Monte Carlo Algorithms

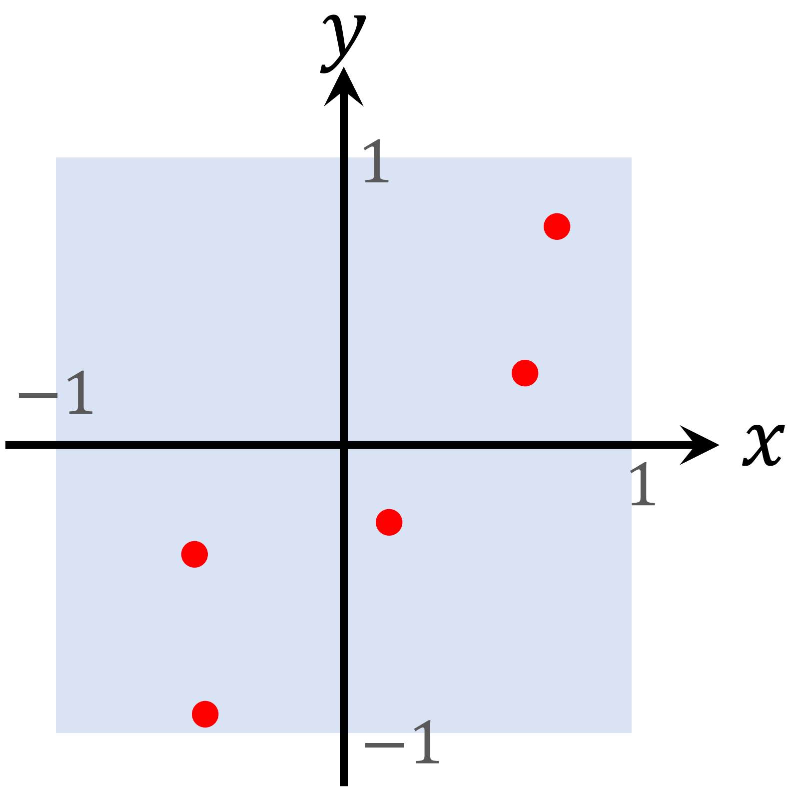
Shusen Wang

Application 1: Calculating Pi

Calculating Pi

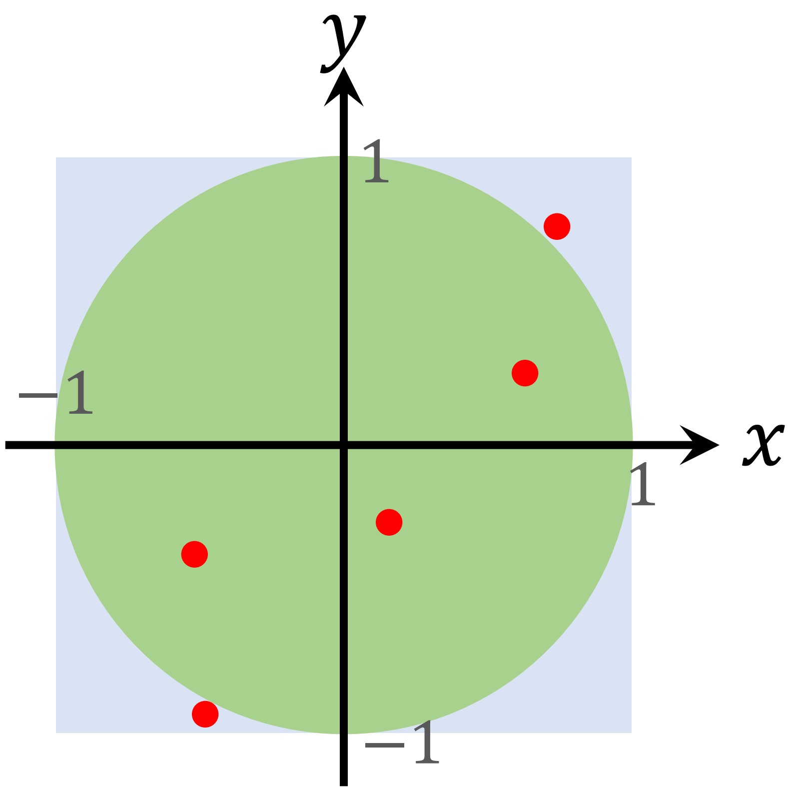
- We already know $\pi \approx 3.141592653589 \ldots$
- Pretend we do not know the value of π.
- Can we find it out (approximately) using a random number generator?

Calculating Pi



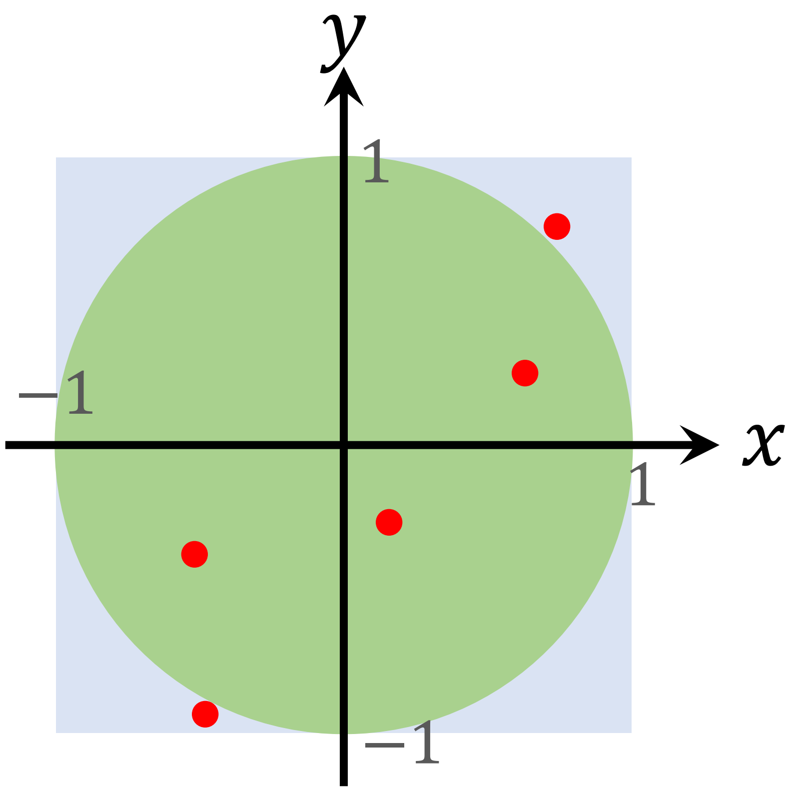
- Assume (x, y) is a point sampled from the square uniformly at random.

Calculating Pi



- Assume (x, y) is a point sampled from the square uniformly at random.
- What is the probability that (x, y) is in the circle?

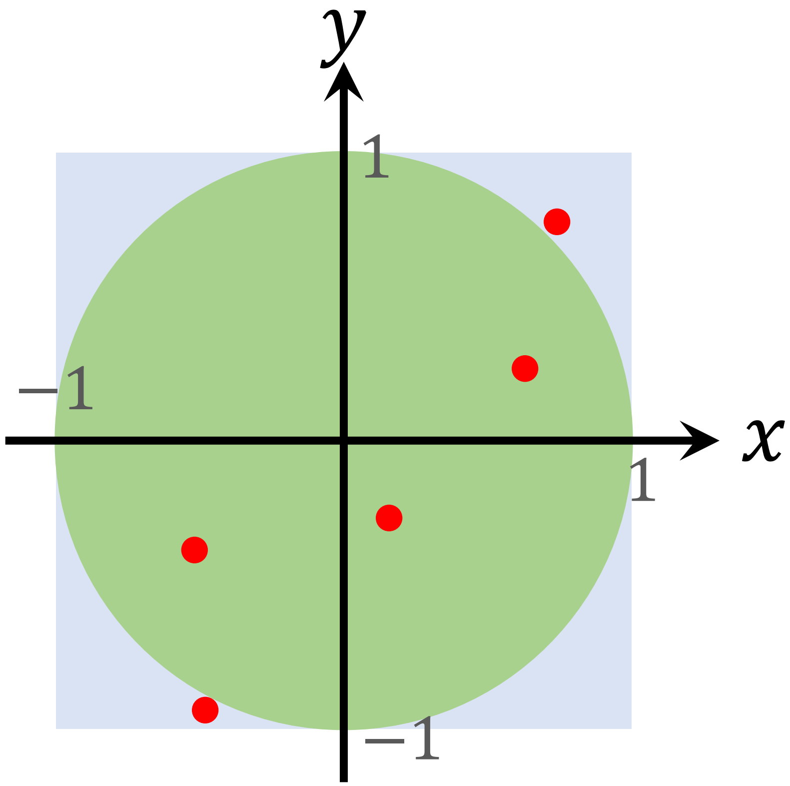
Calculating Pi



- Assume (x, y) is a point sampled from the square uniformly at random.
- What is the probability that (x, y) is in the circle?
- Area of the square is $A_1 = 2^2 = 4$.
- Area of the circle is $A_2 = \pi r^2 = \pi$.
- Probability: $P = \frac{A_2}{A_1} = \frac{\pi}{4}$.

Calculating Pi

- Suppose n points are uniformly sampled from $[-1,1] \times [-1,1]$.
- Then, in expectation, $Pn = \frac{\pi n}{4}$ points are in the circle.



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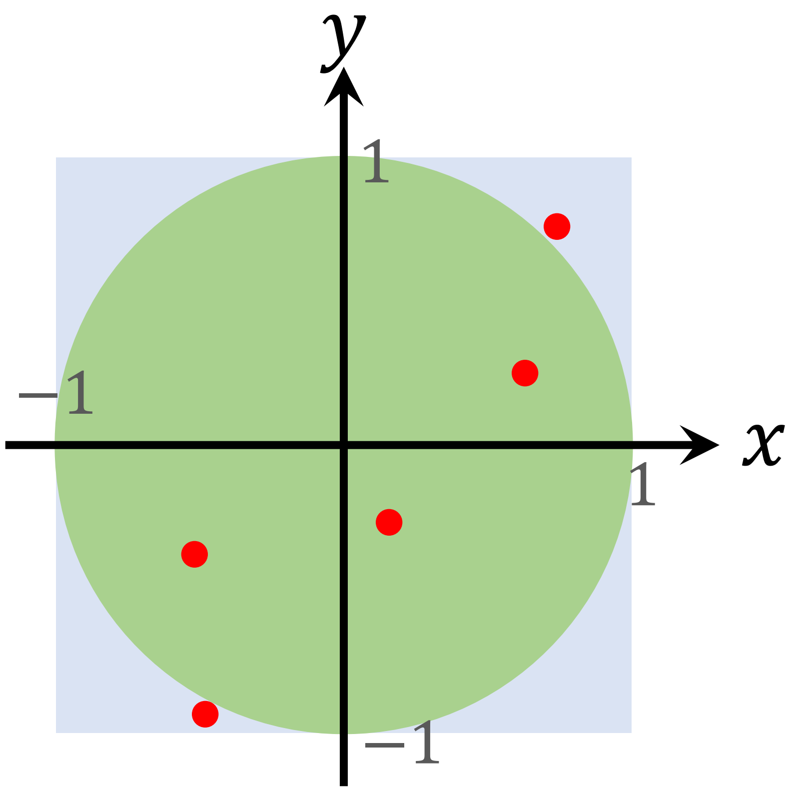

- Given a point (x, y), how do you know whether (x, y) is in the circle?
- If $x^2 + y^2 \leq 1$, then it is in the circle.

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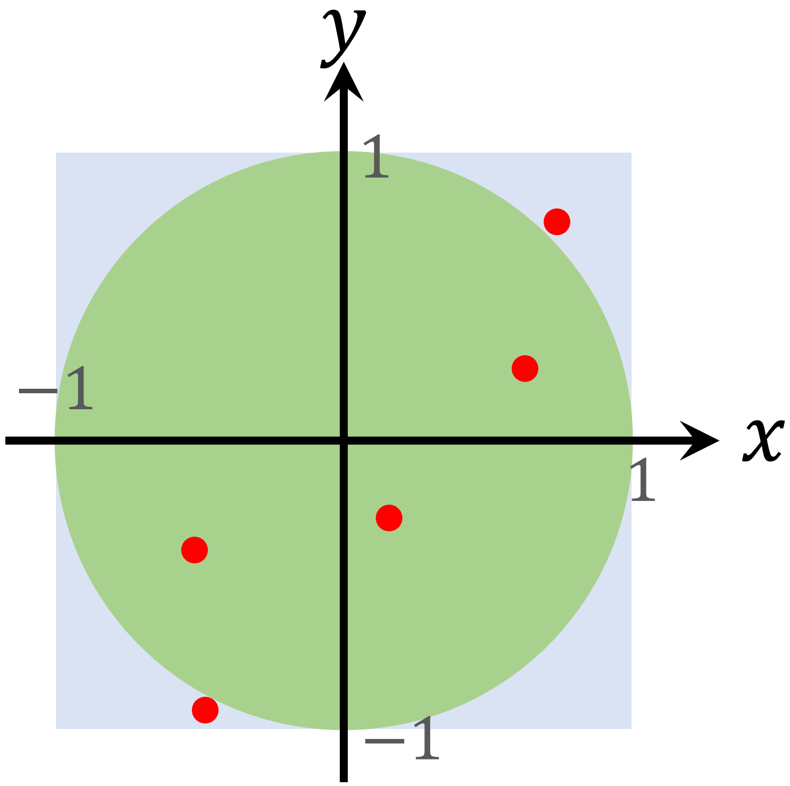

- We found m points in the circle.
- If n is big, then $m \approx \frac{\pi n}{4}$.
- Thus, $\pi \approx \frac{4m}{n}$.

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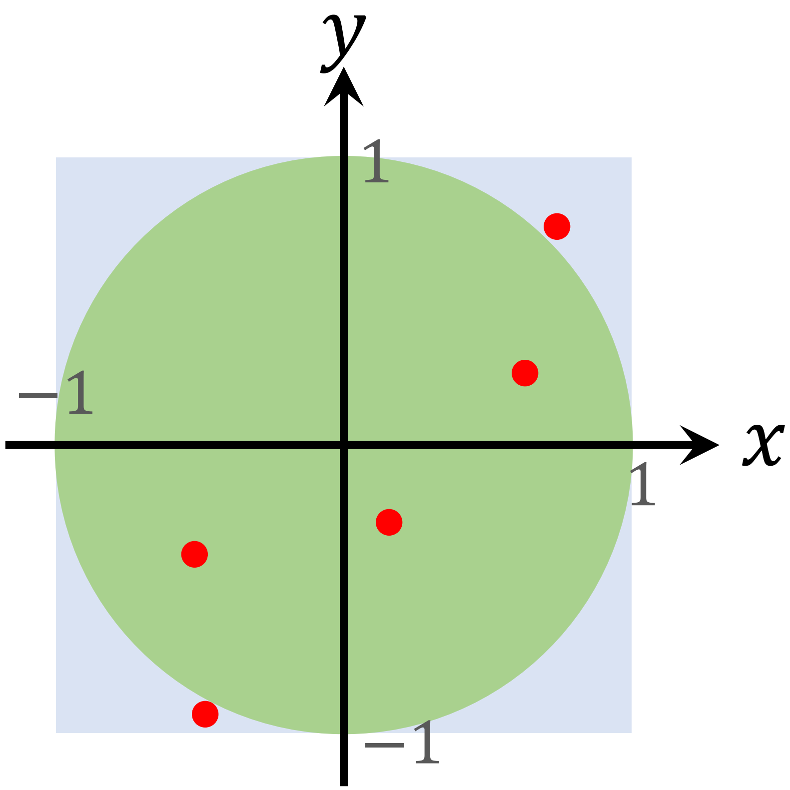

- Law of large numbers:

$$\frac{4m}{n} \to \pi, \quad \text{as } n \to \infty.$$

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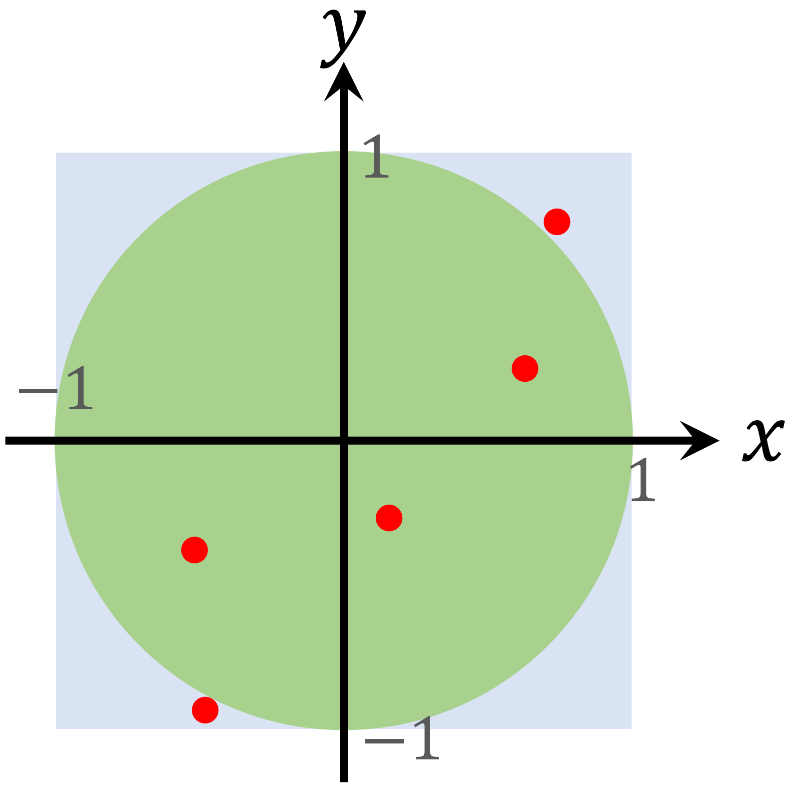

