1 Review of Differential Equations

1.1 Seperation of variables

General First Order ODE:

$$\frac{dy}{dt} + py = g$$

If p and g are constants we can solve this by seperation of variables:

$$\begin{aligned} \frac{dy}{dt} + py &= g \\ \frac{dy}{dt} &= g - py \\ \frac{1}{g-py}dy &= dt \end{aligned}$$

Integrate both sides:

$$\int \frac{1}{g-py}dy = \int 1dt$$

Substitution: Let $u = g - py \implies du = -pdy$

$\implies dy = -\frac{1}{p}du$

$$\begin{aligned} \int \frac{1}{u}\left(-\frac{1}{p}\right)du &= t + C \\ -\frac{1}{p}\ln|u| &= t + C \\ \ln|u| &= -p(t+C) \\ \ln|u| &= -pt + C_{new} \quad (\text{where } C_{new} = -pC) \\ |u| &= e^{-pt+C_{new}} \\ u &= \pm e^{C_{new}}e^{-pt} \end{aligned}$$

Let $C_1 = \pm e^{C_{new}}$. Substitute back $u = g - py$:

$$\begin{aligned} g - py &= C_1e^{-pt} \\ -py &= C_1e^{-pt} - g \\ y &= \frac{C_1e^{-pt}}{-p} + \frac{-g}{-p} \\ \boxed{y(t) = C_2e^{-pt} + \frac{g}{p}} &\quad \text{where } C_2 = -\frac{C_1}{p} \end{aligned}$$

Behavior Analysis

The solution consists of two distinct parts:

$$y(t) = \underbrace{C_2 e^{-pt}}_{\text{Transient Solution}} + \underbrace{\frac{g}{p}}_{\text{Steady State Solution}}$$

- **Transient:** If $p > 0$, this term decays to zero as $t \to \infty$.
- **Steady State:** The value $y(t)$ approaches $\frac{g}{p}$ as time goes on.
- **Stability:** If $p < 0$, the exponential term grows to infinity (Unstable).

Solving for the Initial Condition

Given an initial value $y(0) = y_0$, we can solve for C_2:

$$y(0) = C_2 e^{-p(0)} + \frac{g}{p}$$
$$y_0 = C_2(1) + \frac{g}{p}$$
$$C_2 = y_0 - \frac{g}{p}$$

Substituting this back into the general solution gives the specific solution:

$$\boxed{y(t) = \left(y_0 - \frac{g}{p}\right) e^{-pt} + \frac{g}{p}}$$

Remember, this is for a LINEAR 1ST ORDER ODE WITH CONSTANT COEFFICIENTS.

1.2 Method of integrating Factors

$$\frac{dy}{dt} + p(t)y = q(t)$$

Rewrite to match the form of : Non Homog due to $q(t)$

$$\begin{aligned}
\frac{d}{dt}\left(\mu(t)y\right) &= \mu(t)q(t) \\
\int \frac{d}{dt}\left(\mu(t)y\right)dt' &= \int_0^T \mu(t)q(t)dt' \\
\mu(t)y(t) - \mu(0)y(0) &= \int_0^T \mu(t')q(t')dt' \\
\mu(t)y(t) &= \int_0^T \mu(t')q(t')dt' + \mu(0)y(0) \\
y(t) &= \frac{1}{\mu(t)}\left[\int_0^T \mu(t')q(t')dt' + \mu(0)y(0)\right]
\end{aligned}$$

To find μ, we match both sides of the equation.

$$\frac{d}{dt}\Big(\mu(t)y\Big) = \frac{d\mu}{dt}y + \mu\frac{dy}{dt} \tag{$\bigstar$}$$

Multiply the original equation by μ:

$$\mu\left(\frac{dy}{dt} + p(t)y = q(t)\right)$$

$$\mu\frac{dy}{dt} + \mu p(t)y = \mu q(t) \tag{Δ}$$

Now we can compare $\bigstar$ and Δ

$$\begin{aligned}
\frac{d\mu}{dt} &= \mu p(t) \\
\frac{1}{\mu}d\mu &= p(t)dt \\
\mu &= \boxed{Ce^{\int p(t)dt}}
\end{aligned}$$

