

Re-Entry Series

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1 Review of Differential Equations

1.1 Separation of variables

General First Order ODE:

$$\frac{dy}{dt} + py = g$$

If p and g are constants we can solve this by separation of variables:

$$\begin{aligned}\frac{dy}{dt} + py &= g \\ \frac{dy}{dt} &= g - py \\ \frac{1}{g - py} dy &= dt\end{aligned}$$

Integrate both sides:

$$\int \frac{1}{g - py} dy = \int 1 dt$$

Substitution: Let $u = g - py \implies du = -pdy$

$$\implies dy = -\frac{1}{p} du$$

$$\begin{aligned}\int \frac{1}{u} \left(-\frac{1}{p} \right) du &= t + C \\ -\frac{1}{p} \ln |u| &= t + C \\ \ln |u| &= -p(t + C) \\ \ln |u| &= -pt + C_{new} \quad (\text{where } C_{new} = -pC) \\ |u| &= e^{-pt+C_{new}} \\ u &= \pm e^{C_{new}} e^{-pt}\end{aligned}$$

Let $C_1 = \pm e^{C_{new}}$. Substitute back $u = g - py$:

$$\begin{aligned}g - py &= C_1 e^{-pt} \\ -py &= C_1 e^{-pt} - g \\ y &= \frac{C_1 e^{-pt}}{-p} + \frac{-g}{-p} \\ y(t) &= C_2 e^{-pt} + \frac{g}{p}\end{aligned}$$

where $C_2 = -\frac{C_1}{p}$

Behavior Analysis

The solution consists of two distinct parts:

$$y(t) = \underbrace{C_2 e^{-pt}}_{\text{Transient Solution}} + \underbrace{\frac{g}{p}}_{\text{Steady State Solution}}$$

- **Transient:** If $p > 0$, this term decays to zero as $t \rightarrow \infty$.
- **Steady State:** The value $y(t)$ approaches $\frac{g}{p}$ as time goes on.
- **Stability:** If $p < 0$, the exponential term grows to infinity (Unstable).

Solving for the Initial Condition

Given an initial value $y(0) = y_0$, we can solve for C_2 :

$$\begin{aligned} y(0) &= C_2 e^{-p(0)} + \frac{g}{p} \\ y_0 &= C_2(1) + \frac{g}{p} \\ C_2 &= y_0 - \frac{g}{p} \end{aligned}$$

Substituting this back into the general solution gives the specific solution:

$$y(t) = \left(y_0 - \frac{g}{p} \right) e^{-pt} + \frac{g}{p}$$

Remember, this is for a LINEAR 1ST ORDER ODE WITH CONSTANT COEFFICIENTS.

1.2 Method of integrating Factors

$$\frac{dy}{dt} + p(t)y = q(t)$$

Rewrite to match the form of :

Non Homog due to $q(t)$

$$\begin{aligned} \frac{d}{dt}(\mu(t)y) &= \mu(t)q(t) \\ \int \frac{d}{dt}(\mu(t)y) dt' &= \int_0^T \mu(t)q(t) dt' \\ \mu(t)y(t) - \mu(0)y(0) &= \int_0^T \mu(t')q(t') dt' \\ \mu(t)y(t) &= \int_0^T \mu(t')q(t') dt' + \mu(0)y(0) \\ y(t) &= \frac{1}{\mu(t)} \left[\int_0^T \mu(t')q(t') dt' + \mu(0)y(0) \right] \end{aligned}$$

To find μ , we match both sides of the equation.

$$\frac{d}{dt}(\mu(t)y) = \frac{d\mu}{dt}y + \mu \frac{dy}{dt} \quad (\star)$$

Multiply the original equation by μ :

$$\begin{aligned} \mu \left(\frac{dy}{dt} + p(t)y = q(t) \right) \\ \mu \frac{dy}{dt} + \mu p(t)y = \mu q(t) \end{aligned} \quad (\Delta)$$

