x}$$

Eq. (2) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by 2/5.

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x}$$

Now Eq. (2) applies

$$= \frac{2}{5}(1) = \frac{2}{5}$$

b) $$\lim_{x\to 0} \frac{\tan 2x}{5x} = \lim_{x\to 0} \left(\frac{\sin 2x}{5x} \cdot \frac{1}{\cos 2x}\right)$$

$\tan 2x = \frac{\sin 2x}{\cos 2x}$

$$= \left(\lim_{x\to 0} \frac{\sin 2x}{5x}\right)\left(\lim_{x\to 0} \frac{1}{\cos 2x}\right)$$

$$= \left(\frac{2}{5}\right)\left(\frac{1}{\cos 0}\right) = \frac{2}{5}$$

Part (a) and continuity of $\cos x$

See Fig. 2.38. ❑

Applications

The occurrence of the function $(\sin x)/x$ in calculus is not an isolated event. The function arises in such diverse fields as quantum physics (where it appears in solutions of the wave equation) and electrical engineering (in signal analysis and signal filter design) as well as in the mathematical fields of differential equations and probability theory.

EXAMPLE 12

$$\lim_{t\to(\pi/2)} \frac{\sin\left(t-\frac{\pi}{2}\right)}{t-\frac{\pi}{2}} \qquad \text{Set } \theta = t-(\pi/2). \text{ Then } \theta\to 0 \text{ as } t\to(\pi/2).$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

❑

Exercises 2.4

Derivatives

In Exercises 1–12, find dy/dx.

1. $y = -10x + 3\cos x$

2. $y = \frac{3}{x} + 5\sin x$

3. $y = \csc x - 4\sqrt{x} + 7$

4. $y = x^2\cot x - \frac{1}{x^2}$

5. $y = (\sec x + \tan x)(\sec x - \tan x)$

6. $y = (\sin x + \cos x)\sec x$

7. $y = \frac{\cot x}{1+\cot x}$

8. $y = \frac{\cos x}{1+\sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

11. $y = x^2\sin x + 2x\cos x - 2\sin x$

12. $y = x^2\cos x - 2x\sin x - 2\cos x$

In Exercises 13–16, find ds/dt.

13. $s = \tan t - t$

14. $s = t^2 - \sec t + 1$

15. $s = \frac{1+\csc t}{1-\csc t}$

16. $s = \frac{\sin t}{1-\cos t}$

In Exercises 17–20, find $dr/d\theta$.

17. $r = 4 - \theta^2\sin\theta$

18. $r = \theta\sin\theta + \cos\theta$

19. $r = \sec\theta\csc\theta$

20. $r = (1+\sec\theta)\sin\theta$

In Exercises 21–24, find dp/dq.

21. $p = 5 + \frac{1}{\cot q}$

22. $p = (1+\csc q)\cos q$

23. $p = \frac{\sin q + \cos q}{\cos q}$

24. $p = \frac{\tan q}{1+\tan q}$

25. Find y'' if (a) $y = \csc x$, (b) $y = \sec x$.

26. Find $y^{(4)} = d^4y/dx^4$ if (a) $y = -2\sin x$, (b) $y = 9\cos x$.

Limits

Find the limits in Exercises 27–32.

27. $\lim_{x\to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

28. $\lim_{x\to -\pi/6} \sqrt{1+\cos(\pi\csc x)}$

29. $\lim_{x\to 0} \sec\left[\cos x + \pi\tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$

30. $\lim_{x\to 0} \sin\left(\frac{\pi+\tan x}{\tan x - 2\sec x}\right)$

31. $\lim_{t\to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

32. $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

Find the limits in Exercises 33–48.

33. $\lim_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

34. $\lim_{t\to 0} \frac{\sin kt}{t}$ (k constant)

35. $\lim_{y\to 0} \frac{\sin 3y}{4y}$

36. $\lim_{h\to 0^-} \frac{h}{\sin 3h}$

37. $\lim_{x\to 0} \frac{\tan 2x}{x}$

38. $\lim_{t\to 0} \frac{2t}{\tan t}$

39. $\lim_{x\to 0} \frac{x\csc 2x}{\cos 5x}$

40. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

41. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x\cos x}$

42. $\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$

43. $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t}$

44. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

45. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$

46. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

47. $\lim_{x \to 0} \dfrac{\tan 3x}{\sin 8x}$

48. $\lim_{y \to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

Tangent Lines

In Exercises 49–52, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

49. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$
$x = -\pi,\ 0,\ 3\pi/2$

50. $y = \tan x, \quad -\pi/2 < x < \pi/2$
$x = -\pi/3,\ 0,\ \pi/3$

51. $y = \sec x, \quad -\pi/2 < x < \pi/2$
$x = -\pi/3,\ \pi/4$

52. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$
$x = -\pi/3,\ 3\pi/2$

Do the graphs of the functions in Exercises 53–56 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? You may want to visualize your findings by graphing the functions with a grapher.

53. $y = x + \sin x$

54. $y = 2x + \sin x$

55. $y = x - \cot x$

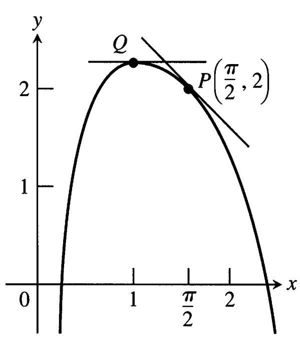
56. $y = x + 2\cos x$

57. Find all points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

58. Find all points on the curve $y = \cot x, 0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

In Exercises 59 and 60, find an equation for (a) the tangent to the curve at P and (b) the horizontal tangent to the curve at Q.

59.



$y = 4 + \cot x - 2\csc x$
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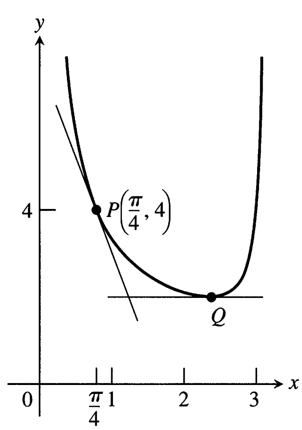
60.



$y = 1 + \sqrt{2}\csc x + \cot x$
(Generated by Mathematica)

Simple Harmonic Motion

The equations in Exercises 61 and 62 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

61. $s = 2 - 2\sin t$

62. $s = \sin t + \cos t$

Theory and Examples

63. Is there a value of c that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \ne 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

64. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$$

continuous at $x = 0$? differentiable at $x = 0$? Give reasons for your answers.

65. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$

66. Find $\dfrac{d^{725}}{dx^{725}}(\sin x)$

67. Derive the formula for the derivative with respect to x of

a) $\sec x$

b) $\csc x$.

68. Derive the formula for the derivative with respect to x of $\cot x$.

Grapher Explorations

69. Graph $y = \cos x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\sin(x + h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? as $h \to 0^-$? What phenomenon is being illustrated here?

70. Graph $y = -\sin x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\cos(x + h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? as $h \to 0^-$? What phenomenon is being illustrated here?

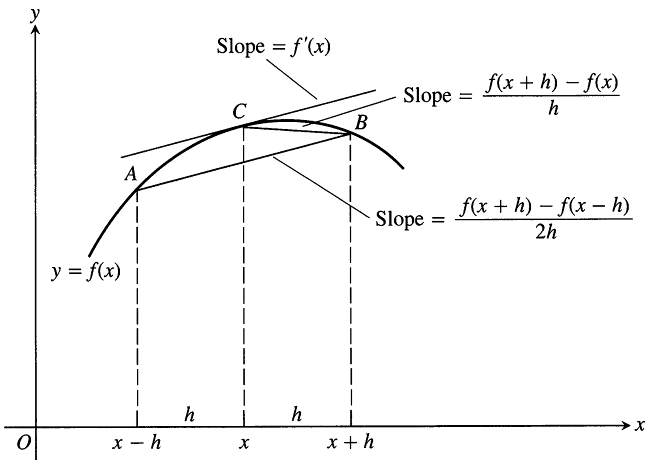
71. *Centered difference quotients.* The **centered difference quotient**

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than Fermat's difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

See the figure below.



a) To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 69 for the same values of h.

b) To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 70 for the same values of h.

72. *A caution about centered difference quotients. (Continuation of Exercise 71.)* The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h\to 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$.

73. Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

74. Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? a largest slope? Is the slope ever positive? Give reasons for your answers.

75. Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

76. *Radians vs. degrees.* What happens to the derivatives of sin x and cos x if x is measured in degrees instead of radians? To find out, take the following steps.

a) With your graphing calculator or computer grapher *in degree mode,* graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h\to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b) With your grapher still in degree mode, estimate

$$\lim_{h\to 0} \frac{\cos h - 1}{h}.$$

c) Now go back to the derivation of the formula for the derivative of sin x in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d) Work through the derivation of the formula for the derivative of cos x using degree-mode limits. What formula do you obtain for the derivative?

e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of sin x and cos x?

2.5 The Chain Rule

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

EXAMPLE 1 The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

Solution We have

$$\frac{dy}{dx} = 6, \qquad \frac{dy}{du} = 2, \qquad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

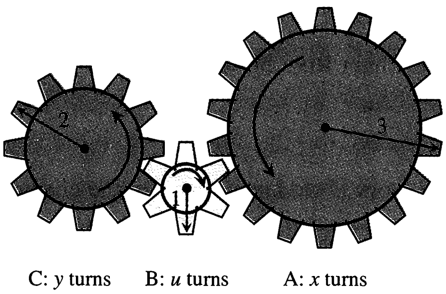
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \qquad \square$$

Is it an accident that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}?$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Fig. 2.39).

Let us try this again on another function.



2.39 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

EXAMPLE 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x\end{aligned}$$

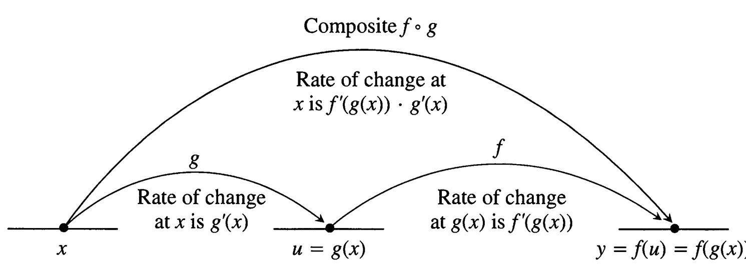
and

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}. \qquad \square$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Fig. 2.40).



2.40 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

Theorem 5

The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{2}$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

and taking the limit as $\Delta x \to 0$. This would work if we knew that Δu, the change in u, was nonzero, but we do not know this. A small change in x could conceivably produce no change in u. The proof requires a different approach, using ideas in Section 3.7. We will return to it when the time comes.

EXAMPLE 3 Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since the derivatives of f and g are

$$f'(u) = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x,$$

the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x) \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

❑

The "Outside-Inside" Rule

It sometimes helps to think about the Chain Rule the following way. If $y = f(g(x))$, Eq. (2) tells us that

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x). \tag{3}$$

In words, Eq. (3) says: To find dy/dx, differentiate the "outside" function f and leave the "inside" $g(x)$ alone; then multiply by the derivative of the inside.

EXAMPLE 4

$$\frac{d}{dx}\overset{\text{outside}}{\sin}\underbrace{(x^2+x)}_{\text{inside}} = \overset{\text{derivative of the outside}}{\cos}\underbrace{(x^2+x)}_{\text{inside left alone}} \cdot \underbrace{(2x+1)}_{\text{derivative of the inside}}$$

❑

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example.

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5-\sin 2t)) \\ &= \sec^2(5-\sin 2t)\cdot\frac{d}{dt}(5-\sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5-\sin 2t \\ &= \sec^2(5-\sin 2t)\cdot\left(0-(\cos 2t)\cdot\frac{d}{dt}(2t)\right) && \text{Derivative of } 5-\sin u \text{ with } u = 2t \\ &= \sec^2(5-\sin 2t)\cdot(-\cos 2t)\cdot 2 \\ &= -2(\cos 2t)\sec^2(5-\sin 2t) \end{aligned}$$

❑

Differentiation Formulas That Include the Chain Rule

Many of the differentiation formulas you will encounter in your scientific work already include the Chain Rule.

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ in the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}. \qquad (4)$$

For example, if u is a differentiable function of x, n is an integer, and $y = u^n$, then the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{du}(u^n)\cdot\frac{du}{dx} \\ &= nu^{n-1}\frac{du}{dx}. && \text{Differentiating } u^n \text{ with respect to } u \text{ itself gives } nu^{n-1}. \end{aligned}$$

Power Chain Rule

If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{5}$$

$\sin^n x$ is short for $(\sin x)^n$, $\quad n \neq -1$.

EXAMPLE 6

a) $$\frac{d}{dx}\sin^5 x = 5\sin^4 x \frac{d}{dx}(\sin x) \qquad \text{Eq. (5) with } u = \sin x,\ n = 5$$
$$= 5\sin^4 x \cos x$$

b) $$\frac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4}\frac{d}{dx}(2x+1) \qquad \text{Eq. (5) with } u = 2x+1,\ n = -3$$
$$= -3(2x+1)^{-4}(2)$$
$$= -6(2x+1)^{-4}$$

c) $$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) \qquad \text{Eq. (5) with } u = 5x^3 - x^4,\ n = 7$$
$$= 7(5x^3 - x^4)^6 (5 \cdot 3x^2 - 4x^3)$$
$$= 7(5x^3 - x^4)^6 (15x^2 - 4x^3)$$

d) $$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} \qquad \text{Eq. (5) with } u = 3x-2,\ n = -1$$
$$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$$
$$= -1(3x-2)^{-2}(3)$$
$$= -\frac{3}{(3x-2)^2}$$

In part (d) we could also have found the derivative with the Quotient Rule. ❑

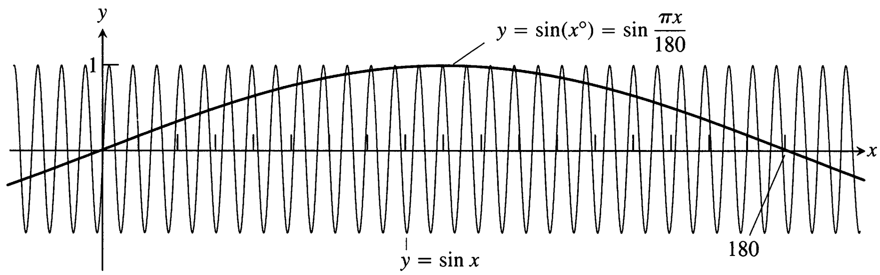
EXAMPLE 7 *Radians vs. degrees*

It is important to remember that the formulas for the derivatives of $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule brings new understanding to the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Fig. 2.41. Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure. ❑



2.41 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$.

⁂ Melting Ice Cubes

In mathematics, we tend to use letters like f, g, x, y, and u for functions and variables. However, other fields use letters like V, for volume, and s, for side, that come from the names of the things being modeled. The letters in the Chain Rule then change too, as in the next example.

EXAMPLE 8 *The melting ice cube*

How long will it take an ice cube to melt?

Solution As with all applications to science, we start with a mathematical model. We assume that the cube retains its cubical shape as it melts. We call its side length s, so its volume is $V = s^3$. We assume that V and s are differentiable functions of time t. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt. In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, however, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.)

Finally, we need at least one more piece of information: How long will it take a specific percentage of the ice cube to melt? We have nothing to guide us unless we make one or more observations, but now let us assume a particular set of conditions in which the cube lost 1/4 of its volume during the first hour. (You could use letters instead of particular numbers: say $n\%$ in r hours. Then your answer would be in terms of n and r.)

Mathematically, we now have the following problem.

Given: $V = s^3$ and $\dfrac{dV}{dt} = -k(6s^2)$

$V = V_0$ when $t = 0$

$V = (3/4)V_0$ when $t = 1$ h

Find: The value of t when $V = 0$

We apply the Chain Rule to differentiate $V = s^3$ with respect to t:

$$\frac{dV}{dt} = 3s^2\frac{ds}{dt}.$$

We set this equal to the given rate, $-k(6s^2)$, to get

$$3s^2\frac{ds}{dt} = -6ks^2$$

$$\frac{ds}{dt} = -2k.$$

The side length is *decreasing* at the constant rate of $2k$ units per hour. Thus, if the initial length of the cube's side is s_0, the length of its side one hour later is $s_1 = s_0 - 2k$. This equation tells us that

$$2k = s_0 - s_1.$$

The melting time is the value of t that makes $2kt = s_0$. Hence,

$$t_{\text{melt}} = \frac{s_0}{2k} = \frac{s_0}{s_0 - s_1} = \frac{1}{1-(s_1/s_0)}.$$

But

$$\frac{s_1}{s_0} = \frac{\left(\frac{3}{4}V_0\right)^{1/3}}{(V_0)^{1/3}} = \left(\frac{3}{4}\right)^{1/3} \approx 0.91.$$

Therefore,

$$t_{\text{melt}} = \frac{1}{1-0.91} \approx 11 \text{ h}.$$

If 1/4 of the cube melts in 1 h, it will take about 10 h more for the rest of it to melt. ❑

If we were natural scientists interested in testing the assumptions on which our mathematical model is based, our next step would be to run a number of experiments and compare their outcomes with the model's predictions. One practical application might lie in analyzing the proposal to tow large icebergs from polar waters to offshore locations near southern California, where the melting ice could provide fresh water. As a first approximation, we might imagine the iceberg to be a large cube or rectangular solid, or perhaps a pyramid. We will say more about mathematical modeling in Section 4.2.

Exercises 2.5

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$

2. $y = 2u^3, \quad u = 8x - 1$

3. $y = \sin u, \quad u = 3x + 1$

4. $y = \cos u, \quad u = -x/3$

5. $y = \cos u, \quad u = \sin x$

6. $y = \sin u, \quad u = x - \cos x$

7. $y = \tan u, \quad u = 10x - 5$

8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$

10. $y = (4 - 3x)^9$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$

12. $y = \left(\frac{x}{2} - 1\right)^{-10}$

13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

14. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

15. $y = \sec(\tan x)$

16. $y = \cot\left(\pi - \frac{1}{x}\right)$

17. $y = \sin^3 x$

18. $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p = \sqrt{3 - t}$

20. $q = \sqrt{2r - r^2}$

21. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$

22. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

23. $r = (\csc\theta + \cot\theta)^{-1}$

24. $r = -(\sec\theta + \tan\theta)^{-1}$

25. $y = x^2 \sin^4 x + x\cos^{-2} x$

26. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$

27. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

28. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$

29. $y = (4x + 3)^4(x + 1)^{-3}$

30. $y = (2x - 5)^{-1}(x^2 - 5x)^6$

31. $h(x) = x\tan(2\sqrt{x}) + 7$

32. $k(x) = x^2\sec\left(\frac{1}{x}\right)$

33. $f(\theta) = \left(\frac{\sin\theta}{1 + \cos\theta}\right)^2$

34. $g(t) = \left(\frac{1 + \cos t}{\sin t}\right)^{-1}$

35. $r = \sin(\theta^2)\cos(2\theta)$

36. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$

37. $q = \sin\left(\frac{t}{\sqrt{t + 1}}\right)$

38. $q = \cot\left(\frac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y = \sin^2(\pi t - 2)$

40. $y = \sec^2 \pi t$

41. $y = (1 + \cos 2t)^{-4}$

42. $y = (1 + \cot(t/2))^{-2}$

43. $y = \sin(\cos(2t - 5))$

44. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

45. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

46. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$

47. $y = \sqrt{1 + \cos(t^2)}$

48. $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$

Find y'' in Exercises 49–52.

49. $y = \left(1 + \frac{1}{x}\right)^3$

50. $y = (1 - \sqrt{x})^{-1}$

51. $y = \frac{1}{9}\cot(3x - 1)$

52. $y = 9\tan\left(\frac{x}{3}\right)$

Finding Numerical Values of Derivatives

In Exercises 53–58, find the value of $(f \circ g)'$ at the given value of x.

53. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

54. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

55. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

56. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

57. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

58. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $2f(x), \quad x = 2$

b) $f(x) + g(x), \quad x = 3$

c) $f(x) \cdot g(x), \quad x = 3$

d) $f(x)/g(x), \quad x = 2$

e) $f(g(x)), \quad x = 2$

f) $\sqrt{f(x)}, \quad x = 2$

g) $1/g^2(x), \quad x = 3$

h) $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

60. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x,

a) $5f(x) - g(x), \quad x = 1$

b) $f(x)g^3(x), \quad x = 0$

c) $\frac{f(x)}{g(x) + 1}, \quad x = 1$

d) $f(g(x)), \quad x = 0$

e) $g(f(x)), \quad x = 0$

f) $(x^{11} + f(x))^{-2}, \quad x = 1$

g) $f(x + g(x)), \quad x = 0$

61. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

62. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Choices in Composition

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 63 and 64.

63. Find dy/dx if $y = x$ by using the Chain Rule with y as a composite of

a) $y = (u/5) + 7$ and $u = 5x - 35$
b) $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

64. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

a) $y = u^3$ and $u = \sqrt{x}$
b) $y = \sqrt{u}$ and $u = x^3$.

Tangents and Slopes

65. a) Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.
b) What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

66. a) Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.
b) Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.
c) For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.
d) The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

Theory, Examples, and Applications

67. *Running machinery too fast.* Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

68. *Temperatures in Fairbanks, Alaska.* The graph in Fig. 2.42 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

a) On what day is the temperature increasing the fastest?
b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

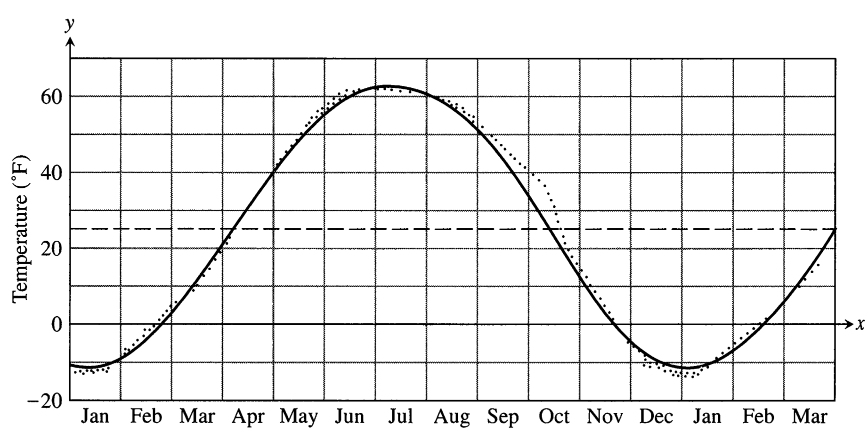
69. The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

70. Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

71. The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

72. A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

73. *Temperature and the period of a pendulum.* For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple



2.42 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

(Exercise 68).

pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

74. Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

75. Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

76. Suppose $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

Using the Chain Rule, show that the power rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 77 and 78.

77. $x^{1/4} = \sqrt{\sqrt{x}}$

78. $x^{3/4} = \sqrt{x\sqrt{x}}$

Grapher Explorations

79. *The derivative of* $\sin 2x$. Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

80. *The derivative of* $\cos(x^2)$. Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

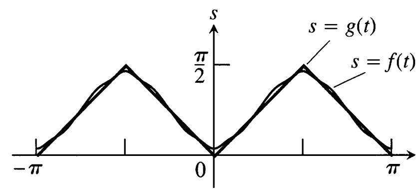
CAS Explorations and Projects

81. As Fig. 2.43 shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t - 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a) Graph dg/dt (where defined) over $[-\pi, \pi]$.
b) Find df/dt.
c) Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



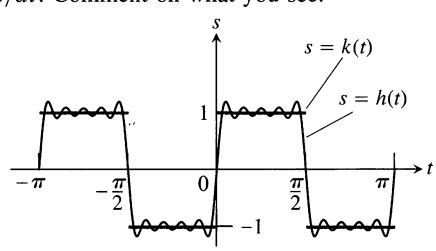
2.43 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 81).

82. (*Continuation of Exercise 81.*) In Exercise 81, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732\sin 2t + 0.4244\sin 6t + 0.25465\sin 10t + 0.18186\sin 14t + 0.14147\sin 18t$$

graphed in Fig. 2.44 approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k.

a) Graph dk/dt (where defined) over $[-\pi, \pi]$.
b) Find dh/dt.
c) Graph dh/dt to see how badly the graph fits the graph of dk/dt. Comment on what you see.



2.44 The approximation of a step function by a trigonometric "polynomial" (Exercise 82).

2.6 Implicit Differentiation and Rational Exponents

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find dy/dx by *implicit differentiation.* This section describes the technique and uses it to extend the Power Rule for differentiation to include all rational exponents.

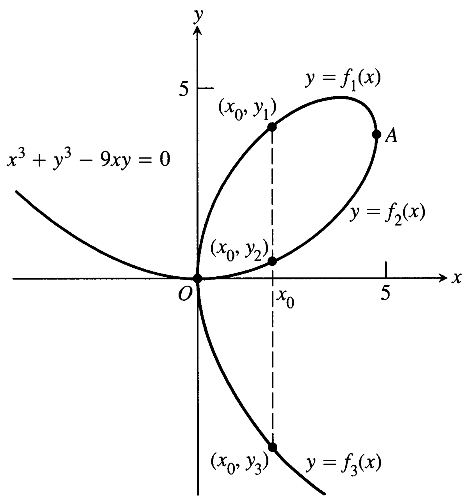
Implicit Differentiation

The graph of the equation $x^3 + y^3 - 9xy = 0$ (Fig. 2.45) has a well-defined slope at nearly every point because it is the union of the graphs of the functions $y = f_1(x)$, $y = f_2(x)$, and $y = f_3(x)$, which are differentiable except at O and A. But how do we find the slope when we cannot conveniently solve the equation to find the functions? The answer is to treat y as a differentiable function of x and differentiate both sides of the equation with respect to x, using the differentiation rules for powers, sums, products, and quotients and the Chain Rule. Then solve for dy/dx in terms of x and y *together* to obtain a formula that calculates the slope at any point (x, y) on the graph from the values of x and y.

The process by which we find dy/dx is called **implicit differentiation.** The phrase derives from the fact that the equation $x^3 + y^3 - 9xy = 0$ defines the functions f_1, f_2, and f_3 that give the graph's slope *implicitly* (i.e., hidden inside the equation), without giving us *explicit* formulas to work with.

When are the functions defined by $F(x, y) = 0$ differentiable?

When may we expect the functions of x defined by an equation of the form $F(x, y) = 0$, where $F(x, y)$ denotes an expression in x and y, to be differentiable? A theorem in advanced calculus guarantees this to be the case if F is continuous (in a sense to be described in Chapter 12) and the first derivatives of F with respect to each variable, with the other held constant, are continuous, and the derivative with respect to y is nonzero. The functions you will encounter in this section all meet these criteria.



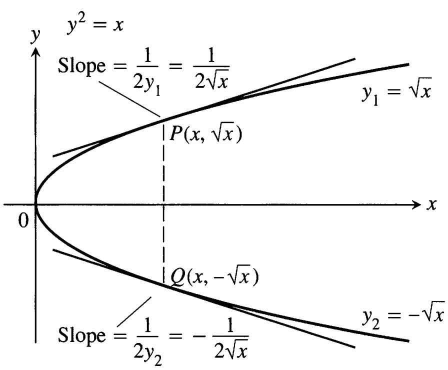
2.45 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. However, the curve can be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium,* dates to Descartes in 1638.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

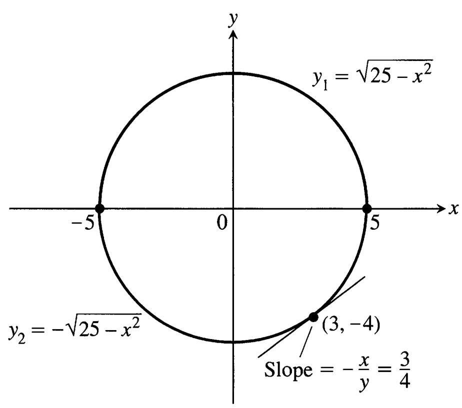
Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Fig. 2.46). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?



2.46 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.



2.47 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

Solving polynomial equations in *x* and *y*

The quadratic formula enables us to solve a second degree equation like $y^2 - 2xy + 3x^2 = 0$ for y in terms of x. There are somewhat more complicated formulas for solving equations of degree three and four. But there are no general formulas for solving equations of degree five or higher. Finding slopes on curves defined by such equations usually requires implicit differentiation.

The answer is yes. To find dy/dx we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

The Chain Rule gives $\frac{d}{dx}y^2 = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.

This one formula gives the derivatives we calculated for *both* of the explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}}, \qquad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$ ❑

EXAMPLE 2 Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Rather it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Fig. 2.47). The point $(3, -4)$ lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}. \tag{1}$$

But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Notice that unlike the slope formula in Eq. (1), which applies only to points below the x-axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope. Notice also that the derivative involves *both* variables x and y, not just the independent variable x. ❑

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

EXAMPLE 3 Find dy/dx if $2y = x^2 + \sin y$.

Implicit Differentiation Takes Four Steps

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx by dividing.

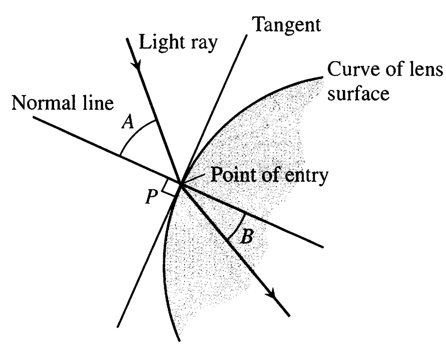
Solution

$$
\begin{aligned}
2y &= x^2 + \sin y \\
\frac{d}{dx}(2y) &= \frac{d}{dx}(x^2 + \sin y) && \text{Differentiate both sides with respect to } x \ldots \\
&= \frac{d}{dx}(x^2) + \frac{d}{dy}(\sin y) \\
2\frac{dy}{dx} &= 2x + \cos y \frac{dy}{dx} && \ldots \text{treating } y \text{ as a function of } x \text{ and using the Chain Rule.} \\
2\frac{dy}{dx} - \cos y \frac{dy}{dx} &= 2x && \text{Collect terms with } dy/dx \ldots \\
(2 - \cos y)\frac{dy}{dx} &= 2x && \ldots \text{and factor out } dy/dx. \\
\frac{dy}{dx} &= \frac{2x}{2 - \cos y} && \text{Solve for } dy/dx \text{ by dividing.}
\end{aligned}
$$

□

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Fig. 2.48). This line is called the *normal* to the surface at the point of entry. In a profile view of a lens like the one in Fig. 2.48, the normal is the line perpendicular to the tangent to the profile curve at the point of entry.



2.48 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

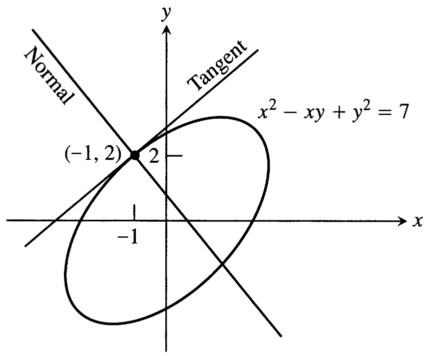
The word *normal*

When analytic geometry was developed in the seventeenth century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The word *normalis,* which scholars used for "perpendicular" in Latin, became *normal* when they discussed geometry in English.

Definition

A line is **normal** to a curve at a point if it is perpendicular to the curve's tangent there. The line is called the **normal** to the curve at that point.

The profiles of lenses are often described by quadratic curves. When they are, we can use implicit differentiation to find the tangents and normals.



2.49 The graph of $x^2 - xy + y^2 = 7$ is an ellipse. Example 4 shows how to find equations for the tangent and normal lines at the point $(-1, 2)$.

EXAMPLE 4 Find the tangent and normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$ (Fig. 2.49).

Solution We first use implicit differentiation to find dy/dx:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7) \qquad \text{Differentiate both sides with respect to } x, \ldots$$

$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0 \qquad \ldots \text{ treating } xy \text{ as a product and } y \text{ as a function of } x.$$

$$(2y - x)\frac{dy}{dx} = y - 2x \qquad \text{Collect terms.}$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}. \qquad \text{Solve for } dy/dx.$$

We then evaluate the derivative at $(x, y) = (-1, 2)$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y - 2x}{2y - x}\right|_{(-1,2)} = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{4}{5}.$$

The tangent to the curve at $(-1, 2)$ is the line

$$y = 2 + \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}.$$

The normal to the curve at $(-1, 2)$ is

$$y = 2 - \frac{5}{4}(x - (-1))$$

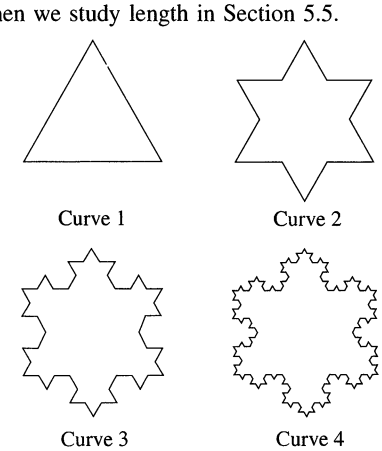
$$y = -\frac{5}{4}x + \frac{3}{4}.$$

❑

Helga von Koch's snowflake curve (1904)

Start with an equilateral triangle, calling it curve 1. On the middle third of each side, build an equilateral triangle pointing outward. Then erase the interiors of the old middle thirds. Call the expanded curve curve 2. Now put equilateral triangles, again pointing outward, on the middle thirds of the sides of curve 2. Erase the interiors of the old middle thirds to make curve 3. Repeat the process, as shown, to define an infinite sequence of plane curves. The limit curve of the sequence is Koch's snowflake curve.

The snowflake curve is too rough to have a tangent at any point. In other words, the equation $F(x, y) = 0$ defining the curve does not define y as a differentiable function of x or x as a differentiable function of y at any point. We will encounter the snowflake again when we study length in Section 5.5.



Using Implicit Differentiation to Find Derivatives of Higher Order

Implicit differentiation can also produce derivatives of higher order.

EXAMPLE 5 Find d^2y/dx^2 if $2x^3 - 3y^2 = 7$.

Solution To start, we differentiate both sides of the equation with respect to x to find $y' = dy/dx$:

$$2x^3 - 3y^2 = 7$$

$$\frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(7)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y} \qquad (\text{if } y \neq 0).$$

We differentiate the equation $x^2 - yy' = 0$ again to find y'':

$$\frac{d}{dx}(x^2 - yy') = \frac{d}{dx}(0)$$

$$2x - y'y' - yy'' = 0 \qquad \text{Product Rule with } u = y,\ v = y'$$

$$yy'' = 2x - (y')^2$$

$$y'' = \frac{2x}{y} - \frac{(y')^2}{y} \qquad (y \neq 0).$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y:

$$y'' = \frac{2x}{y} - \frac{(x^2/y)^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3} \qquad (y \neq 0).$$

❑

Rational Powers of Differentiable Functions

We know that the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

holds when n is an integer. We can now show that it holds when n is any rational number.

Theorem 6
Power Rule for Rational Powers

If n is a rational number, then x^n is differentiable at every interior point x of the domain of x^{n-1}, and

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3}$$

Proof Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then

$$y^q = x^p.$$

This equation is an algebraic combination of powers of x and y, so the advanced theorem we mentioned at the beginning of the section assures us that y is a differentiable function of x. Since p and q are integers (for which we already have the Power Rule), we can differentiate both sides of the equation implicitly with respect to x and obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}. \tag{4}$$

If $y \neq 0$, we can then divide both sides of Eq. (4) by qy^{q-1} to solve for dy/dx, obtaining

$$\begin{aligned}\frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} && \text{Eq. (4) divided by } qy^{q-1}\\ &= \frac{p}{q}\cdot\frac{x^{p-1}}{(x^{(p/q)})^{q-1}} && y = x^{p/q}\end{aligned}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} \qquad \frac{p}{q}(q-1) = p - \frac{p}{q}$$

$$= \frac{p}{q} \cdot x^{(p-1)-(p-p/q)} \qquad \text{A law of exponents}$$

$$= \frac{p}{q} \cdot x^{(p/q)-1}.$$

This proves the rule. ❑

EXAMPLE 6

a) $$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad \text{Eq. (3) with } n = \frac{1}{2}$$

function defined for $x \geq 0$

derivative defined only for $x > 0$

b) $$\frac{d}{dx}(x^{1/5}) = \frac{1}{5}x^{-4/5} \qquad \text{Eq. (3) with } n = \frac{1}{5}$$

function defined for all x

derivative not defined at $x = 0$ ❑

A version of the Power Rule with a built-in application of the Chain Rule states that if n is a rational number, u is differentiable at x, and $(u(x))^{n-1}$ is defined, then u^n is differentiable at x, and

$$\boxed{\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad (5)}$$

EXAMPLE 7

a) $$\frac{d}{dx}(1-x^2)^{1/4} = \frac{1}{4}(1-x^2)^{-3/4}(-2x) \qquad \text{Eq. (5) with } u = 1 - x^2 \text{ and } n = 1/4$$

function defined on $[-1, 1]$

$$= \frac{-x}{2(1-x^2)^{3/4}}$$

derivative defined only on $(-1, 1)$

b) $$\frac{d}{dx}(\cos x)^{-1/5} = -\frac{1}{5}(\cos x)^{-6/5}\frac{d}{dx}(\cos x)$$

$$= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

$$= \frac{1}{5}\sin x(\cos x)^{-6/5}$$ ❑

Exercises 2.6

Derivatives of Rational Powers

Find dy/dx in Exercises 1–10.

1. $y = x^{9/4}$
2. $y = x^{-3/5}$
3. $y = \sqrt[3]{2x}$
4. $y = \sqrt[4]{5x}$
5. $y = 7\sqrt{x+6}$
6. $y = -2\sqrt{x-1}$
7. $y = (2x+5)^{-1/2}$
8. $y = (1-6x)^{2/3}$
9. $y = x(x^2+1)^{1/2}$
10. $y = x(x^2+1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11–18.

11. $s = \sqrt[7]{t^2}$
12. $r = \sqrt[4]{\theta^{-3}}$
13. $y = \sin\left[(2t+5)^{-2/3}\right]$
14. $z = \cos\left[(1-6t)^{2/3}\right]$
15. $f(x) = \sqrt{1-\sqrt{x}}$
16. $g(x) = 2(2x^{-1/2}+1)^{-1/3}$
17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$
18. $k(\theta) = (\sin(\theta+5))^{5/4}$

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 19–32.

19. $x^2y + xy^2 = 6$
20. $x^3 + y^3 = 18xy$
21. $2xy + y^2 = x + y$
22. $x^3 - xy + y^3 = 1$
23. $x^2(x-y)^2 = x^2 - y^2$
24. $(3xy+7)^2 = 6y$
25. $y^2 = \dfrac{x-1}{x+1}$
26. $x^2 = \dfrac{x-y}{x+y}$
27. $x = \tan y$
28. $x = \sin y$
29. $x + \tan(xy) = 0$
30. $x + \sin y = xy$
31. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
32. $y^2 \cos\left(\dfrac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33–36.

