1 Review of Differential Equations

1.1 Seperation of variables

General First Order ODE:

$$\frac{dy}{dt} + py = g$$

If p and g are constants we can solve this by seperation of variables:

$$\begin{aligned}
\frac{dy}{dt} + py &= g \\
\frac{dy}{dt} &= g - py \\
\frac{1}{g-py}dy &= dt
\end{aligned}$$

Integrate both sides:

$$\int \frac{1}{g-py}dy = \int 1dt$$

Substitution: Let $u = g - py \implies du = -pdy$

$\implies dy = -\frac{1}{p}du$

$$\begin{aligned}
\int \frac{1}{u}\left(-\frac{1}{p}\right) du &= t + C \\
-\frac{1}{p}\ln|u| &= t + C \\
\ln|u| &= -p(t+C) \\
\ln|u| &= -pt + C_{new} \quad (\text{where } C_{new} = -pC) \\
|u| &= e^{-pt+C_{new}} \\
u &= \pm e^{C_{new}}e^{-pt}
\end{aligned}$$

Let $C_1 = \pm e^{C_{new}}$. Substitute back $u = g - py$:

$$\begin{aligned}
g - py &= C_1 e^{-pt} \\
-py &= C_1 e^{-pt} - g \\
y &= \frac{C_1 e^{-pt}}{-p} + \frac{-g}{-p} \\
\boxed{y(t) = C_2 e^{-pt} + \frac{g}{p}} &\quad \text{where } C_2 = -\frac{C_1}{p}
\end{aligned}$$

Behavior Analysis

The solution consists of two distinct parts:

$$y(t) = \underbrace{C_2 e^{-pt}}_{\text{Transient Solution}} + \underbrace{\frac{g}{p}}_{\text{Steady State Solution}}$$

- **Transient:** If $p > 0$, this term decays to zero as $t \to \infty$.
- **Steady State:** The value $y(t)$ approaches $\frac{g}{p}$ as time goes on.
- **Stability:** If $p < 0$, the exponential term grows to infinity (Unstable).

Solving for the Initial Condition

Given an initial value $y(0) = y_0$, we can solve for C_2:

$$\begin{aligned} y(0) &= C_2 e^{-p(0)} + \frac{g}{p} \\ y_0 &= C_2(1) + \frac{g}{p} \\ C_2 &= y_0 - \frac{g}{p} \end{aligned}$$

Substituting this back into the general solution gives the specific solution:

$$\boxed{y(t) = \left(y_0 - \frac{g}{p}\right) e^{-pt} + \frac{g}{p}}$$

Remember, this is for a LINEAR 1ST ORDER ODE WITH CONSTANT COEFFICIENTS.

1.2 Method of integrating Factors

$$\frac{dy}{dt} + p(t)y = q(t)$$

Rewrite to match the form of : Non Homog due to $q(t)$

$$\begin{aligned}
\frac{d}{dt}\left(\mu(t)y\right) &= \mu(t)q(t) \\
\int \frac{d}{dt}\left(\mu(t)y\right) dt' &= \int_0^T \mu(t)q(t)dt' \\
\mu(t)y(t) - \mu(0)y(0) &= \int_0^T \mu(t')q(t')dt' \\
\mu(t)y(t) &= \int_0^T \mu(t')q(t')dt' + \mu(0)y(0) \\
y(t) &= \frac{1}{\mu(t)}\left[\int_0^T \mu(t')q(t')dt' + \mu(0)y(0)\right]
\end{aligned}$$

To find μ, we match both sides of the equation.

$$\frac{d}{dt}\Big(\mu(t)y\Big) = \frac{d\mu}{dt}y + \mu\frac{dy}{dt} \tag{$\bigstar$}$$

