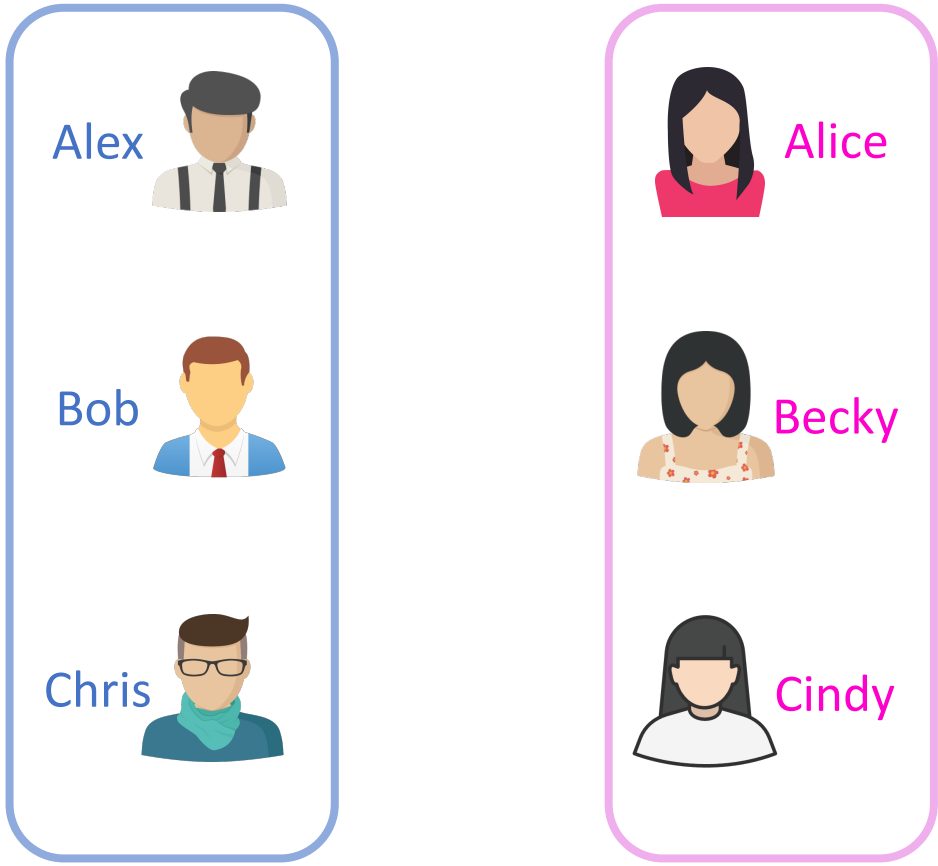
Stable Marriage Problem

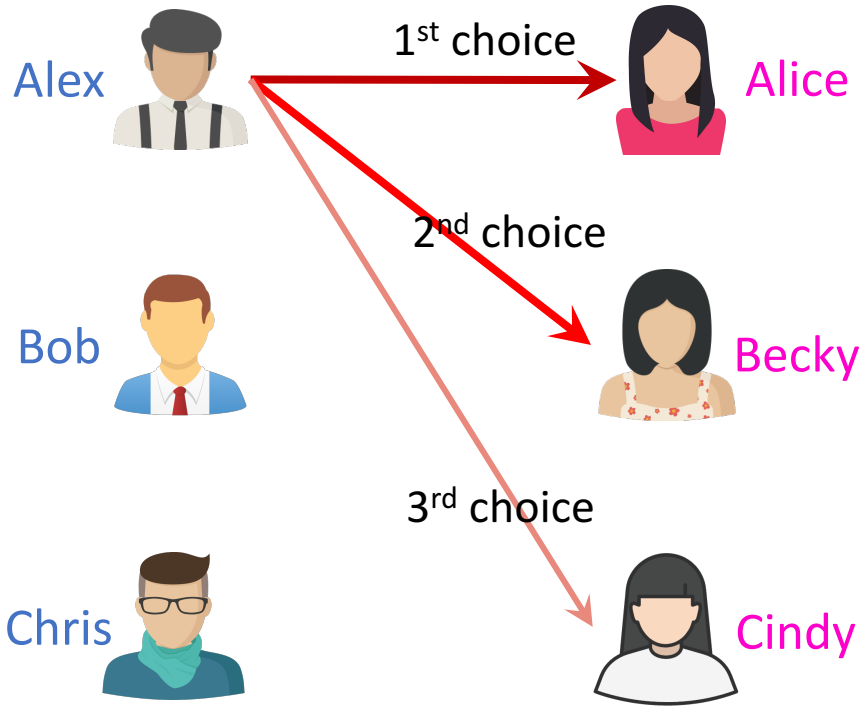
Shusen Wang

Bipartite Matching



Bipartite Matching

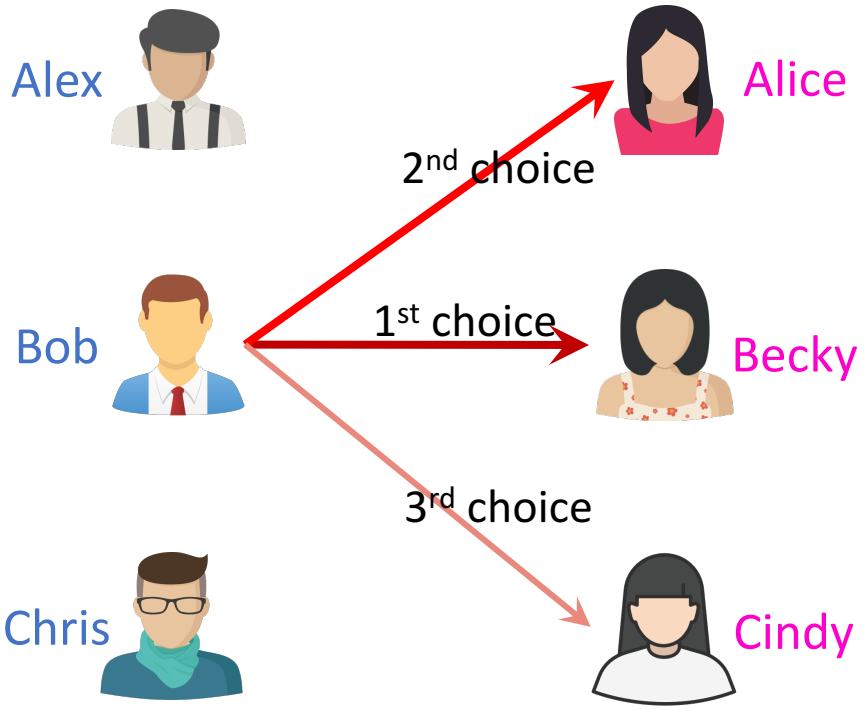
Alice > Becky > Cindy



Bipartite Matching

Alice > Becky > Cindy

Becky > Alice > Cindy

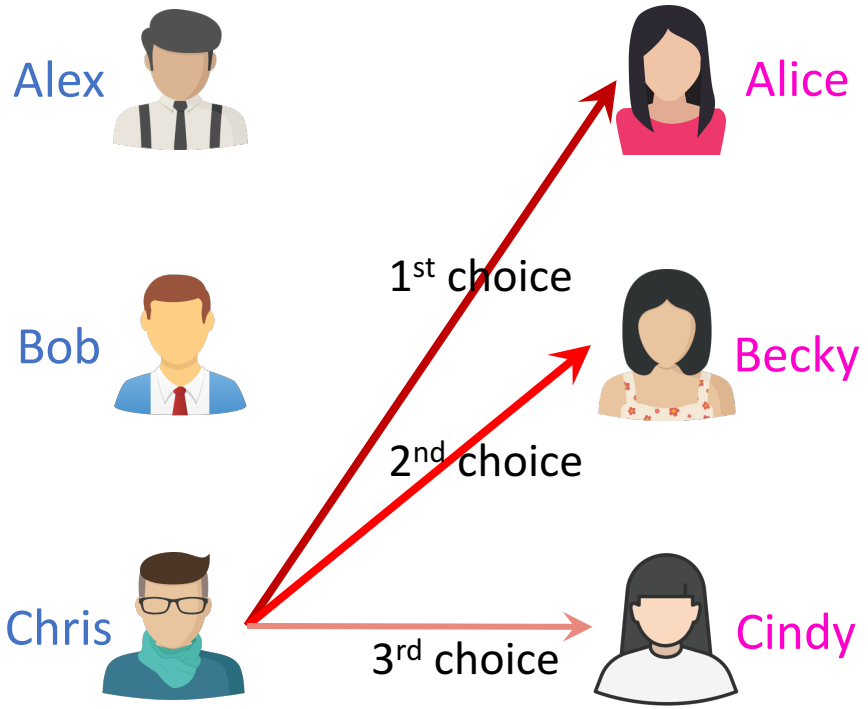


Bipartite Matching

Alice > Becky > Cindy

Becky > Alice > Cindy

Alice > Becky > Cindy

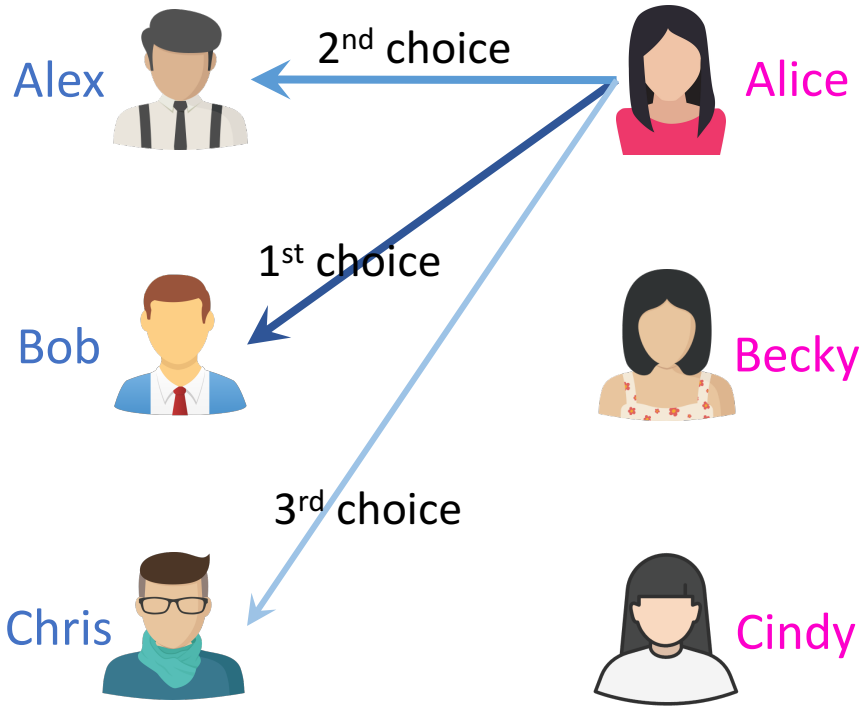


Bipartite Matching

Alice > Becky > Cindy

Becky > Alice > Cindy

Alice > Becky > Cindy



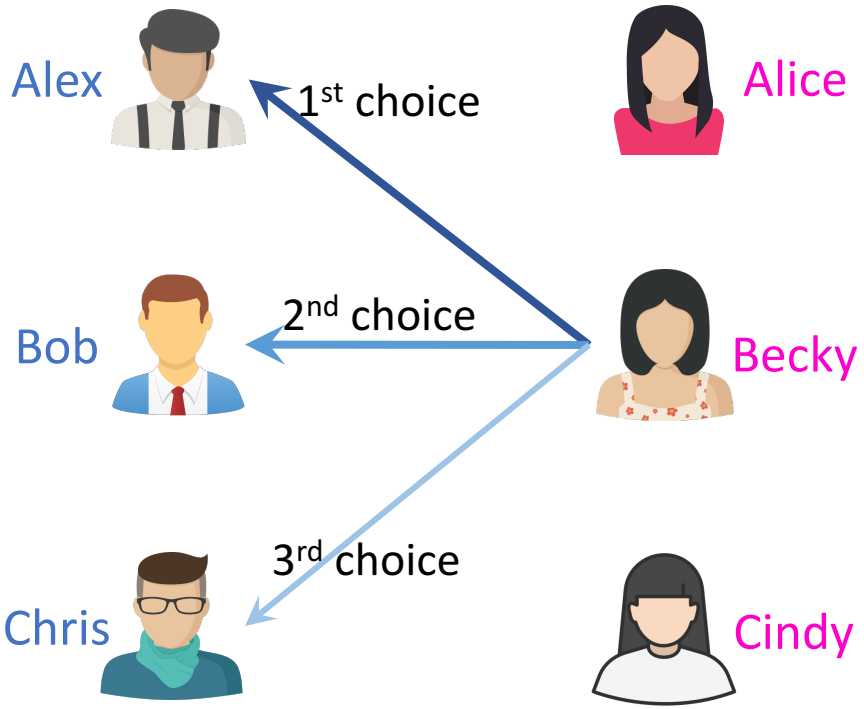
Bob > Alex > Chris

Bipartite Matching

Alice > Becky > Cindy

Becky > Alice > Cindy

Alice > Becky > Cindy



Bob > Alex > Chris

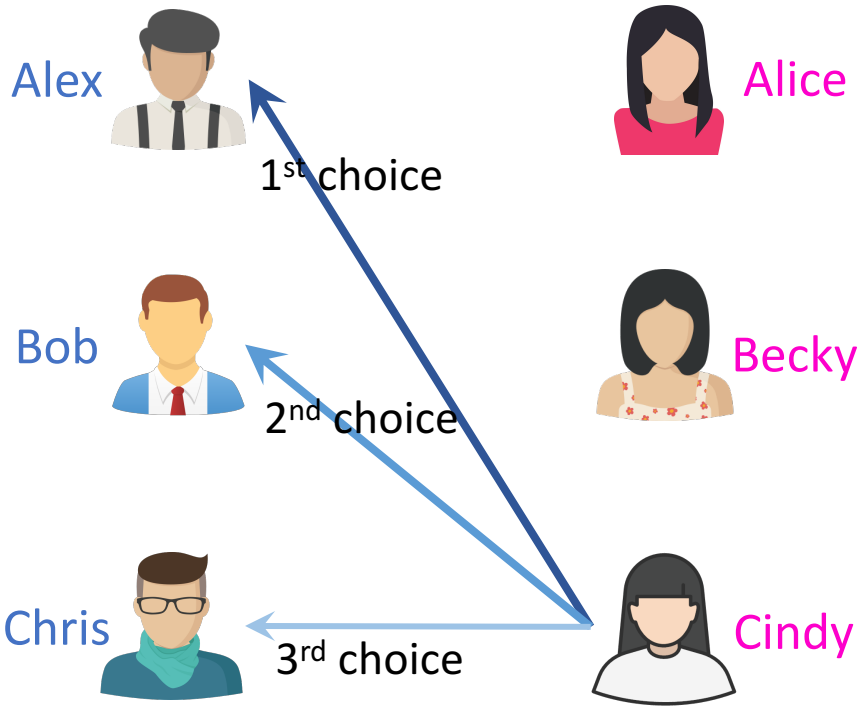
Alex > Bob > Chris

Bipartite Matching

Alice > Becky > Cindy

Becky > Alice > Cindy

Alice > Becky > Cindy



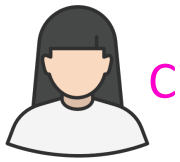
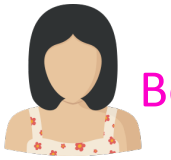
Bob > Alex > Chris

Alex > Bob > Chris