33. $\theta^{1/2} + r^{1/2} = 1$
34. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$
35. $\sin(r\theta) = \dfrac{1}{2}$
36. $\cos r + \cos\theta = r\theta$

Higher Derivatives

In Exercises 37–42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$
38. $x^{2/3} + y^{2/3} = 1$
39. $y^2 = x^2 + 2x$
40. $y^2 - 2x = 1 - 2y$
41. $2\sqrt{y} = x - y$
42. $xy + y^2 = 1$
43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

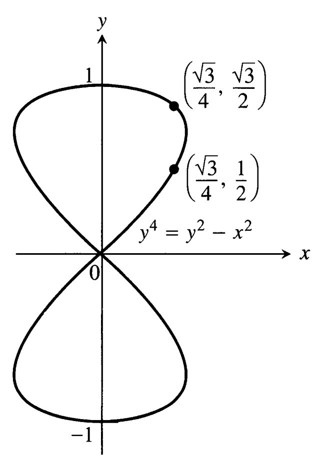
Slopes, Tangents, and Normals

In Exercises 45 and 46, find the slope of the curve at the given points.

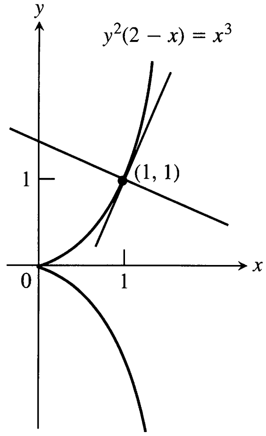
45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
46. $(x^2+y^2)^2 = (x-y)^2$ at $(1, 0)$ and $(1, -1)$

In Exercises 47–56, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

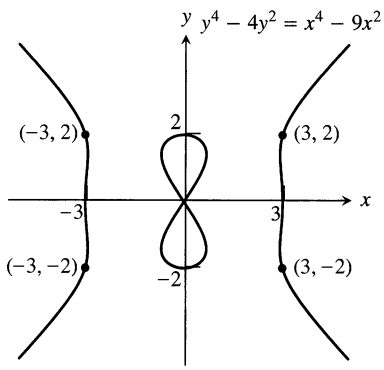
47. $x^2 + xy - y^2 = 1, \quad (2, 3)$
48. $x^2 + y^2 = 25, \quad (3, -4)$
49. $x^2y^2 = 9, \quad (-1, 3)$
50. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$
51. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$
52. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$
53. $2xy + \pi \sin y = 2\pi, \quad (1, \pi/2)$
54. $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$
55. $y = 2\sin(\pi x - y), \quad (1, 0)$
56. $x^2 \cos^2 y - \sin y = 0, \quad (0, \pi)$
57. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
58. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?
59. *The eight curve.* Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



60. *The cissoid of Diocles (from about 200 B.C.).* Find equations for the tangent and normal to the cissoid of Diocles $y^2(2-x)=x^3$ at $(1, 1)$.



61. *The devil's curve (Gabriel Cramer [the Cramer of Cramer's rule], 1750).* Find the slopes of the devil's curve $y^4-4y^2=x^4-9x^2$ at the four indicated points.



62. *The folium of Descartes.* (See Fig. 2.45.)

a) Find the slope of the folium of Descartes, $x^3+y^3-9xy=0$ at the points (4, 2) and (2, 4).

b) At what point other than the origin does the folium have a horizontal tangent?

c) Find the coordinates of the point A in Fig. 2.45, where the folium has a vertical tangent.

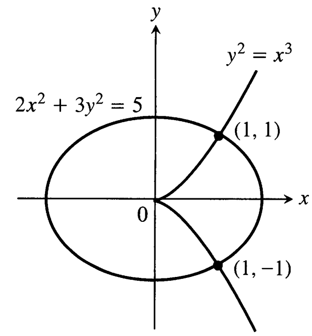
Theory and Examples

63. Which of the following could be true if $f''(x)=x^{-1/3}$?

a) $f(x)=\frac{3}{2}x^{2/3}-3$ **b)** $f(x)=\frac{9}{10}x^{5/3}-7$

c) $f'''(x)=-\frac{1}{3}x^{-4/3}$ **d)** $f'(x)=\frac{3}{2}x^{2/3}+6$

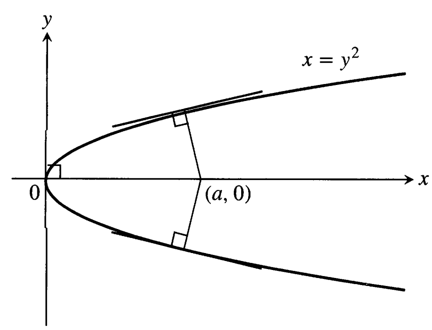
64. Is there anything special about the tangents to the curves $2x^2+3y^2=5$ and $y^2=x^3$ at the points $(1, \pm 1)$? Give reasons for your answer.



65. The line that is normal to the curve $x^2+2xy-3y^2=0$ at $(1, 1)$ intersects the curve at what other point?

66. Find the normals to the curve $xy+2x-y=0$ that are parallel to the line $2x+y=0$.

67. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x=y^2$ shown here, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



68. What is the geometry behind the restrictions on the domains of the derivatives in Example 6 and Example 7(a)?

In Exercises 69 and 70 find both dy/dx (treating y as a function of x) and dx/dy (treating x as a function of y). How do dy/dx and dx/dy seem to be related? Can you explain the relationship geometrically in terms of the graphs?

69. $xy^3+x^2y=6$ **70.** $x^3+y^2=\sin^2 y$

Grapher Explorations

71. a) Given that $x^4+4y^2=1$, find dy/dx two ways: (1) by solving for y and differentiating the resulting functions in the usual way and (2) by implicit differentiation. Do you get the same result each way?

b) Solve the equation $x^4 + 4y^2 = 1$ for y and graph the resulting functions together to produce a complete graph of the equation $x^4 + 4y^2 = 1$. Then add the graphs of the first derivatives of these functions to your display. Could you have predicted the general behavior of the derivative graphs from looking at the graph of $x^4 + 4y^2 = 1$? Could you have predicted the general behavior of the graph of $x^4 + 4y^2 = 1$ by looking at the derivative graphs? Give reasons for your answers.

72. a) Given that $(x-2)^2 + y^2 = 4$, find dy/dx two ways: (1) by solving for y and differentiating the resulting functions with respect to x and (2) by implicit differentiation. Do you get the same result each way?

b) Solve the equation $(x-2)^2 + y^2 = 4$ for y and graph the resulting functions together to produce a complete graph of the equation $(x-2)^2 + y^2 = 4$. Then add the graphs of the functions' first derivatives to your picture. Could you have predicted the general behavior of the derivative graphs from looking at the graph of $(x-2)^2 + y^2 = 4$? Could you have predicted the general behavior of the graph of $(x-2)^2 + y^2 = 4$ by looking at the derivative graphs? Give reasons for your answers.

CAS Explorations and Projects

Use a CAS to perform the following steps in Exercises 73–80.

a) Plot the equation with the implicit plotter of CAS. Check to see that the given point P satisfies the equation.

b) Using implicit differentiation find a formula for the derivative dy/dx and evaluate it at the given point P.

c) Use the slope found in part (b) to define the equation of the tangent line to the curve at P. Then plot the implicit curve and tangent line together on a single graph.

73. $x^3 - xy + y^3 = 7, \quad P(2, 1)$

74. $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

75. $y^2 + y = \dfrac{2+x}{1-x}, \quad P(0, 1)$

76. $y^3 + \cos xy = x^2, \quad P(1, 0)$

77. $x + \tan\left(\dfrac{y}{x}\right) = 2, \quad P\left(1, \dfrac{\pi}{4}\right)$

78. $xy^3 + \tan(x+y) = 1, \quad P\left(\dfrac{\pi}{4}, 0\right)$

79. $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

80. $x\sqrt{1+2y} + y = x^2, \quad P(1, 0)$

2.7 Related Rates of Change

How rapidly will the fluid level inside a vertical cylindrical storage tank drop if we pump the fluid out at the rate of 3000 L/min?

A question like this asks us to calculate a rate that we cannot measure directly from a rate that we can. To do so, we write an equation that relates the variables involved and differentiate it to get an equation that relates the rate we seek to the rate we know.

EXAMPLE 1 *Pumping out a tank*

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

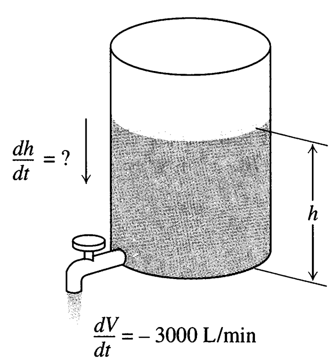
Solution We draw a picture of a partially filled vertical cylindrical tank, calling its radius r and the height of the fluid h (Fig. 2.50). Call the volume of the fluid V.

As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. We are told that

$$\frac{dV}{dt} = -3000. \qquad \text{We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.}$$

We are asked to find

$$\frac{dh}{dt}. \qquad \text{How fast will the fluid level drop?}$$



2.50 The cylindrical tank in Example 1.

Reminder

Rates of change are represented by derivatives. If a quantity is increasing, its derivative with respect to time is positive; if a quantity is decreasing, its derivative is negative.

To find dh/dt, we first write an equation that relates h to V. The equation depends on the units chosen for V, r, and h. With V in liters and r and h in meters, the appropriate equation for the cylinder's volume is

$$V = 1000\pi r^2 h$$

because a cubic meter contains 1000 liters.

Since V and h are differentiable functions of t, we can differentiate both sides of the equation $V = 1000\pi r^2 h$ with respect to t to get an equation that relates dh/dt to dV/dt:

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}. \qquad r \text{ is a constant.}$$

We substitute the known value $dV/dt = -3000$ and solve for dh/dt:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}. \tag{1}$$

The fluid level will drop at the rate of $3/(\pi r^2)$ m/min. ❑

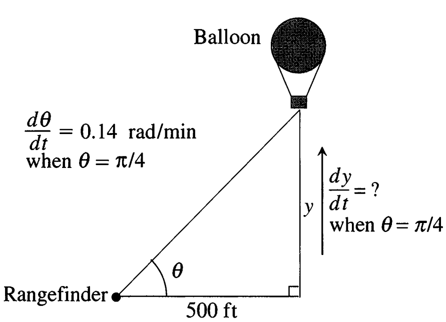
Equation (1) shows how the rate at which the fluid level drops depends on the tank's radius. If r is small, dh/dt will be large; if r is large, dh/dt will be small.

If $r = 1$ m: $\quad \frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95$ m/min $= -95$ cm/min

If $r = 10$ m: $\quad \frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095$ m/min $= -0.95$ cm/min

EXAMPLE 2 *A rising balloon*

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



2.51 The balloon in Example 2.

Solution We answer the question in six steps.

Step 1: *Draw a picture and name the variables and constants* (Fig. 2.51). The variables in the picture are

$\theta =$ the angle the range finder makes with the ground (radians)

$y =$ the height of the balloon (feet).

We let t represent time and assume θ and y to be differentiable functions of t.

The one constant in the picture is the distance from the range finder to the lift-off point (500 ft). There is no need to give it a special symbol.

Step 2: *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

Step 3: *Write down what we are asked to find.* We want dy/dt when $\theta = \pi/4$.

Step 4: *Write an equation that relates the variables y and θ.*

$$\frac{y}{500} = \tan\theta, \qquad \text{or} \qquad y = 500\tan\theta$$

Step 5: *Differentiate with respect to t using the Chain Rule.* The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500 \sec^2 \theta \frac{d\theta}{dt}$$

Step 6: *Evaluate with* $\theta = \pi/4$ *and* $d\theta/dt = 0.14$ *to find* dy/dt.

$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = (1000)(0.14) = 140 \qquad \sec \frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ❑

Strategy for Solving Related Rate Problems

1. *Draw a picture and name the variables and constants.* Use t for time. Assume all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variable whose rate you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

EXAMPLE 3 *A highway chase*

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We carry out the steps of the basic strategy.

Step 1: *Picture and variables.* We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Fig. 2.52). We let t represent time and set

x = position of car at time t,

y = position of cruiser at time t,

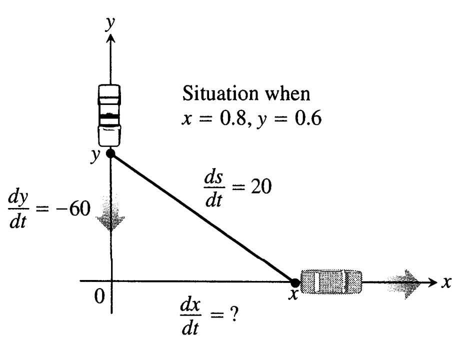
s = distance between car and cruiser at time t.

We assume x, y, and s to be differentiable functions of t.

Step 2: *Numerical information.* At the instant in question,

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

(dy/dt is negative because y is decreasing.)



2.52 Figure for Example 3.

Step 3: *To find:* $\dfrac{dx}{dt}$

Step 4: *How the variables are related:* $s^2 = x^2 + y^2$ Pythagorean theorem
(The equation $s = \sqrt{x^2 + y^2}$ would also work.)

Step 5: *Differentiate with respect to t.*

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \text{Chain Rule}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Step 6: *Evaluate, with* $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, *and solve for dx/dt.*

$$20 = \underbrace{\frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}}_{1}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$20 = 0.8\frac{dx}{dt} - 36$$

$$\frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ❑

EXAMPLE 4 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution We carry out the steps of the basic strategy.

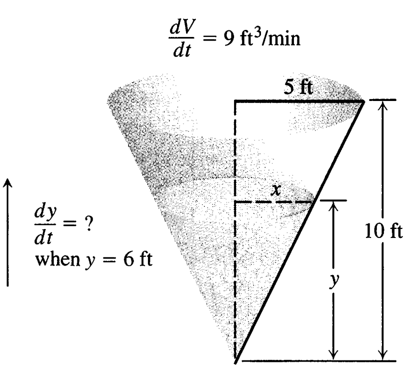
Step 1: *Picture and variables.* We draw a picture of a partially filled conical tank (Fig. 2.53). The variables in the problem are

V = volume (ft^3) of water in the tank at time t (min),

x = radius (ft) of the surface of the water at time t,

y = depth (ft) of water in the tank at time t.

We assume V, x, and y to be differentiable functions of t. The constants are the dimensions of the tank.



2.53 The conical tank in Example 4.

Step 2: *Numerical information.* At the time in question,

$$y = 6 \text{ ft}, \qquad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

Step 3: *To find:* $\dfrac{dy}{dt}$.

Step 4: *How the variables are related.*

$$V = \frac{1}{3}\pi x^2 y \qquad \text{Cone volume formula} \qquad (2)$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. Using similar triangles (Fig. 2.53) gives us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10}, \qquad \text{or} \qquad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3. \tag{3}$$

Step 5: *Differentiate with respect to t.* We differentiate Eq. (3), getting

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}. \tag{4}$$

We then solve for dy/dt to express the rate we want (dy/dt) in terms of the rate we know (dV/dt):

$$\frac{dy}{dt} = \frac{4}{\pi y^2}\frac{dV}{dt}.$$

Step 6: *Evaluate, with $y = 6$ and $dV/dt = 9$.*

$$\frac{dy}{dt} = \frac{4}{\pi(6)^2}\cdot 9 = \frac{1}{\pi} \approx 0.32 \text{ ft/min}$$

At the moment in question, the water level is rising at about 0.32 ft/min. ❑

Exercises 2.7

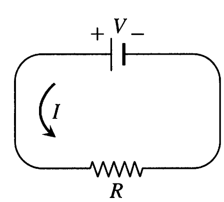
1. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.

2. Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates dS/dt to dr/dt.

3. The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   - **a)** How is dV/dt related to dh/dt if r is constant?
   - **b)** How is dV/dt related to dr/dt if h is constant?
   - **c)** How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

4. The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
   - **a)** How is dV/dt related to dh/dt if r is constant?
   - **b)** How is dV/dt related to dr/dt if h is constant?
   - **c)** How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

5. *Changing voltage.* The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in seconds.



   - **a)** What is the value of dV/dt?
   - **b)** What is the value of dI/dt?
   - **c)** What equation relates dR/dt to dV/dt and dI/dt?
   - **d)** Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing?

6. The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current i (amperes) by the equation $P = Ri^2$.
   - **a)** How are dP/dt, dR/dt, and di/dt related if none of P, R, and i are constant?
   - **b)** How is dR/dt related to di/dt if P is constant?

7. Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$

be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.

a) How is ds/dt related to dx/dt if y is constant?
b) How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
c) How is dx/dt related to dy/dt if s is constant?

8. If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.

a) Assuming that x, y, and z are differentiable functions of t, how is ds/dt related to dx/dt, dy/dt, and dz/dt?
b) How is ds/dt related to dy/dt and dz/dt if x is constant?
c) How are dx/dt, dy/dt, and dz/dt related if s is constant?

9. The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is

$$A = \frac{1}{2}ab\sin\theta.$$

a) How is dA/dt related to $d\theta/dt$ if a and b are constant?
b) How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
c) How is dA/dt related to $d\theta/dt$, da/dt, and db/dt if none of a, b, and θ are constant?

10. *Heating a plate.* When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

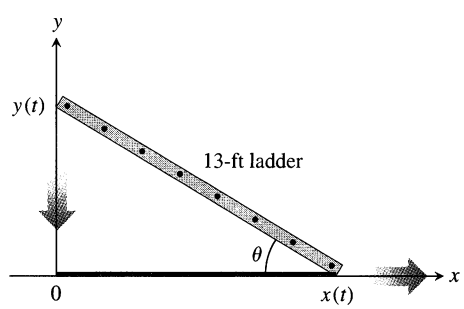
11. *Changing dimensions in a rectangle.* The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

12. *Changing dimensions in a rectangular box.* Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \qquad \frac{dy}{dt} = -2 \text{ m/sec}, \qquad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

13. *A sliding ladder.* A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.



a) How fast is the top of the ladder sliding down the wall then?
b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
c) At what rate is the angle θ between the ladder and the ground changing then?

14. *Commercial air traffic.* Two commercial airplanes are flying at 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

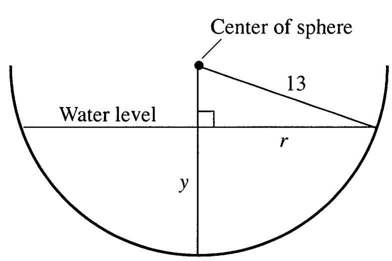
15. *Flying a kite.* A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

16. *Boring a cylinder.* The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

17. *A growing sand pile.* Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in cm/min.

18. *A draining conical reservoir.* Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. (a) How fast is the water level falling when the water is 5 m deep? (b) How fast is the radius of the water's surface changing then? Answer in cm/min.

19. *A draining hemispherical reservoir.* Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.



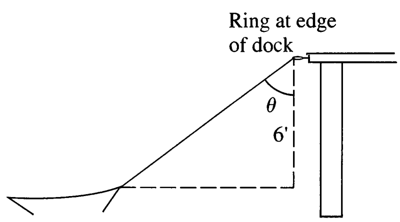
a) At what rate is the water level changing when the water is 8 m deep?
b) What is the radius r of the water's surface when the water is y m deep?

c) At what rate is the radius r changing when the water is 8 m deep?

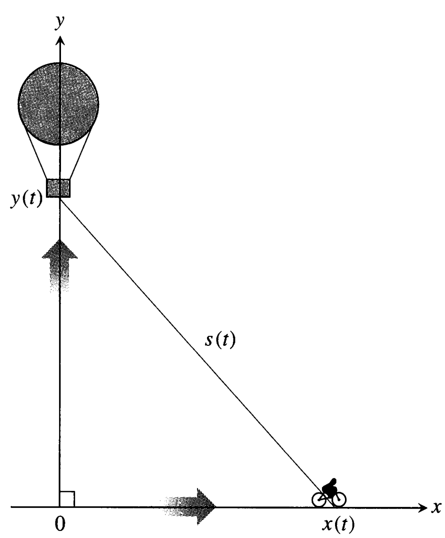
20. *A growing raindrop.* Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

21. *The radius of an inflating balloon.* A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

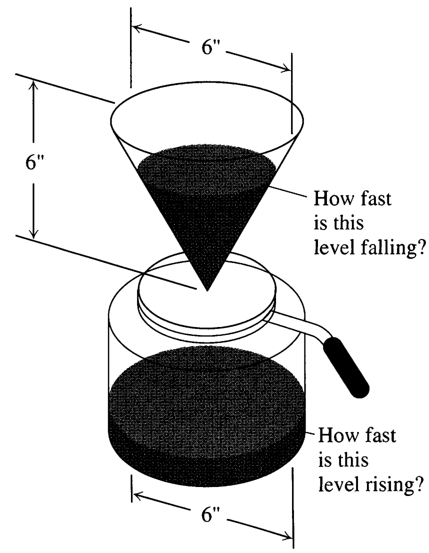
22. *Hauling in a dinghy.* A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec. (a) How fast is the boat approaching the dock when 10 ft of rope are out? (b) At what rate is angle θ changing then (see the figure)?



23. *A balloon and a bicycle.* A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later?



24. *Making coffee.* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. (a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep? (b) How fast is the level in the cone falling then?



25. *Cardiac output.* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

26. *Cost, revenue, and profit.* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (everything in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$

b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$

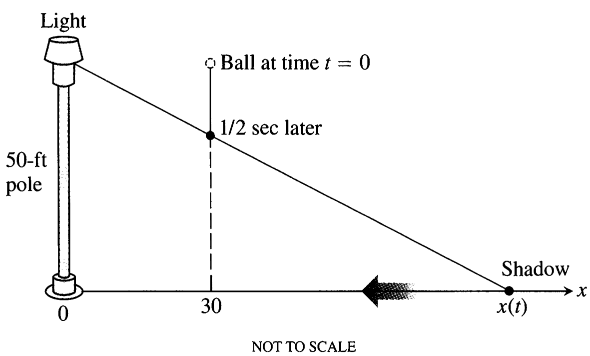
27. *Moving along a parabola.* A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

28. *Moving along another parabola.* A particle moves from right to left along the parabola $y = \sqrt{-x}$ in such a way that its x-coordinate (measured in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$?

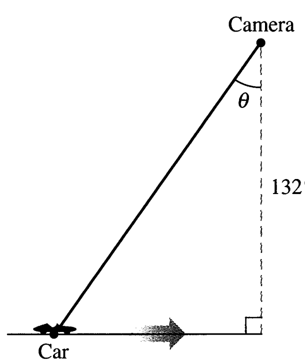
29. *Motion in the plane.* The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

30. *A moving shadow.* A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

31. *Another moving shadow.* A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



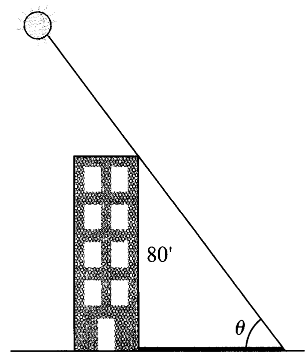
32. You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



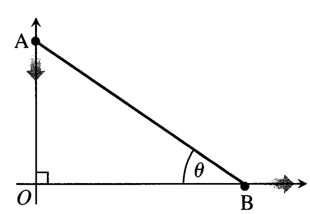
33. *A melting ice layer.* A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

34. *Highway patrol.* A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

35. *A building's shadow.* On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



36. *Walkers.* A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec; B moves away from the intersection 1 m/sec. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.

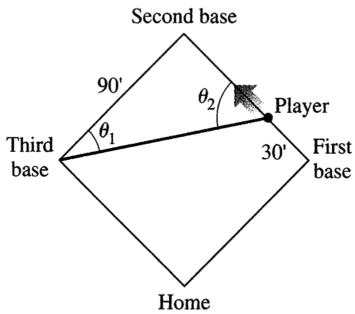


37. A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

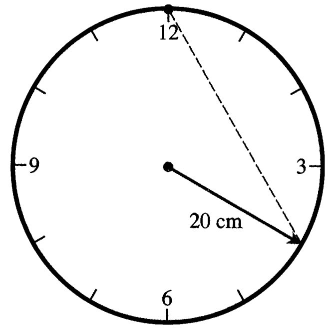
    a) At what rate is the player's distance from third base changing when the player is 30 ft from first base?

    b) At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?

c) The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?



38. *A second hand.* At what rate is the distance between the tip of the second hand and the 12 o'clock mark changing when the second hand points to 4 o'clock?



39. *Ships.* Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

CHAPTER 2 QUESTIONS TO GUIDE YOUR REVIEW

1. What is the derivative of a function f? How is its domain related to the domain of f? Give examples.

2. What role does the derivative play in defining slopes, tangents, and rates of change?

3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?

4. What does it mean for a function to be differentiable on an open interval? on a closed interval?

5. How are derivatives and one-sided derivatives related?

6. Describe geometrically when a function typically does *not* have a derivative at a point.

7. How is a function's differentiability at a point related to its continuity there, if at all?

8. Could the unit step function

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

possibly be the derivative of some other function on $[-1, 1]$? Explain.

9. What rules do you know for calculating derivatives? Give some examples.

10. Explain how the three formulas

a) $\dfrac{d}{dx}(x^n) = nx^{n-1}$,

b) $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$,

c) $\dfrac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \dfrac{du_1}{dx} + \dfrac{du_2}{dx} + \cdots + \dfrac{du_n}{dx}$

enable us to differentiate any polynomial.

11. What formula do we need, in addition to the three listed in question 10, to differentiate rational functions?

12. What is a second derivative? a third derivative? How many derivatives do the functions you know have? Give examples.

13. What is the relationship between a function's average and instantaneous rates of change? Give an example.

14. How do derivatives arise in the study of motion? What can you learn about a body's motion along a line by examining the derivatives of the body's position function? Give examples.

15. How can derivatives arise in economics?

16. Give examples of still other applications of derivatives.

17. What is the value of $\lim_{\theta \to 0}(\sin\theta)/\theta$? Does it matter whether θ is measured in degrees or radians? Explain.

18. What do the limits $\lim_{h \to 0}(\sin h)/h$ and $\lim_{h \to 0}(\cos h - 1)/h$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?

19. Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of tan x, cot x, sec x, and csc x? What *are* the derivatives of these functions?

20. At what points are the six basic trigonometric functions continuous? How do you know?

21. What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.

22. If u is a differentiable function of x, how do you find $(d/dx)(u^n)$ if n is an integer? if n is a rational number? Give examples.

23. What is implicit differentiation? When do you need it? Give examples.

24. How do related rate problems arise? Give examples.

25. Outline a strategy for solving related rate problems. Illustrate with an example.

CHAPTER 2 PRACTICE EXERCISES

Derivatives of Functions

Find the derivatives of the functions in Exercises 1–36.

1. $y = x^5 - 0.125x^2 + 0.25x$
2. $y = 3 - 0.7x^3 + 0.3x^7$
3. $y = x^3 - 3(x^2 + \pi^2)$
4. $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi + 1}$
5. $y = (x+1)^2(x^2+2x)$
6. $y = (2x-5)(4-x)^{-1}$
7. $y = (\theta^2 + \sec\theta + 1)^3$
8. $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$
9. $s = \dfrac{\sqrt{t}}{1+\sqrt{t}}$
10. $s = \dfrac{1}{\sqrt{t} - 1}$
11. $y = 2\tan^2 x - \sec^2 x$
12. $y = \dfrac{1}{\sin^2 x} - \dfrac{2}{\sin x}$
13. $s = \cos^4(1-2t)$
14. $s = \cot^3\left(\dfrac{2}{t}\right)$
15. $s = (\sec t + \tan t)^5$
16. $s = \csc^5(1 - t + 3t^2)$
17. $r = \sqrt{2\theta}\sin\theta$
18. $r = 2\theta\sqrt{\cos\theta}$
19. $r = \sin\sqrt{2\theta}$
20. $r = \sin(\theta + \sqrt{\theta + 1})$
21. $y = \dfrac{1}{2}x^2 \csc\dfrac{2}{x}$
22. $y = 2\sqrt{x}\sin\sqrt{x}$
23. $y = x^{-1/2}\sec(2x)^2$
24. $y = \sqrt{x}\csc(x+1)^3$
25. $y = 5\cot x^2$
26. $y = x^2 \cot 5x$
27. $y = x^2\sin^2(2x^2)$
28. $y = x^{-2}\sin^2(x^3)$
29. $s = \left(\dfrac{4t}{t+1}\right)^{-2}$
30. $s = \dfrac{-1}{15(15t-1)^3}$
31. $y = \left(\dfrac{\sqrt{x}}{1+x}\right)^2$
32. $y = \left(\dfrac{2\sqrt{x}}{2\sqrt{x}+1}\right)^2$
33. $y = \sqrt{\dfrac{x^2+x}{x^2}}$
34. $y = 4x\sqrt{x + \sqrt{x}}$
35. $r = \left(\dfrac{\sin\theta}{\cos\theta - 1}\right)^2$
36. $r = \left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)^2$

In Exercises 37–48, find dy/dx.

37. $y = (2x+1)\sqrt{2x+1}$
38. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5}$
39. $y = \dfrac{3}{(5x^2 + \sin 2x)^{3/2}}$
40. $y = (3 + \cos^3 3x)^{-1/3}$
41. $xy + 2x + 3y = 1$
42. $x^2 + xy + y^2 - 5x = 2$
43. $x^3 + 4xy - 3y^{4/3} = 2x$
44. $5x^{4/5} + 10y^{6/5} = 15$
45. $\sqrt{xy} = 1$
46. $x^2y^2 = 1$
47. $y^2 = \dfrac{x}{x+1}$
48. $y^2 = \sqrt{\dfrac{1+x}{1-x}}$

In Exercises 49 and 50, find dp/dq.

49. $p^3 + 4pq - 3q^2 = 2$
50. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 51 and 52, find dr/ds.

51. $r\cos 2s + \sin^2 s = \pi$
52. $2rs - r - s + s^2 = -3$

53. Find d^2y/dx^2 by implicit differentiation:
    a) $x^3 + y^3 = 1$
    b) $y^2 = 1 - \dfrac{2}{x}$

54. **a)** By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.
    b) Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

55. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the first derivatives of the following combinations at the given value of x.

a) $5f(x) - g(x), \quad x = 1$

b) $f(x)g^3(x), \quad x = 0$

c) $\dfrac{f(x)}{g(x)+1}, \quad x = 1$

d) $f(g(x)), \quad x = 0$

e) $g(f(x)), \quad x = 0$

f) $(x + f(x))^{3/2}, \quad x = 1$

g) $f(x + g(x)), \quad x = 0$

56. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivatives of the following combinations at the given value of x.

a) $\sqrt{x}f(x), \quad x = 1$

b) $\sqrt{f(x)}, \quad x = 0$

c) $f\left(\sqrt{x}\right), \quad x = 1$

d) $f(1 - 5\tan x), \quad x = 0$

e) $\dfrac{f(x)}{2 + \cos x}, \quad x = 0$

f) $10\sin\left(\dfrac{\pi x}{2}\right) f^2(x), \quad x = 1$

57. Find the value of dy/dt at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

58. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

59. Find the value of dw/ds at $s = 0$ if $w = \sin(\sqrt{r} - 2)$ and $r = 8\sin(s + \pi/6)$.

60. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

61. If $y^3 + y = 2\cos x$, find the value of d^2y/dx^2 at the point $(0, 1)$.

62. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point $(8, 8)$.

Derivative Definition

In Exercises 63 and 64, find the derivative using the definition.

63. $f(t) = \dfrac{1}{2t+1}$

64. $g(x) = 2x^2 + 1$

65. a) Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

b) Is f continuous at $x = 0$?

c) Is f differentiable at $x = 0$?

Give reasons for your answers.

66. a) Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

b) Is f continuous at $x = 0$?

c) Is f differentiable at $x = 0$?

Give reasons for your answers.

67. a) Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

b) Is f continuous at $x = 1$?

c) Is f differentiable at $x = 1$?

Give reasons for your answers.

68. For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

a) continuous at $x = 0$?

b) differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangents, and Normals

69. Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

70. Are there any points on the curve $y = x - 1/(2x)$ where the slope is 3? If so, find them.

71. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

72. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

73. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

a) perpendicular to the line $y = 1 - (x/24)$;

b) parallel to the line $y = \sqrt{2} - 12x$.

74. Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

75. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

76. Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.

77. The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C.

78. Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).

79. For what value of c is the curve $y = c/(x+1)$ tangent to the line through the points (0, 3) and (5, −2)?

80. Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

In Exercises 81–86, find equations for the lines that are tangent and normal to the curve at the given point.

81. $x^2 + 2y^2 = 9, \quad (1, 2)$

82. $x^3 + y^2 = 2, \quad (1, 1)$

83. $xy + 2x - 5y = 2, \quad (3, 2)$

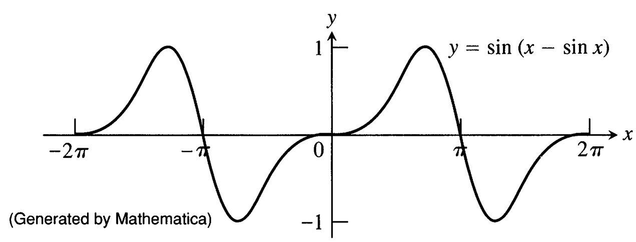
84. $(y - x)^2 = 2x + 4, \quad (6, 2)$

85. $x + \sqrt{xy} = 6, \quad (4, 1)$

86. $x^{3/2} + 2y^{3/2} = 17, \quad (1, 4)$

87. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points (1, 1) and (1, −1).

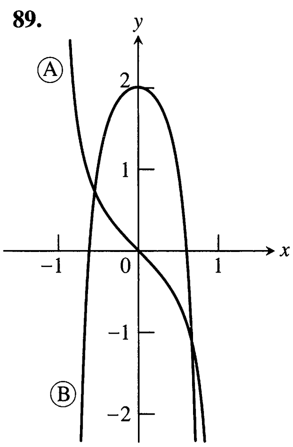
88. The graph below suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the x-axis. Does it? Give reasons for your answer.



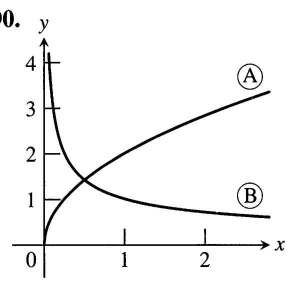
Analyzing Graphs

Each of the figures in Exercises 89 and 90 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

89.

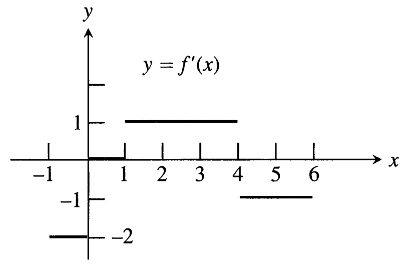


90.



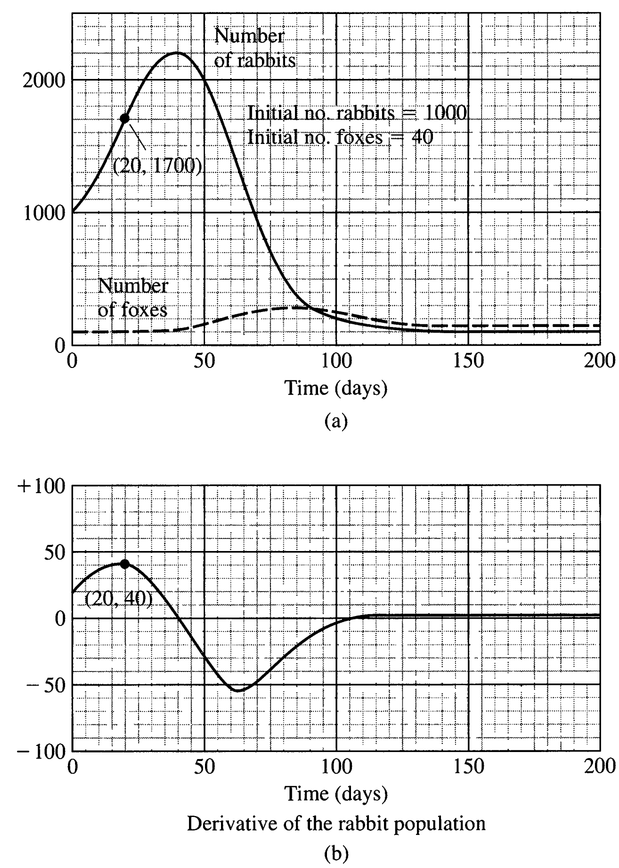
91. Use the following information to graph the function $y = f(x)$ for $-1 \le x \le 6$.

i) The graph of f is made of line segments joined end to end.
ii) The graph starts at the point $(-1, 2)$.
iii) The derivative of f, where defined, agrees with the step function shown here.



92. Repeat Exercise 91, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 93 and 94 are about the graphs in Fig. 2.54. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic



2.54 Rabbits and foxes in an arctic predator-prey food chain. (Source: *Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, Mass, 1975, p. 86.)

population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 2.54(b) shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 4 in Section 2.1.

93. a) What is the value of the derivative of the rabbit population in Fig. 2.54 when the number of rabbits is largest? smallest?

b) What is the size of the rabbit population in Fig. 2.54 when its derivative is largest? smallest?

94. In what units should the slopes of the rabbit and fox population curves be measured?

Limits

Find the limits in Exercises 95–104.

95. $\lim\limits_{s\to 0} \dfrac{\sin(s/2)}{s/3}$

96. $\lim\limits_{\theta\to -\pi} \dfrac{\sin^2(\theta+\pi)}{\theta+\pi}$

97. $\lim\limits_{x\to 0} \dfrac{\sin x}{2x^2-x}$

98. $\lim\limits_{x\to 0} \dfrac{3x-\tan 7x}{2x}$

99. $\lim\limits_{r\to 0} \dfrac{\sin r}{\tan 2r}$

100. $\lim\limits_{\theta\to 0} \dfrac{\sin(\sin\theta)}{\theta}$

101. $\lim\limits_{\theta\to(\pi/2)^-} \dfrac{4\tan^2\theta+\tan\theta+1}{\tan^2\theta+5}$

102. $\lim\limits_{\theta\to 0^+} \dfrac{1-2\cot^2\theta}{5\cot^2\theta-7\cot\theta-8}$

103. $\lim\limits_{x\to 0} \dfrac{x\sin x}{2-2\cos x}$

104. $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 105 and 106 to be continuous at the origin.

105. $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

106. $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

107. Is there any value of k that will make

$$f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

continuous at $x = 0$? If so, what is it? Give reasons for your answer.

108. a) GRAPHER Graph the function

$$f(x) = \begin{cases} \dfrac{x^2}{\sin^2 2x}, & x \neq 0 \\ c, & x = 0. \end{cases}$$

b) Find a value of c that makes f continuous at $x = 0$. Justify your answer.