- Law of large numbers:

$$\frac{4m}{n} \to \pi, \quad \text{as } n \to \infty.$$

- Concentration bound:

$$\left|\frac{4m}{n} - \pi\right| = O\left(\frac{1}{\sqrt{n}}\right).$$

Calculating Pi

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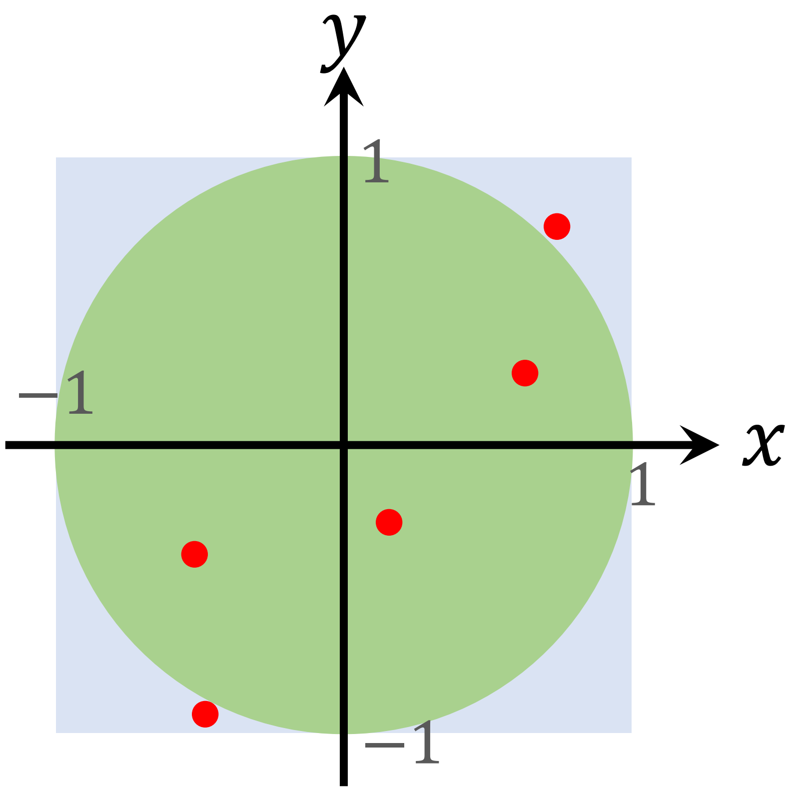

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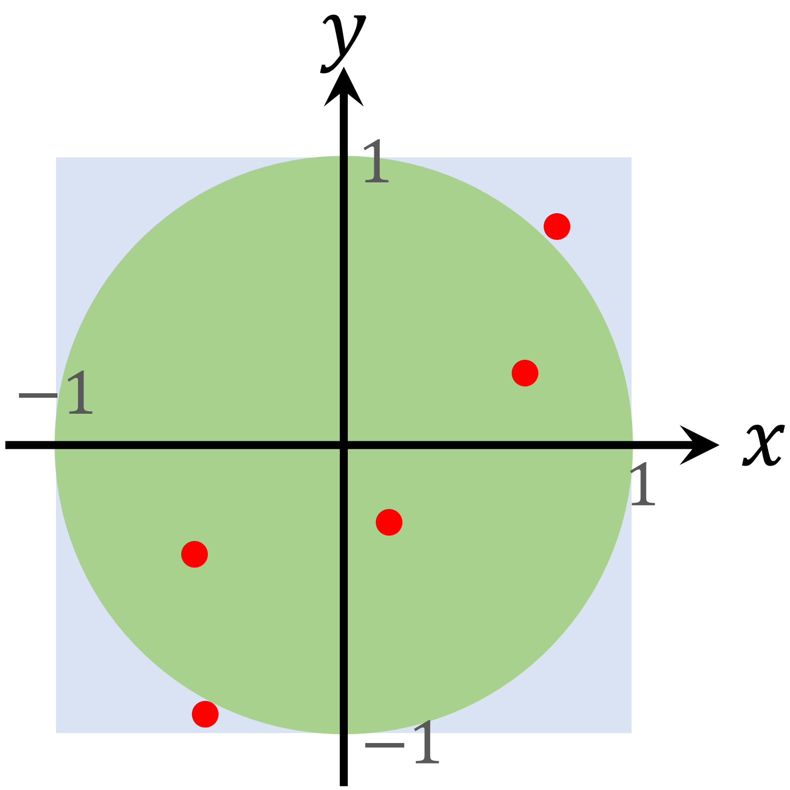

Algorithm

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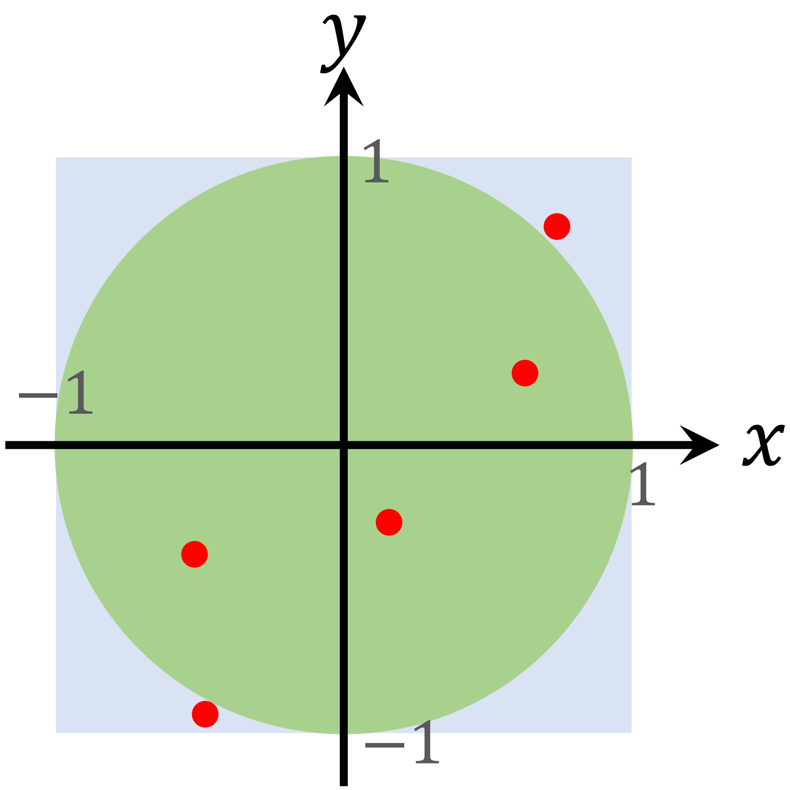


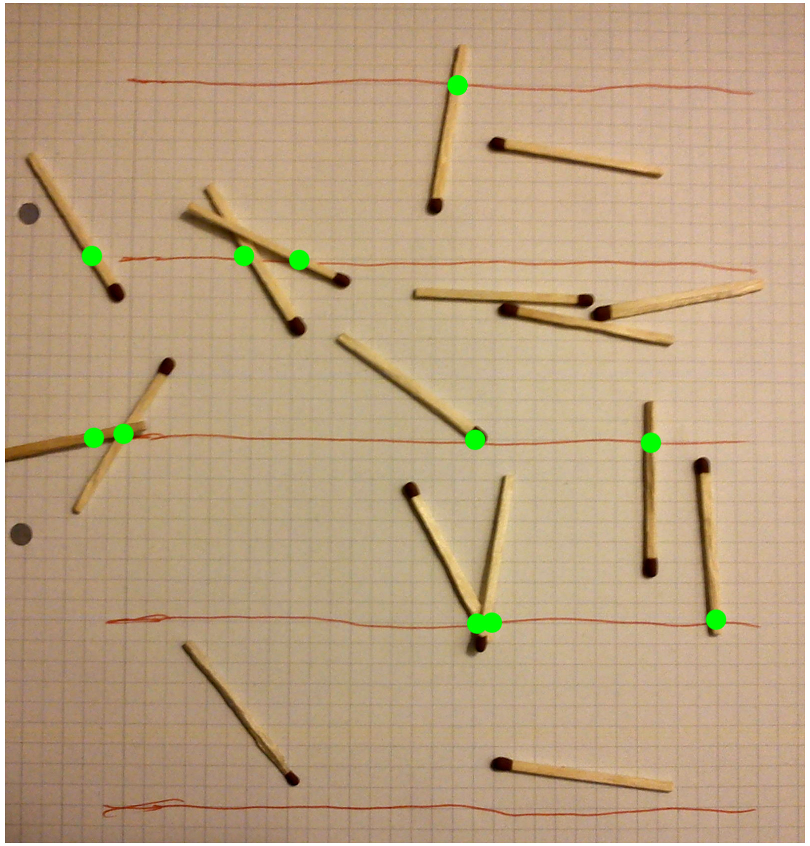
Algorithm

1. User specifies a big n; reset counter $m = 0$.
2. For $i = 1$ to n:
   a) Randomly generate $x \in [-1, 1]$.
   b) Randomly generate $y \in [-1, 1]$.
   c) If $x^2 + y^2 \leq 1$, then $m \leftarrow m + 1$.
3. Return $\pi \approx \frac{4m}{n}$.

Application 2: Buffon's Needle Problem

Buffon's Needle Problem

Buffon, 1707 – 1788
French scientist



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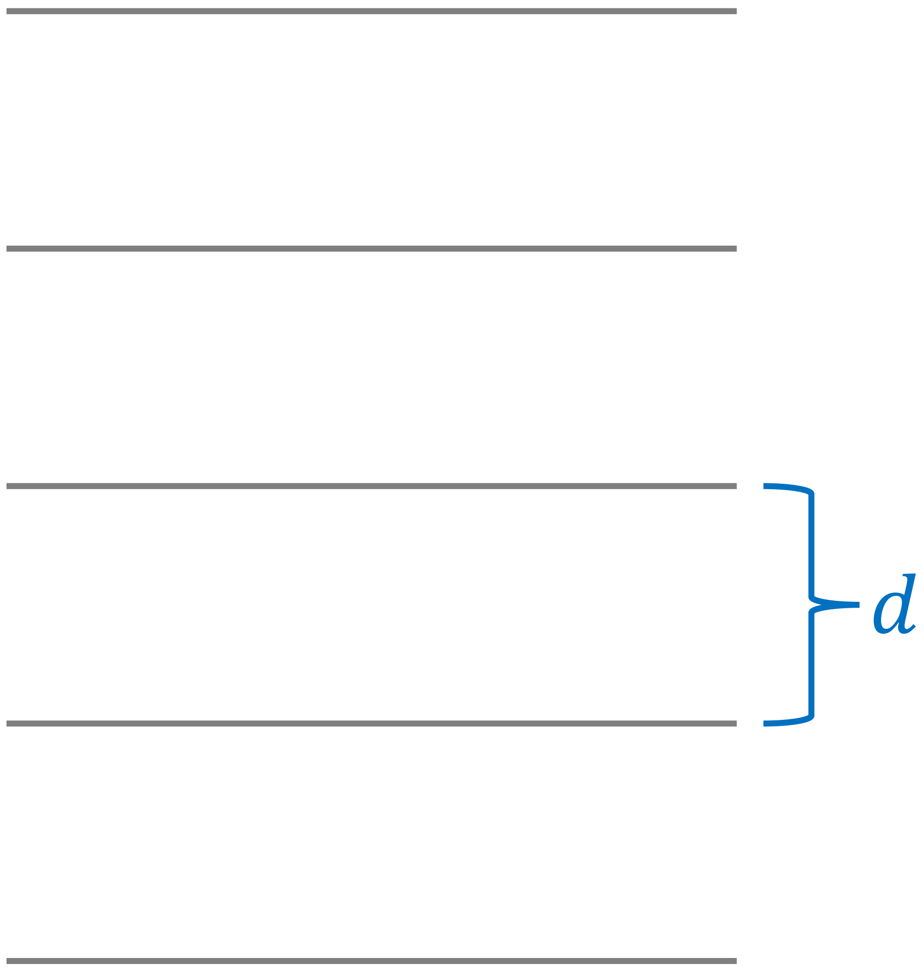
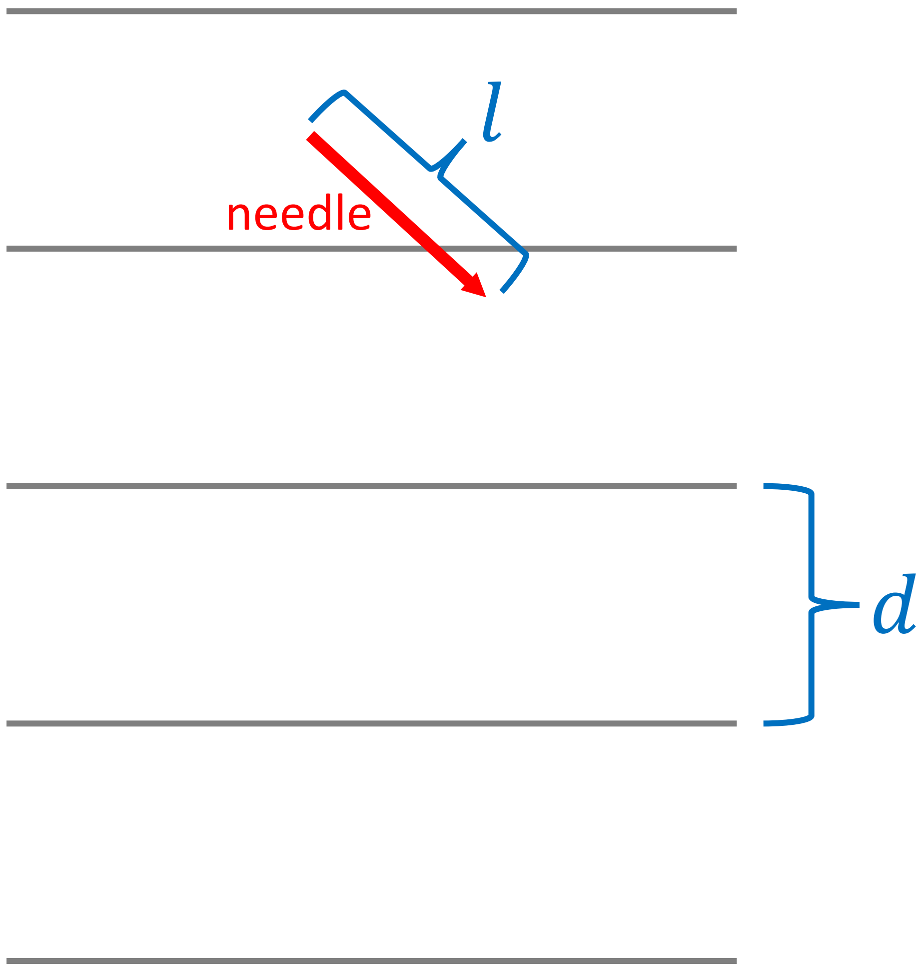


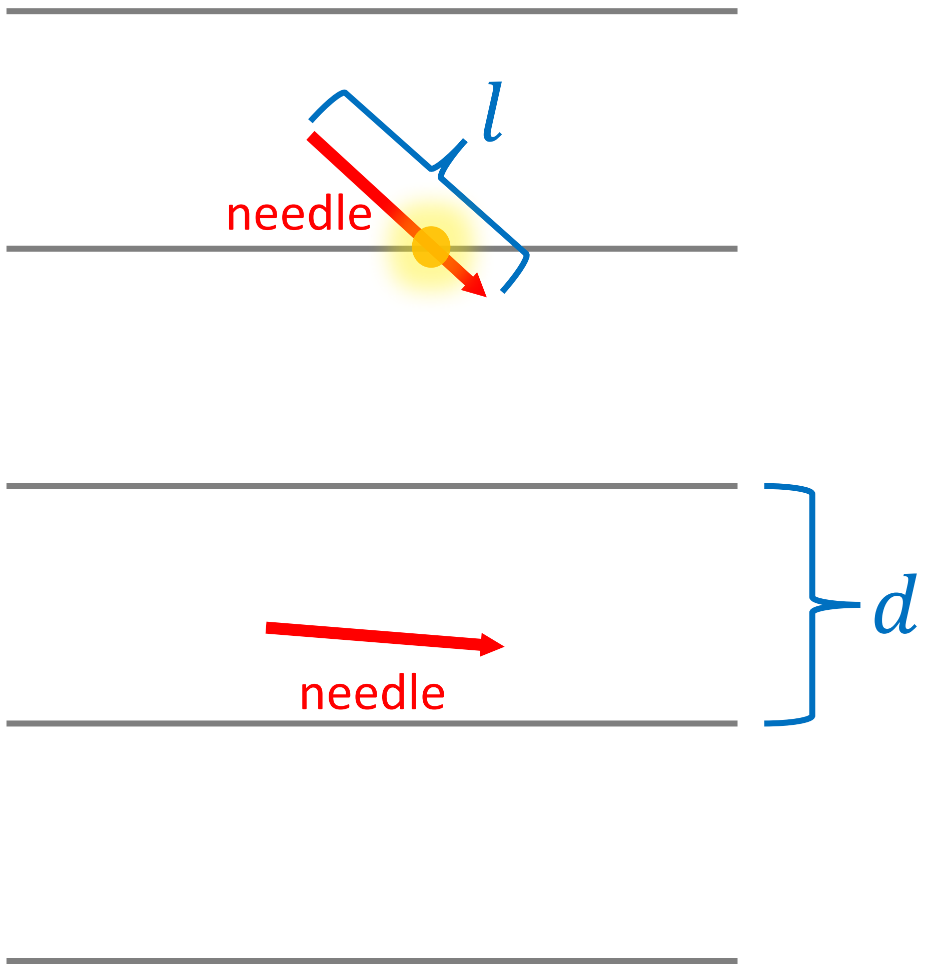
- The parallel lines have distance d.
- Needles have length l.

