Disjoint Sets

Shusen Wang

Disjoint Sets (Math)

- Disjoint sets are those whose pairwise intersections are empty sets.
- Disjoint sets:
  - $\mathcal{S}_1 = \{0, 3\}$,
  - $\mathcal{S}_2 = \{1, 2, 7\}$,
  - $\mathcal{S}_3 = \{4, 5, 6\}$.

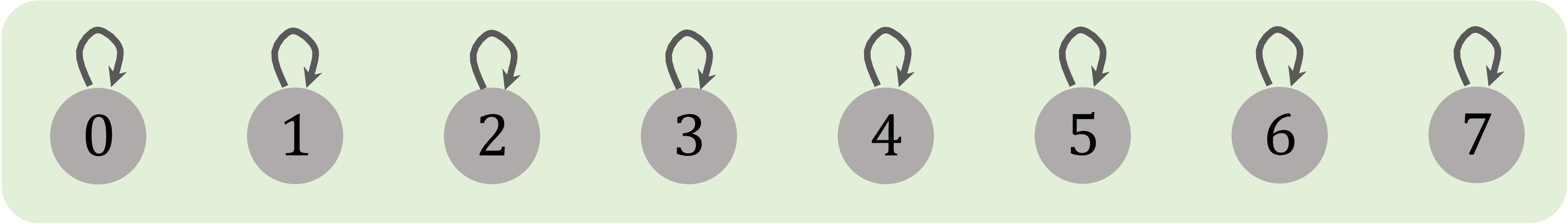
Disjoint Sets (Math)

- Disjoint sets are those whose pairwise intersections are empty sets.
- Not disjoint sets:
  - $\mathcal{S}_1 = \{0, 3\}$,
  - $\mathcal{S}_2 = \{1, 2, 3\}$,
  - $\mathcal{S}_3 = \{4, 5, 6\}$.
  - Because the intersection is not empty: $\mathcal{S}_1 \cap \mathcal{S}_2 = \{3\}$.

Data Structure

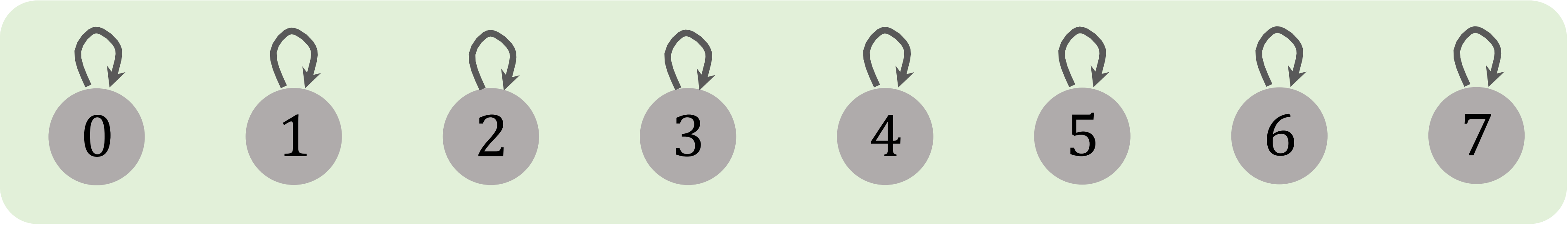
Initialization

Element: 0 1 2 3 4 5 6 7



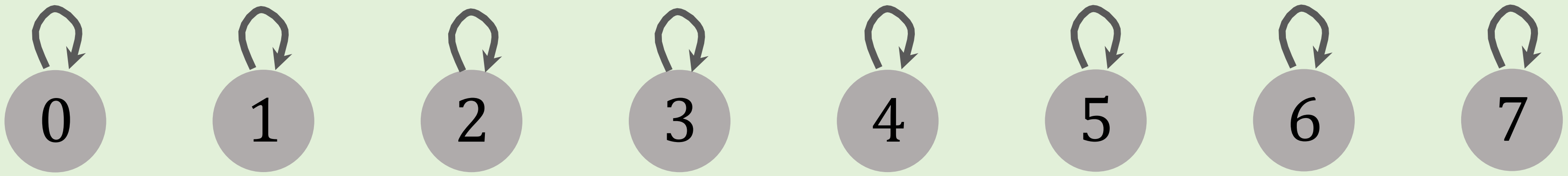
Initialization

Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



Sets Representation

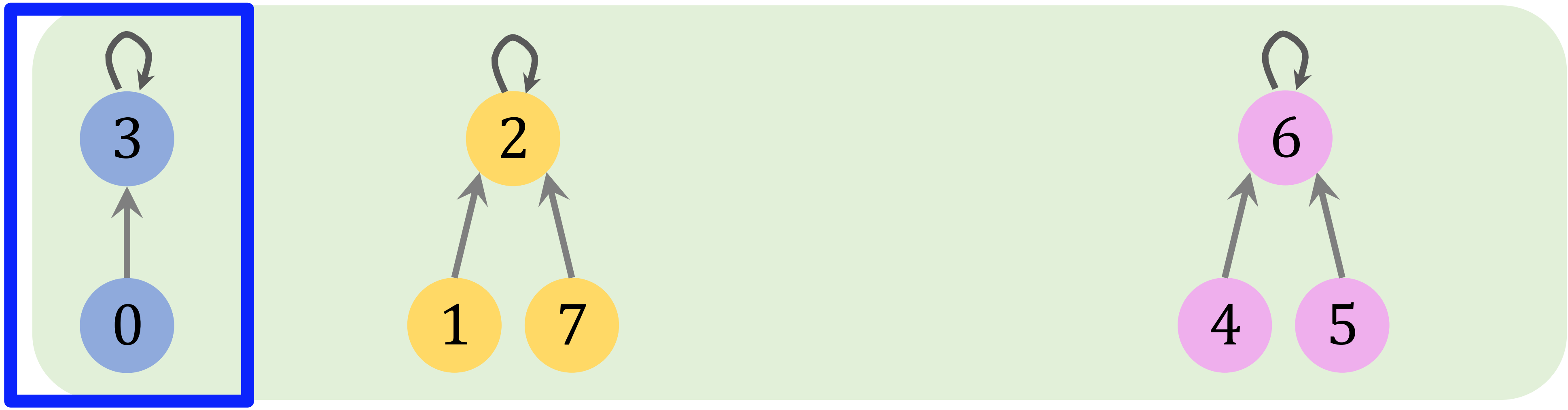
Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



Representing sets $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$.

Sets Representation

Element:	0	1	2	3	4	5	6	7
Parent:	3	2	-1	-1	6	6	-1	2



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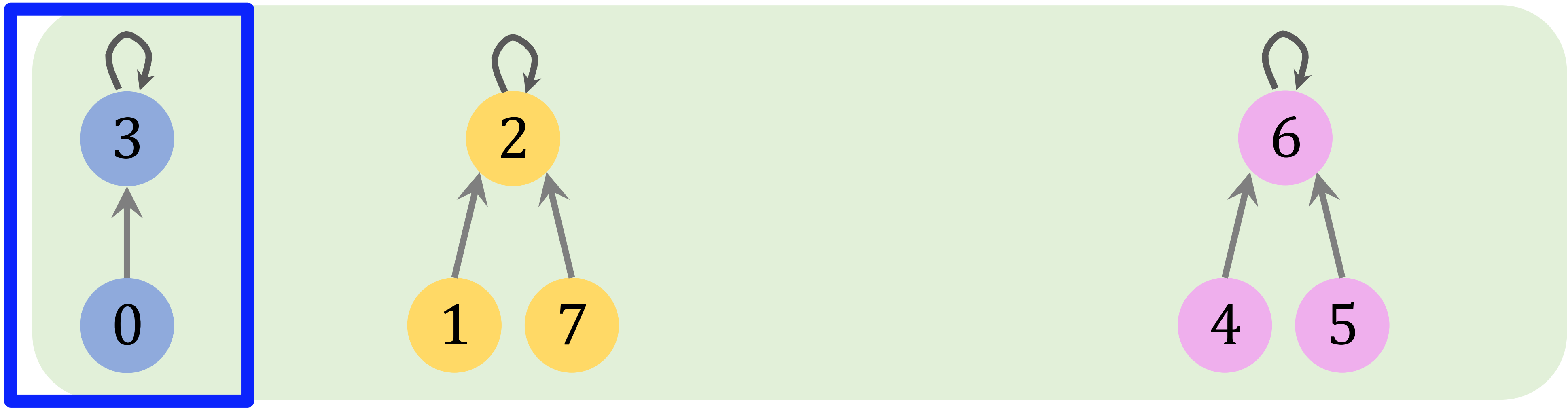

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Sets Representation

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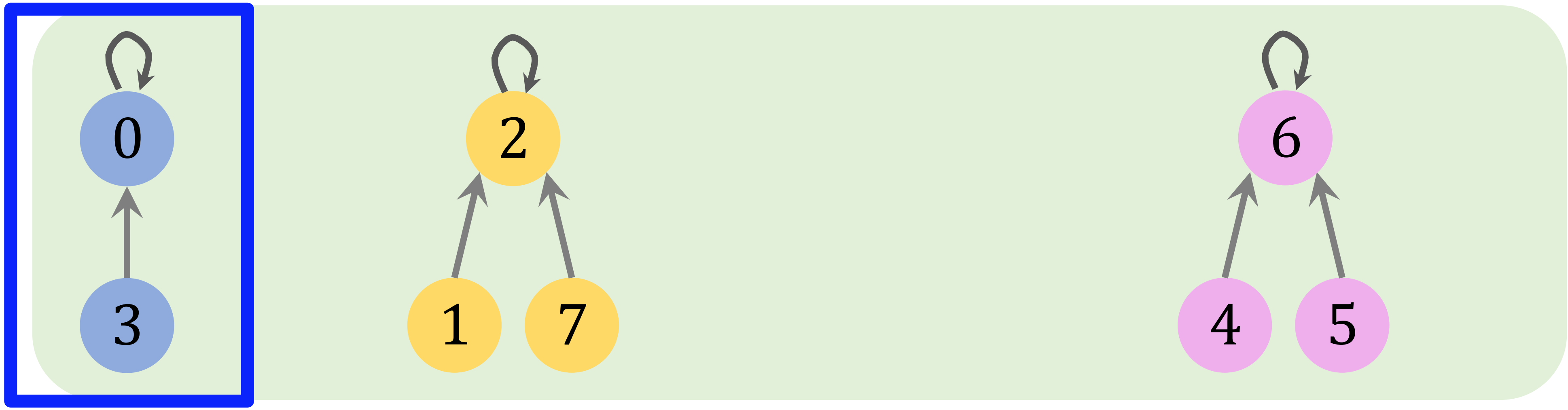
Parent:	-1	2	-1	0	6	6	-1	2



