3.1 - Plane Wall

3.1.1 - Temp Distribution

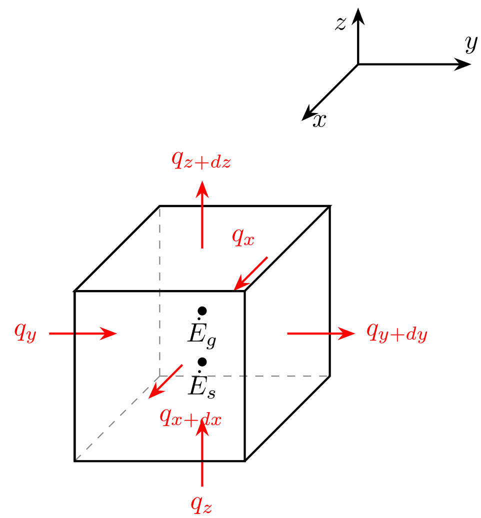


Figure 1: Differential Control Volume.

Taylor Series Expansions:

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$
$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$
$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

Energy Generation:

$$\dot{E}_g = \dot{q} dx dy dz$$

Energy Storage:

$$\dot{E}_{st} = \rho C_p \frac{\partial T}{\partial t} dx dy dz$$

Conservation of Energy:

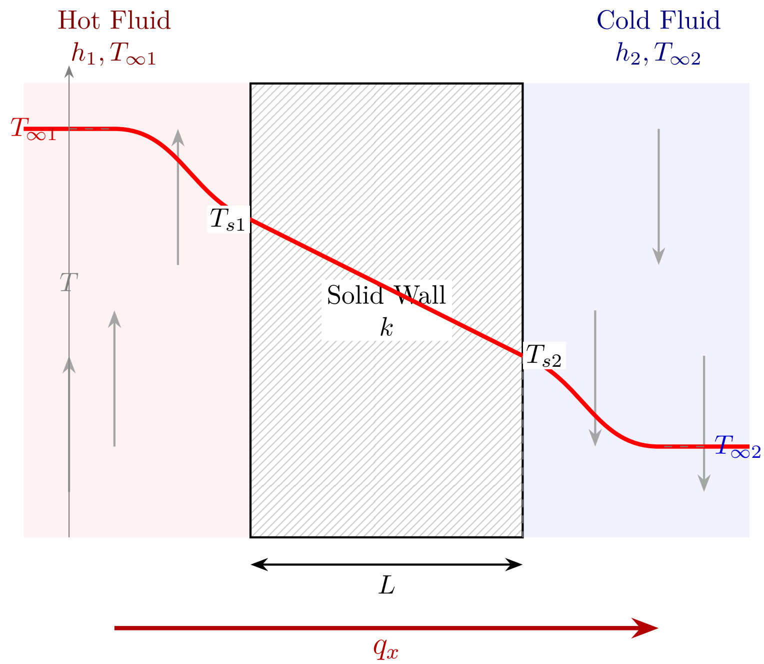
$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$
$$(q_x + q_y + q_z) + \dot{E}_g - (q_{x+dx} + q_{y+dy} + q_{z+dz}) = \dot{E}_{st}$$

Applying this and doing some algebra will get you the general heat equation.

General Heat Eq

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t}$$

1D Conduction



Finding the heat xfer rate:

1. Start with the conditions in the wall.
2. Solve the general heat equation with the appropriate boundary conditions to find temp distribution.
3. Find the conduction Q xfer rate

Lets start with the general heat equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t}$$

For our 1D wall with no heat source and steady state, this reduces to:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

We will solve this to find $T(x)$ or "Temp Distribution"
Assume k is constant through the wall.

$$\int \frac{d}{dx}\left(k\frac{dT}{dx}\right)dx = \int 0dx$$
$$k\frac{dT}{dx} = A$$
$$\frac{dT}{dx} = \frac{A}{k}$$
$$\frac{dT}{dx} = C_1 \qquad \text{where } C_1 = \frac{A}{k}$$
$$dT = C_1dx$$
$$\int dTdx = \int C_1dx$$
$$T(x) = C_1x + C_2$$