Now we can compare \star and Δ

$$\begin{aligned} \frac{d\mu}{dt} &= \mu p(t) \\ \frac{1}{\mu} d\mu &= p(t) dt \\ \mu &= \boxed{Ce^{\int p(t) dt}} \end{aligned}$$

Note on the Constant C

We can choose $C = 1$ because any constant cancels out.

why:

Let $\mu = Ce^{\int p(t)dt}$. Substitute into general solution:

$$y(t) = \frac{1}{Ce^{\int pdt}} \left[\int Ce^{\int pdt} q(t) dt + C_{int} \right]$$

Factor out C from the integral:

$$y(t) = \frac{1}{Ce^{\int pdt}} \cdot C \left[\int e^{\int pdt} q(t) dt + \frac{C_{int}}{C} \right]$$

Cancel the C terms:

$$y(t) = \frac{1}{e^{\int pdt}} \left[\int e^{\int pdt} q(t) dt + C_{new} \right]$$

The constant C in μ is redundant.

Big Must: Standard Form

The derivation assumes the coefficient of $\frac{dy}{dt}$ is **1**.

If given a general form:

$$a(t) \frac{dy}{dt} + b(t)y = g(t)$$

You **must** divide by $a(t)$ first:

$$\frac{dy}{dt} + \frac{b(t)}{a(t)}y = \frac{g(t)}{a(t)}$$

Identify your terms:

$$p(t) = \frac{b(t)}{a(t)}, \quad q(t) = \frac{g(t)}{a(t)}$$

Algorithm Summary**1. Standardize:**

Ensure $\frac{dy}{dt}$ has a coefficient of 1.

2. Identify:

Find $p(t)$ and $q(t)$.

3. Calculate μ :

$$\mu(t) = e^{\int p(t)dt} \quad (\text{ignore } + C).$$

4. Solve:

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)q(t)dt + C \right].$$

1.3 2nd Order Linear Differential Equations

A general form of a 2nd order ODE:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

or

$$F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

A quick note of Implicit, Explicit and Homogeneity

When we say $F\left(x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right)$, the **Container** F holds all combinations of said $x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ that add up to 0. Please note, this doesn't mean its **Homogeneous** in how it's presented here. More on this in a little.

For the other form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Here we are saying, "if you give me current time, position and velocity, I'll be able to calculate acceleration"

Back to **Homogeneous or not**.

In the form $F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$ we have only moved all things in the container to one side.
Using my ridiculous example from before,

Imagine an Implicit Function F representing the ingredients of a lunch equation:

$$F\left(x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

This is **Implicit** because the ingredients are all mixed together in the bucket F .

To check if it is **Homogeneous**, we try to isolate the sandwich mechanics (y, y', y'') from the external ingredients:

$$\underbrace{\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y}_{\text{The Bread}} = \underbrace{\text{PB} + \text{Jelly}}_{\text{External Forcing Function } f(t)}$$

- If $\text{PB} = 0$ and $\text{Jelly} = 0$, the equation is **Homogeneous** (Just Bread).
- If $\text{PB} \neq 0$ or $\text{Jelly} \neq 0$, the equation is **Non-Homogeneous** (A Sandwich exists).

The Linear Case

$$\frac{d^2y}{dt^2} = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

if $g(t) = 0$;

$$\begin{aligned}\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y &= g(t) \\ \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y &= 0\end{aligned}$$

This is now homogeneous and we can move forward with a characteristic equation.

The Characteristic Equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

change the $\frac{d}{dt}$ terms to r's

$$r^2y + p(t)ry + q(t)r^0y = 0$$

we can now change the coefficients to manageable variables

$$\begin{aligned}\left[ar^2 + br + c \right] y &= 0 \\ \underbrace{ar^2 + br + c = 0}_{\text{Characteristic equation}}\end{aligned}$$

→ find the roots r_1 and r_2 of the characteristic

$$(r - r_1)(r - r_2)y = 0$$

This equation is satisfied when:

$$(r - r_1)y = 0 \quad \text{or} \quad (r - r_2)y = 0$$

So the

$$\begin{array}{ll} (r - r_1)y = 0 & (r - r_2)y = 0 \\ \left(\frac{d}{dt} - r_1\right)y = 0 & \left(\frac{d}{dt} - r_2\right)y = 0 \\ \frac{dy}{dt} = r_1 y_1 & \frac{dy}{dt} = r_2 y_2 \\ y_1 = C_1 e^{r_1 t} & y_2 = C_2 e^{r_2 t} \end{array}$$

general solution for this would be:

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

C_1 and C_2 will both be found by initial conditions.