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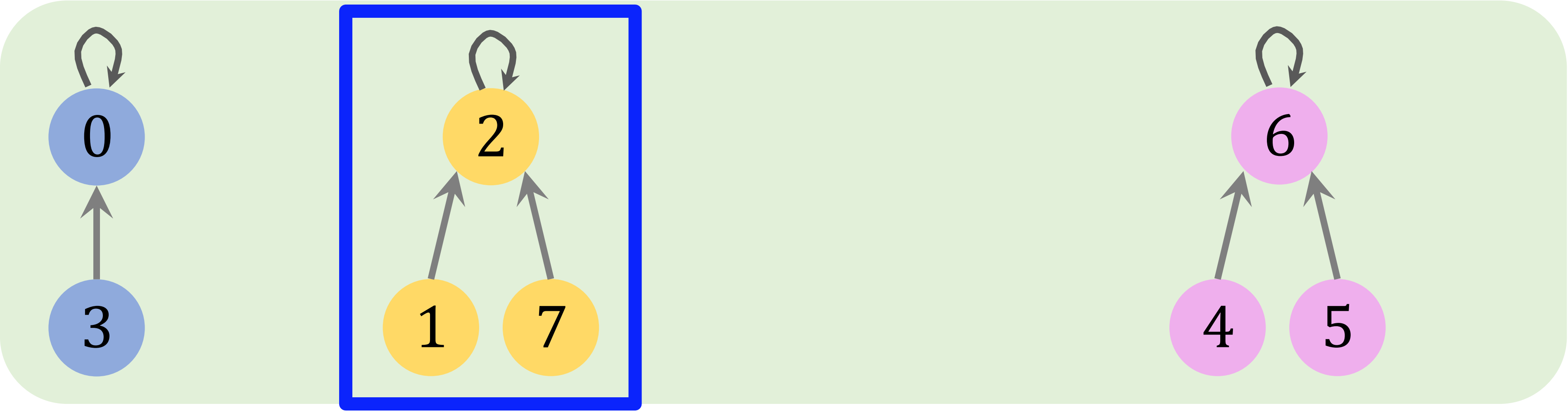

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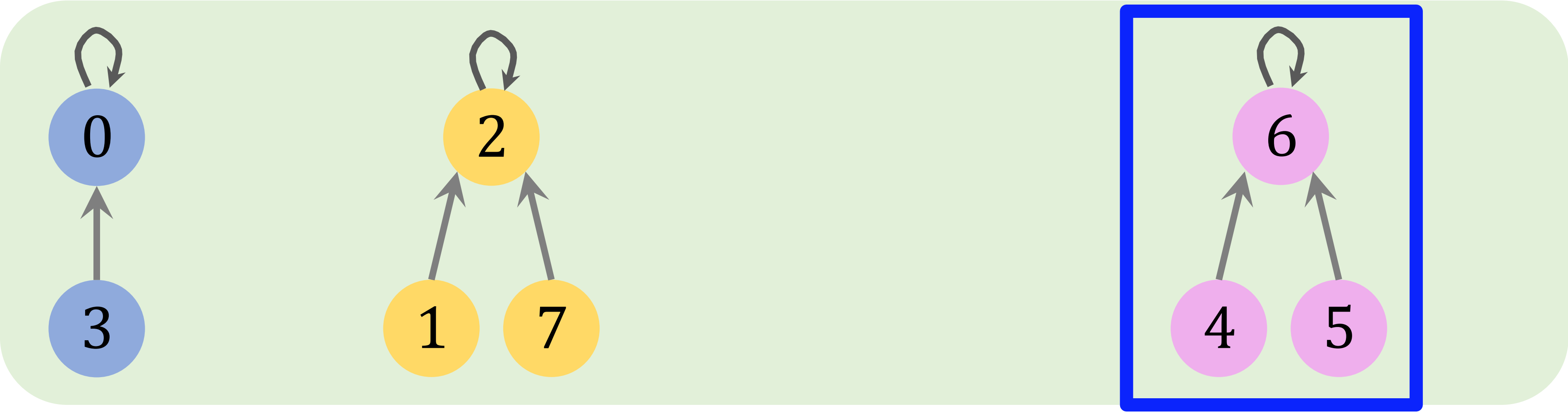

Representing sets $\mathcal{S}_1 = \{0, 3\}$, $\mathcal{S}_2 = \{1, 2, 7\}$, and $\mathcal{S}_3 = \{4, 5, 6\}$.

Operations

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

Operation 1: Are elements i and j belong to the same set?

- Find the roots of i and j.
- If they have the same root, then return TRUE.
- Otherwise, return False.

Operations

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

Operation 1: Are elements i and j belong to the same set?

Operation 2: Merge (union) sets $\mathcal{S}_1$ and $\mathcal{S}_2$.

- Find the root of $\mathcal{S}_1$; denote it by r_1.
- Find the root of $\mathcal{S}_2$; denote it by r_2.
- Make r_1 a child of r_2.

Find Root

Current State

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

Current State

Element:	0	1	2	3	4	5	6	7
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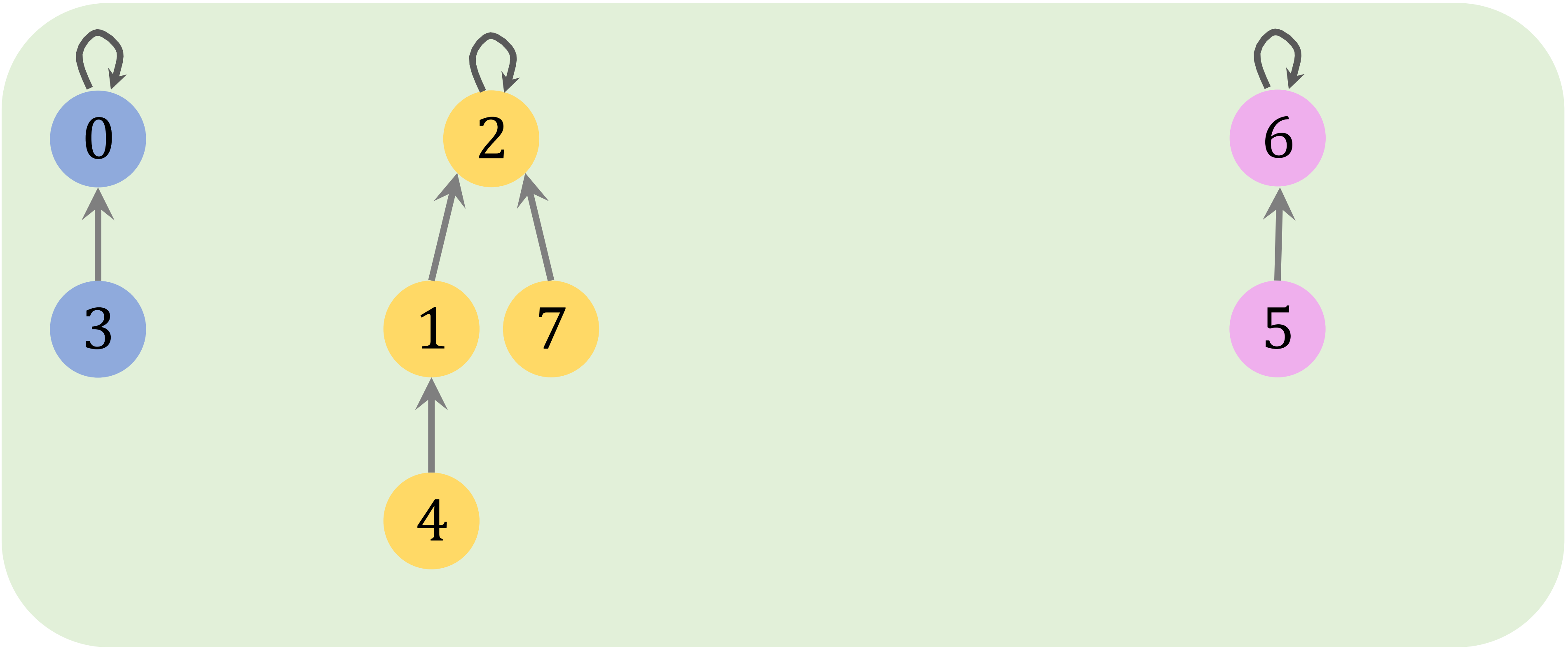


What is the root of 2?

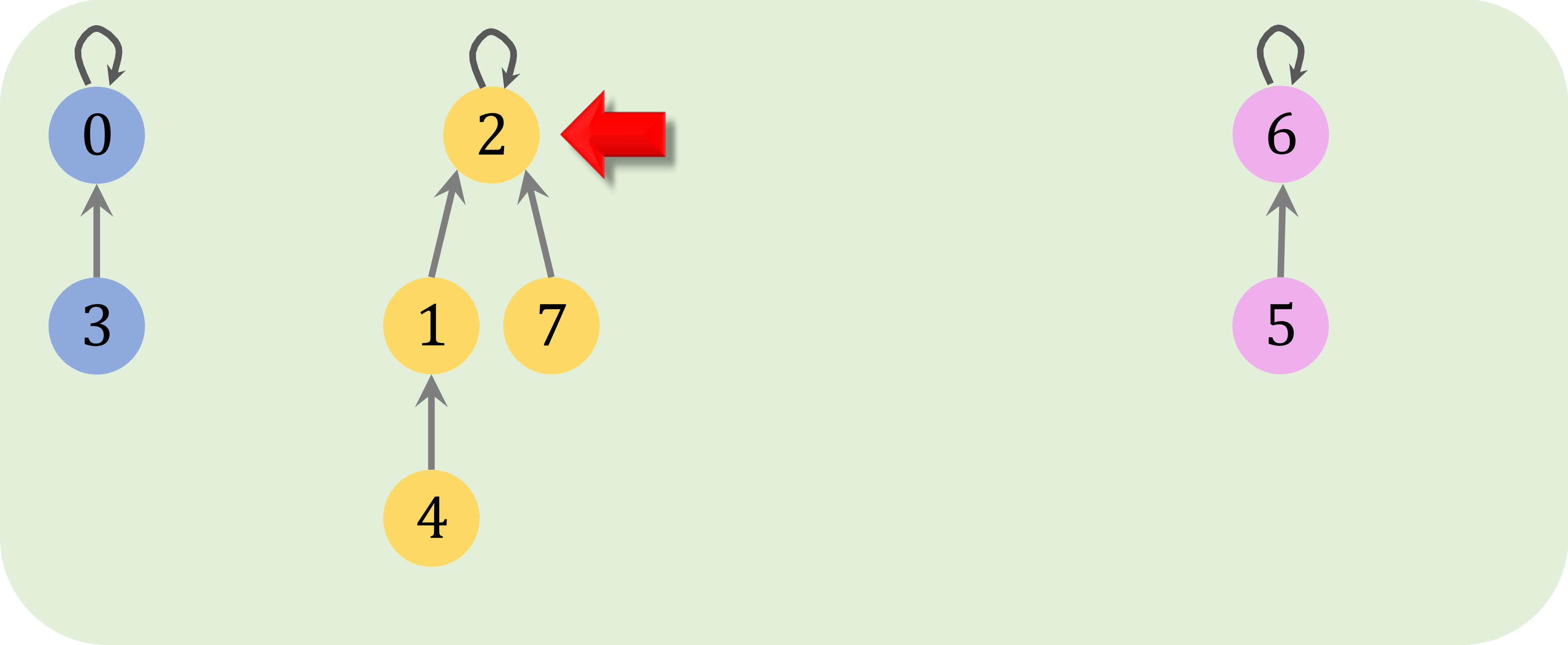
Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

The root of 2 is 2 itself.