Buffon's Needle Problem



- The parallel lines have distance d.
- Needles have length l.
- Randomly throw a needle; the needle may or may not lie across a line.

Buffon's Needle Problem



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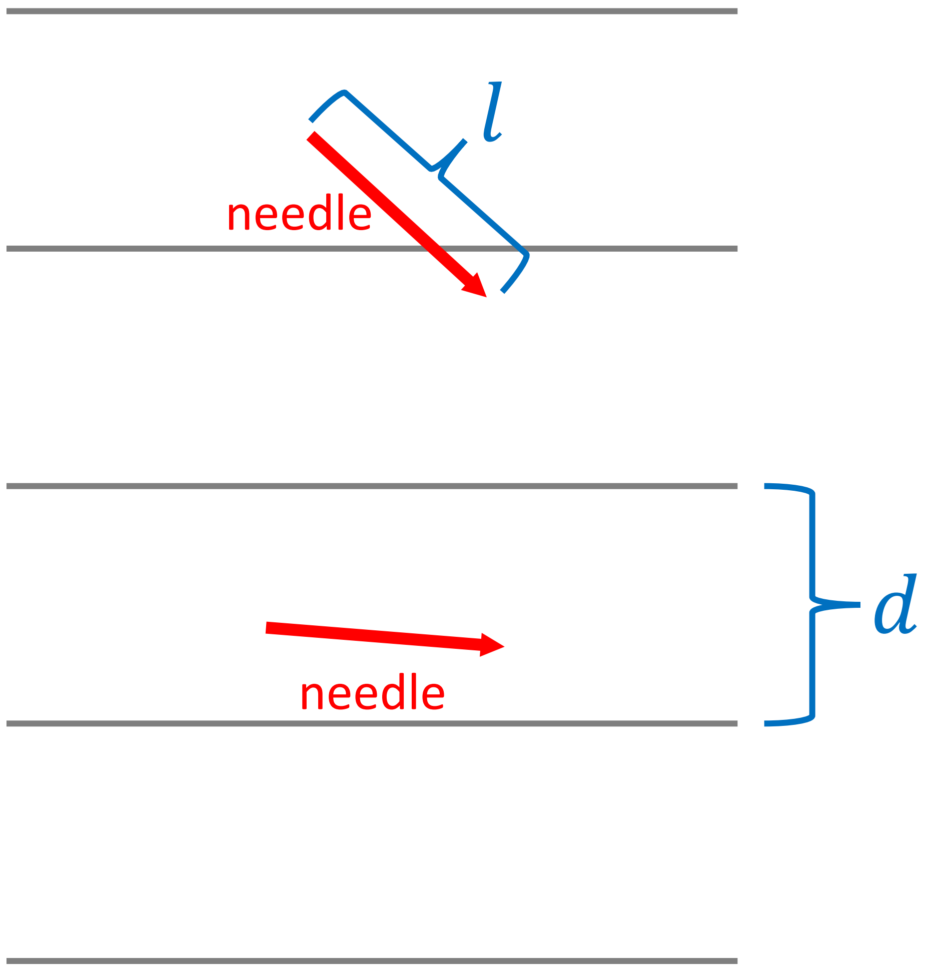
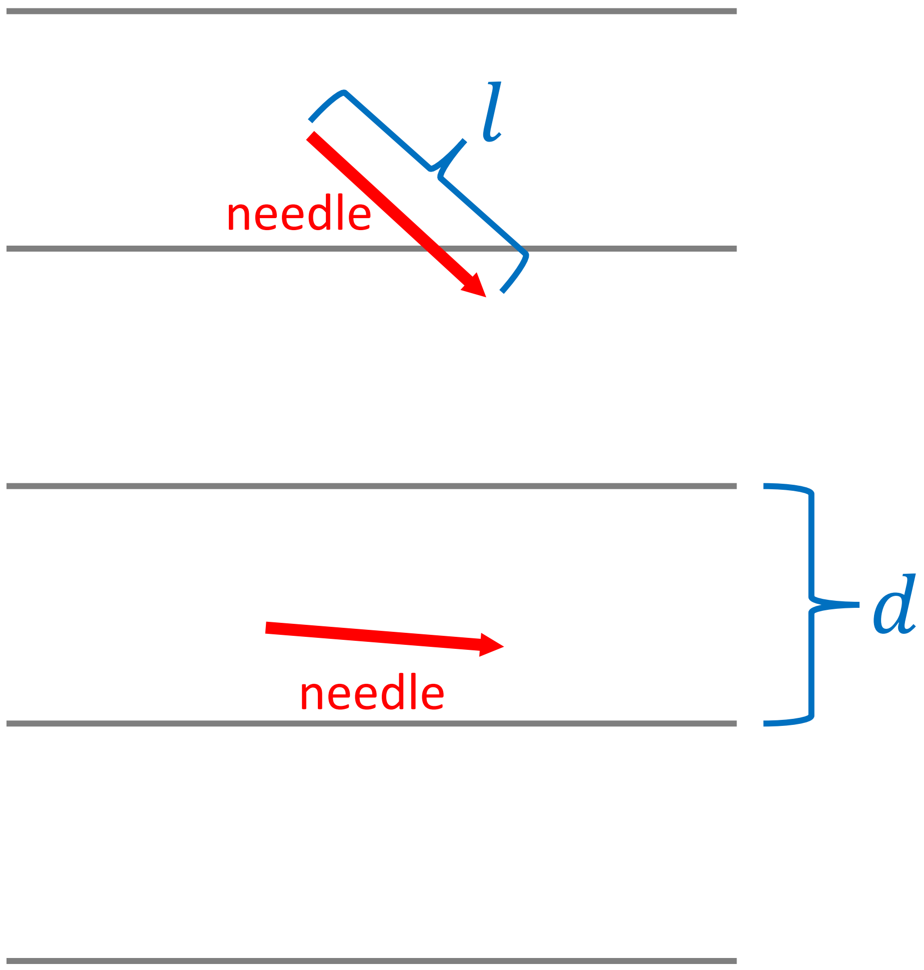


- The parallel lines have distance d.
- Needles have length l.
- Randomly throw a needle; the needle may or may not lie across a line.

With probability $\boxed{P = \frac{2l}{\pi d}}$, they are across.

- It can be proved using integral.

Buffon's Needle Problem



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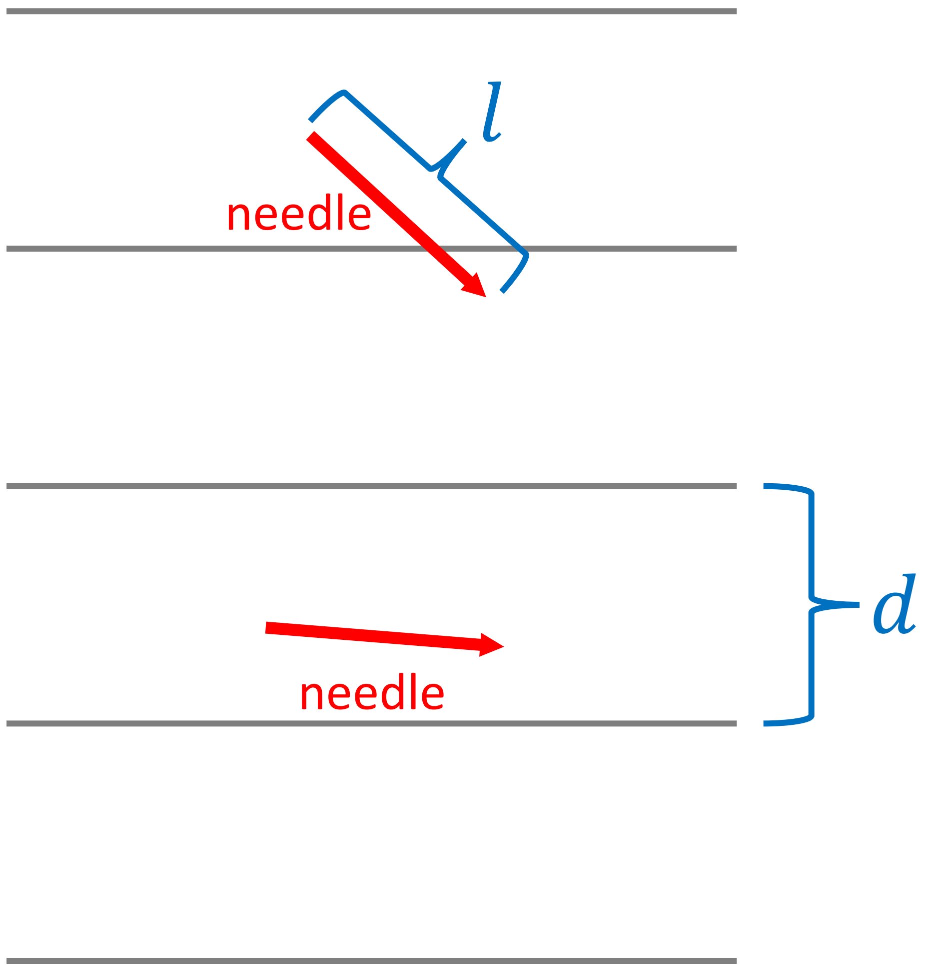
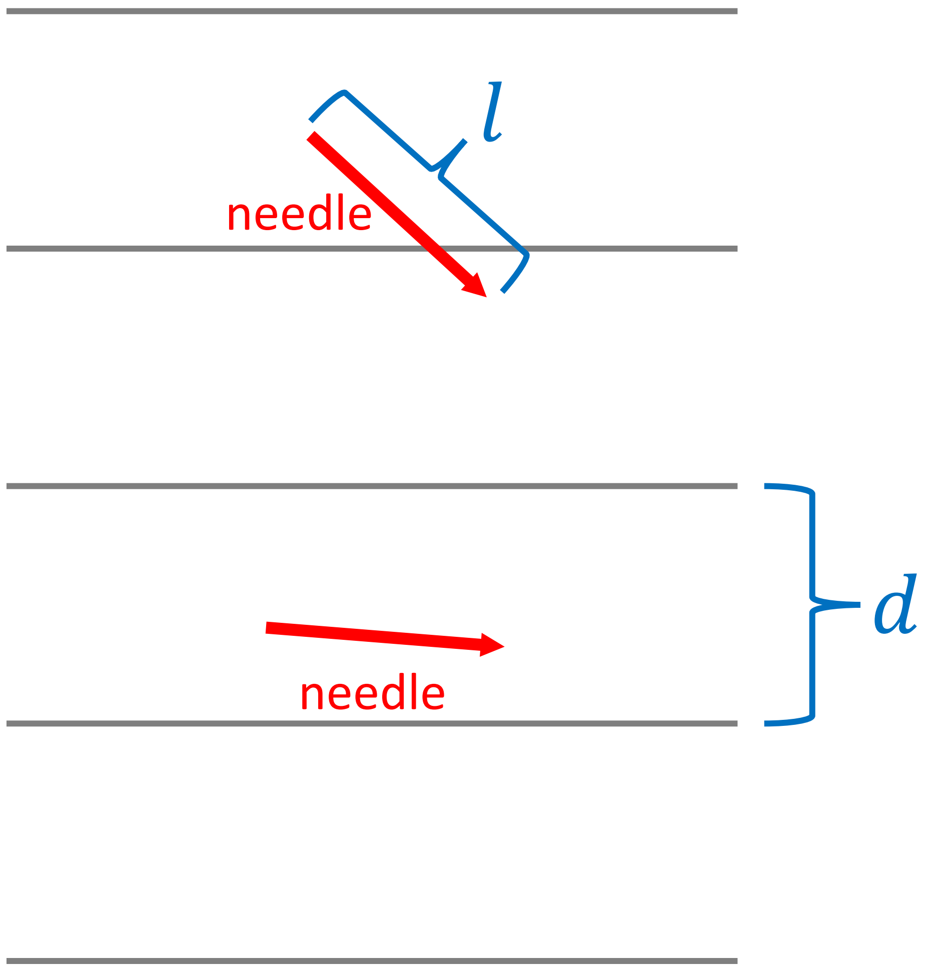


With probability $P = \frac{2l}{\pi d}$, they are across.

- Randomly throw a total of n needles.
- In expectation, $\boxed{\boxed{Pn} = \frac{2ln}{\pi d}}$ needles lie across the lines.

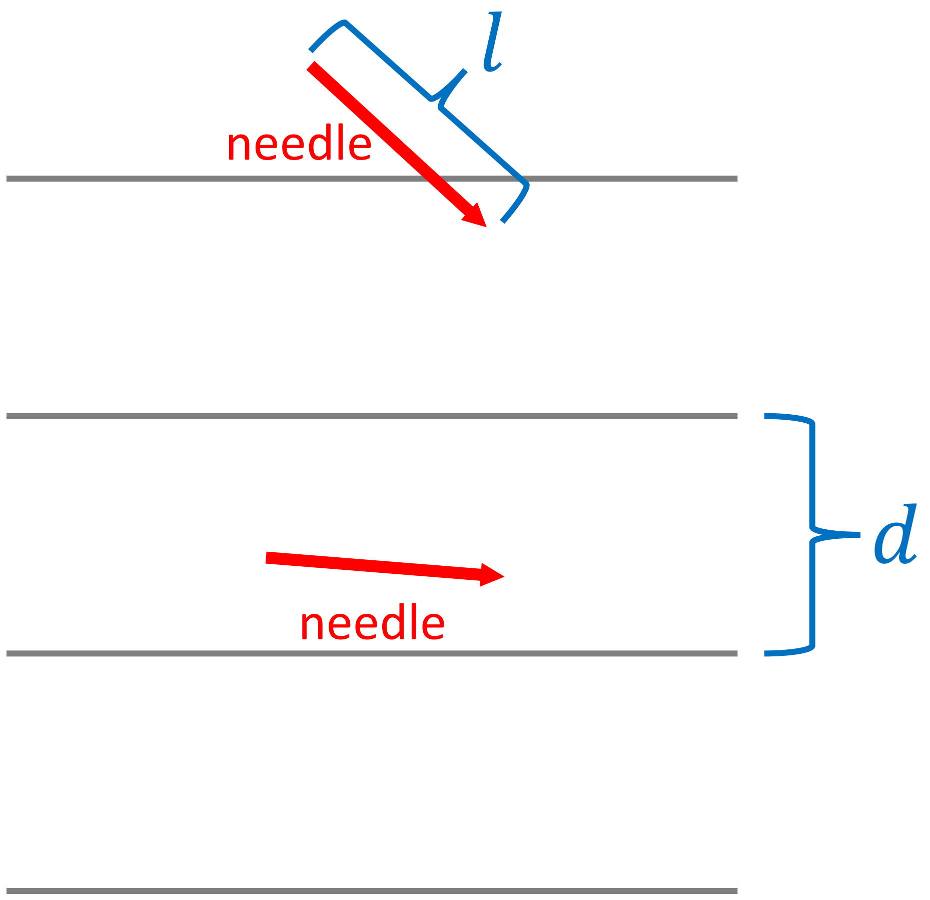
Buffon's Needle Problem



With probability $P = \frac{2l}{\pi d}$, they are across.

- Randomly throw a total of n needles.
- In expectation, $Pn = \frac{2ln}{\pi d}$ needles lie across the lines.
- Actually observe m needles across the lines.
- If n is big, $m \approx \frac{2\,l\,n}{\pi\,d}$.