Alex > Bob > Chris

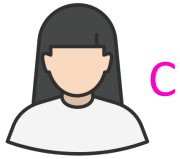
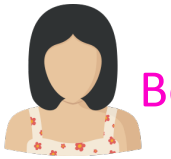
Stable Marriage

Is this a stable marriage?

Alice > Becky > Cindy	Alex		❤	 Cindy	Alex > Bob > Chris
Becky > Alice > Cindy	Bob		❤	 Becky	Alex > Bob > Chris
Alice > Becky > Cindy	Chris		❤	 Alice	Bob > Alex > Chris

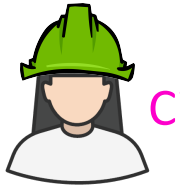
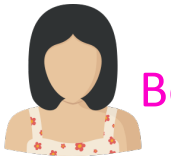
If a man and a woman (who are not spouses) prefer each other over their current spouses, then the marriage is not stable.

Is this a stable marriage?

Alice > Becky > Cindy	Alex 	♥	 Cindy	Alex > Bob > Chris
Becky > Alice > Cindy	Bob 	♥	 Becky	Alex > Bob > Chris
Alice > Becky > Cindy	Chris 	♥	 Alice	Bob > Alex > Chris

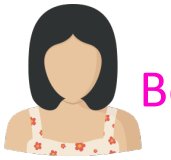
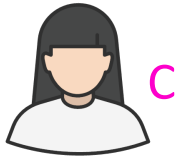
- Alex prefers Alice over his wife, Cindy.
- Alice prefers Alex over her husband, Chris.

This is not a stable marriage!

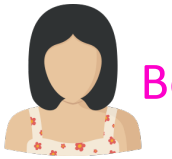
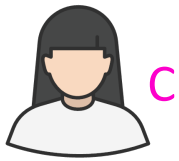
