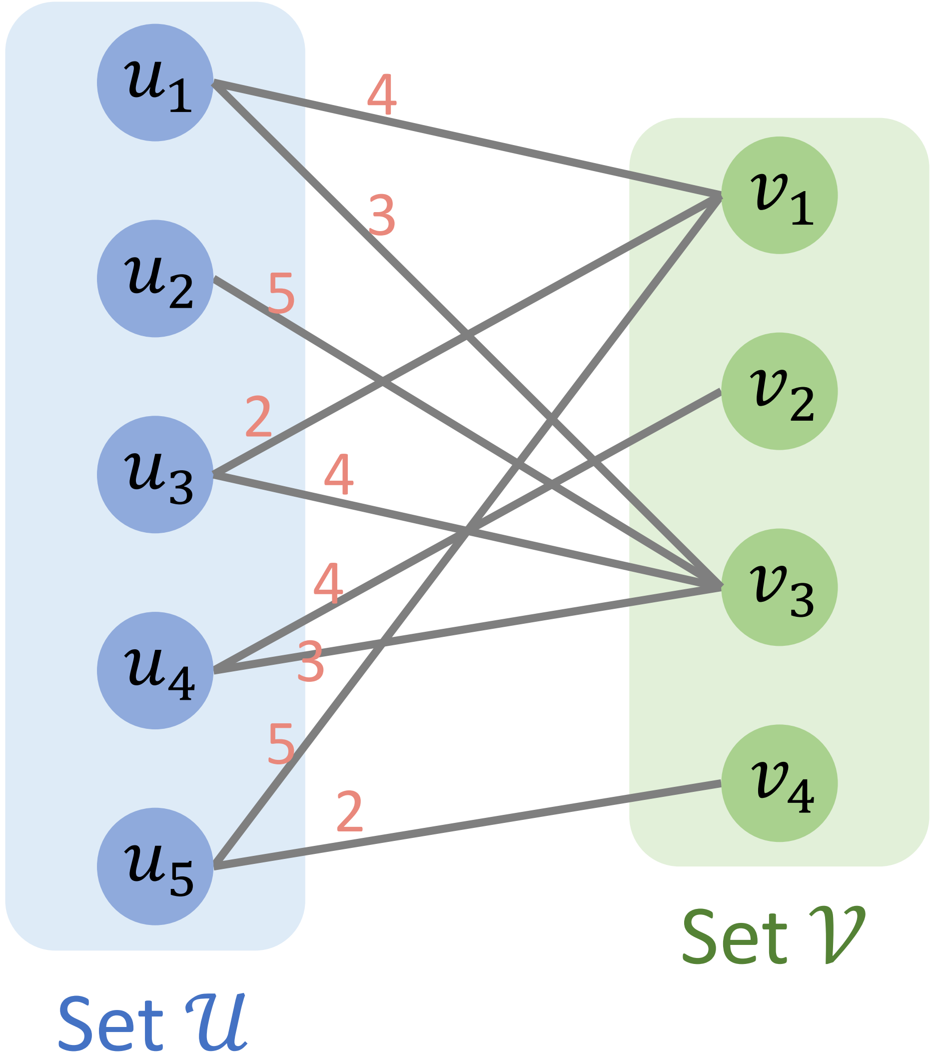
Maximum-Weight Bipartite Matching

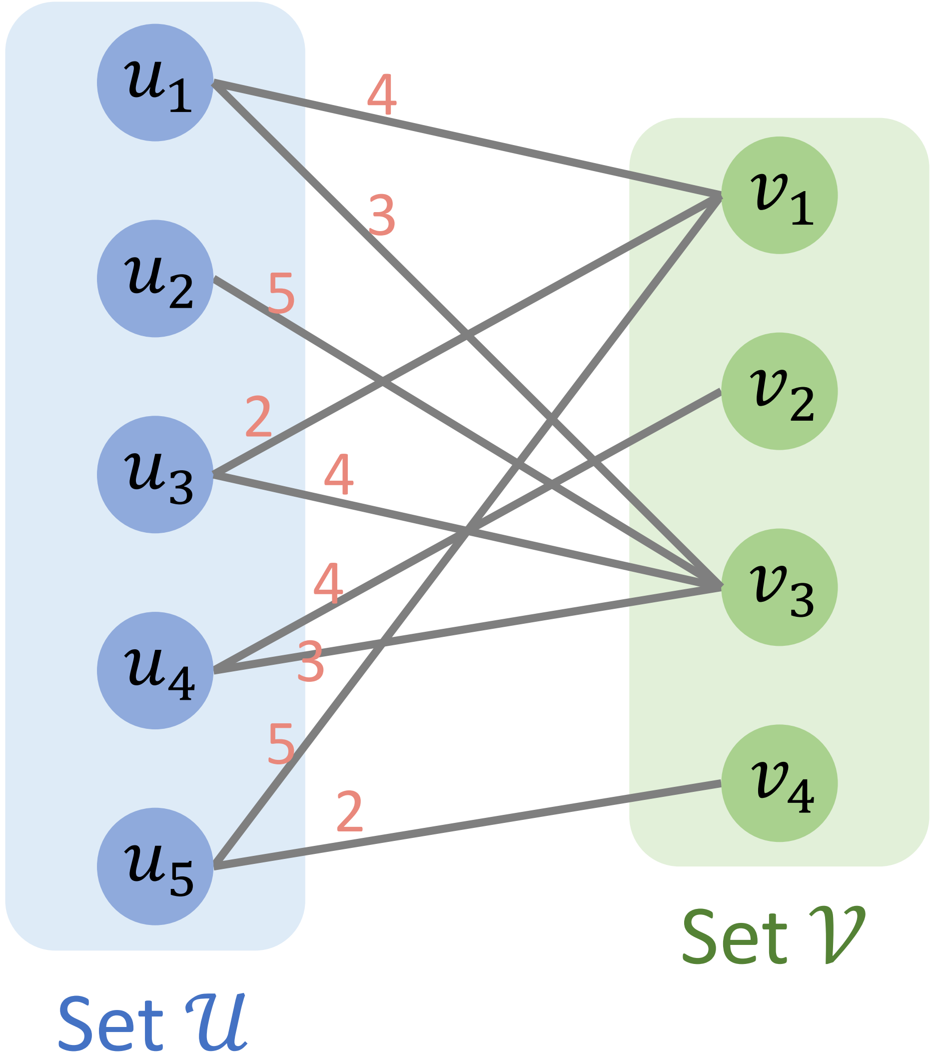
Shusen Wang

Weighted Bipartite Graph



- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Edges have weights: w_{ij}.