Buffon's Needle Problem



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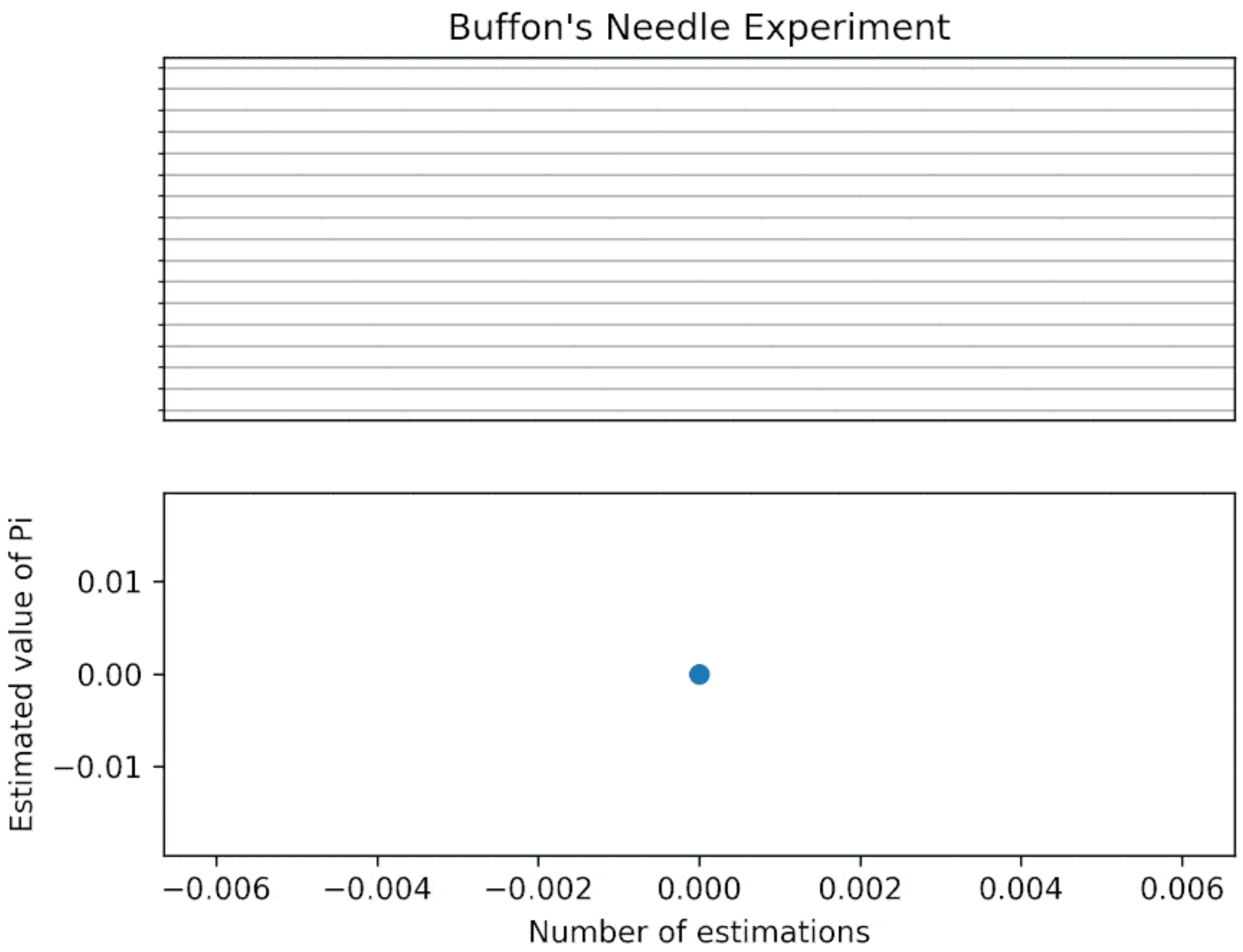
- Actually observe m needles across the lines.
- If n is big, $m \approx \frac{2\,l\,n}{\pi\,d}$.
- Thus, $\pi \approx \frac{2\,l\,n}{d\,m}$.

Buffon's Needle Problem

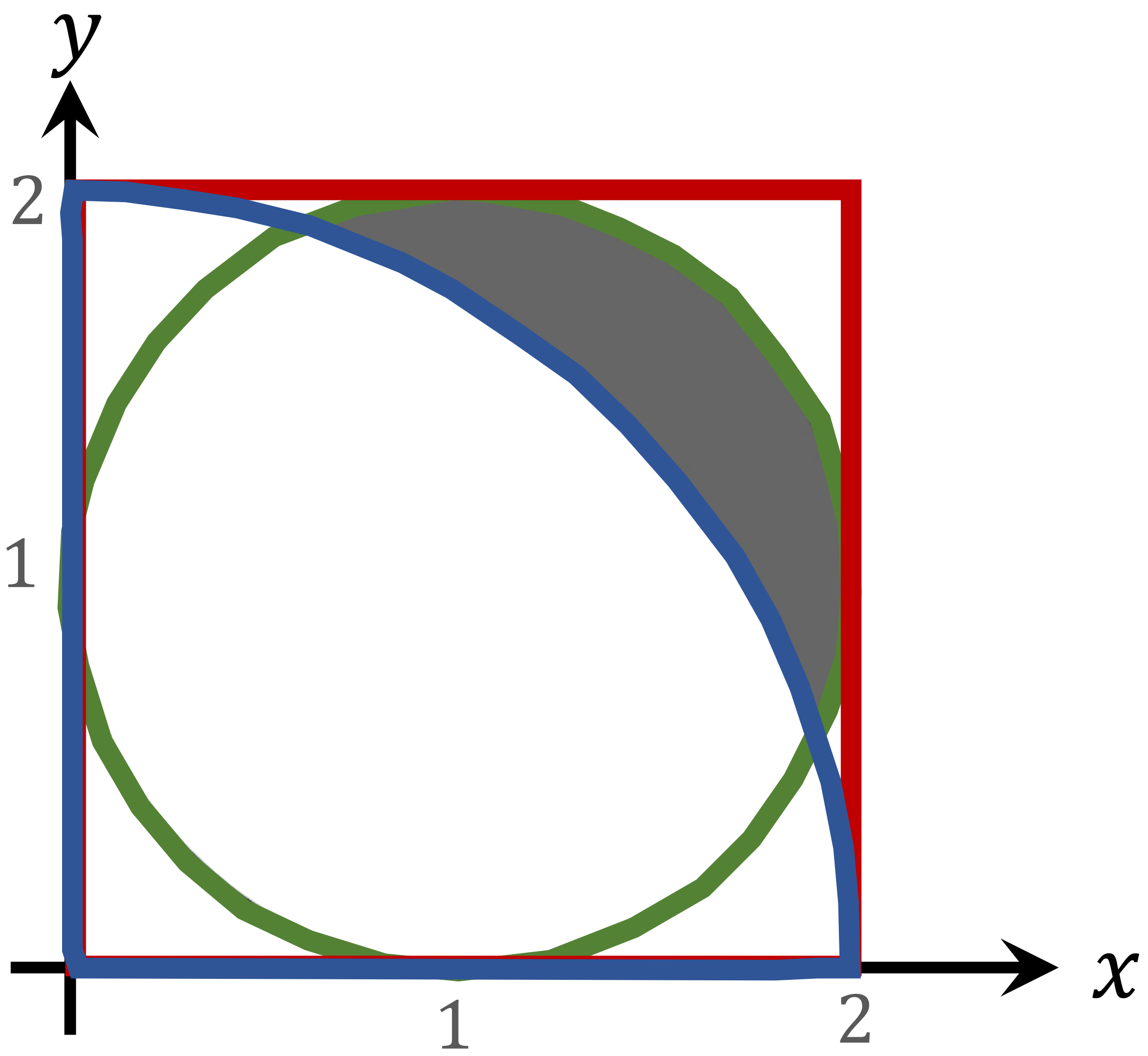
Researcher	Year	$n =$	$m =$	Estimate of π
Wolf	1850	5000	2532	3.1596
Smith	1855	3204	1218	3.1554
De Morgan	1860	600	382	3.137
Fox	1884	1030	489	3.1595
Lazzerini	1901	3408	1808	3.1415929
Reina	1925	2520	859	3.1795

Buffon's Needle Problem



Application 3: Area of A Region

What is the area of the grey region?



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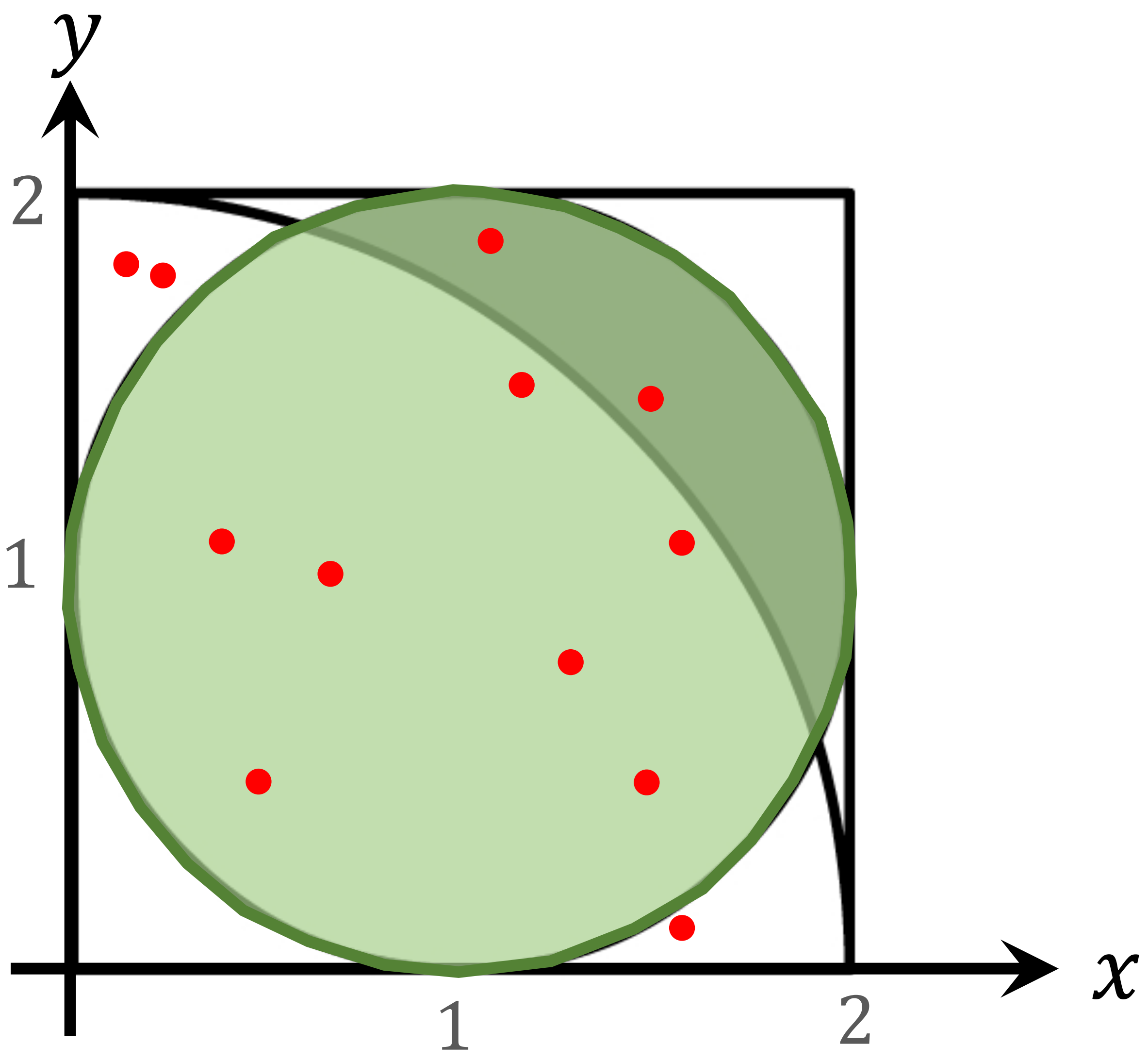

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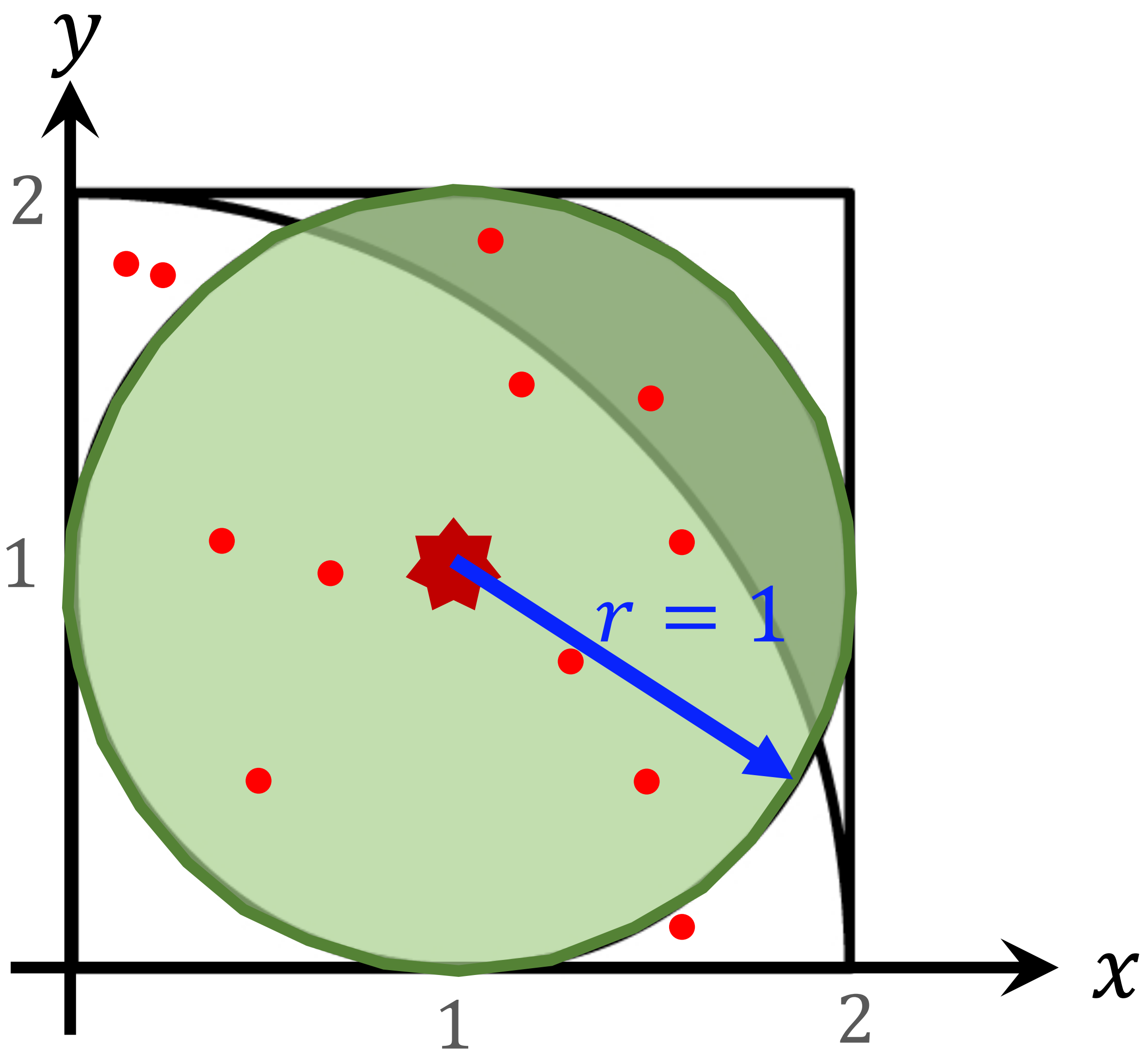