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

The root of 2 is 2 itself.

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2



What is the root of 4?

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

What is the root of 4?

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

What is the root of 4?

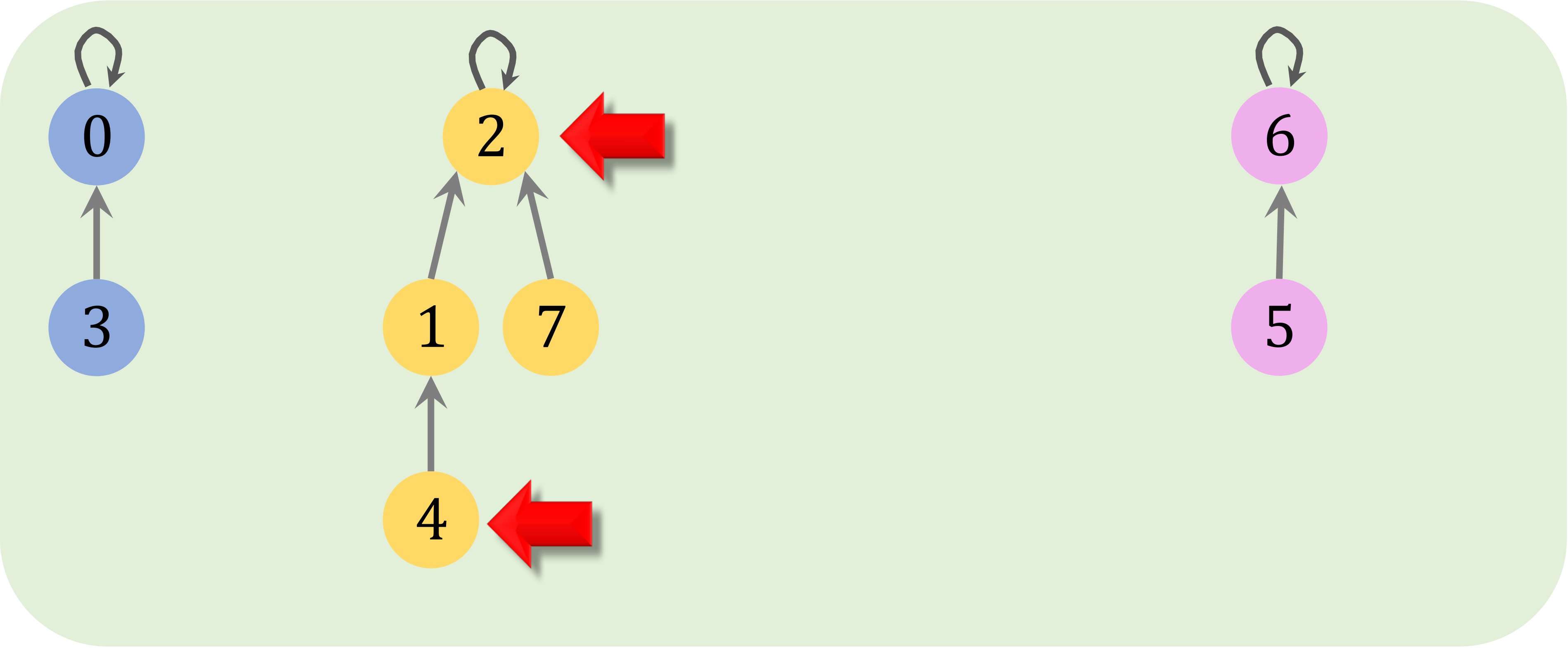
Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

The root of 4 is 2.

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

The root of 4 is 2.

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2



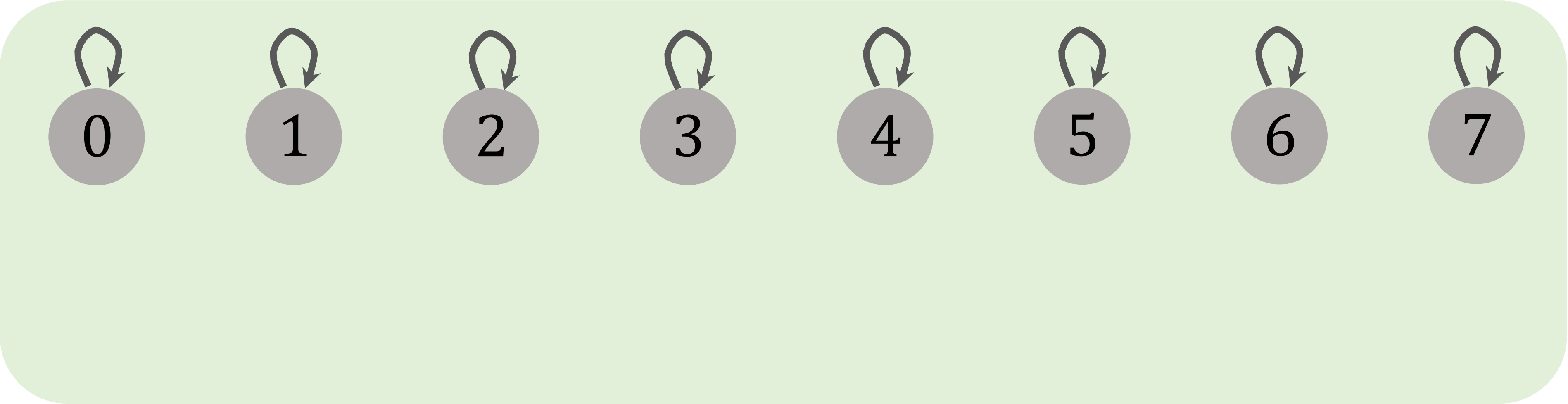
Are two elements in the same set?

- **Inputs:** two elements (indices) i and j.
- **Question:** Are they in the same set?
- **Solution:**
  1. $r_i \leftarrow$ the root of i.
  2. $r_j \leftarrow$ the root of i.
  3. If r_i and r_j are the same, then i and j are in the same set.
  4. Otherwise, i and j are in different sets.

Union Operation

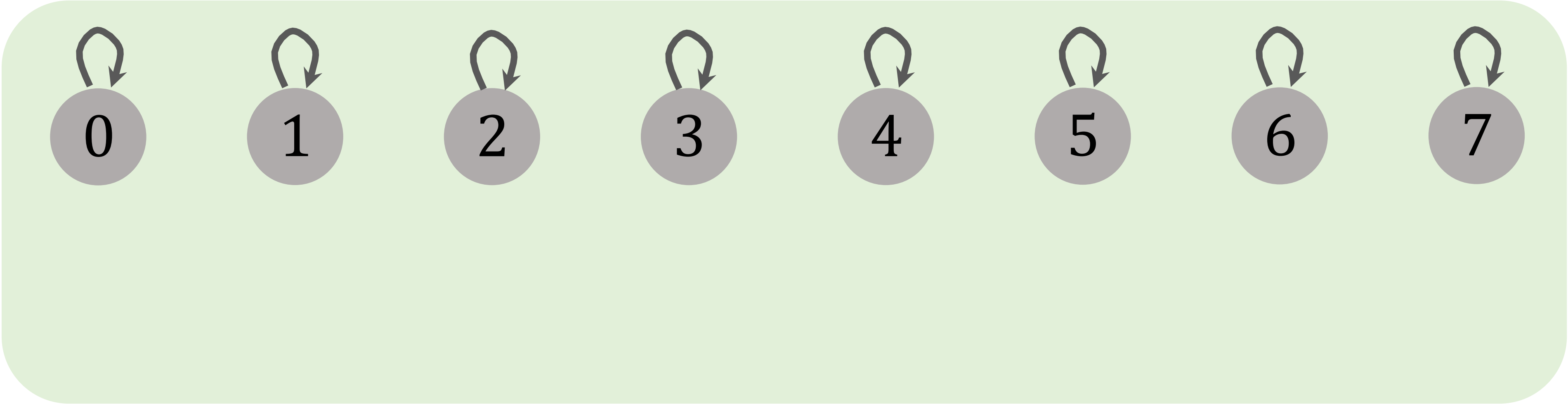
Initial State

Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



Union(1, 2)

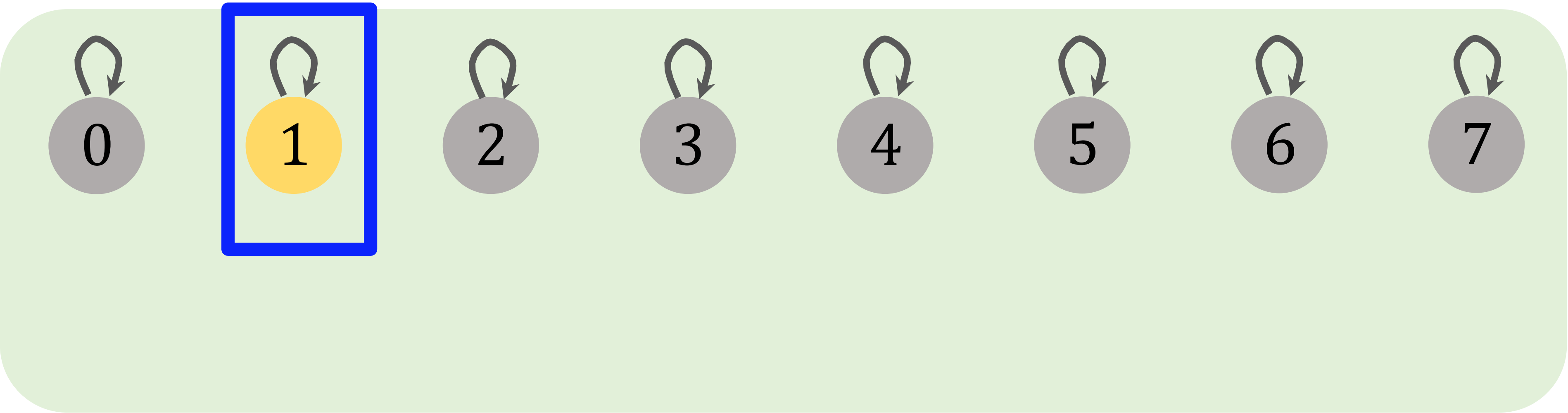
Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