Related Rates

109. The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi rh$.

a) How is dS/dt related to dr/dt if h is constant?

b) How is dS/dt related to dh/dt if r is constant?

c) How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

d) How is dr/dt related to dh/dt if S is constant?

110. The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2+h^2}$.

a) How is dS/dt related to dr/dt if h is constant?

b) How is dS/dt related to dh/dt if r is constant?

c) How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

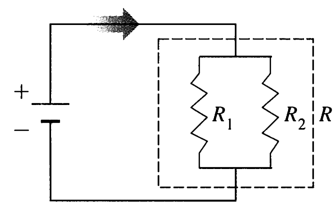
111. The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

112. The volume of a cube is increasing at the rate of $1200\ \text{cm}^3/\text{min}$ at the instant its edges are 20 cm long. At what rate are the edges changing at that instant?

113. If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

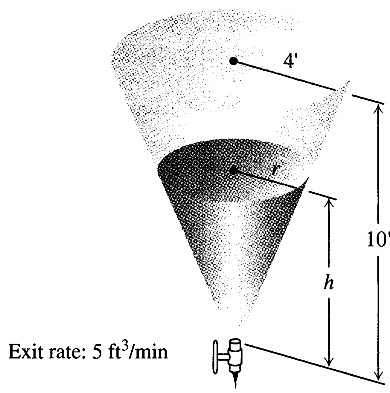


114. The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2+X^2}$. If R is increasing at 3 ohms/sec and X is decreasing at 2 ohms/sec, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?

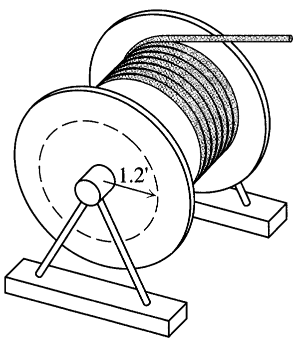
115. The coordinates of a particle moving in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)?

116. A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

117. Water drains from the conical tank shown in Fig. 2.55 at the rate of $5\ \text{ft}^3/\text{min}$. (a) What is the relation between the variables h and r in the figure? (b) How fast is the water level dropping when $h = 6$ ft?



2.55 The conical tank in Exercise 117.

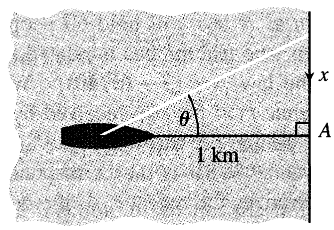


2.56 The television cable in Exercise 118.

118. As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see Fig. 2.56). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (rad/sec) the spool is turning when the layer of radius 1.2 ft is being unwound.

119. The figure below shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

a) How fast is the light moving along the shore when it reaches point A?
b) How many revolutions per minute is 0.6 rad/sec?



120. Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

CHAPTER 2 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of θ. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of θ, not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ, the resulting new equation will also be an identity.

Differentiate the following to show that the resulting equations hold for all θ.

a) $\sin 2\theta = 2\sin\theta\cos\theta$
b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

2. If the identity $\sin(x + a) = \sin x\cos a + \cos x\sin a$ is differentiated with respect to x, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. a) Find values for the constants a, b, and c that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

a) Find values for b and c that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b\sin x + c\cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

b) For the determined values of a, b, and c, what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. a) Show that $y = \sin x$, $y = \cos x$, and $y = a\cos x + b\sin x$ (a and b constants) all satisfy the equation

$$y'' + y = 0.$$

b) How would you modify the functions in (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

5. *An osculating circle.* Find the values of h, k, and a that make the circle $(x-h)^2 + (y-k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari* meaning "to kiss"). We will encounter them again in Chapter 11.

6. *Marginal revenue.* A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This is the fare that maximizes the revenue, so the bus company should probably rethink its fare policy.)

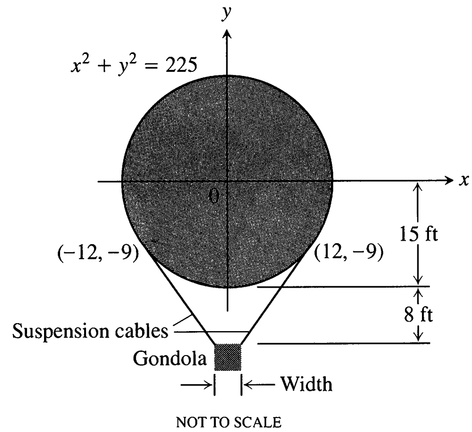
7. *Industrial production*

   a) Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

      Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

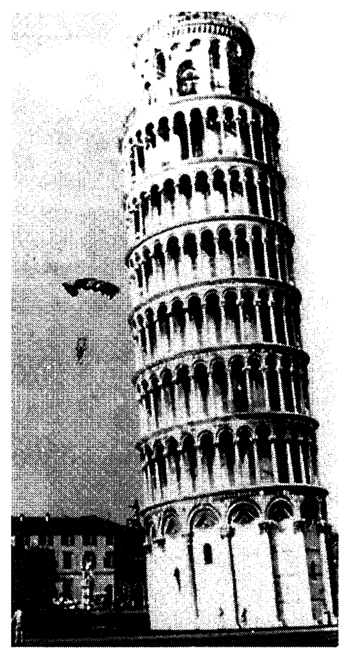
   b) Suppose that the labor force in (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

8. The designer of a 30-ft-diameter spherical hot-air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon (Fig. 2.57). Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?



2.57 The balloon and gondola in Exercise 8.

9. *Pisa by parachute.* The accompanying photograph shows Mike McCarthy parachuting from the top of the Tower of Pisa on August 5, 1988. Make a rough sketch to show the shape of the graph of his speed during the jump.



Mike McCarthy of London jumped from the Tower of Pisa and then opened his parachute in what he said was a world record low-level parachute jump of 179 feet. Source: *Boston Globe*, Aug. 6, 1988.

10. The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \pi/4).$$

   a) What is the particle's starting position ($t = 0$)?

   b) What are the points farthest to the left and right of the origin reached by the particle?

   c) Find the particle's velocity and acceleration at the points in question (b).

   d) When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

11. On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In t seconds after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.

a) How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?

b) On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in t seconds. About how long will it take the paper clip to reach its maximum height and how high will it go?

12. At time t sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

13. A particle of constant mass m moves along the x-axis. Its velocity v and position x satisfy the equation

$$\frac{1}{2}m\,(v^2 - v_0{}^2) = \frac{1}{2}k(x_0{}^2 - x^2),$$

where k, v_0, and x_0 are constants. Show that whenever $v \neq 0$,

$$m\,\frac{dv}{dt} = -kx.$$

14. a) Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

b) What is the geometric significance of the result in (a)?

15. Find all values of the constants m and b for which the function

$$y = \begin{cases} \sin x & \text{for } x < \pi \\ mx + b & \text{for } x \geq \pi, \end{cases}$$

is (a) continuous at $x = \pi$; (b) differentiable at $x = \pi$.

16. Does the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

have a derivative at $x = 0$? Explain.

17. a) For what values of a and b will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of x?

b) Discuss the geometry of the resulting graph of f.

18. a) For what values of a and b will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of x?

b) Discuss the geometry of the resulting graph of g.

19. Is there anything special about the derivative of an odd differentiable function of x? Give reasons for your answer.

20. Is there anything special about the derivative of an even differentiable function of x? Give reasons for your answer.

21. *A surprising result.* Suppose that the functions f and g are defined throughout an open interval containing the point x_0, that f is differentiable at x_0, that $f(x_0) = 0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0. This shows, for example, that while $|x|$ is not differentiable at $x = 0$, the product $x|x|$ *is* differentiable at $x = 0$.

22. *(Continuation of Exercise 21.)* Use the result of Exercise 21 to show that the following functions are differentiable at $x = 0$.

a) $|x|\sin x$ **b)** $x^{2/3}\sin x$ **c)** $\sqrt[3]{x}(1 - \cos x)$

d) $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

23. Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

derived at $x = 0$? continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

24. Suppose that a function f satisfies the following conditions for all real values of x and y:

i) $f(x + y) = f(x) \cdot f(y)$;

ii) $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

25. *The generalized product rule.* Use mathematical induction (Appendix 1) to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then y is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx}u_2 \cdots u_n + u_1\frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1}\frac{du_n}{dx}.$$

26. *Leibniz's rule for higher order derivatives of products.* Leibniz's rule for higher order derivatives of products of differentiable functions says that

a) $$\frac{d^2(uv)}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2},$$

b) $$\frac{d^3(uv)}{dx^3} = \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3},$$

c) $$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

The equations in (a) and (b) are special cases of the equation in (c). Derive the equation in (c) by mathematical induction, using the fact that

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

CHAPTER 3

Applications of Derivatives

OVERVIEW This chapter shows how to draw conclusions from derivatives. We use derivatives to find extreme values of functions, to predict and analyze the shapes of graphs, to find replacements for complicated formulas, to determine how sensitive formulas are to errors in measurement, and to find the zeros of functions numerically. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in Chapter 4.

3.1 Extreme Values of Functions

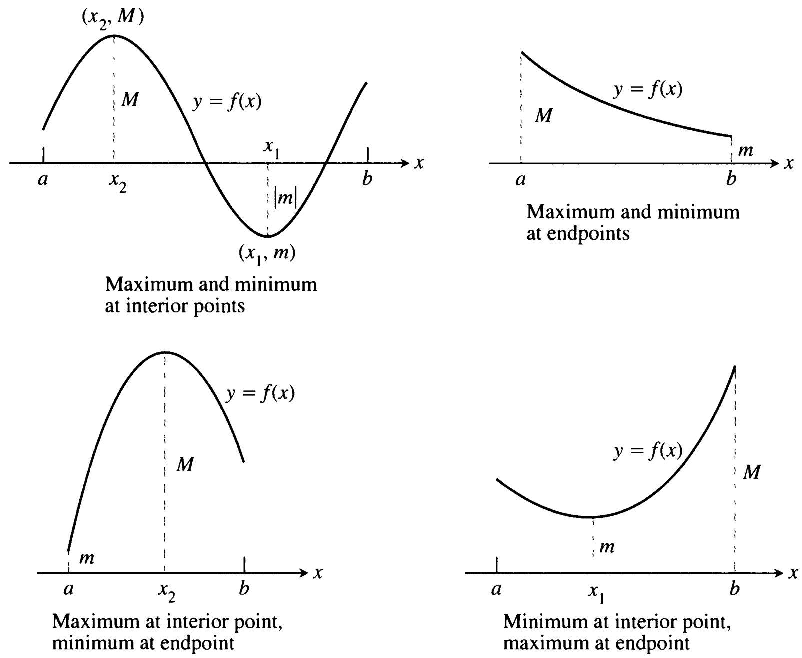
This section shows how to locate and identify extreme values of continuous functions.

The Max-Min Theorem

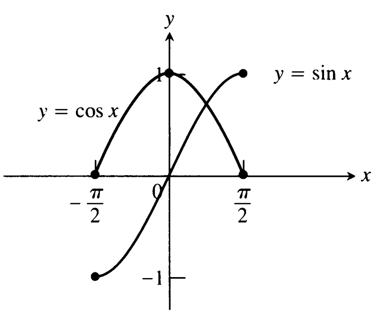
A function that is continuous at every point of a closed interval has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function, and we will see the role they play in problem solving (this chapter) and in the development of the integral calculus (Chapters 4 and 5).

Theorem 1
The Max-Min Theorem for Continuous Functions
If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in I (Fig. 3.1 on the following page).

The proof of Theorem 1 requires a detailed knowledge of the real number system and we will not give it here.



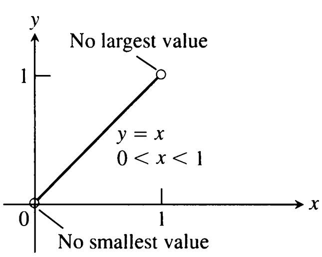
3.1 Typical arrangements of a continuous function's absolute maxima and minima on a closed interval $[a, b]$.



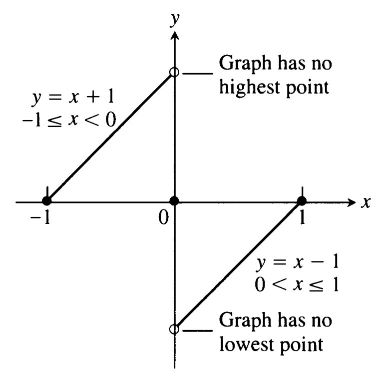
3.2 Figure for Example 1.

EXAMPLE 1 On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Fig. 3.2). ❑

As Figs. 3.3 and 3.4 show, the requirements that the interval be closed and the function continuous are key ingredients of Theorem 1. Without them, the conclusion of the theorem need not hold.



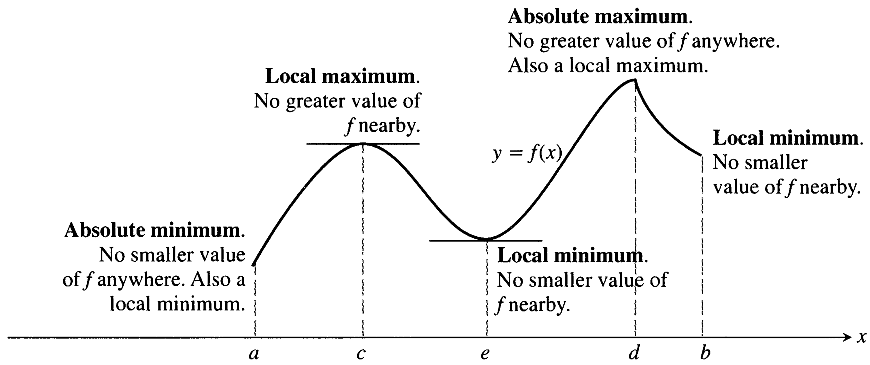
3.3 On an open interval, a continuous function need not have either a maximum or a minimum value. The function $f(x) = x$ has neither a largest nor a smallest value on $(0, 1)$.



3.4 Even a single point of discontinuity can keep a function from having either a maximum or a minimum value on a closed interval. The function

$$y = \begin{cases} x + 1, & -1 \le x < 0 \\ 0, & x = 0 \\ x - 1, & 0 < x \le 1 \end{cases}$$

is continuous at every point of $[-1,1]$ except $x = 0$, yet its graph over $[-1,1]$ has neither a highest nor a lowest point.



3.5 How to classify maxima and minima.

Local vs. Absolute (Global) Extrema

Figure 3.5 shows a graph with five extreme points. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby.* The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

Definition

Absolute Extreme Values

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in } D.$$

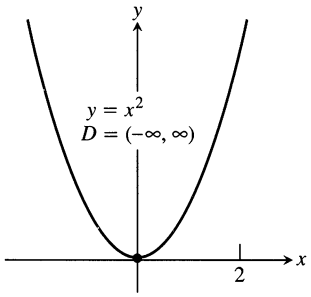
Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema.

Functions with the same defining rule can have different extrema, depending on the domain.

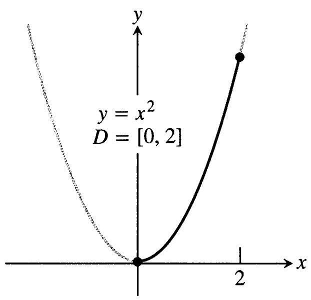
EXAMPLE 2 (See Fig. 3.6.)

	Function rule	**Domain D**	**Absolute extrema on D (if any)**
a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
b)	$y = x^2$	$[0, 2]$	Absolute maximum of $(2)^2 = 4$ at $x = 2$. Absolute minimum of 0 at $x = 0$.
c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
d)	$y = x^2$	$(0, 2)$	No absolute extrema.

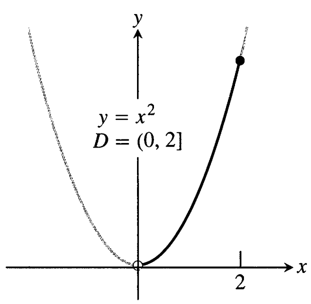
❑



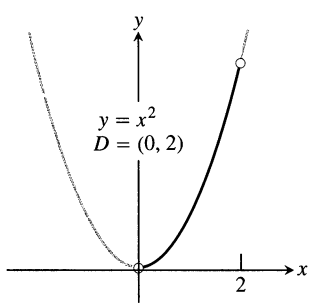
(a) abs min only



(b) abs max and min



(c) abs max only



(d) no abs max or min

3.6 Graphs for Example 2.

Definition
Local Extreme Values

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a **local maximum** or **local minimum** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. In Fig. 3.5, the function f has local maxima at c and d and local minima at a, e, and b.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

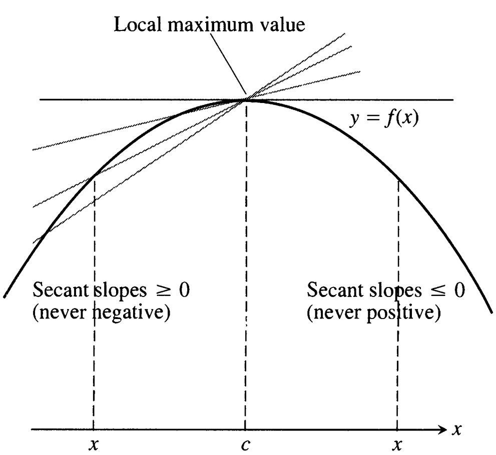
Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

Theorem 2
The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$



3.7 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To show that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Fig. 3.7) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \qquad (1)$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \tag{2}$$

Together, (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in (1) and (2). ❑

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

Definition

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Summary

The only domain points where a function can assume extreme values are critical points and endpoints.

Most quests for extreme values call for finding the absolute extrema of a continuous function on a closed interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. These points are often so few in number that we can simply list them and calculate the corresponding function values to see what the largest and smallest are.

How to Find the Absolute Extrema of a Continuous Function f on a Closed Interval

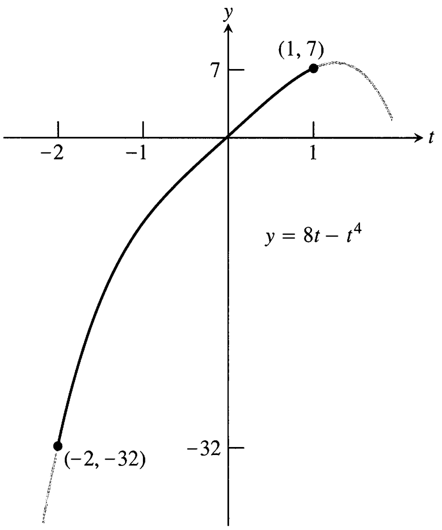
1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 3 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value:	$f(0) = 0$
Endpoint values:	$f(-2) = 4$
	$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ❑



3.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 4).

EXAMPLE 4 Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

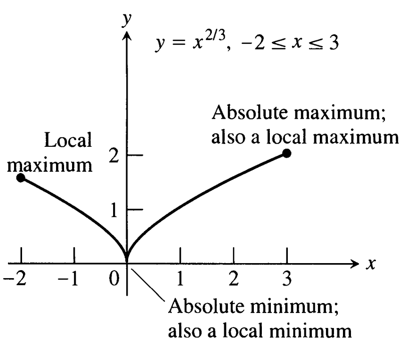
Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$\begin{aligned} 8 - 4t^3 &= 0 \\ t^3 &= 2 \\ t &= 2^{1/3}, \end{aligned}$$

a point not in the given domain. The function's local extrema therefore occur at the endpoints, where we find

$$\begin{aligned} g(-2) &= -32 \qquad \text{(Absolute minimum)} \\ g(1) &= 7. \qquad \text{(Absolute maximum)} \end{aligned}$$

See Fig. 3.8. ❑



3.9 The extreme values of $h(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 5).

EXAMPLE 5 Find the absolute extrema of $h(x) = x^{2/3}$ on $[-2, 3]$.

Solution The first derivative

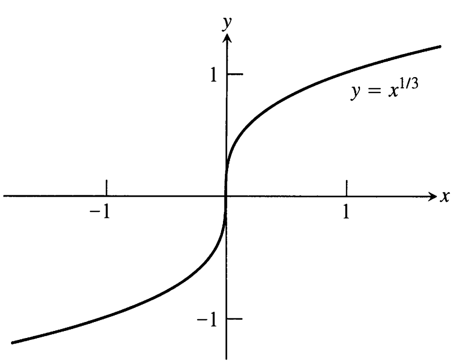
$$h'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

has no zeros but is undefined at $x = 0$. The values of h at this one critical point and at the endpoints $x = -2$ and $x = 3$ are

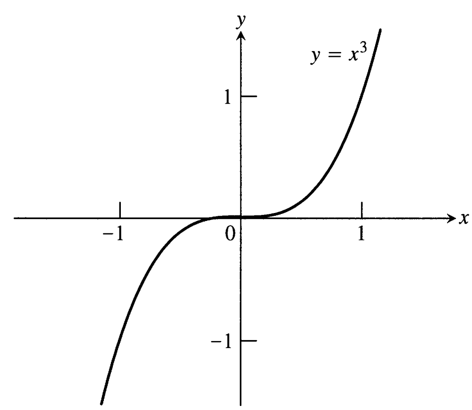
$$\begin{aligned} h(0) &= 0 \\ h(-2) &= (-2)^{2/3} = 4^{1/3} \\ h(3) &= (3)^{2/3} = 9^{1/3}. \end{aligned}$$

The absolute maximum value is $9^{1/3}$, assumed at $x = 3$; the absolute minimum is 0, assumed at $x = 0$ (Fig. 3.9). ❑

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figures 3.10 and 3.11 illustrate this for interior points, and Exercise 34 asks you for a function that fails to assume an extreme value at an endpoint of its domain.



3.10 $f(x) = x^{1/3}$ has no extremum at $x = 0$, even though $f'(x) = (1/3)x^{-2/3}$ is undefined at $x = 0$.



3.11 $g(x) = x^3$ has no extremum at $x = 0$ even though $g'(x) = 3x^2$ is zero at $x = 0$.

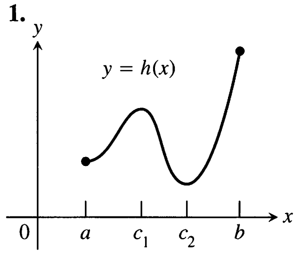
As we will see in Section 3.3, we can determine the behavior of a function f at a critical point c by further examining f', but we must look beyond what f' does at c itself.

Exercises 3.1

Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

1.



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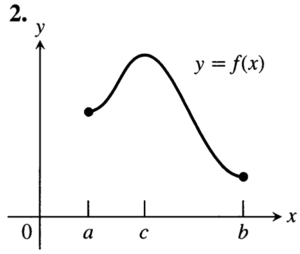

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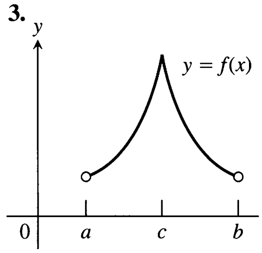

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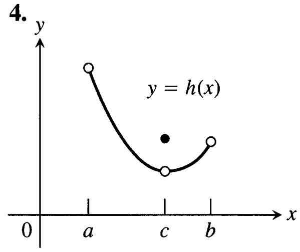
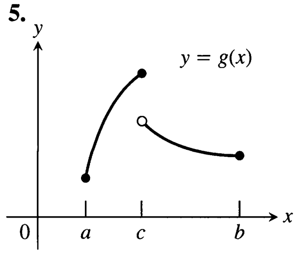
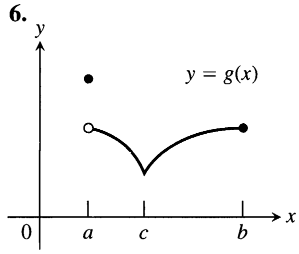


5.



6.



Absolute Extrema on Closed Intervals

In Exercises 7–22, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

7. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

8. $f(x) = -x - 4, \quad -4 \le x \le 1$

9. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

10. $f(x) = 4 - x^2, \quad -3 \le x \le 1$

11. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

12. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

13. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

14. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

15. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

16. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

17. $f(\theta) = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$

18. $f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

19. $g(x) = \csc x, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$

20. $g(x) = \sec x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$

21. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

22. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 23–26, find the function's absolute maximum and minimum values and say where they are assumed.

23. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

24. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

25. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

26. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Local Extrema in the Domain

In Exercises 27 and 28, find the values of any local maxima and minima the functions may have on the given domains, and say where they are assumed. Which extrema, if any, are absolute for the given domain?

27. **a)** $f(x) = x^2 - 4, \quad -2 \le x \le 2$

b) $g(x) = x^2 - 4, \quad -2 \le x < 2$

c) $h(x) = x^2 - 4, \quad -2 < x < 2$

d) $k(x) = x^2 - 4, \quad -2 \le x < \infty$

e) $l(x) = x^2 - 4, \quad 0 < x < \infty$

28. **a)** $f(x) = 2 - 2x^2, \quad -1 \leq x \leq 1$

b) $g(x) = 2 - 2x^2, \quad -1 < x \leq 1$

c) $h(x) = 2 - 2x^2, \quad -1 < x < 1$

d) $k(x) = 2 - 2x^2, \quad -\infty < x \leq 1$

e) $l(x) = 2 - 2x^2, \quad -\infty < x < 0$

Theory and Examples

29. The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

30. Why can't the conclusion of Theorem 2 be expected to hold if c is an endpoint of the function's domain?

31. If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

32. If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

33. We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

34. Give an example of a function defined on $[0, 1]$ that has neither a local maximum nor a local minimum value at 0.

CAS Explorations and Projects

In Exercises 35–40, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps:

a) Plot the function over the interval to see general behavior there.

b) Find the interior points where $f' = 0$. (In some exercises you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.

c) Find the interior points where f' does not exist.

d) Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

e) Find the function's absolute extreme values on the interval and identify where they occur.

35. $f(x) = x^4 - 8x^2 + 4x + 2, \quad \left[-\frac{20}{25}, \frac{64}{25}\right]$

36. $f(x) = -x^4 + 4x^3 - 4x + 1, \quad \left[-\frac{3}{4}, 3\right]$

37. $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

38. $f(x) = 2 + 2x - 3x^{2/3}, \quad \left[-1, \frac{10}{3}\right]$

39. $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

40. $f(x) = x^{3/4} - \sin x + \frac{1}{2}, \quad [0, 2\pi]$

3.2 The Mean Value Theorem

When the French mathematician Michel Rolle published his theorem in 1691, his goal was to show that between every two zeros of a polynomial function there always lies a zero of the polynomial we now know to be the function's derivative. (The modern version of the theorem is not restricted to polynomials.)

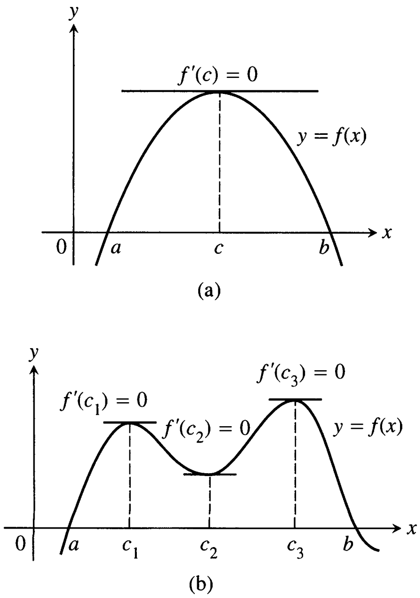
Rolle distrusted the new methods of calculus, however, and spent a great deal of time and energy denouncing their use and attacking l'Hôpital's all too popular (he felt) calculus book. It is ironic that Rolle is known today only for his inadvertent contribution to a field he tried to suppress.

If a body falls freely from rest near the surface of the earth, its position t seconds into the fall is $s = 4.9t^2$ m. From this we deduce that the body's velocity and acceleration are $v = ds/dt = 9.8t$ m/sec and $a = d^2s/dt^2 = 9.8$ m/sec^2. But suppose we started with the body's acceleration. Could we work backward to find its velocity and displacement functions?

What we are really asking here is what functions can have a given derivative. More generally, we might ask what kind of function can have a particular *kind* of derivative. What kind of function has a positive derivative, for instance, or a negative derivative, or a derivative that is always zero? We answer these questions by applying corollaries of the Mean Value Theorem.

Rolle's Theorem

There is strong geometric evidence that between any two points where a differentiable curve crosses the x-axis there is a point on the curve where the tangent is horizontal. A 300-year-old theorem of Michel Rolle (1652–1719) assures us that this is indeed the case.



3.12 Rolle's theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses the *x*-axis. It may have just one (a), or it may have more (b).

Theorem 3
Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b) = 0,$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

See Fig. 3.12.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

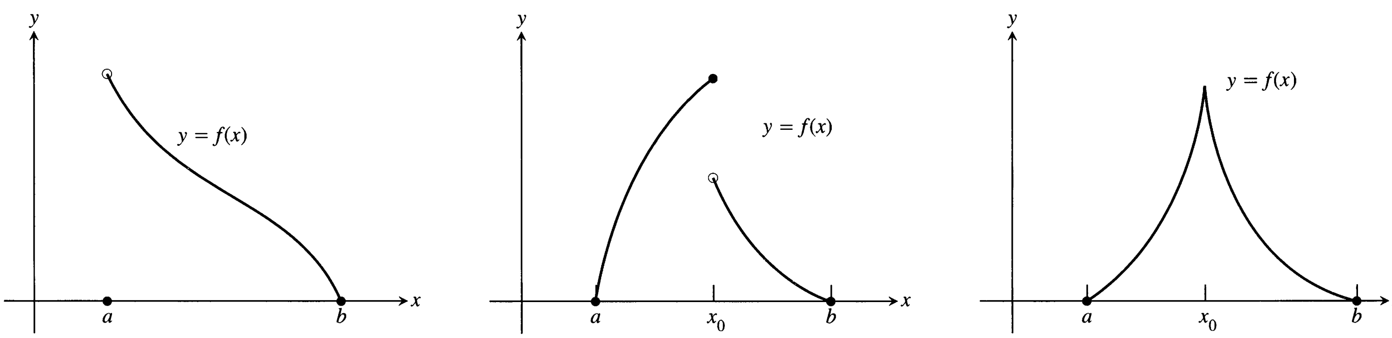
1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c inside the interval, then $f'(c) = 0$ by Theorem 2 in Section 3.1, and we have found a point for Rolle's theorem.

If both maximum and minimum are at a or b, then f is constant, $f' = 0$, and c can be taken anywhere in the interval. This completes the proof. ❑

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Fig. 3.13).



(a) Discontinuous at an endpoint

(b) Discontinuous at an interior point

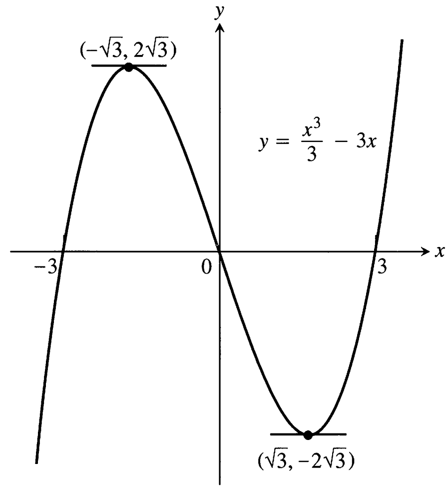
(c) Continuous on $[a, b]$ but not differentiable at some interior point

3.13 No horizontal tangent.

EXAMPLE 1 The polynomial function

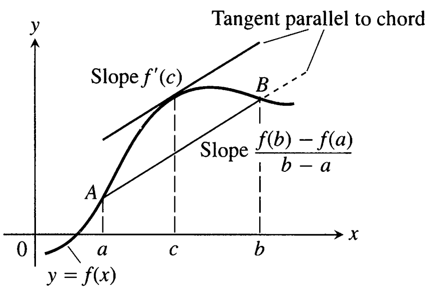
$$f(x) = \frac{x^3}{3} - 3x$$

graphed in Fig. 3.14 (on the following page) is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$. Since $f(-3) = f(3) = 0$, Rolle's



3.14 As predicted by Rolle's theorem, this curve has horizontal tangents between the points where it crosses the x-axis (Example 1).

theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact, $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$. ❑



3.15 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

The Mean Value Theorem

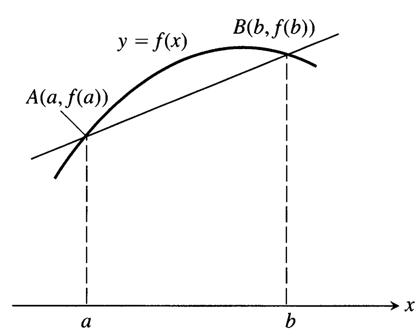
The Mean Value Theorem is a slanted version of Rolle's theorem (Fig. 3.15). There is a point where the tangent is parallel to chord AB.

Theorem 4
The Mean Value Theorem

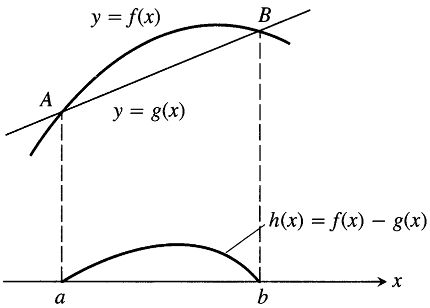
Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

Proof We picture the graph of f as a curve in the plane and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$ (see Fig. 3.16). The line is the graph of the



3.16 The graph of f and the chord AB over the interval $[a, b]$.



3.17 The chord *AB* in Fig. 3.16 is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

(point–slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$

Figure 3.17 shows the graphs of f, g, and h together.

The function h satisfies the hypotheses of Rolle's theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore, $h' = 0$ at some point c in (a, b). This is the point we want for Eq. (1).

To verify Eq. (1), we differentiate both sides of Eq. (3) with respect to x and then set $x = c$:

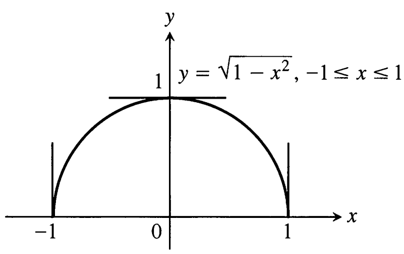
$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3) . . .}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{. . . with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

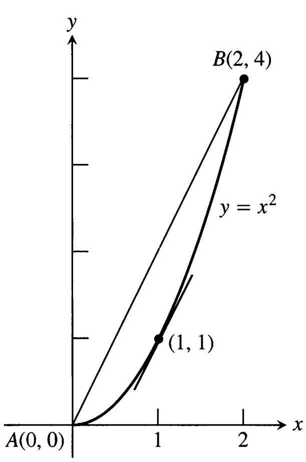
which is what we set out to prove. ❑



3.18 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

Notice that the hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. Continuity at a and b is enough (Fig. 3.18).

We usually do not know any more about the number c than the theorem tells, which is that c exists. In a few cases we can satisfy our curiosity about the identity of c, as in the next example. However, our ability to identify c is the exception rather than the rule, and the importance of the theorem lies elsewhere.



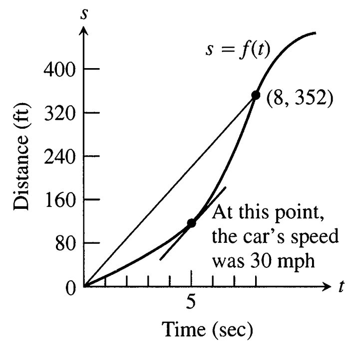
3.19 As we find in Example 2, $c = 1$ is where the tangent is parallel to the chord.

EXAMPLE 2 The function $f(x) = x^2$ (Fig. 3.19) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this (exceptional) case we can identify c by solving the equation $2c = 2$ to get $c = 1$. ❑

Physical Interpretations

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly 30 mph (44 ft/sec) (Fig. 3.20). ❑



3.20 Distance vs. elapsed time for the car in Example 3.

Corollaries and Some Answers

At the beginning of the section, we asked what kind of function has a zero derivative. The first corollary of the Mean Value Theorem provides the answer.

Corollary 1
Functions with Zero Derivatives Are Constant
If $f'(x) = 0$ at each point of an interval I, then $f(x) = C$ for all x in I, where C is a constant.

We know that if a function f has a constant value on an interval I, then f is differentiable on I and $f'(x) = 0$ for all x in I. Corollary 1 provides the converse.

Proof of Corollary 1 We want to show that f has a constant value on I. We do so by showing that if x_1 and x_2 are any two points in I, then $f(x_1) = f(x_2)$.

Suppose that x_1 and x_2 are two points in I, numbered from left to right so that $x_1 < x_2$. Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$, and hence continuous at every point as well. Therefore,

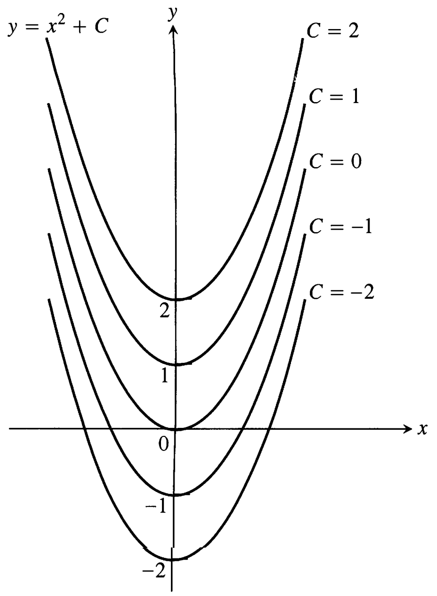
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2. Since $f' = 0$ throughout I, this equation translates successively into

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \quad f(x_2) - f(x_1) = 0, \quad \text{and} \quad f(x_1) = f(x_2).$$

❑

At the beginning of the section, we also asked if we could work backward from the acceleration of a body falling freely from rest to find the body's velocity and displacement functions. The answer is yes, and it is a consequence of the next corollary.



3.21 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives can differ only by a vertical shift. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Corollary 2
Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there exists a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof At each point x in I the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on I (Corollary 1). That is, $f(x) - g(x) = C$ on I, so $f(x) = g(x) + C$. ❑

Corollary 2 says that functions can have identical derivatives on an interval only if their values on the interval have a constant difference. We know, for instance, that the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$. Any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Fig. 3.21).

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through (0, 2)):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The formula for f is $f(x) = -\cos x + 3$. ❑

Finding Velocity and Position from Acceleration

Here is how to find the velocity and displacement functions of a body falling freely from rest with acceleration 9.8 m/sec^2.

We know that $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C \tag{4}$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. How about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $h(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C \tag{5}$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \qquad \text{and} \qquad C = 0.$$

The position function must be $s(t) = 4.9t^2$.

The ability to find functions from their rates of change is one of the great powers we gain from calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 4. We will continue the story there.

Increasing Functions and Decreasing Functions

At the beginning of the section we asked what kinds of functions have positive derivatives or negative derivatives. The answer, provided by the Mean Value Theorem's third corollary, is this: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

Definitions

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \quad \Rightarrow \quad f(x_2) < f(x_1)$.

