

Disjoint Sets

Shusen Wang

Disjoint Sets (Math)

- Disjoint sets are those whose pairwise intersections are empty sets.
- Example: $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$ are disjoint sets.
- Example: $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 3\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$ are not disjoint sets. (Because an intersection is not empty: $\mathcal{S}_1 \cap \mathcal{S}_2 = \{3\}$.)

Data Structure

Initialization

Element:

0

1

2

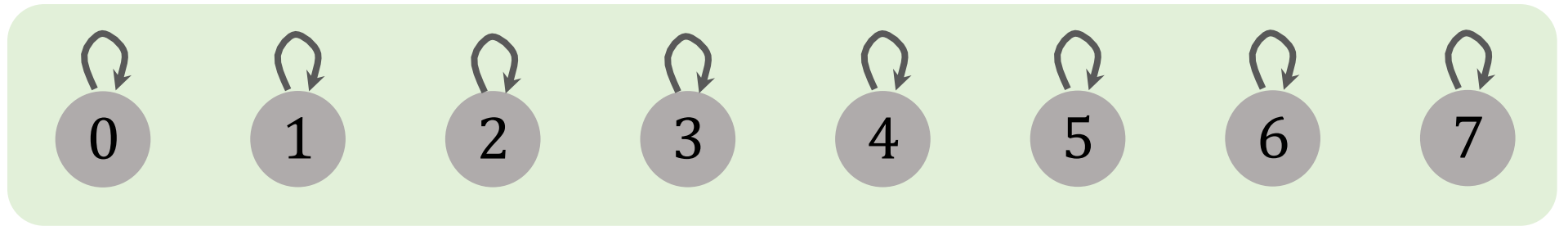
3

4

5

6

7



Initialization

Element:

0

1

2

3

4

5

6

7

Parent:

-1

-1

-1

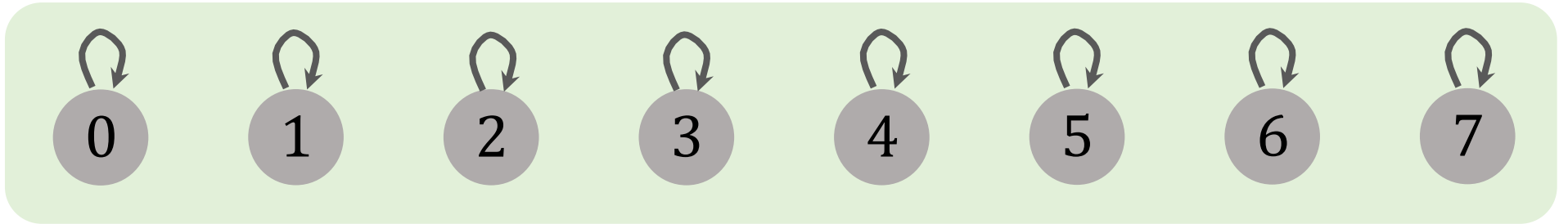
-1

-1

-1

-1

-1



- Initially, all the parents are -1 (which means no parent.)

Sets Representation

Element:

0

1

2

3

4

5

6

7

Parent:

-1

-1

-1

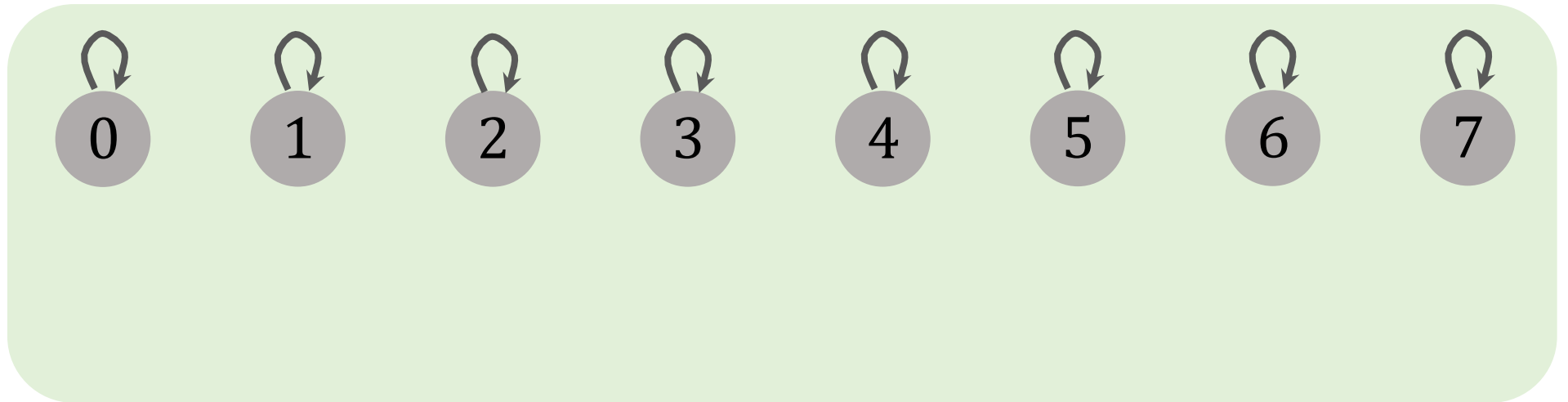
-1

-1

-1

-1

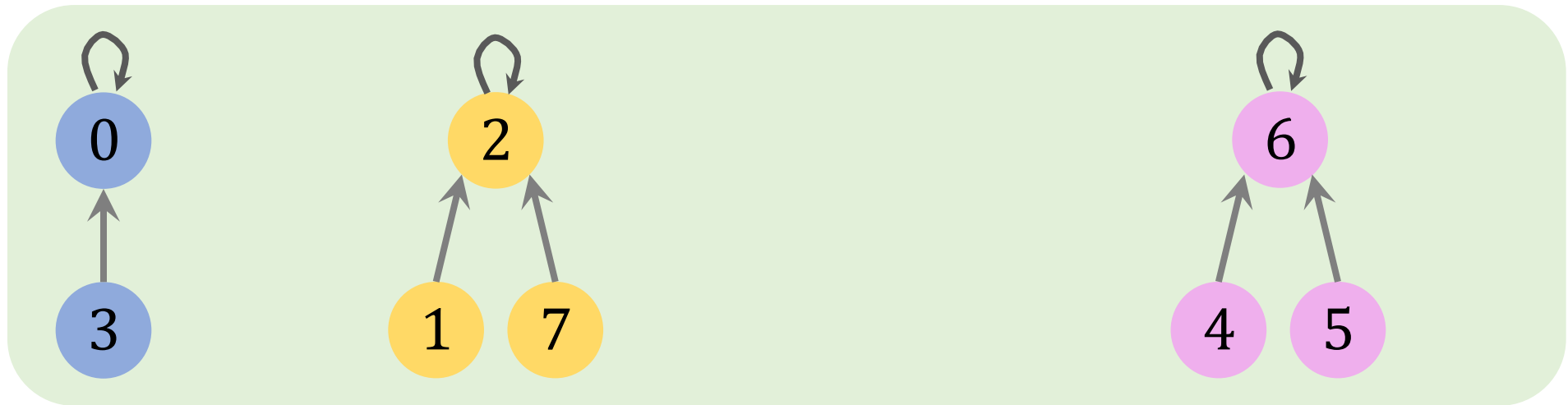
-1



Representing sets $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$.

Sets Representation

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2



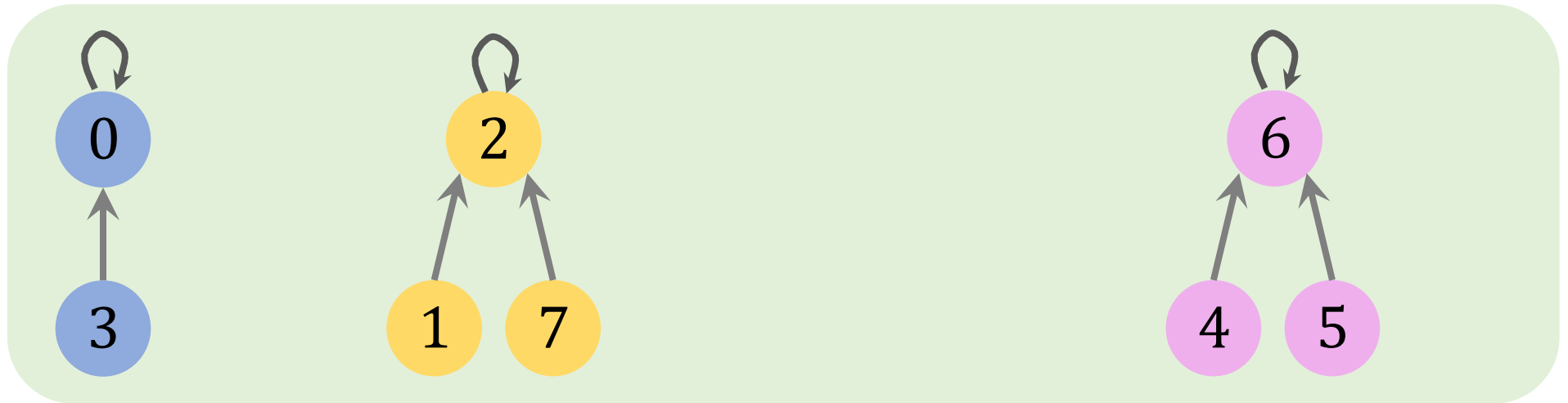
Representing sets $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$.

Sets Representation

Element: 0 1 2 3 4 5 6 7

Parent:

-1	2	-1	0	6	6	-1	2
----	---	----	---	---	---	----	---



Representing sets $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$.

Operations

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

0

6

6

-1

2

Operation 1: Are elements i and j belong to the same set?

- Find the roots of i and j .
- If they have the same root, then return TRUE.
- Otherwise, return False.

Operations

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

0

6

6

-1

2

Operation 1: Are elements i and j belong to the same set?

Operation 2: Merge (union) sets \mathcal{S}_1 and \mathcal{S}_2 .

- Find the root of \mathcal{S}_1 ; denote it by r_1 .
- Find the root of \mathcal{S}_2 ; denote it by r_2 .
- Make r_1 a child of r_2 .