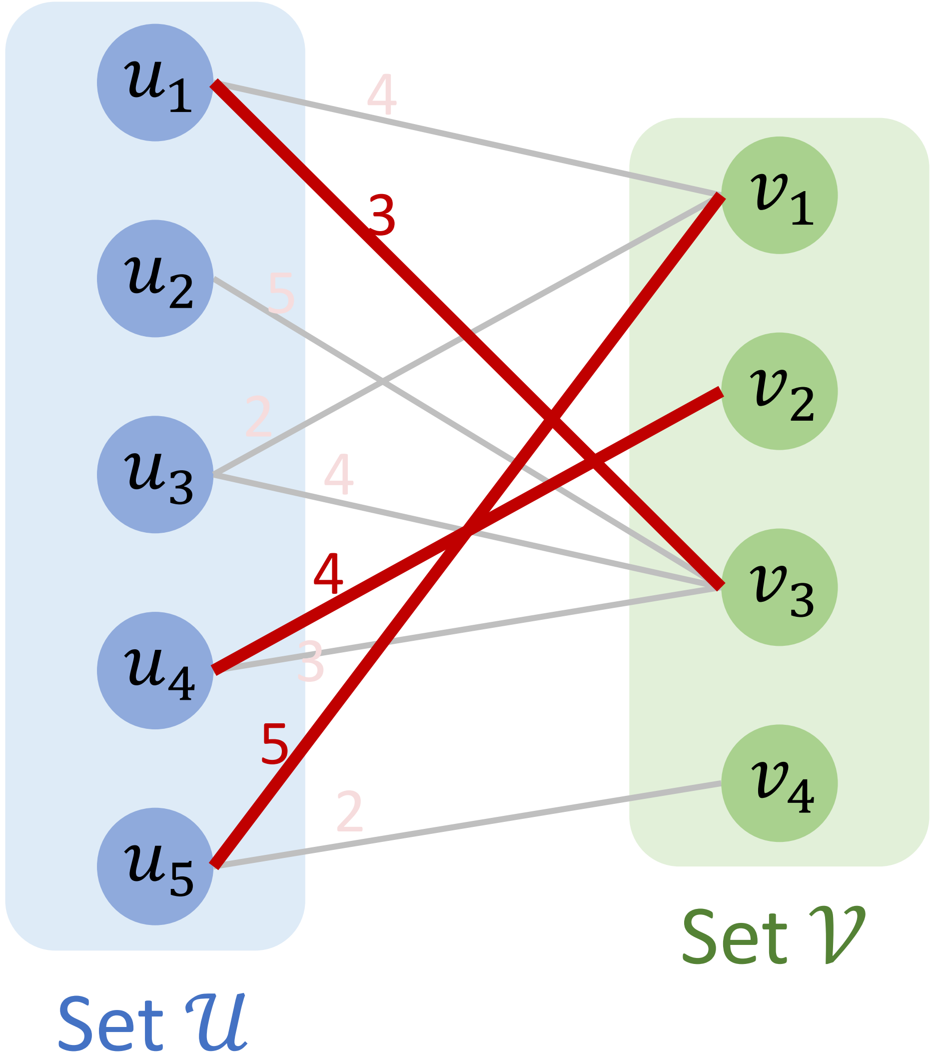
Weighted Bipartite Graph



- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Edges have weights: w_{ij}.
- Adjacency matrix:

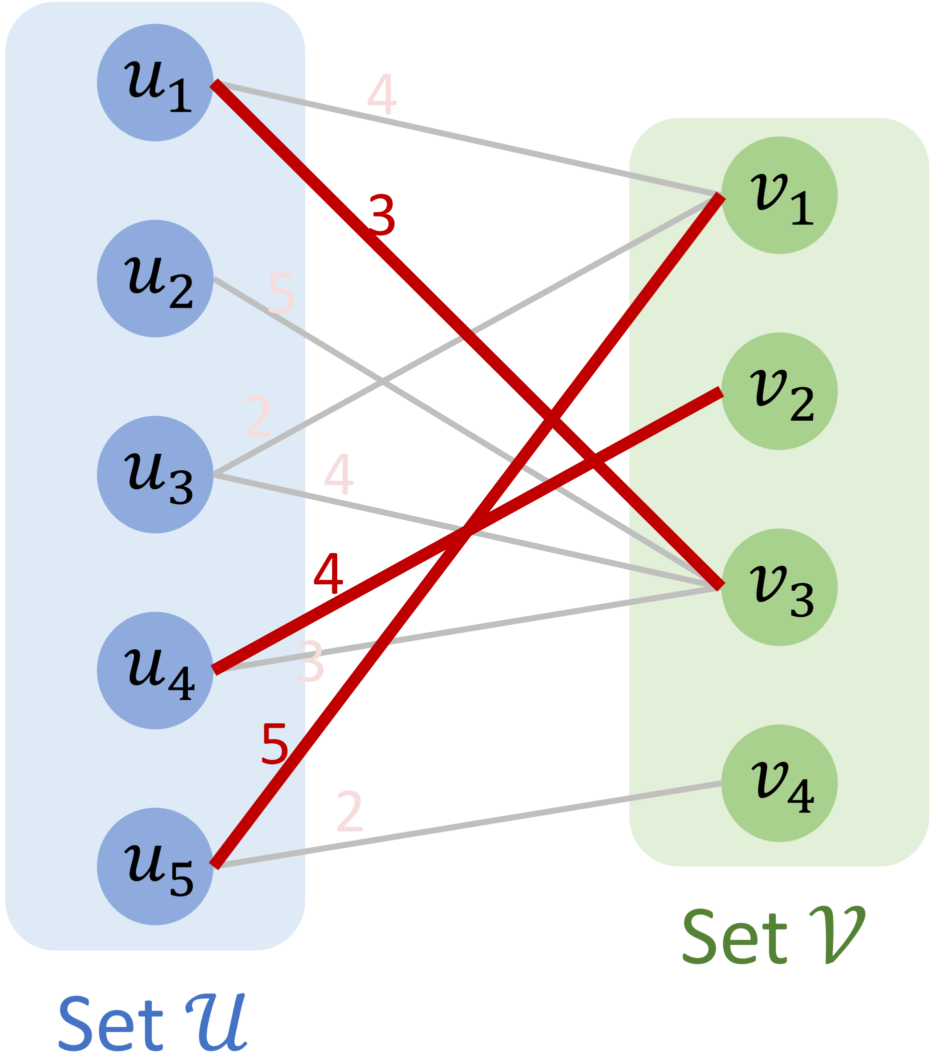
	v_1	v_2	v_3	v_4
u_1	**4**	0	**3**	0
u_2	0	0	**5**	0
u_3	**2**	0	**4**	0
u_4	0	**4**	**3**	0
u_5	**5**	0	0	**2**

Bipartite Matching in Weighted Graph



- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Matching is a subset of edges without common vertices.
- Denote the matching by set $\mathcal{S} \subseteq \mathcal{E}$.