1. Find the root of 1: $r_1 = 1$.

Union(1, 2)

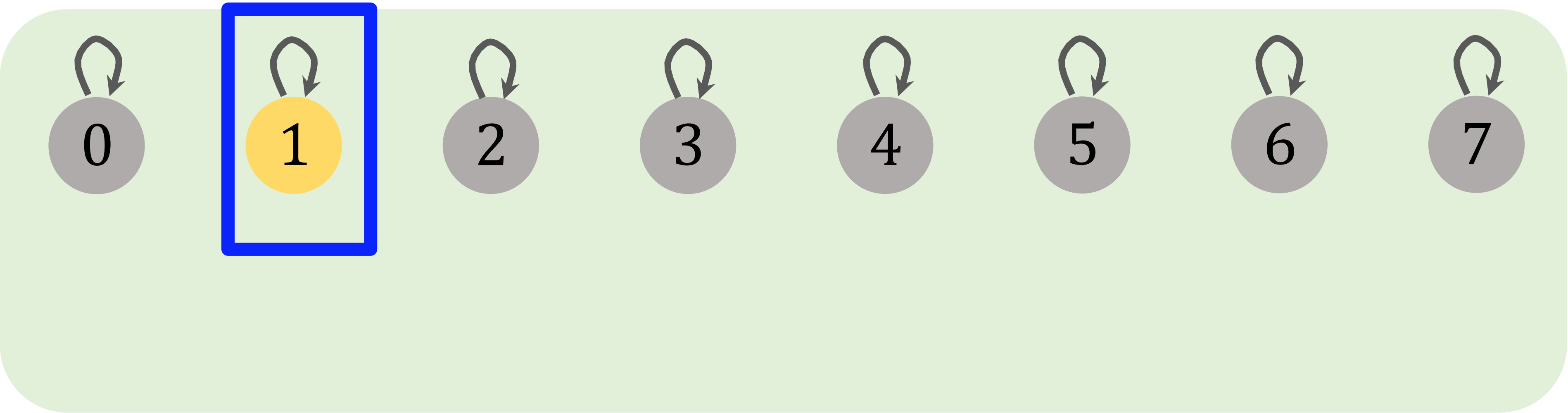
Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



1. Find the root of 1: $r_1 = 1$.

Union(1, 2)

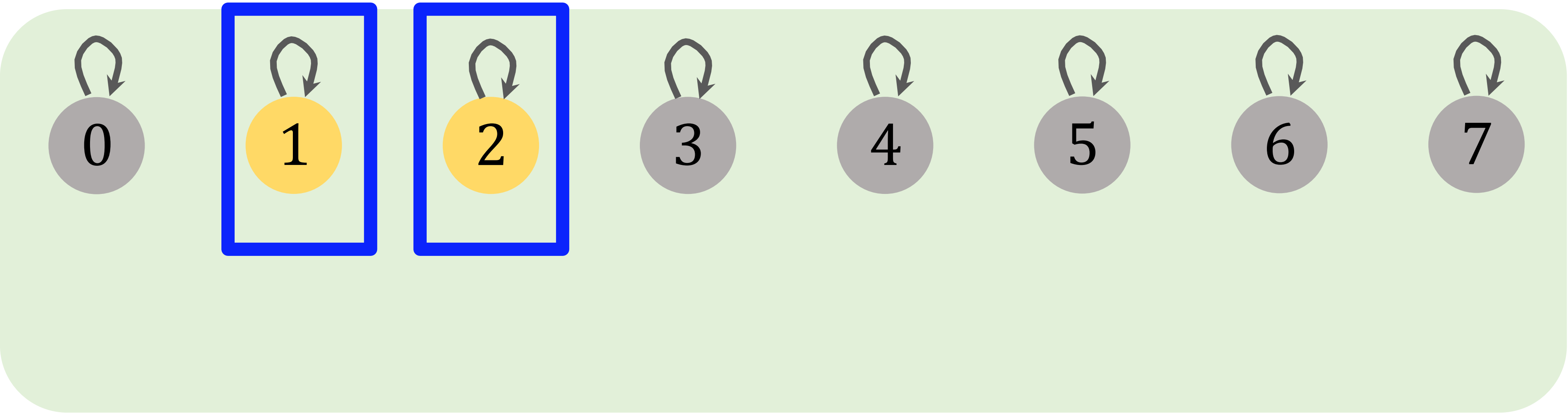
Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.

Union(1, 2)

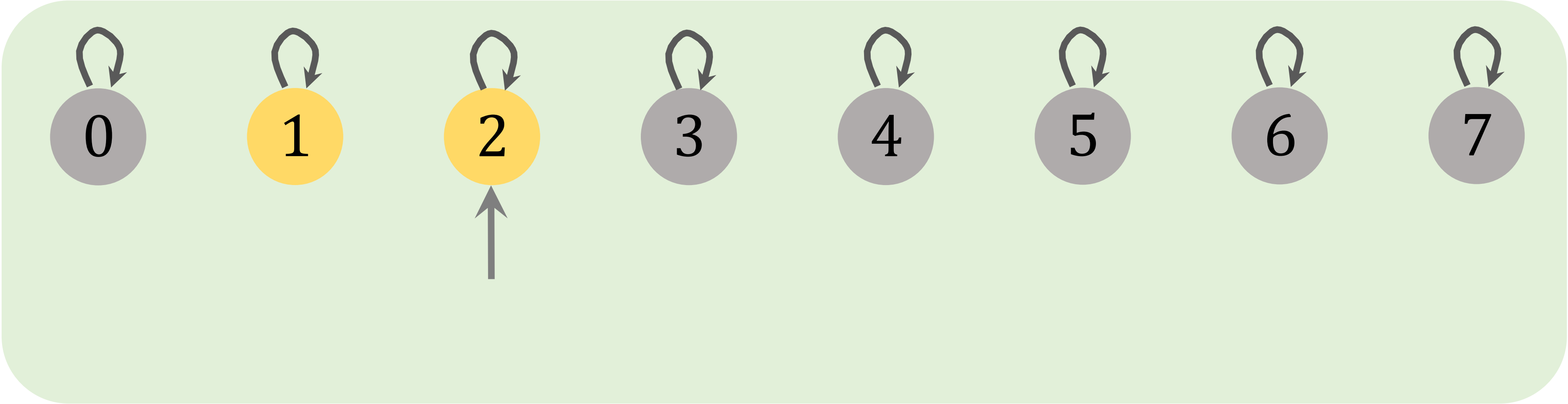
Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.

Union(1, 2)

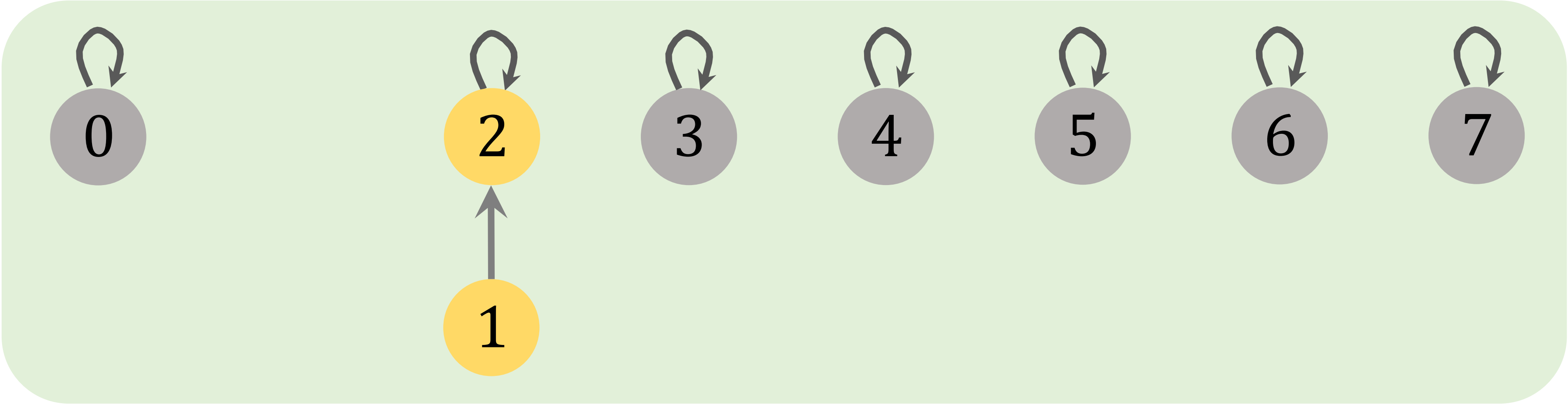
Element:	0	1	2	3	4	5	6	7
Parent:	-1	-1	-1	-1	-1	-1	-1	-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.
3. Make r_2 the parent of r_1.

Union(1, 2)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	-1	-1	-1	-1	-1



1. Find the root of 1: $r_1 = 1$.
2. Find the root of 2: $r_2 = 2$.
3. Make r_2 the parent of r_1: `parent[`r_1`] =` r_2.

After many unions…

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

Representing sets $\boxed{\mathcal{S}_1 = \{0, 3\}}$, $\boxed{\mathcal{S}_2 = \{1, 2, 7\}}$, and $\boxed{\mathcal{S}_3 = \{4, 5, 6\}}$.

Union(4, 5)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

Union(4, 5)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

1. Find the root of 4: $\boxed{r_4 = 6.}$

Union(4, 5)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

1. Find the root of 4: $\boxed{r_4 = 6.}$
2. Find the root of 5: $\boxed{r_5 = 6.}$
3. They have the same root; thus no operation.