- If a point (x, y) is in the circle, it must satisfy

$$(x-1)^2 + (y-1)^2 \leq 1.$$

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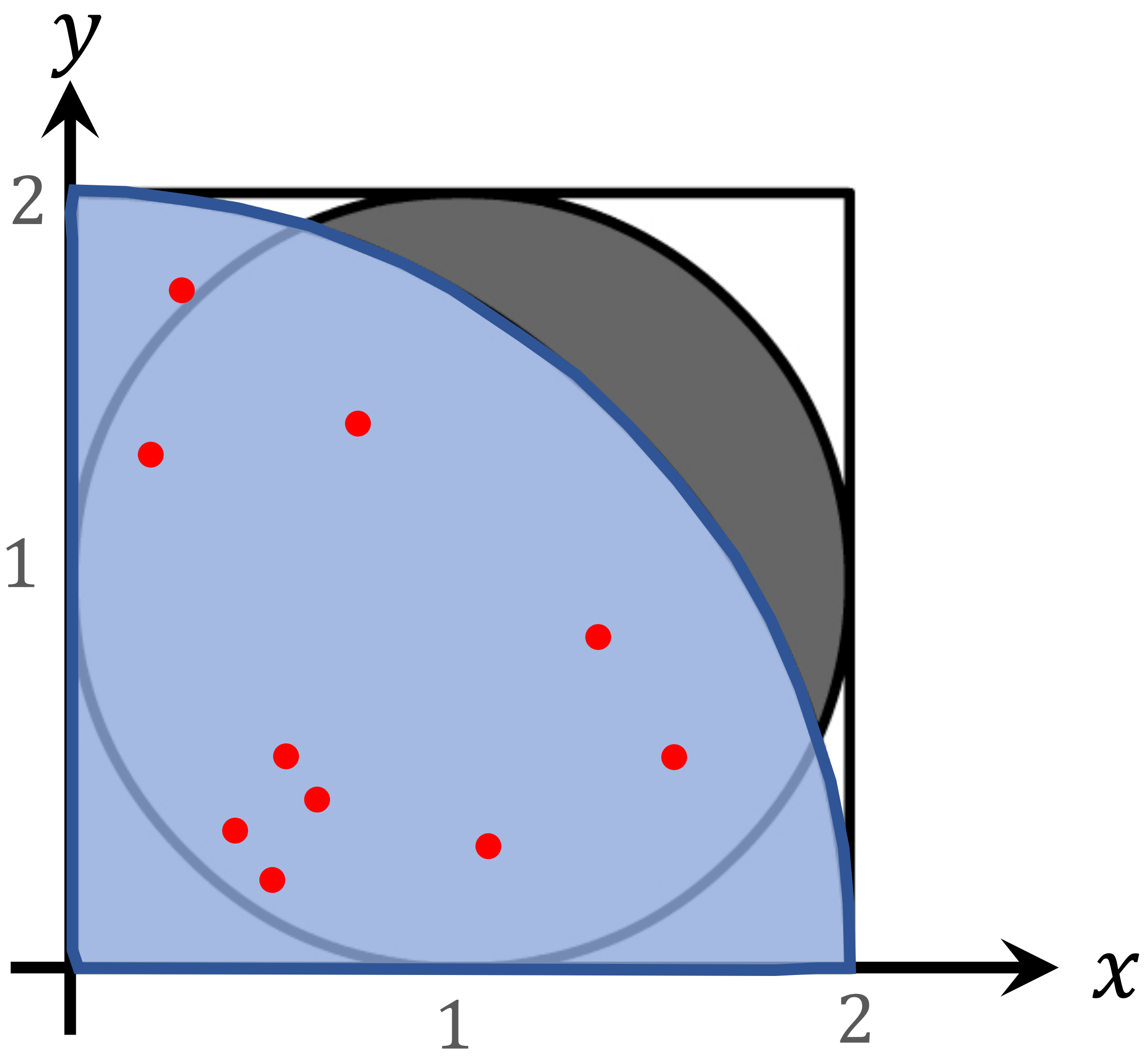


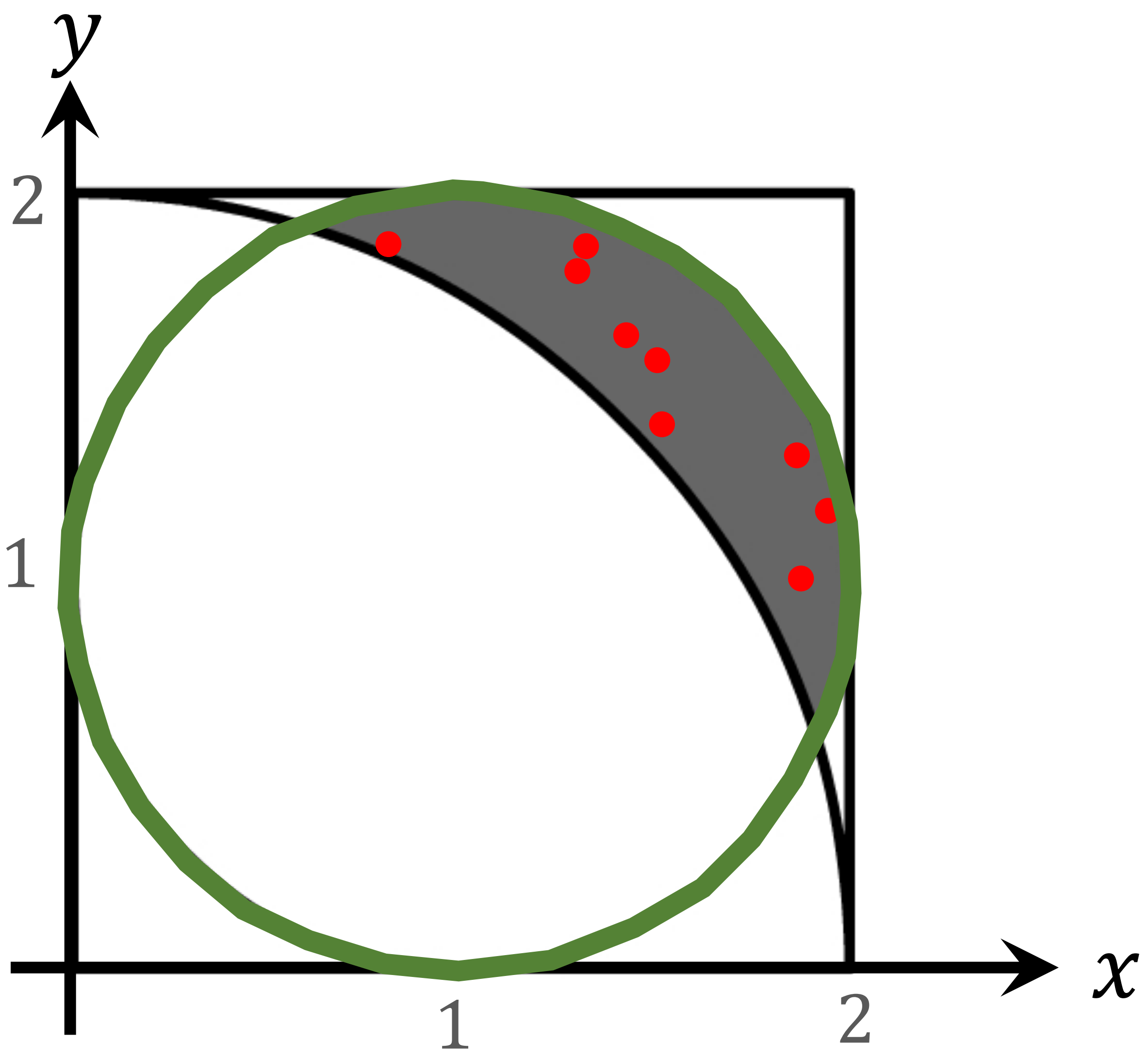
- If a point (x, y) is in the circle, it must satisfy

$$(x-1)^2 + (y-1)^2 \le 1.$$

- If a point (x, y) is in the quarter circle, it must satisfy

$$x^2 + y^2 \le 2^2.$$

What is the area of the grey region?



- A point (x, y) in the grey region satisfies both of

  1. $\boxed{(x-1)^2 + (y-1)^2 \leq 1,}$
  2. $x^2 + y^2 > 2^2.$

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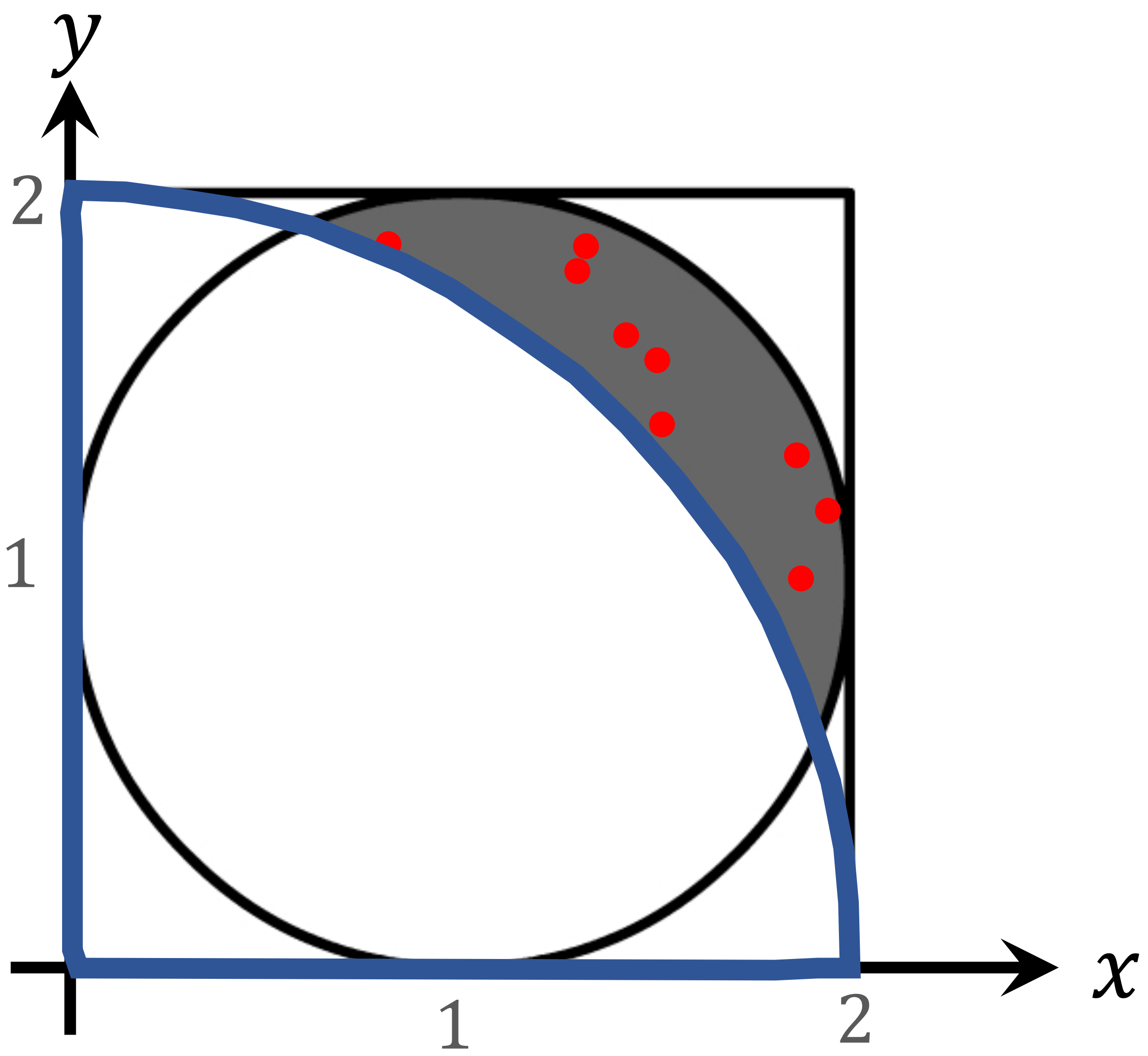

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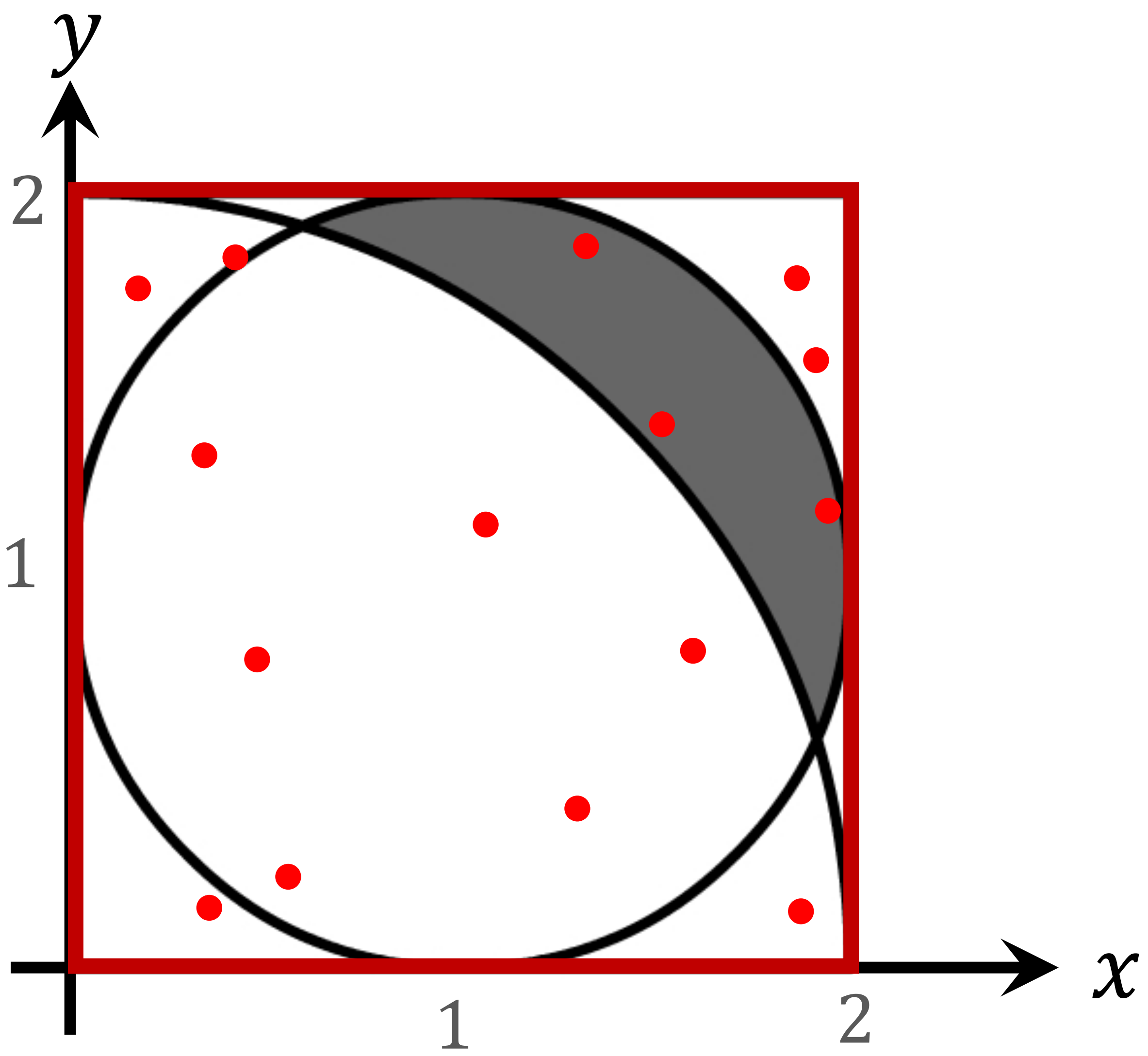
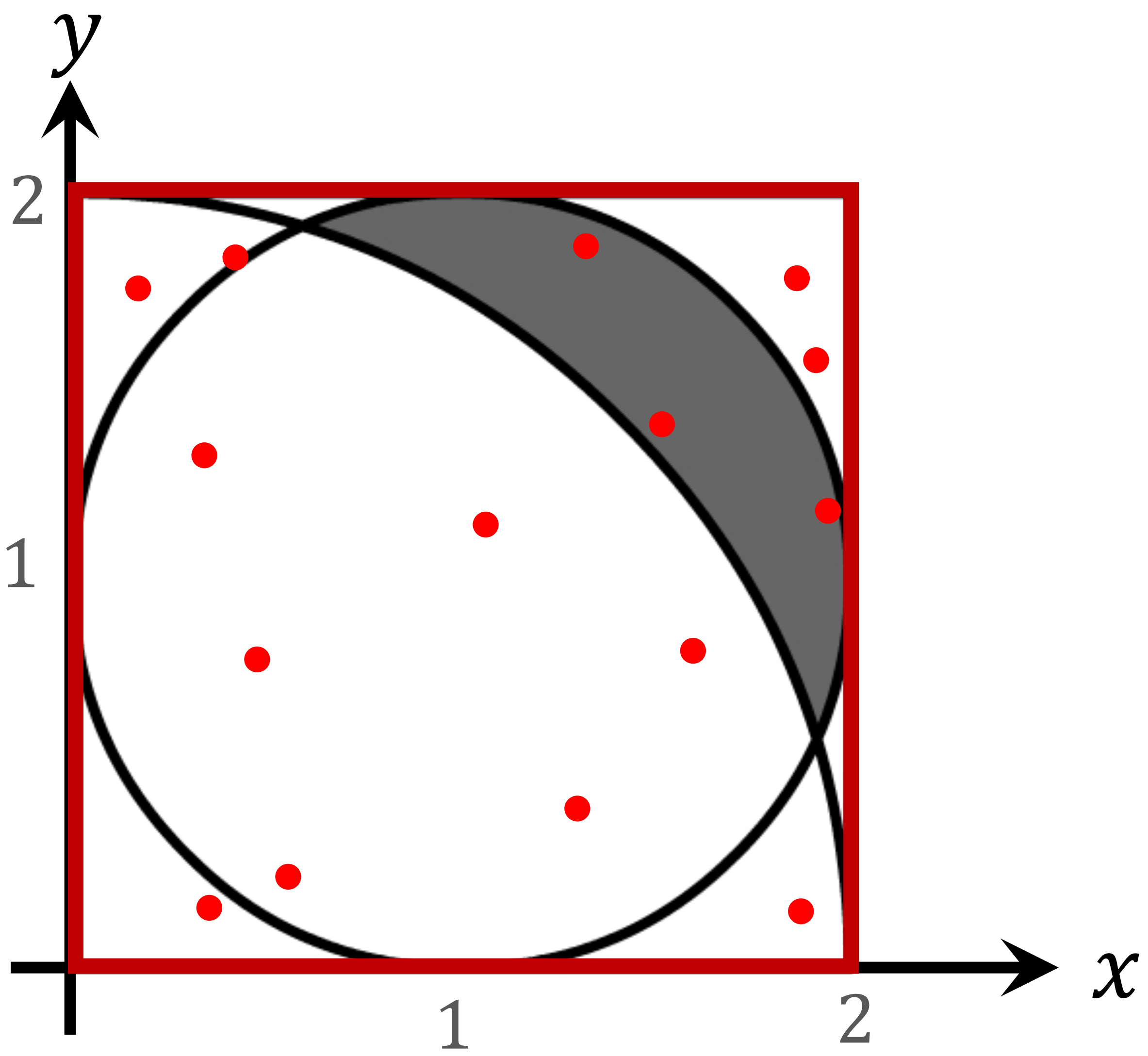