Corollary 3
The First Derivative Test for Increasing and Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

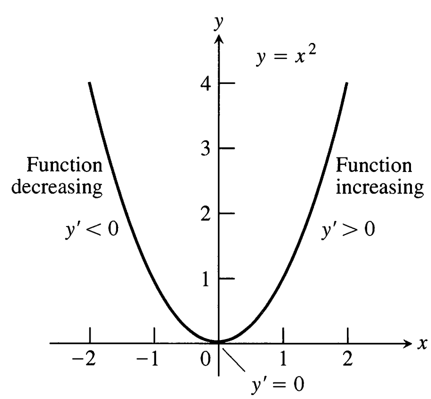
If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof Let x_1 and x_2 be two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{6}$$

for some c between x_1 and x_2. The sign of the right-hand side of Eq. (6) is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b), and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ❑

EXAMPLE 5 The function $f(x) = x^2$ decreases on $(-\infty, 0)$, where $f'(x) = 2x < 0$. It increases on $(0, \infty)$, where $f'(x) = 2x > 0$ (Fig. 3.22). ❑



3.22 The graph for Example 5.

Exercises 3.2

Finding c in the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–4.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \dfrac{1}{x}, \quad \left[\dfrac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x-1}, \quad [1, 3]$

Checking and Using Hypotheses

Which of the functions in Exercises 5–8 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

5. $f(x) = x^{2/3}, \quad [-1, 8]$
6. $f(x) = x^{4/5}, \quad [0, 1]$
7. $f(x) = \sqrt{x(1-x)}, \quad [0, 1]$
8. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
9. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

10. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

11. **a)** Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

    i) $y = x^2 - 4$
    ii) $y = x^2 + 8x + 15$
    iii) $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$
    iv) $y = x^3 - 33x^2 + 216x = x(x-9)(x-24)$

    b) Use Rolle's theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ there lies a zero of

$$nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

12. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.
13. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
14. Show that a cubic polynomial can have at most three real zeros.

Theory and Examples

15. Show that at some instant during a 2-h automobile trip the car's speedometer reading will equal the average speed for the trip.
16. *Temperature change.* It took 14 sec for a thermometer to rise from -19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at exactly 8.5°C/sec.
17. Suppose that f is differentiable on $[0, 1]$ and that its derivative is never zero. Show that $f(0) \ne f(1)$.
18. Show that $|\sin b - \sin a| \le |b - a|$ for any numbers a and b.
19. Suppose that f is differentiable on $[a, b]$ and that $f(b) < f(a)$. Can you then say anything about the values of f' on $[a, b]$?
20. Suppose that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel.
21. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

    a) Show that $f(x) \ge 1$ for all x.
    b) Must $f'(1) = 0$? Explain.

22. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.
23. *A surprising graph.* Graph the function

$$f(x) = \sin x \sin(x+2) - \sin^2(x+1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

24. If the graphs of two functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.
25. **a)** Show that $g(x) = 1/x$ decreases on every interval in its domain.

b) If the conclusion in (a) is really true, how do you explain the fact that $g(1) = 1$ is actually greater than $g(-1) = -1$?

26. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answer.

27. CALCULATOR Use the inequalities in Exercise 26 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \leq x \leq 0.1$ and $f(0) = 1$.

28. CALCULATOR Use the inequalities in Exercise 26 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \leq x \leq 0.1$ and $f(0) = 2$.

29. *The geometric mean of a and b.* The **geometric mean** of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval $[a, b]$ of positive numbers is $c = \sqrt{ab}$.

30. *The arithmetic mean of a and b.* The **arithmetic mean** of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

Finding Functions from Derivatives

31. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

32. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

33. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a) $f(0) = 0$ b) $f(1) = 0$ c) $f(-2) = 3$.

34. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 35–40, find all possible functions with the given derivative.

35. a) $y' = x$ b) $y' = x^2$ c) $y' = x^3$

36. a) $y' = 2x$
b) $y' = 2x - 1$
c) $y' = 3x^2 + 2x - 1$

37. a) $y' = -\dfrac{1}{x^2}$

b) $y' = 1 - \dfrac{1}{x^2}$

c) $y' = 5 + \dfrac{1}{x^2}$

38. a) $y' = \dfrac{1}{2\sqrt{x}}$

b) $y' = \dfrac{1}{\sqrt{x}}$

c) $y' = 4x - \dfrac{1}{\sqrt{x}}$

39. a) $y' = \sin 2t$ b) $y' = \cos \dfrac{t}{2}$

c) $y' = \sin 2t + \cos \dfrac{t}{2}$

40. a) $y' = \sec^2 \theta$ b) $y' = \sqrt{\theta}$

c) $y' = \sqrt{\theta} - \sec^2 \theta$

In Exercises 41–44, find the function with the given derivative whose graph passes through the point P.

41. $f'(x) = 2x - 1, \quad P(0, 0)$

42. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

43. $r'(\theta) = 8 - \csc^2 \theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

44. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

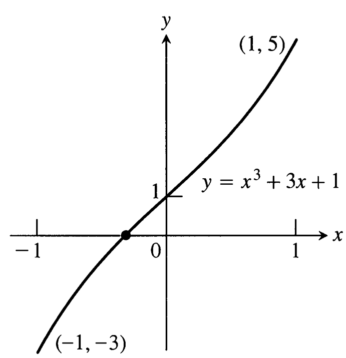
Counting Zeros

When we solve an equation $f(x) = 0$ numerically, we usually want to know beforehand how many solutions to look for in a given interval. With the help of Corollary 3 we can sometimes find out.

Suppose that

1. f is continuous on $[a, b]$ and differentiable on (a, b),
2. $f(a)$ and $f(b)$ have opposite signs,
3. $f' > 0$ on (a, b) or $f' < 0$ on (a, b).

Then f has exactly one zero between a and b: It cannot have more than one because it is either increasing on $[a, b]$ or decreasing on $[a, b]$. Yet it has at least one, by the Intermediate Value Theorem (Section 1.5). For example, $f(x) = x^3 + 3x + 1$ has exactly one zero on $[-1, 1]$ because f is differentiable on $[-1, 1]$, $f(-1) = -3$ and $f(1) = 5$ have opposite signs, and $f'(x) = 3x^2 + 3 > 0$ for all x (Fig. 3.23).



3.23 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here between −1 and 0.

Show that the functions in Exercises 45–52 have exactly one zero in the given interval.

45. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

46. $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$

47. $g(t) = \sqrt{t} + \sqrt{1+t} - 4, \quad (0, \infty)$

48. $g(t) = \dfrac{1}{1-t} + \sqrt{1+t} - 3.1, \quad (-1, 1)$

49. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

50. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

51. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

52. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

CAS Exploration

53. *Rolle's original theorem*

a) Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1,$ and 2.

b) Graph f and its derivative f' together. How is what you see related to Rolle's original theorem? (See the marginal note on Rolle.)

c) Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon?

d) How would you state and prove Rolle's original theorem in light of what we know today?

3.3

The First Derivative Test for Local Extreme Values

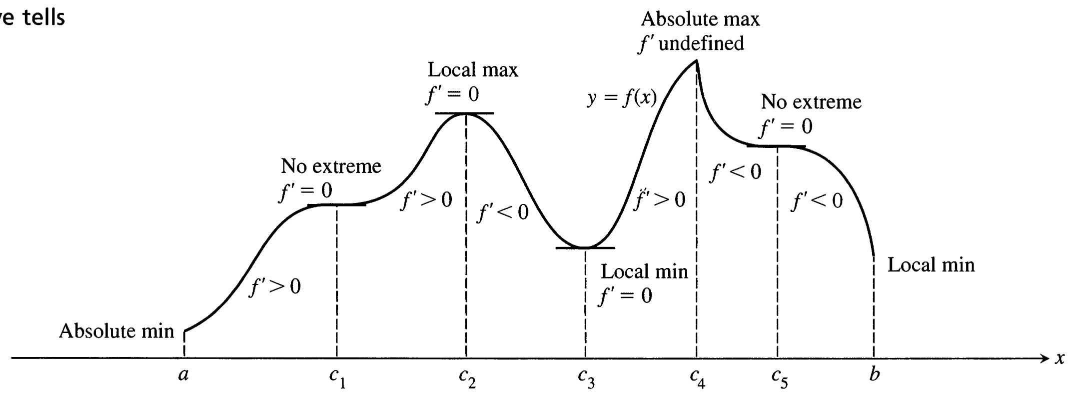
This section shows how to test a function's critical points for the presence of local extreme values.

The Test

As we see once again in Fig. 3.24, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in the point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

These observations lead to a test for the presence of local extreme values.



3.24 A function's first derivative tells how the graph rises and falls.

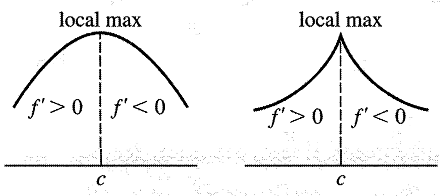
Theorem 5

The First Derivative Test for Local Extreme Values

The following test applies to a continuous function $f(x)$.

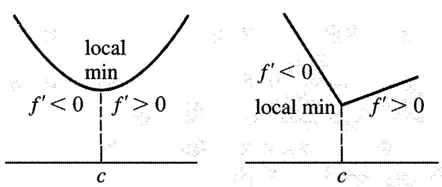
At a critical point c:

1. If f' changes from positive to negative at c $(f' > 0$ for $x < c$ and $f' < 0$ for $x > c)$, then f has a local maximum value at c.



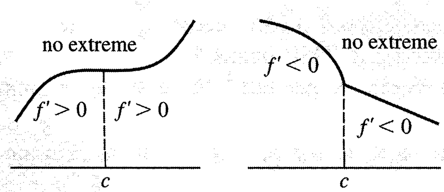
(a) $f'(c) = 0$ (b) $f'(c)$ undefined

2. If f' changes from negative to positive at c $(f' < 0$ for $x < c$ and $f' > 0$ for $x > c)$, then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

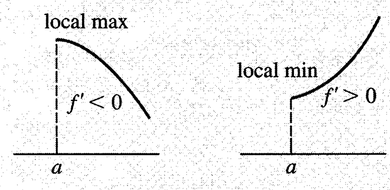
3. If f' does not change sign at c $(f'$ has the same sign on both sides of $c)$, then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

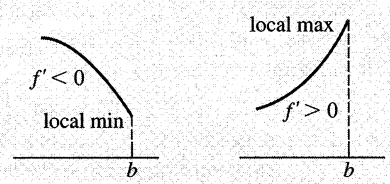
At a left endpoint a:

If $f' < 0$ $(f' > 0)$ for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ $(f' > 0)$ for $x < b$, then f has a local minimum (maximum) value at b.

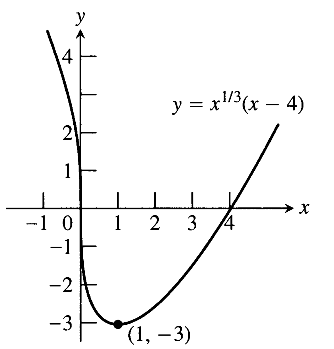


EXAMPLE 1 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is defined for all real numbers and is continuous (Fig. 3.25).



3.25 The graph of $y = x^{1/3}(x - 4)$ (Example 1).

The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x-1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in f's domain, so the critical points, $x = 0$ and $x = 1$, are the only places where f might have an extreme value of any kind.

These critical points divide the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f both between and at the critical points. We can display the information in a picture like the following.

Sign of $\frac{4}{3x^{2/3}}$:	$+$		$+$		$+$
Sign of $(x-1)$:	$-$		$-$		$+$
Sign of $f'(x) = \frac{4}{3x^{2/3}}(x-1)$:	$-$		$-$		$+$
x		0		1	
Change in f:	$\searrow$		$\searrow$		$\nearrow$
Extreme values:		no extreme		local min	

To make the picture, we marked the critical points on the x-axis, noted the sign of each factor of f' on the intervals between the points, and "multiplied" the signs of the factors to find the sign of f'. We then applied Corollary 3 of the Mean Value Theorem to determine that f decreases ($\searrow$) on $(-\infty, 0)$, decreases on $(0, 1)$, and increases ($\nearrow$) on $(1, \infty)$. Theorem 5 tells us that f has no extreme at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum because the function's values fall toward it from the left and rise away from it on the right. Figure 3.25 shows this value in relation to the function's graph. ❑

EXAMPLE 2 Find the intervals on which

$$g(x) = -x^3 + 12x + 5, \qquad -3 \le x \le 3$$

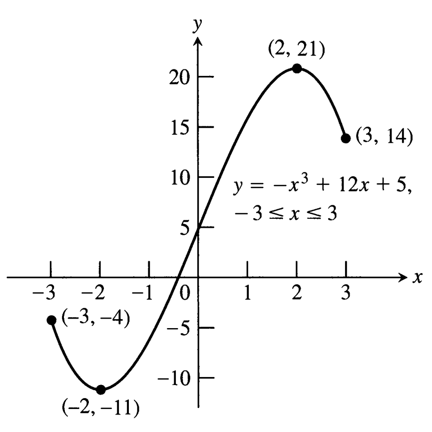
is increasing and decreasing. Where does the function assume extreme values and what are these values?

Solution The function f is continuous on its domain, $[-3, 3]$ (Fig. 3.26). The first derivative

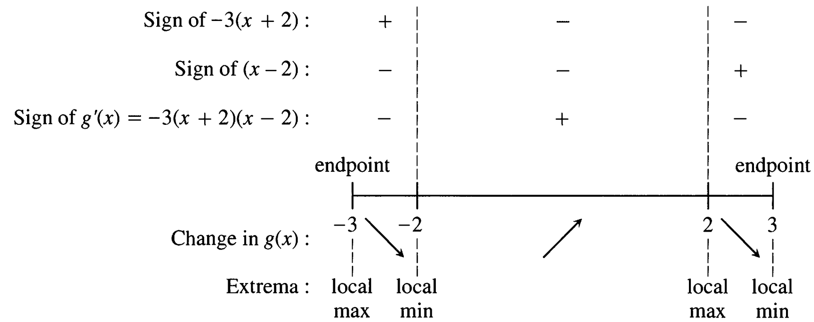
$$g'(x) = -3x^2 + 12 = -3(x^2 - 4)$$

$$= -3(x + 2)(x - 2),$$

defined at all points of $[-3, 3]$, is zero at $x = -2$ and $x = 2$. These critical points divide the domain of g into intervals on which g' is either positive or negative. We analyze the behavior of g by picturing the sign pattern of g':



3.26 The graph of $g(x) = -x^3 + 12x + 5$, $-3 \le x \le 3$ (Example 2).



We conclude that g has local maxima at $x = -3$ and $x = 2$ and local minima at $x = -2$ and $x = 3$. The corresponding values of $g(x) = -x^3 + 12x + 5$ are

Local maxima: $\quad g(-3) = -4, \quad g(2) = 21$

Local minima: $\quad g(-2) = -11, \quad g(3) = 14.$

Since g is defined on a closed interval, we also know that $g(-2)$ is the absolute minimum and $g(2)$ is the absolute maximum. Figure 3.26 shows these values in relation to the function's graph. ❑

Exercises 3.3

Analyzing *f* Given *f*

Answer the following questions about the functions whose derivatives are given in Exercises 1–8:

a) What are the critical points of f?
b) On what intervals is f increasing or decreasing?
c) At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x-1)$
2. $f'(x) = (x-1)(x+2)$
3. $f'(x) = (x-1)^2(x+2)$
4. $f'(x) = (x-1)^2(x+2)^2$
5. $f'(x) = (x-1)(x+2)(x-3)$
6. $f'(x) = (x-7)(x+1)(x+5)$
7. $f'(x) = x^{-1/3}(x+2)$
8. $f'(x) = x^{-1/2}(x-3)$

Extremes of Given Functions

In Exercises 9–28:

a) Find the intervals on which the function is increasing and decreasing.
b) Then identify the function's local extreme values, if any, saying where they are taken on.
c) Which, if any, of the extreme values are absolute?
d) GRAPHER You may wish to support your findings with a graphing calculator or computer grapher.

9. $g(t) = -t^2 - 3t + 3$
10. $g(t) = -3t^2 + 9t + 5$
11. $h(x) = -x^3 + 2x^2$
12. $h(x) = 2x^3 - 18x$
13. $f(\theta) = 3\theta^2 - 4\theta^3$
14. $f(\theta) = 6\theta - \theta^3$
15. $f(r) = 3r^3 + 16r$
16. $h(r) = (r+7)^3$
17. $f(x) = x^4 - 8x^2 + 16$
18. $g(x) = x^4 - 4x^3 + 4x^2$
19. $H(t) = \frac{3}{2}t^4 - t^6$
20. $K(t) = 15t^3 - t^5$
21. $g(x) = x\sqrt{8 - x^2}$
22. $g(x) = x^2\sqrt{5 - x}$
23. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$
24. $f(x) = \dfrac{x^3}{3x^2 + 1}$
25. $f(x) = x^{1/3}(x+8)$
26. $g(x) = x^{2/3}(x+5)$
27. $h(x) = x^{1/3}(x^2 - 4)$
28. $k(x) = x^{2/3}(x^2 - 4)$

Extremes on Half-Open Intervals

In Exercises 29–36:

a) Identify the function's local extreme values in the given domain, and say where they are assumed.
b) Which of the extreme values, if any, are absolute?
c) GRAPHER You may wish to support your findings with a graphing calculator or computer grapher.

29. $f(x) = 2x - x^2, \quad -\infty < x \le 2$
30. $f(x) = (x+1)^2, \quad -\infty < x \le 0$

31. $g(x) = x^2 - 4x + 4, \quad 1 \le x < \infty$

32. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$

33. $f(t) = 12t - t^3, \quad -3 \le t < \infty$

34. $f(t) = t^3 - 3t^2, \quad -\infty < t \le 3$

35. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \le x < \infty$

36. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$

Graphing Calculator or Computer Grapher

In Exercises 37–40:

a) Find the local extrema of each function on the given interval, and say where they are assumed.

b) GRAPHER Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

37. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \le x \le 2\pi$

38. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$

39. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

40. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

Theory and Examples

Show that the functions in Exercises 41 and 42 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

41. $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

42. $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

43. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

a) $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;
b) $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;
c) $f'(x) > 0$ for $x \ne 1$;
d) $f'(x) < 0$ for $x \ne 1$.

44. Sketch the graph of a differentiable function $y = f(x)$ that has

a) a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;
b) a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;
c) local maxima at $(1, 1)$ and $(3, 3)$;
d) local minima at $(1, 1)$ and $(3, 3)$.

45. Sketch the graph of a continuous function $y = g(x)$ such that

a) $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;
b) $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

46. Sketch the graph of a continuous function $y = h(x)$ such that

a) $h(0) = 0$, $-2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$;
b) $h(0) = 0$, $-2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

47. As x moves from left to right through the point $c = 2$, is the graph of $f(x) = x^3 - 3x + 2$ rising, or is it falling? Give reasons for your answer.

48. Find the intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

3.4 Graphing with y' and y''

In Section 3.1, we saw the role played by the first derivative in locating a function's extreme values. A function can have extreme values only at the endpoints of its domain and at its critical points. We also saw that critical points do not necessarily yield extreme values. In Section 3.2, we saw that almost all the information about a differentiable function is contained in its derivative. To recover the function completely, the only additional information we need is the value of the function at any one single point. If a function's derivative is $2x$ and the graph passes through the origin, the function must be x^2. If a function's derivative is $2x$ and the graph passes through the point $(0, 4)$, the function must be $x^2 + 4$.

In Section 3.3, we extended our ability to recover information from a function's first derivative by showing how to use it to tell exactly what happens at a critical point. We can tell whether there really is an extreme value there or whether the graph just continues to rise or fall.

In the present section, we show how to determine the way the graph of a function $y = f(x)$ bends or turns. We know that the information must be contained in y', but how do we find it? The answer, for functions that are twice differentiable except perhaps at isolated points, is to differentiate y'. Together y' and y'' tell us the shape of the function's graph. We will see in Chapter 4 how this enables us to sketch solutions of differential equations and initial value problems.

Concavity

As you can see in Fig. 3.27, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we come in from the left toward the origin along the curve, the curve turns to our right and falls below its tangents. As we leave the origin, the curve turns to our left and rises above its tangents.

To put it another way, the slopes of the tangents decrease as the curve approaches the origin from the left and increase as the curve moves from the origin into the first quadrant.

Definition

The graph of a differentiable function $y = f(x)$ is **concave up** on an interval where y' is increasing and **concave down** on an interval where y' is decreasing.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to conclude that y' increases if $y'' > 0$ and decreases if $y'' < 0$.

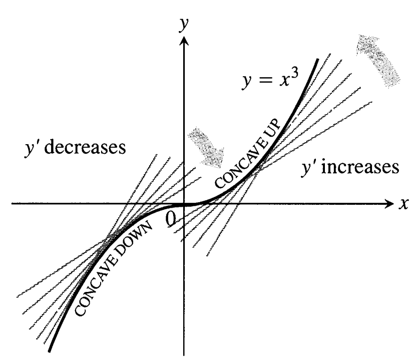
The Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on an interval I.

1. If $y'' > 0$ on I, the graph of f over I is concave up.
2. If $y'' < 0$ on I, the graph of f over I is concave down.

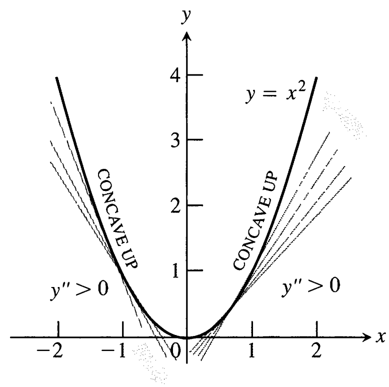
EXAMPLE 1

a) The curve $y = x^3$ (Fig. 3.27) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.



3.27 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

b) The parabola $y = x^2$ (Fig. 3.28) is concave up on every interval because $y'' = 2 > 0$.



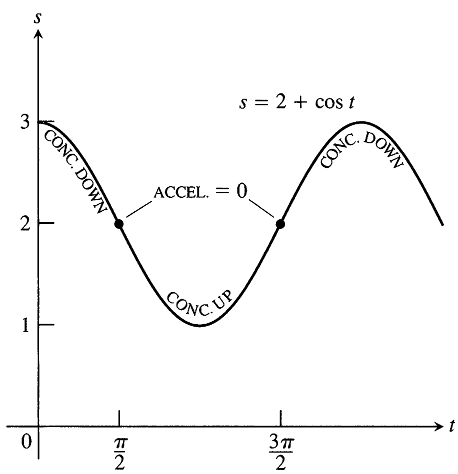
3.28 The graph of $f(x) = x^2$ on any interval is concave up. ❑

Points of Inflection

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points where the concavity changes.

Definition

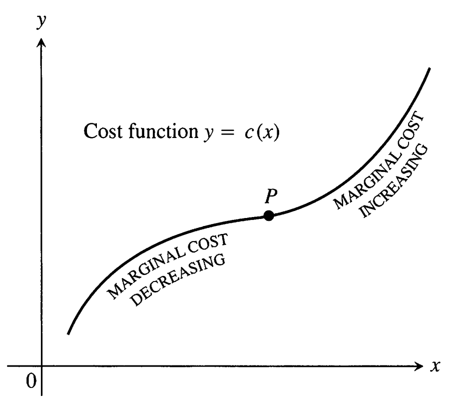
A point where the graph of a function has a tangent line and where the concavity changes is called a **point of inflection.**



3.29 The motion in Example 2.

Thus a point of inflection on a curve is a point where y'' is positive on one side and negative on the other. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined.

> On the graph of a twice-differentiable function, $y'' = 0$ at a point of inflection.



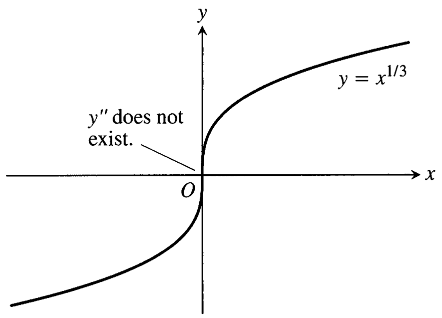
3.30 The point of inflection on a typical cost curve separates the interval of decreasing marginal cost from the interval of increasing marginal cost. This is the point where the marginal cost is smallest (Example 3).

EXAMPLE 2 *Simple harmonic motion*

The graph of $s = 2 + \cos t$, $t \geq 0$ (Fig. 3.29), changes concavity at $t = \pi/2, 3\pi/2, \ldots$, where the acceleration $s'' = -\cos t$ is zero. ❑

EXAMPLE 3 *Marginal cost*

Inflection points have applications in some areas of economics. Suppose that $y = c(x)$ is the total cost of producing x units of something (Fig. 3.30). The point of inflection at P is then the point at which the marginal cost (the approximate cost of producing one more unit) changes from decreasing to increasing. ❑



3.31 A point where y'' fails to exist can be a point of inflection.

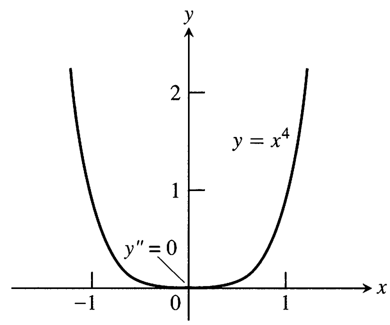
EXAMPLE 4 *An inflection point where y'' does not exist*

The curve $y = x^{1/3}$ has a point of inflection at $x = 0$ (Fig. 3.31), but y'' does not exist there.

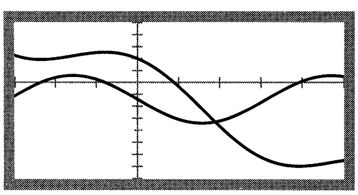
$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}$$ ❑

EXAMPLE 5 *No inflection where $y'' = 0$*

The curve $y = x^4$ has no inflection point at $x = 0$ (Fig. 3.32). Even though $y'' = 12x^2$ is zero there, it does not change sign.



3.32 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there. ❑

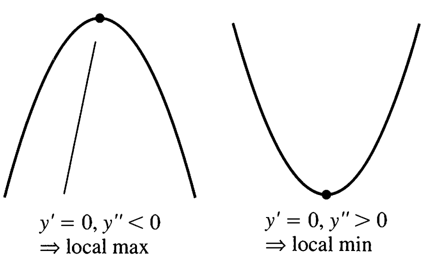


The graph of $y = 2\cos x - \sqrt{2}\,x$ and its first derivative.

Technology *Graphing a Function with Its Derivatives* When we graph a function $y = f(x)$, it may be difficult to identify the inflection points exactly by zooming in. Try it on the curve $y = 2\cos x - \sqrt{2}x$, $-\pi \leq x \leq 3\pi/2$. Adding the graph of f' to the display can help to identify inflection points more closely, but the strongest visual evidence comes from graphing f and f'' together. It is interesting to watch all three functions, f, f', and f'', being graphed simultaneously.

The Second Derivative Test for Local Extreme Values

Instead of examining y' for sign changes at a critical point, we can sometimes use the following test to determine the presence of a local extremum.



The Second Derivative Test for Local Extreme Values

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Notice that the test requires us to know y'' only at c itself, and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $y'' = 0$ or if y'' does not exist. When this happens, use the first derivative test for local extreme values.

Graphing with y' and y''

We now apply what we have learned to sketch the graphs of functions.

Testing the critical points in Example 6

As a quick test to see if any of the critical points are local extreme values, we could try the second derivative test.

At $x = 3$, $y'' > 0$:
We now know that this point is definitely a local minimum.

At $x = 0$, $y'' = 0$:
Test fails, and so we will need to check the signs of y' to know whether this point gives a local extreme value.

EXAMPLE 6 Graph the function

$$y = x^4 - 4x^3 + 10.$$

Solution

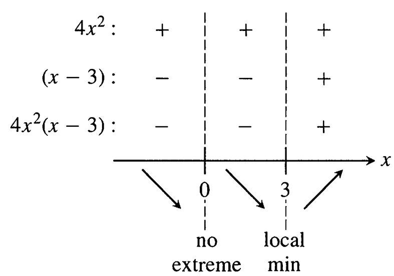
Step 1: *Find y' and y''.*

$$\begin{aligned} y &= x^4 - 4x^3 + 10 \\ y' &= 4x^3 - 12x^2 = 4x^2(x-3) \\ y'' &= 12x^2 - 24x = 12x(x-2) \end{aligned}$$

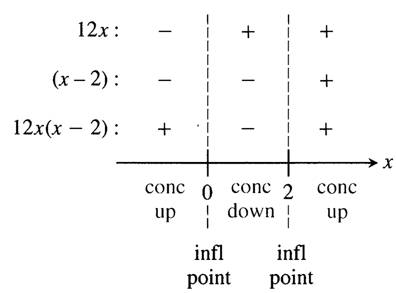
Critical points: $x = 0$, $x = 3$

Possible inflection points: $x = 0$, $x = 2$

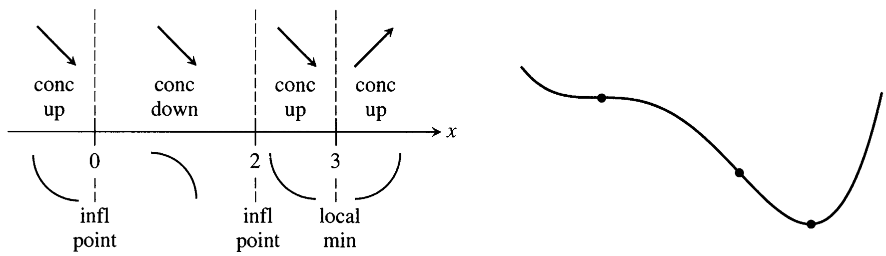
Step 2: *Rise and fall.* Sketch the sign pattern for y' and use it to describe the behavior of y.



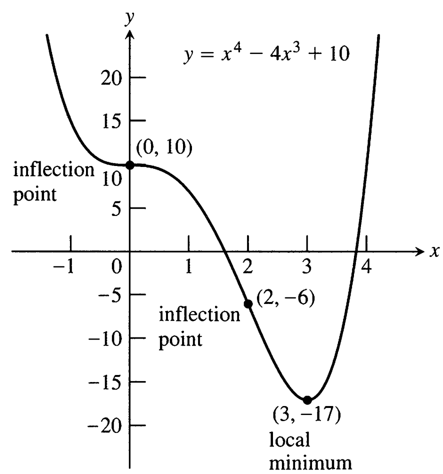
Step 3: *Concavity.* Sketch the sign pattern for y'' and use it to describe the way the graph bends.



Step 4: *Summary and general shape.* Summarize the information from steps 2 and 3. Show the shape over each interval. Then combine the shapes to show the curve's general form.



Step 5: *Specific points and curve.* Plot the curve's intercepts (if convenient) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape in step 4 as a guide to sketch the curve. (Plot additional points as needed.)



The steps in Example 6 give a general procedure for graphing by hand.

Strategy for Graphing $y = f(x)$

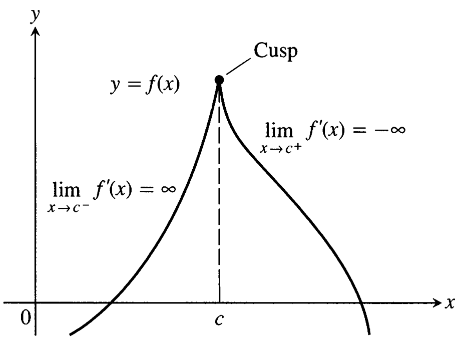
1. Find y' and y''.
2. Find the rise and fall of the curve.
3. Determine the concavity of the curve.
4. Make a summary and show the curve's general shape.
5. Plot specific points and sketch the curve.

EXAMPLE 7 Graph $y = x^{5/3} - 5x^{2/3}$.

Cusps

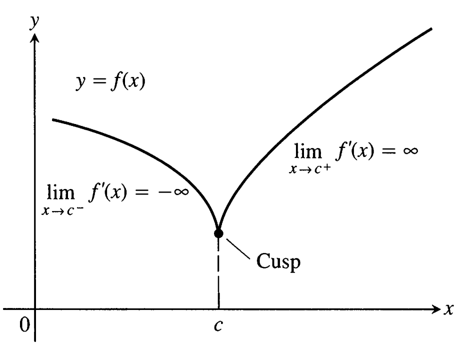
The graph of a continuous function $y = f(x)$ has a *cusp* at a point $x = c$ if the concavity is the same on both sides of c and either

1. $\lim_{x\to c^-} f'(x) = \infty$ and $\lim_{x\to c^+} f'(x) = -\infty$



or

2. $\lim_{x\to c^-} f'(x) = -\infty$ and $\lim_{x\to c^+} f'(x) = \infty$.



A cusp can be either a local maximum (1) or a local minimum (2).

Solution

Step 1: *Find y' and y''.*

$$y = x^{5/3} - 5x^{2/3} = x^{2/3}(x-5)$$
$$y' = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2)$$
$$y'' = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1)$$

The x-intercepts are at $x = 0$ and $x = 5$.

Critical points: $x = 0$, $x = 2$

Possible inflection points: $x = 0$, $x = -1$

Step 2: *Rise and fall.*

	$x < 0$	$0 < x < 2$	$x > 2$
$\frac{5}{3}x^{-1/3}$:	−	+	+
$(x-2)$:	−	−	+
$y' = \frac{5}{3}x^{-1/3}(x-2)$:	+	−	+
	↗	↘	↗

At $x = 0$: (y' doesn't exist), local max. At $x = 2$: ($y' = 0$), local min.

Step 3: *Concavity.*

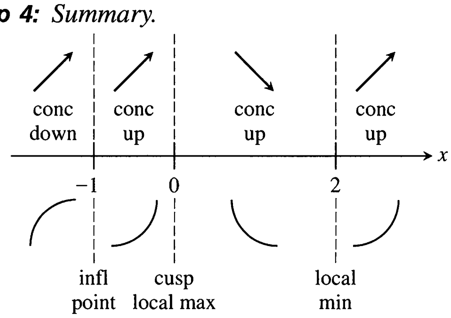
	$x < -1$	$-1 < x < 0$	$x > 0$
$\frac{10}{9}x^{-4/3}$:	+	+	+
$(x+1)$:	−	+	+
$y'' = \frac{10}{9}x^{-4/3}(x+1)$:	−	+	+
	conc down	conc up	conc up

At $x = -1$: infl point.

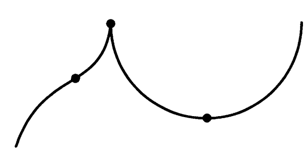
From the sign pattern for y'', we see that there is an inflection point at $x = -1$, but not at $x = 0$. However, knowing that

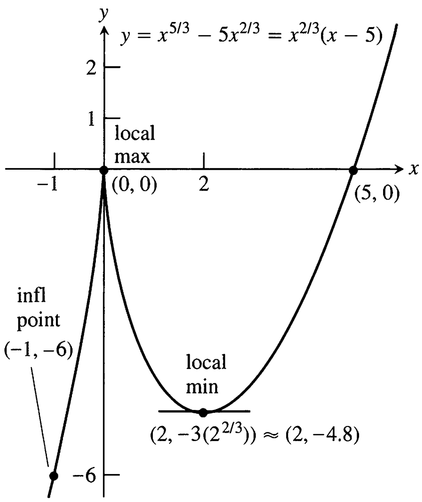
1. the function $y = x^{5/3} - 5x^{2/3}$ is continuous,
2. $y' \to \infty$ as $x \to 0^-$ and $y' \to -\infty$ as $x \to 0^+$ (see the formula for y' in step 2), and
3. the concavity does not change at $x = 0$ (step 3) tells us that the graph has a *cusp* at $x = 0$.

Step 4: *Summary.*



General shape.



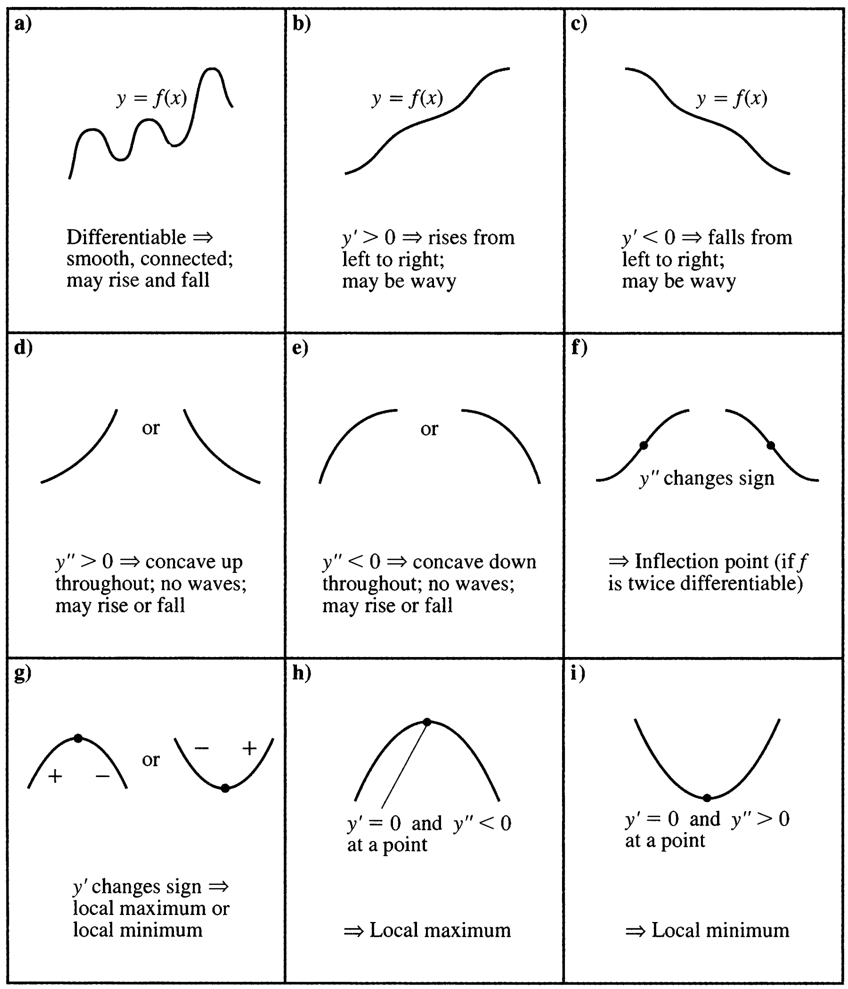


Step 5: *Specific points and curve.* See the figure to the left. ❑

Learning About Functions from Derivatives

Pause for a moment to see how remarkable the conclusions in Examples 6 and 7 really are. In each case, we have been able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining y'. We can find where the graph rises and falls and where the local extremes are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. The only information we cannot get from the derivative is how to place the graph in the xy-plane. That requires evaluating the formula for f at various points. Or so it seems. But as we saw in Section 3.2, even *that* is nearly superfluous. All we really need, in addition to y', is the value of f at a single point.

