**Math 286 (Bronski/Tyson)**

**Putzer's Method** Last time we saw how to exponentiate any matrix which is similar to a diagoanl matrix: if $D = U^{-1}AU$ (and thus $A = Ue^{tD}U^{-1}$ then $e^{tA} = Ue^{tD}U^{-1}$.

Unfortunately there are some matrices which are **not** similar to a diagonal matrices. These are (some) matrices with repeated eigenvalues, for instance

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

There is a more general form (the Jordan canonical form) into which every matrix can be transformed, but the algebra involved is rather lengthy. Instead we will present an alternative method, called Putzer's method, for which one only needs to know the eigenvalues for the matrix, and be able to solve a sequence of first order differential equations.

**Theorem 1** (Putzer's Algorithm)**.** *Assume that $A$ has $n$ eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_n$ (with multiple eigenvalues listed according to their* **algebraic** *multiplicity.) The order in which the eigenvalues are listed does not matter, but must be used consistently throughout the procedure. Define the following matrices.*

$$\begin{aligned}
&M_0 = I \\
&M_1 = (A - \lambda_1 I)M_0 = (A - \lambda_1 I) \\
&M_2 = (A - \lambda_2 I)M_1 = (A - \lambda_2 I)(A - \lambda_1 I) \\
&M_3 = (A - \lambda_3 I)(A - \lambda_2 I)(A - \lambda_1 I) \\
&M_{n-1} = (A - \lambda_{n-1} I)(A - \lambda_{n-2} I) \ldots (A - \lambda_1 I)
\end{aligned}$$

*(Note: You should find that $M_n = 0$) Define the following sequence of functions*

$$\begin{aligned}
&\frac{dr_0}{dt} = \lambda_1 r_0 \qquad r_0(0) = 1 \\
&\frac{dr_1}{dt} = \lambda_2 r_1 + r_0(t) \qquad r_1(0) = 0 \\
&\frac{dr_2}{dt} = \lambda_3 r_2 + r_1(t) \qquad r_2(0) = 0 \\
&\vdots \\
&\frac{dr_{n-1}}{dt} = \lambda_n r_{n-1} + r_{n-2} \qquad r_k(0) = 0
\end{aligned}$$

*so the $k^{th}$ function is the righthand side in the computation of the $(k+1)^{st}$ function. Then the matrix exponential is given by*

$$e^{tA} = \sum_{i=0}^{n-1} r_i(t) M_i$$

This algorithm **always** works and does not require one to compute eigenvectors, but may be less efficient for large matrices.

**Example 1.** *We begin with an example that is not diagonalizable:*
$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *This matrix has $\lambda = 1$ as a double eigenvalue so we have $\lambda_1 = 1, \lambda_2 = 1$ As far as the matrices go we have*

$$M_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$M_1 = (A - I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$M_2 = (A - I)(A - I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

*Note that $M_2$ is not needed for the algorithm but computing it is a good check, since it should come out to be zero.*

*The function $r_0(t)$ solves*

$$\frac{dr_0}{dt} = r_0 \qquad r_0(0) = 1$$

*giving $r_0(t) = e^t$. The function $r_1(t)$ solves*

$$\frac{dr_1}{dt} = r_1 + r_0(t) = r_1 + e^t \qquad r_1(0) = 0$$

*Which has the solution $r_1(t) = te^t$. Thus the matrix exponential is given by*

$$e^{tA} = e^t I + te^t(A - I) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

Let us do some more complicated examples:

**Example 2.** *Compute* $e^{tA}$ *where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ *To start note that the eigenvalues are given by* $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 3$.

**Example 3.** *Compute $e^{tA}$ using Putzer's method, where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$

*To start note that the eigenvalues are given by* $\lambda_1 = 1, \lambda_2 = 1 + i, \lambda_3 = 1 - i$