Alice > Becky > Cindy	Alex 	💔	 Cindy	Alex > Bob > Chris
Becky > Alice > Cindy	Bob 	❤️	 Becky	Alex > Bob > Chris
Alice > Becky > Cindy	Chris 	💔	 Alice	Bob > Alex > Chris

- Alex prefers Alice over his wife, Cindy.
- Alice prefers Alex over her husband, Chris.

This is a stable marriage

Alice > Becky > Cindy	Alex		❤		Alice	Bob > Alex > Chris
Becky > Alice > Cindy	Bob		❤		Becky	Alex > Bob > Chris
Alice > Becky > Cindy	Chris		❤		Cindy	Alex > Bob > Chris

This is another stable marriage

Alice > Becky > Cindy	Alex 	❤	 Becky	Alex > Bob > Chris
Becky > Alice > Cindy	Bob 	❤	 Alice	Bob > Alex > Chris
Alice > Becky > Cindy	Chris 	❤	 Cindy	Alex > Bob > Chris

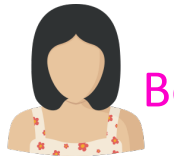
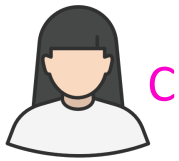
Stable marriage is not unique.

Gale-Shapley Algorithm

Reference:

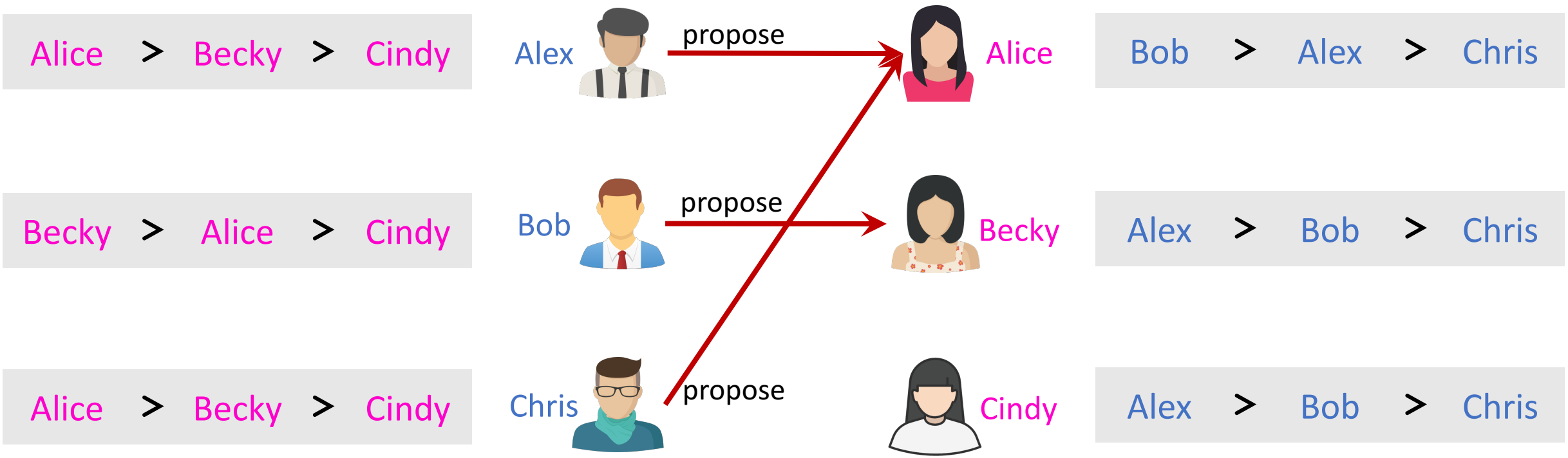
- David Gale and Lloyd S. Shapley. College Admissions and the Stability of Marriage. *American Mathematical Monthly*, 69 (1): 9–14, 1962.