Example:

$$\begin{aligned} y'' + 4y' + 4y &= 0 && (\text{original}) \\ r^2 + 4r + 4 &= 0 \\ (r + 2)(r + 2) &= 0 \\ (r + 2)^2 &= 0 \end{aligned}$$

We know that one linear independence is of the form:

$$y_1 = C_1 e^{-2t}$$

for the other solution we can assume:

$$y_2 = \nu(t) e^{-2t}$$

lets find $\nu(t)$ by plugging this back into the original ODE

But first we need to find y'_2 and y''_2

$$\begin{aligned} y_2 &\Rightarrow \nu(t)e^{-2t} \\ y'_2 &= \frac{d}{dt}\nu(t)e^{-2t} + \nu(t)\frac{d}{dt}e^{-2t} \\ &= \nu'(t)e^{-2t} - 2\nu(t)e^{-2t} \\ y''_2 &= \frac{d}{dt}\nu'(t)e^{-2t} - 2\nu(t)\frac{d}{dt}e^{-2t} \\ &= \nu''(t)e^{-2t} - 4\nu'(t)e^{-2t} + 4\nu(t)e^{-2t} \end{aligned}$$

now into the original:

$$y''_2 + 4y'_2 + 4y_2 = 0$$

$$\begin{aligned} \nu''(t)e^{-2t} - 4\nu'(t)e^{-2t} + 4\nu(t)e^{-2t} + 4(\nu'(t)e^{-2t} - 2\nu(t)e^{-2t}) + 4(\nu(t)e^{-2t}) &= 0 \\ \nu''(t)e^{-2t} - 4\nu'(t)e^{-2t} + 4\nu(t)e^{-2t} + 4\nu'(t)e^{-2t} - 8\nu(t)e^{-2t} + 4\nu(t)e^{-2t} &= 0 \\ \nu''(t) &= 0 \end{aligned}$$

now integrate to get $\nu(t)$

$$\nu(t) = C_1t + C_2$$

to form the basis needed, we just take the simplest form of the equation for $\nu(t)$, where $C_1 = 1$ and $C_2 = 0$

$$\nu(t) = t$$

Now we have both parts of the general solution

$$y = C_1e^{-2t} + C_2te^{-2t}$$

Note on Complex Roots

If the characteristic equation $ar^2 + br + c = 0$ yields complex roots:

$$r_{1,2} = \lambda \pm i\mu$$

The general solution takes the form:

$$y(t) = e^{\lambda t} \left(C_1 \cos(\mu t) + C_2 \sin(\mu t) \right)$$

Why? This comes from Euler's Identity:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Applying this to our root $r = \lambda + i\mu$:

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t} \cdot e^{i(\mu t)} \\ &= e^{\lambda t} (\cos \mu t + i \sin \mu t) \end{aligned}$$

(The imaginary terms form the oscillation μ , and the real part λ forms the growth/decay envelope).

1.4 Non-Homogeneous 2nd order ODE

Method of Undetermined Coefficients

Example:

$$y'' - 3y' - 4y = 3e^{2t}$$

Step 1: Find the homogeneous (y_h) solution via the characteristic eqn:

$$\begin{aligned} r^2 - 3r - 4 &= 0 \\ (r - 4)(r + 1) &= 0 \\ r &= 4, -1 \\ y_h(t) &= C_1 e^{4t} + C_2 e^{-t} \end{aligned}$$

Now we make an **Ansatz** (guess) for the particular solution $y_p(t)$ based on the forcing function $3e^{2t}$.