Note on the Constant C

We can choose $C = 1$ because any constant cancels out.

why:

Let $\mu = Ce^{\int p(t)dt}$. Substitute into general solution:

$$y(t) = \frac{1}{Ce^{\int pdt}}\left[\int Ce^{\int pdt}q(t)dt + C_{int}\right]$$

Factor out C from the integral:

$$y(t) = \frac{1}{Ce^{\int pdt}} \cdot C\left[\int e^{\int pdt}q(t)dt + \frac{C_{int}}{C}\right]$$

Cancel the C terms:

$$y(t) = \frac{1}{e^{\int pdt}}\left[\int e^{\int pdt}q(t)dt + C_{new}\right]$$

The constant C in μ is redundant.

Big Must: Standard Form

The derivation assumes the coefficient of $\frac{dy}{dt}$ is **1**.

If given a general form:

$$a(t)\frac{dy}{dt} + b(t)y = g(t)$$

You **must** divide by $a(t)$ first:

$$\frac{dy}{dt} + \frac{b(t)}{a(t)}y = \frac{g(t)}{a(t)}$$

Identify your terms:

$$p(t) = \frac{b(t)}{a(t)}, \quad q(t) = \frac{g(t)}{a(t)}$$

Algorithm Summary

1. **Standardize:**

   Ensure $\frac{dy}{dt}$ has a coefficient of 1.
2. **Identify:**

   Find $p(t)$ and $q(t)$.
3. **Calculate $\boldsymbol{\mu}$:**

   $\mu(t) = e^{\int p(t)dt}$ (ignore $+C$).
4. **Solve:**

   $$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right].$$

1.3 2nd Order Linear Differential Equations

A general form of a 2nd order ODE:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

or

$$F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

A quick note of Implicit, Explicit and Homogeneity

When we say $F\left(x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right)$, the **Container** F holds all combinations of said $x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ that add add to 0. Please note, this doesn't mean its Homogeneous in how its presented here. More on this in a little.

For the other form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Here we are saying, "if you give me current time, position and velocity, Ill be able to calculate acceleration"

Back to **Homgeneous or not**.
In the form $F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$ we have only moved all things in the container to one side. Using my ridiculous example from before,

Imagine an Implicit Function F representing the ingredients of a lunch equation:

$$F\left(x, y, \text{PB}, \text{Jelly}, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

This is **Implicit** because the ingredients are all mixed together in the bucket F.

To check if it is **Homogeneous**, we try to isolate the sandwich mechanics (y, y', y'') from the external ingredients:

$$\underbrace{\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y}_{\text{The Bread}} = \underbrace{\text{PB} + \text{Jelly}}_{\text{External Forcing Function } f(t)}$$

- If $\text{PB} = 0$ and $\text{Jelly} = 0$, the equation is **Homogeneous** (Just Bread).
- If $\text{PB} \neq 0$ or $\text{Jelly} \neq 0$, the equation is **Non-Homogeneous** (A Sandwich exists).

The Linear Case

$$\frac{d^2y}{dt^2} = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

if $g(t) = 0$;

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$
$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

This is now homogeneous and we can move forward with a characteristic equation.