- Area of the square: $A_1 = 2^2 = 4.$

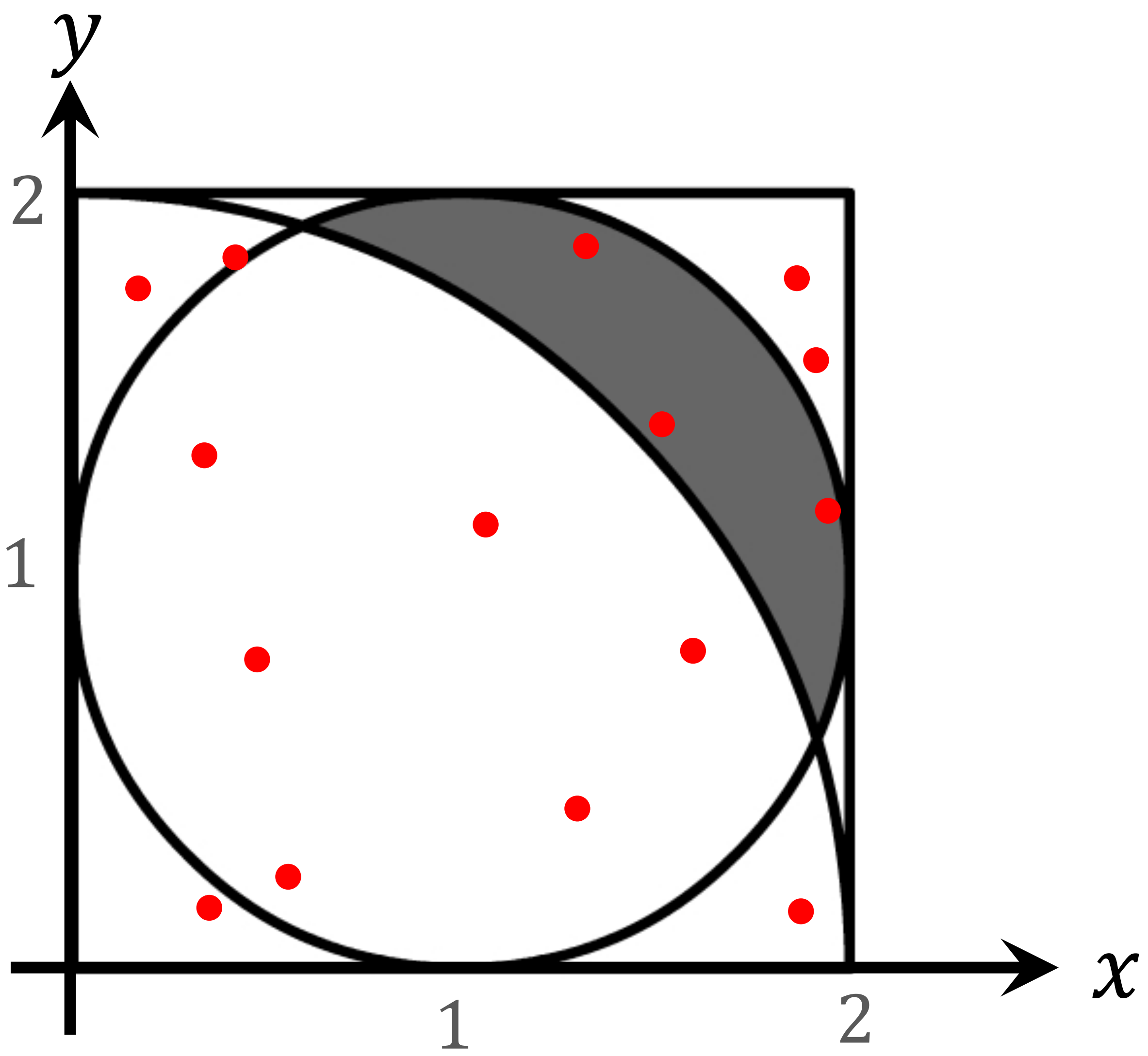
What is the area of the grey region?



- Area of the square: $A_1 = 2^2 = 4$.
- Area of the grey region: $\boxed{A_2}$.
- A point uniformly sampled from the square falls in the grey region w.p.

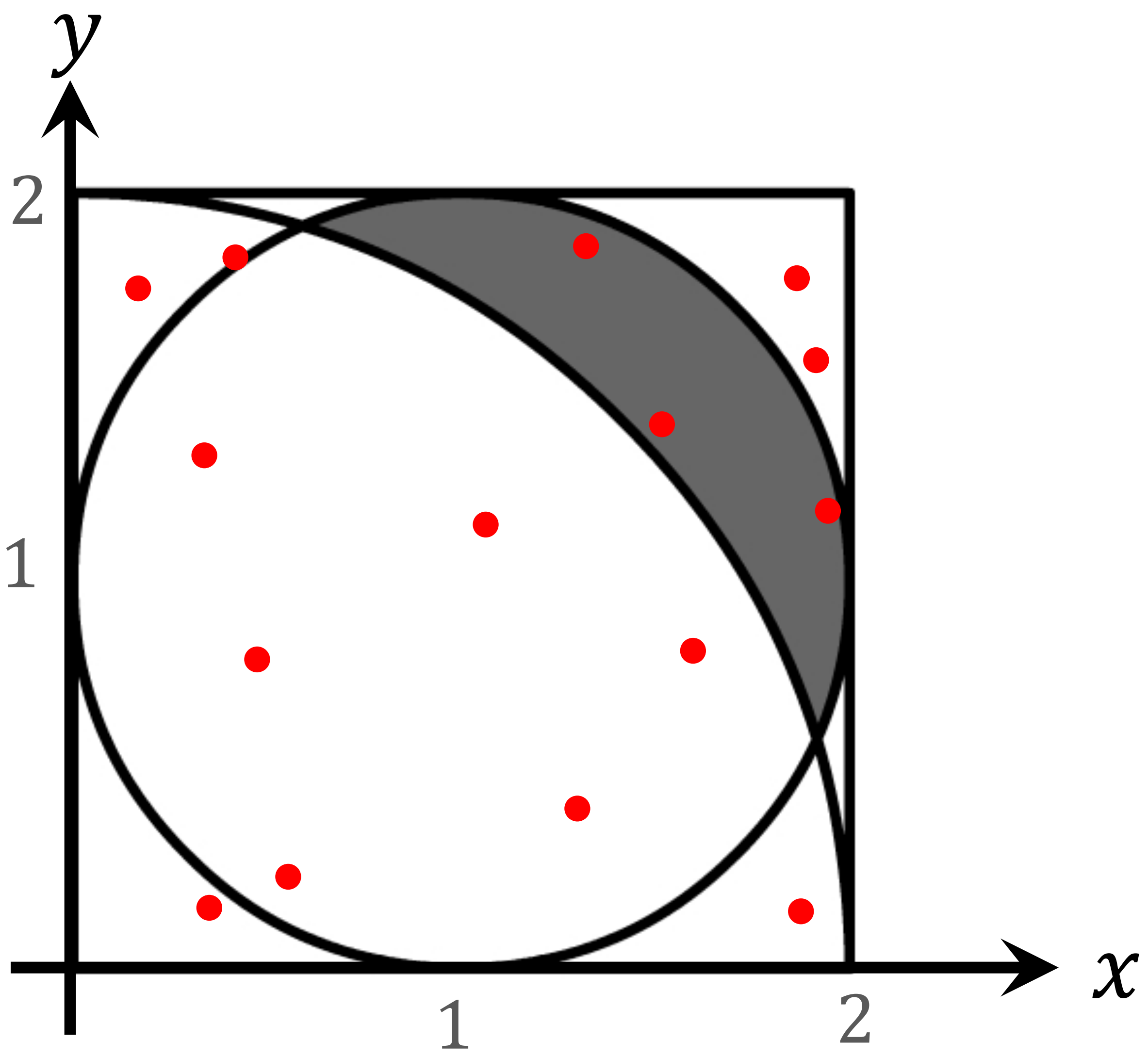
$$P = \frac{A_2}{A_1} = \frac{A_2}{4}.$$

What is the area of the grey region?



- Uniformly sample n points from the square $[0,2]\times[0,2]$.
- In expectation, $\boxed{nP} = \boxed{\frac{n\,A_2}{4}}$ points fall in the grey region.

What is the area of the grey region?



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- We actually observe m points in the grey region.
- If n is big, then $\boxed{m \approx \frac{n\,A_2}{4}}$
- Thus, $\boxed{A_2 \approx \frac{4m}{n}}$.

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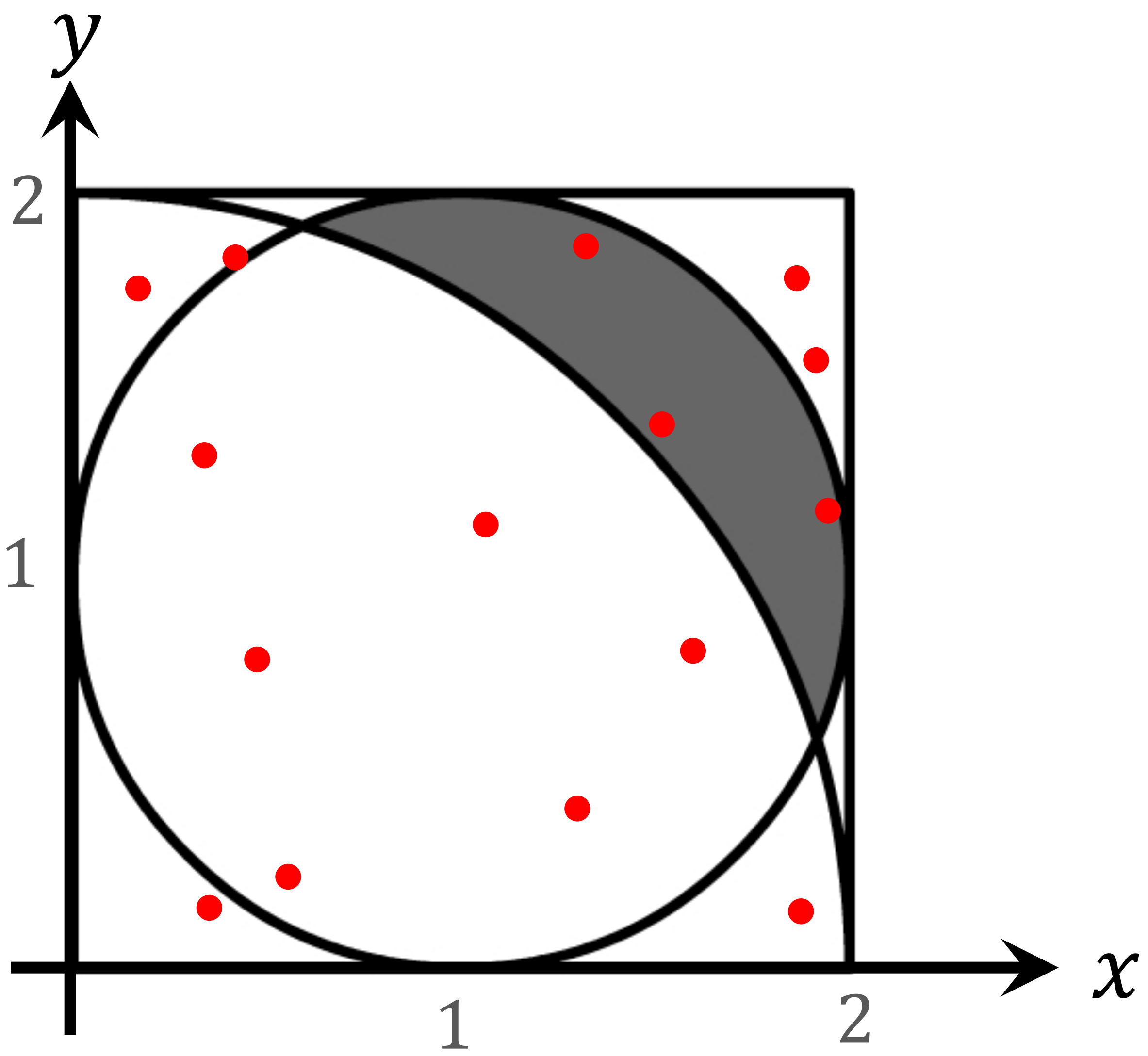

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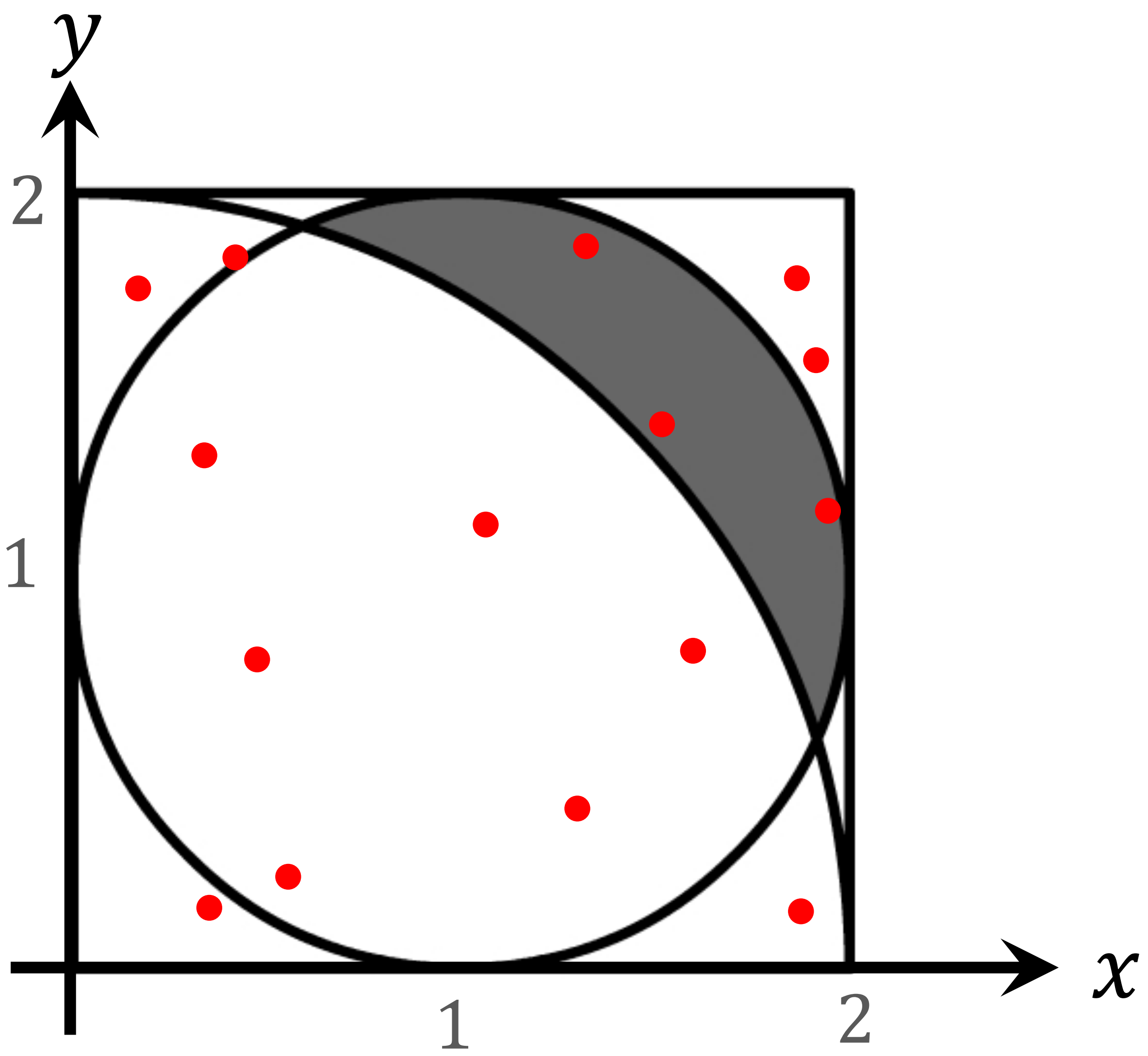