Find Root

Current State

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

0

1

6

-1

2

Current State

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

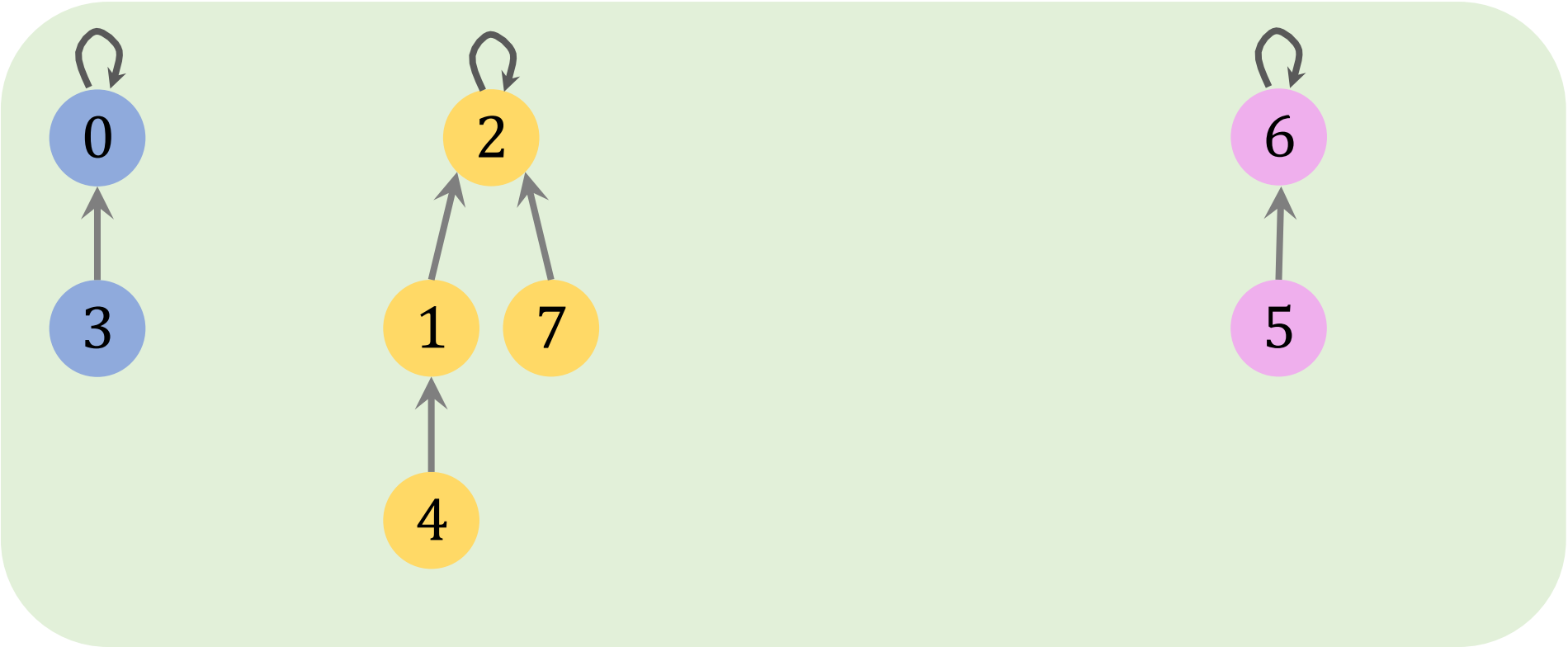
0

1

6

-1

2



What is the root of 2?

Element:

0

1

2



3

4

5

6

7

Parent:

-1

2

-1

0

1

6

-1

2



The root of **2** is **2** itself.

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

0

1

6

-1

2



What is the root of 4?

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

0

1

6

-1

2



What is the root of 4?

Element: 0 1 ← 2 3 4 5 6 7

Parent:

-1	2	-1	0	1 ←	6	-1	2
----	---	----	---	-----	---	----	---



What is the root of 4?

Element: 0 1 2 ← 3 4 5 6 7

Parent:

-1	2 ←	-1	0	1	6	-1	2
----	-----	----	---	---	---	----	---



The root of 4 is 2.

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

0

1

6

-1

2



Are two elements in the same set?

- **Inputs:** two elements (indices) i and j .
- **Question:** Are they in the same set?
- **Solution:**
 1. $r_i \leftarrow$ the root of i .
 2. $r_j \leftarrow$ the root of j .
 3. If r_i and r_j are the same, then i and j are in the same set.
 4. Otherwise, i and j are in different sets.

Union Operation

Initial State

Element:

0

1

2

3

4

5

6

7

Parent:

-1

-1

-1

-1

-1

-1

-1

-1



Union(1, 2)

Element:

0

1



2

3

4

5

6

7

Parent:

-1

-1

-1

-1

-1

-1

-1

-1



1. Find the root of 1: $r_1 = 1$.

Union(1, 2)

Element:

0

1



2

3

4

5

6

7

Parent:

-1

-1

-1

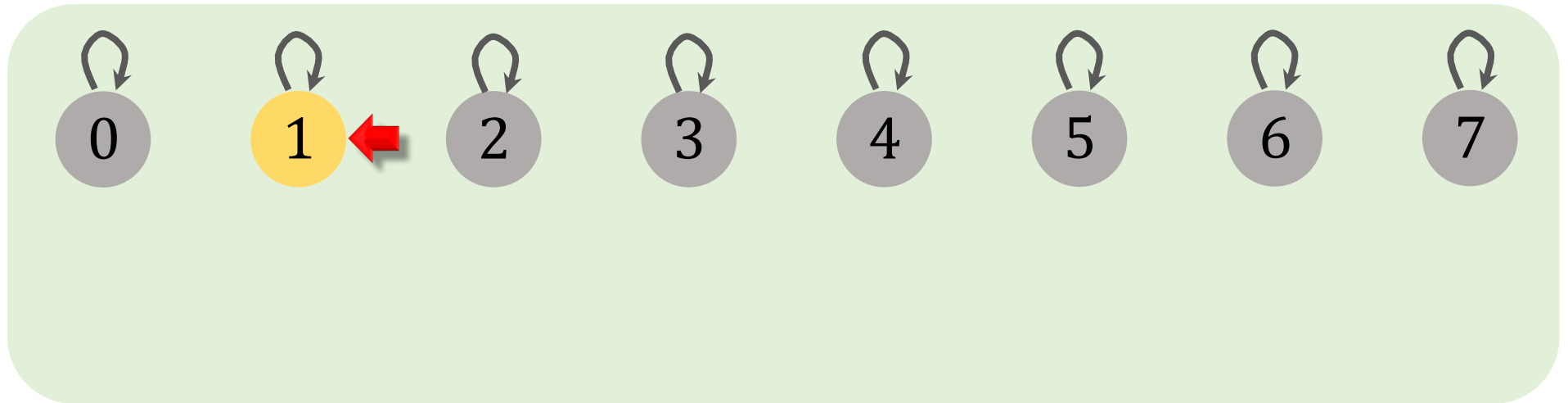
-1

-1

-1

-1

-1



1. Find the root of 1: $r_1 = 1$.

Union(1, 2)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

-1

-1

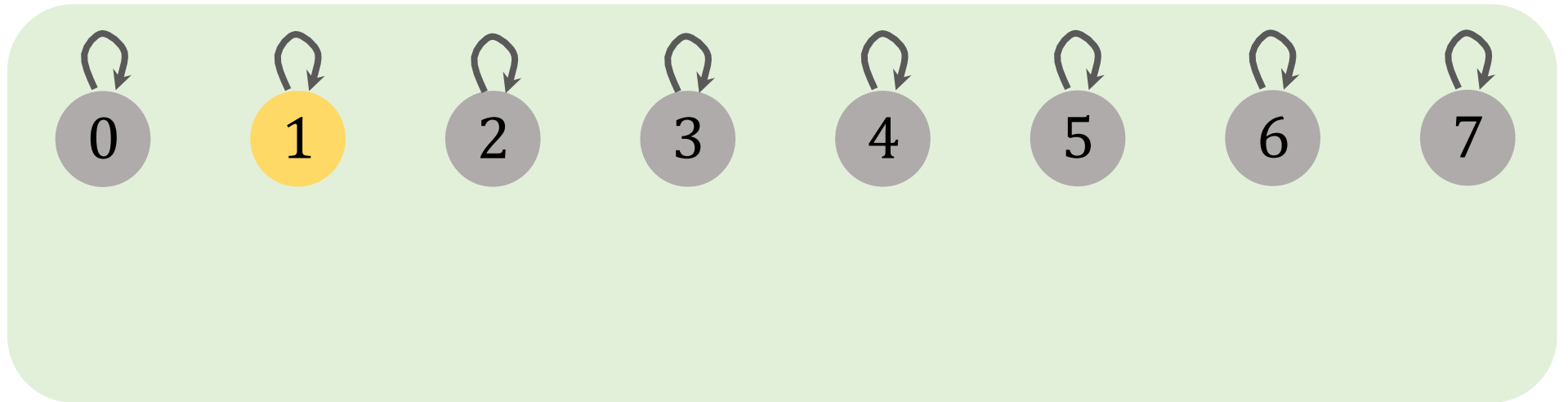
-1

-1

-1

-1

-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.

Union(1, 2)

Element:

0

1

2 ←

3

4

5

6

7

Parent:

-1

-1

-1

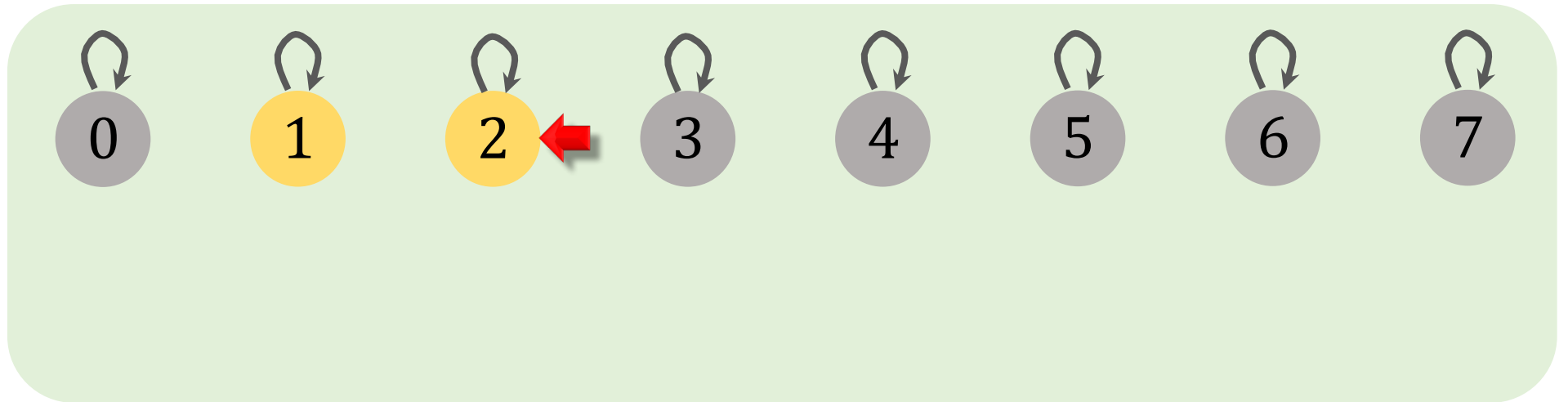
-1

-1

-1

-1

-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.