Bipartite Matching in Weighted Graph



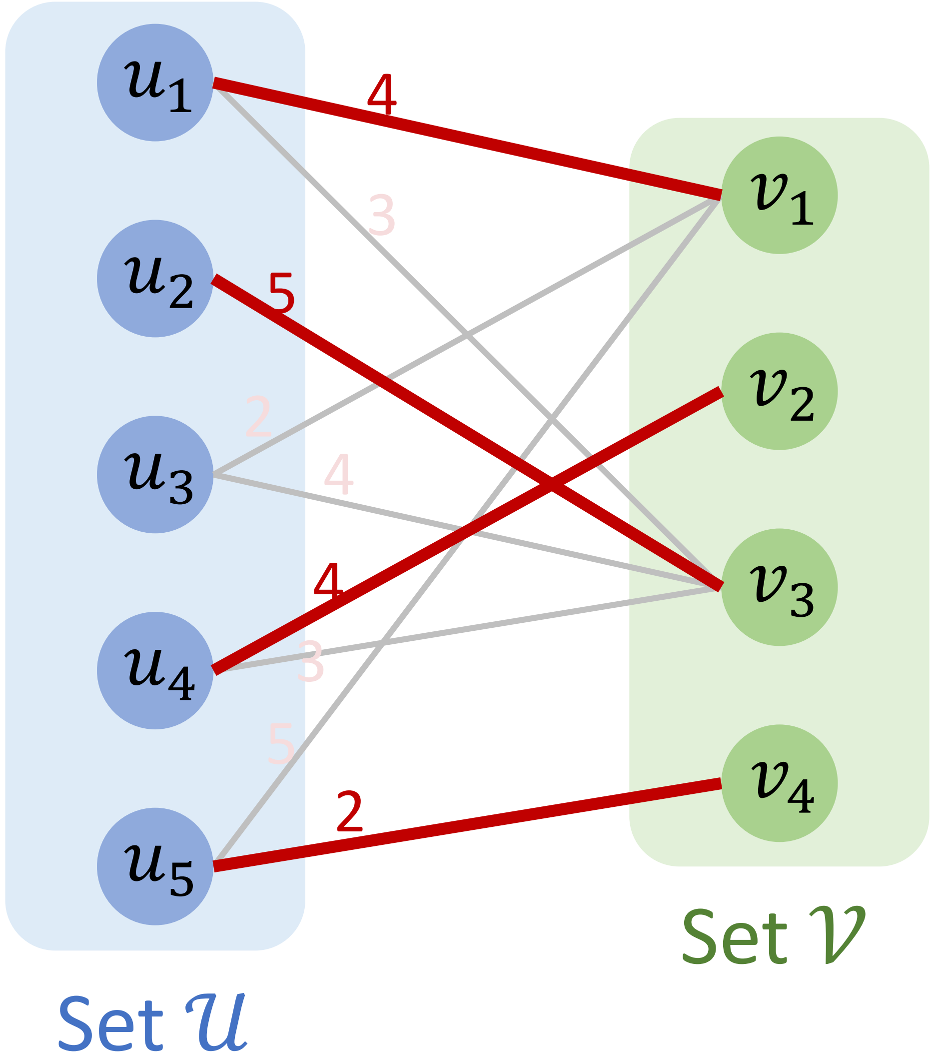
- Sum of weights in matching $\mathcal{S}$:

$$f(\mathcal{S}) = \sum_{(u,v)\in\mathcal{S}} w_{uv}.$$

- In this example,

$$f(\mathcal{S}) = 3 + 4 + 5 = 12.$$

Bipartite Matching in Weighted Graph



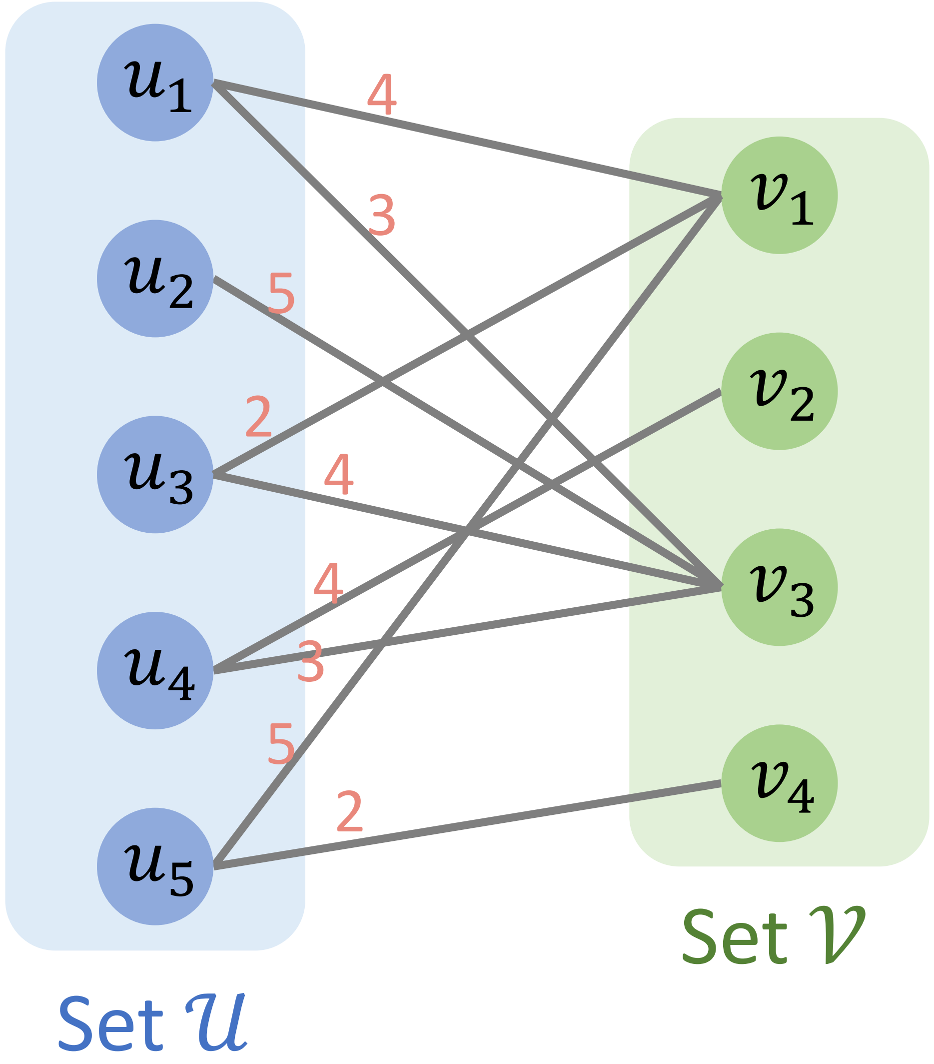
- Sum of weights in matching $\mathcal{S}$:

$$f(\mathcal{S}) = \sum_{(u,v)\in\mathcal{S}} w_{uv}.$$

- In this example,

$$f(\mathcal{S}) = 4 + 5 + 4 + 2 = 20.$$

Maximum-Weight Bipartite Matching



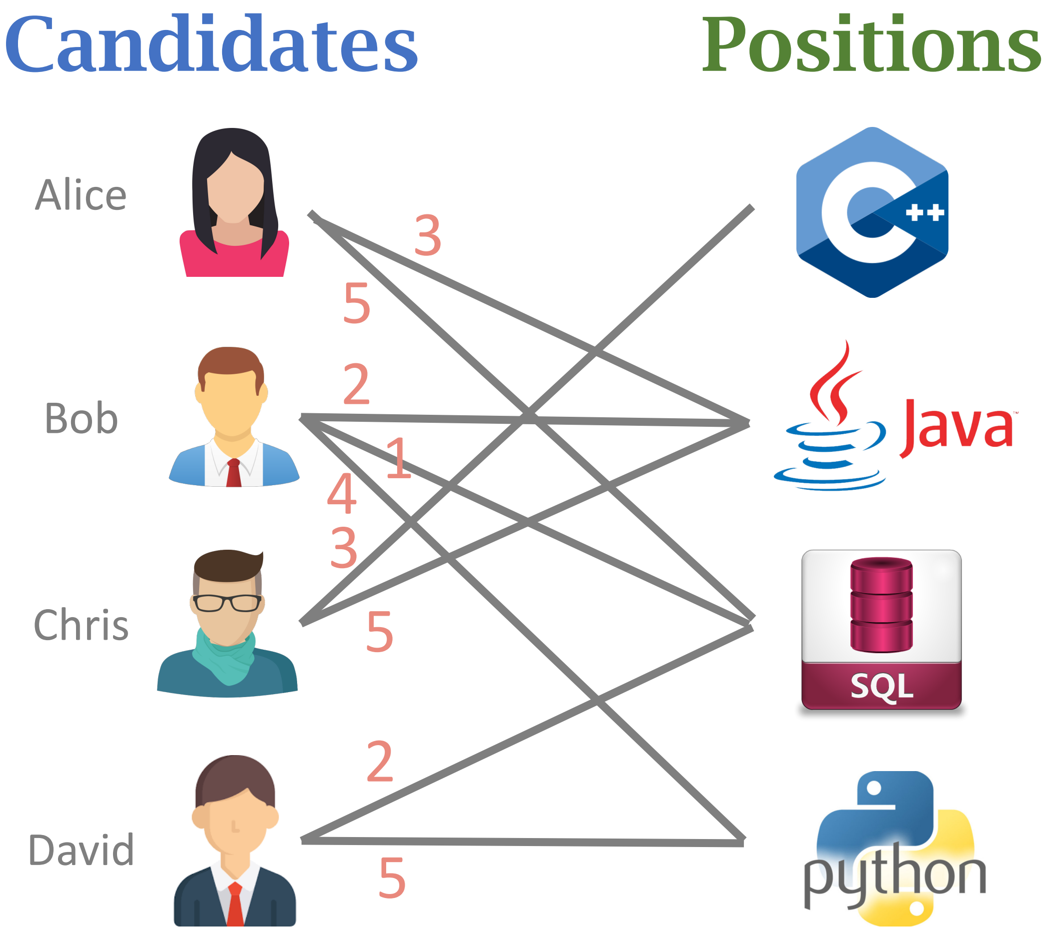
- Sum of weights in matching $\mathcal{S}$:

$$f(\mathcal{S}) = \sum_{(u,v)\in\mathcal{S}} w_{uv}.$$

- **Objective:** Finding matching $\mathcal{S}$ that has the maximum weight:

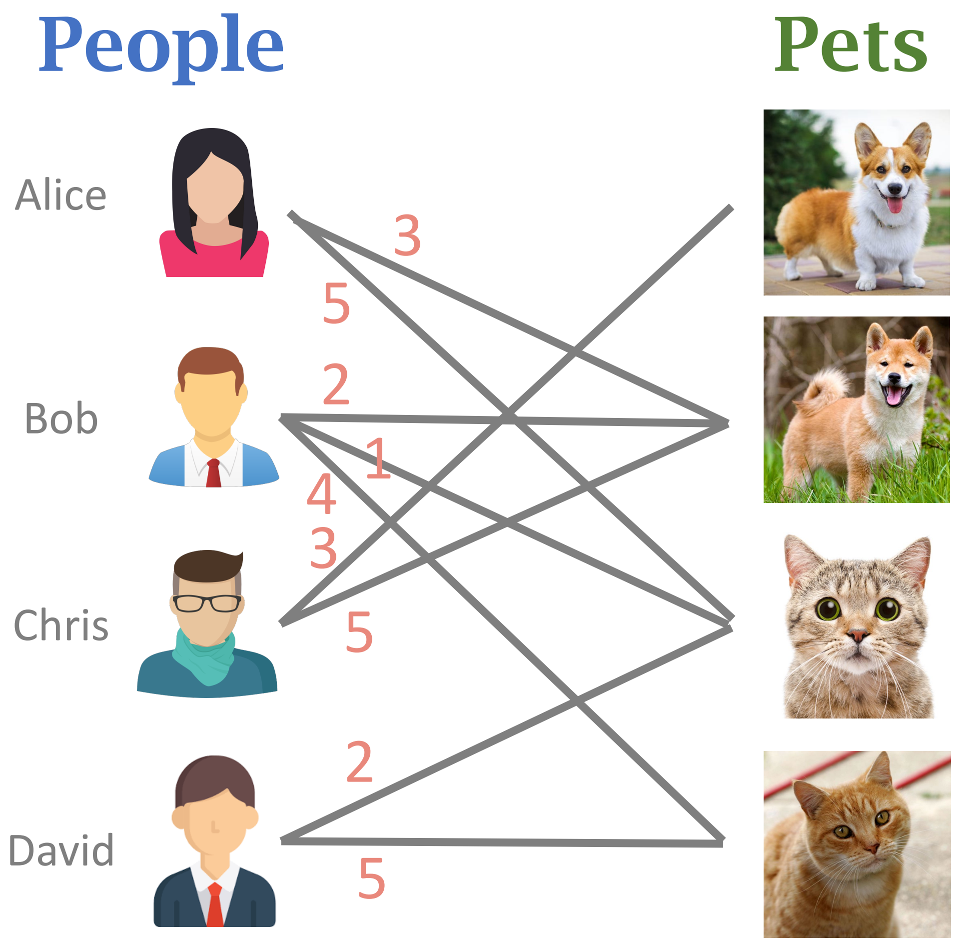
$$\max_{\mathcal{S}} f(\mathcal{S}).$$

Application 1: Match candidates and positions



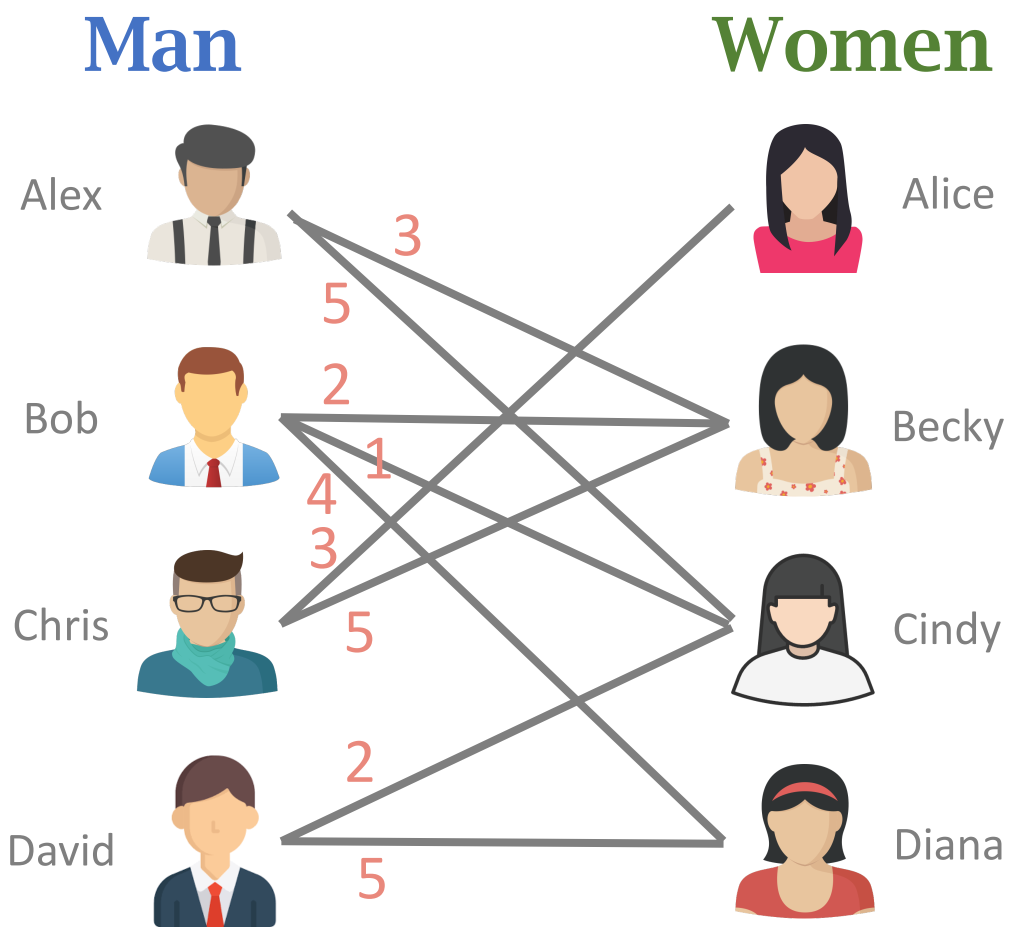
- Edge weights quantify candidates' skills.
- Maximize the weights of matching.
  (Match the right person with the right job position to maximize the company's interest.)

Application 2: Pet adoptions



- Edge weight quantifies how much a person loves a pet.
- Maximize the weights of matching. (Maximize people's happiness.)

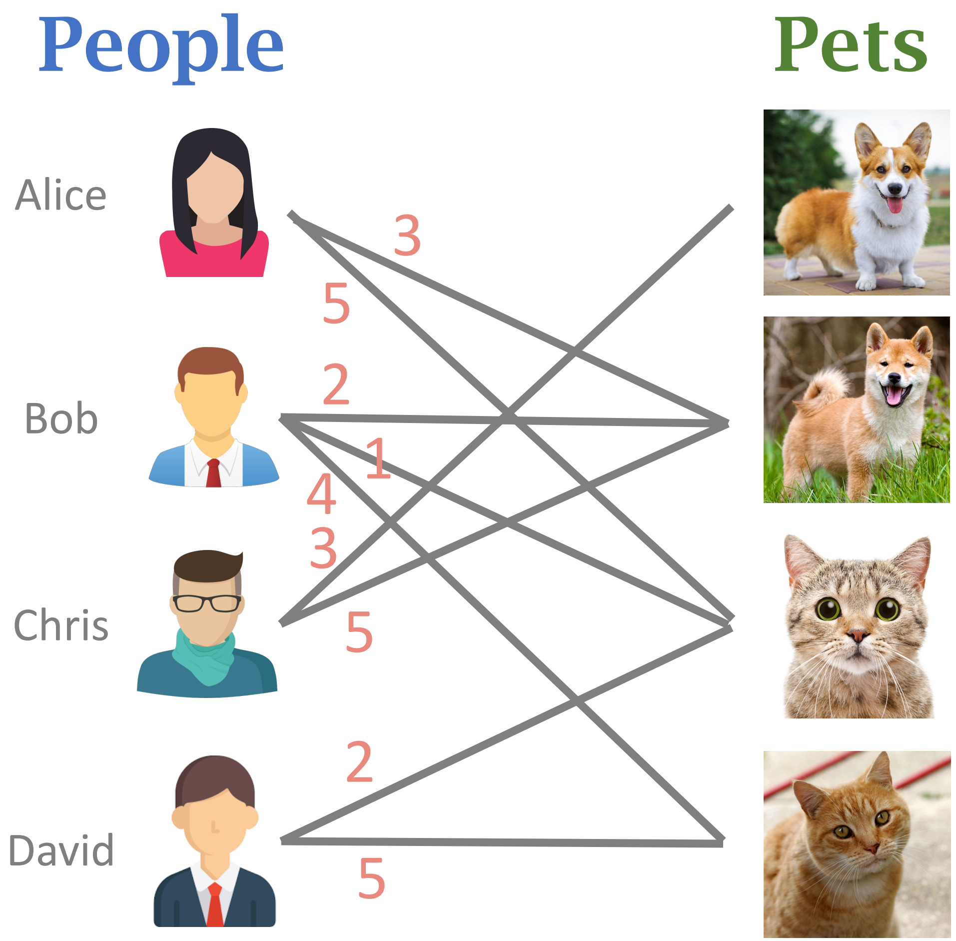
Application 3: Dating



- Edge weights quantify how well two persons match (e.g., similar hoppy).
- Maximize the weights of matching. (Maximize the change of success.)