What Derivatives Tell Us About Graphs



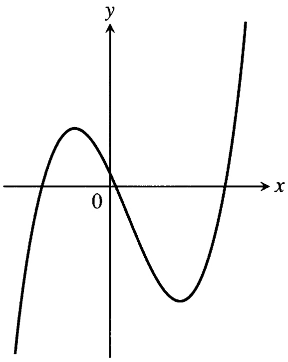
Exercises 3.4

Analyzing Graphed Functions

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

1.

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$$



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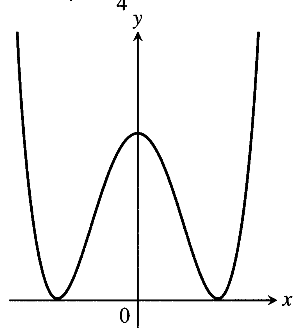
2.

$$y = \frac{x^4}{4} - 2x^2 + 4$$



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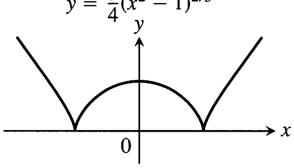
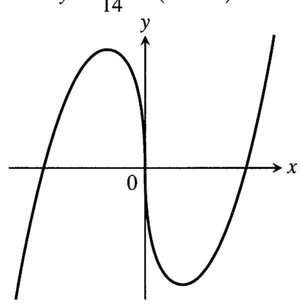
3.

$$y = \frac{3}{4}(x^2 - 1)^{2/3}$$



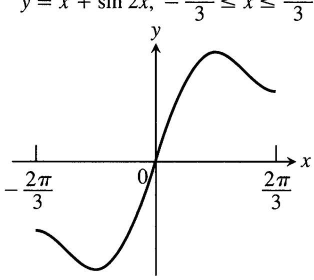
4.

$$y = \frac{9}{14}x^{1/3}(x^2 - 7)$$



5.

$$y = x + \sin 2x, \ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$$



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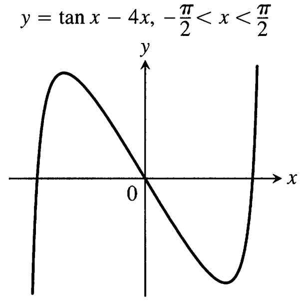
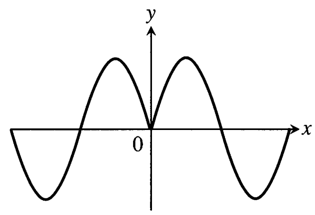
6.

$$y = \tan x - 4x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$$



7.

$$y = \sin|x|, -2\pi \le x \le 2\pi$$



NOT TO SCALE

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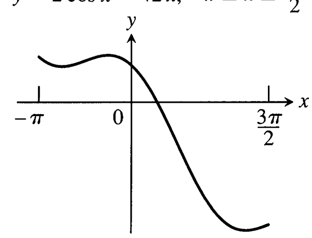
8.

$$y = 2\cos x - \sqrt{2}\,x, -\pi \le x \le \frac{3\pi}{2}$$



Graphing Equations

Use the steps of the graphing procedure on page 214 to graph the equations in Exercises 9–40. Include the coordinates of any local extreme points and inflection points.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = x^{1/5}$

26. $y = x^{3/5}$

27. $y = x^{2/5}$

28. $y = x^{4/5}$

29. $y = 2x - 3x^{2/3}$

30. $y = 5x^{2/5} - 2x$

31. $y = x^{2/3}\left(\frac{5}{2} - x\right)$

32. $y = x^{2/3}(x - 5)$

33. $y = x\sqrt{8 - x^2}$

34. $y = (2 - x^2)^{3/2}$

35. $y = \dfrac{x^2 - 3}{x - 2}, \quad x \ne 2$

36. $y = \dfrac{x^3}{3x^2 + 1}$

37. $y = |x^2 - 1|$

38. $y = |x^2 - 2x|$

39. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

40. $y = \sqrt{|x - 4|}$

Sketching the General Shape Knowing y'

Each of Exercises 41–62 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use steps 2–4 of the graphing procedure on page 214 to sketch the general shape of the graph of f.

41. $y' = 2 + x - x^2$

42. $y' = x^2 - x - 6$

43. $y' = x(x-3)^2$

44. $y' = x^2(2-x)$

45. $y' = x(x^2 - 12)$

46. $y' = (x-1)^2(2x+3)$

47. $y' = (8x - 5x^2)(4-x)^2$

48. $y' = (x^2 - 2x)(x-5)^2$

49. $y' = \sec^2 x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

50. $y' = \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

51. $y' = \cot \dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

52. $y' = \csc^2 \dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

53. $y' = \tan^2 \theta - 1, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

54. $y' = 1 - \cot^2 \theta, \quad 0 < \theta < \pi$

55. $y' = \cos t, \quad 0 \le t \le 2\pi$

56. $y' = \sin t, \quad 0 \le t \le 2\pi$

57. $y' = (x+1)^{-2/3}$

58. $y' = (x-2)^{-1/3}$

59. $y' = x^{-2/3}(x-1)$

60. $y' = x^{-4/5}(x+1)$

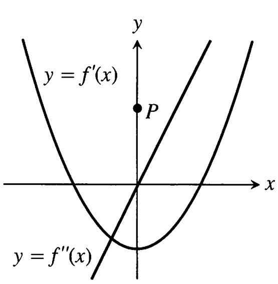
61. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

62. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

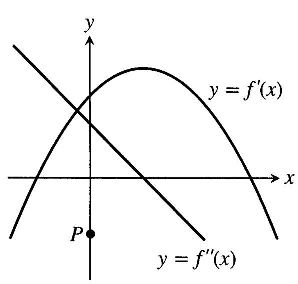
Sketching y from Graphs of y' and y''

Each of Exercises 63–66 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

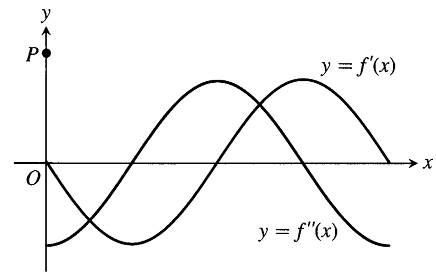
63.



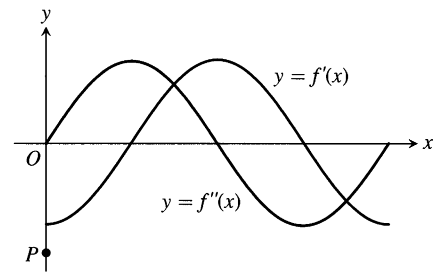
64.



65.

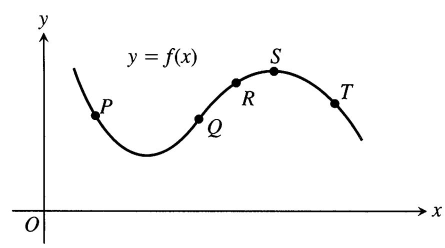


66.



Theory and Examples

67. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



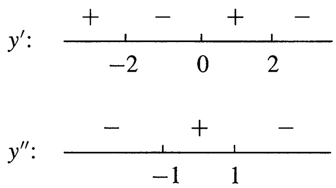
68. Sketch a smooth connected curve $y = f(x)$ with

$$\begin{aligned} &f(-2) = 8, && f'(2) = f'(-2) = 0, \\ &f(0) = 4, && f'(x) < 0 \quad \text{for} \quad |x| < 2, \\ &f(2) = 0, && f''(x) < 0 \quad \text{for} \quad x < 0, \\ &f'(x) > 0 \quad \text{for} \quad |x| > 2, && f''(x) > 0 \quad \text{for} \quad x > 0. \end{aligned}$$

69. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

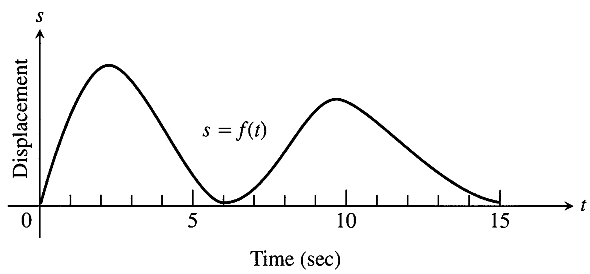
x	y	**Derivatives**
$x < 2$		$y' < 0,\quad y'' > 0$
2	1	$y' = 0,\quad y'' > 0$
$2 < x < 4$		$y' > 0,\quad y'' > 0$
4	4	$y' > 0,\quad y'' = 0$
$4 < x < 6$		$y' > 0,\quad y'' < 0$
6	7	$y' = 0,\quad y'' < 0$
$x > 6$		$y' < 0,\quad y'' < 0$

70. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 2)$ and whose first two derivatives have the following sign patterns:

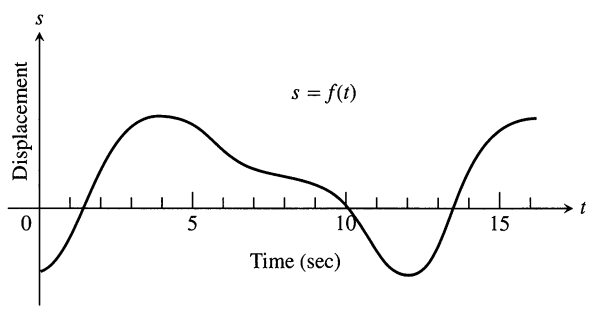


Velocity and acceleration. The graphs in Exercises 71 and 72 show the position $s = f(t)$ of a body moving back and forth on a coordinate line. (a) When is the body moving away from the origin? toward the origin? At approximately what times is the (b) velocity equal to zero? (c) acceleration equal to zero? (d) When is the acceleration positive? negative?

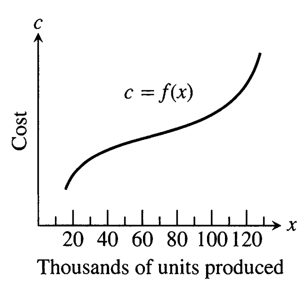
71.



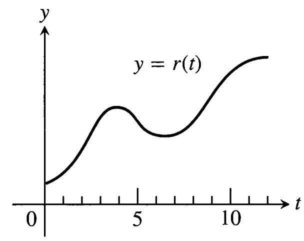
72.



73. *Marginal cost.* The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



74. The accompanying graph shows the monthly revenue of the Widget Corporation for the last twelve years. During approximately what time intervals was the marginal revenue increasing? decreasing?



75. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

76. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

77. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

78. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

79. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

80. *Horizontal tangents.* True, or false? Explain.

a) The graph of every polynomial of even degree (largest exponent even) has at least one horizontal tangent.

b) The graph of every polynomial of odd degree (largest exponent odd) has at least one horizontal tangent.

81. *Parabolas*

a) Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, $a \neq 0$.

b) When is the parabola concave up? concave down? Give reasons for your answers.

82. Is it true that the concavity of the graph of a twice-differentiable function $y = f(x)$ changes every time $f''(x) = 0$? Give reasons for your answer.

83. *Quadratic curves.* What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c$, $a \neq 0$? Give reasons for your answer.

84. *Cubic curves.* What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d$, $a \neq 0$? Give reasons for your answer.

Grapher Explorations

In Exercises 85–88, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

85. $y = x^5 - 5x^4 - 240$

86. $y = x^3 - 12x^2$

87. $y = \dfrac{4}{5}x^5 + 16x^2 - 25$

88. $y = \dfrac{x^4}{4} - \dfrac{x^3}{3} - 4x^2 + 12x + 20$

89. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

90. Graph $f(x) = x\cos x$ and its second derivative together for $0 \le x \le 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

91. a) On a common screen, graph $f(x) = x^3 + kx$ for $k = 0$ and nearby positive and negative values of k. How does the value of k seem to affect the shape of the graph?

b) Find $f'(x)$. As you will see, $f'(x)$ is a quadratic function of x. Find the discriminant of the quadratic (the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$). For what values of k is the discriminant positive? zero? negative? For what values of k does f' have two zeros? one or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

c) Experiment with other values of k. What appears to happen as $k \to -\infty$? as $k \to \infty$?

92. a) On a common screen, graph $f(x) = x^4 + kx^3 + 6x^2$, $-1 \le x \le 4$ for $k = -4$, and some nearby values of k. How does the value of k seem to affect the shape of the graph?

b) Find $f''(x)$. As you will see, $f''(x)$ is a quadratic function of x. What is the discriminant of this quadratic (see Exercise 91b)? For what values of k is the discriminant positive? zero? negative? For what values of k does $f''(x)$ have two zeros? one or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

93. a) Graph $y = x^{2/3}(x^2 - 2)$ for $-3 \le x \le 3$. Then use calculus to confirm what the screen shows about concavity, rise, and fall. (Depending on your grapher, you may have to enter $x^{2/3}$ as $(x^2)^{1/3}$ to obtain a plot for negative values of x.)

b) Does the curve have a cusp at $x = 0$, or does it just have a corner with different right-hand and left-hand derivatives?

94. a) Graph $y = 9x^{2/3}(x - 1)$ for $-0.5 \le x \le 1.5$. Then use calculus to confirm what the screen shows about concavity, rise, and fall. What concavity does the curve have to the left of the origin? (Depending on your grapher, you may have to enter $x^{2/3}$ as $(x^2)^{1/3}$ to obtain a plot for negative values of x.)

b) Does the curve have a cusp at $x = 0$, or does it just have a corner with different right-hand and left-hand derivatives?

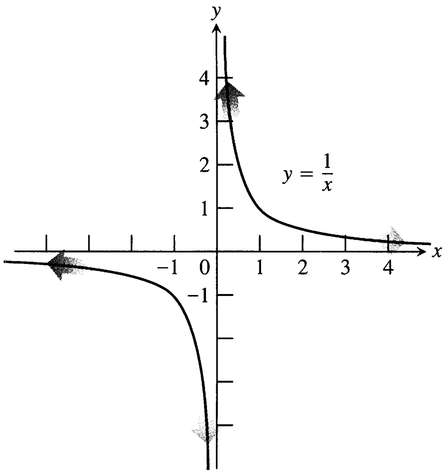
95. Does the curve $y = x^2 + 3\sin 2x$ have a horizontal tangent near $x = -3$? Give reasons for your answer.

3.5 Limits as $x \to \pm\infty$, Asymptotes, and Dominant Terms

In this section, we analyze the graphs of rational functions (quotients of polynomial functions), as well as other functions with interesting limit behavior as $x \to \pm\infty$. Among the tools we use are asymptotes and dominant terms.

Limits as $x \to \pm\infty$

The function $f(x) = 1/x$ is defined for all $x \neq 0$ (Fig. 3.33). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \pm\infty$.



3.33 The graph of $y = 1/x$.

The symbol infinity (∞)

As always, the symbol ∞ does not represent a real number and we cannot use it in arithmetic in the usual way.

Definitions

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

The strategy for calculating limits of functions as $x \to \pm\infty$ is similar to the one for finite limits in Section 1.2. There, we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying a theorem about limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x\to\pm\infty} k = k \qquad \text{and} \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the latter and leave the former to Exercises 87 and 88.

EXAMPLE 1 Show that

a) $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$ b) $\displaystyle\lim_{x\to-\infty} \frac{1}{x} = 0.$

Solution

a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

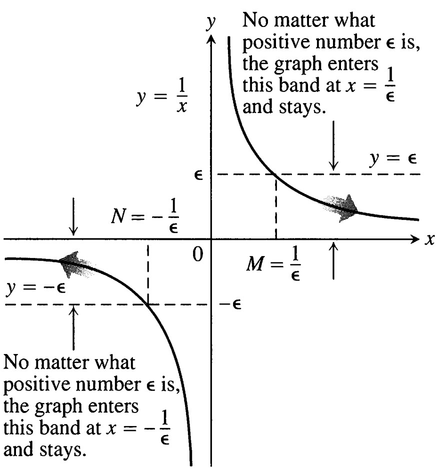
$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Fig. 3.34). This proves $\lim_{x\to\infty} (1/x) = 0$.

b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Fig. 3.34). This proves $\lim_{x\to-\infty} (1/x) = 0$. ❑



3.34 The geometry behind the argument in Example 1.

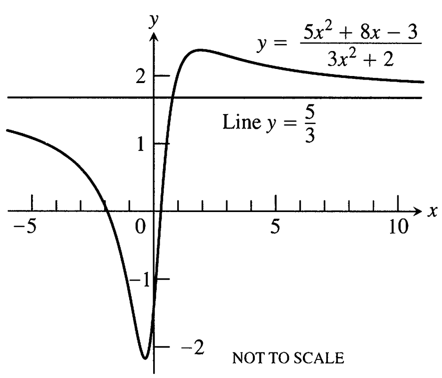
The following theorem enables us to build on Eqs. (1) to calculate other limits.

Theorem 6
Properties of Limits as $x \to \pm\infty$

The following rules hold if $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$ (L and M real numbers).

1. *Sum Rule:* $\lim\limits_{x\to\pm\infty} [f(x) + g(x)] = L + M$
2. *Difference Rule:* $\lim\limits_{x\to\pm\infty} [f(x) - g(x)] = L - M$
3. *Product Rule:* $\lim\limits_{x\to\pm\infty} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule:* $\lim\limits_{x\to\pm\infty} kf(x) = kL$ (any number k)
5. *Quotient Rule:* $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$
6. *Power Rule:* If m and n are integers, then $\lim\limits_{x\to\pm\infty} [f(x)]^{m/n} = L^{m/n}$ provided $L^{m/n}$ is a real number.

These properties are just like the properties in Theorem 1, Section 1.2, and we use them the same way.



3.35 The function in Example 3.

EXAMPLE 2

a) $$\lim_{x\to\infty}\left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$
$$= 5 + 0 = 5 \qquad \text{Known values}$$

b) $$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$
$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product Rule}$$
$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known values}$$
❑

Limits of Rational Functions as $x \to \pm\infty$

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator. What happens then depends on the degrees of the polynomials involved.

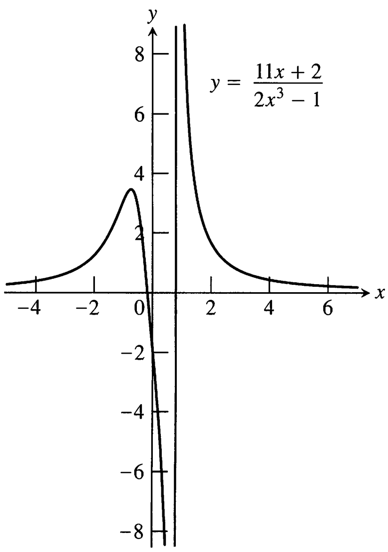
The **degree** of the polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

$a_n \neq 0$, is n, the largest exponent.

EXAMPLE 3 *Numerator and denominator of same degree*

$$\lim_{x\to\infty}\frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty}\frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} \qquad \text{Divide numerator and denominator by } x^2.$$
$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} \qquad \text{See Fig. 3.35.}$$
❑



3.36 The graph of the function in Example 4. The graph approaches the x-axis as $|x|$ increases.

EXAMPLE 4 *Degree of numerator less than degree of denominator*

$$\lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$$

Divide numerator and denominator by x^3.

$$= \frac{0+0}{2-0} = 0$$

See Fig. 3.36. ❑

EXAMPLE 5 *Degree of numerator greater than degree of denominator*

a) $$\lim_{x\to-\infty} \frac{2x^2-3}{7x+4} = \lim_{x\to-\infty} \frac{2x-(3/x)}{7+(4/x)}$$

Divide numerator and denominator by x.

$$= -\infty$$

The numerator now approaches $-\infty$ while the denominator approaches 7, so the ratio $\to -\infty$. See Fig. 3.37.

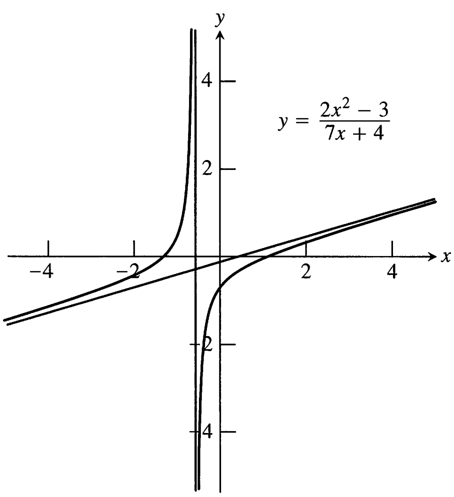
b) $$\lim_{x\to-\infty} \frac{-4x^3+7x}{2x^2-3x-10} = \lim_{x\to-\infty} \frac{-4x+(7/x)}{2-(3/x)-(10/x^2)}$$

Divide numerator and denominator by x^2.

$$= \infty$$

Numerator $\to \infty$. Denominator $\to 2$. Ratio $\to \infty$.

❑



3.37 The function in Example 5(a).

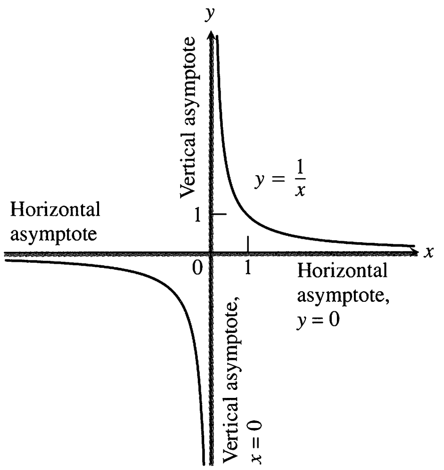
Examples 3–5 reveal a pattern for finding limits of rational functions as $x \to \pm\infty$.

1. If the numerator and the denominator have the same degree, the limit is the ratio of the polynomials' leading coefficients (Example 3).
2. If the degree of the numerator is less than the degree of the denominator, the limit is zero (Example 4).
3. If the degree of the numerator is greater than the degree of the denominator, the limit is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator as $|x|$ becomes large (Example 5).

The **leading coefficient** of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, is a_n, the coefficient of the highest-powered term.

Summary for Rational Functions

1. If deg (f) = deg (g), $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{a_n}{b_n}$, the ratio of the leading coefficients of f and g.
2. If deg (f) < deg (g), $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$.
3. If deg (f) > deg (g), $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty$, depending on the signs of numerator and denominator.



3.38 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

Horizontal and Vertical Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

EXAMPLE 6 The coordinate axes are asymptotes of the curve $y = 1/x$ (Fig. 3.38). The x-axis is an asymptote of the curve on the right because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

The y-axis is an asymptote of the curve both above and below because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Notice that the denominator is zero at $x = 0$ and the function is undefined. ❑

Definitions

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

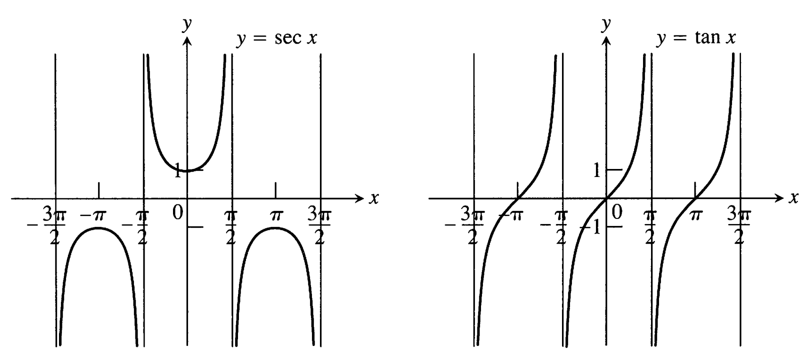
A line $x = a$ is a **vertical asymptote** of the graph if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

EXAMPLE 7 The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Fig. 3.39).

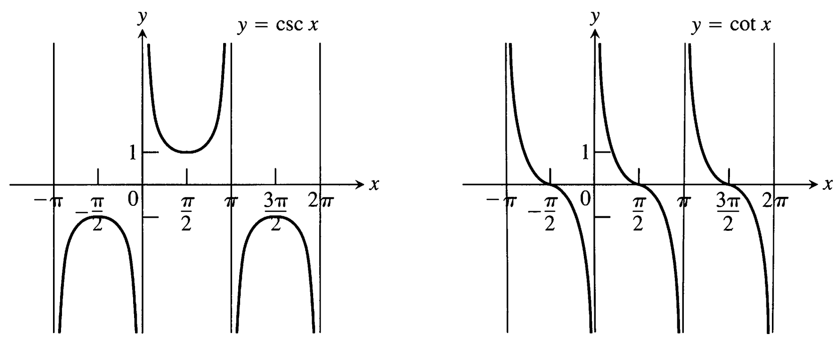


3.39 The graphs of sec x and tan x (Example 7).

The graphs of

$$y = \csc x = \frac{1}{\sin x} \qquad \text{and} \qquad y = \cot x = \frac{\cos x}{\sin x}$$

have vertical asymptotes at integer multiples of π, where $\sin x = 0$ (Fig. 3.40).



3.40 The graphs of $\csc x$ and $\cot x$ (Example 7). ❑

EXAMPLE 8 Find the asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$, where the denominator is zero.

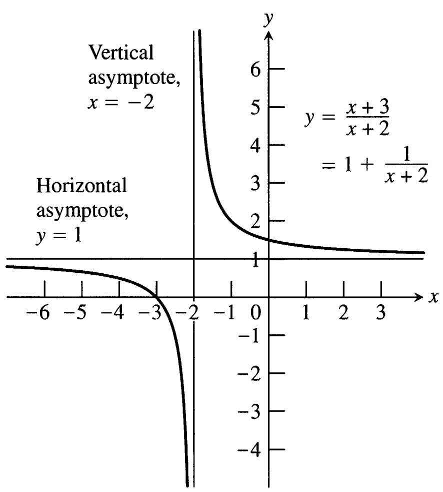
The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x+2)$ into $(x+3)$.

$$\begin{array}{r} 1 \\ x+2 \overline{)\, x+3} \\ \underline{x+2} \\ 1 \end{array}$$

This enables us to rewrite y:

$$y = 1 + \frac{1}{x+2}$$

From this we see that the curve in question is the graph of $y = 1/x$ shifted 1 unit up and 2 units left (Fig. 3.41). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ❑



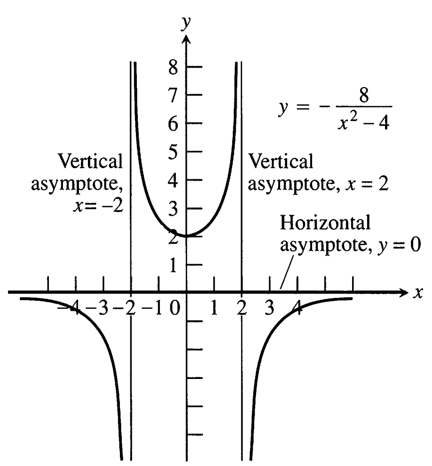
3.41 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x+3)/(x+2)$ (Example 8).

EXAMPLE 9 Find the asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

The behavior as $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is an asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Fig. 3.42).



3.42 The graph of $y = -8/(x^2 - 4)$ (Example 9). Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

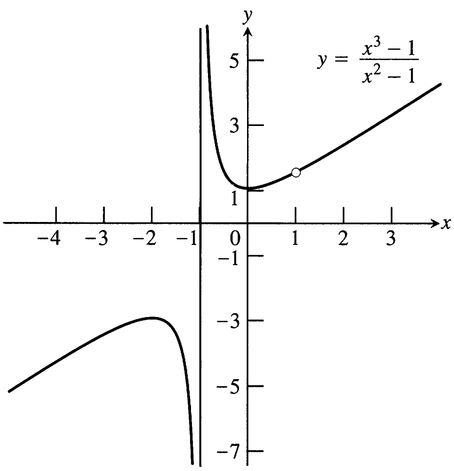
The behavior as $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is an asymptote both from the right and from the left. By symmetry, the same holds for the line $x = -2$.

There are no other asymptotes because f has a finite limit at every other point. ❑

We might be tempted at this point to say that rational functions have vertical asymptotes where their denominators are zero. That is nearly true, but not quite. What is true is that rational functions *reduced to lowest terms* have vertical asymptotes where their denominators are zero.



3.43 The graph of $f(x) = (x^3 - 1)/(x^2 - 1)$ has one vertical asymptote, not two. The discontinuity at $x = 1$ is removable.

EXAMPLE 10 *A removable discontinuity at a zero of the denominator*

The graph of

$$f(x) = \frac{x^3 - 1}{x^2 - 1}$$

has a vertical asymptote at $x = -1$ but not at $x = 1$. Since

$$\frac{x^3 - 1}{x^2 - 1} = \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \frac{x^2 + x + 1}{x + 1},$$

the function has a finite limit (3/2) as $x \to 1$ and the discontinuity is removable (Fig. 3.43). ❑

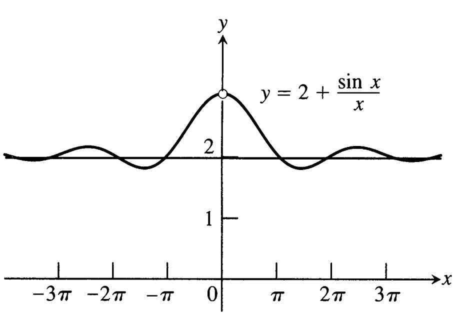
The Sandwich Theorem (Section 1.2, Theorem 4) also holds for limits as $x \to \pm\infty$. Here is a typical application.

EXAMPLE 11 Using the Sandwich Theorem, find the asymptotes of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to 0$, where the denominator is zero.

The behavior as $x \to 0$. We know that $\lim_{x\to 0} (\sin x)/x = 1$, so there is no asymptote at the origin.



3.44 A curve may cross one of its asymptotes infinitely often (Example 11).

The behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|,$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is an asymptote of the curve on both left and right (Fig. 3.44). ❑

Oblique Asymptotes

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique asymptote,** that is, a linear asymptote that is neither vertical nor horizontal.

EXAMPLE 12 Find the asymptotes of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

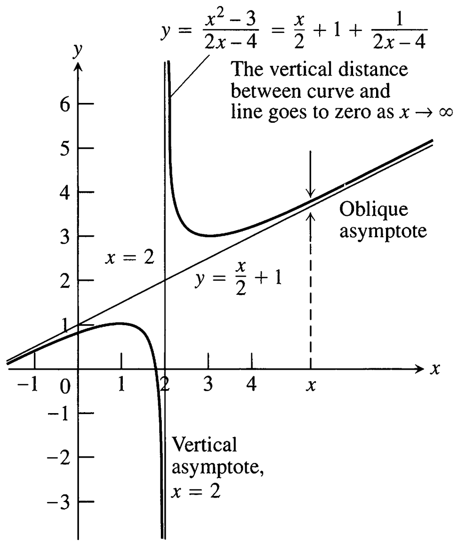
Solution We are interested in the behavior as $x \to \pm\infty$ and also as $x \to 2$, where the denominator is zero. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r@{}l}
 & \frac{x}{2} + 1 \\
2x - 4 \big) & x^2 - 3 \\
 & \underline{x^2 - 2x} \\
 & \quad\;\; 2x - 3 \\
 & \quad\;\; \underline{2x - 4} \\
 & \qquad\quad\;\; 1
\end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{\text{linear}} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder}}. \tag{2}$$

Since $\lim_{x \to 2^+} f(x) = \infty$ and $\lim_{x \to 2^-} f(x) = -\infty$, the line $x = 2$ is a two-sided asymptote. As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to (x/2) + 1$. The line $y = (x/2) + 1$ is an asymptote both to the right and to the left (Fig. 3.45). ❑



3.45 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ (Example 12).

Graphing with Asymptotes and Dominant Terms

Of all the observations we can make quickly about the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Example 12, probably the most useful is that

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}.$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{for } x \text{ numerically large}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{for } x \text{ near 2}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when x is numerically large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x is numerically large, and we say that $1/(2x - 4)$ dominates when x is near 2. **Dominant terms** like these are the key to predicting a function's behavior.

EXAMPLE 13 Graph the function

$$y = \frac{x^3 + 1}{x}.$$

Solution We investigate symmetry, dominant terms, asymptotes, rise, fall, extreme values, and concavity.

Step 1: *Symmetry.* There is none.

Step 2: *Find any dominant terms and asymptotes.* We write the rational function as a polynomial plus remainder:

$$y = x^2 + \frac{1}{x}. \tag{3}$$

For $|x|$ large, $y \approx x^2$. For x near zero, $y \approx 1/x$.

Equation (3) reveals a vertical asymptote at $x = 0$, where the denominator of the remainder is zero.

Step 3: *Find y' and analyze the function's critical points. Where does the curve rise and fall?*

The first derivative

$$y' = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} \qquad \text{From Eq. (3)}$$

is undefined at $x = 0$ and zero when

$$\begin{aligned} 2x - \frac{1}{x^2} &= 0 \\ 2x^3 - 1 &= 0 \\ x^3 &= \frac{1}{2} \\ x &= \frac{1}{\sqrt[3]{2}} \approx 0.8. \end{aligned}$$

$2x^3 - 1$:	−		−		+
x^2:	+		+		+
$y' = \frac{2x^3 - 1}{x^2}$:	−		−		+
x	↘	0	↘	0.8	↗
		no extreme value		local min	

Step 4: *Find y'' and determine the curve's concavity.* The second derivative

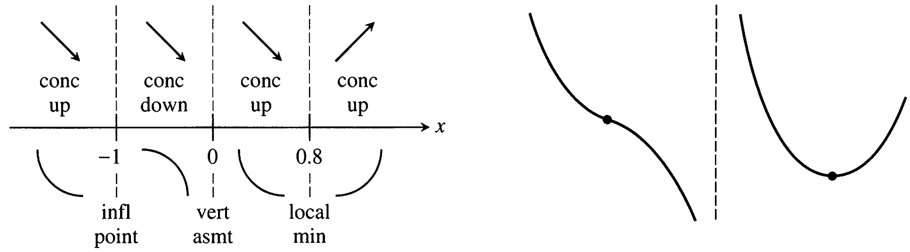
$$y'' = 2 + \frac{2}{x^3} = \frac{2x^3 + 2}{x^3}$$

is undefined at $x = 0$ and zero when

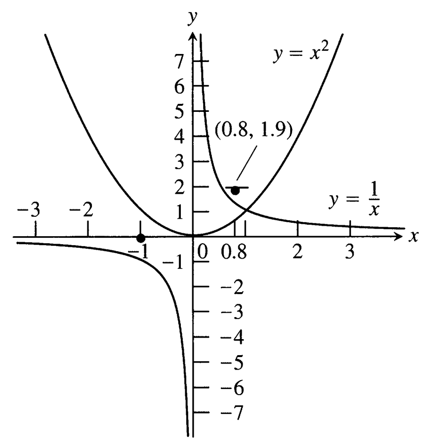
$$\begin{aligned} 2 + \frac{2}{x^3} &= 0 \\ 2x^3 + 2 &= 0 \\ x^3 &= -1 \\ x &= -1. \end{aligned}$$

$2x^3 + 2$:	−		+		+
x^3:	−		−		+
$y'' = \frac{2x^3 + 2}{x^3}$:	+		−		+
x	conc up	−1	conc down	0	conc up
		infl point			

Step 5: *Summarize the information from the preceding steps and sketch the curve's general shape.*

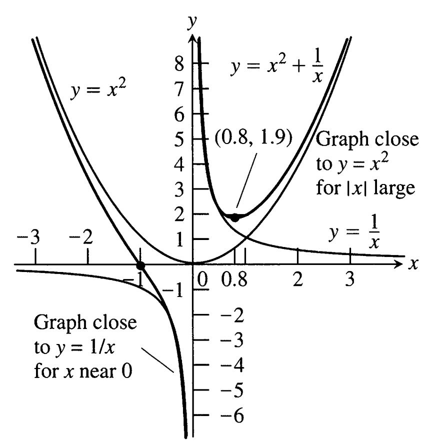


Step 6: *Plot the curve's intercepts, mark any horizontal tangents, and graph the dominant terms.* See Fig. 3.46. This provides a framework for graphing the curve.



3.46 The dominant terms and horizontal tangent provide a framework for graphing the function.

Step 7: *Now add the final curve to your figure, using the framework and the curve's general shape as guides.* See Fig. 3.47. ❑



3.47 The function, graphed with the aid of the framework in Fig. 3.46.

Hidden Behavior

Sometimes graphing f' or f'' will suggest where to zoom in on a computer generated graph of f to reveal behavior hidden in the grapher's original picture.

Checklist for Graphing a Function $y = f(x)$

1. Look for symmetry.
   Is the function even? odd?
2. Is the function a shift of a known function?
3. Analyze dominant terms.
   Divide rational functions into polynomial + remainder.
4. Check for asymptotes and removable discontinuities.
   Is there a zero denominator at any point?
   What happens as $x \to \pm\infty$?
5. Compute f' and solve $f' = 0$. Identify critical points and determine intervals of rise and fall.
6. Compute f'' to determine concavity and inflection points.
7. Sketch the graph's general shape.
8. Evaluate f at special values (endpoints, critical points, intercepts).
9. Graph f, using dominant terms, general shape, and special points for guidance.

Exercises 3.5

Calculating Limits as $x \to \pm\infty$

In Exercises 1–6, find the limit of each function (a) as $x \to \infty$ and (b) as $x \to -\infty$. (You may wish to visualize your answer with a grapher.)

1. $f(x) = \dfrac{2}{x} - 3$

2. $f(x) = \pi - \dfrac{2}{x^2}$

3. $g(x) = \dfrac{1}{2 + (1/x)}$

4. $g(x) = \dfrac{1}{8 - (5/x^2)}$

5. $h(x) = \dfrac{-5 + (7/x)}{3 - (1/x^2)}$

6. $h(x) = \dfrac{3 - (2/x)}{4 + \left(\sqrt{2}/x^2\right)}$

Find the limits in Exercises 7–10.

7. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

8. $\lim\limits_{\theta\to-\infty} \dfrac{\cos\theta}{3\theta}$

9. $\lim\limits_{t\to-\infty} \dfrac{2 - t + \sin t}{t + \cos t}$

10. $\lim\limits_{r\to\infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 11–24, find the limit of each rational function (a) as $x \to \infty$ and (b) as $x \to -\infty$.

11. $f(x) = \dfrac{2x + 3}{5x + 7}$

12. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

13. $f(x) = \dfrac{x + 1}{x^2 + 3}$

14. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

15. $f(x) = \dfrac{1 - 12x^3}{4x^2 + 12}$

16. $g(x) = \dfrac{1}{x^3 - 4x + 1}$

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

18. $g(x) = \dfrac{3x^2 - 6x}{4x - 8}$

19. $f(x) = \dfrac{2x^5 + 3}{-x^2 + x}$

20. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

21. $g(x) = \dfrac{x^4}{x^3 + 1}$

22. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

23. $h(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

24. $h(x) = \dfrac{-x^4}{x^4 - 7x^3 + 7x^2 + 9}$

Limits with Noninteger or Negative Powers

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 25–30.

25. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

26. $\lim\limits_{x\to\infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

27. $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

28. $\lim\limits_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

29. $\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

30. $\lim_{x\to-\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

Inventing Graphs from Values and Limits

In Exercises 31–34, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

31. $f(0) = 0$, $f(1) = 2$, $f(-1) = -2$, $\lim_{x\to-\infty} f(x) = -1$, and $\lim_{x\to\infty} f(x) = 1$

32. $f(0) = 0$, $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = 2$, and $\lim_{x\to 0^-} f(x) = -2$

33. $f(0) = 0$, $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 1^-} f(x) = \lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to 1^+} f(x) = -\infty$, and $\lim_{x\to -1^-} f(x) = -\infty$

34. $f(2) = 1$, $f(-1) = 0$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to-\infty} f(x) = 1$

Inventing Functions

In Exercises 35–38, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

35. $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 2^-} f(x) = \infty$, and $\lim_{x\to 2^+} f(x) = \infty$

36. $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$

37. $\lim_{x\to-\infty} h(x) = -1$, $\lim_{x\to\infty} h(x) = 1$, $\lim_{x\to 0^-} h(x) = -1$, and $\lim_{x\to 0^+} h(x) = 1$

38. $\lim_{x\to\pm\infty} k(x) = 1$, $\lim_{x\to 1^-} k(x) = \infty$, and $\lim_{x\to 1^+} k(x) = -\infty$

Graphing Rational Functions

Graph the rational functions in Exercises 39–66. Include the graphs and equations of the asymptotes and dominant terms.

39. $y = \frac{1}{x-1}$

40. $y = \frac{1}{x+1}$

41. $y = \frac{1}{2x+4}$

42. $y = \frac{-3}{x-3}$

43. $y = \frac{x+3}{x+2}$

44. $y = \frac{2x}{x+1}$

45. $y = \frac{2x^2+x-1}{x^2-1}$

46. $y = \frac{x^2-49}{x^2+5x-14}$

47. $y = \frac{x^2-1}{x}$

48. $y = \frac{x^2+4}{2x}$

49. $y = \frac{x^4+1}{x^2}$

50. $y = \frac{x^3+1}{x^2}$

51. $y = \frac{1}{x^2-1}$

52. $y = \frac{x^2}{x^2-1}$

53. $y = -\frac{x^2-2}{x^2-1}$

54. $y = \frac{x^2-4}{x^2-2}$

55. $y = \frac{x^2}{x-1}$

56. $y = -\frac{x^2}{x+1}$

57. $y = \frac{x^2-4}{x-1}$

58. $y = -\frac{x^2-4}{x+1}$

59. $y = \frac{x^2-x+1}{x-1}$

60. $y = -\frac{x^2-x+1}{x-1}$

61. $y = \frac{x^3-3x^2+3x-1}{x^2+x-2}$

62. $y = \frac{x^3+x-2}{x-x^2}$

63. $y = \frac{x}{x^2-1}$

64. $y = \frac{x-1}{x^2(x-2)}$

65. $y = \frac{8}{x^2+4}$ (Agnesi's witch)

66. $y = \frac{4x}{x^2+4}$ (Newton's serpentine)

Grapher Explorations

Graph the curves in Exercises 67–72 and explain the relation between the curve's formula and what you see.

67. $y = \frac{x}{\sqrt{4-x^2}}$

68. $y = \frac{-1}{\sqrt{4-x^2}}$

69. $y = x^{2/3} + \frac{1}{x^{1/3}}$

70. $y = 2\sqrt{x} + \frac{2}{\sqrt{x}} - 3$

71. $y = \sin\left(\frac{\pi}{x^2+1}\right)$

72. $y = -\cos\left(\frac{\pi}{x^2+1}\right)$

Graphing Terms

Each of the functions in Exercises 73–76 is given as the sum or difference of two terms. First graph the terms (with the same set of axes). Then, using these graphs as guides, sketch in the graph of the function.

73. $y = \sec x + \frac{1}{x}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

74. $y = \sec x - \frac{1}{x^2}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

75. $y = \tan x + \frac{1}{x^2}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

76. $y = \frac{1}{x} - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Theory and Examples

77. Let $f(x) = (x^3 + x^2)/(x^2 + 1)$. Show that there is a value of c for which $f(c)$ equals

a) -2 **b)** $\cos 3$ **c)** $5{,}000{,}000$.

EXAMPLE 6

a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x\, dx$

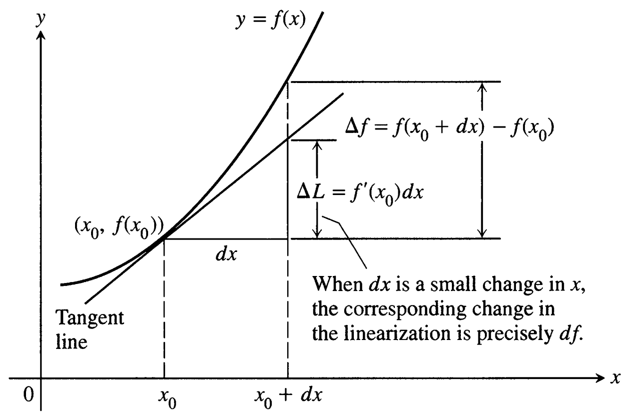
b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d\,(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ❑

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point x_0 and we want to predict how much this value will change if we move to a nearby point $x_0 + dx$. If dx is small, f and its linearization L at x_0 will change by nearly the same amount. Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

In the notation of Fig. 3.65, the change in f is

$$\Delta f = f(x_0 + dx) - f(x_0).$$



3.65 If dx is small, the change in the linearization of f is nearly the same as the change in f.

The corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + dx) - L(x_0)\\ &= \underbrace{f(x_0) + f'(x_0)\Big[(x_0 + dx) - x_0\Big]}_{L(x_0+dx)} - \underbrace{f(x_0)}_{L(x_0)=f(x_0)}\\ &= f'(x_0)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: When df is evaluated at $x = x_0$, $df = \Delta L$, the change in the linearization of f corresponding to the change dx.

The Differential Estimate of Change

Let $f(x)$ be differentiable at $x = x_0$. The approximate change in the value of f when x changes from x_0 to $x_0 + dx$ is

$$df = f'(x_0)\,dx.$$

Grapher Explorations—"Seeing" Limits at Infinity

Sometimes a change of variable can change an unfamiliar expression into one whose limit we know how to find. For example,

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{\theta\to 0^+} \sin\theta \qquad \text{Substitute } \theta = 1/x$$
$$= 0.$$

This suggests a creative way to "see" limits at infinity. Describe the procedure and use it to picture and determine limits in Exercises 103–108.

103. $\displaystyle\lim_{x\to\pm\infty} x\sin\frac{1}{x}$

104. $\displaystyle\lim_{x\to-\infty} \frac{\cos(1/x)}{1+(1/x)}$

105. $\displaystyle\lim_{x\to\pm\infty} \frac{3x+4}{2x-5}$

106. $\displaystyle\lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x}$

107. $\displaystyle\lim_{x\to\pm\infty} \left(3+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)$

108. $\displaystyle\lim_{x\to\infty} \left(\frac{3}{x^2}-\cos\frac{1}{x}\right)\left(1+\sin\frac{1}{x}\right)$

3.6 Optimization

To optimize something means to maximize or minimize some aspect of it. What is the size of the most profitable production run? What is the least expensive shape for an oil can? What is the stiffest beam we can cut from a 12-inch log? In the mathematical models in which we use functions to describe the things that interest us, we usually answer such questions by finding the greatest or smallest value of a differentiable function.

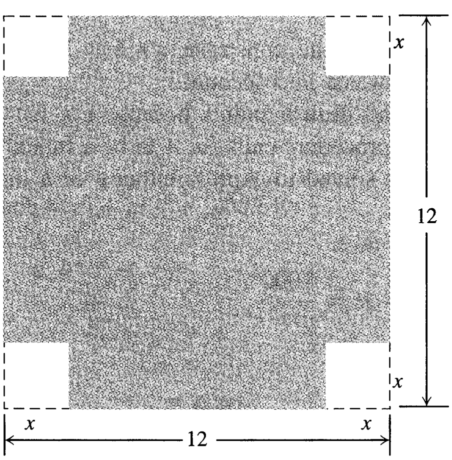
Examples from Business and Industry

EXAMPLE 1 *Metal fabrication*

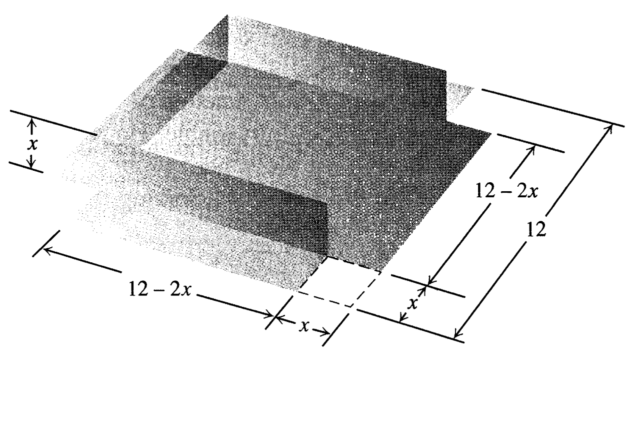
An open-top box is to be made by cutting small congruent squares from the corners of a 12-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Fig. 3.48). In the figure, the corner squares are x inches on a side. The volume of the box is a function of this variable:

$$V(x) = x(12-2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

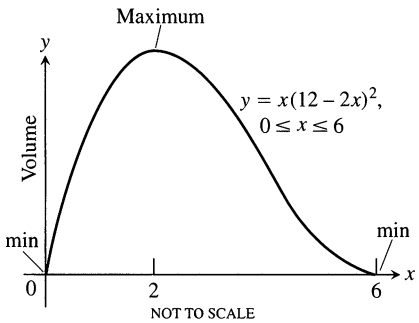


(a)



(b)

3.48 An open box made by cutting the corners from a square sheet of tin.



3.49 The volume of the box in Fig. 3.48 graphed as a function of *x*.

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Fig. 3.49) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical-point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \quad V(6) = 0.$

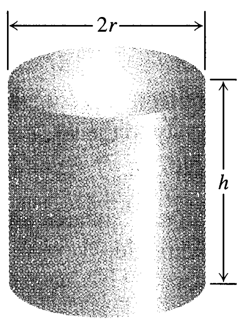
The maximum volume is 128 in^3. The cut-out squares should be 2 in. on a side. ❑

EXAMPLE 2 *Product design*

You have been asked to design a 1-L oil can shaped like a right circular cylinder. What dimensions will use the least material?

Solution We picture the can as a right circular cylinder with height h and diameter $2r$ (Fig. 3.50). If r and h are measured in centimeters and the volume is expressed as 1000 cm^3, then r and h are related by the equation

$$\pi r^2 h = 1000. \qquad 1 \text{ L} = 1000 \text{ cm}^3 \tag{1}$$



3.50 This 1-L can uses the least material when $h = 2r$ (Example 2).

How shall we interpret the phrase "least material"? One possibility is to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area

$$A = \underbrace{2\pi r^2}_{\text{cylinder ends}} + \underbrace{2\pi r h}_{\text{cylinder wall}} \tag{2}$$

as small as possible while satisfying the constraint $\pi r^2 h = 1000$. (Exercise 18 describes one way we might take waste into account.)

We are not quite ready to find critical points because Eq. (2) gives A as a function of two variables and our procedure calls for A to be a function of a single variable. However, Eq. (1) can be solved to express either r or h in terms of the other.

Solving for h is easier, so we take

$$h = \frac{1000}{\pi r^2}.$$

This changes the formula for A to

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.$$

For small r (a tall thin container, like a pipe), the term $2000/r$ dominates and A is large. For larger r (a short wide container, like a pizza pan), the term $2\pi r^2$

dominates and A is again large. If A has a minimum, it must be at a value of r that is neither too large nor too small.

Since A is differentiable throughout its domain $(0, \infty)$ and the domain has no endpoints, A can have a minimum only where $dA/dr = 0$.

$$A = 2\pi r^2 + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} \qquad \text{Find } dA/dr.$$

$$4\pi r - \frac{2000}{r^2} = 0 \qquad \text{Set it equal to 0.}$$

$$4\pi r^3 = 2000 \qquad \text{Solve for } r.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \qquad \text{Critical point}$$

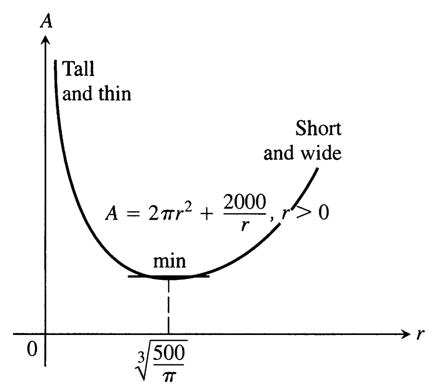
So something happens at $r = \sqrt[3]{500/\pi}$, but what?

If the domain of A were a closed interval, we could find out by evaluating A at this critical point and the endpoints and comparing the results. But the domain is not a closed interval, so we must learn what is happening at $r = \sqrt[3]{500/\pi}$ by determining the shape of A's graph. We can do this by investigating the second derivative, d^2A/dr^2:

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}.$$

The second derivative is positive throughout the domain of A. The value of A at $r = \sqrt[3]{500/\pi}$ is therefore an absolute minimum because the graph of A is concave up (Fig. 3.51).



3.51 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

When

$$r = \sqrt[3]{500/\pi},$$

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{500/\pi} = 2r. \qquad \text{After some arithmetic} \quad (3)$$

Equation (3) tells us that the most efficient can has its height equal to its diameter. With a calculator we find

$$r \approx 5.42 \text{ cm}, \qquad h \approx 10.84 \text{ cm}. \qquad \square$$

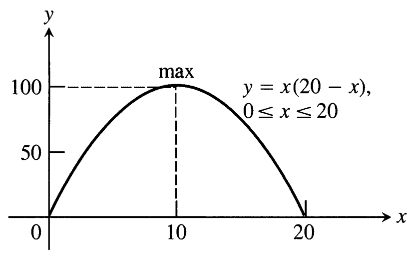
Strategy for Solving Max-Min Problems

1. *Read the problem.* Read the problem until you understand it. What is unknown? What is given? What is sought?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression.
4. *Identify the unknown. Write an equation for it.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints.* Use what you know about the shape of the function's graph and the physics of the problem. Use the first and second derivatives to identify and classify critical points (where $f' = 0$ or does not exist).

Examples from Mathematics

EXAMPLE 3 *Products of numbers*

Find two positive numbers whose sum is 20 and whose product is as large as possible.



3.52 The product of x and $(20 - x)$ reaches a maximum value of 100 when $x = 10$ (Example 3).

Solution If one number is x, the other is $(20 - x)$. Their product is

$$f(x) = x(20 - x) = 20x - x^2.$$

We want the value or values of x that make $f(x)$ as large as possible. The domain of f is the closed interval $0 \leq x \leq 20$.

We evaluate f at the critical points and endpoints. The first derivative,

$$f'(x) = 20 - 2x,$$

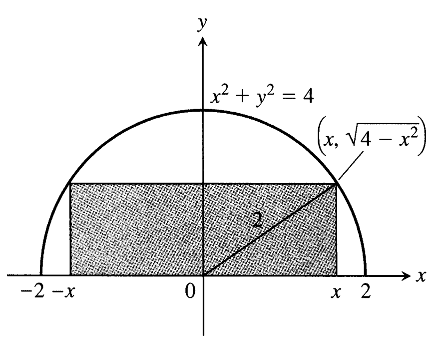
is defined at every point of the interval $0 \leq x \leq 20$ and is zero only at $x = 10$. Listing the values of f at this one critical point and the endpoints gives

Critical-point value: $f(10) = 20(10) - (10)^2 = 100$

Endpoint values: $f(0) = 0, \quad f(20) = 0.$

We conclude that the maximum value is $f(10) = 100$. The corresponding numbers are $x = 10$ and $(20 - 10) = 10$ (Fig. 3.52). ❑

EXAMPLE 4 *Geometry*



3.53 The rectangle and semicircle in Example 4.

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution To describe the dimensions of the rectangle, we place the circle and rectangle in the coordinate plane (Fig. 3.53). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

Length: $2x$ Height: $\sqrt{4 - x^2}$ Area: $2x \cdot \sqrt{4 - x^2}$.

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our mathematical goal is now to find the absolute maximum value of the continuous function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain [0, 2]. We do this by examining the values of A at the critical points and endpoints. The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\begin{aligned}
\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} &= 0 \\
-2x^2 + 2(4 - x^2) &= 0 \qquad \text{Multiply both sides by } \sqrt{4 - x^2}. \\
8 - 4x^2 &= 0 \\
x^2 &= 2 \\
x &= \pm\sqrt{2}.
\end{aligned}$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

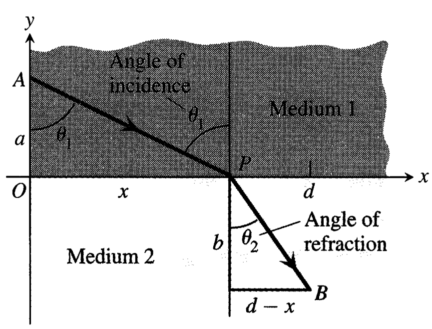
Critical-point value: $A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - 2} = 4$
Endpoint values: $A(0) = 0, \qquad A(2) = 0.$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ❑

Fermat's Principle and Snell's Law

The speed of light depends on the medium through which it travels and tends to be slower in denser media. In a vacuum, it travels at the famous speed $c = 3 \times 10^8$ m/sec, but in the earth's atmosphere it travels slightly slower than that, and in glass slower still (about two-thirds as fast).

Fermat's principle in optics states that light always travels from one point to another along the quickest route. This observation enables us to predict the path light will take when it travels from a point in one medium (air, say) to a point in another medium (say, glass or water).



3.54 A light ray refracted (deflected from its path) as it passes from one medium to another. θ_1 is the angle of incidence and θ_2 is the angle of refraction.

EXAMPLE 5 Find the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 across a straight boundary to a point B in a medium where the speed of light is c_2.

Solution Since light traveling from A to B will do so by the quickest route, we look for a path that will minimize the travel time.

We assume that A and B lie in the xy-plane and that the line separating the two media is the x-axis (Fig. 3.54).

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Hence the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B. From the formula distance equals rate times time, we have

$$\text{time} = \frac{\text{distance}}{\text{rate}}.$$

The time required for light to travel from A to P is therefore

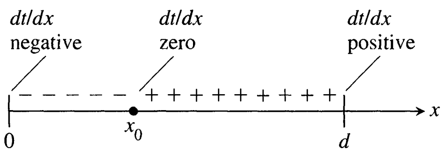
$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2+x^2}}{c_1}.$$

From P to B the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2+(d-x)^2}}{c_2}.$$

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2+x^2}}{c_1} + \frac{\sqrt{b^2+(d-x)^2}}{c_2}. \tag{4}$$



3.55 The sign pattern of *dt/dx* in Example 5.

Equation (4) expresses t as a differentiable function of x whose domain is $[0, d]$, and we want to find the absolute minimum value of t on this closed interval. We find

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{(d-x)}{c_2\sqrt{b^2+(d-x)^2}}. \tag{5}$$

In terms of the angles θ_1 and θ_2 in Fig. 3.54,

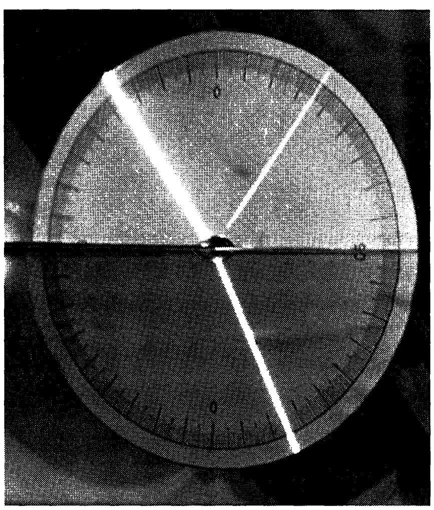
$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}. \tag{6}$$

We can see from Eq. (5) that $dt/dx < 0$ at $x = 0$ and $dt/dx > 0$ at $x = d$. Hence, $dt/dx = 0$ at some point x_0 in between (Fig. 3.55). There is only one such point because dt/dx is an increasing function of x (Exercise 52). At this point,

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

This equation is **Snell's law** or the **law of refraction.**

We conclude that the path the ray of light follows is the one described by Snell's law. Figure 3.56 shows how this works for air and water. ❑



3.56 For air and water at room temperature, the light velocity ratio is 1.33 and Snell's law becomes $\sin\theta_1 = 1.33\sin\theta_2$. In this laboratory photograph, $\theta_1 = 35.5°, \theta_2 = 26°$, and $(\sin 35.5°/\sin 26°) \approx 0.581/0.438 \approx 1.33$, as predicted.

This photograph also illustrates that angle of reflection = angle of incidence (Exercise 39).

Cost and Revenue in Economics

Here we want to point out two of the many places where calculus makes a contribution to economic theory. The first has to do with the relationship between profit, revenue (money received), and cost.

Suppose that

$r(x) =$ the revenue from selling x items

$c(x) =$ the cost of producing the x items

$p(x) = r(x) - c(x) =$ the profit from selling x items.

The marginal revenue and cost at this production level (x items) are

$$\frac{dr}{dx} = \text{marginal revenue}$$

$$\frac{dc}{dx} = \text{marginal cost.}$$

The first theorem is about the relationship of the profit p to these derivatives.

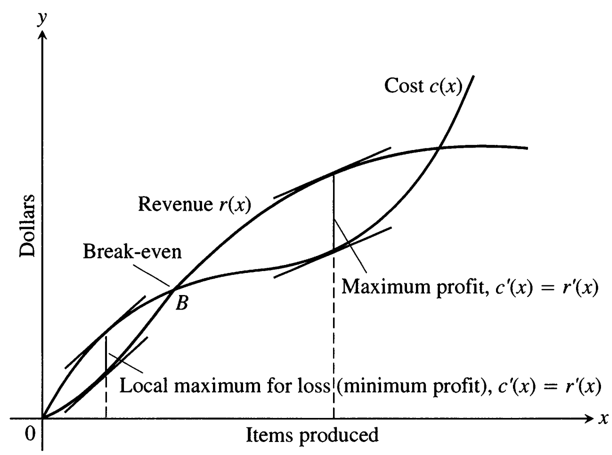
Theorem 7

Maximum profit (if any) occurs at a production level at which marginal revenue equals marginal cost.

Proof We assume that $r(x)$ and $c(x)$ are differentiable for all $x > 0$, so if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies

$$r'(x) - c'(x) = 0 \qquad \text{or} \qquad r'(x) = c'(x).$$

This concludes the proof (Fig. 3.57).



3.57 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of market saturation and rising labor and material costs) and production levels become unprofitable again.

□

What guidance do we get from Theorem 7? We know that a production level at which $p'(x) = 0$ need not be a level of maximum profit. It might be a level of minimum profit, for example. But if we are making financial projections for our company, we should look for production levels at which marginal cost seems to equal marginal revenue. If there is a most profitable production level, it will be one of these.

Developing a physical law

In developing a physical law, we typically observe an effect, measure values and list them in a table, and then try to find a rule by which one thing can be connected with another. The Alexandrian Greek Claudius Ptolemy (c. 100–c. 170 A.D.) tried to do this for the refraction of light by water. He made a table of angles of incidence and corresponding angles of refraction, with values very close to the ones we find for air and water today.

Angle in air (degrees)	Ptolemy's angle in water (degrees)	Modern angle in water (degrees)
10	8	7.5
20	15.5	15
30	22.5	22
40	28	29
50	35	35
60	40.5	40.5
70	45	45
80	50	47.6

The rule that connected these angles, however, eluded him, as it did everyone else for the next 1400 years. The Dutch mathematician Willebrord Snell (1580–1626) found it in 1621.

Finding a rule is nice, but the real glory of science is finding a way of thinking that makes the rule evident. Fermat discovered it around 1650. His idea was this: Of all the paths light might take to get from one point to another, it follows the path that takes the shortest time. In Example 5, you see how this principle leads to Snell's law. The derivation we give is Fermat's own.

For more on marginal revenue and cost, see the end of Section 2.3.

EXAMPLE 6 The cost and revenue functions at American Gadget are

$$r(x) = 9x \quad \text{and} \quad c(x) = x^3 - 6x^2 + 15x,$$

where x represents thousands of gadgets. Is there a production level that will maximize American Gadget's profit? If so, what is it?

Solution

$$r(x) = 9x, \qquad c(x) = x^3 - 6x^2 + 15x$$ Find $r'(x)$ and $c'(x)$.

$$r'(x) = 9, \qquad c'(x) = 3x^2 - 12x + 15$$

$$3x^2 - 12x + 15 = 9$$ Set them equal.

$$3x^2 - 12x + 6 = 0$$ Rearrange.

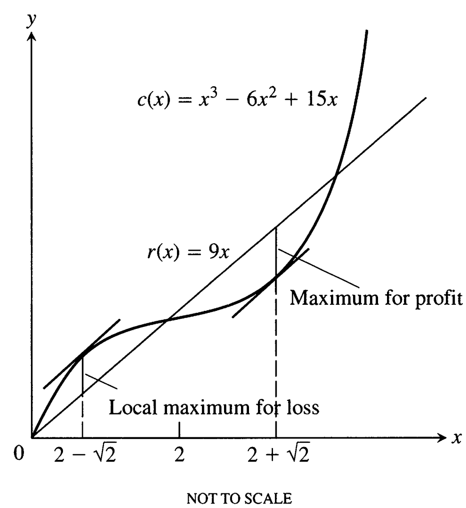
$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2}$$ Solve for x with the quadratic formula.

$$= \frac{4 \pm 2\sqrt{2}}{2}$$

$$= 2 \pm \sqrt{2}$$

The possible production levels for maximum profit are $x = 2 + \sqrt{2}$ thousand units and $x = 2 - \sqrt{2}$ thousand units. A quick glance at the graphs in Fig. 3.58 or at the corresponding values of r and c shows $x = 2 + \sqrt{2}$ to be a point of maximum profit and $x = 2 - \sqrt{2}$ to be a local maximum for loss.



3.58 The cost and revenue curves for Example 6. ❑

Another way to look for optimal production levels is to look for levels that minimize the average cost of the units produced. The next theorem helps us to find them.

Theorem 8

The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

Proof We start with

$$c(x) = \text{cost of producing } x \text{ items, } x > 0$$
$$\frac{c(x)}{x} = \text{average cost of producing } x \text{ items,}$$

assumed differentiable.

If the average cost can be minimized, it will be at a production level at which

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$
$$\frac{xc'(x) - c(x)}{x^2} = 0 \qquad \text{Quotient Rule}$$
$$xc'(x) - c(x) = 0 \qquad \text{Multiplied by } x^2$$
$$\underbrace{c'(x)}_{\substack{\text{marginal}\\ \text{cost}}} = \underbrace{\frac{c(x)}{x}}_{\substack{\text{average}\\ \text{cost}}}.$$

This completes the proof. ❑

Again we have to be careful about what Theorem 8 does and does not say. It does not say that there is a production level of minimum average cost—it says where to look to see if there is one. Look for production levels at which average cost and marginal cost are equal. Then check to see if any of them gives a minimum average cost.

EXAMPLE 7 The cost function at American Gadget is $c(x) = x^3 - 6x^2 + 15x$ (x in thousands of units). Is there a production level that minimizes average cost? If so, what is it?

Solution We look for levels at which average cost equals marginal cost.

Cost: $c(x) = x^3 - 6x^2 + 15x$

Marginal cost: $c'(x) = 3x^2 - 12x + 15$

Average cost: $\dfrac{c(x)}{x} = x^2 - 6x + 15$

$$3x^2 - 12x + 15 = x^2 - 6x + 15 \qquad \text{MC} = \text{AC}$$
$$2x^2 - 6x = 0$$
$$2x(x - 3) = 0$$
$$x = 0 \qquad \text{or} \qquad x = 3$$

Since $x > 0$, the only production level that might minimize average cost is $x = 3$ thousand units.

We check the derivatives:

$$\frac{c(x)}{x} = x^2 - 6x + 15 \qquad \text{Average cost}$$

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0.$$

The second derivative is positive, so $x = 3$ gives an absolute minimum. ❑

Modeling Discrete Phenomena with Differentiable Functions

In case you are wondering how we can use differentiable functions $c(x)$ and $r(x)$ to describe the cost and revenue that come from producing a number of items x, which can only be an integer, here is the rationale.

When x is large, we can reasonably fit the cost and revenue data with smooth curves $c(x)$ and $r(x)$ that are defined not only at integer values of x but at the values in between. Once we have these differentiable functions, which are supposed to behave like the real cost and revenue when x is an integer, we can apply calculus to draw conclusions about their values. We then translate these mathematical conclusions into inferences about the real world that we hope will have predictive value. When they do, as is the case with the economic theory here, we say that the functions give a good model of reality.

What do we do when our calculus tells us that the best production level is a value of x that isn't an integer, as it did in Example 6 when it said that $x = 2 + \sqrt{2}$ thousand units would be the production level for maximum profit? The practical answer is to use the nearest convenient integer. For $x = 2 + \sqrt{2}$ thousand, we might use 3414, or perhaps 3410 or 3420 if we ship in boxes of 10.

Exercises 3.6

If you have a grapher, this is a good place to use it. We have included some specific grapher exercises but there is something to be learned from graphing in most of the other exercises as well.

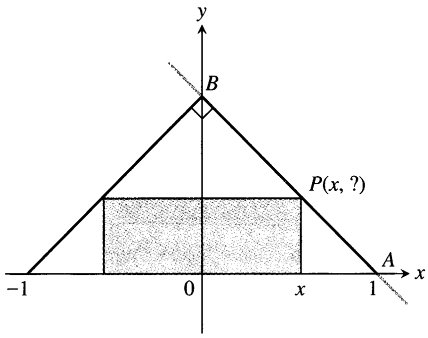
Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

Applications in Geometry

1. A sector shaped like a slice of pie is cut from a circle of radius r. The outer circular arc of the sector has length s. If the sector's total perimeter $(2r + s)$ is to be 100 m, what values of r and s will maximize the sector's area?
2. What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?
3. What is the smallest perimeter possible for a rectangle whose area is 16 in^2?
4. Show that among all rectangles with a given perimeter, the one with the largest area is a square.
5. The figure shown here shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a) Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b) Express the area of the rectangle in terms of x.

c) What is the largest area the rectangle can have?



6. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

7. You are planning to make an open rectangular box from an 8-by-15-in. piece of cardboard by cutting squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way?

8. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

9. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose?

10. A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

11. *The lightest steel holding tank.* Your iron works has contracted to design and build a 500-ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding $\frac{1}{2}$-in.-thick stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible. What dimensions do you tell the shop to use?

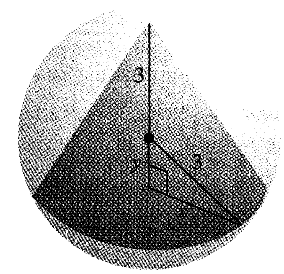
12. *Catching rainwater.* An 1125-ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy. If the cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

13. You are designing a poster to contain 50 in^2 of printing with margins of 4 in. each at top and bottom and 2 in. at each side. What overall dimensions will minimize the amount of paper used?

14. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



15. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (*Hint:* $A = (1/2)ab\sin\theta$.)

16. Find the largest possible value of $s = 2x + y$ if x and y are side lengths in a right triangle whose hypotenuse is $\sqrt{5}$ units long.

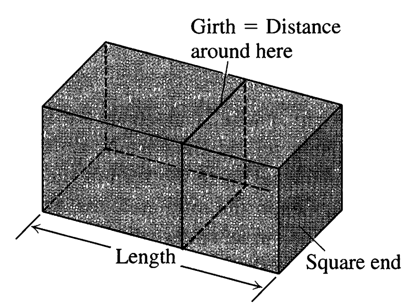
17. What are the dimensions of the lightest (least material) open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

18. You are designing 1000-cm^3 right circular cylindrical cans whose manufacture will take waste into account. There is no waste in cutting the aluminum for the sides, but the tops and bottoms of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used by each can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2 the ratio of h to r for the most economical cans was 2 to 1. What is the ratio now?

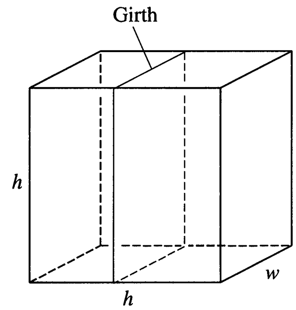
19. a) The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



b) GRAPHER Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length, and compare what you see with your answer in (a).

20. (*Continuation of Exercise 19.*) Suppose that instead of having a box with square ends you have a box with square sides so that

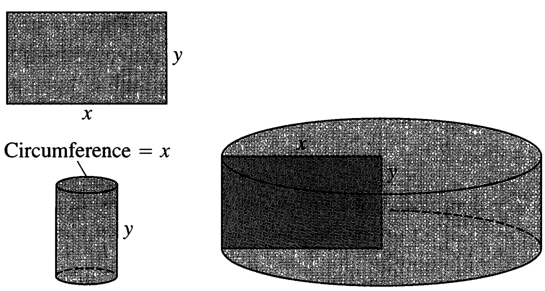
its dimensions are h by h by w and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



21. Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into the cylinder shown here (a). What values of x and y give the largest volume?

b) The rectangular sheet of perimeter 36 cm and dimensions x by y is to be revolved about one of the sides of length y to sweep out the cyclinder shown here (b). What values of x and y give the largest volume?



(a) (b)

22. A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

23. *Circle vs. square*

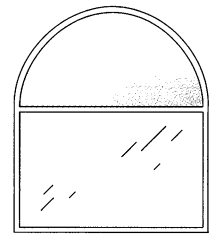
a) A 4-m length of wire is available for making a circle and a square. How should the wire be distributed between the two shapes to maximize the sum of the enclosed areas?

b) GRAPHER Graph the total area enclosed by the wire as a function of the circle's radius. Reconcile what you see with your answer in (a).

c) GRAPHER Now graph the total area enclosed by the wire as a function of the square's side length. Again, reconcile what you see with your answer in (a).

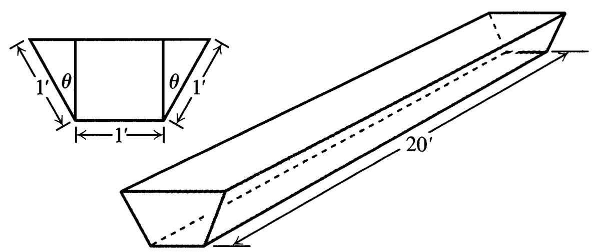
24. If the sum of the surface areas of a cube and a sphere is held constant, what ratio of an edge of the cube to the radius of the sphere will make the sum of the volumes (a) as small as possible, (b) as large as possible?

25. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass while the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

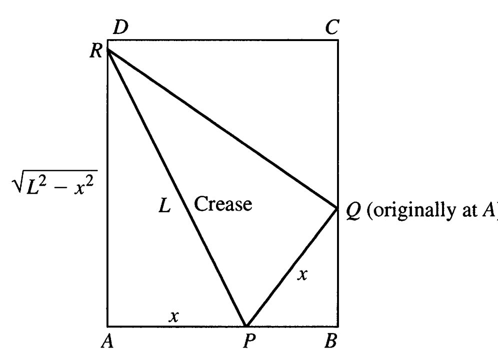


26. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

27. The trough here is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



28. A rectangular sheet of $8\frac{1}{2}$-by-11-in. paper shown here is placed on a flat surface, and one of the corners is placed on the opposite longer edge. The other corners are held in their original positions. With all four corners now held fixed, the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L.



a) Try it with paper.
b) Show that $L^2 = 2x^3/(2x - 8.5)$.
c) What value of x minimizes L^2?
d) CALCULATOR Find the minimum value of L to the nearest tenth of an inch.
e) GRAPHER Graph L as a function of x and compare what you see with your answer in (d).

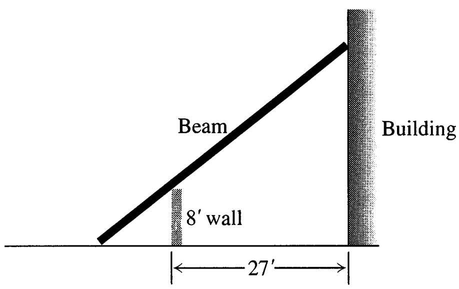
Physical Applications

29. The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \quad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

30. CALCULATOR The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



31. *The strength of a beam.* The strength S of a rectangular wooden beam is proportional to its width w times the square of its depth d.

a) Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.
b) GRAPHER Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in (a).
c) GRAPHER On the same screen, or on a separate screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in (a). What would be the effect of changing to some other value of k? Try it.

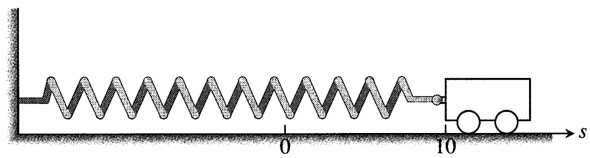
32. *The stiffness of a beam.* The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

a) Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter log.
b) GRAPHER Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in (a).
c) GRAPHER On the screen, or on a separate screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in (a). What would be the effect of changing to some other value of k? Try it.

33. Suppose that at any given time t (sec) the current i (amp) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

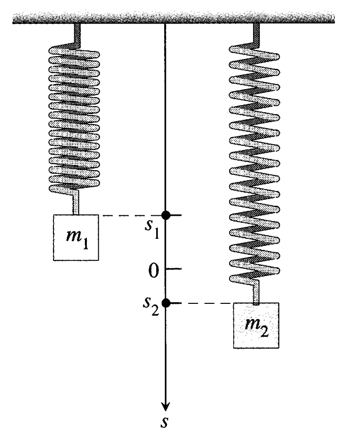
34. A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10\cos \pi t$.

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?
b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



35. Two masses hanging side by side from springs have positions $s_1 = 2\sin t$ and $s_2 = \sin 2t$, respectively.

a) At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2\sin t \cos t$.)
b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)



36. The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$.

a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?
b) What is the farthest apart the particles ever get?
c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

37. Suppose that at time $t \ge 0$ the position of a particle moving on the x-axis is $x = (t - 1)(t - 4)^4$.

a) When is the particle at rest?
b) During what time interval does the particle move to the left?