It needs to satisfy the ODE:

$$y_p''(t) - 3y_p'(t) - 4y_p(t) = 3e^{2t}$$

Since the forcing function is an exponential, we guess a matching form:

$$\begin{aligned} y_p(t) &= Ae^{2t} \\ y_p'(t) &= 2Ae^{2t} \\ y_p''(t) &= 4Ae^{2t} \end{aligned}$$

Substitute these back into the original ODE:

$$\begin{aligned} 4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) &= 3e^{2t} \\ (4 - 6 - 4)Ae^{2t} &= 3e^{2t} \\ -6A &= 3 \\ A &= -\frac{1}{2} \end{aligned}$$

So the general solution is:

$$y(t) = y_h(t) + y_p(t)$$

$y(t) = C_1 e^{4t} + C_2 e^{-t} - \frac{1}{2} e^{2t}$

What happens if we change things a little??

lets say we instead had,

$$y'' - 3y' - 4y = 3e^{-t}$$

The y_c part would still be $C_1 e^{4t} + C_2 e^{-t}$

Notice that the y_c we chose is no longer linearly independent from the y_p we are going to have. So what we need to do now is change the ansatz to Ate^{2t} .

We need y_p to be **Linearly Independent** from the homogeneous solution y_c .

- **The Conflict:** If our guess y_p appears in y_c , then y_p is **Linearly Dependent** (it is just a copy of the natural response). Plugging it into the ODE will yield $0 = g(t)$, which is impossible.
- **The Fix:** Multiplying by t breaks this dependency. The functions e^{-t} and te^{-t} are Linearly Independent, allowing the math to balance.

Summary Table: Standard Guesses

When the forcing function $g(t)$ is one of the standard forms, use the corresponding trial solution $y_p(t)$.

Forcing Function $g(t)$	Trial Solution Form $y_p(t)$
C (Constant)	A
$at^n + \dots + a_0$ (Polynomial)	$A_n t^n + \dots + A_1 t + A_0$
Ce^{rt}	Ae^{rt}
$C \cos(\omega t)$ or $C \sin(\omega t)$	$A \cos(\omega t) + B \sin(\omega t)$
$e^{rt} \cos(\omega t)$	$e^{rt} (A \cos(\omega t) + B \sin(\omega t))$

Resonance

Important: If any term in your guess $y_p(t)$ is *already* a part of the homogeneous solution $y_c(t)$, you must multiply your guess by t^s , where s is the smallest integer ($1, 2, \dots$) that makes the term unique.

- **Example:** $y'' - y' = 3$.

- Homogeneous: $y_c = c_1 + c_2 e^t$.
- Naive Guess for $g(t) = 3$: $y_p = A$.
- **Problem:** The constant A is absorbed by c_1 .
- **Fix:** Multiply by $t \implies [y_p = At]$.

Example

$$y'' - 2y' + y = e^x + x$$

Step 1: Homogeneous Solution

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r - 1)^2 &= 0 \implies r = 1, 1 \text{ (Repeated!)} \\ y_h &= C_1 e^x + \underbrace{C_2 x e^x}_{\text{Repeated root rule}} \end{aligned}$$

Step 2: Particular Solution (y_p) The forcing function is $g(x) = e^x + x$. We split this into two parts:

1. Part A (e^x):

- Initial guess: Ae^x .
- Conflict: Overlaps with $C_1 e^x$. \rightarrow Try Axe^x .
- Conflict: Overlaps with $C_2 xe^x$. \rightarrow Try $Ax^2 e^x$.
- **Final Guess A:** $y_{p1} = Ax^2 e^x$

2. Part B (x):

- This is a degree 1 polynomial.
- **Final Guess B:** $y_{p2} = Bx + C$

Combined Guess:

$$\begin{aligned} y_p &= Ax^2 e^x + Bx + C \\ y'_p &= A(2xe^x + x^2 e^x) + B \\ y''_p &= A(2e^x + 2xe^x + 2xe^x + x^2 e^x) = Ae^x(x^2 + 4x + 2) \end{aligned}$$