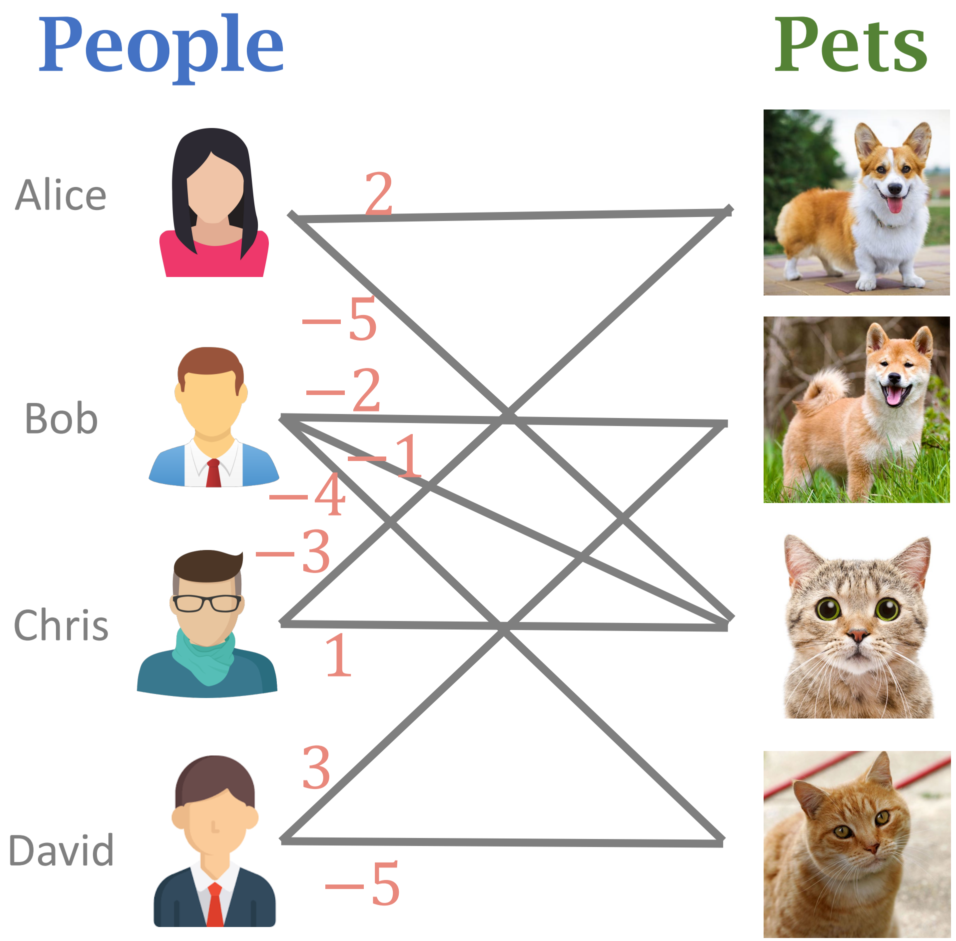
Maximum Matching ⟺ Minimum Matching

Maximum Matching



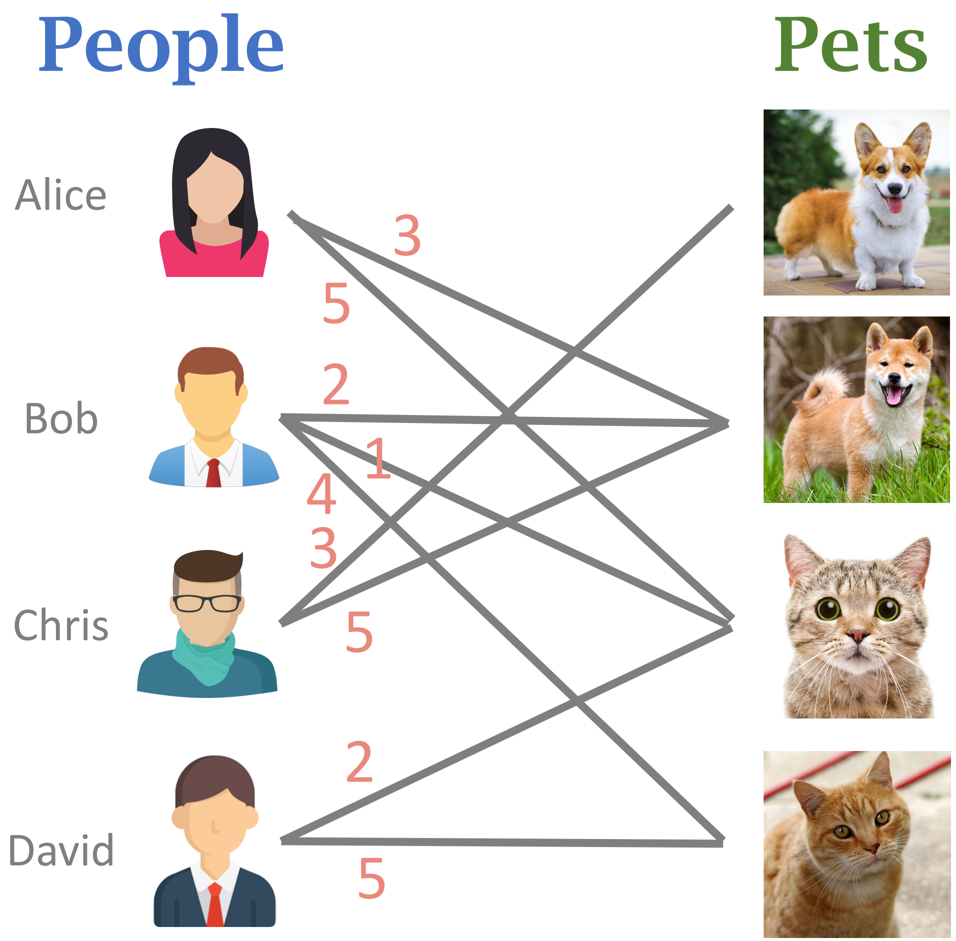
- Adopting a pet can bring happiness to people.
- A weight quantifies how much a person loves a pet.
- Maximize the weights of matching. (Maximize people's happiness.)