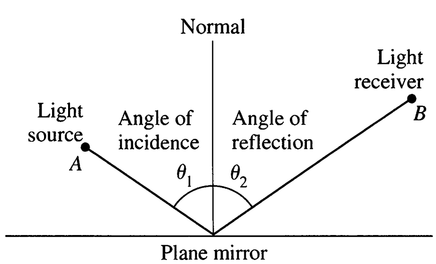
c) What is the fastest the particle goes while moving to the left?

d) GRAPHER Graph x as a function of t for $0 \le t \le 6$. Graph dx/dt over the same interval, in another color if possible. Compare the graphs with one another and with your answers in (a)–(c).

38. At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour—a nautical mile is 2000 yd) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.

a) Start counting time with $t = 0$ at noon and express the distance s between the ships as a function of t.
b) How rapidly was the distance between the ships changing at noon? One hour later?
c) CALCULATOR The visibility that day was 5 nautical miles. Did the ships ever sight each other?
d) GRAPHER Graph s and ds/dt together as functions of t for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in (b) and (c).
e) The graph of ds/dt looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that ds/dt approaches a limiting value at $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

39. Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Figure 3.59 shows light from a source A reflected by a plane mirror to a receiver at point B. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



3.59 In studies of light reflection, the angles of incidence and reflection are measured from the line normal to the reflecting surface. Exercise 39 asks you to show that if light obeys Fermat's "least-time" principle, then $\theta_1 = \theta_2$.

40. *Tin pest.* Metallic tin, when kept below 13°C for a while, becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw the tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of the reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product
$a =$ the amount of substance at the beginning
$k =$ a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?

Mathematical Applications

41. Is the function $f(x) = x^2 - x + 1$ ever negative? Explain.

42. You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

a) Explain why you need consider values of x only in the interval $[0, 2\pi]$.
b) Is f ever negative? Explain.

43. Find the points on the curve $y = \sqrt{x}$ nearest the point $(c, 0)$

a) if $c \ge 1/2$
b) if $c < 1/2$.

44. What value of a makes $f(x) = x^2 + (a/x)$ have (a) a local minimum of $x = 2$; (b) a point of inflection at $x = 1$?

45. What values of a and b make

$$f(x) = x^3 + ax^2 + bx$$

have (a) a local maximum at $x = -1$ and a local minimum at $x = 3$; (b) a local minimum at $x = 4$ and a point of inflection at $x = 1$?

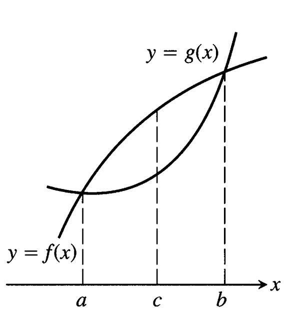
46. Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of a.

47. a) The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.
b) GRAPHER Graph the function and compare what you see with your answer in (a).

48. a) The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.
b) GRAPHER Graph the function and compare what you see with your answer in (a).

49. How close does the curve $y = \sqrt{x}$ come to the point $(1/2, 16)$?

50. Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



51. Show that if a, b, c, and d are positive integers, then

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \geq 16.$$

52. *The derivative dt/dx in Example 5*

a) Show that

$$f(x) = \frac{x}{\sqrt{a^2+x^2}}$$

is an increasing function of x.

b) Show that

$$g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

is a decreasing function of x.

c) Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{d-x}{c_2\sqrt{b^2+(d-x)^2}}$$

is an increasing function of x.

Medicine

53. *Sensitivity to medicine (Continuation of Exercise 50, Section 2.2).* Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and C is a constant.

54. *How we cough*

a) When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \quad \frac{r_0}{2} \leq r \leq r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea.

Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that x-ray photographs confirm that the trachea contracts about this much during a cough.

b) GRAPHER Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \leq r \leq 0.5$. Compare what you see to the claim that v is at a maximum when $r = (2/3)r_0$.

Business and Economics

55. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by $n = a/(x-c) + b(100-x)$, where a and b are certain positive constants. What selling price will bring a maximum profit?

56. You operate a tour service that offers the following rates:

a) \$200 per person if 50 people (the minimum number to book the tour) go on the tour.

b) For each additional person, up to a maximum of 80 people total, everyone's charge is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

57. *The best quantity to order.* One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

58. (*Continuation of Exercise 57.*) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

59. Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

60. Suppose $c(x) = x^3 - 20x^2 + 20{,}000\,x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

3.7

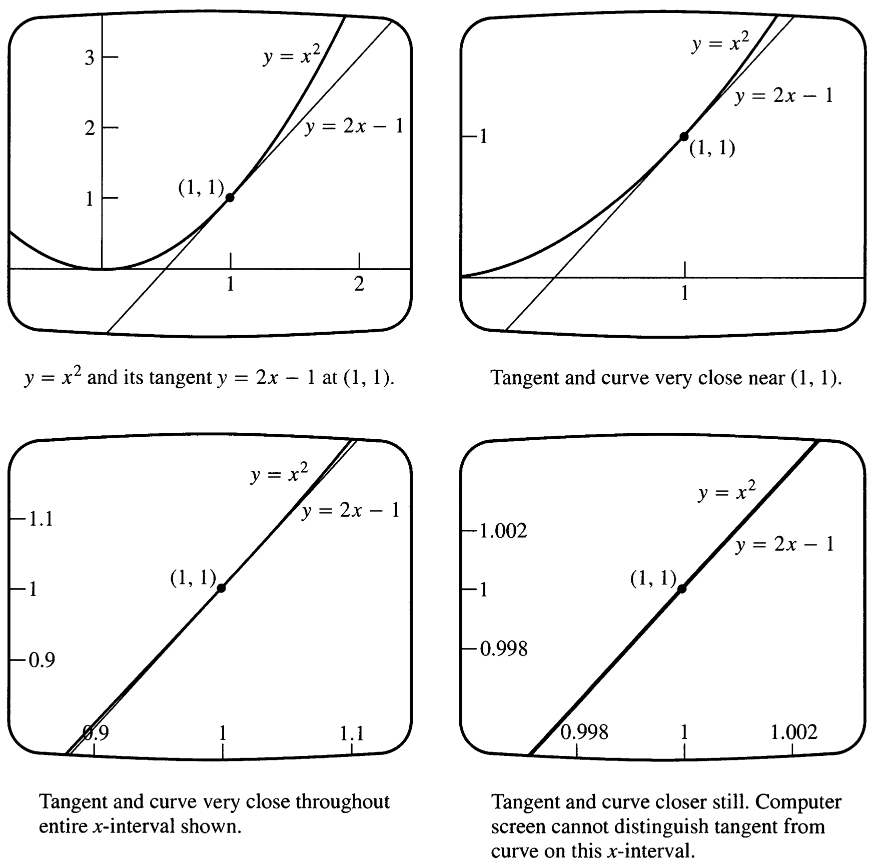
Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations.* They are based on tangent lines.

We introduce new variables dx and dy and define them in a way that gives new meaning to the Leibniz notation dy/dx. We will use dy to estimate error in measurement and sensitivity to change.

Linear Approximations

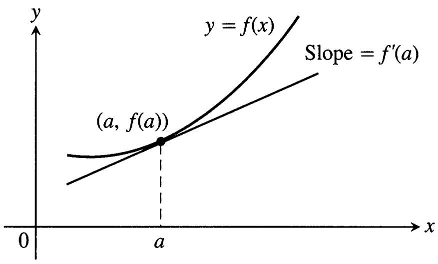
As you can see in Fig. 3.60, the tangent to a curve $y = f(x)$ lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line give a good approximation to the y-values on the curve.



3.60 **The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.**

In the notation of Fig. 3.61, the tangent passes through the point $(a, f(a))$, so its point–slope equation is

$$y = f(a) + f'(a)(x - a).$$



3.61 The equation of the tangent line is $y = f(a) + f'(a)(x - a)$.

Thus, the tangent is the graph of the function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as the line remains close to the graph of f, $L(x)$ gives a good approximation to $f(x)$.

Definitions

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a) \tag{1}$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

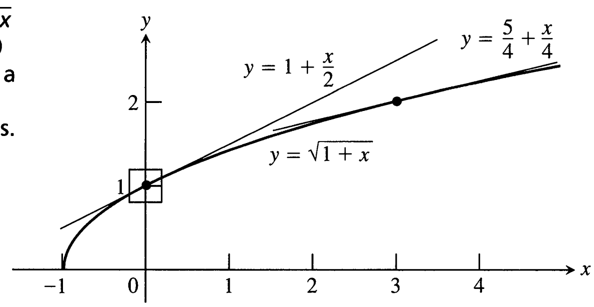
Solution We evaluate Eq. (1) for f at $a = 0$. With

$$f'(x) = \frac{1}{2}(1+x)^{-1/2},$$

we have $f(0) = 1$, $f'(0) = 1/2$, and

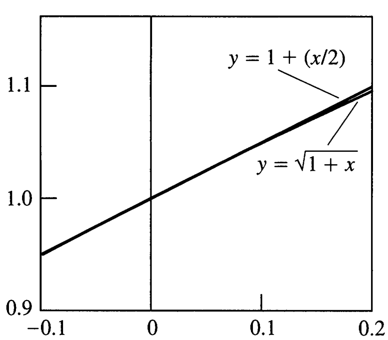
$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Fig. 3.62.



3.62 The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.63 shows a magnified view of the small window about 1 on the y-axis.

❑



3.63 Magnified view of the window in Fig. 3.62.

The approximation $\sqrt{1+x} \approx 1 + (x/2)$ (Fig. 3.63) gives

$$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10, \qquad \text{Accurate to 2 decimals}$$

$$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025, \qquad \text{Accurate to 3 decimals}$$

$$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250. \qquad \text{Accurate to 5 decimals}$$

Do not be misled by these calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1+x}$ for x close to 0 and can tolerate the small amount of error involved, we can work with $1+(x/2)$ instead. Of course, we then need to know how much error there is. We will touch on this toward the end of the section but will not have the full story until Chapter 8.

A linear approximation normally loses accuracy away from its center. As Fig. 3.62 suggests, the approximation $\sqrt{1+x} \approx 1+(x/2)$ will probably be too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

Solution We evaluate Eq. (1) for f at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1+x)^{-1/2}\Big|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x-3) = \frac{5}{4} + \frac{x}{4}.$$

❑

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

EXAMPLE 3 The most important linear approximation for roots and powers is

$$(1+x)^k \approx 1 + kx \qquad (x \approx 0;\ \text{any number } k) \tag{2}$$

(Exercise 20). This approximation, good for values of x sufficiently close to zero, has broad application.

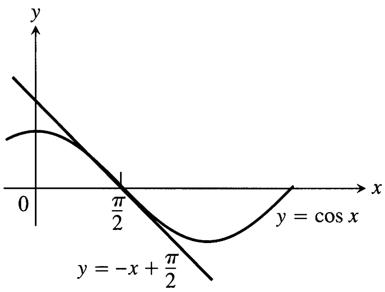
Approximation ($x \approx 0$)	**Source: Eq. (2) with ...**
$\sqrt{1+x} \approx 1 + \dfrac{x}{2}$	$k = 1/2$
$\dfrac{1}{1-x} = (1-x)^{-1} \approx 1 + (-1)(-x) = 1 + x$	$k = -1$; $-x$ in place of x
$\sqrt[3]{1+5x^4} = (1+5x^4)^{1/3} \approx 1 + \dfrac{1}{3}(5x^4) = 1 + \dfrac{5}{3}x^4$	$k = 1/3$; $5x^4$ in place of x
$\dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \approx 1 + \left(-\dfrac{1}{2}\right)\left(-x^2\right) = 1 + \dfrac{x^2}{2}$	$k = -1/2$; $-x^2$ in place of x

❑

Common linear approximations, $x \approx 0$

$$\sin x \approx x$$
$$\cos x \approx 1$$
$$\tan x \approx x$$
$$(1+x)^k \approx 1 + kx$$

(See the Exercises.)



3.64 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$.

EXAMPLE 4 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Fig. 3.64).

Solution With

$$f(\pi/2) = \cos(\pi/2) = 0 \qquad \text{and} \qquad f'(\pi/2) = -\sin(\pi/2) = -1,$$

we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

❑

Differentials

Definitions

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

The meaning of dx and dy

In most contexts, the differential dx of the independent variable is its change Δx, but we do not impose this restriction on the definition.

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.

EXAMPLE 5 Find dy if

a) $y = x^5 + 37x$ **b)** $y = \sin 3x$.

Solution

a) $dy = (5x^4 + 37)\,dx$ **b)** $dy = (3\cos 3x)\,dx$ ❑

If $dx \neq 0$ and we divide both sides of the equation $dy = f'(x)\,dx$ by dx, we obtain the familiar equation

$$\frac{dy}{dx} = f'(x).$$

This equation says that when $dx \neq 0$, we can regard the derivative dy/dx as a quotient of differentials.

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, and call df the **differential** of f. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv,$$

obtained by multiplying both sides by dx (Table 3.1).

Table 3.1 Formulas for differentials

$dc = 0$
$d(cu) = c\,du$
$d(u + v) = du + dv$
$d(uv) = u\,dv + v\,du$
$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$
$d(u^n) = nu^{n-1}\,du$
$d(\sin u) = \cos u\,du$
$d(\cos u) = -\sin u\,du$
$d(\tan u) = \sec^2 u\,du$
$d(\cot u) = -\csc^2 u\,du$
$d(\sec u) = \sec u \tan u\,du$
$d(\csc u) = -\csc u \cot u\,du$

EXAMPLE 6

a) $d(\tan 2x) = \sec^2(2x)\,d(2x) = 2\sec^2 2x\,dx$

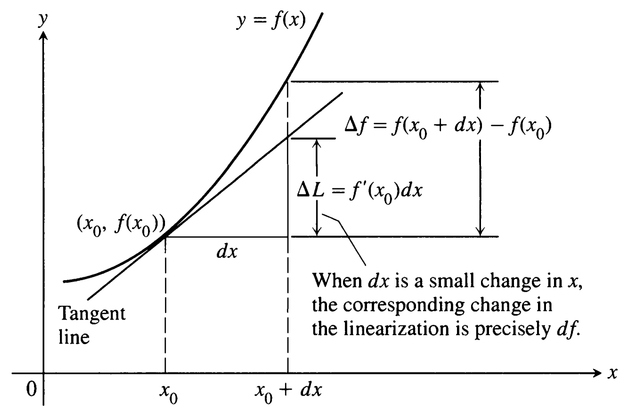
b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d\,(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ❑

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point x_0 and we want to predict how much this value will change if we move to a nearby point $x_0 + dx$. If dx is small, f and its linearization L at x_0 will change by nearly the same amount. Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

In the notation of Fig. 3.65, the change in f is

$$\Delta f = f(x_0 + dx) - f(x_0).$$



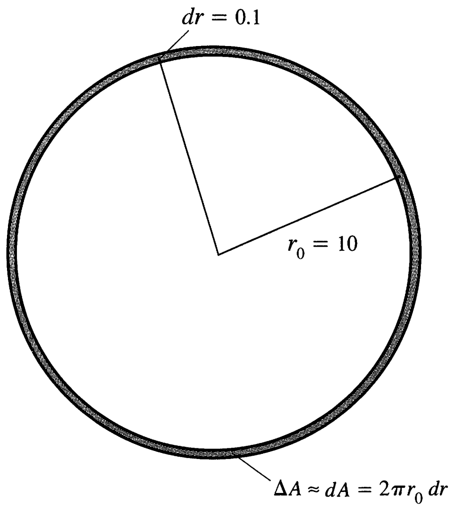
3.65 If dx is small, the change in the linearization of f is nearly the same as the change in f.

The corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + dx) - L(x_0)\\ &= \underbrace{f(x_0) + f'(x_0)\Big[(x_0 + dx) - x_0\Big]}_{L(x_0+dx)} - \underbrace{f(x_0)}_{L(x_0)=f(x_0)}\\ &= f'(x_0)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: When df is evaluated at $x = x_0$, $df = \Delta L$, the change in the linearization of f corresponding to the change dx.

> **The Differential Estimate of Change**
>
> Let $f(x)$ be differentiable at $x = x_0$. The approximate change in the value of f when x changes from x_0 to $x_0 + dx$ is
>
> $$df = f'(x_0)\,dx.$$



3.66 When dr is small compared with r_0, as it is when $dr = 0.1$ and $r_0 = 10$, the differential $dA = 2\pi r_0 dr$ gives a good estimate of ΔA (Example 7).

EXAMPLE 7 The radius r of a circle increases from $r_0 = 10$ m to 10.1 m (Fig. 3.66). Estimate the increase in the circle's area A by calculating dA. Compare this with the true change ΔA.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(r_0)\,dr = 2\pi r_0\,dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = \underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}}.$$

❑

Absolute, Relative, and Percentage Change

As we move from x_0 to a nearby point $x_0 + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(x_0 + dx) - f(x_0)$	$df = f'(x_0)\,dx$
Relative change	$\dfrac{\Delta f}{f(x_0)}$	$\dfrac{df}{f(x_0)}$
Percentage change	$\dfrac{\Delta f}{f(x_0)} \times 100$	$\dfrac{df}{f(x_0)} \times 100$

EXAMPLE 8 The estimated percentage change in the area of the circle in Exercise 7 is

$$\frac{dA}{A(r_0)} \times 100 = \frac{2\pi}{100\pi} \times 100 = 2\%.$$

❑

EXAMPLE 9 *The earth's surface area*

Suppose the earth were a perfect sphere and we determined its radius to be 3959 $\pm$ 0.1 miles. What effect would the tolerance of $\pm$ 0.1 have on our estimate of the earth's surface area?

Solution The surface area of a sphere of radius r is $S = 4\pi r^2$. The uncertainty in the calculation of S that arises from measuring r with a tolerance of dr miles is about

$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r\,dr.$$

With $r = 3959$ and $dr = 0.1$, our estimate of S could be off by as much as

$$dS = 8\pi(3959)(0.1) \approx 9950 \text{ mi}^2,$$

to the nearest square mile, which is about the area of the state of Maryland. ❑

If we underestimated the radius of the earth by 528 ft during a calculation of the earth's surface area, we would leave out an area the size of the state of Maryland.

EXAMPLE 10 About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

Solution We want any inaccuracy in our measurement to be small enough to make the corresponding increment ΔS in the surface area satisfy the inequality

$$|\Delta S| \leq \frac{1}{100} S = \frac{4\pi r^2}{100}.$$

We replace ΔS in this inequality with

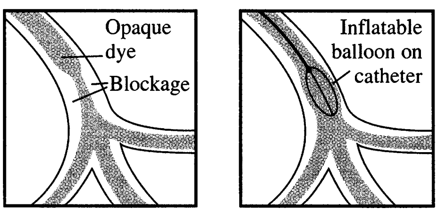
$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r\, dr.$$

This gives

$$|8\pi r dr| \leq \frac{4\pi r^2}{100}, \qquad \text{or} \qquad |dr| \leq \frac{1}{8\pi r} \cdot \frac{4\pi r^2}{100} = \frac{1}{2}\frac{r}{100}.$$

We should measure r with an error dr that is no more than 0.5% of the true value. ❑

Angiography: An opaque dye is injected into a partially blocked artery to make the inside visible under x-rays. This reveals the location and severity of the blockage.



Angioplasty: A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

EXAMPLE 11 *Unclogging arteries*

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-*zoy*") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

Solution The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr} dr = 4kr^3 dr.$$

Hence,

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4\frac{dr}{r}. \qquad \text{Dividing by } V = kr^4$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow. ❑

Sensitivity

The equation $df = f'(x)\,dx$ tells how sensitive the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater is the effect of a given change dx.

EXAMPLE 12 You want to calculate the height of a bridge from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculation be to a 0.1-sec error in measuring the time?

Solution The size of ds in the equation

$$ds = 32t\, dt$$

depends on how big t is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later, at $t = 5$ sec, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

❑

The Error in the Approximation $\Delta f \approx df$

Let $f(x)$ be differentiable at $x = x_0$ and suppose that Δx is an increment of x. We have two ways to describe the change in f as x changes from x_0 to $x_0 + \Delta x$:

The true change: $\quad \Delta f = f(x_0 + \Delta x) - f(x_0)$

The differential estimate: $\quad df = f'(x_0)\Delta x.$

How well does df approximate Δf?

We measure the approximation error by subtracting df from Δf:

$$\begin{aligned}
\text{Approximation error } &= \Delta f - df \\
&= \Delta f - f'(x_0)\Delta x \\
&= \underbrace{f(x_0 + \Delta x) - f(x_0)}_{\Delta f} - f'(x_0)\Delta x \\
&= \underbrace{\left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right)}_{\text{Call this part } \epsilon} \Delta x \\
&= \epsilon \cdot \Delta x.
\end{aligned}$$

As $\Delta x \to 0$, the difference quotient

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

approaches $f'(x_0)$ (remember the definition of $f'(x_0)$), so the quantity in parentheses becomes a very small number (which is why we called it ϵ). In fact, $\epsilon \to 0$ as $\Delta x \to 0$. When Δx is small, the approximation error $\epsilon\, \Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\substack{\text{true} \\ \text{change}}} = \underbrace{f'(x_0)\Delta x}_{\substack{\text{estimated} \\ \text{change}}} + \underbrace{\epsilon\, \Delta x}_{\text{error}}$$

While we do not know exactly how small the error is and will not be able to make much progress on this front until Chapter 8, there is something worth noting here, namely the *form* taken by the equation.

> If $y = f(x)$ is differentiable at $x = x_0$, and x changes from x_0 to $x_0 + \Delta x$, the change Δy in f is given by an equation of the form
>
> $$\Delta y = f'(x_0)\Delta x + \epsilon\, \Delta x \tag{3}$$
>
> in which $\epsilon \to 0$ as $\Delta x \to 0$.

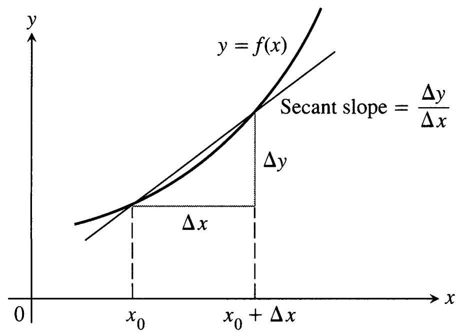
Surprising as it may seem, just knowing the form of Eq. (3) enables us to bring the proof of the Chain Rule to a successful conclusion.

Proof of the Chain Rule

You may recall our saying in Section 2.5 that the proof we wanted to give for the Chain Rule depended on ideas in Section 3.7, the present section. We were referring to Eq. (3), and here is the proof:

Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the composite $y = f(g(x))$ is a differentiable function of x. More precisely, if g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composite is differentiable at x_0 and

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$



3.67 The graph of y as a function of x. The derivative of y with respect to x at $x = x_0$ is $\lim_{\Delta x \to 0} \Delta y/\Delta x$.

Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y. As you can see in Fig. 3.67,

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

so our goal is to show that this limit is $f'(g(x_0)) \cdot g'(x_0)$.

By Eq. (3),

$$\Delta u = g'(x_0)\Delta x + \epsilon_1 \Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2\, \Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$. Notice also that $\Delta u \to 0$ as $\Delta x \to 0$. Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2\, g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since ϵ_1 and ϵ_2 go to zero as Δx goes to zero, three of the four terms on the right vanish in the limit, leaving

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

This concludes the proof. ❑

⁂ The Conversion of Mass to Energy

Here is an example of how the approximation

$$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{1}{2}x^2 \tag{4}$$

from Example 3 is used in an applied problem.

Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{5}$$

where the "rest mass" m_0 represents the mass of a body that is not moving and c is the speed of light, which is about 300,000 km/sec. When v is very small compared with c, v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)$$

(Eq. 4 with $x = v/c$) to write

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right). \tag{6}$$

Equation (6) expresses the increase in mass that results from the added velocity v.

In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the body, and if we rewrite Eq. (6) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0 (0)^2 = \Delta(\mathrm{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\mathrm{KE}). \tag{7}$$

In other words, the change in kinetic energy $\Delta(\mathrm{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$.

With c equal to 3×10^8 m/sec, Eq. (7) becomes

$$\Delta(\mathrm{KE}) \approx 90{,}000{,}000{,}000{,}000{,}000\ \Delta m \text{ joules} \qquad \text{mass in kilograms}$$

and we see that a small change in mass can create a large change in energy. The energy released by exploding a 20-kiloton atomic bomb, for instance, is the result of converting only 1 gram of mass to energy. The products of the explosion weigh only 1 gram less than the material exploded. A U.S. penny weighs about 3 grams.

Exercises 3.7

Finding Linearizations

In Exercises 1–6, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^4$ at $x = 1$

2. $f(x) = x^{-1}$ at $x = 2$

3. $f(x) = x^3 - x$ at $x = 1$
4. $f(x) = x^3 - 2x + 3$ at $x = 2$
5. $f(x) = \sqrt{x}$ at $x = 4$
6. $f(x) = \sqrt{x^2 + 9}$ at $x = -4$

You want linearizations that will replace the functions in Exercises 7–12 over intervals that include the given points x_0. To make your subsequent work as simple as possible, you want to center each linearization not at x_0 but at a nearby integer $x = a$ at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

7. $f(x) = x^2 + 2x, \quad x_0 = 0.1$
8. $f(x) = x^{-1}, \quad x_0 = 0.6$
9. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -0.9$
10. $f(x) = 1 + x, \quad x_0 = 8.1$
11. $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$
12. $f(x) = \dfrac{x}{x+1}, \quad x_0 = 1.3$

Linearizing Trigonometric Functions

In Exercises 13–16, find the linearization of f at $x = a$. Then graph the linearization and f together.

13. $f(x) = \sin x$ at (a) $x = 0$, (b) $x = \pi$
14. $f(x) = \cos x$ at (a) $x = 0$, (b) $x = -\pi/2$
15. $f(x) = \sec x$ at (a) $x = 0$, (b) $x = -\pi/3$
16. $f(x) = \tan x$ at (a) $x = 0$, (b) $x = \pi/4$

The Approximation $(1 + x)^k \approx 1 + kx$

17. Use the formula $(1 + x)^k \approx 1 + kx$ to find linear approximations of the following functions for values of x near zero.

 a) $f(x) = (1 + x)^2$ b) $f(x) = \dfrac{1}{(1+x)^5}$

 c) $g(x) = \dfrac{2}{1-x}$ d) $g(x) = (1 - x)^6$

 e) $h(x) = 3(1 + x)^{1/3}$ f) $h(x) = \dfrac{1}{\sqrt{1+x}}$

18. *Faster than a calculator.* Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate

 a) $(1.0002)^{50}$ b) $\sqrt[3]{1.009}$.

19. Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x+1}$ and $\sin x$?

20. We know from the Power Rule that the equation

$$\frac{d}{dx}(1 + x)^k = k(1 + x)^{k-1}$$

holds for every rational number k. In Chapter 6, we will show that it holds for every irrational number as well. Assuming this result for now, show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$ for any number k.

Derivatives in Differential Form

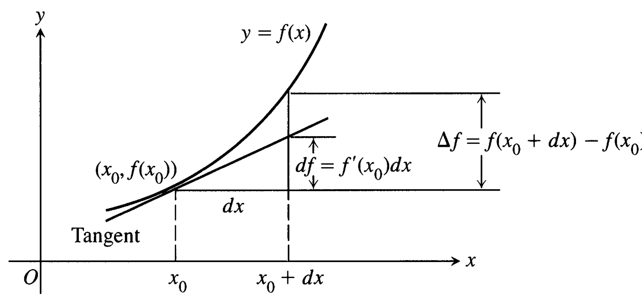
In Exercises 21–32, find dy.

21. $y = x^3 - 3\sqrt{x}$
22. $y = x\sqrt{1 - x^2}$
23. $y = \dfrac{2x}{1 + x^2}$
24. $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$
25. $2y^{3/2} + xy - x = 0$
26. $xy^2 - 4x^{3/2} - y = 0$
27. $y = \sin(5\sqrt{x})$
28. $y = \cos(x^2)$
29. $y = 4\tan(x^3/3)$
30. $y = \sec(x^2 - 1)$
31. $y = 3\csc(1 - 2\sqrt{x})$
32. $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

Approximation Error

In Exercises 33–38, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a) the change $\Delta f = f(x_0 + dx) - f(x_0)$;
b) the value of the estimate $df = f'(x_0)\,dx$; and
c) the approximation error $|\Delta f - df|$.



33. $f(x) = x^2 + 2x, \quad x_0 = 0, \quad dx = 0.1$
34. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$
35. $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$
36. $f(x) = x^4, \quad x_0 = 1, \quad dx = 0.1$
37. $f(x) = x^{-1}, \quad x_0 = 0.5, \quad dx = 0.1$
38. $f(x) = x^3 - 2x + 3, \quad x_0 = 2, \quad dx = 0.1$

Differential Estimates of Change

In Exercises 39–44, write a differential formula that estimates the given change in volume or surface area.

39. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$
40. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$
41. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

42. The change in the lateral surface area $S = \pi r\sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change

43. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change

44. The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change

Applications

45. The radius of a circle is increased from 2.00 to 2.02 m.

a) Estimate the resulting change in area.

b) Express the estimate in (a) as a percentage of the circle's original area.

46. The diameter of a tree was 10 in. During the following year, the circumference grew 2 in. About how much did the tree's diameter grow? the tree's cross-section area?

47. The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

48. About how accurately should you measure the side of a square to be sure of calculating the area within 2% of its true value?

49. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

50. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.

51. The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated from a measurement of h and must be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

52. a) About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank within 5% of the true amount?

53. A manufacturer contracts to mint coins for the federal government. How much variation dr in the radius of the coins can be tolerated if the coins are to weigh within 1/1000 of their ideal weight? Assume that the thickness does not vary.

54. *(Continuation of Example 11.)* By what percentage should r be increased to increase V by 50%?

55. *(Continuation of Example 12.)* Show that a 5% error in measuring t will cause about a 10% error in calculating s from the equation $s = 16t^2$.

56. *The effect of flight maneuvers on the heart.* The amount of work done in a unit of time by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g and the equation takes the simplified form

$$W = a + \frac{b}{g} \qquad (a,\ b \text{ constant}). \tag{8}$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2$ ft/sec^2, with the effect the same change dg would have on Earth, where $g = 32$ ft/sec^2. You use Eq. (8) to find the ratio of dW_{moon} to dW_{Earth}. What do you conclude?

57. *Sketching the change in a cube's volume.* The volume $V = x^3$ of a cube with edges of length x increases by an amount ΔV when x increases by an amount Δx. Show with a sketch how to represent ΔV geometrically as the sum of the volumes of

a) three slabs of dimensions x by x by Δx;

b) three bars of dimensions x by Δx by Δx;

c) one cube of dimensions Δx by Δx by Δx.

The differential formula $dV = 3x^2 dx$ estimates the change in V with the three slabs.

58. *Measuring the acceleration of gravity.* When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

a) With L held constant and g as the independent variable, calculate dT and use it to answer (b) and (c).

b) If g increases, will T increase, or decrease? Will a pendulum clock speed up, or slow down? Explain.

c) A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec^2 to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

Theory and Examples

59. Show that the approximation of $\sqrt{1+x}$ by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x\to 0} \frac{\sqrt{1+x}}{1+(x/2)} = 1.$$

60. Show that the approximation of $\tan x$ by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

61. Suppose that the graph of a differentiable function $f(x)$ has a horizontal tangent at $x = a$. Can anything be said about the linearization of f at $x = a$? Give reasons for your answer.

62. *Reading derivatives from graphs.* The idea that differentiable curves flatten out when magnified can be used to estimate the values of the derivatives of functions at particular points. We magnify the curve until the portion we see looks like a straight line through the point in question, and then we use the screen's coordinate grid to read the slope of the curve as the slope of the line it resembles.

a) To see how the process works, try it first with the function $y = x^2$ at $x = 1$. The slope you read should be 2.

b) Then try it with the curve $y = e^x$ at $x = 1$, $x = 0$, and $x = -1$. In each case, compare your estimate of the derivative with the value of e^x at the point. What pattern do you see? Test it with other values of x. Chapter 6 will explain what is going on.

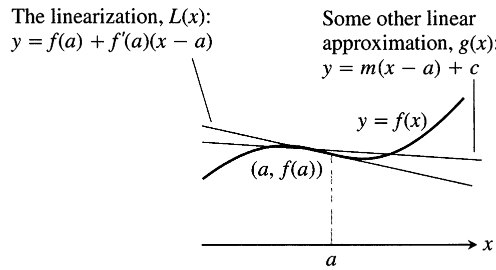
63. *Linearizations at inflection points.* As Fig. 3.64 suggests, linearizations fit particularly well at inflection points. You will understand why if you do Exercise 40 in Section 8.10 later in the book. As another example, graph *Newton's serpentine,* $f(x) = 4x/(x^2 + 1)$, together with its linearizations at $x = 0$ and $x = \sqrt{3}$.

64. *The linearization is the best linear approximation.* (This is why we use the linearization.) Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

1. $E(a) = 0$ — The approximation error is zero at $x = a$.

2. $\displaystyle\lim_{x \to a} \frac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



65. CALCULATOR Enter 2 in your calculator and take successive square roots by pressing the square root key repeatedly (or raising the displayed number repeatedly to the 0.5 power). What pattern do you see emerging? Explain what is going on. What happens if you take successive tenth roots instead?

66. CALCULATOR Repeat Exercise 65 with 0.5 in place of 2 as the original entry. What happens now? Can you use any positive number x in place of 2? Explain what is going on.

CAS Explorations and Projects

In Exercises 67–70, you will use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval I. Perform the following steps:

a) Plot the function f over I.

b) Find the linearization L of the function at the point a.

c) Plot f and L together on a single graph.

d) Plot the absolute error $|f(x) - L(x)|$ over I and find its maximum value.

e) From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \Rightarrow |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5$, 0.1, and 0.01. Then check graphically to see if your δ-estimate holds true.

67. $f(x) = x^3 + x^2 - 2x, \quad [-1, 2], \quad a = 1$

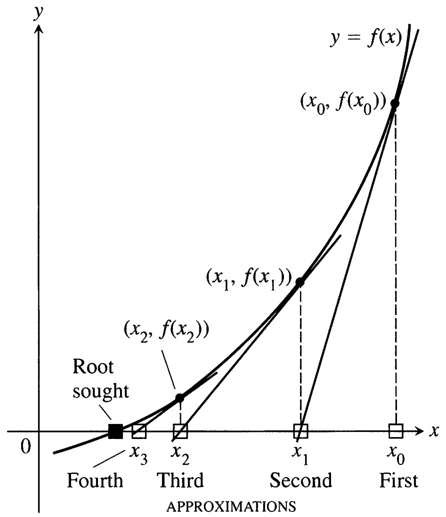
68. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad \left[-\dfrac{3}{4}, 1\right], \quad a = \dfrac{1}{2}$

69. $f(x) = x^{2/3}(x - 2), \quad [-2, 3], \quad a = 2$

70. $f(x) = \sqrt{x} - \sin x, \quad [0, 2\pi], \quad a = 2$

3.8 Newton's Method

We know simple formulas for solving linear and quadratic equations, and there are somewhat more complicated formulas for cubic and quartic equations (equations of degree three and four). At one time it was hoped that similar formulas might be found for quintic and higher degree equations, but the Norwegian mathematician



3.68 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

Neils Henrik Abel (1802–1829) showed that no formulas like these are possible for polynomial equations of degree greater than four.

When exact formulas for solving an equation $f(x) = 0$ are not available, we can turn to numerical techniques from calculus to approximate the solutions we seek. One of these techniques is *Newton's method* or, as it is more accurately called, the *Newton-Raphson method.* It is based on the idea of using tangent lines to replace the graph of $y = f(x)$ near the points where f is zero. Once again, linearization is the key to solving a practical problem.

The Theory

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of f crosses the x-axis (Fig. 3.68).

The initial estimate, x_0, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point where the tangent meets the x-axis x_1. The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence. We continue on, using each approximation to generate the next, until we are close enough to the root to stop.

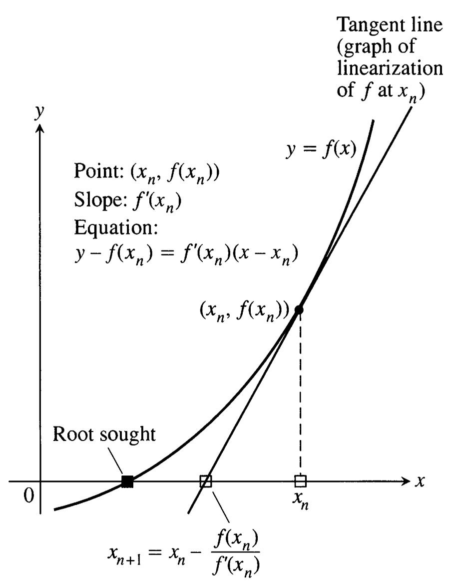
We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n, the point–slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n) \tag{1}$$

(Fig. 3.69). We find where the tangent crosses the x-axis by setting y equal to 0 in this equation and solving for x, giving, in turn,

$$\begin{aligned} 0 - f(x_n) &= f'(x_n)(x - x_n) && \text{Eq. (1) with } y = 0 \\ -f(x_n) &= f'(x_n)x - f'(x_n)x_n \\ f'(x_n)x &= f'(x_n)x_n - f(x_n) \\ x &= x_n - \frac{f(x_n)}{f'(x_n)}. && \text{Assuming } f'(x_n) \neq 0 \end{aligned}$$

This value of x is the next approximation, x_{n+1}.



3.69 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

The Strategy for Newton's Method

1. Guess a first approximation to a root of the equation $f(x) = 0$. A graph of $y = f(x)$ will help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (f'(x_n) \neq 0) \tag{2}$$

where $f'(x_n)$ is the derivative of f at x_n.

The Practice

In our first example we find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.

EXAMPLE 1 Find the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Eq. (2) becomes

$$x_{n+1} = x_n - \frac{x_n{}^2 - 2}{2x_n}.$$

To use our calculator efficiently, we rewrite this equation in a form that uses fewer arithmetic operations:

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

The equation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

enables us to go from each approximation to the next with just a few keystrokes. With the starting value $x_0 = 1$, we get the results in the first column of the following table. (To 5 decimal places, $\sqrt{2} = 1.41421$.)

	Error	**Number of correct figures**
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

❑

Newton's method is the method used by most calculators to calculate roots because it converges so fast (more about this later). If the arithmetic in the table in Example 1 had been carried to 13 decimal places instead of 5, then going one step further would have given $\sqrt{2}$ correctly to more than 10 decimal places.

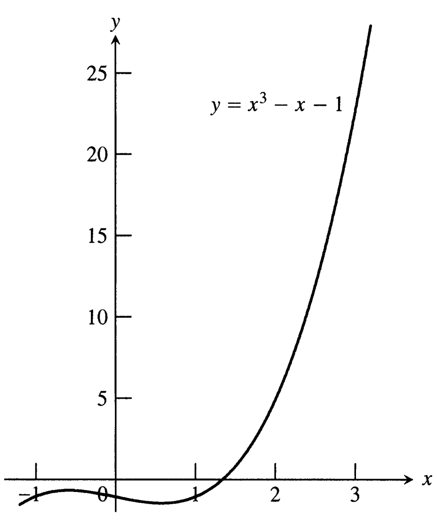
Algorithm and iteration

It is customary to call a specified sequence of computational steps like the one in Newton's method an *algorithm.* When an algorithm proceeds by repeating a given set of steps over and over, using the answer from the previous step as the input for the next, the algorithm is called *iterative* and each repetition is called an *iteration.* Newton's method is one of the really fast iterative techniques for finding roots.