Multiply the original equation by μ:

$$\mu\left(\frac{dy}{dt} + p(t)y = q(t)\right)$$

$$\mu\frac{dy}{dt} + \mu p(t)y = \mu q(t) \tag{Δ}$$

Now we can compare $\bigstar$ and Δ

$$\begin{aligned}
\frac{d\mu}{dt} &= \mu p(t) \\
\frac{1}{\mu}d\mu &= p(t)dt \\
\mu &= \boxed{Ce^{\int p(t)dt}}
\end{aligned}$$

Note on the Constant C

We can choose $C = 1$ because any constant cancels out.

why:

Let $\mu = Ce^{\int p(t)dt}$. Substitute into general solution:

$$y(t) = \frac{1}{Ce^{\int pdt}} \left[\int Ce^{\int pdt} q(t)dt + C_{int} \right]$$

Factor out C from the integral:

$$y(t) = \frac{1}{Ce^{\int pdt}} \cdot C \left[\int e^{\int pdt} q(t)dt + \frac{C_{int}}{C} \right]$$

Cancel the C terms:

$$y(t) = \frac{1}{e^{\int pdt}} \left[\int e^{\int pdt} q(t)dt + C_{new} \right]$$

The constant C in μ is redundant.

Big Must: Standard Form

The derivation assumes the coefficient of $\frac{dy}{dt}$ is **1**.

If given a general form:

$$a(t)\frac{dy}{dt} + b(t)y = g(t)$$

You **must** divide by $a(t)$ first:

$$\frac{dy}{dt} + \frac{b(t)}{a(t)}y = \frac{g(t)}{a(t)}$$

Identify your terms:

$$p(t) = \frac{b(t)}{a(t)}, \quad q(t) = \frac{g(t)}{a(t)}$$

Algorithm Summary

1. **Standardize:**

   Ensure $\frac{dy}{dt}$ has a coefficient of 1.
2. **Identify:**

   Find $p(t)$ and $q(t)$.
3. **Calculate $\boldsymbol{\mu}$:**

   $\mu(t) = e^{\int p(t)dt} \quad$ (ignore $+C$).
4. **Solve:**

   $$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)q(t)dt + C \right].$$

1.3 2nd Order Linear Differential Equations

A general form of a 2nd order ODE:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

or

$$F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

A quick note of Implicit, Explicit and Homogeneity

When we say $F\left(x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right)$, the **Container** F holds all combinations of said $x, y, PB, Jelly, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ that add add to 0. Please note, this doesn't mean its Homogeneous in how its presented here. More on this in a little.

For the other form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Here we are saying, "if you give me current time, position and velocity, Ill be able to calculate acceleration"

Back to **Homgeneous or not**.
In the form $F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$ we have only moved all things in the container to one side. Using my ridiculous example from before,

Imagine an Implicit Function F representing the ingredients of a lunch equation:

$$F\left(x, y, \text{PB}, \text{Jelly}, t, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

This is **Implicit** because the ingredients are all mixed together in the bucket F.
To check if it is **Homogeneous**, we try to isolate the sandwich mechanics (y, y', y'') from the external ingredients:

$$\underbrace{\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y}_{\text{The Bread}} = \underbrace{\text{PB} + \text{Jelly}}_{\text{External Forcing Function } f(t)}$$

- If $\text{PB} = 0$ and $\text{Jelly} = 0$, the equation is **Homogeneous** (Just Bread).
- If $\text{PB} \neq 0$ or $\text{Jelly} \neq 0$, the equation is **Non-Homogeneous** (A Sandwich exists).

The Linear Case

$$\frac{d^2y}{dt^2} = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

if $g(t) = 0$;

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$
$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

This is now homogeneous and we can move forward with a characteristic equation.

The Characteristic Equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

change the $\frac{d}{dt}$ terms to r's

$$r^2y + p(t)ry + q(t)r^0y = 0$$

we can now change the coefficients to manageable variables

$$\left[ar^2 + br + c\right]y = 0$$
$$\underbrace{ar^2 + br + c = 0}_{\text{Characteristic equation}}$$

$\rightarrow$ find the roots r_1 and r_2 of the characteristic