Minimum Matching



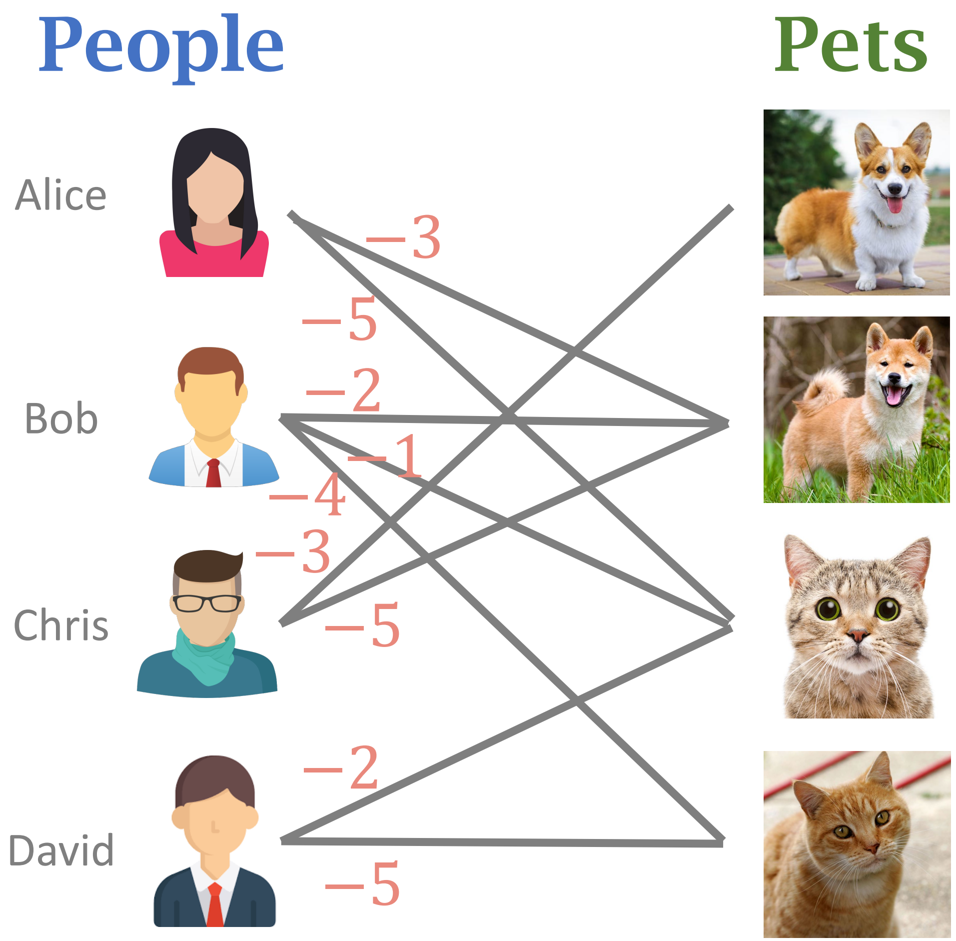
- Adopting a pet can cost time and money.
- A weight quantifies how a person dislike a pet.
- Minimize the weights of matching. (Maximize people's happiness.)

Maximum Matching ⟺ Minimum Matching



- If we have an algorithm for finding minimum matching.
- Then we can use it for finding maximum matching.

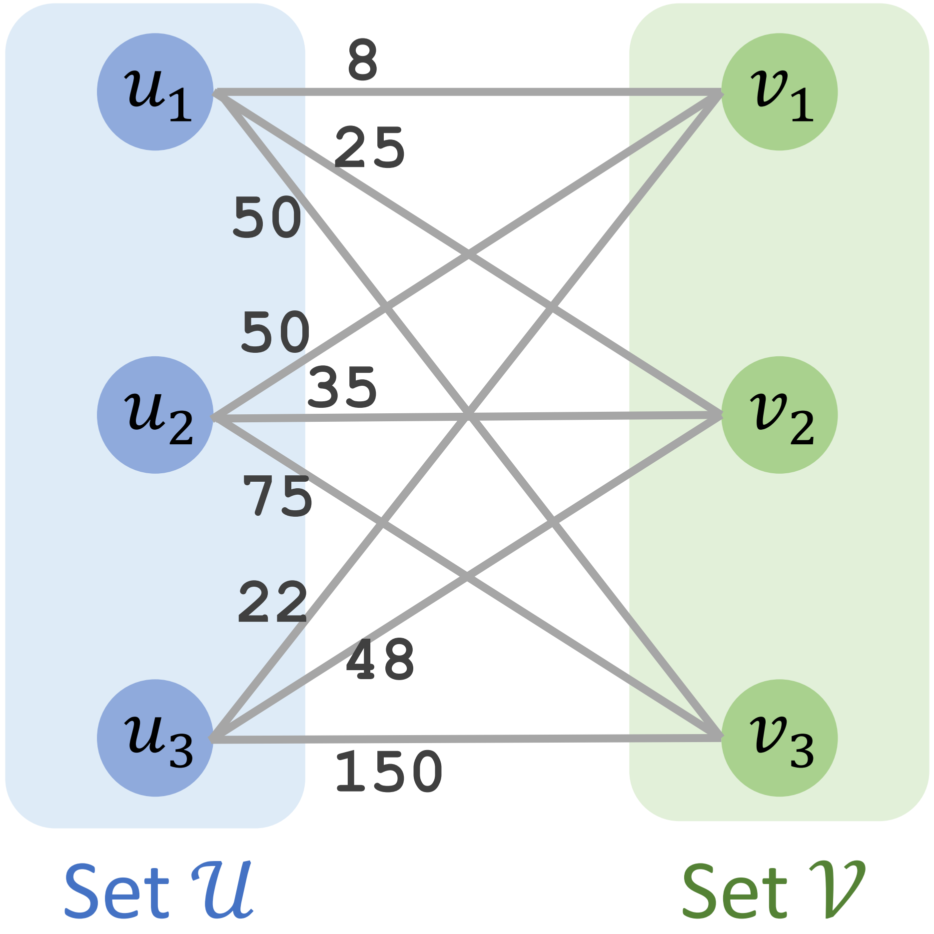
Maximum Matching ⟷ Minimum Matching



- If we have an algorithm for finding **minimum matching**.
- Then we can use it for finding **maximum matching**.
    1. Flip the signs of all the weights.
    2. Run the minimum matching algorithm.

