Practice #5: Seismic tomography using ray theory

Ge162: Introduction to Seismology, Jeroen Tromp

Assigned: February 28, 2007 — Due: March 7, 2007

The objective of this assignment is to design a synthetic seismic tomography experiment and to familiarize yourselves with the fundamental aspects of a tomographic inversion: model parameterization, construction of the design matrix for the forward problem, and solving the least-squares problem using damping. The practical aspects of the assignment are entirely based on Matlab, and I have provided several scripts that you will need in a reference directory. From a local folder on your computer, copy the reference directory:

```
cp -r /net/denali/home2/carltape/classes/tromp_ge162/hw2_matlab/ .
```

The source-receiver geometry for this problem is shown in Figure 1.

I have listed some relevant references for additional information.

1. **Forward problem: constructing the design matrix.**

   (a) *Model parameterization.* A tomographic model is typically described in terms of a perturbation in wave speed structure from some reference model, i.e., $\delta \ln c = \delta c / c$. This function may be expanded in terms of a set of M basis functions:

   $$\delta \ln c(\mathbf{x}) = \sum_{k=1}^{M} \delta m_k \, B_k(\mathbf{x}), \tag{1}$$

   where δm_k, $k = 1, \ldots, M$, represent the perturbed model coefficients. Here we choose to use the spherical spline basis functions of *Wang and Dahlen* (1995); *Wang et al.* (1998).

   First, familiarize yourself with these basis functions by running the Matlab script `test_spline_vals.m`, which generates a randomly chosen location for a spherical spline basis function of grid order q. Change the value for q and see what happens. An example of a $q = 8$ basis function is shown in Figure 2b.

   What is the value of the $q = 8$ spline function that is centered at (longitude, latitude) = $(-117^\circ,\ 34^\circ)$, evaluated at the point $(-117.05^\circ,\ 34.25^\circ)$?

   (b) *Formulate an entry of the design matrix.* From the class notes, the ray theory travel-time perturbation for the ith source-receiver combination is given by

   $$\delta T_i = - \int_{\mathrm{ray}_i} c^{-1} \, \delta \ln c \, \mathrm{d}s, \tag{2}$$

   where $\mathrm{d}s$ denotes a segment of the ith ray. Show how this leads to $\mathbf{d} = \mathbf{G} \boldsymbol{\delta} \mathbf{m}$, where the data vector is given by

   $$\mathbf{d} = (\delta T_1, \ldots, \delta T_i, \ldots, \delta T_N)^T, \tag{3}$$

   and write the expression for the G_{ik} element of the design matrix $\mathbf{G}$.

(c) *Compute an entry of the design matrix.* Now compute the $N \times M$ design matrix $\mathbf{G}$ in Matlab. I have provided a template program `compute_Gik_ray.m` for you to start with. Take into account the following details:

- Assume a homogeneous reference phase speed model with $c_0 = 3500$ m/s.
- Because the reference model is homogeneous, the ray paths are simply great circles. Use the built-in Matlab function `gcwaypts` to compute the ray paths, and use approximately a few hundred points to effectively describe the ray path. I have provided the function `arcdist.m`, which computes the great-circle distance (in degrees) between two points on a sphere.
- Use the source–receiver ordering (see also Figure 1b) given by

  ```
  i = (isrc-1)*nrec + irec
  ```

  Here index i is the row index (or measurement index), `isrc` is the source index, and `irec` is the receiver index. For example, row $i = 3293$ of $\mathbf{G}$ (and $\mathbf{d}$) corresponds to the pairing of source 25 with receiver 125.
- Use $M = 286$ spherical spline basis functions of order $q = 8$; the center-points are in the data file `con_lonlat_q08.dat`.

What is the value of G_{ik} with $i = 126$ and $k = 204$? What does each row of $\mathbf{G}$ correspond to? What does each column of $\mathbf{G}$ correspond to?

(d) Save the design matrix (e.g., `save Gik`), so that you can readily load it (e.g., `load Gik`) for the next problem.

2. **Inverse problem: solving for the unknown velocity structure.**

At this point, you should have saved your design matrix. Now open the template file `tomography_hw.m`.

(a) Write the symbolic expression for the solution $\boldsymbol{\delta}\mathbf{m}$ to the least squares problem $\mathbf{G}\boldsymbol{\delta}\mathbf{m} = \mathbf{d}$ in the case where $\mathbf{G}^T\mathbf{G}$ is full rank. List the dimensions of $\boldsymbol{\delta}\mathbf{m}$, $\mathbf{d}$, $\mathbf{G}$, $\mathbf{G}^T\mathbf{G}$, and $\mathbf{G}^T\mathbf{d}$.

(b) In order to do the inverse problem, you will need the data vector of traveltime measurements (Eq. 3). Load the column vector `measure_vec.dat` into Matlab, and check that it has dimension 3300×1.

(c) Compute $\boldsymbol{\delta}\mathbf{m}$ using the formula above. What do you get, and why is this the case?