Union(1, 2)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

-1

-1

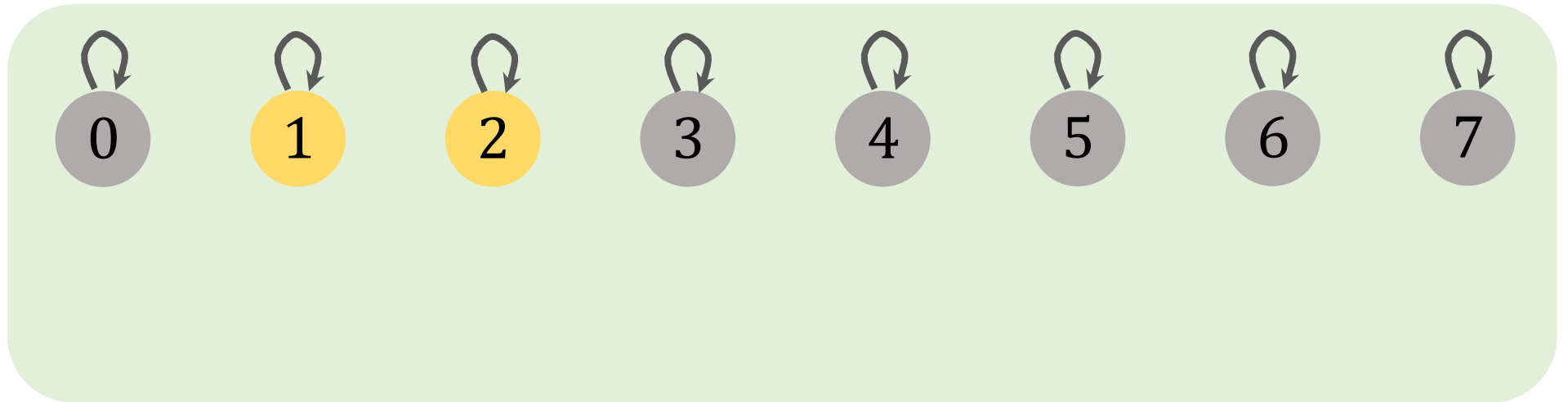
-1

-1

-1

-1

-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.
3. Make r_2 the parent of r_1 .

Union(1, 2)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

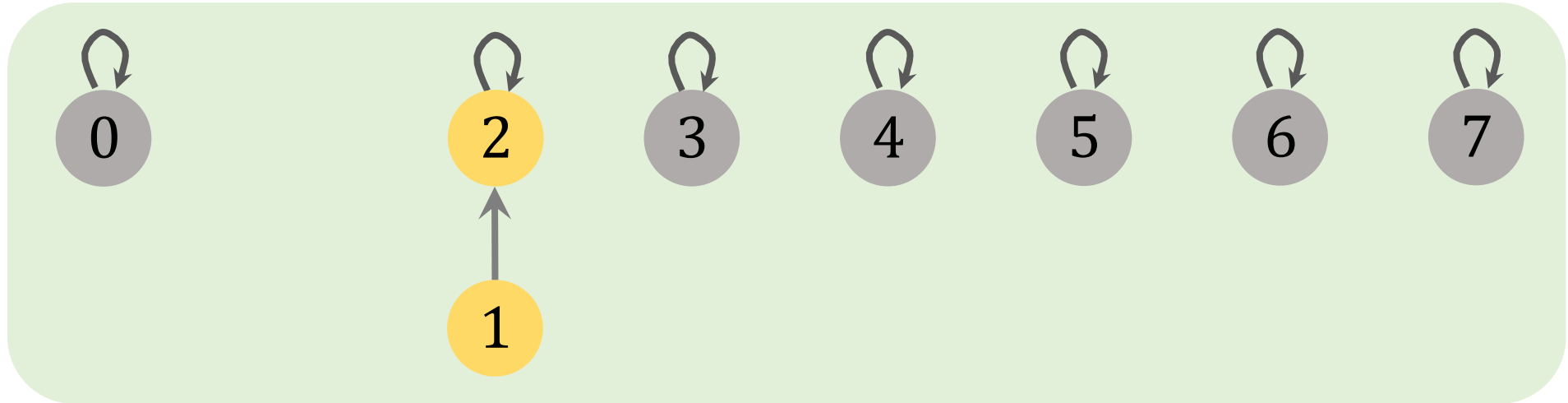
-1

-1

-1

-1

-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.
3. Make r_2 the parent of r_1 : $\text{parent}[r_1] = r_2$.

After many unions...

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

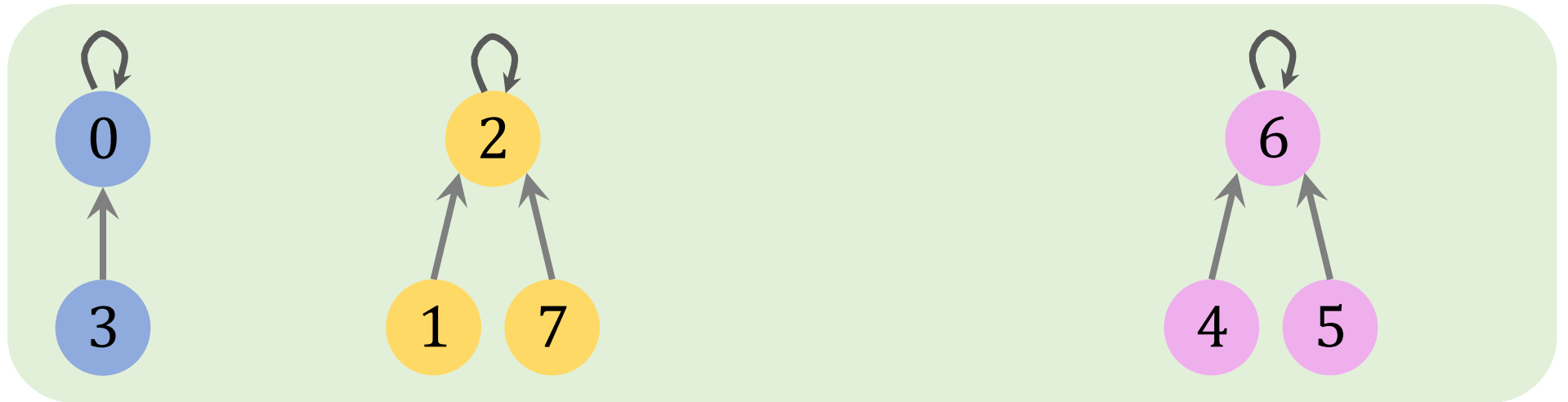
0

6

6

-1

2



Representing sets $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$.