Gale-Shapley Algorithm

Alice > Becky > Cindy	Alex 	 Alice	Bob > Alex > Chris
Becky > Alice > Cindy	Bob 	 Becky	Alex > Bob > Chris
Alice > Becky > Cindy	Chris 	 Cindy	Alex > Bob > Chris

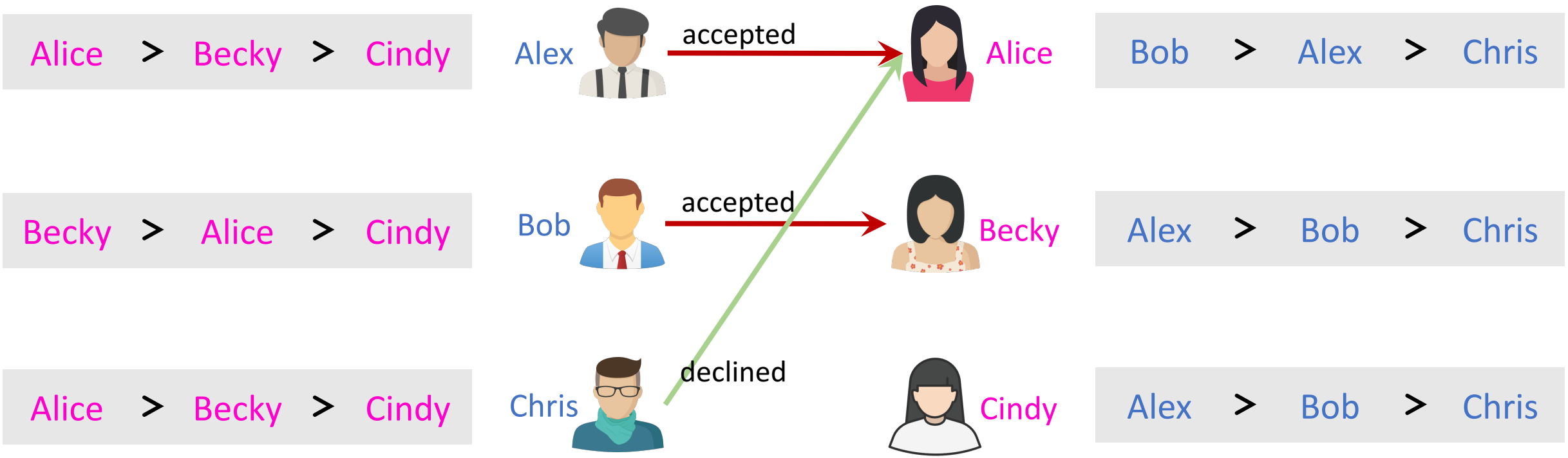
Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Gale-Shapley Algorithm



Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Gale-Shapley Algorithm



Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

~~Alice~~ > Becky > Cindy

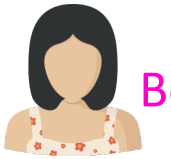
Alex 

 Alice

Bob > Alex > Chris

~~Becky~~ > Alice > Cindy

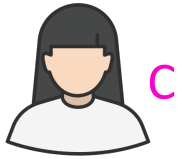
Bob 

 Becky

Alex > Bob > Chris

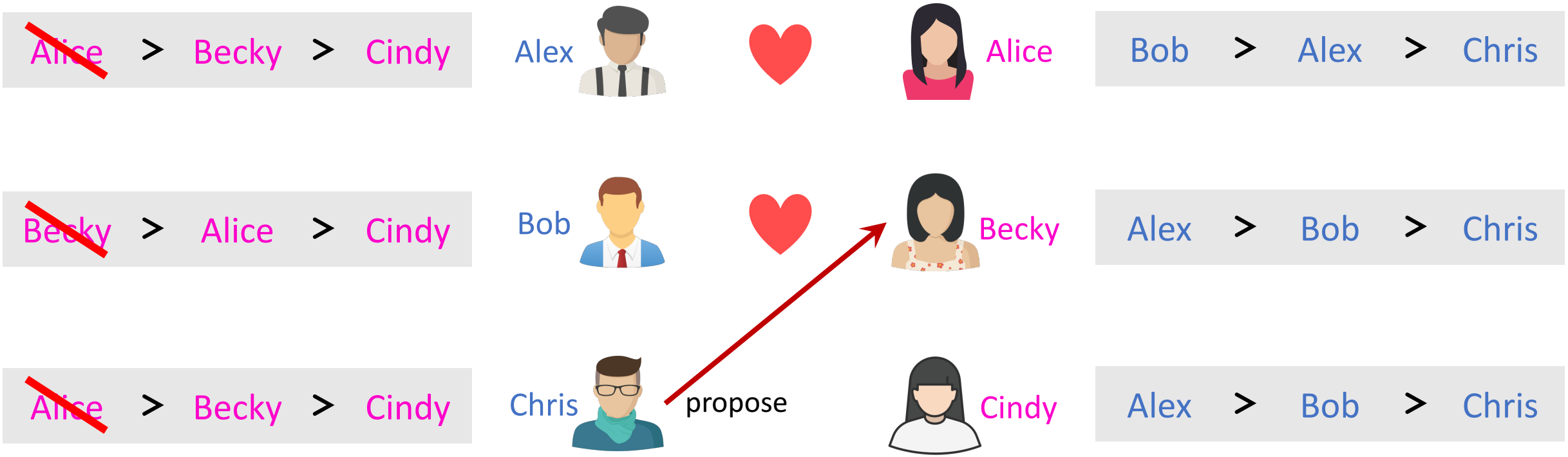
~~Alice~~ > Becky > Cindy

Chris 

 Cindy

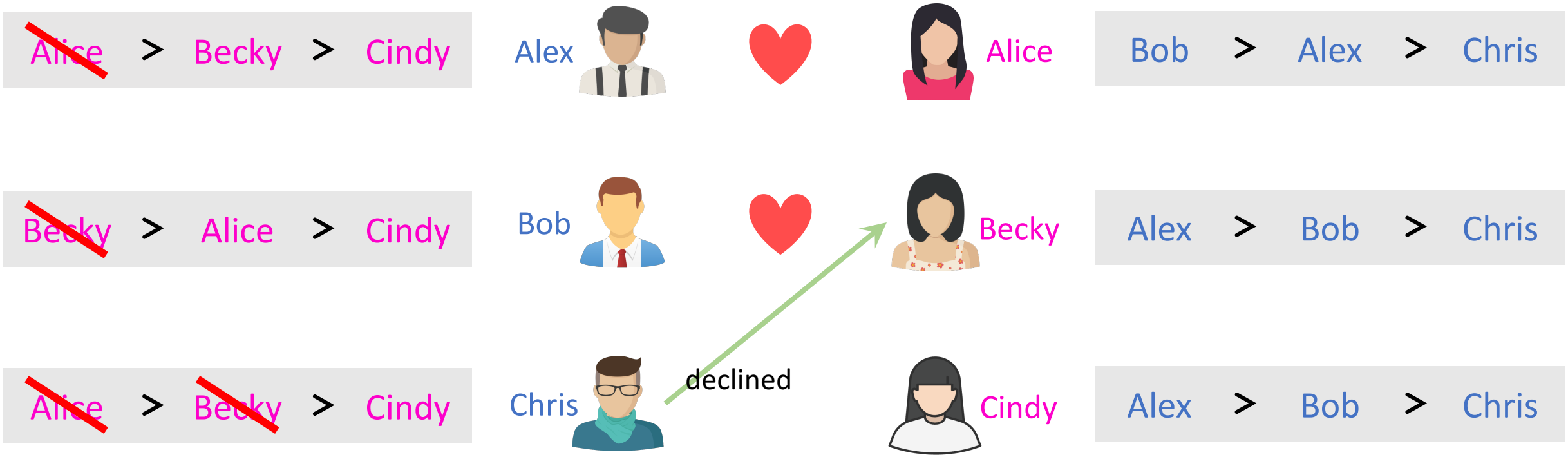
Alex > Bob > Chris

Gale-Shapley Algorithm



Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Gale-Shapley Algorithm



Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

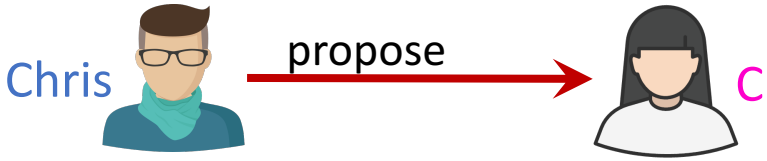
~~Alice~~ > Becky > Cindy

Bob > Alex > Chris

~~Becky~~ > Alice > Cindy

Alex > Bob > Chris

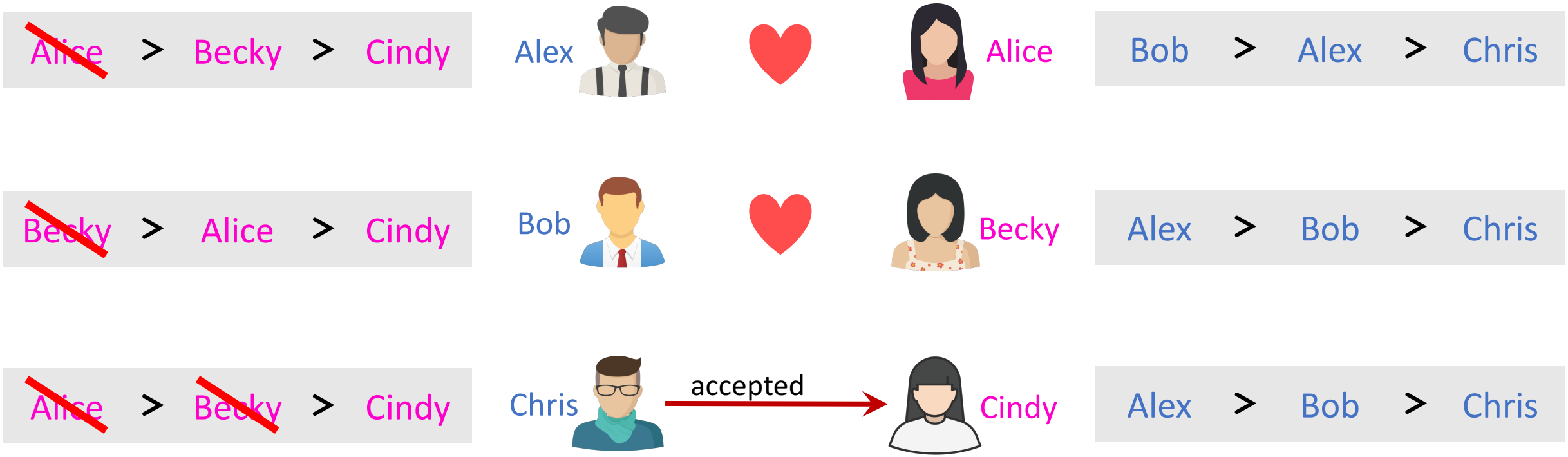
~~Alice~~ > ~~Becky~~ > Cindy



Alex > Bob > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Gale-Shapley Algorithm



Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

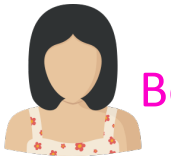
Gale-Shapley Algorithm

~~Alice~~ > Becky > Cindy

Alex  ❤  Alice

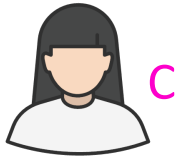
Bob > Alex > Chris

~~Becky~~ > Alice > Cindy

Bob  ❤  Becky

Alex > Bob > Chris

~~Alice~~ > ~~Becky~~ > Cindy

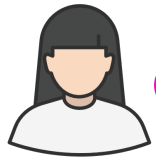
Chris  ❤  Cindy

Alex > Bob > Chris

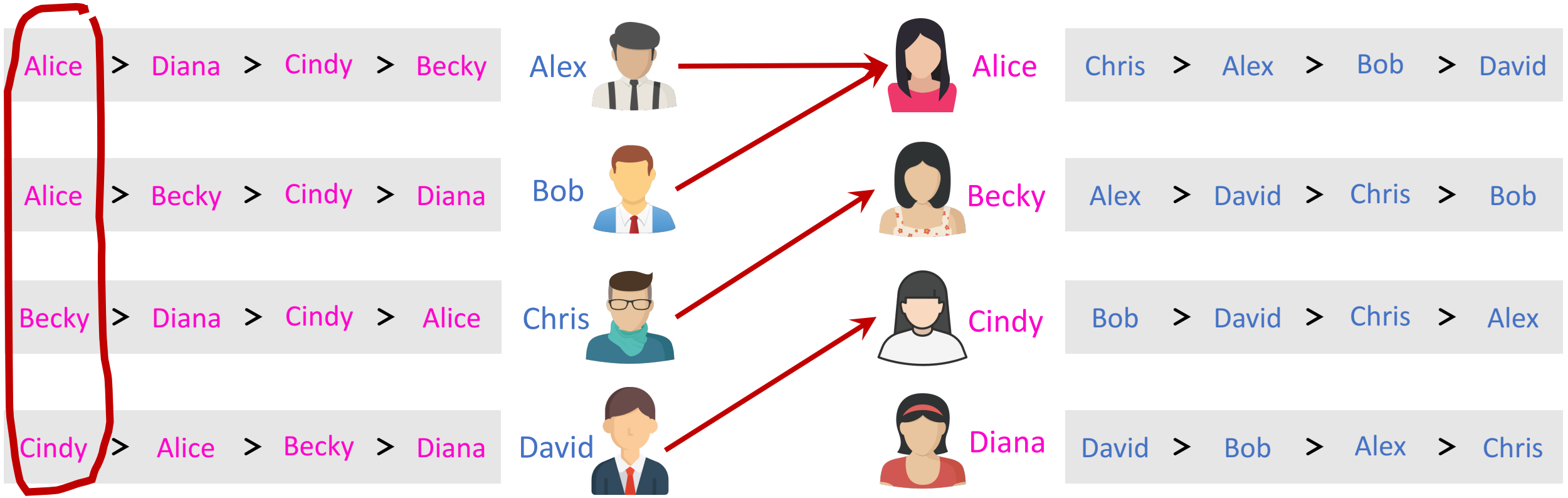
Stop if everyone is married.