(d) If $\mathbf{G}^T\mathbf{G}$ is not full rank, then a damping parameter λ may be introduced to stabilize the inversion. One possible solution to the problem is:

$$\boldsymbol{\delta}\mathbf{m} = \left(\mathbf{G}^T\mathbf{G} + \lambda^2\mathbf{I}\right)\mathbf{G}^T\mathbf{d}. \tag{4}$$

Compute $\boldsymbol{\delta}\mathbf{m}$ for a range of λ values: for example, try

```
lamvec = 10.^linspace( log10(minlam), log10(maxlam), numlam);
```

with $\lambda_{\min} = 0.1$ and $\lambda_{\max} = 40.0$.

(e) Plot some of these model vectors by expanding them in the spherical spline basis functions, use the same color scale for each plot (type `colormap(seis)` after each figure is done, if you want the `seis` color scale from GMT), and also include the locations of the sources in the plots. I have pre-computed the necessary matrix, $\mathbf{B}$, in `tomography_hw.m`, such that the function value $\delta \ln c(\phi_i, \theta_i)$ is computed by the dot product $\mathbf{B}(i,:) \cdot \mathbf{m}$. Comment on any trends that you see in phase speed maps that you produce.

(f) Now, perform an inversion using only the first five events; this requires using a reduced form of your original design matrix. Use the same damping parameters as before, and show and discuss your results.

(g) What are two ways to stabilize the inverse problem, given total control over the experimental design? (Hint: How can we construct a matrix $\mathbf{G}^T\mathbf{G}$ with fewer zeros?)

References

Menke, W., *Geophysical Data Analysis: Discrete Inverse Theory*, Academic Press, San Diego, Calif., 1989.

Tape, C., Q. Liu, and J. Tromp, Finite-frequency tomography using adjoint methods—Methodology and examples using membrane surface waves, *Geophys. J. Int.*, *168*, 1105–1129, 2007.

Tarantola, A., *Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM, Philadelphia, Penn., 2005.

Wang, Z., and F. A. Dahlen, Spherical-spline parameterization of three-dimensional Earth, *Geophys. Res. Lett.*, *22*, 3099–3102, 1995.

Wang, Z., J. Tromp, and G. Ekström, Global and regional surface-wave inversion: A spherical-spline parameterization, *Geophys. Res. Lett.*, *25*, 207–210, 1998.