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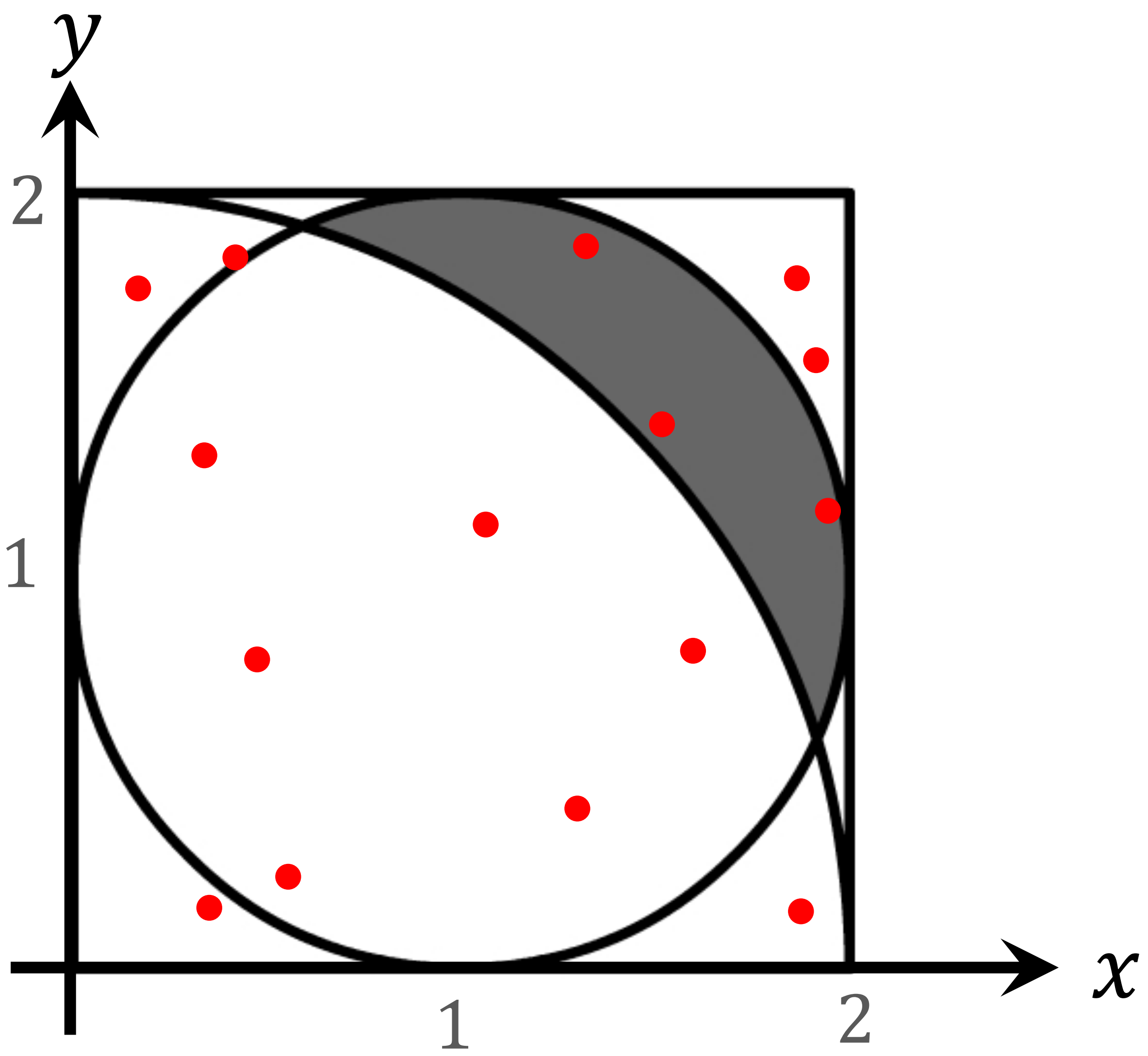

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   c) If both of the following conditions are satisfied, then $m \leftarrow m + 1$:
      i. $(x - 1)^2 + (y - 1)^2 \leq 1$.
      ii. $x^2 + y^2 > 2^2$.
3. Return $\text{area} \approx \frac{4m}{n}$.

Application 4: Integration

Integration

- We are given a function, e.g. $\boxed{f(x) = \frac{1}{1+\sin(x)\cdot(\log_{\mathrm{e}} x)^2}.}$
- Calculate the integral: $I = \underline{\int_{0.8}^{3} f(x)\, dx}.$

Integration

- We are given a function, e.g., $f(x) = \frac{1}{1+\sin(x)\cdot(\log_{\mathrm{e}} x)^2}$.
- Calculate the integral: $I = \int_{0.8}^{3} f(x)\, dx$.
- If $f(x)$ is very involved, there is no way to analytically calculate the integral.
- Using Monte Carlo to approximate the integral.

Monte Carlo Integration (Univariate)

Task: Given a univariate function $f(x)$, calculate $\boxed{I = \int_a^b f(x)\,dx}$

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Task: Given a univariate function $f(x)$, calculate $I = \int_a^b f(x)\, dx$.

1. Draw n samples from $[a, b]$ uniformly at random; denote them by $\underline{x_1, \cdots, x_n}$.

Monte Carlo Integration (Univariate)

Task: Given a univariate function $f(x)$, calculate $I = \int_a^b f(x)\, dx$.

1. Draw n samples from $[a, b]$ uniformly at random; denote them by $x_1, \cdots, x_n$.
2. Calculate $\boxed{Q_n} = \boxed{(b-a)} \cdot \boxed{\frac{1}{n}\sum_{i=1}^{n}} \boxed{f(x_i)}$

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2. Calculate $Q_n = (b - a) \cdot \frac{1}{n} \sum_{i=1}^{n} f(x_i)$.
3. Return $\boxed{Q_n}$ as an approximation to the integral $\boxed{I = \int_a^b f(x)\, dx}$

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3. Return Q_n as an approximation to the integral $I = \int_a^b f(x)\, dx$.

Theory: Law of large numbers guarantees $\boxed{Q_n \to I}$, as $\boxed{n \to \infty}$

Monte Carlo Integration (Univariate): Example

Task: Given function $\boxed{f(x) = \frac{1}{1+\sin(x)\cdot(\log_e x)^2},}$ calculate $\boxed{I = \int_{0.8}^{3} f(x)\,dx}$

Monte Carlo Integration (Univariate): Example

Task: Given function $f(x) = \frac{1}{1+\sin(x)\cdot(\log_e x)^2}$, calculate $I = \int_{0.8}^{3} f(x)\, dx$.

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Monte Carlo Integration (Univariate): Example

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1. Draw n samples from $[0.8, 3]$ uniformly at random; denote them by $x_1, \cdots, x_n$.
2. Calculate $Q_n = \boxed{2.2} \boxed{\frac{1}{n}\sum_{i=1}^{n}} \boxed{f(x_i)}$

Monte Carlo Integration (Univariate): Example

Task: Given function $f(x) = \frac{1}{1+\sin(x)\cdot(\log_e x)^2}$, calculate $I = \int_{0.8}^{3} f(x)\, dx$.

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2. Calculate $Q_n = 2.2 \cdot \frac{1}{n} \sum_{i=1}^{n} f(x_i)$.
3. Return Q_n as an approximation to the integral $I = \int_{0.8}^{3} f(x)\, dx$.

Monte Carlo Integration (Multivariate)

Task: Given a multivariate function $f(\mathbf{x})$, calculate $I = \int_\Omega f(\mathbf{x})\, d\mathbf{x}$.

$\mathbf{x} \in \mathbb{R}^d$ is a vector

Ω is a subset of $\mathbb{R}^d$

Monte Carlo Integration (Multivariate)

Task: Given a multivariate function $f(\mathbf{x})$, calculate $I = \int_\Omega f(\mathbf{x})\, d\mathbf{x}$.

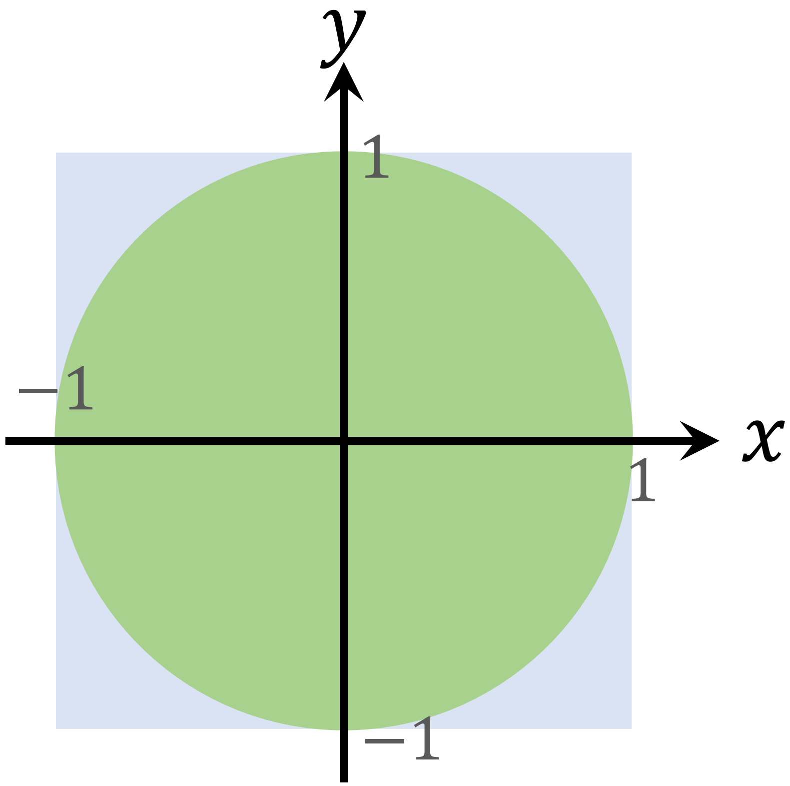
1. Draw n samples from set Ω uniformly at random; denote them by $\underline{\mathbf{x}_1, \cdots, \mathbf{x}_n}$.
2. Calculate $V = \underline{\int_\Omega d\mathbf{x}}$.

Monte Carlo Integration (Multivariate)

Task: Given a multivariate function $f(\mathbf{x})$, calculate $I = \int_\Omega f(\mathbf{x})\, d\mathbf{x}$.

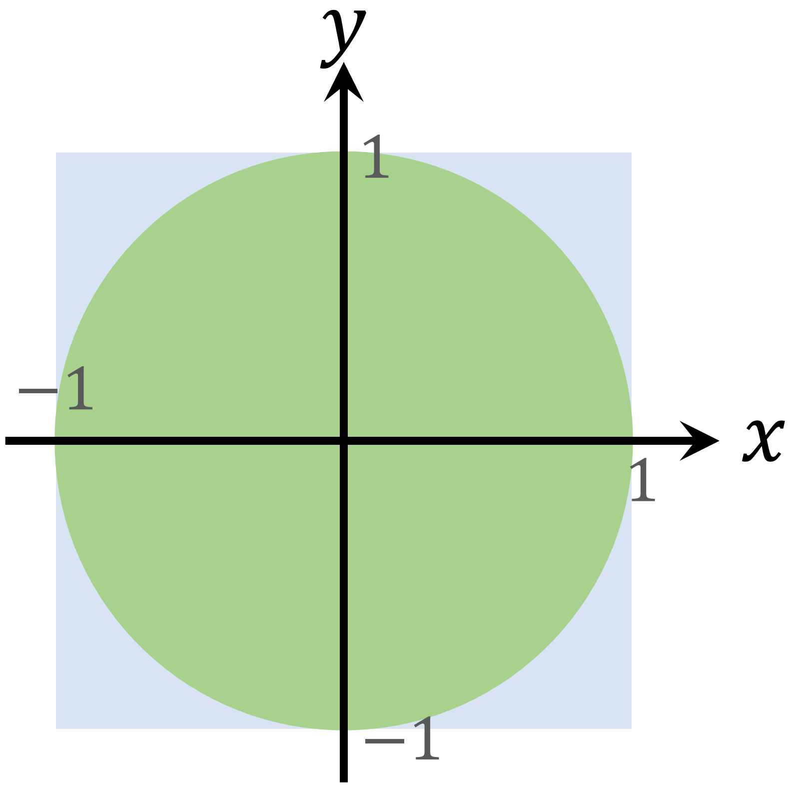
1. Draw n samples from set Ω uniformly at random; denote them by $\mathbf{x}_1, \cdots, \mathbf{x}_n$.
2. Calculate $V = \int_\Omega d\mathbf{x}$.
3. Calculate $\boxed{Q_n} = \boxed{V}\, \boxed{\frac{1}{n}\sum_{i=1}^{n}}\, \boxed{f(\mathbf{x}_i)}$
4. Return Q_n as an approximation to the integral $I = \int_\Omega f(\mathbf{x})\, d\mathbf{x}$.

Monte Carlo Integration: Bivariate Example



- Let $\boxed{f(x,y)} = \begin{cases} 1, \text{if } x^2 + y^2 \le 1; \\ 0, \quad \text{otherwise.} \end{cases}$

Monte Carlo Integration: Bivariate Example



- Let $f(x, y) = \begin{cases} 1, \text{if } x^2 + y^2 \leq 1; \\ 0, \quad \text{otherwise.} \end{cases}$
- Let $\Omega = [-1, 1] \times [-1, 1]$.
- What is $I = \int_{\Omega} f(x, y)\, dx\, dy$?

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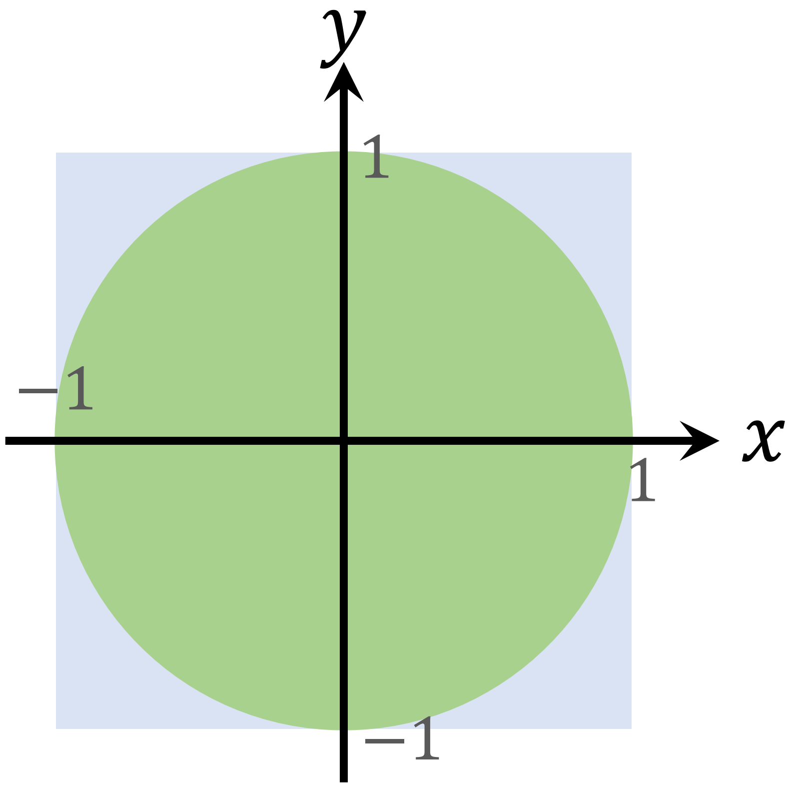

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- Let $\Omega = [-1,1]\times[-1,1]$.
- What is $I = \int_\Omega f(x,y)\, dx\, dy$?
- I is the area of the circle:

$$\boxed{I = \pi r^2 = \pi.}$$

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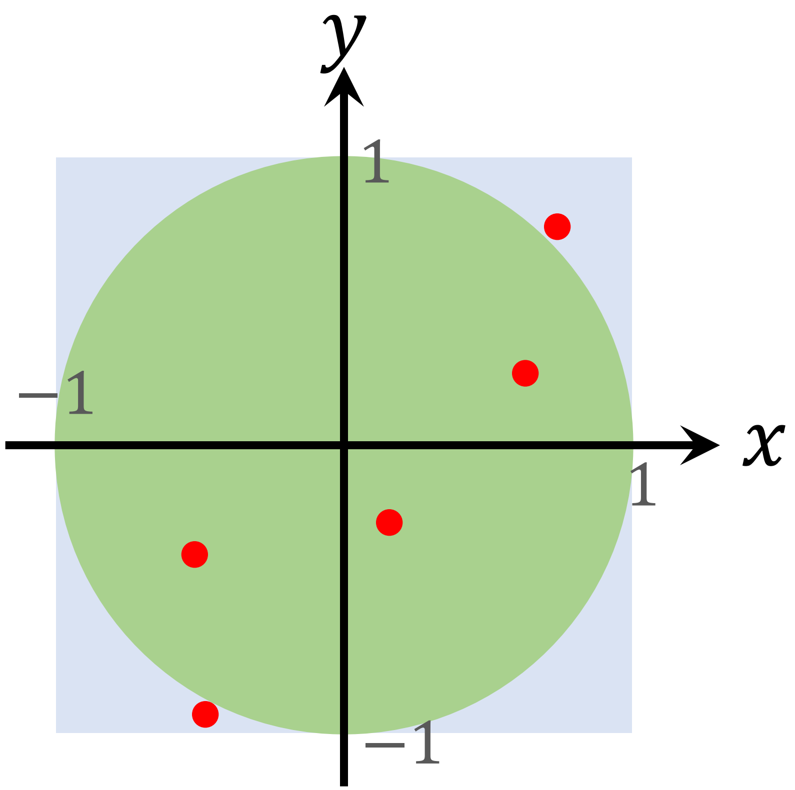