Union(4, 7)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

Union(4, 7)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

1. Find the root of 4: $\boxed{r_4 = 6.}$

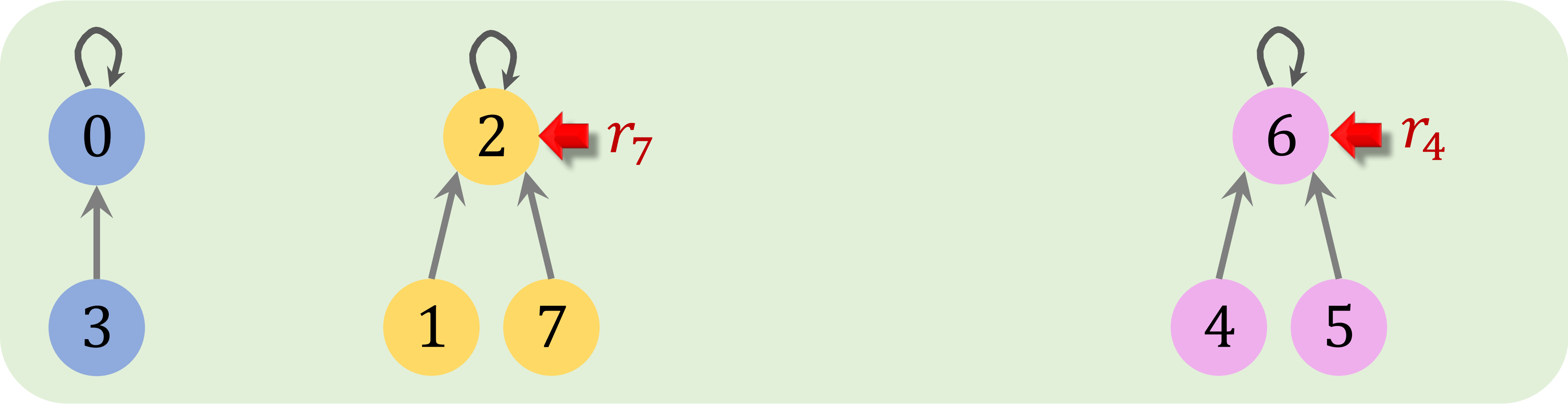
Union(4, 7)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2

1. Find the root of 4: $\boxed{r_4 = 6.}$
2. Find the root of 7: $\boxed{r_7 = 2.}$

Union(4, 7)

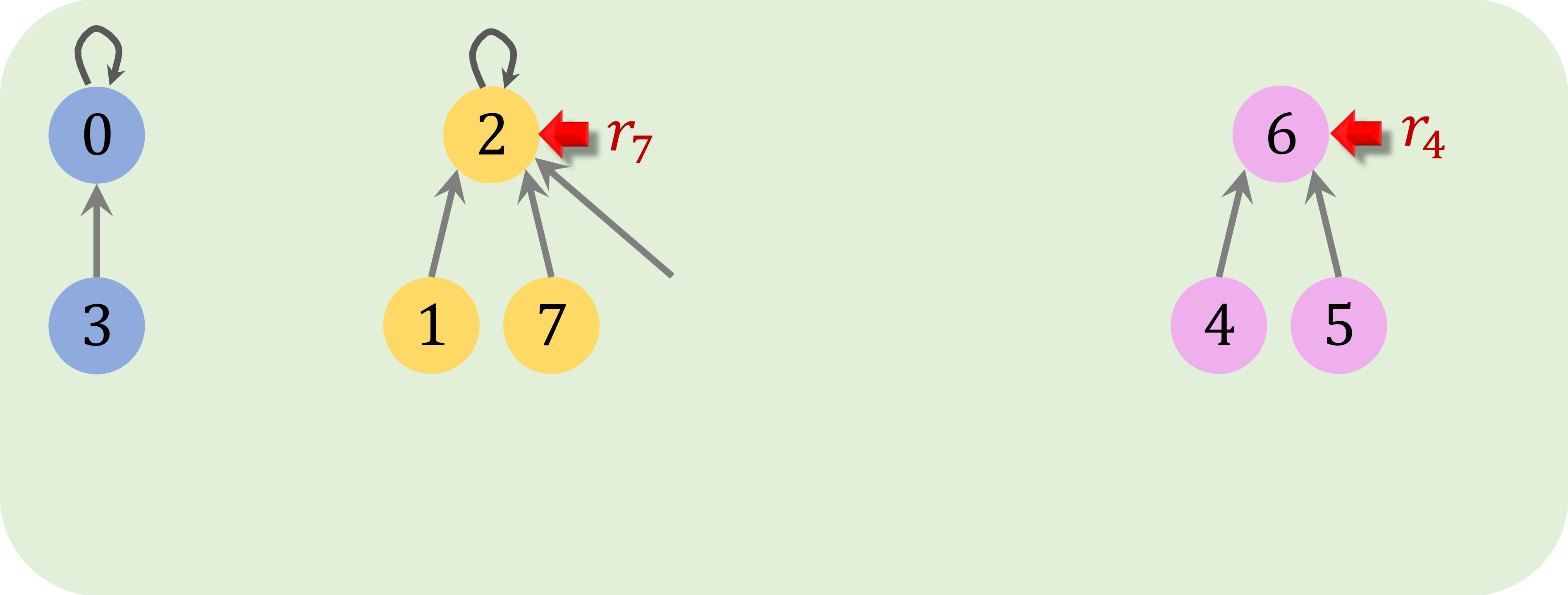
Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2



1. Find the root of 4: $\boxed{r_4 = 6.}$
2. Find the root of 7: $\boxed{r_7 = 2.}$
3. Make r_7 the parent of r_4: `parent[`r_4`] = `r_7.

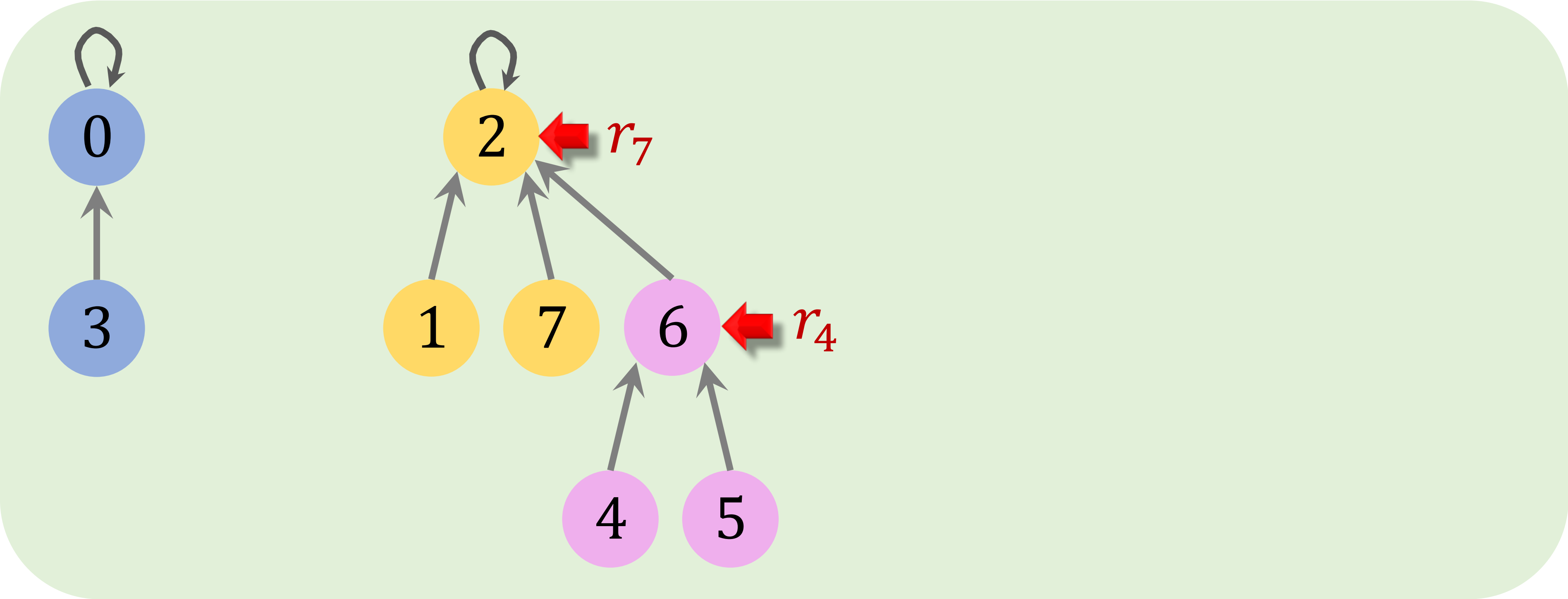
Union(4, 7)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2



Union(4, 7)