What about non-constant k terms?

$$\frac{d}{dx}(k(t)\frac{dT}{dx}) = 0$$
$$\int \frac{d}{dx}k(t)\frac{dT}{dx}dx = \int 0dx$$
$$k(T)\frac{dT}{dx} = C_0 \longleftarrow \text{Still Constant flux}$$
$$k(T)dT = C_0dx$$
$$\int k(T)dT = \int C_0dx$$
$$\int k_0(1+\beta T)dT = \int C_0dx$$
$$k_0\int(1+\beta T)dT = C_0x + C_1$$
$$k_0\left[T + \frac{\beta T^2}{2}\right] = C_0x + C_1$$
$$x(t) = \frac{T + \frac{\beta}{2}T^2}{1+\frac{\beta}{2}} \qquad \text{(plotted below)}$$

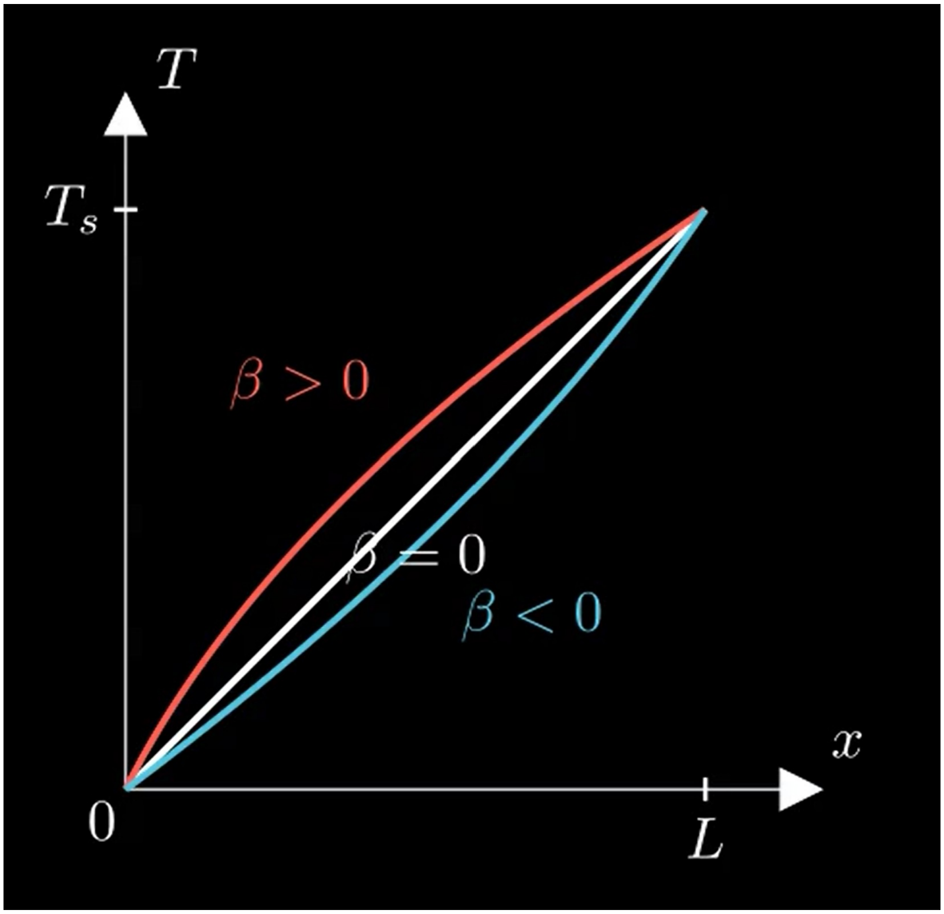


Figure 2: with BC $T(x=0)=0$ and $T(x=L)=T_s$

0.1 numerical view of the work

$$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0}$$

So with the $T(x)$ that we found, we can see that we still need the constants. Lets get those:

$$T(x) = C_1 x + C_2$$

Apply the boundary conditions

$T(x=0)=T_{s1}$ and $T(x=L)=T_{s2}$

$$\begin{aligned} T(x=0) \quad &\textbf{First} \\ T(x=0) &= C_1(x=0)+C_2 \\ T_{s1} &= C_1 0 + C_2 \\ T_{s1} &= C_2 \end{aligned}$$