Union(4, 5)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

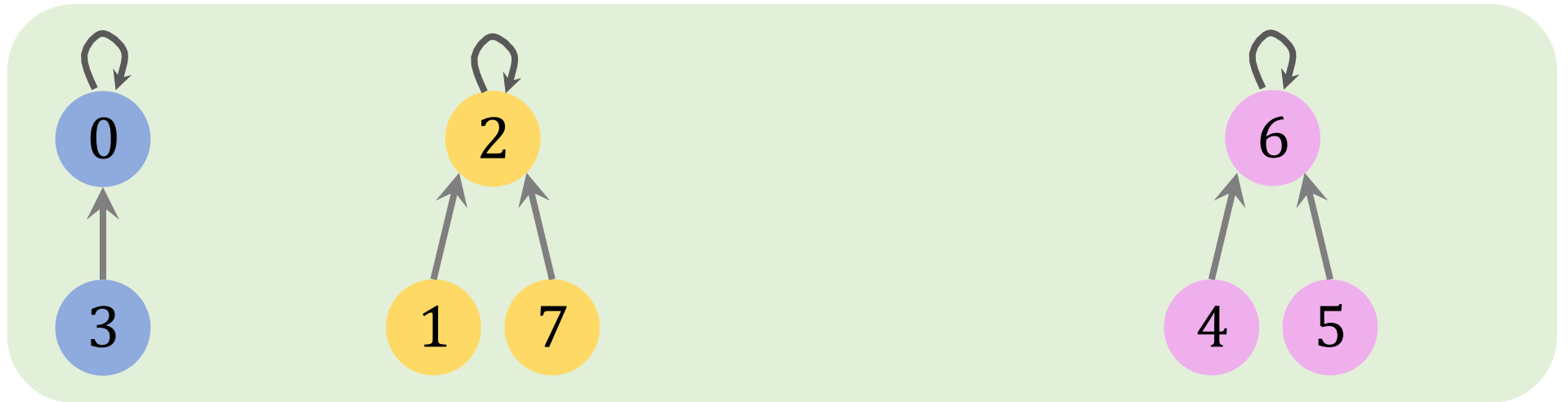
0

6

6

-1

2



Union(4, 5)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

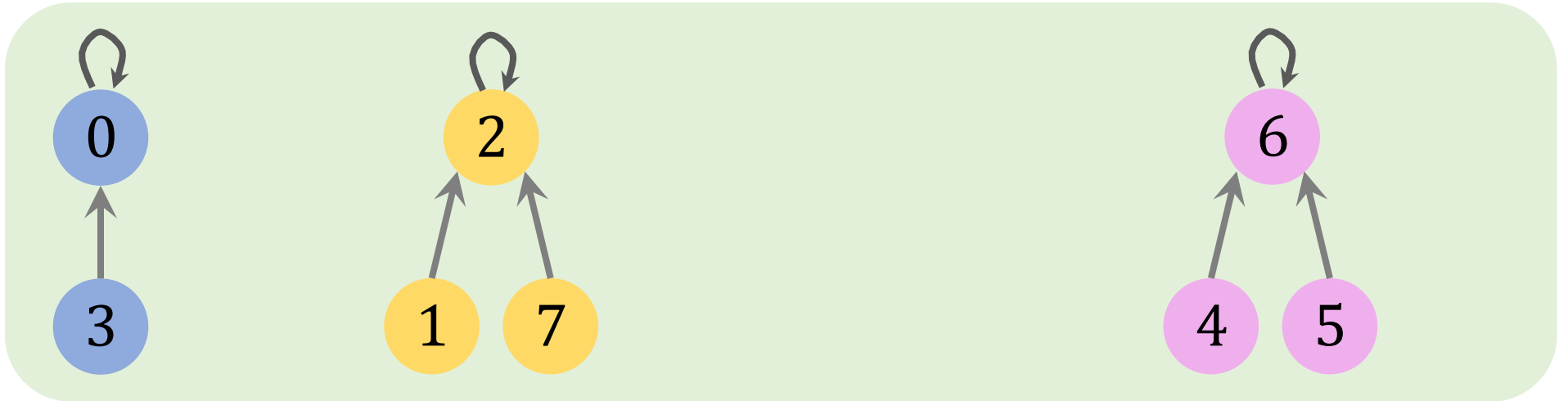
0

6

6

-1

2



1. Find the root of 4: $r_4 = 6$.

Union(4, 5)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

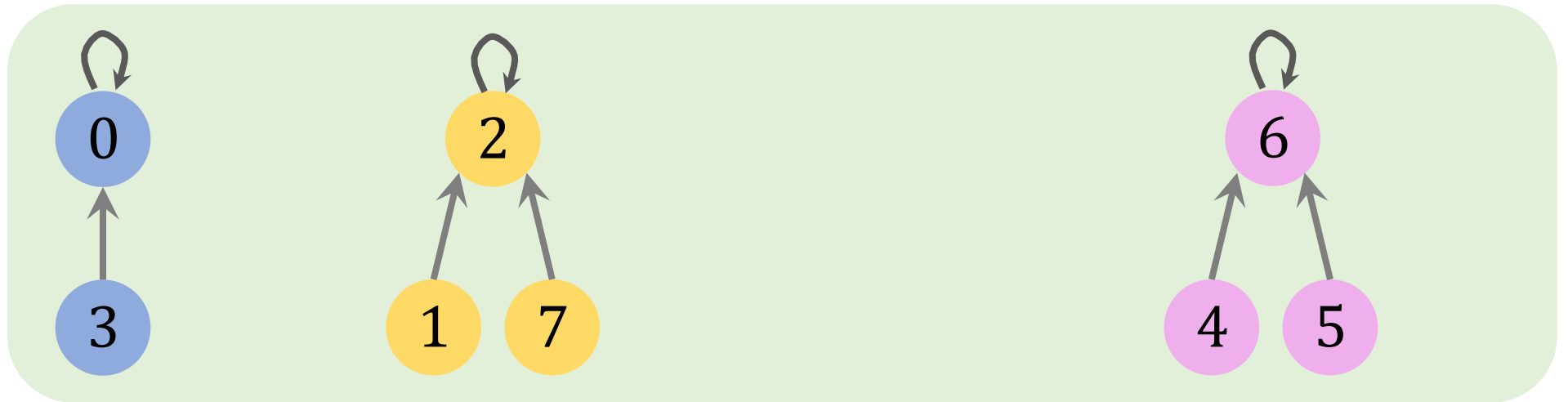
0

6

6

-1

2



1. Find the root of 4: $r_4 = 6$.
2. Find the root of 5: $r_5 = 6$.
3. They have the same root; thus no operation.

Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

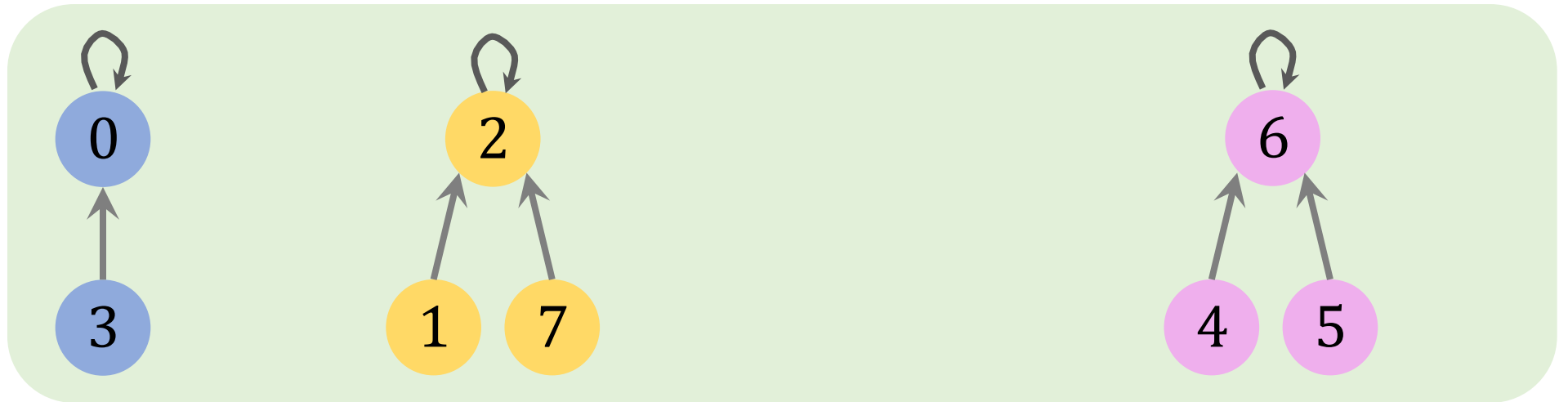
0

6

6

-1

2



Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

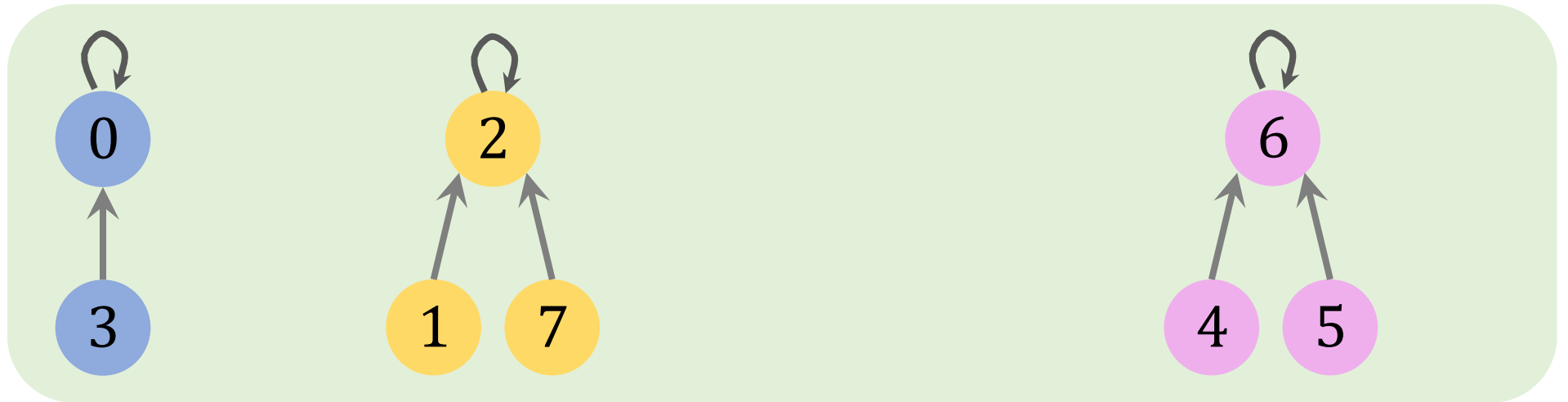
0

6

6

-1

2



1. Find the root of 4: $r_4 = 6$.

Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

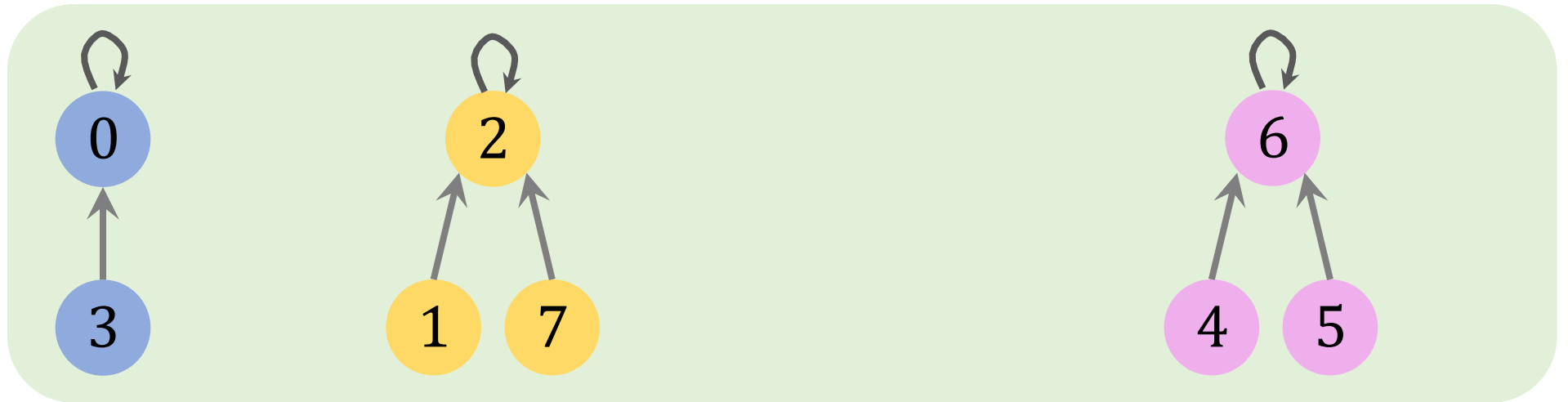
0

6

6

-1

2



1. Find the root of 4: $r_4 = 6$.
2. Find the root of 7: $r_7 = 2$.

Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

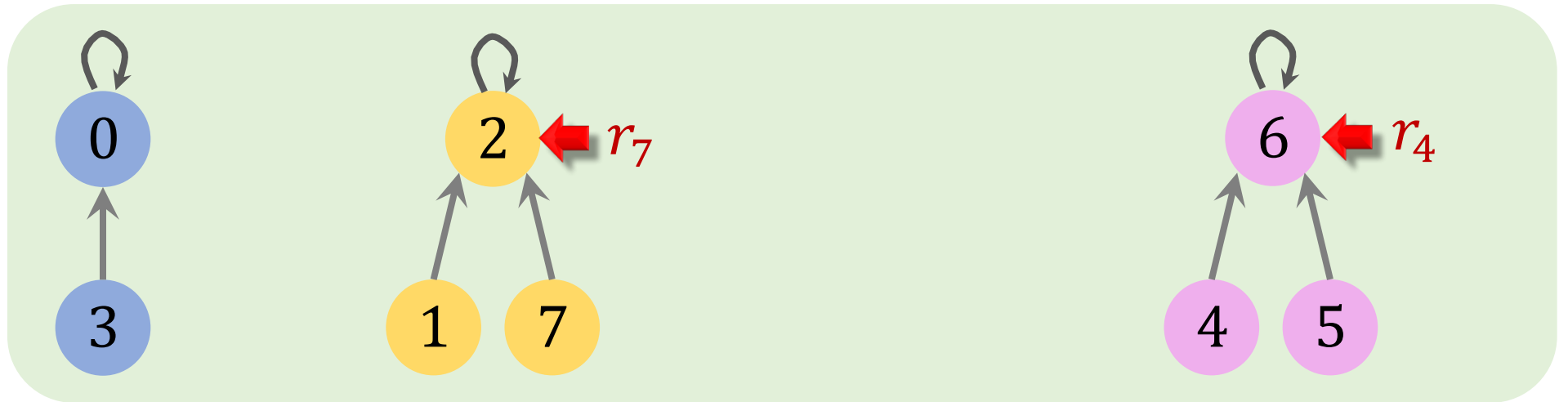
0

6

6

-1

2



1. Find the root of 4: $r_4 = 6$.
2. Find the root of 7: $r_7 = 2$.
3. Make r_7 the parent of r_4 : $\text{parent}[r_4] = r_7$.

Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

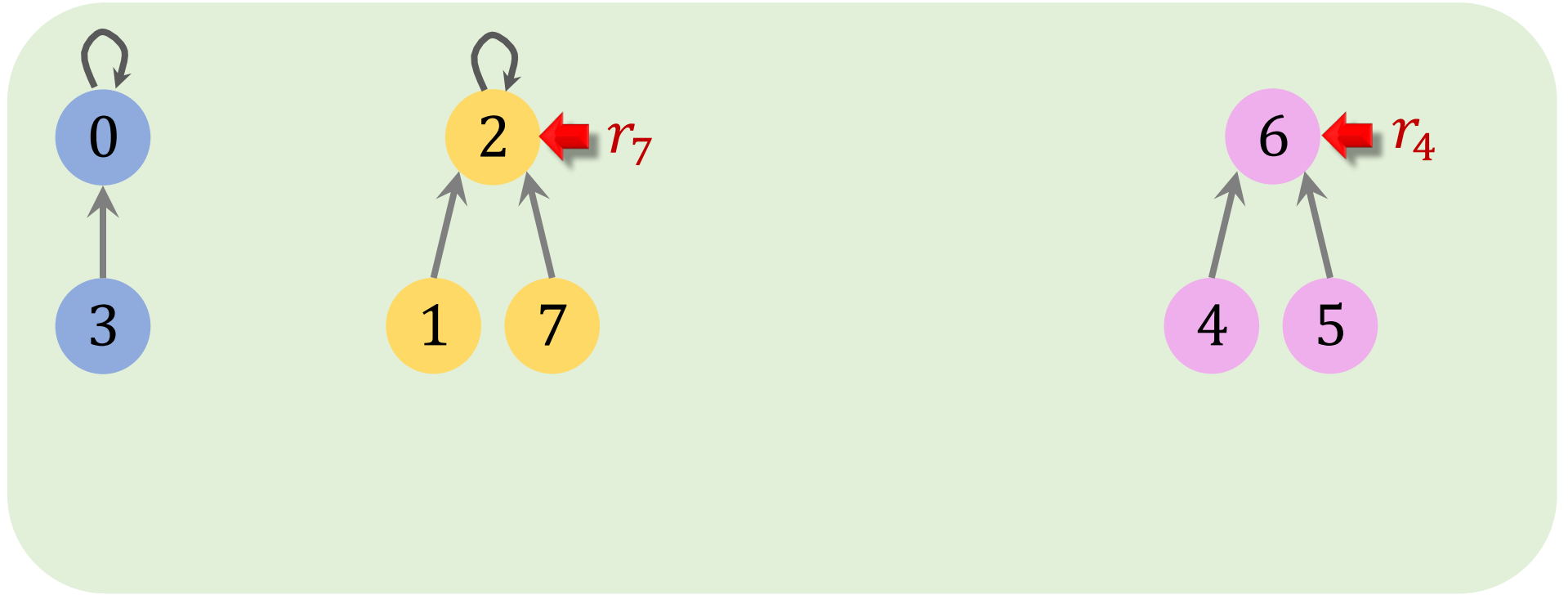
0

6

6

-1

2



Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

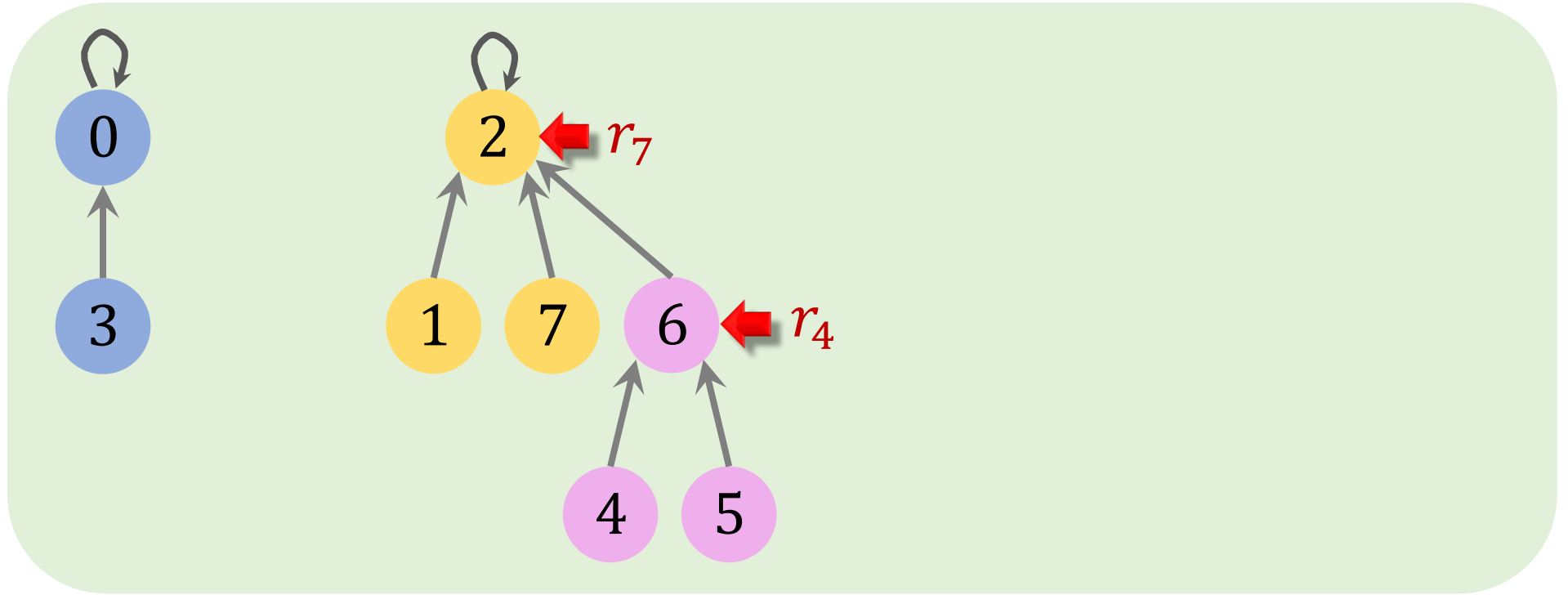
0

6

6

-1

2



Union(4, 7)