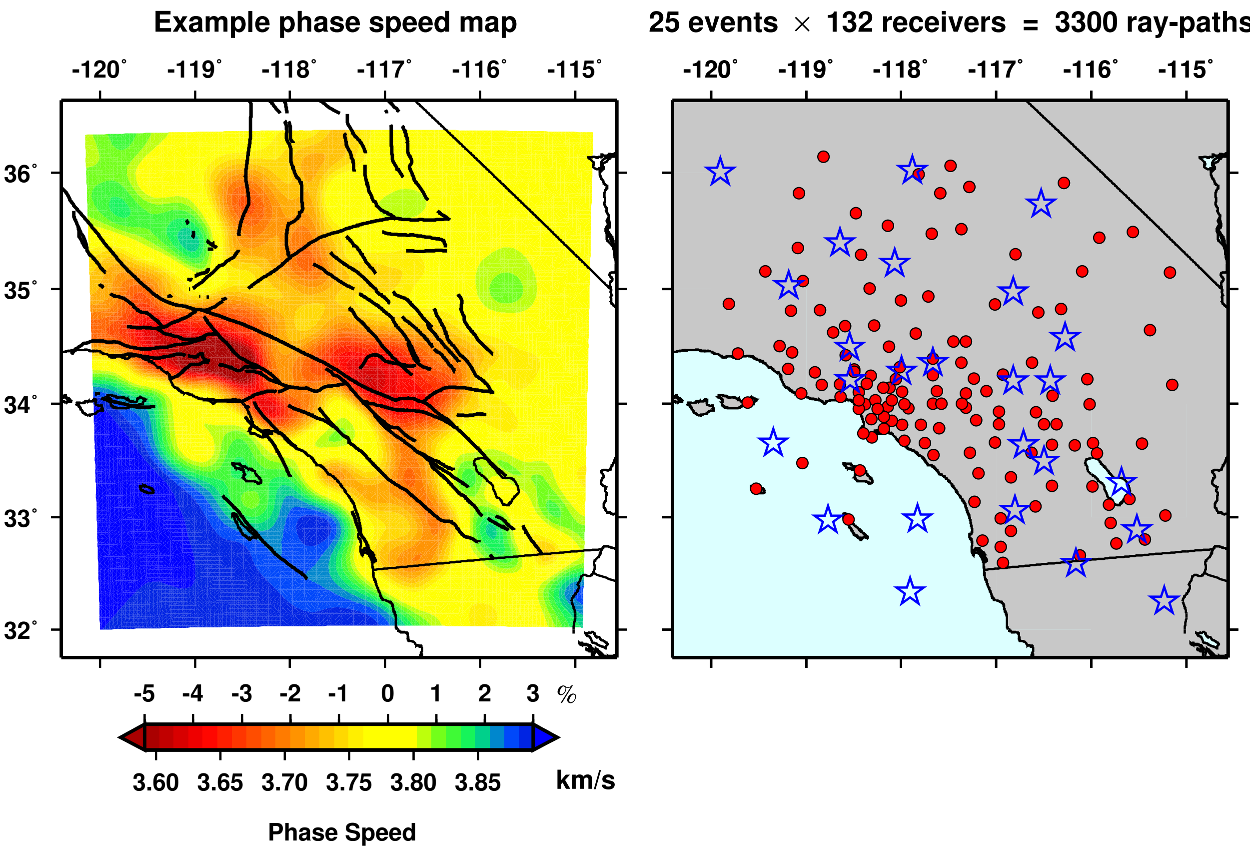

Figure 1: *Left*: Example phase velocity map and faults for Southern California. The phase velocity map is for Rayleigh waves with periods of 20 seconds. *Right*: Source–receiver geometry for the problem in this homework. The ☆ symbols denote the locations of 25 earthquakes; the ○ symbols denote the locations of 132 broadband receivers in the Southern California Seismic Network.

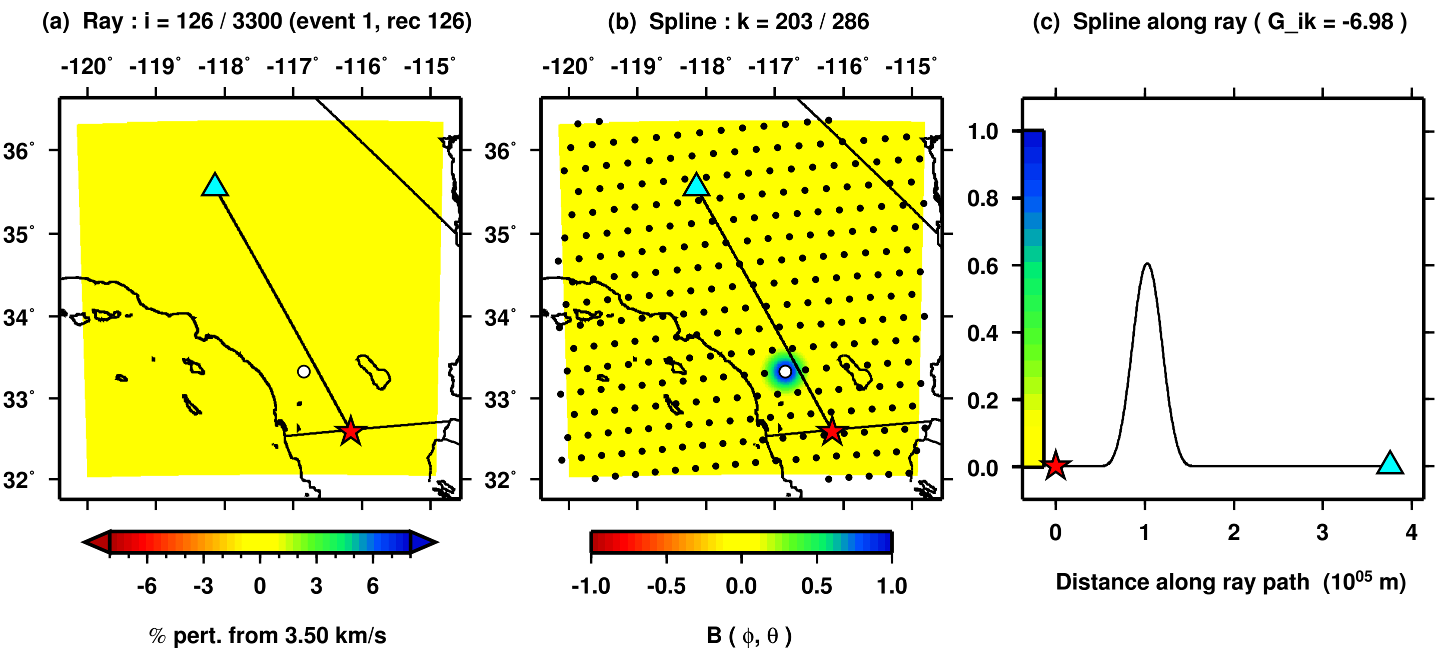

Figure 2: Example computation for an element, G_{ik}, of the design matrix $\mathbf{G}$, using rays. The row index i is the source-receiver combination, the column index k is the basis function index. The source is denoted by the ☆, the receiver is denoted by the △, and the ∘ shows the center-point of the spherical spline in (b). (a) Ray path for event number 1 and receiver number 126, corresponding to the $i = 126$ index of the $N = 3300$ ray paths. (b) $B_{203}(\mathbf{x})$, the spherical spline basis function for index $k = 203$. Also shown are the center-points of the $M = 286$ spherical splines. (c) Spline B_{203} evaluated along the ray path. The value of the phase speed for the reference model is constant, so $G_{ik} = (-1/c) \int_{\mathrm{ray}_i} B_k \, \mathrm{d}s$. In this example $G_{ik} = -1/(3500\,\mathrm{m\,s^{-1}})\;(2.45 \times 10^4\,\mathrm{m}) = -6.98$ s.