$$* \; P(A \mid B, C) = \frac{P(B \mid A, C)\, P(B \mid C)}{P(B \mid C)}$$

Smoothing

$$P(x_k \mid z_{1:t}) = P(x_k \mid z_{1:k}, z_{k+1:t})$$

$$= \frac{P(x_k \mid z_{1:k}) \times P(z_{k+1:t} \mid x_k, z_{1:t})}{P(z_{k+1:t} \mid z_{1:t})}$$

$$= \eta\, P(x_k \mid z_{1:k})\, P(z_{k+1:t} \mid x_k, z_{1:k}) \quad \text{sensor independence}$$

$$= \eta\, \underbrace{P(x_k \mid z_{1:k})}_{\text{filtering}}\, \underbrace{P(z_{k+1:t} \mid x_k)}_{\text{backward}}$$

Filtering

$$P(x_{t+1} \mid z_{1:t+1}) = P(x_{t+1} \mid z_{t+1}, z_{1:t})$$

$$= \frac{P(z_{t+1} \mid x_{t+1}, z_{1:t})\, P(x_{t+1} \mid z_{1:t})}{P(z_{t+1} \mid z_{1:t})}$$

$$= \eta\, P(z_{t+1} \mid x_{t+1}, z_{1:t})\, P(x_{t+1} \mid z_{1:t}) \quad \text{sensor independence}$$

$$= \eta\, P(z_{t+1} \mid x_{t+1})\, P(x_{t+1} \mid z_{1:t}) \quad \text{total probability}$$

$$= \eta\, P(z_{t+1} \mid x_{t+1}) \sum_{x_t} P(x_{t+1} \mid x_t, z_{1:t})\, P(x_t \mid z_{1:t}) \quad \text{sensor independence}$$

$$= \eta\, P(z_{t+1} \mid x_{t+1}) \sum_{x_t} P(x_{t+1} \mid x_t)\, P(x_t \mid z_{1:t})$$

Backward

$$P(z_{k+1:t} \mid x_t) = P(z_{k+1:t} \mid x_t) \quad \text{Total probability}$$

$$= \sum_{x_{k+1}} P(z_{k+1:t} \mid x_k, x_{k+1})\, P(x_{k+1} \mid x_k)$$

$$= \sum_{x_{k+1}} P(z_{k+1}, z_{k+2:t} \mid x_{k+1})\, P(x_{k+1} \mid x_k) \quad \text{sensor independence}$$

$$= \sum_{x_{k+1}} \underbrace{P(z_{k+1} \mid x_{k+1})}_{\text{sensor model}}\, \underbrace{P(x_{k+2:t} \mid x_{k+1})}_{\text{recursive call}}\, \underbrace{P(x_{k+1} \mid x_k)}_{\text{motion model}}$$

$$\begin{bmatrix} P(X_{t+1}=1 \mid X_t=1) & P(X_{t+1}=1 \mid X_t=0) \\ P(X_{t+1}=0 \mid X_t=1) & P(X_{t+1}=0 \mid X_t=0) \end{bmatrix} = T^T$$

if $Z=1$: $\begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix}$

else $Z=0$: $\begin{bmatrix} 0.1 \\ 0.7 \end{bmatrix}$

$\boxed{\text{Filtering}}$

if $Z=1$

$$P_{t+1} = \eta \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} \otimes T^T \cdot P_t$$

elif $Z=0$

$$P_{t+1} = \eta \begin{bmatrix} 0.1 \\ 0.7 \end{bmatrix} \otimes T^T \cdot P_t$$

$\otimes$ Hadamard Product

$\boxed{\text{Backward}}$

if $Z=1$

$$P_{t+1} = T^T \times \left(\begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} \otimes P_t \right)$$

elif $Z=0$

$$P_{(t+1)} = T^T \times \left(\begin{bmatrix} 0.1 \\ 0.7 \end{bmatrix} \otimes P_t \right)$$

$\otimes$ Hadamard Product

$\boxed{\text{Smoothing}}$

Filter Result $= \alpha_t$

Backward Result $= \beta_t$

$$P_t = \eta \, (\alpha_t \otimes \beta_t)$$

$\otimes$ Hadamard Product