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	-1	2



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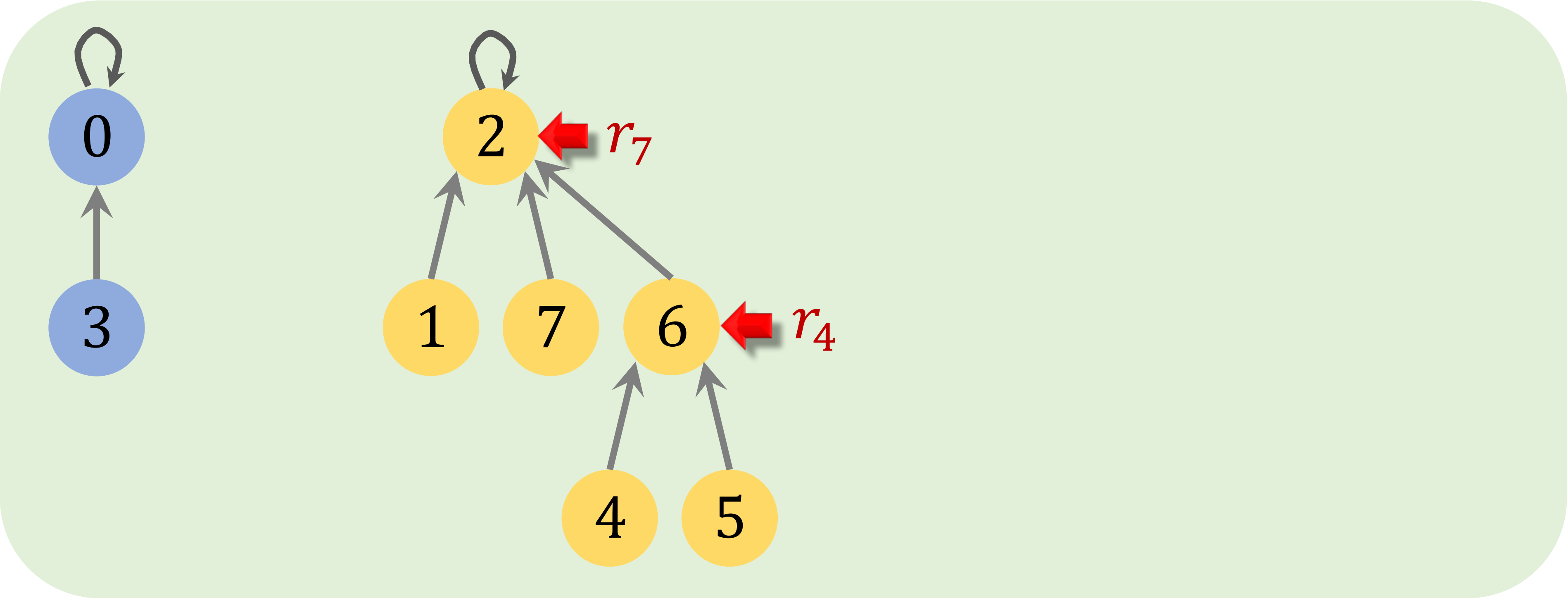
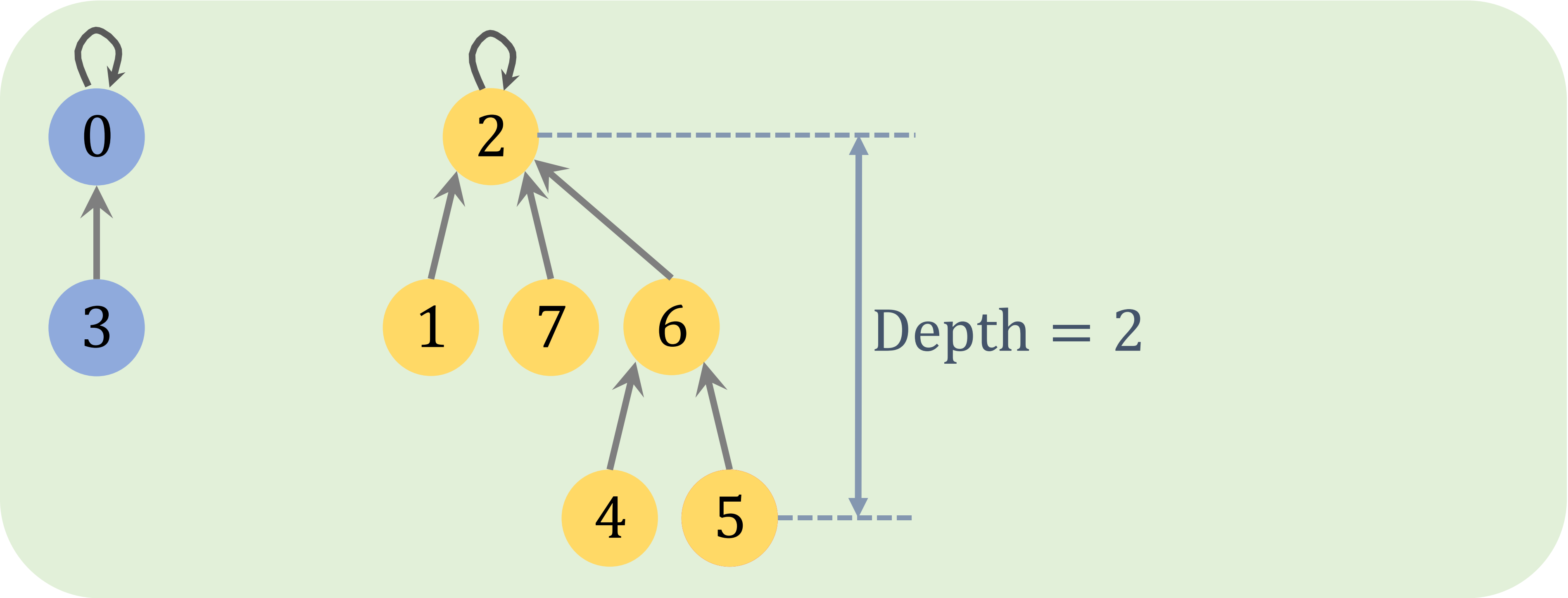


Time Complexities

Time complexity of finding root

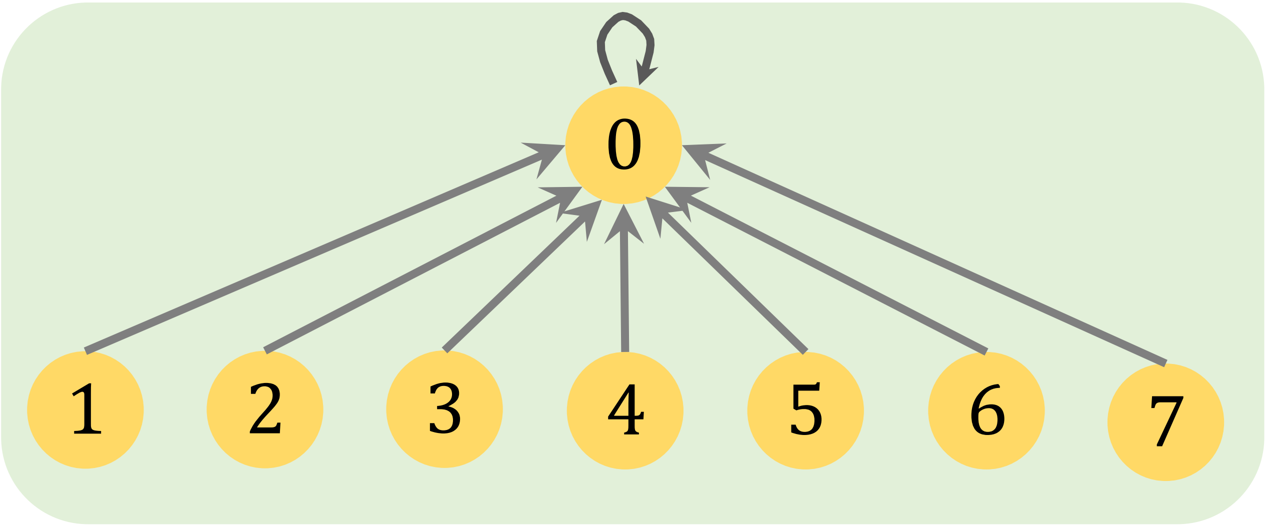
Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	6	6	2	2



Time complexities

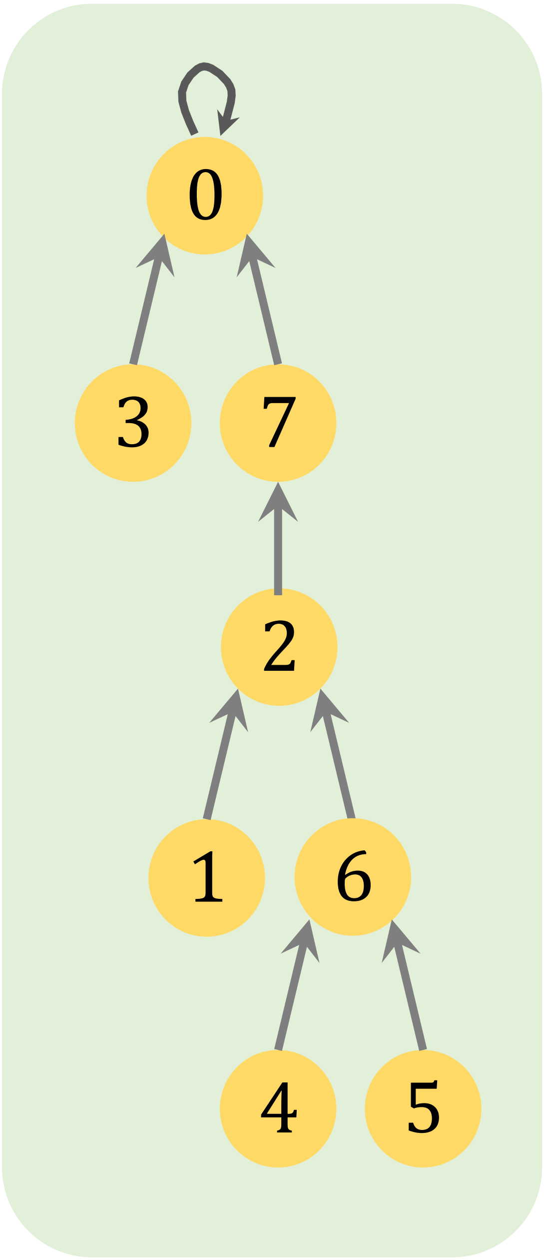
- Time complexity of finding root is the depth of the vertex.
- Time complexity of union is the depth of the tree.
- In the worst case, it can be $O(n)$. (Here, n is the total number of elements.)

Question: How to reduce the time complexity?



Good

Bad

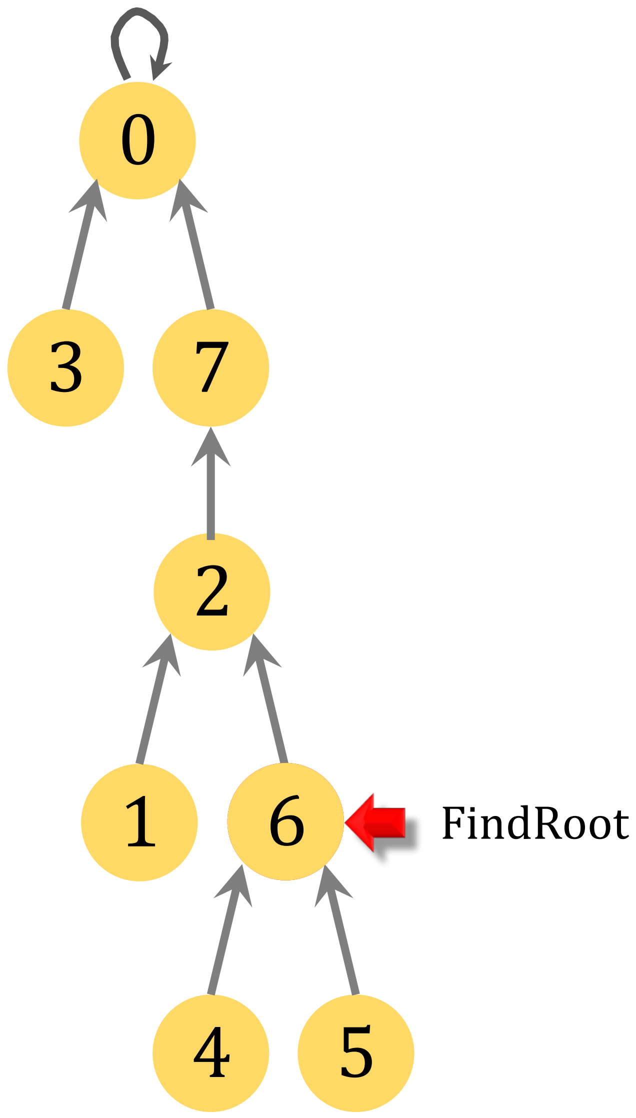


Path Compression

Path Compression

- After executing finding root, we know the root of an element:

$$r_i = \text{FindRoot}(i).$$



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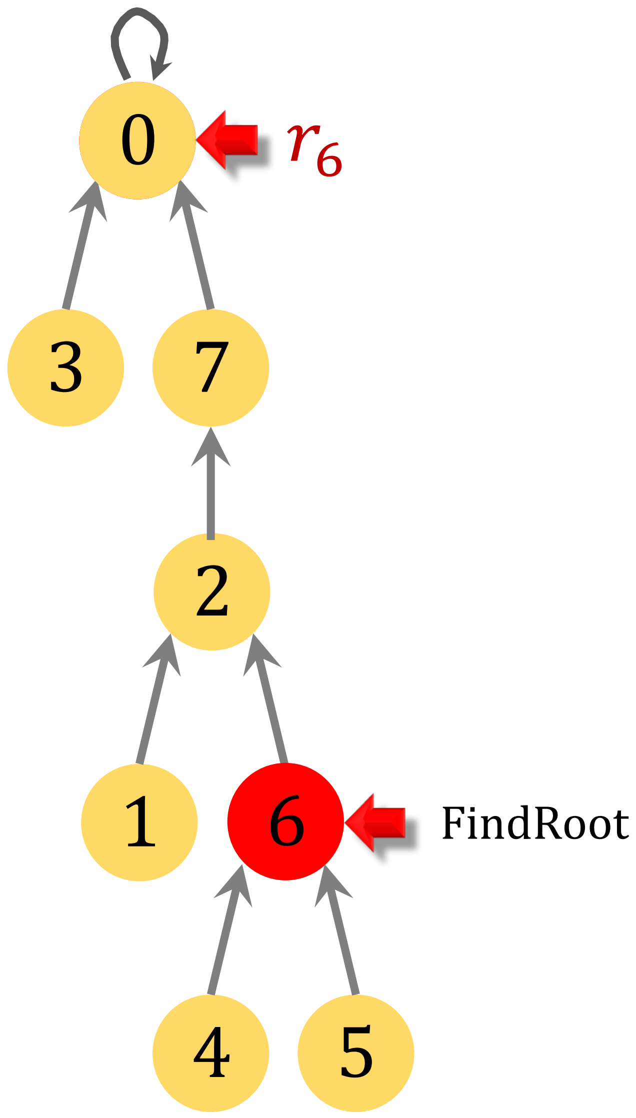

Path Compression

- After executing finding root, we know the root of an element:

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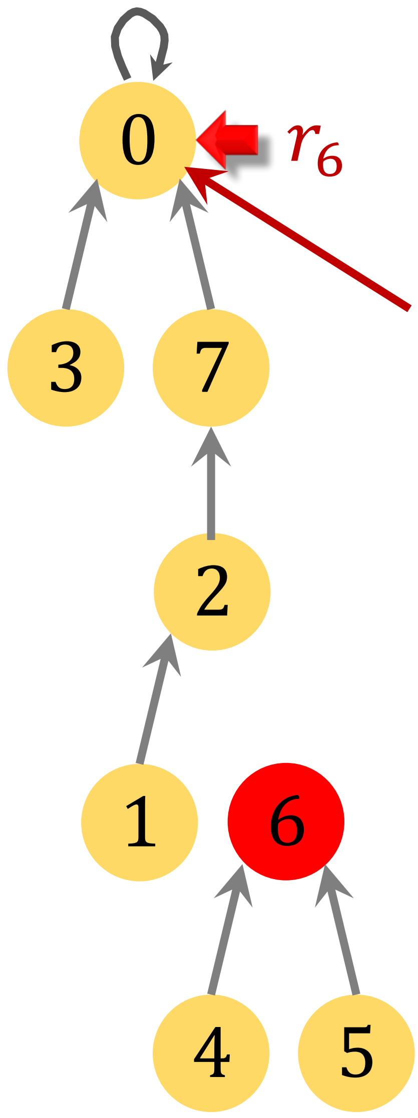
- Let r_i be the parent of index i.



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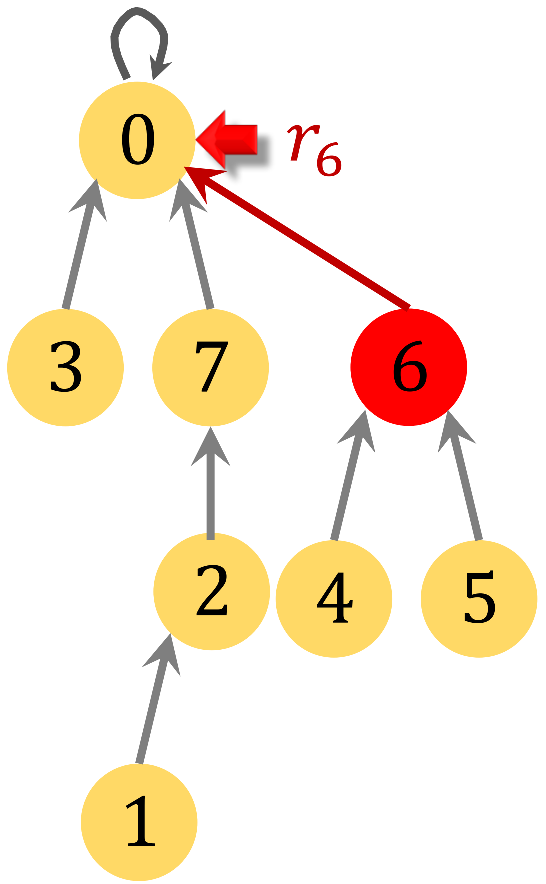
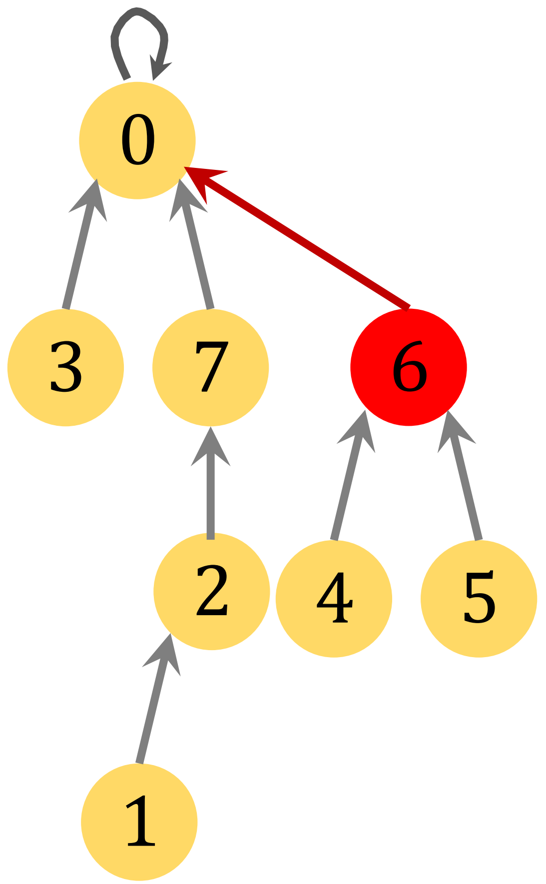


Path Compression

- After executing finding root, we know the root of an element:

$$r_i = \text{FindRoot}(i).$$

- Let r_i be the parent of index i.
- Path compression reduces the depth without extra cost.



Summary

Disjoint Sets Data Structure

- **Motivation:** Tracking a set of elements partitioned into disjoint (non-overlapping) subsets.
- **Operation 1:** Union (merge) two sets.
- **Operation 2:** Given two elements, tell whether they belong to the same set.
- **Time complexity:** Depths of the trees.
- **Path compression:** Reduce the depth of trees.

Questions

Q1: Plot the disjoint sets as trees

Element:	0	1	2	3	4	5	6	7
Parent:	-1	2	-1	0	1	6	-1	2

Q1: Plot the disjoint sets as trees

Element:	0	1	2	3	4	5	6	7
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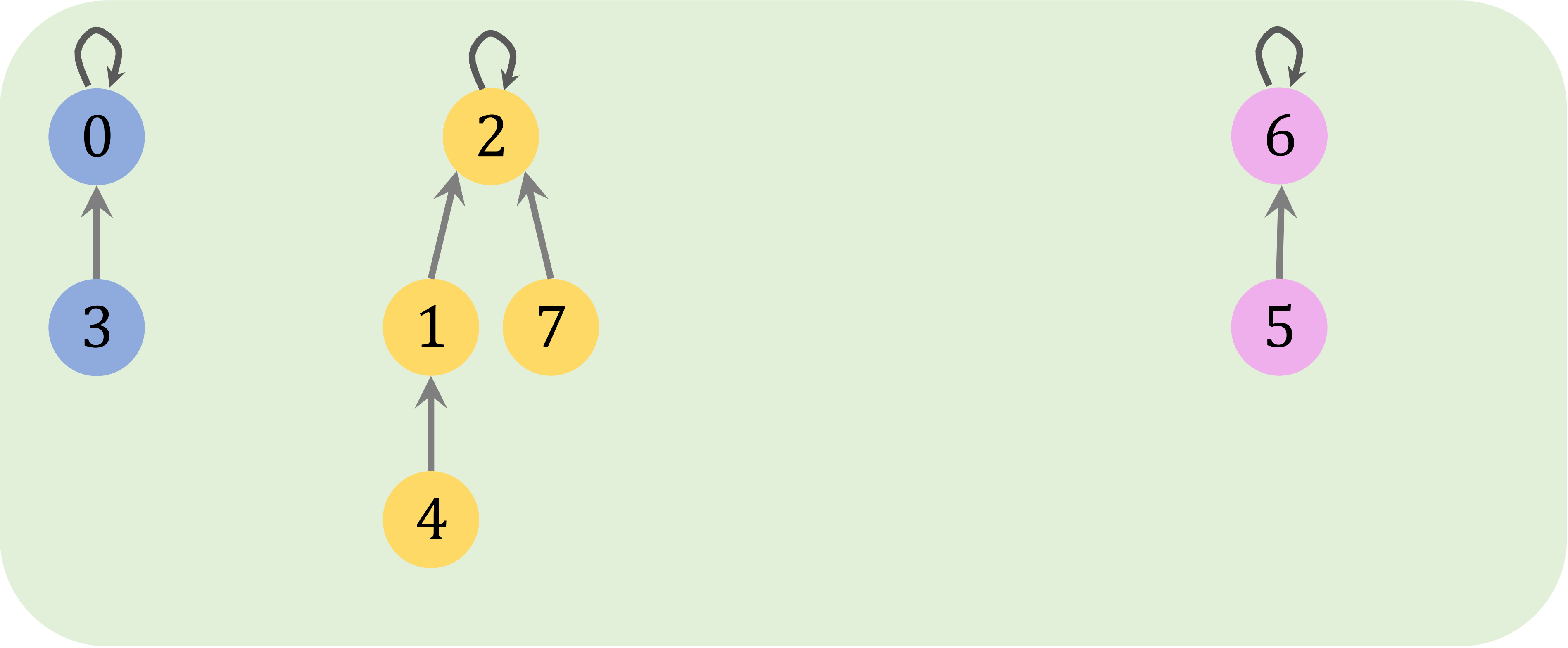


Q1: Plot the disjoint sets as trees

Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

?