Another Example

Preference Lists

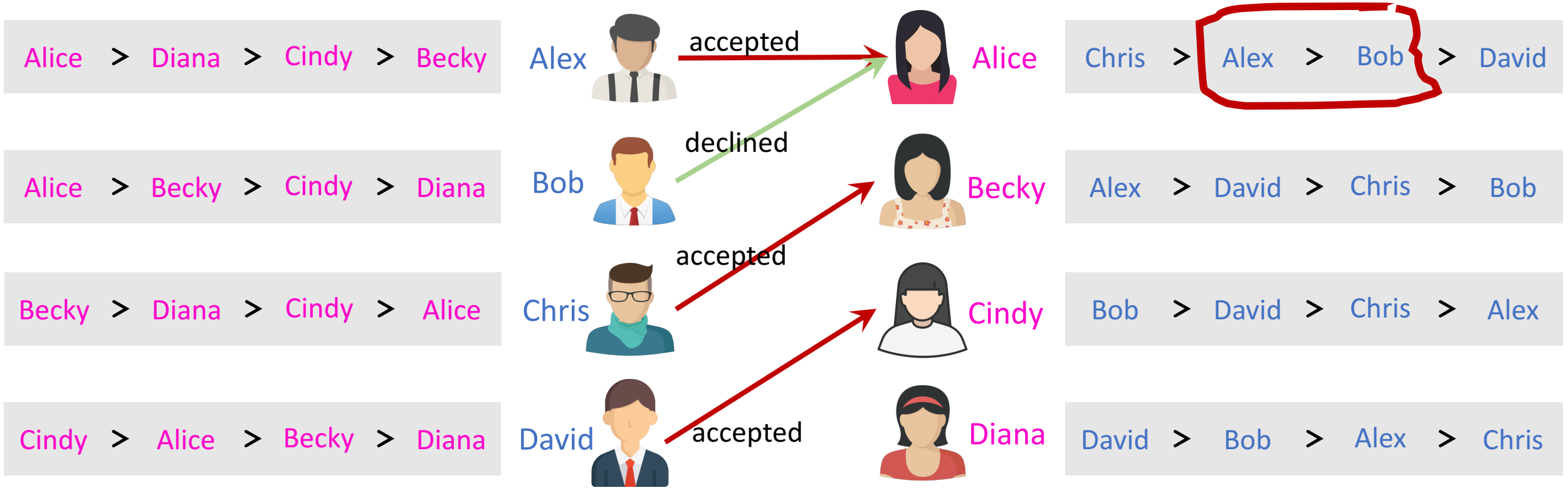
Alice > Diana > Cindy > Becky	Alex			Alice	Chris > Alex > Bob > David
Alice > Becky > Cindy > Diana	Bob			Becky	Alex > David > Chris > Bob
Becky > Diana > Cindy > Alice	Chris			Cindy	Bob > David > Chris > Alex
Cindy > Alice > Becky > Diana	David			Diana	David > Bob > Alex > Chris

Gale-Shapley Algorithm



Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

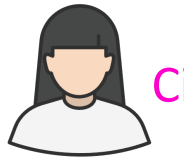
Gale-Shapley Algorithm



Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

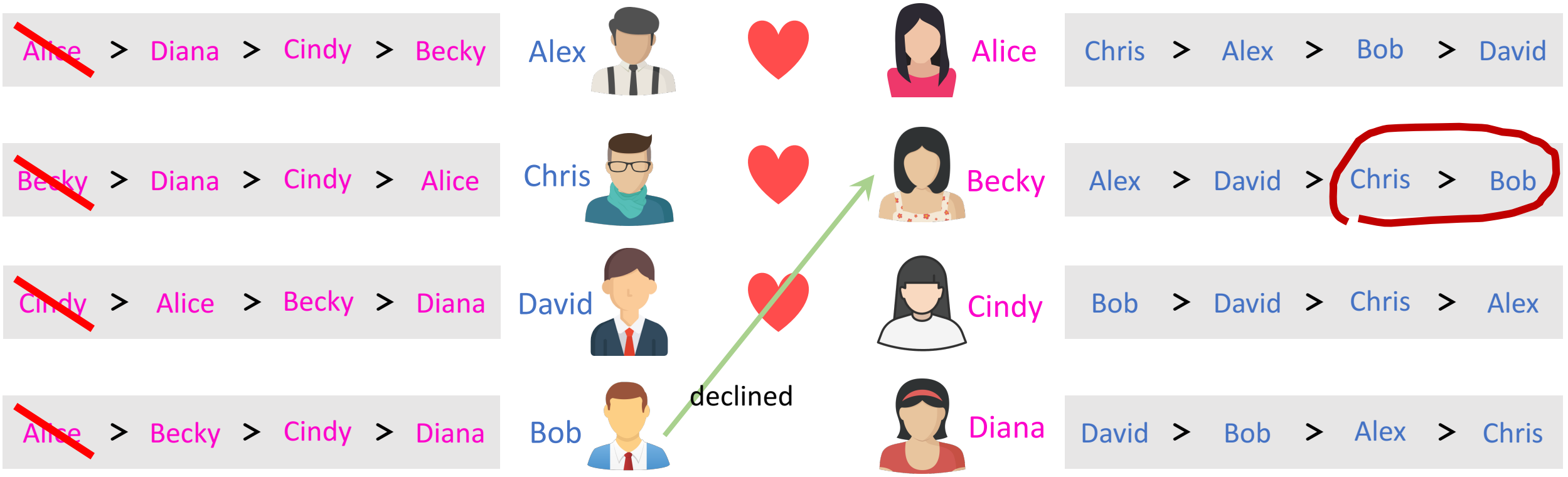
~~Alice~~	>	Diana	>	Cindy	>	Becky	Alex 	❤	 Alice	Chris > Alex > Bob > David
~~Becky~~	>	Diana	>	Cindy	>	Alice	Chris 	❤	 Becky	Alex > David > Chris > Bob
~~Cindy~~	>	Alice	>	Becky	>	Diana	David 	❤	 Cindy	Bob > David > Chris > Alex
~~Alice~~	>	Becky	>	Cindy	>	Diana	Bob 		 Diana	David > Bob > Alex > Chris

Gale-Shapley Algorithm

~~Alice~~	>	Diana	>	Cindy	>	Becky	Alex			Alice	Chris > Alex > Bob > David
~~Becky~~	>	Diana	>	Cindy	>	Alice	Chris			Becky	Alex > David > Chris > Bob
~~Cindy~~	>	Alice	>	Becky	>	Diana	David			Cindy	Bob > David > Chris > Alex
~~Alice~~	>	Becky	>	Cindy	>	Diana	Bob			Diana	David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

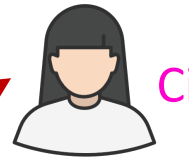
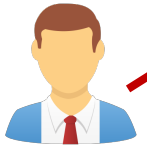
Gale-Shapley Algorithm