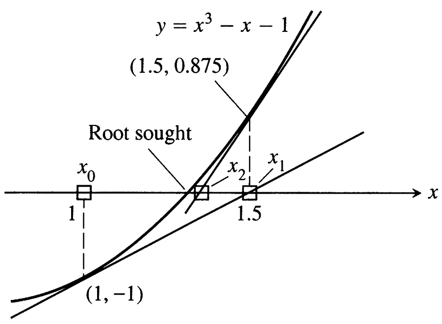
EXAMPLE 2 Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? The graph of f (Fig. 3.70) shows a single root, located between $x = 1$ and $x = 2$. We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 3.2 and Fig. 3.71.

At $n = 5$ we come to the result $x_6 = x_5 = 1.3247\ 17957$. When $x_{n+1} = x_n$, Eq. (2) shows that $f(x_n) = 0$. We have found a solution of $f(x) = 0$ to 9 decimals.



3.70 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis between $x = 1$ and $x = 2$.

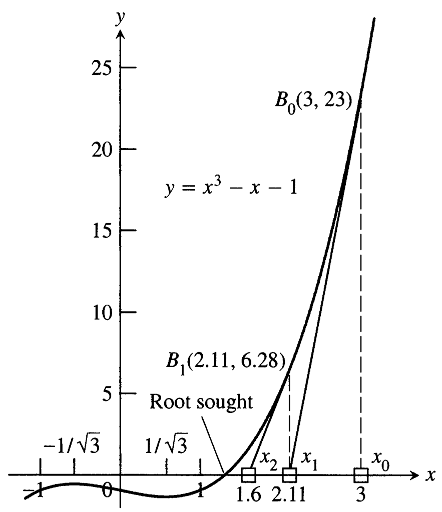


3.71 The first three x-values in Table 3.2.

Table 3.2 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68293	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.0437E-9	4.2646 32997	1.3247 17957

The equation $x^3 - x - 1 = 0$ is the equation we solved graphically in Section 1.5. Notice how much more rapidly and accurately we find the solution here. ❑



3.72 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root.

In Fig. 3.72, we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x-axis, but the tangent at B_0 crosses the x-axis at about (2.11, 0), so x_1 is still an improvement over x_0. If we use Eq. (2) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we confirm the 9-place solution $x_6 = x_5 = 1.3247\ 17957$ in six steps.

The curve in Fig. 3.72 has a local maximum at $x = -1/\sqrt{3}$ and a local minimum at $x = +1/\sqrt{3}$. We would not expect good results from Newton's method if we were to start with x_0 between these points, but we can start any place to the right of $x = 1/\sqrt{3}$ and get the answer. It would not be very clever to do so, but we could even begin far to the right of B_0, for example with $x_0 = 10$. It takes a bit longer, but the process still converges to the same answer as before.

Convergence Is Usually Assured

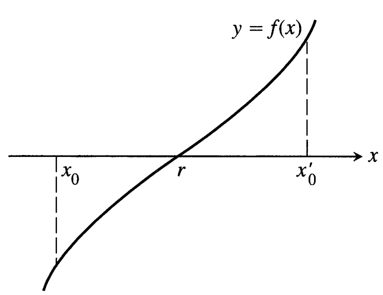
In practice, Newton's method usually converges with impressive speed, but since this is not guaranteed you must test that convergence is actually taking place. One way to do this would be to begin by graphing the function to find a good starting value for x_0. It is important to test that you are getting closer to a zero of the function, by evaluating $|f(x_n)|$, and to check that the method is converging, by evaluating $|x_n - x_{n+1}|$.

Theory does provide some help, however. A theorem from advanced calculus says that if

$$\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| < 1 \tag{3}$$

for all x in an interval about a root r, then the method will converge to r for any starting value x_0 in that interval. In practice, the theorem is somewhat hard to apply and convergence is evaluated by calculating $f(x_n)$ and $|x_n - x_{n+1}|$.

Inequality (3) is a *sufficient* but not a necessary condition. The method can and does converge in some cases where there is no interval about r on which the inequality holds. Newton's method always converges if the curve $y = f(x)$ is convex ("bulges") toward the x-axis in the interval between x_0 and the root sought. See Fig. 3.73.



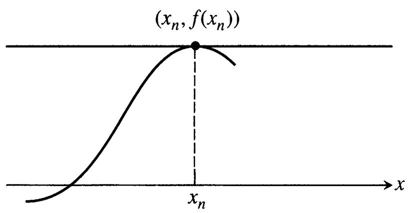
3.73 Newton's method will converge to r from either starting point.

Under favorable circumstances, the speed with which Newton's method converges to r is expressed by the advanced calculus formula

$$\underbrace{|x_{n+1} - r|}_{\text{error } e_{n+1}} \le \frac{\max |f''|}{2 \min |f'|} |x_n - r|^2 = \text{constant} \cdot \underbrace{|x_n - r|^2}_{\text{error } e_n}, \tag{4}$$

where max and min refer to the maximum and minimum values in an interval surrounding r. The formula says that the error in step $n + 1$ is no greater than a constant times the square of the error in step n. This may not seem like much, but think of what it says. If the constant is less than or equal to 1, and $|x_n - r| < 10^{-3}$, then $|x_{n+1} - r| < 10^{-6}$. *In a single step* the method moves from three decimal places of accuracy to six!

The results in (3) and (4) both assume that f is "nice." In the case of (4), this means that f has only a single root at r, so that $f'(r) \neq 0$. If f has a multiple root at r, the convergence may be slower.



3.74 If $f'(x_n) = 0$, there is no intersection point to define x_{n+1}.

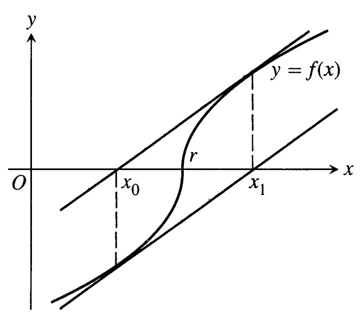
But Things Can Go Wrong

Newton's method stops if $f'(x_n) = 0$ (Fig. 3.74). In that case, try a new starting point. Of course, f and f' may have a common root. To detect whether this is so, you could first find the solutions of $f'(x) = 0$ and check f at those values. Or you could graph f and f' together.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \ge r, \end{cases} \tag{5}$$

the graph will be like the one in Fig. 3.75. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

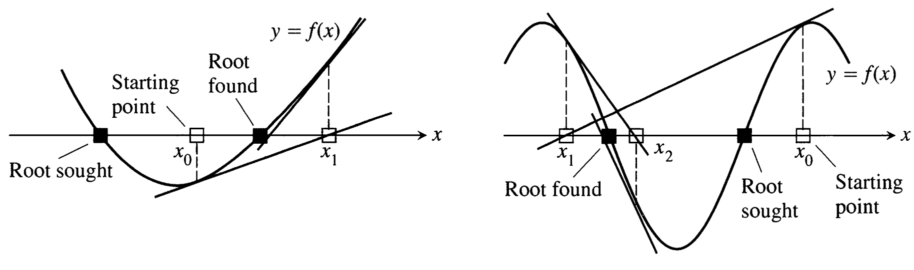


3.75 Newton's method fails to converge.

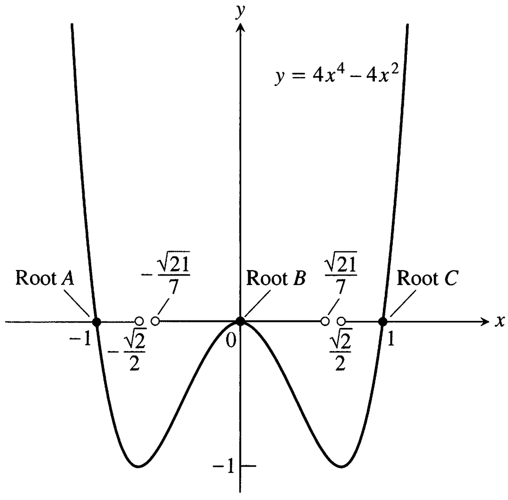
If Newton's method does converge, it converges to a root. In theory, that is. In practice, there are situations in which the method appears to converge but there is no root there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 3.76 shows two ways this can happen.

The solution then is to use everything you know about the curve—from graphs drawn by computer or from calculus-based analysis—to get a feeling for the shape of the curve near r and to choose an x_0 close to r. Use Newton's method and test its convergence as you go along. The chances are you will have no problems.

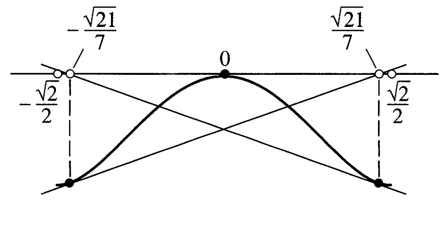


3.76 Newton's method may miss the root you want if you start too far away.

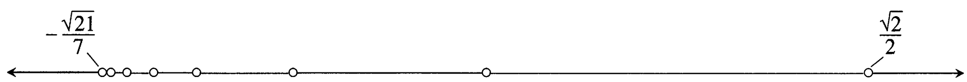


(a)

3.77 (a) Starting values in $(-\infty, -\sqrt{2}/2)$, $(-\sqrt{21}/7, \sqrt{21}/7)$, and $(\sqrt{2}/2, \infty)$ lead respectively to roots *A*, *B*, and *C*. (b) The values $x = \pm\sqrt{21}/7$ lead only to each other. (c) Between $\sqrt{21}/7$ and $\sqrt{2}/2$ there are infinitely many open intervals of points attracted to *A* alternating with open intervals of points attracted to *C*. This behavior is mirrored in the interval $(-\sqrt{2}/2, -\sqrt{21}/7)$.



(b)



(c)

⁂ Chaos in Newton's Method

The process of finding roots by Newton's method can be chaotic, meaning that for some equations the final outcome can be extremely sensitive to the starting value's location.

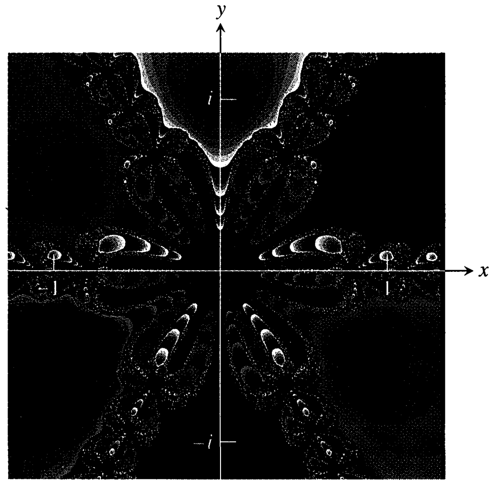
The equation $4x^4 - 4x^2 = 0$ is a case in point (Fig. 3.77a). Starting values in the blue zone on the x-axis lead to root *A*. Starting values in the black lead to root *B*, and starting values in the red zone lead to root *C*. The points $\pm\sqrt{2}/2$ give horizontal tangents. The points $\pm\sqrt{21}/7$ "cycle," each leading to the other, and back (Fig. 3.77b).

The interval between $\sqrt{21}/7$ and $\sqrt{2}/2$ contains infinitely many open intervals of points leading to root *A*, alternating with intervals of points leading to root *C* (Fig. 3.77c). The boundary points separating consecutive intervals (there are infinitely many) do not lead to roots, but cycle back and forth from one to another.

Here is where the "chaos" is truly manifested. As we select points that approach $\sqrt{21}/7$ from the right it becomes increasingly difficult to distinguish which lead to root *A* and which to root *C*. On the same side of $\sqrt{21}/7$, we find arbitrarily close together points whose ultimate destinations are far apart.

If we think of the roots as "attractors" of other points, the coloring in Fig. 3.77 shows the intervals of the points they attract (the "intervals of attraction"). You might think that points between roots *A* and *B* would be attracted to either *A* or *B*, but, as we see, that is not the case. Between *A* and *B* there are infinitely many intervals of points attracted to *C*. Similarly, between *B* and *C* lie infinitely many intervals of points attracted to *A*.

We encounter an even more dramatic example of chaotic behavior when we apply Newton's method to solve the complex-number equation $z^6 - 1 = 0$. It has six solutions: 1, -1, and the four numbers $\pm(1/2) \pm (\sqrt{3}/2)i$. As Fig. 3.78 (on the following page) suggests, each of the six roots has infinitely many "basins"



3.78 This computer-generated initial value portrait uses color to show where different points in the complex plane end up when they are used as starting values in applying Newton's method to solve the equation $z^6 - 1 = 0$. Red points go to 1, green points to $(1/2) + (\sqrt{3}/2)i$, dark blue points to $(-1/2) + (\sqrt{3}/2)i$, and so on. Starting values that generate sequences that do not arrive within 0.1 units of a root after 32 steps are colored black.

of attraction in the complex plane (Appendix 3). Starting points in red basins are attracted to the root 1, those in the green basin to the root $(1/2) + (\sqrt{3}/2)i$, and so on. Each basin has a boundary whose complicated pattern repeats without end under successive magnifications.

Exercises 3.8

Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.

3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.

4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

7. CALCULATOR At what value(s) of x does $\cos x = 2x$?

8. CALCULATOR At what value(s) of x does $\cos x = -x$?

9. CALCULATOR Use the Intermediate Value Theorem from Section 1.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to 5 decimal places.

10. CALCULATOR Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

Theory, Examples, and Applications

11. Suppose your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to x_1 and later approximations?

12. You plan to estimate $\pi/2$ to 5 decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

13. *Oscillation.* Show that if $h > 0$, applying Newton's method to
$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$
leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

14. *Approximations that get worse and worse.* Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$, and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

15. a) Explain why the following four statements ask for the same information:

i) Find the roots of $f(x) = x^3 - 3x - 1$.
ii) Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.
iii) Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.
iv) Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

a) CALCULATOR Use Newton's method to find the two negative zeros of $f(x) = x^3 - 3x - 1$ to 5 decimal places.

b) GRAPHER Graph $f(x) = x^3 - 3x - 1$ for $-2 \le x \le 2.5$. Use ZOOM and TRACE to estimate the zeros of f to 5 decimal places.

c) GRAPHER Graph $g(x) = 0.25x^4 - 1.5x^2 - x + 5$. Use ZOOM and TRACE with appropriate rescaling to find, to 5 decimal places, the values of x where the graph has horizontal tangents.

16. *Locating a planet.* To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to 5 decimal places.) Remember to use radians.

17. *Finding an ion concentration.* While trying to find the acidity of a saturated solution of magnesium hydroxide in hydrochloric acid, you derive the equation

$$\frac{3.64 \times 10^{-11}}{[H_3O^+]^2} = [H_3O^+] + 3.6 \times 10^{-4}$$

for the hydronium ion concentration $[H_3O^+]$. To find the value of $[H_3O^+]$, you set $x = 10^4[H_3O^+]$ and convert the equation to

$$x^3 + 3.6x^2 - 36.4 = 0.$$

You then solve this by Newton's method. What do you get for x? (Make it good to 2 decimal places.) For $[H_3O^+]$?

18. Show that Newton's method cannot converge to a point $x = c$ where the function's graph has an upward pointing cusp above the x-axis like the one in the margin on p. 215.

Computer or Programmable Calculator

Exercises 19–28 require a computer or programmable calculator.

19. The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

20. Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

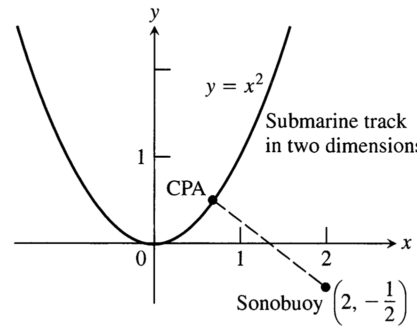
21. a) How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?
b) Use Newton's method to find them.

22. a) Does $\cos 3x$ ever equal x?
b) Use Newton's method to find where.

23. Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

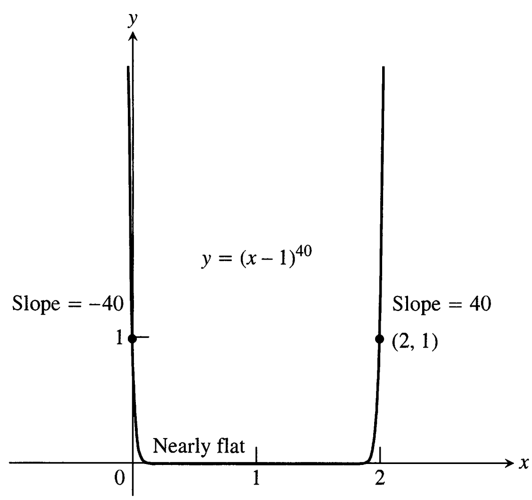
24. *The sonobuoy problem.* In submarine location problems it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on a parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

a) Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.
b) Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.



(Source: *The Contraction Mapping Principle,* by C. O. Wilde, UMAP Unit 326, Arlington, MA, COMAP, Inc.)

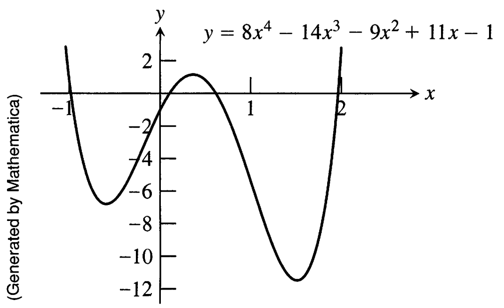
25. *Curves that are nearly flat at the root.* Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$.



26. *Finding a root different from the one sought.* All three roots of $f(x) = 4x^4 - 4x^2$ can be found by starting Newton's method near $x = \sqrt{21}/7$. Try it. See Fig. 3.77.

27. Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



28. *Chaos in Newton's method.* If you have a computer or a calculator that can be programmed to do complex-number arithmetic, experiment with Newton's method to solve the equation $z^6 - 1 = 0$. The recursion relation to use is

$$z_{n+1} = z_n - \frac{z_n^6 - 1}{6z_n^5} \quad \text{or} \quad z_{n+1} = \frac{5}{6}z_n + \frac{1}{6z_n^5}.$$

Try these starting values (among others): 2, i, $\sqrt{3} + i$.

CHAPTER 3 QUESTIONS TO GUIDE YOUR REVIEW

1. What can be said about the values of a function that is continuous on a closed interval?
2. What does it mean for a function to have a local extreme value on its domain? An absolute extreme value? How are local and absolute extreme values related, if at all? Give examples.
3. What is the First Derivative Theorem for Local Extreme Values? How does it lead to a procedure for finding a function's local extreme values?
4. How do you find the absolute extrema of a continuous function on a closed interval? Give examples.
5. What are the hypotheses and conclusion of Rolle's theorem? Are the hypotheses really necessary? Explain.
6. What are the hypotheses and conclusion of the Mean Value Theorem? What physical interpretations might the theorem have?
7. State the Mean Value Theorem's three corollaries.
8. How can you sometimes identify a function $f(x)$ by knowing f' and knowing the value of f at a point $x = x_0$? Give an example.
9. What is the First Derivative Test for Local Extreme Values? Give examples of how it is applied.
10. How do you test a twice-differentiable function to determine where its graph is concave up or concave down? Give examples.
11. What is an inflection point? Give an example. What physical significance do inflection points sometimes have?
12. What is the Second Derivative Test for Local Extreme Values? Give examples of how it is applied.
13. What do the derivatives of a function tell you about the shape of its graph?
14. List the steps you would take to graph a polynomial function. Illustrate with an example.
15. What is a cusp? Give examples.
16. What exactly do $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$ mean? Give examples.
17. What are $\lim_{x\to\pm\infty} k$ (k a constant) and $\lim_{x\to\pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.
18. How do you find the limit of a rational function as $x \to \pm\infty$? What are the three basic possibilities? Give examples.
19. List the steps you would take to graph a rational function. Illustrate with an example.
20. Outline a general strategy for solving max-min problems. Give examples.
21. What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of f at a for the linearization to exist? How are linearizations used? Give examples.
22. If x moves from x_0 to a nearby value $x_0 + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.
23. How do you estimate the error in a linear approximation? Give an example.
24. Describe Newton's method for solving equations. Give an example. What is the theory behind the method? What are some of the things to watch out for when you use the method?

CHAPTER 3 PRACTICE EXERCISES

Existence of Extreme Values

1. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.

2. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.

3. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

4. Find values of a and b such that the function
$$f(x) = \frac{ax + b}{x^2 - 1}$$
has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

5. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.

6. a) Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.
   b) How is this consistent with Theorem 2 in Section 3.1? Give reasons for your answer.

7. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Max-Min Theorem for continuous functions? Why?

8. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Max-Min Theorem for continuous functions? Why?

9. Grapher A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.
   a) Graph f over the interval $-2.5 \le x \le 2.5$. Where does the graph appear to have local extreme values or points of inflection?
   b) Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.
   c) Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

   The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

   (Source: *Uses of Technology in the Mathematics Curriculum,* by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under National Science Foundation Grant USE-8950044.)

10. (*Continuation of Exercise 9.*)
   a) Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \le x \le 2$. Where does the graph appear to have local extreme values or points of inflection?
   b) Show that f has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.
   c) Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

11. a) Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.
   b) How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

12. a) Show that $y = \tan\theta$ increases on every interval in its domain.
   b) If the conclusion in (a) is really correct, how do you explain the fact that $\tan \pi = 0$ is less than $\tan(\pi/4) = 1$?

13. a) Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.
   b) CALCULATOR Find the solution to as many decimal places as you can.

14. a) Show that $f(x) = x/(x + 1)$ increases on every interval in its domain.
   b) Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

15. CALCULATOR As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 h. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover one acre to the depth of one foot. A cubic foot holds 7.48 gal.)

16. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

17. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though

$$\frac{x}{x+1} \neq -\frac{1}{x+1}.$$

Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

18. Calculate the first derivatives of $f(x) = x^2/(x^2+1)$ and $g(x) = -1/(x^2+1)$. What can you conclude about the graphs of these functions?

Graphs and Graphing

Graph the curves in Exercises 19–28.

19. $y = x^2 - (x^3/6)$

20. $y = x^3 - 3x^2 + 3$

21. $y = -x^3 + 6x^2 - 9x + 3$

22. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

23. $y = x^3(8 - x)$

24. $y = x^2(2x^2 - 9)$

25. $y = x - 3x^{2/3}$

26. $y = x^{1/3}(x - 4)$

27. $y = x\sqrt{3 - x}$

28. $y = x\sqrt{4 - x^2}$

Each of Exercises 29–34 gives the first derivative of a function $y = f(x)$. (a) At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? (b) Sketch the general shape of the graph.

29. $y' = 16 - x^2$

30. $y' = x^2 - x - 6$

31. $y' = 6x(x+1)(x-2)$

32. $y' = x^2(6 - 4x)$

33. $y' = x^4 - 2x^2$

34. $y' = 4x^2 - x^4$

GRAPHER In Exercises 35–38, graph each function. Then use the function's first derivative to explain what you see.

35. $y = x^{2/3} + (x-1)^{1/3}$

36. $y = x^{2/3} + (x-1)^{2/3}$

37. $y = x^{1/3} + (x-1)^{1/3}$

38. $y = x^{2/3} - (x-1)^{1/3}$

Sketch the graphs of the functions in Exercises 39–46.

39. $y = \dfrac{x+1}{x-3}$

40. $y = \dfrac{2x}{x+5}$

41. $y = \dfrac{x^2+1}{x}$

42. $y = \dfrac{x^2-x+1}{x}$

43. $y = \dfrac{x^3+2}{2x}$

44. $y = \dfrac{x^4-1}{x^2}$

45. $y = \dfrac{x^2-4}{x^2-3}$

46. $y = \dfrac{x^2}{x^2-4}$

Using the graphs of the dominant terms as a guide, sketch the graphs of the equations in Exercises 47 and 48.

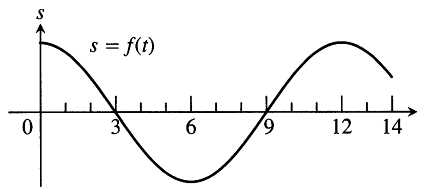
47. $y = \csc x - \dfrac{1}{x^2}, \quad 0 < x < \pi$

48. $y = \tan x - \dfrac{2}{x}, \quad -\dfrac{\pi}{2} < x < 0$

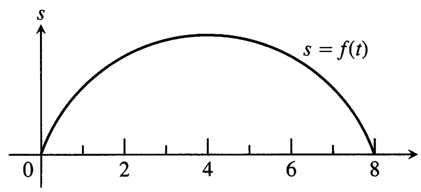
Drawing Conclusions About Motion from Graphs

Each of the graphs in Exercises 49 and 50 is the graph of the position function $s = f(t)$ of a body moving on a coordinate line (t represents time). At approximately what times (if any) is each body's (a) velocity equal to zero? (b) acceleration equal to zero? During approximately what time intervals does the body move (c) forward? (d) backward?

49.



50.



Limits

Find the limits in Exercises 51–60.

51. $\displaystyle\lim_{x\to\infty} \frac{2x+3}{5x+7}$

52. $\displaystyle\lim_{x\to-\infty} \frac{2x^2+3}{5x^2+7}$

53. $\displaystyle\lim_{x\to-\infty} \frac{x^2-4x+8}{3x^3}$

54. $\displaystyle\lim_{x\to\infty} \frac{1}{x^2-7x+1}$

55. $\displaystyle\lim_{x\to-\infty} \frac{x^2-7x}{x+1}$

56. $\displaystyle\lim_{x\to\infty} \frac{x^4+x^3}{12x^3+128}$

57. $\displaystyle\lim_{x\to\infty} \frac{\sin x}{\lfloor x\rfloor}$ (If you have a grapher, try graphing the function for $-5 \le x \le 5$.)

58. $\displaystyle\lim_{\theta\to\infty} \frac{\cos\theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to "see" the limit at infinity.)

59. $\displaystyle\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

60. $\displaystyle\lim_{x\to\infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

Optimization

61. The sum of two nonnegative numbers is 36. Find the numbers if (a) the difference of their square roots is to be as large as possible, (b) the sum of their square roots is to be as large as possible.

62. The sum of two nonnegative numbers is 20. Find the numbers

a) if the product of one number and the square root of the other is to be as large as possible;

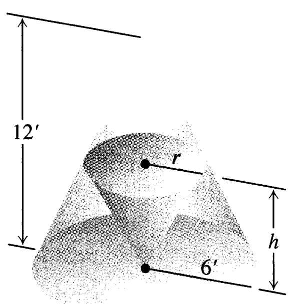
b) if one number plus the square root of the other is to be as large as possible.

63. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

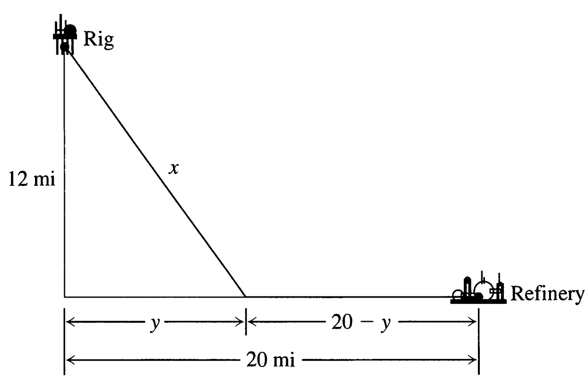
64. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

65. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

66. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



67. A drilling rig 12 mi offshore is to be connected by a pipe to a refinery onshore, 20 mi down the coast from the rig. If underwater pipe costs \$50,000 per mile and land-based pipe costs \$30,000 per mile, what values of x and y give the least expensive connection?



68. An athletic field is to be built in the shape of a rectangle x units long capped by semicircular regions of radius r at the two ends. The field is to be bounded by a 400-m racetrack. What values of x and r give the rectangle the largest possible area?

69. Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on grade A tires is twice your profit on grade B tires. Find the most profitable number of each kind of tire to make.

70. Suppose a manufacturer can sell x items a week for a revenue of $r = 200x - 0.01x^2$ cents, and it costs $c = 50x + 20{,}000$ cents to make x items. Is there a most profitable number of items to make each week? If so, what is it? Explain.

Linearization

71. Find the linearizations of

a) $\tan x$ at $x = -\pi/4$ b) $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

72. We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \quad \text{and} \quad \tan x \approx x$$

to get

$$\frac{1}{1 + \tan x} \approx 1 - x.$$

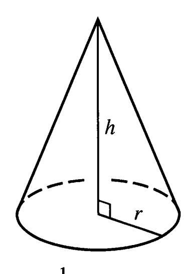
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

73. Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

74. Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1+x} - 3.1$ at $x = 0$.

Differential Estimates of Change

75. Write a formula that estimates the change that occurs in the volume of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change.



$V = \frac{1}{3}\pi r^2 h$

$S = \pi r \sqrt{r^2 + h^2}$

(Lateral surface area)

76. Write a formula that estimates the change that occurs in the lateral surface area of a cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.

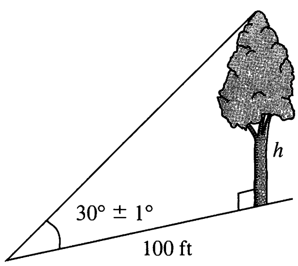
Applications of Differentials

77. a) How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

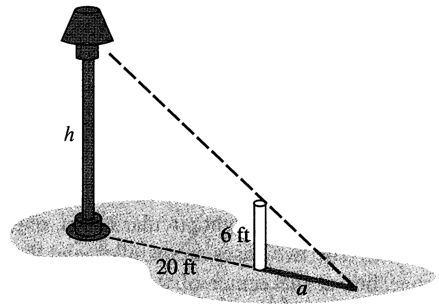
b) Suppose the edge is measured with the accuracy required in (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that would result from using the edge measurement.

78. The circumference of a great circle of a sphere is measured as 10 cm with a possible error of 0.4 cm. The measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of (a) the radius, (b) the surface area, and (c) the volume.

79. To find the height of a tree, you measure the angle from the ground to the treetop from a point 100 ft away from the base. The best figure you can get with the equipment at hand is $30° \pm 1°$. About how much error could the tolerance of $\pm 1°$ create in the calculated height? Remember to work in radians.



80. To find the height of a lamppost, you stand a 6-ft pole 20 ft from the lamp and measure the length a of its shadow. The figure you get for a is 15 ft, give or take an inch. Calculate the height of the lamppost from the value $a = 15$ and estimate the possible error in the result.



Newton's Method

81. CALCULATOR Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval [2, 3] and use Newton's method to find it.

82. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval [3, 4] and use Newton's method to find it.

CHAPTER 3 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.

2. Is it true that a discontinuous function cannot have both an absolute maximum and an absolute minimum value on a closed interval? Give reasons for your answer.

3. Can you conclude anything about the extreme values of a continuous function on an open interval? on a half-open interval? Give reasons for your answer.

4. Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x-1)(x-2)^2(x-3)^3(x-4)^4$$

to identify the points where f has local maximum and minimum values.

5. a) Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x+1)(x-2)^2.$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

b) Suppose that the first derivative of $y = f(x)$ is

$$y' = 6x(x+1)(x-2).$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

6. If $f'(x) \le 2$ for all x, what is the most the values of f can increase on [0, 6]? Give reasons for your answer.

7. Suppose that f is continuous on $[a, b]$ and that c is an interior point of the interval. Show that if $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. a) Show that $-1/2 \le x/(1+x^2) \le 1/2$ for every value of x.

b) Suppose that f is a function whose derivative is $f'(x) = x/(1+x^2)$. Use the result in (a) to show that

$$|f(b) - f(a)| \le \frac{1}{2}|b - a|$$

for any a and b.

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. Let $h = fg$ be the product of two differentiable functions of x.

a) If f and g are positive, with local maxima at $x = a$, and if f' and g' change sign at a, does h have local maximum at a?

b) If the graphs of f and g have inflection points at $x = a$, does the graph of h have an inflection point at a?

In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

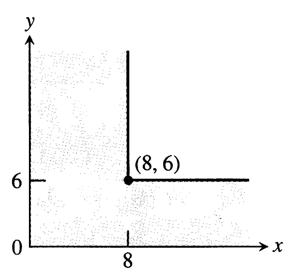
11. Use the following information to find the values of a, b, and c in the formula $f(x) = (x+a)/(bx^2+cx+2)$.

i) The values of a, b, and c are either 0 or 1.
ii) The graph of f passes through the point $(-1, 0)$.
iii) The line $y = 1$ is an asymptote of the graph of f.

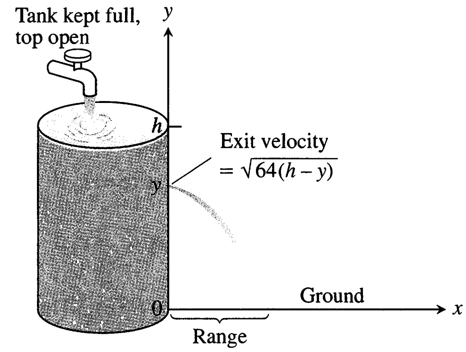
12. For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

13. Points A and B lie at the ends of a diameter of a unit circle and point C lies on the circumference. Is it true that the perimeter of triangle ABC is largest when the triangle is isosceles? How do you know?

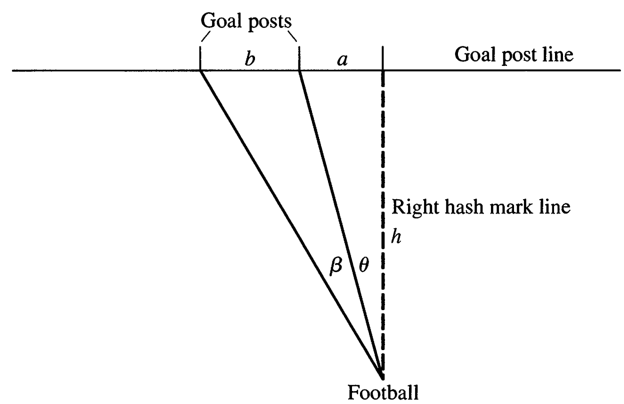
14. *The ladder problem.* What is the approximate length (ft) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



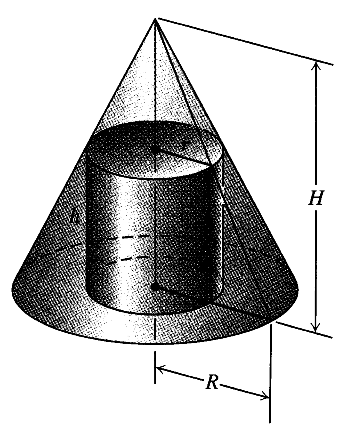
15. You want to bore a hole in the side of the tank shown here at a height that will make the stream of water coming out hit the ground as far from the tank as possible. If you drill the hole near the top, where the pressure is low, the water will exit slowly but spend a relatively long time in the air. If you drill the hole near the bottom, the water will exit at a higher velocity but have only a short time to fall. Where is the best place, if any, for the hole? (*Hint:* How long will it take an exiting particle of water to fall from height y to the ground?)



16. An American football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are b feet apart and that the hash mark line is a distance $a > 0$ feet from the right goal post. (See the accompanying figure.) Find the distance h from the goal post line that gives the kicker his largest angle β. Assume the football field is flat.



17. *A max-min problem with a variable answer.* Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H, as shown here. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$.



18. Find the smallest value of the positive constant m that will make $mx - 1 + (1/x)$ greater than or equal to zero for all positive values of x.

19. *The second derivative test.* The second derivative test for local maxima and minima (Section 3.4) says:

a) f has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$;

b) f has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

To prove statement (a), let $\epsilon = (1/2)|f''(c)|$. Then use the fact that

$$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h}$$

to conclude that for some $\delta > 0$,

$$0 < |h| < \delta \Rightarrow \frac{f'(c+h)}{h} < f''(c) + \epsilon < 0.$$

Thus $f'(c+h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

20. *Schwarz's inequality*

a) Show that if $a > 0$, then $f(x) = ax^2 + 2bx + c \geq 0$ for all (real) x if and only if $b^2 \leq ac$.

b) Derive **Schwarz's inequality,**

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2)(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2),$$

by applying what you learned in (a) to the sum

$$(a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2.$$

c) Show that equality holds in Schwarz's inequality only if there exists a real number x that makes a_ix equal $-b_i$ for every value of i from 1 to n.

21. *The period of a clock pendulum.* The period T of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where T is measured in seconds, $g = 32.2$ ft/sec^2, and L, the length of the pendulum, is measured in feet. Find approximately

a) the length of a clock pendulum whose period is $T = 1$ sec;

b) the change dT in T if the pendulum in (a) is lengthened 0.01 ft; and

c) the amount the clock gains or loses in a day as a result of the period's changing by the amount dT found in (b).

22. *Estimating reciprocals without division.* You can estimate the value of the reciprocal of a number a without ever dividing by a if you apply Newton's method to the function $f(x) = (1/x) - a$. For example, if $a = 3$, the function involved is $f(x) = (1/x) - 3$.

a) Graph $y = (1/x) - 3$. Where does the graph cross the x-axis?

b) Show that the recursion formula in this case is

$$x_{n+1} = x_n(2 - 3x_n),$$

so there is no need for division.

End Behavior Models

We call the function $y = 0$ an end behavior model for $f(x) = 1/x$ in the sense that $y = 0$ is a simpler function that behaves virtually the same way for $|x|$ large.

Definition

The function g is an **end behavior model** for f if

1. $\lim_{x\to\pm\infty} f/g = 1$ when $g(x) \neq 0$ for $|x|$ large, or
2. $\lim_{x\to\pm\infty} f(x) = 0$ when $g(x) = 0$.

For instance, $g(x) = 2$ is an end behavior model for $f(x) = 2 + (\sin x)/x$.

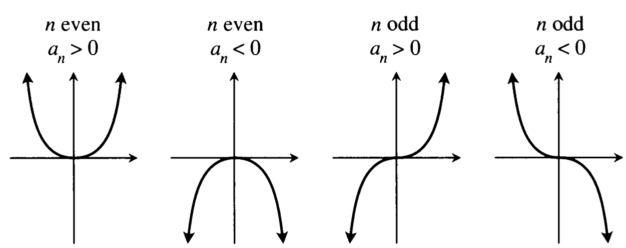
23. Show that $y = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 5x + 1$.

24. *Polynomial end behavior*

a) Show that $y = a_nx^n$ is an end behavior model for

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0. \quad (a_n \neq 0)$$

b) Then show that for $n \geq 1$ there are only four types of polynomial end behavior models.



c) *Polynomials of odd degree.* Show that the polynomial function

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad (a_n \neq 0)$$

has at least one zero if n is odd and $n > 1$.