The Characteristic Equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

change the $\frac{d}{dt}$ terms to r's

$$r^2y + p(t)ry + q(t)r^0y = 0$$

we can now change the coefficients to manageable variables

$$\left[ar^2 + br + c\right]y = 0$$
$$\underbrace{ar^2 + br + c = 0}_{\text{Characteristic equation}}$$

$\rightarrow$ find the roots r_1 and r_2 of the characteristic

$$(r - r_1)(r - r_2)y = 0$$

This equation is satisfied when:

$$(r - r_1)y = 0 \quad \textbf{or} \quad (r - r_2)y = 0$$

So the

$$\begin{aligned} (r - r_1)y &= 0 \\ (\frac{d}{dt} - r_1)y &= 0 \\ \frac{dy}{dt} &= r_1 y_1 \\ y_1 &= C_1 e^{r_1 t} \end{aligned}$$

$$\begin{aligned} (r - r_2)y &= 0 \\ (\frac{d}{dt} - r_2)y &= 0 \\ \frac{dy}{dt} &= r_2 y_2 \\ y_2 &= C_2 e^{r_2 t} \end{aligned}$$

general solution for this would be:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

C_1 and C_2 will both be found by initial conditions.

Example:

$$\begin{aligned} y'' + 4y' + 4y &= 0 && \text{(original)} \\ r^2 + 4r + 4 &= 0 \\ (r+2)(r+2) &= 0 \\ (r+2)^2 &= 0 \end{aligned}$$

We know that one linear independence is of the form:

$$y_1 = C_1 e^{-2t}$$

for the other solution we can assume:

$$y_2 = \nu(t) e^{-2t}$$

lets find $\nu(t)$ by plugging this back into the original ODE

But first we need to find y_2' and y_2''

$$\begin{aligned}
y_2 &= \nu(t)e^{-2t} \\
y_2' &= \frac{d}{dt}\nu(t)e^{-2t} + \nu(t)\frac{d}{dt}e^{-2t} \\
&= \nu'(t)e^{-2t} - 2\nu(t)e^{-2t} \\
y_2'' &= \frac{d}{dt}\nu'(t)e^{-2t} - 2\nu(t)\frac{d}{dt}e^{-2t} \\
&= \nu''(t)e^{-2t} - 4\nu'(t)e^{-2t} + 4\nu(t)e^{-2t}
\end{aligned}$$

now into the original:

$$y_2'' + 4y_2' + 4y = 0$$

$$\begin{aligned}
\nu''(t)e^{-2t} - 4\nu'(t)e^{-2t} + 4\nu(t)e^{-2t} + 4(\nu'(t)e^{-2t} - 2\nu(t)e^{-2t}) + 4(\nu(t)e^{-2t}) &= 0 \\
\nu''(t)e^{-2t} - 4\nu'(t)e^{-2t} + 4\nu(t)e^{-2t} + 4\nu'(t)e^{-2t} - 8\nu(t)e^{-2t}) + 4\nu(t)e^{-2t}) &= 0 \\
\nu''(t) &= 0
\end{aligned}$$

now integrate to get $\nu(t)$

$$v(t) = C_1 t + C_2$$

to form the basis needed, we just take the simplest form of the equation for $\nu(t)$, where $C_1 = 1$ and $C_2 = 0$

$$\nu(t) = t$$

Now we have both parts of the general solution

$$y = C_1 e^{-2t} + C_2 t e^{-2t}$$

Note on Complex Roots

If the characteristic equation $ar^2 + br + c = 0$ yields complex roots:

$$r_{1,2} = \lambda \pm i\mu$$

The general solution takes the form:

$$y(t) = e^{\lambda t}\Big(C_1 \cos(\mu t) + C_2 \sin(\mu t)\Big)$$

Why? This comes from Euler's Identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Applying this to our root $r = \lambda + i\mu$:

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t} \cdot e^{i(\mu t)} \\ &= e^{\lambda t}(\cos\mu t + i\sin\mu t) \end{aligned}$$

(The imaginary terms form the oscillation μ, and the real part λ forms the growth/decay envelope).

1.4 Non-Homogeneous 2nd order ODE

Method of Undetermined Coefficients

Example:

$$y'' - 3y' - 4y = 3e^{2t}$$

Step 1: Find the homogeneous (y_h) solution via the characteristic eqn:

$$\begin{aligned} r^2 - 3r - 4 &= 0 \\ (r-4)(r+1) &= 0 \\ r &= 4, -1 \\ y_h(t) &= C_1 e^{4t} + C_2 e^{-t} \end{aligned}$$

Now we make an **Ansatz** (guess) for the particular solution $y_p(t)$ based on the forcing function $3e^{2t}$.