Q2: Find roots

Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

1. Find the root of element 0.
2. Find the root of element 4.
3. Find the root of element 8.

Q3: Are elements in the same set?

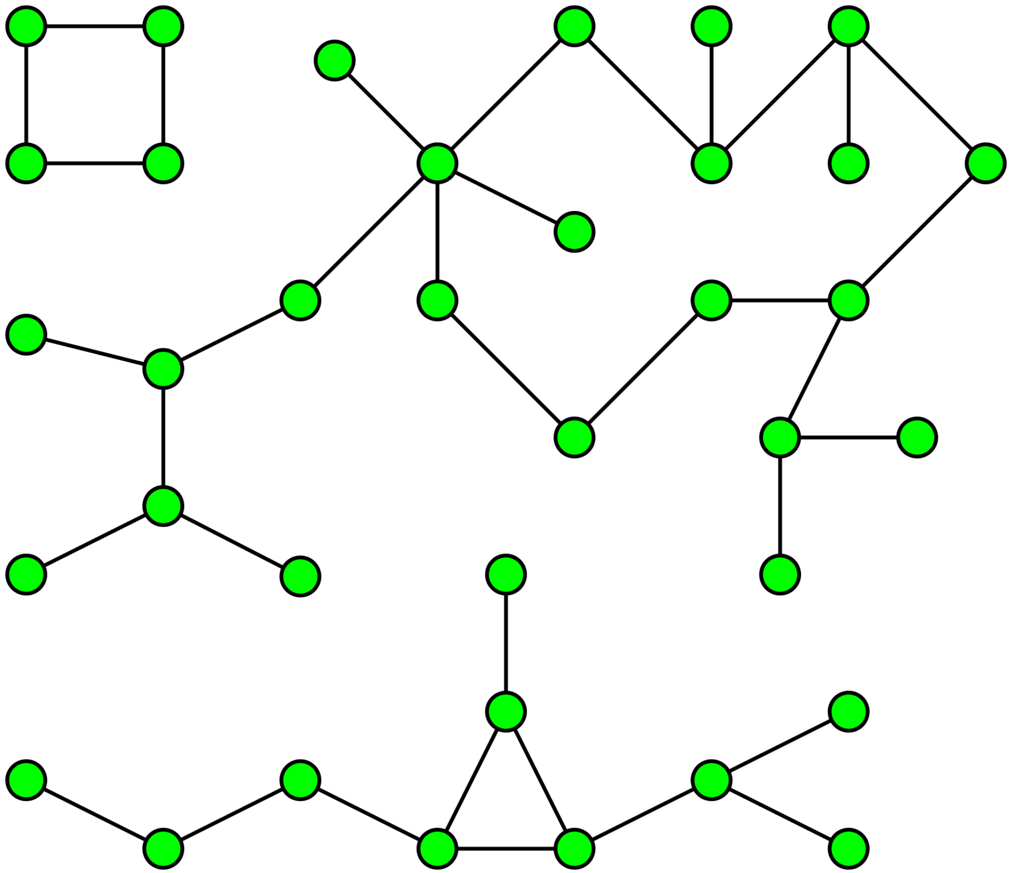
Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

1. Are elements 0 and 1 in the same set?
2. Are elements 0 and 2 in the same set?
3. Are elements 5 and 6 in the same set?
4. Are elements 8 and 9 in the same set?

Q4: How many disjoint sets?

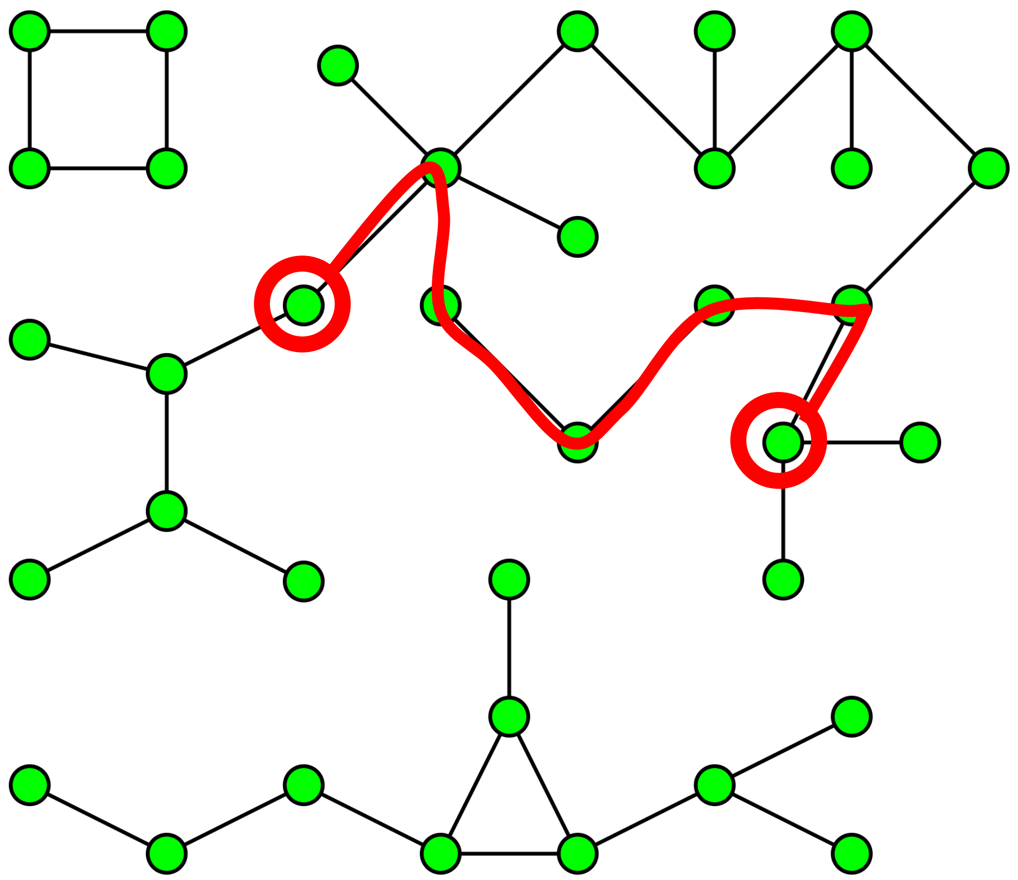
Element:	0	1	2	3	4	5	6	7	8	9
Parent:	2	-1	7	-1	0	1	3	-1	1	1

Q5: Graph Connectivity



- A graph is said to be connected if there is a path between every pair of vertex.
- Given an undirected graph, decide whether it is connected.

Q5: Graph Connectivity



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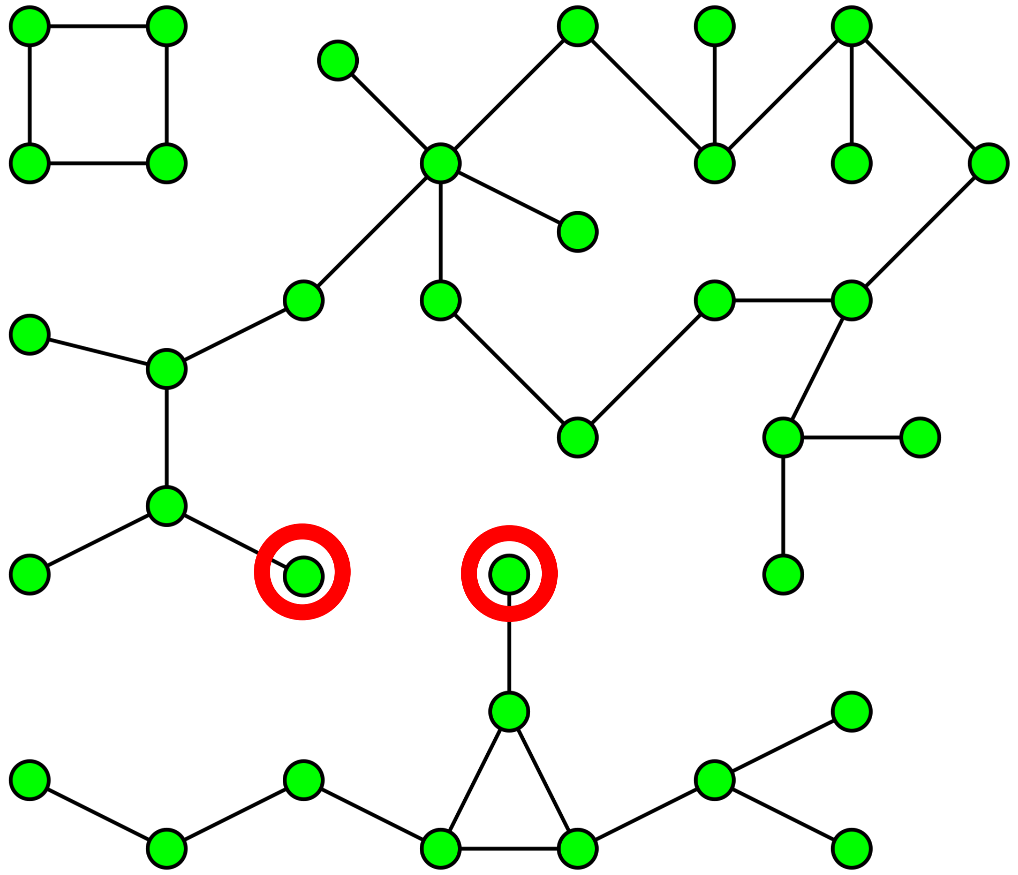

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Thank You!