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

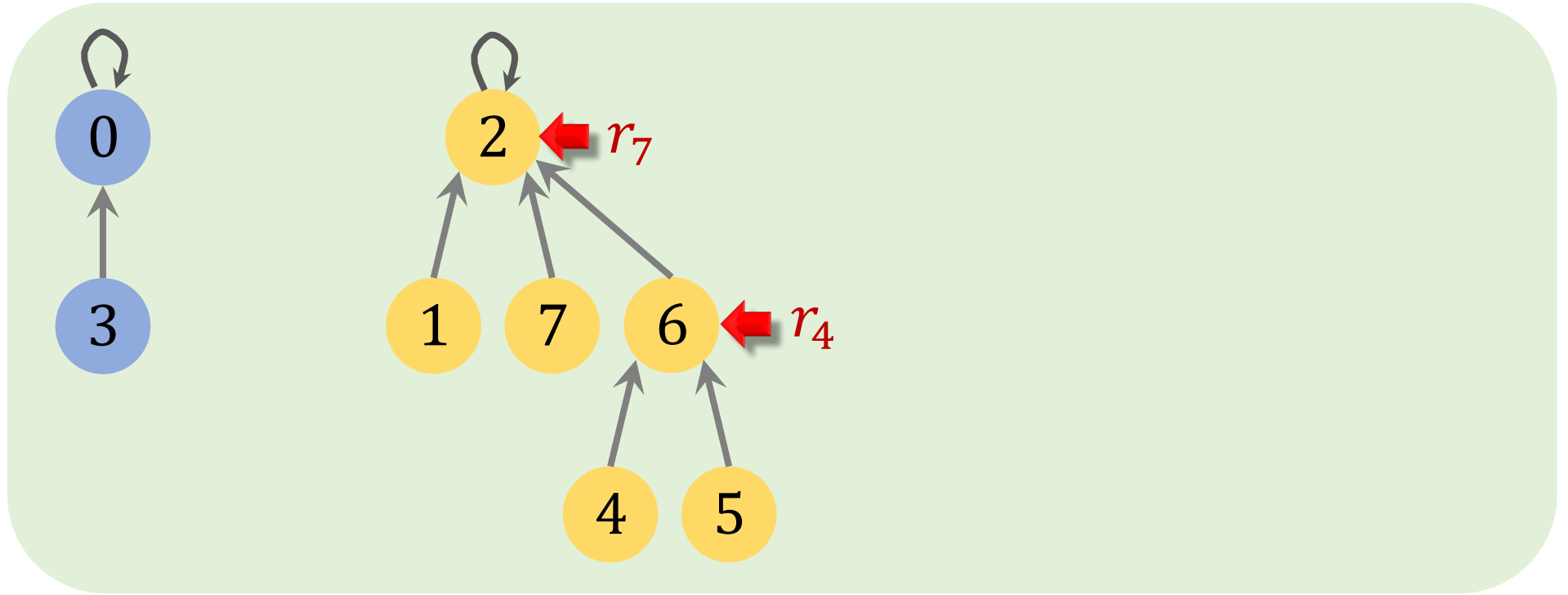
0

6

6

2

2



Time Complexities

Time complexity of finding root

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

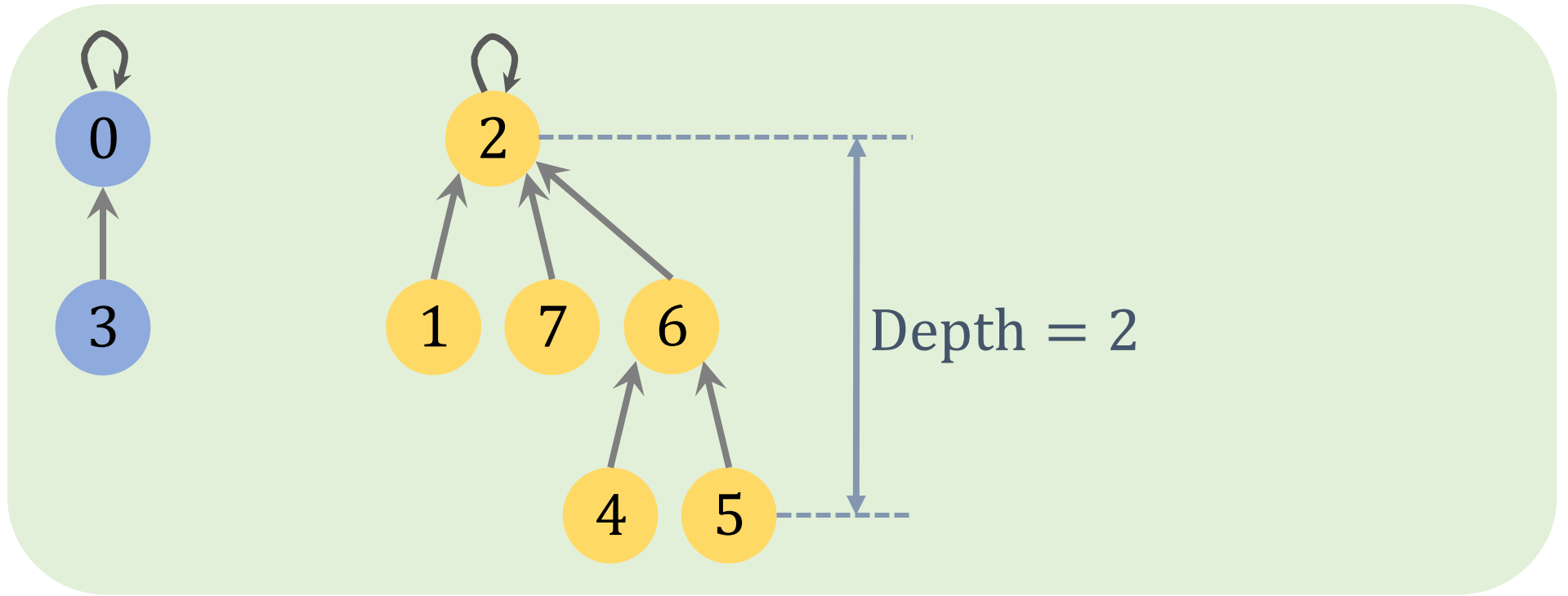
0

6

6

2

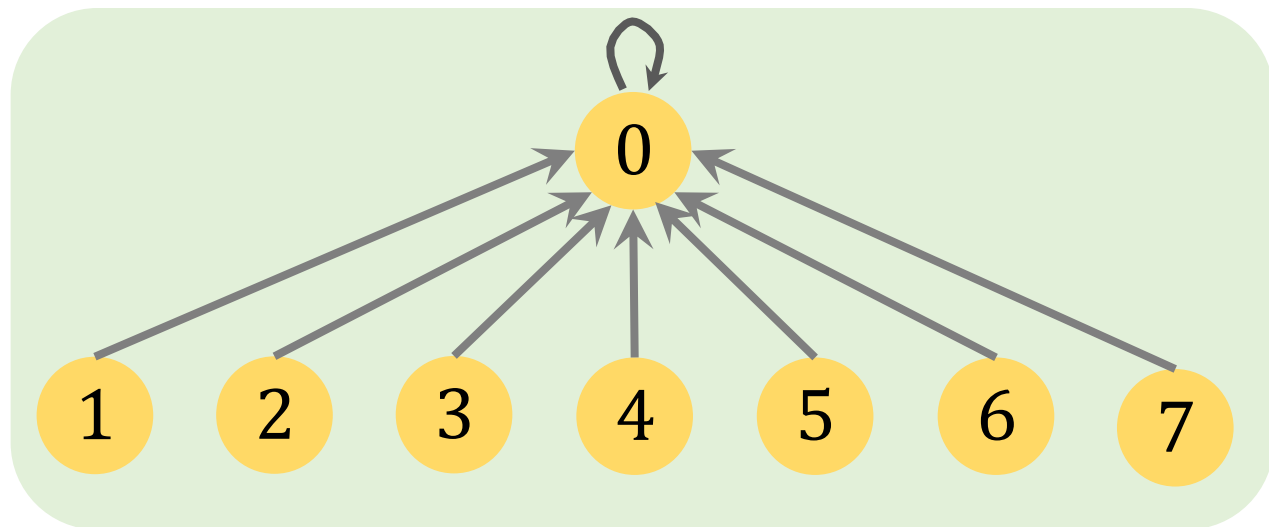
2



Time complexities

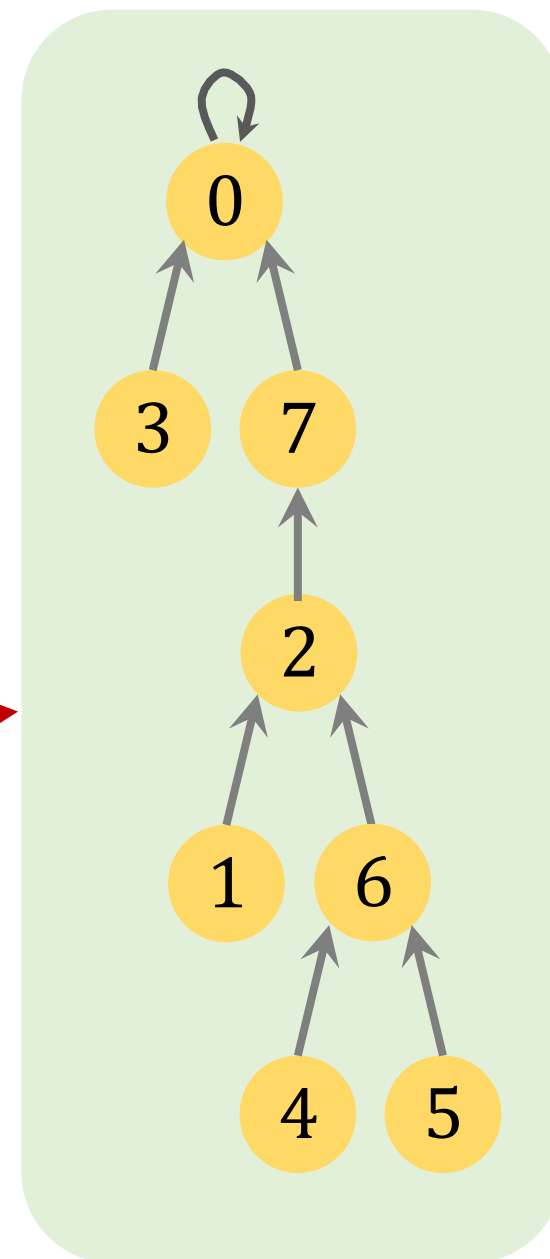
- Time complexity of **finding root** is the depth of the vertex.
- Time complexity of **union** is the same.
- In the **worst case**, it can be $O(n)$. (Here, n is the total number of elements.)

Question: How to reduce the time complexity?



↑
Good

Bad

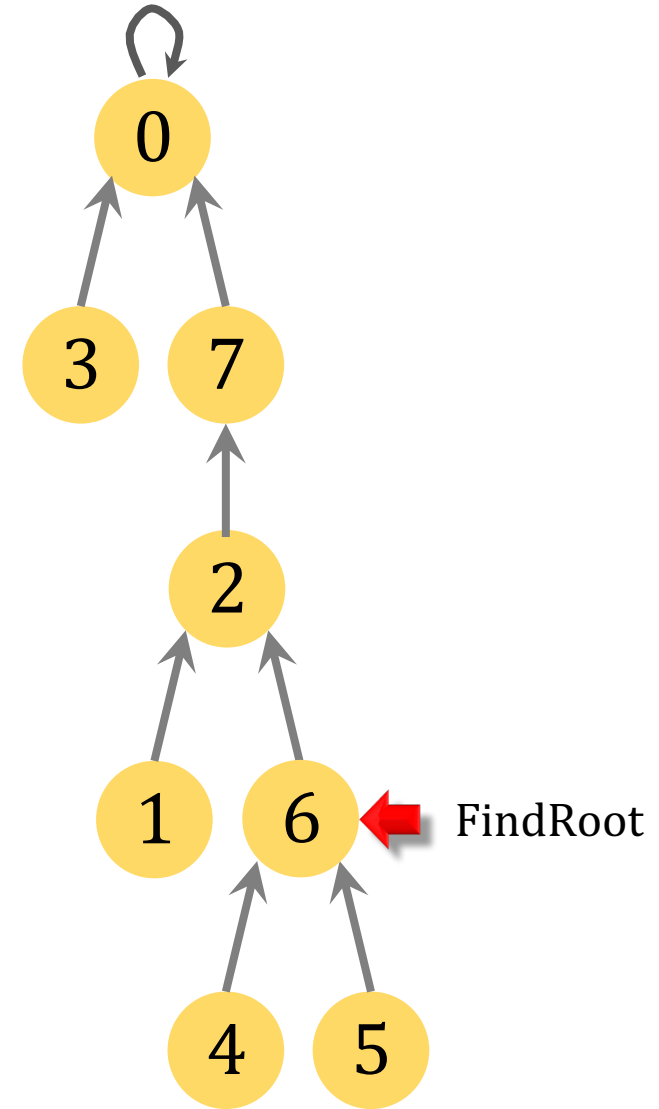


Path Compression

How to reduce the depth?