It needs to satisfy the ODE:

$$y_p''(t) - 3y_p'(t) - 4y_p(t) = 3e^{2t}$$

Since the forcing function is an exponential, we guess a matching form:

$$\begin{aligned} y_p(t) &= Ae^{2t} \\ y_p'(t) &= 2Ae^{2t} \\ y_p''(t) &= 4Ae^{2t} \end{aligned}$$

Substitute these back into the original ODE:

$$\begin{aligned} 4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) &= 3e^{2t} \\ (4-6-4)Ae^{2t} &= 3e^{2t} \\ -6A &= 3 \\ A &= -\frac{1}{2} \end{aligned}$$

So the general solution is:

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = C_1e^{4t} + C_2e^{-t} - \frac{1}{2}e^{2t}}$$

What happens if we change things a little??
lets say we instead had,

$$y'' - 3y' - 4y = 3e^{-t}$$

The y_c part would still be $C_1e^{4t} + C_2e^{-t}$

Notice that the y_c we chose is no longer linearly independent from the y_p we are going to have. So what we need to do now is change the ansatz to Ate^{2t}.
We need y_p to be **Linearly Independent** from the homogeneous solution y_c.

- **The Conflict:** If our guess y_p appears in y_c, then y_p is **Linearly Dependent** (it is just a copy of the natural response). Plugging it into the ODE will yield $0 = g(t)$, which is impossible.
- **The Fix:** Multiplying by t breaks this dependency. The functions e^{-t} and te^{-t} are Linearly Independent, allowing the math to balance.

Summary Table: Standard Guesses

When the forcing function $g(t)$ is one of the standard forms, use the corresponding trial solution $y_p(t)$.

Forcing Function $g(t)$	**Trial Solution Form** $y_p(t)$
C (Constant)	A
$at^n + \cdots + a_0$ (Polynomial)	$A_nt^n + \cdots + A_1t + A_0$
Ce^{rt}	Ae^{rt}
$C\cos(\omega t)$ or $C\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
$e^{rt}\cos(\omega t)$	$e^{rt}(A\cos(\omega t) + B\sin(\omega t))$

Resonance

Important: If any term in your guess $y_p(t)$ is *already* a part of the homogeneous solution $y_c(t)$, you must multiply your guess by t^s, where s is the smallest integer $(1, 2, \dots)$ that makes the term unique.

- **Example:** $y'' - y' = 3$.
  - Homogeneous: $y_c = c_1 + c_2e^t$.
  - Naive Guess for $g(t) = 3$: $y_p = A$.
  - **Problem:** The constant A is absorbed by c_1.
  - **Fix:** Multiply by $t \implies \boxed{y_p = At}$.

Example

$$y'' - 2y' + y = e^x + x$$

Step 1: Homogeneous Solution

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \quad \implies r = 1, 1 \text{ (Repeated!)} \\ y_h &= C_1 e^x + \underbrace{C_2 x e^x}_{\text{Repeated root rule}} \end{aligned}$$

Step 2: Particular Solution (y_p) The forcing function is $g(x) = e^x + x$. We split this into two parts:

1. **Part A (e^x):**
   - Initial guess: Ae^x.
   - Conflict: Overlaps with $C_1 e^x$. $\rightarrow$ Try Axe^x.
   - Conflict: Overlaps with $C_2 x e^x$. $\rightarrow$ Try $Ax^2 e^x$.
   - **Final Guess A:** $y_{p1} = Ax^2 e^x$
2. **Part B (x):**
   - This is a degree 1 polynomial.
   - **Final Guess B:** $y_{p2} = Bx + C$