$$
\begin{aligned}
T(x = L) \quad & \textbf{Second} \\
T(x = L) &= C_1(x = L) + C_2 \\
T_{s2} &= C_1 L + T_{s1} \\
C_1 &= \frac{T_{s2} - T_{s1}}{L}
\end{aligned}
$$

with C_1 and C_2 we can finsish T(x):

$$T(x) = \frac{T_{s2} - T_{s1}}{L} x + T_{s1}$$

1. 1D
2. Steady State
3. No Source/Sink
4. k is constant

Find Conduction Heat transfer

Now that we have $T(x)$ "the temp distr WRT x"
we can now move to conduction heat xfer rate.

$$T(x) = \frac{T_{s2} - T_{s1}}{L} x + T_{s1}$$

Apply Fourier's Law

$$
\begin{aligned}
q_x &= -kA_\perp \frac{dT}{dn} \\
q_x &= -kA_\perp \frac{d}{dx}\left[\frac{T_{s2} - T_{s1}}{L} x + T_{s1}\right] \\
q_x &= -kA_\perp \frac{T_{s2} - T_{s1}}{L} \frac{d}{dx} x + \underbrace{\cancel{\frac{d}{dx} T_{s1}}}_{0} \\
q_x &= \frac{-kA_\perp}{L}(T_{s2} - T_{s1})
\end{aligned}
$$

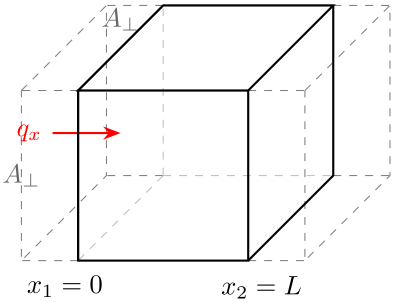


Figure 3: This shows the normal area $A_\perp$ is normal to the heat.

Find Heat Flux Rate

$$q_x'' = \frac{Q_x}{A}$$
$$= \frac{\frac{-kA_\perp}{L}(T_{s2} - T_{s1})}{A}$$
$$q_x'' = \frac{-k}{L}(T_{s2} - T_{s1})$$

3.1.2 - Thermal Resistance

First, some circuit stuff

3.1.2 - Thermal Resistance

First, some circuit stuff. There is a direct analogy between the flow of electricity and the flow of heat.

Electrical System	**Thermal System**
Current flow, I [Amps]	Heat transfer rate, q_x [Watts]
Voltage potential, ΔV [Volts]	Temperature difference, ΔT [K]
Electrical Resistance, R_e [Ω]	Thermal Resistance, R_{th} [K/W]
Ohm's Law: $I = \dfrac{\Delta V}{R_e}$	**Fourier's Law:** $q_x = \dfrac{\Delta T}{R_{th}}$

Based on this analogy, we can define the **Conduction Resistance** for a plane wall. From our previous equation for heat rate:

$$q_x = \frac{kA}{L}(T_{s1} - T_{s2})$$
$$q_x = \frac{T_{s1} - T_{s2}}{\left(\frac{L}{kA}\right)}$$

Note on Sign Convention:
You might expect to see $\Delta T = T_{final} - T_{initial}$ $(T_{s2} - T_{s1})$. However, heat transfer always flows from **High Temp** to **Low Temp**.

Fourier's Law includes a negative sign ($q = -k \dots$) to correct for this natural gradient. In the resistance analogy, we simplify this by defining the driving potential as:

$$\text{Potential} = T_{high} - T_{low} = T_{s1} - T_{s2}$$

This ensures our heat rate q_x is a positive value flowing in the direction of the arrow, exactly like Voltage Drop $(V_{high} - V_{low})$ in a circuit.