- After executing **finding root**, we know the root of an element:

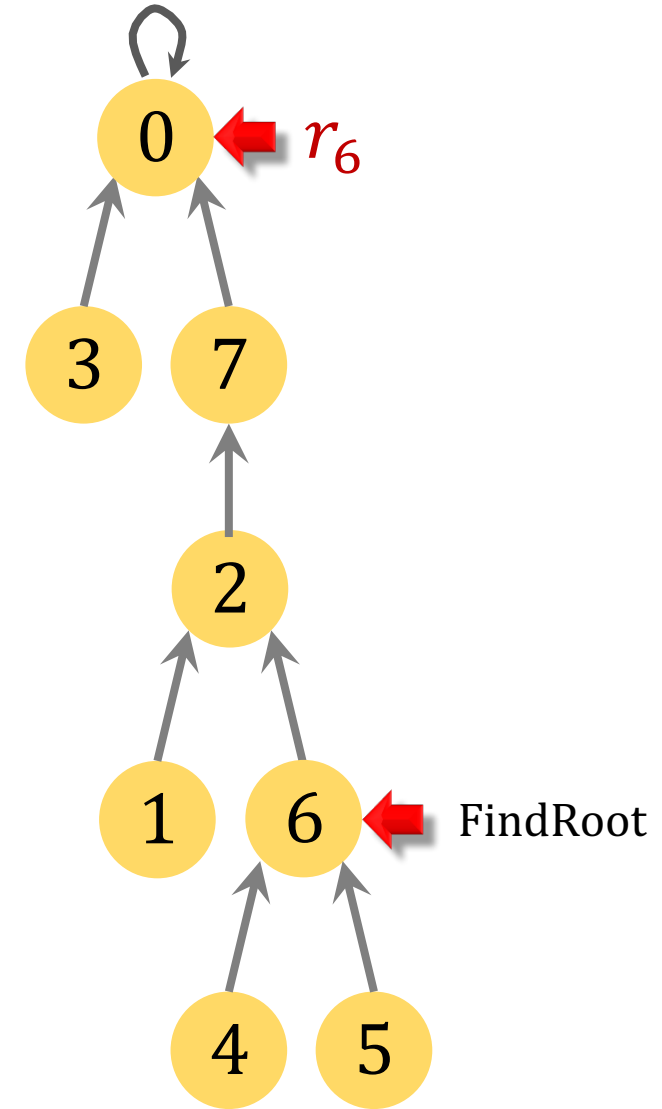
$$r_i = \text{FindRoot}(i).$$



How to reduce the depth?

- After executing **finding root**, we know the root of an element:

$$r_i = \text{FindRoot}(i).$$

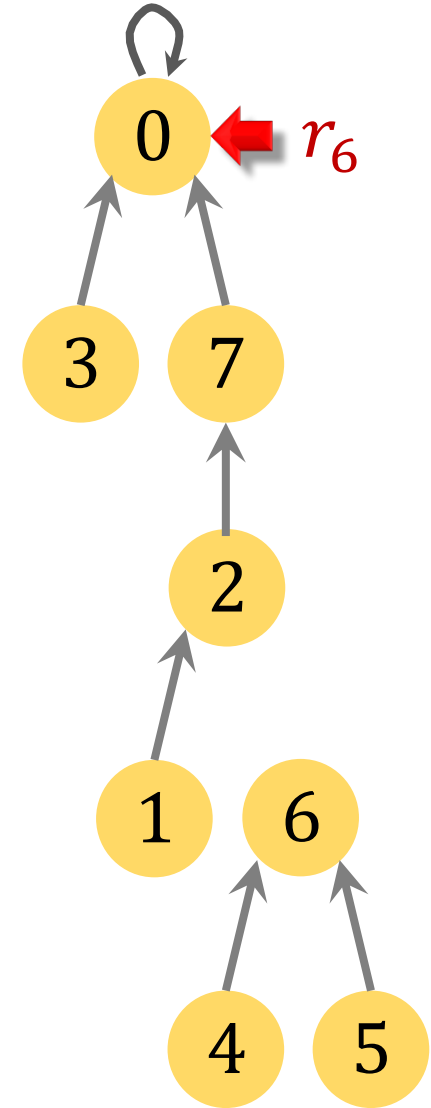


How to reduce the depth?

- After executing **finding root**, we know the root of an element:

$$r_i = \text{FindRoot}(i).$$

- Let r_i be the parent of index i .

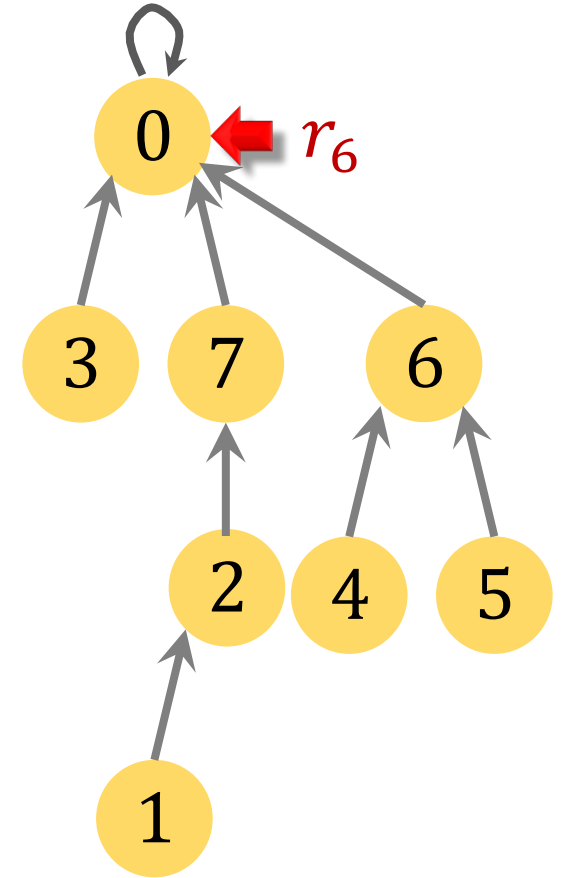


How to reduce the depth?

- After executing **finding root**, we know the root of an element:

$$r_i = \text{FindRoot}(i).$$

- Let r_i be the parent of index i .

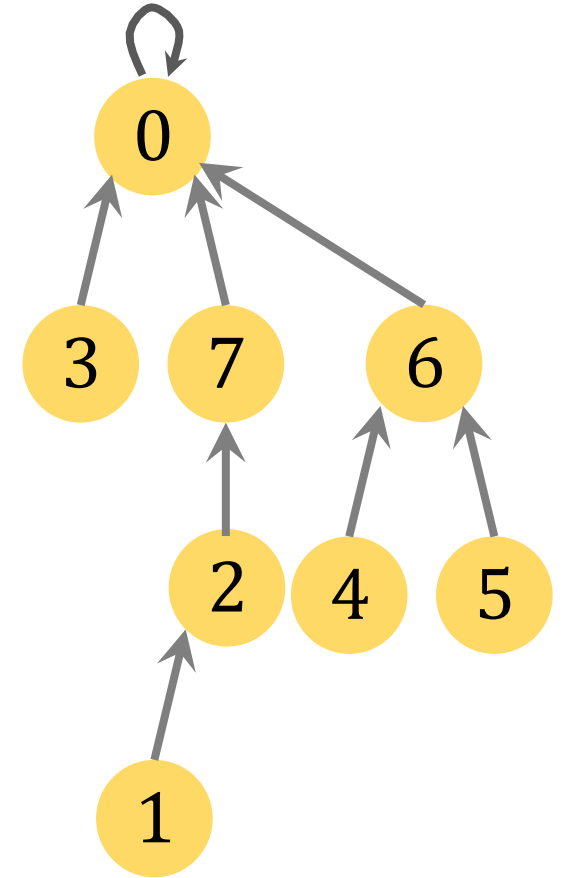


How to reduce the depth?

- After executing **finding root**, we know the root of an element:

$$r_i = \text{FindRoot}(i).$$

- Let r_i be the parent of index i .
- **Path compression** reduces the depth without extra cost.



Summary

Disjoint Sets Data Structure

- **Motivation:** Tracking a set of elements partitioned into a number of disjoint (non-overlapping) subsets.
- **Operation 1:** Union (merge) two sets.
- **Operation 2:** Given two elements, tell whether they belong to the same set.
- **Time complexity:** Depth of the trees.
- **Path compression:** Reduce the depth of trees.

Questions

Q1: Plot the disjoint sets as trees

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

Q1: Plot the disjoint sets as trees

Element:

0

1

2

3

4

5

6

7

Parent:

-1

2

-1

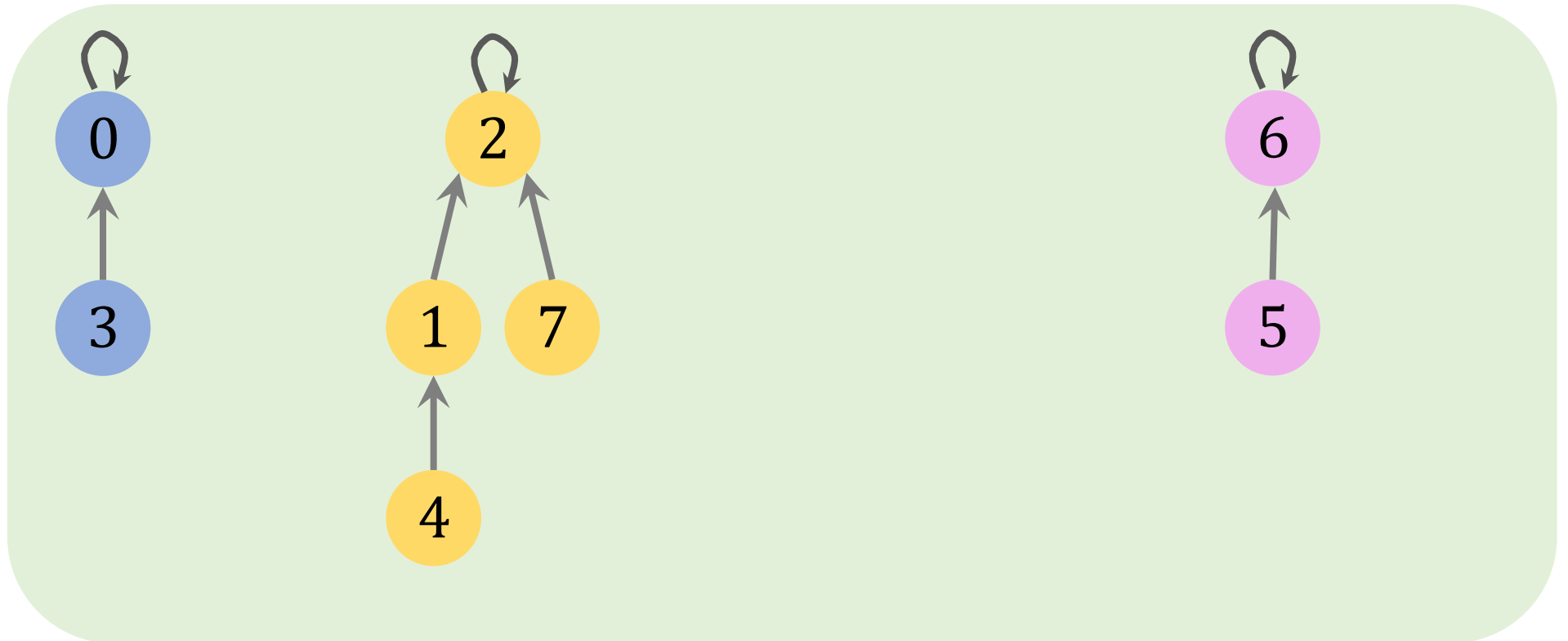
0

1

6

-1

2



Q1: Plot the disjoint sets as trees

Element: 0 1 2 3 4 5 6 7 8 9

Parent:

2	-1	7	-1	0	1	3	-1	1	1
---	----	---	----	---	---	---	----	---	---



?