Substitute these into the original ODE ($y'' - 2y' + y = e^x + x$):

$$\left[Ae^x(x^2 + 4x + 2) \right] - 2 \left[Ae^x(x^2 + 2x) + B \right] + \left[Ax^2 e^x + Bx + C \right] = e^x + x$$

Step 4: Solving for Coefficients

Group the terms by type (e^x terms, x terms, and constants):

$$\text{Exp terms: } Ae^x \left[(x^2 + 4x + 2) - 2(x^2 + 2x) + x^2 \right] = Ae^x[2]$$

$$\text{Poly terms: } Bx + (C - 2B)$$

Set equal to the right hand side ($1e^x + 1x + 0$):

$$2Ae^x = 1e^x \implies 2A = 1 \implies A = \frac{1}{2}$$

$$Bx = 1x \implies B = 1$$

$$C - 2B = 0 \implies C - 2(1) = 0 \implies C = 2$$

Step 5: General Solution

Combine y_h and the solved y_p :

$$y(x) = y_h + y_p$$

$$y(x) = C_1 e^x + C_2 x e^x + \frac{1}{2} x^2 e^x + x + 2$$

1.5 Variation of Parameters (Lagrange)

Use Case: When the forcing function $r(x)$ is not amenable to Undetermined Coefficients (e.g., $\sec x, \tan x, \ln x$), or coefficients are not constant.

Standard Form Assumption:

$$y'' + f(x)y' + g(x)y = r(x)$$

Warning: The coefficient of y'' must be 1. If it is $x^2 y''$, divide by x^2 first!

Step 1: The Homogeneous Foundation

First, find the complementary solution (y_h) satisfying $y'' + fy' + gy = 0$:

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

Step 2: Variation of Constants

We replace the constants C_1, C_2 with unknown functions $u(x), v(x)$:

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

Step 3: Deriving the System

To find u and v , we need two equations.

1. The Constraint (Avoiding u'', v''): Differentiating y_p initially gives terms with u' and v' . To simplify, we choose:

$$u'y_1 + v'y_2 = 0$$

This simplifies the first derivative to: $y'_p = uy'_1 + vy'_2$.

2. The Second Derivative & Substitution: Differentiate y'_p again to get y''_p :

$$y''_p = (u'y'_1 + uy''_1) + (v'y'_2 + vy''_2)$$

Substitute y_p, y'_p, y''_p into the ODE ($y'' + Py' + Qy = r$):

$$\underbrace{[(u'y'_1 + uy''_1) + (v'y'_2 + vy''_2)]}_{y''} + P \underbrace{[uy'_1 + vy'_2]}_{y'} + Q \underbrace{[uy_1 + vy_2]}_y = r$$

Group by u and v . Since y_1, y_2 are homogeneous solutions, their terms sum to zero:

$$u \underbrace{(y''_1 + Py'_1 + Qy_1)}_0 + v \underbrace{(y''_2 + Py'_2 + Qy_2)}_0 + (u'y'_1 + v'y'_2) = r$$

This leaves the second equation:

$$u'y'_1 + v'y'_2 = r(x)$$

Step 4: Solve with Wronskian

We solve the system for u', v' using Cramer's Rule:

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1$$

The formulas for the derivatives are:

$$u'(x) = \frac{-y_2(x)r(x)}{W}, \quad v'(x) = \frac{y_1(x)r(x)}{W}$$

Step 5: Integrate

Integrate u', v' to find the functions, then plug back into the Ansatz:

$$y_p(x) = \left(\int u'(x)dx \right) y_1(x) + \left(\int v'(x)dx \right) y_2(x)$$

2 Last time

1st order ODEs	2nd Order ODEs
Seperation of variabels	Homog, Constant Coefficient → Charcteristic eqn

3 Taylor Series