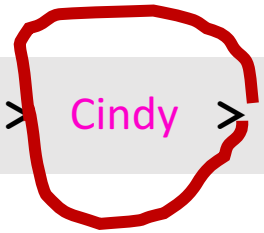


Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

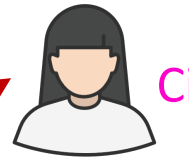
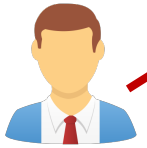
Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex 	♥	 Alice	Chris > Alex > Bob > David
~~Becky~~ > Diana > Cindy > Alice	Chris 	♥	 Becky	Alex > David > Chris > Bob
~~Cindy~~ > Alice > Becky > Diana	David 	♥	 Cindy	Bob > David > Chris > Alex
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 		 Diana	David > Bob > Alex > Chris



Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

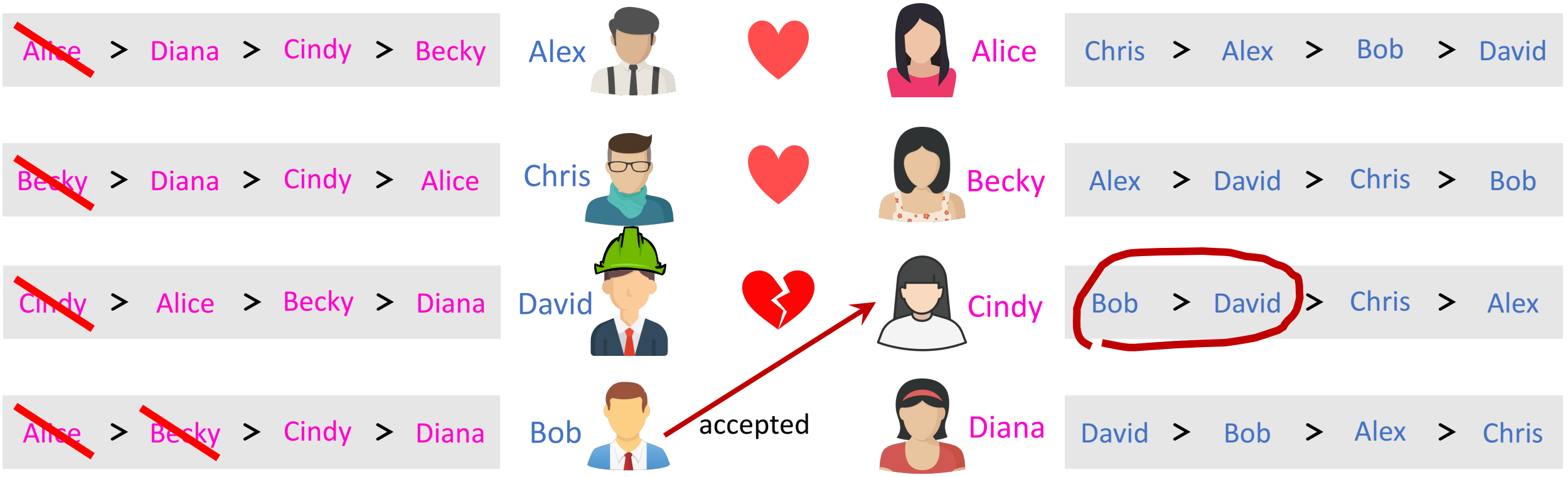
Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex 	♥	 Alice	Chris > Alex > Bob > David
~~Becky~~ > Diana > Cindy > Alice	Chris 	♥	 Becky	Alex > David > Chris > Bob
~~Cindy~~ > Alice > Becky > Diana	David 	♥	 Cindy	Bob > David > Chris > Alex
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 		 Diana	David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

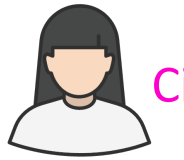
Gale-Shapley Algorithm



Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex 	♥	 Alice	Chris > Alex > Bob > David
~~Becky~~ > Diana > Cindy > Alice	Chris 	♥	 Becky	Alex > David > Chris > Bob
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 	♥	 Cindy	Bob > David > Chris > Alex
~~Cindy~~ > Alice > Becky > Diana	David 		 Diana	David > Bob > Alex > Chris

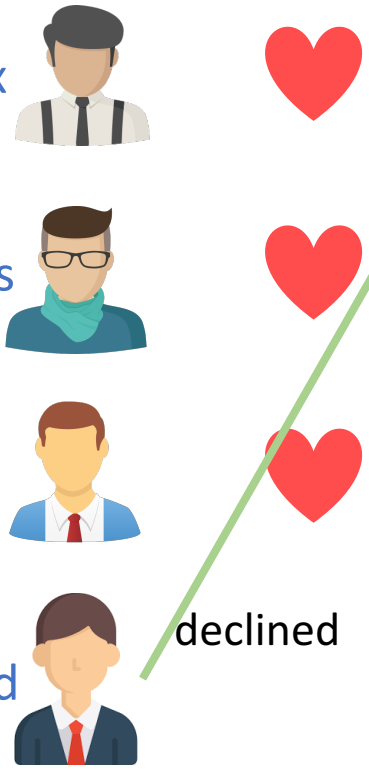
Gale-Shapley Algorithm

~~Alice~~	>	Diana	>	Cindy	>	Becky	Alex		Alice	Chris	>	Alex	>	Bob	>	David
~~Becky~~	>	Diana	>	Cindy	>	Alice	Chris		Becky	Alex	>	David	>	Chris	>	Bob
~~Alice~~	>	~~Becky~~	>	Cindy	>	Diana	Bob		Cindy	Bob	>	David	>	Chris	>	Alex
~~Cindy~~	>	Alice	>	Becky	>	Diana	David		Diana	David	>	Bob	>	Alex	>	Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Gale-Shapley Algorithm

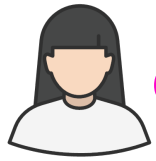
~~Alice~~	>	Diana	>	Cindy	>	Becky	Alex	Alice	Chris > Alex > Bob > David
~~Becky~~	>	Diana	>	Cindy	>	Alice	Chris	Becky	Alex > David > Chris > Bob
~~Alice~~	>	~~Becky~~	>	Cindy	>	Diana	Bob	Cindy	Bob > David > Chris > Alex
~~Cindy~~	>	Alice	>	Becky	>	Diana	David	Diana	David > Bob > Alex > Chris



Step 2: A woman accepts her best offer.