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$$(x_1, y_1), \cdots, (x_n, y_n).$$

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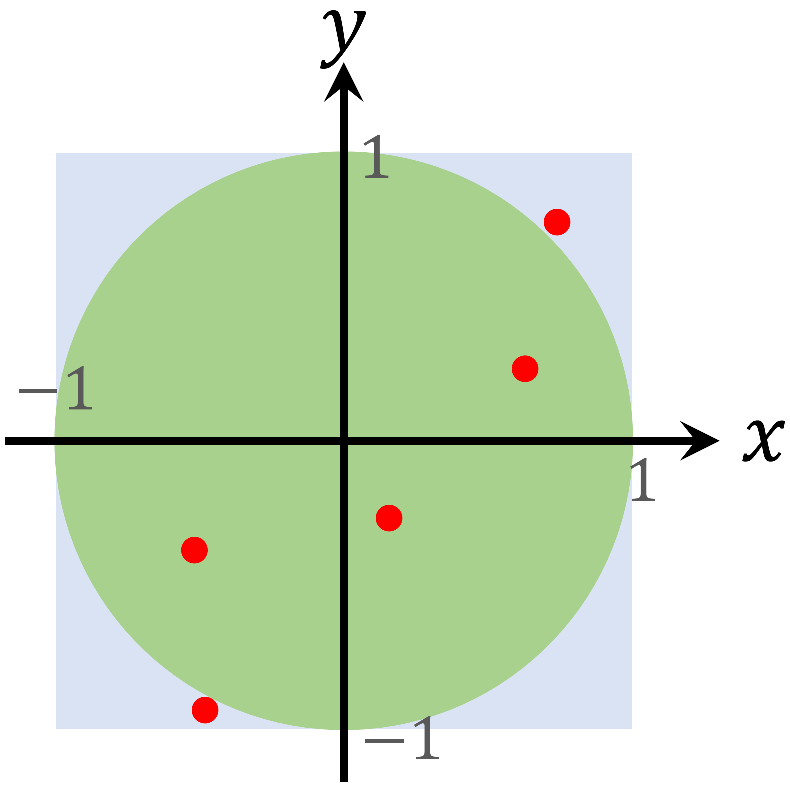

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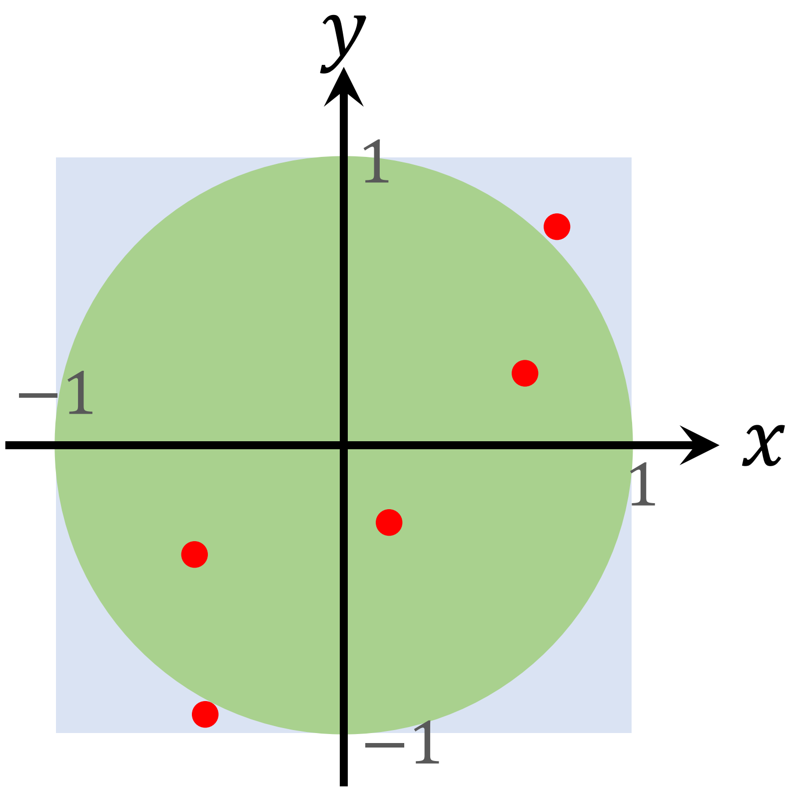

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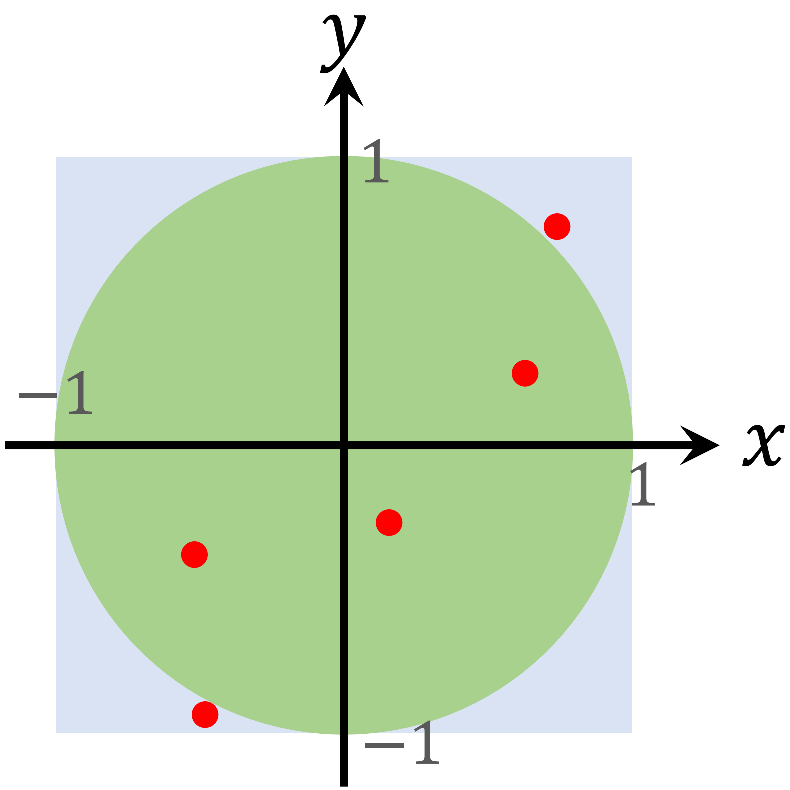

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3. Calculate $Q_n = V \cdot \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i)$.

4. Return $\boxed{Q_n}$ as an approximation to the integral $\boxed{\pi = \int_\Omega f(x, y)\, dx\, dy}$

Application 5: Estimate of Expectation

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- Let X be a d-dimensional random vector.
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  - E.g., PDF of uniform distribution is $p(x) = \frac{1}{t}$, for $x \in [0, t]$.
  - E.g., PDF of univariate Gaussian is $\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]}$.

Expectation

- Let X be a d-dimensional random vector.
- Let $p(\mathbf{x})$ be a probability density function (PDF).
- Let $\boxed{f(\mathbf{x})}$ be any function of vector variable.
- **Expectation**: $\underline{\mathbb{E}_{X\sim p}[f(X)]} = \underline{\int_{\mathbb{R}^d} f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x}}.$

Monte Carlo Estimate of Expectation

Task: Estimate the expectation $\mathbb{E}_{X\sim p}[f(X)] = \int_{\mathbb{R}^d} f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x}$.

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3. Return Q_n as an approximation to $\mathbb{E}_{X\sim p}[f(X)]$.

Monte Carlo and Beyond

Monte Carlo

Casino de Monte-Carlo, Monaco

- The term “Monte Carlo method" was firstly introduced in 1947 by Nicholas Metropolis.

Reference

- Metropolis. The beginning of the Monte Carlo method. *Los Alamos Science*, 125–130, 1987.

Monte Carlo Algorithms

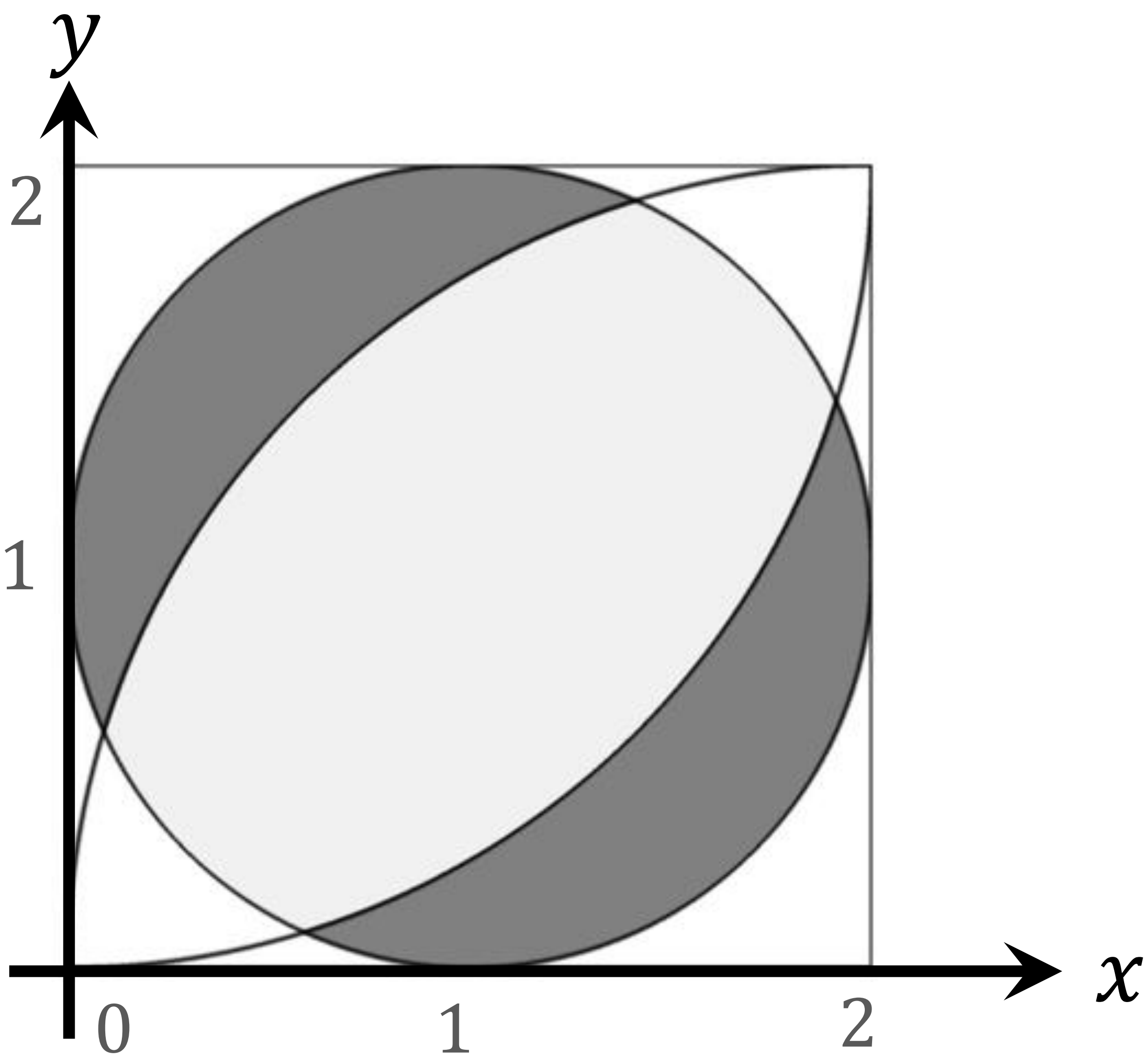
- Monte Carlo refers to algorithms that rely on repeated random sampling to obtain numerical results.
- The output of Monte Carlo algorithms can be incorrect.
  - In all of our examples, the algorithms' outputs are incorrect.
  - But they are close to the correct solution.

Monte Carlo Algorithms

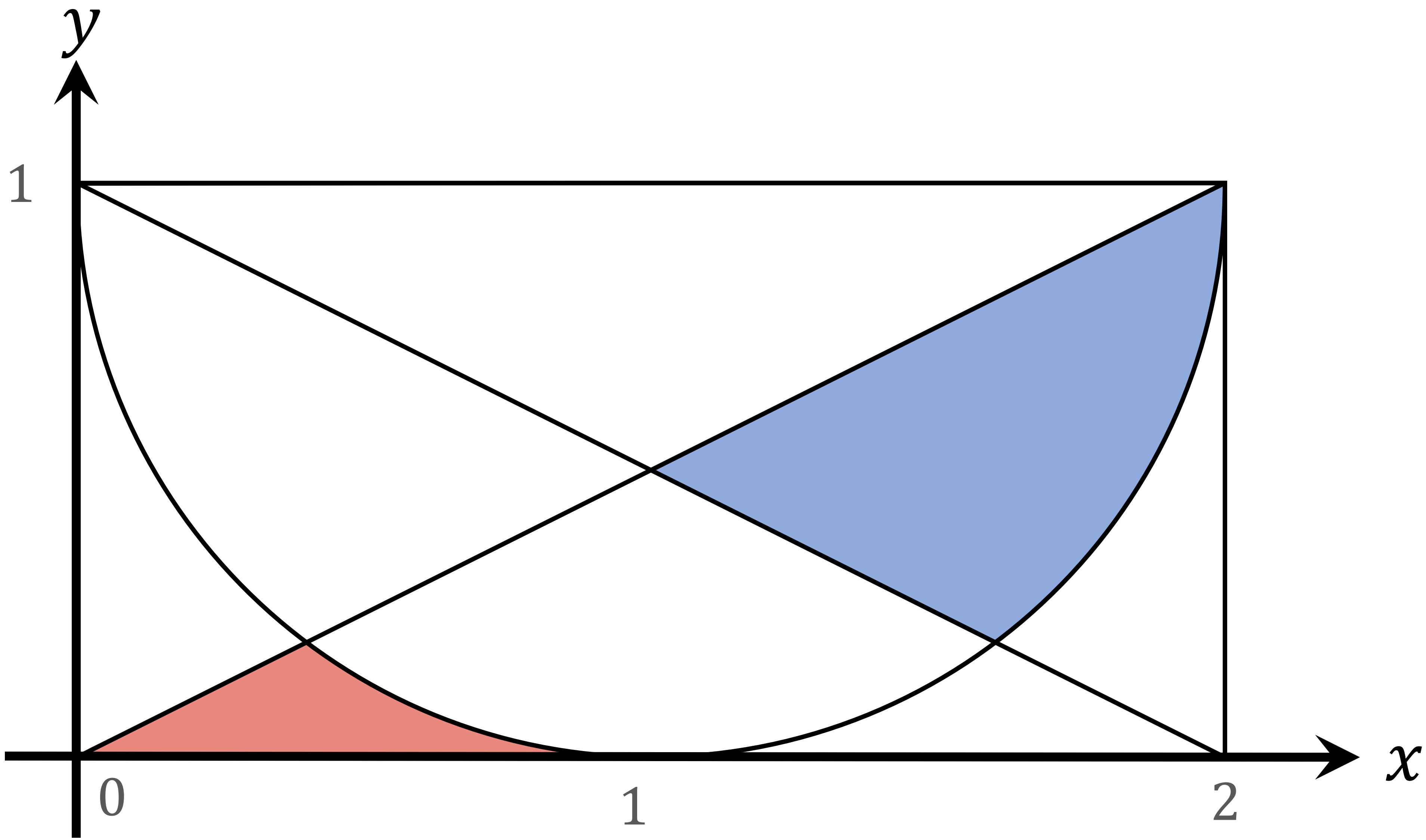
- Monte Carlo refers to algorithms that rely on repeated random sampling to obtain numerical results.
- The output of Monte Carlo algorithms can be incorrect.
- Las Vegas algorithms are those always produce the correct answers.
  - E.g., random quicksort.
- Atlantic City algorithms are polynomial-time randomized algorithms that answer correctly w.p. greater than 75%.

Question

What is the area of the grey region?



What are the areas of the red and blue regions?



Thank you!