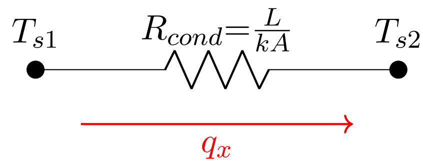
Comparing this to the "Thermal Ohm's Law" ($q = \Delta T/R$), the denominator represents the resistance:

$$\boxed{R_{t,cond} = \frac{T_{s1} - T_{s2}}{q_x} = \frac{L}{kA}}$$

(Thermal Resistance for Plane Wall Conduction)

The Thermal Circuit

We can now draw the wall as a resistor between two temperature nodes:



What about convection?
Using Newtons law of cooling

$$q = hA(T_s - T_\infty)$$
$$\frac{T_s - T_\infty}{q} = \frac{1}{hA}$$
$$\boxed{R_{conv} = \frac{T_s - T_\infty}{q} = \frac{1}{hA}}$$

(convective resistance, High to low)

The equivalent thermal circuit

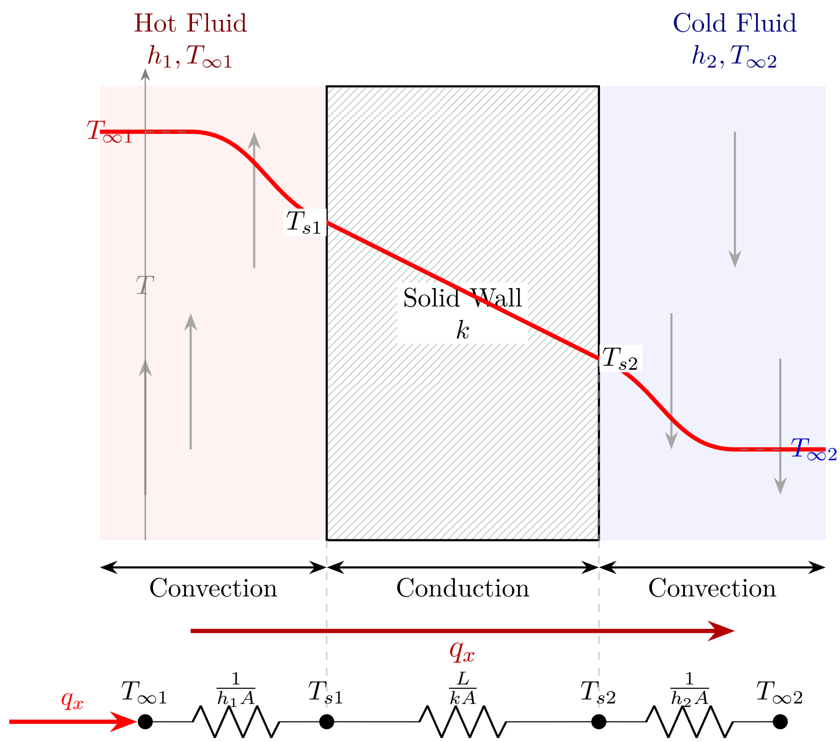


Figure 4: Temperature distribution and equivalent thermal circuit.

$$q_x = \frac{T_\infty - T_{s1}}{\frac{1}{hA}} = \frac{T_{s1} - T_{s2}}{\frac{L}{kA}} = \frac{T_{s2} - T_\infty}{\frac{1}{hA}}$$

Notice that we have a total ΔT limit from $T_{\infty 1}$ and $T_{\infty 2}$ and R_{tot} is a sum of resistors in series.

$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

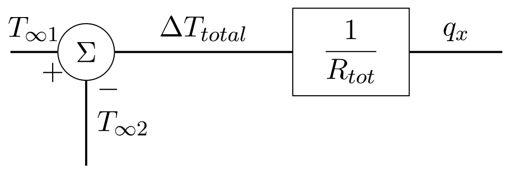
so

$$q_x = \frac{T_{\infty 1} - T_{\infty 2}}{R_{tot}}$$

Block Diagram Representation

a control (block) diagram represents the system response

- **Input:** The driving potential difference (ΔT)
- **System Gain:** The total thermal conductance ($1/R_{tot}$)
- **Output:** The resulting heat flow (q_x)



This helps visualize that heat transfer is a linear system where the flow q_x is directly proportional to the temperature difference, scaled by the system's "Conductance" ($UA = 1/R_{tot}$).