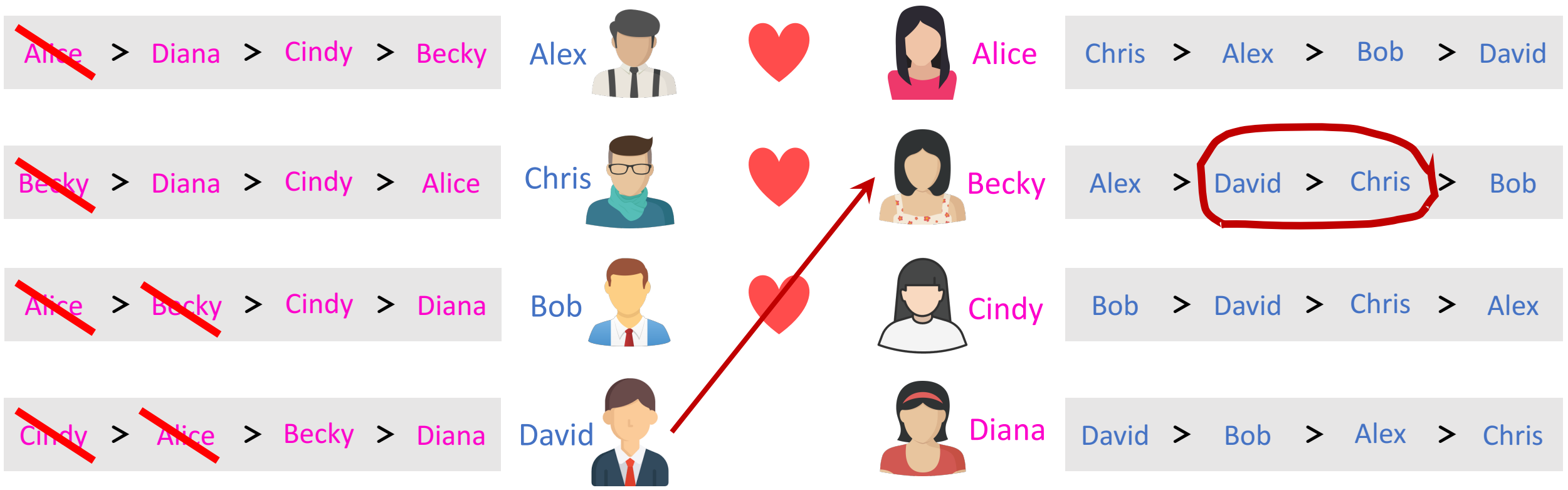
- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex			Alice	Chris > Alex > Bob > David
~~Becky~~ > Diana > Cindy > Alice	Chris			Becky	Alex > David > Chris > Bob
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob			Cindy	Bob > David > Chris > Alex
~~Cindy~~ > ~~Alice~~ > Becky > Diana	David			Diana	David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

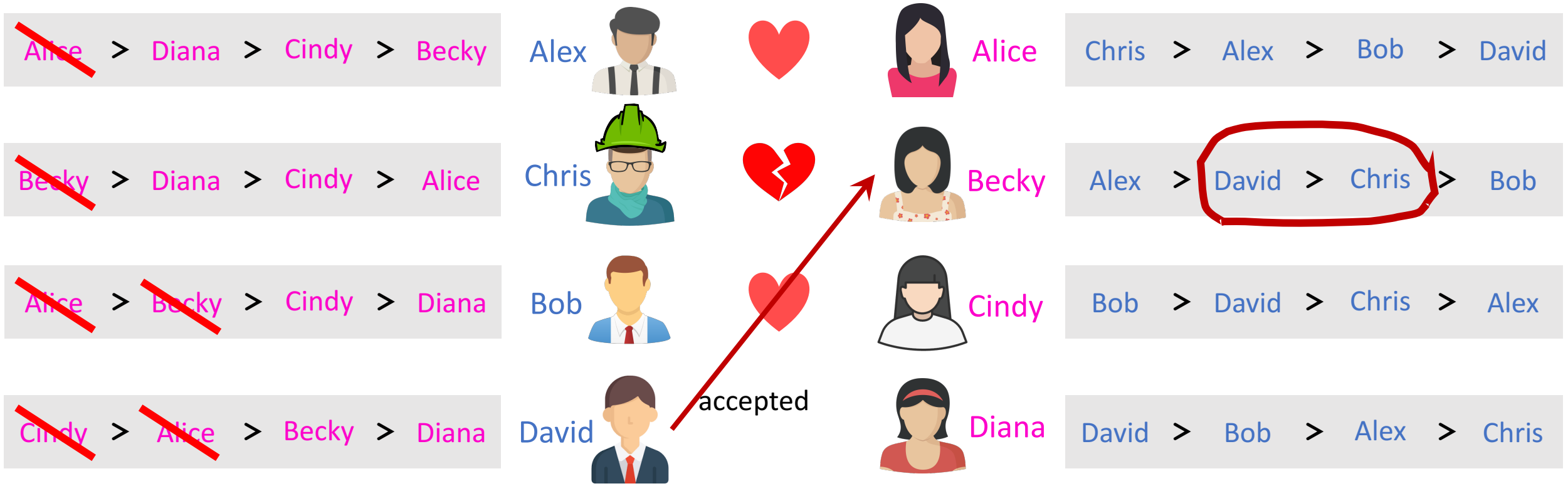
Gale-Shapley Algorithm



Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

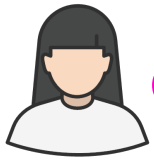
Gale-Shapley Algorithm



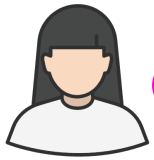
Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

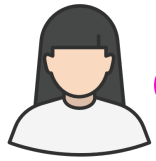
~~Alice~~ > Diana > Cindy > Becky	Alex 			Alice	Chris > Alex > Bob > David
~~Cindy~~ > ~~Alice~~ > Becky > Diana	David 			Becky	Alex > David > Chris > Bob
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 			Cindy	Bob > David > Chris > Alex
~~Becky~~ > Diana > Cindy > Alice	Chris 			Diana	David > Bob > Alex > Chris

Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex 	♥	 Alice	Chris > Alex > Bob > David
~~Cindy~~ > ~~Alice~~ > Becky > Diana	David 	♥	 Becky	Alex > David > Chris > Bob
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 	♥	 Cindy	Bob > David > Chris > Alex
~~Becky~~ > Diana > Cindy > Alice	Chris 	→	 Diana	David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

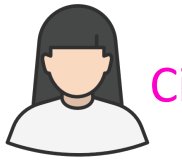
Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex 		 Alice	Chris > Alex > Bob > David
~~Cindy~~ > ~~Alice~~ > Becky > Diana	David 		 Becky	Alex > David > Chris > Bob
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 		 Cindy	Bob > David > Chris > Alex
~~Becky~~ > Diana > Cindy > Alice	Chris  accepted →		 Diana	David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.

- If the woman finds better matching than her current spouse, then divorce.

Gale-Shapley Algorithm

~~Alice~~ > Diana > Cindy > Becky	Alex 	♥	 Alice	Chris > Alex > Bob > David
~~Cindy~~ > ~~Alice~~ > Becky > Diana	David 	♥	 Becky	Alex > David > Chris > Bob
~~Alice~~ > ~~Becky~~ > Cindy > Diana	Bob 	♥	 Cindy	Bob > David > Chris > Alex
~~Becky~~ > Diana > Cindy > Alice	Chris 	♥	 Diana	David > Bob > Alex > Chris

Stop if everyone is married.

Summary

Stable Marriage Problem

- Directed weighted bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
  - $\mathcal{U}$: a set of men.
  - $\mathcal{V}$: a set of women.
  - The weights are the orders of preference.
- The numbers of men and women are the same: $|\mathcal{U}| = |\mathcal{V}| = n$.
- If a man and a woman prefer each other over their current spouses, then the marriage is not stable.

Gale-Shapley Algorithm

1. Every unmarried man proposes to a woman who is his most preferred among those he has not proposed to.
2. Every woman accepts her best offer.
   - A bad offer is better than no offer.
   - If the offer is better than her current spouse, then divorce.
3. Stop if everyone is married; otherwise, repeat Steps 1 and 2.

Time complexity: $O(n^2)$

Thank You!