Combined Guess:

$$\begin{aligned} y_p &= Ax^2 e^x + Bx + C \\ y_p' &= A(2xe^x + x^2 e^x) + B \\ y_p'' &= A(2e^x + 2xe^x + 2xe^x + x^2 e^x) = Ae^x(x^2 + 4x + 2) \end{aligned}$$

Substitute these into the original ODE ($y'' - 2y' + y = e^x + x$):

$$\Big[Ae^x(x^2 + 4x + 2)\Big] - 2\Big[Ae^x(x^2 + 2x) + B\Big] + \Big[Ax^2 e^x + Bx + C\Big] = e^x + x$$

Step 4: Solving for Coefficients

Group the terms by type (e^x terms, x terms, and constants):

$$\begin{aligned} \textbf{Exp terms:} \quad & Ae^x\Big[(x^2 + 4x + 2) - 2(x^2 + 2x) + x^2\Big] = Ae^x[2] \\ \textbf{Poly terms:} \quad & Bx + (C - 2B) \end{aligned}$$

Set equal to the right hand side ($1e^x + 1x + 0$):

$$2Ae^x = 1e^x \implies 2A = 1 \implies \boxed{A = \frac{1}{2}}$$

$$Bx = 1x \implies \boxed{B = 1}$$

$$C - 2B = 0 \implies C - 2(1) = 0 \implies \boxed{C = 2}$$

Step 5: General Solution

Combine y_h and the solved y_p:

$$y(x) = y_h + y_p$$
$$\boxed{y(x) = C_1 e^x + C_2 x e^x + \frac{1}{2}x^2 e^x + x + 2}$$

1.5 Variation of Parameters (Lagrange)

Use Case: When the forcing function $r(x)$ is not amendable to Undetermined Coefficients (e.g., $\sec x, \tan x, \ln x$), or coefficients are not constant.

Standard Form Assumption:

$$y'' + f(x)y' + g(x)y = r(x)$$

Warning: The coefficient of y'' must be 1. If it is $x^2 y''$, divide by x^2 first!

Step 1: The Homogeneous Foundation

First, find the complementary solution (y_h) satisfying $y'' + fy' + gy = 0$:

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

Step 2: Variation of Constants

We replace the constants C_1, C_2 with unknown functions $u(x), v(x)$:

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

Step 3: Deriving the System

To find u and v, we need two equations.

1. The Constraint (Avoiding u'', v''): Differentiating y_p initially gives terms with u' and v'. To simplify, we choose:

$$\boxed{u'y_1 + v'y_2 = 0}$$

This simplifies the first derivative to: $y_p' = uy_1' + vy_2'$.

2. The Second Derivative & Substitution: Differentiate y_p' again to get y_p'':

$$y_p'' = (u'y_1' + uy_1'') + (v'y_2' + vy_2'')$$

Substitute y_p, y_p', y_p'' into the ODE ($y'' + Py' + Qy = r$):

$$\underbrace{[(u'y_1' + uy_1'') + (v'y_2' + vy_2'')]}_{y''} + P\underbrace{[uy_1' + vy_2']}_{y'} + Q\underbrace{[uy_1 + vy_2]}_{y} = r$$

Group by u and v. Since y_1, y_2 are homogeneous solutions, their terms sum to zero:

$$u(\underbrace{y_1'' + Py_1' + Qy_1}_{0}) + v(\underbrace{y_2'' + Py_2' + Qy_2}_{0}) + (u'y_1' + v'y_2') = r$$

This leaves the second equation:

$$\boxed{u'y_1' + v'y_2' = r(x)}$$

Step 4: Solve with Wronskian

We solve the system for u', v' using Cramer's Rule:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

The formulas for the derivatives are:

$$u'(x) = \frac{-y_2(x)r(x)}{W}, \quad v'(x) = \frac{y_1(x)r(x)}{W}$$

Step 5: Integrate

Integrate u', v' to find the functions, then plug back into the Ansatz:

$$y_p(x) = \left(\int u'(x)dx\right) y_1(x) + \left(\int v'(x)dx\right) y_2(x)$$