Q2: Find roots

Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

1. Find the root of element 0.
2. Find the root of element 4.
3. Find the root of element 8.

Q3: Are elements in the same set?

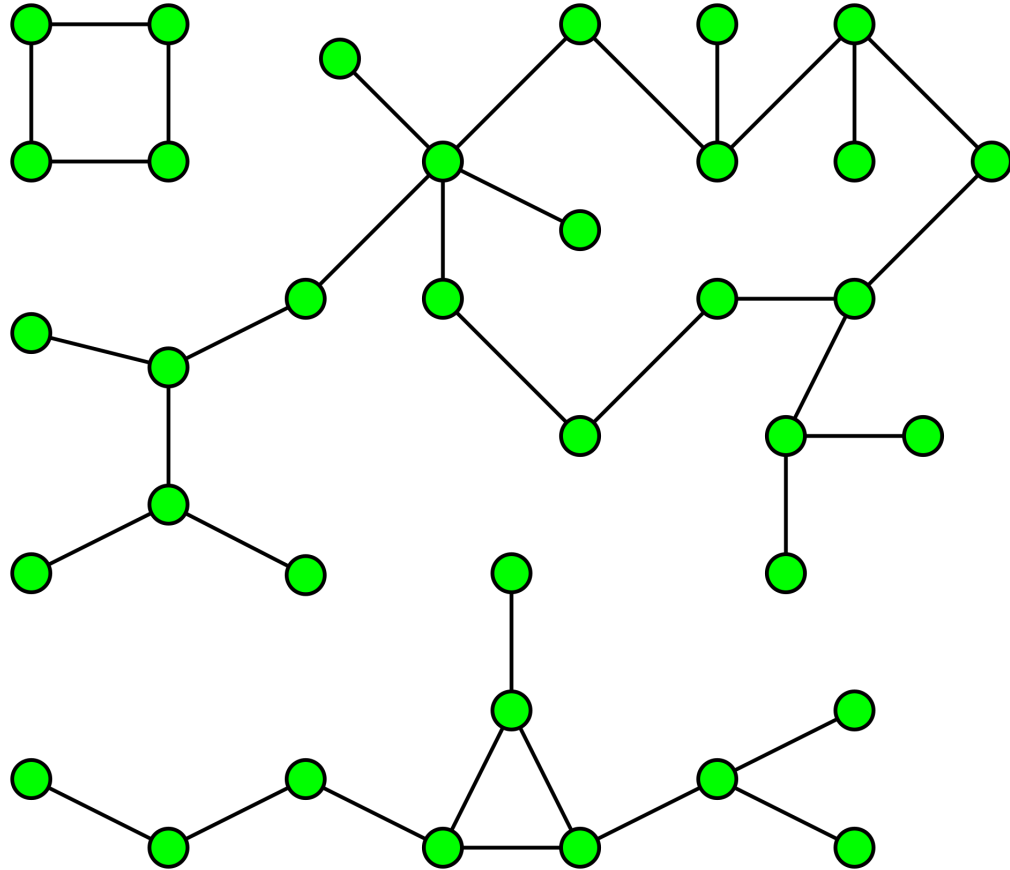
Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

1. Are elements 0 and 1 in the same set?
2. Are elements 0 and 2 in the same set?
3. Are elements 5 and 6 in the same set?
4. Are elements 8 and 9 in the same set?

Q4: How many disjoint sets?

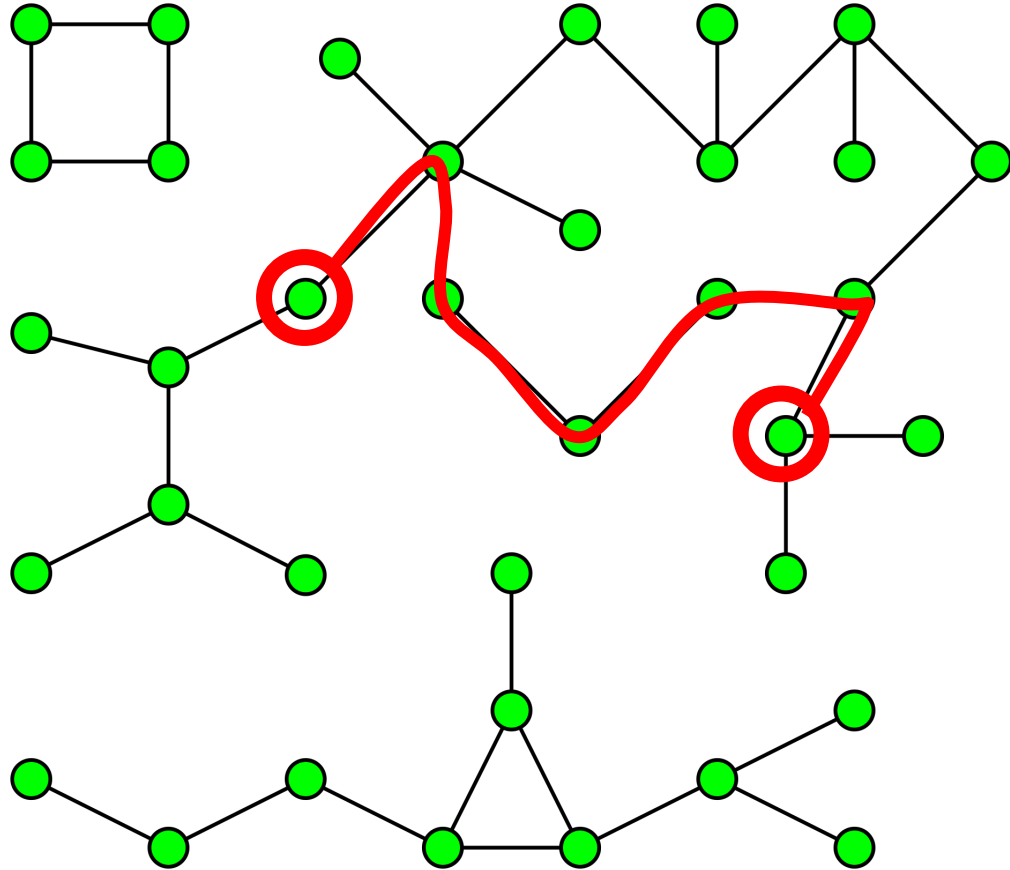
Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

Q5: Graph Connectivity



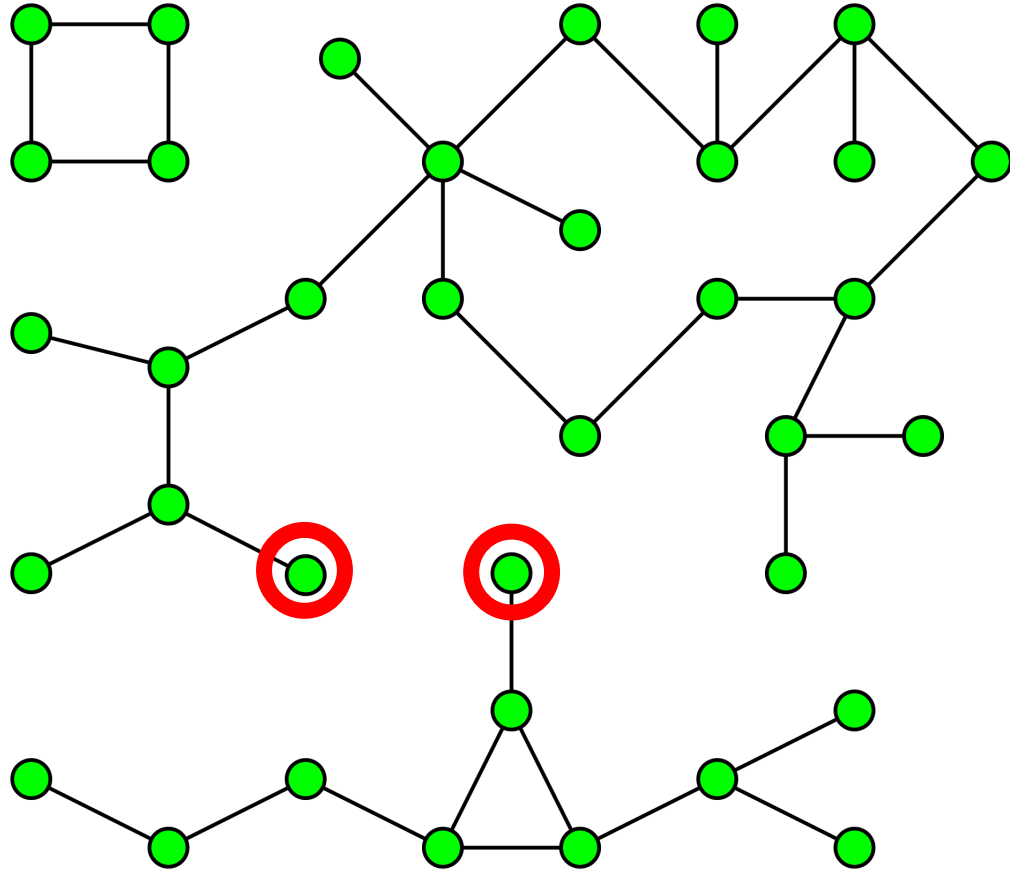
- A graph is said to be **connected** if there is a path between every pair of vertex.
- Given an undirected graph, decide whether it is connected.

Q5: Graph Connectivity



- A graph is said to be **connected** if there is a path between every pair of vertex.
- Given an undirected graph, decide whether it is connected.

Q5: Graph Connectivity



- A graph is said to be **connected** if there is a path between every pair of vertex.
- Given an undirected graph, decide whether it is connected.

Thank You!