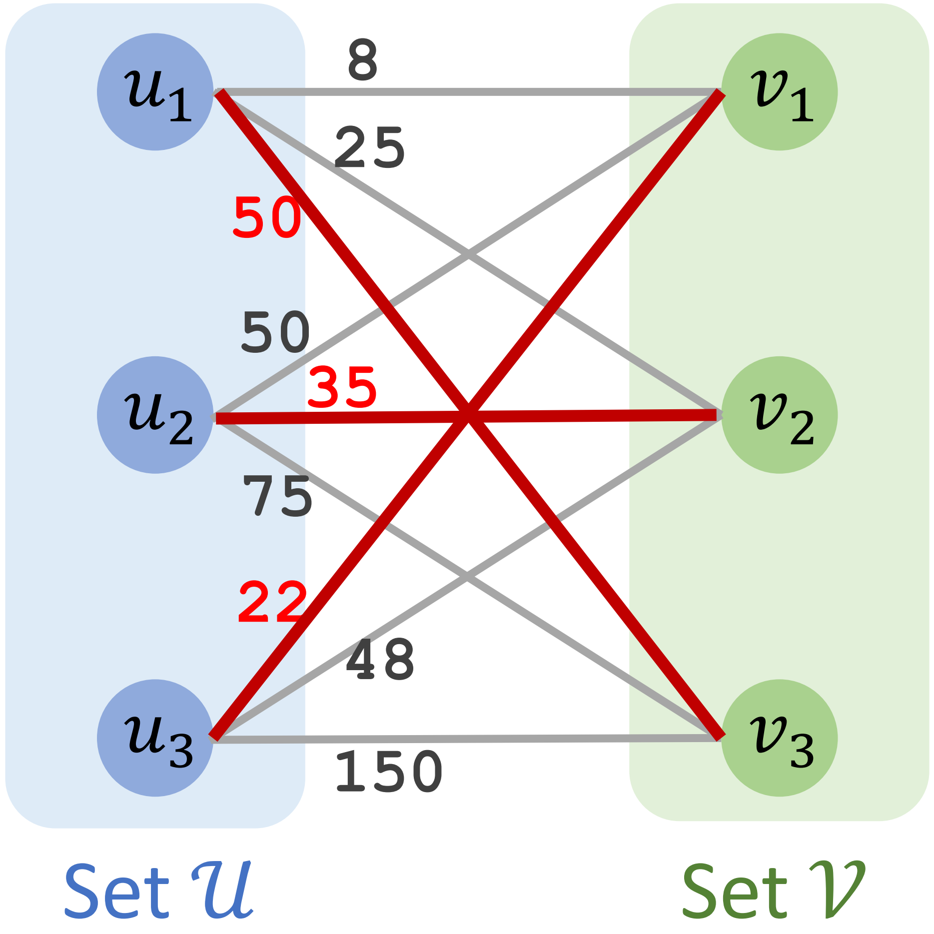
Hungarian Algorithm for Minimum-Weight Bipartite Matching

Minimum-Weight Bipartite Matching



	v_1	v_2	v_3
u_1	8	25	50
u_2	50	35	75
u_3	22	48	150

Minimum-Weight Bipartite Matching



	v_1	v_2	v_3
u_1	8	25	50
u_2	50	35	75
u_3	22	48	150

The minimum sum of weight is $50 + 35 + 22 = 107$.

Hungarian Algorithm

- Hungarian algorithm is for finding the minimum-weight bipartite matching.
- In the graph, the cardinality of $\mathcal{U}$ and $\mathcal{V}$ must be the same:

$$|\mathcal{U}| = |\mathcal{V}| = n.$$

- Time complexity: $O(n^3)$.

	v_1	v_2	v_3
u_1	8	25	50
u_2	50	35	75
u_3	22	48	150

Reference:

- Harold W. Kuhn. The Hungarian Method for the assignment problem. *Naval Research Logistics Quarterly*, 2: 83–97, 1955.

Hungarian Algorithm

- Step 1: Subtract row minima.

	v_1	v_2	v_3
u_1	8	25	50
u_2	50	35	75
u_3	22	48	150

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.

	v_1	v_2	v_3
u_1	8 −8	25 −8	50 −8
u_2	50 −35	35 −35	75 −35
u_3	22 −22	48 −22	150 −22

Hungarian Algorithm

- Step 1: Subtract row minima.
    - Subtract the smallest entry of each row from all the entries in the row.
    - The minimum of the row is equal to 0.

	v_1	v_2	v_3
u_1	0	17	42
u_2	15	0	40
u_3	0	26	128

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.
  - The minimum of the row is equal to 0.
- Step 2: Subtract column minima.
  - Subtract the smallest entry of each column from all the entries in the column.

	v_1	v_2	v_3
u_1	0 -0	17 -0	42 -40
u_2	15 -0	0 -0	40 -40
u_3	0 -0	26 -0	128 -40

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.
  - The minimum of the row is equal to 0.
- Step 2: Subtract column minima.
  - Subtract the smallest entry of each column from all the entries in the column.
  - The minimum of the column is equal to 0.

	v_1	v_2	v_3
u_1	0	17	2
u_2	15	0	0
u_3	0	26	88

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
  - Use either horizontal or vertical lines.
  - Minimize the total number of lines.

	v_1	v_2	v_3
u_1	0	17	2
u_2	15	0	0
u_3	0	26	88

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
  - Use either horizontal or vertical lines.
  - Minimize the total number of lines.

	v_1	v_2	v_3
u_1	~~0~~	~~17~~	~~2~~
u_2	~~15~~	~~0~~	~~0~~
u_3	~~0~~	~~26~~	~~88~~

- The number of line is 3.
- It is NOT the minimum.

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
  - Use either horizontal or vertical lines.
  - Minimize the total number of lines.

	v_1	v_2	v_3
u_1	0	17	2
u_2	15	0	0
u_3	0	26	88

- The number of line is 2.
- It is the minimum.

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
  - If n lines are required, the algorithm stops.
  - If less than n lines are required, continue with Step 3C.

- The number of line is 2.
- Number of vertices is $n = 3$.
- Thus continue to Step 3C.

	v_1	v_2	v_3
u_1	0	17	2
u_2	15	0	0
u_3	0	26	88

Hungarian Algorithm

Step 3: Repeat the following:

- Step 3A: Cover all zeros with a minimum number of lines.
- Step 3B: Decide whether to stop.
- Step 3C: Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.

	v_1	v_2	v_3
u_1	0	17	2
u_2	15	0	0
u_3	0	26	88

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.

	v_1	v_2	v_3
u_1	0	17	2 =k
u_2	15	0	0
u_3	0	26	88

The smallest element that is not covered is $k = 2$.

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.

	v_1	v_2	v_3
u_1	0	17 -2	2 -2
u_2	15	0	0
u_3	0	26 -2	88 -2

The smallest element that is not covered is $k = 2$.

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
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- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.

$$\begin{array}{c|ccc|} & v_1 & v_2 & v_3 \\ \hline u_1 & 0 & 15 & 0 \\ u_2 & 15 & 0 & 0 \\ u_3 & 0 & 24 & 86 \\ \hline \end{array}$$

Hungarian Algorithm

Step 3: Repeat the following:

- Step 3A: Cover all zeros with a minimum number of lines.
- Step 3B: Decide whether to stop.
- Step 3C: Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.
  - Add k to all elements that are covered twice.

	v_1	v_2	v_3
u_1	0	15	0
u_2	15 +2	0	0
u_3	0	24	86

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
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  - Add k to all elements that are covered twice.

	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.

	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.

	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

At least 3 lines are needed.

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
  - If n lines are required, the algorithm stops.

The algorithm stops.

	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

Output the matching

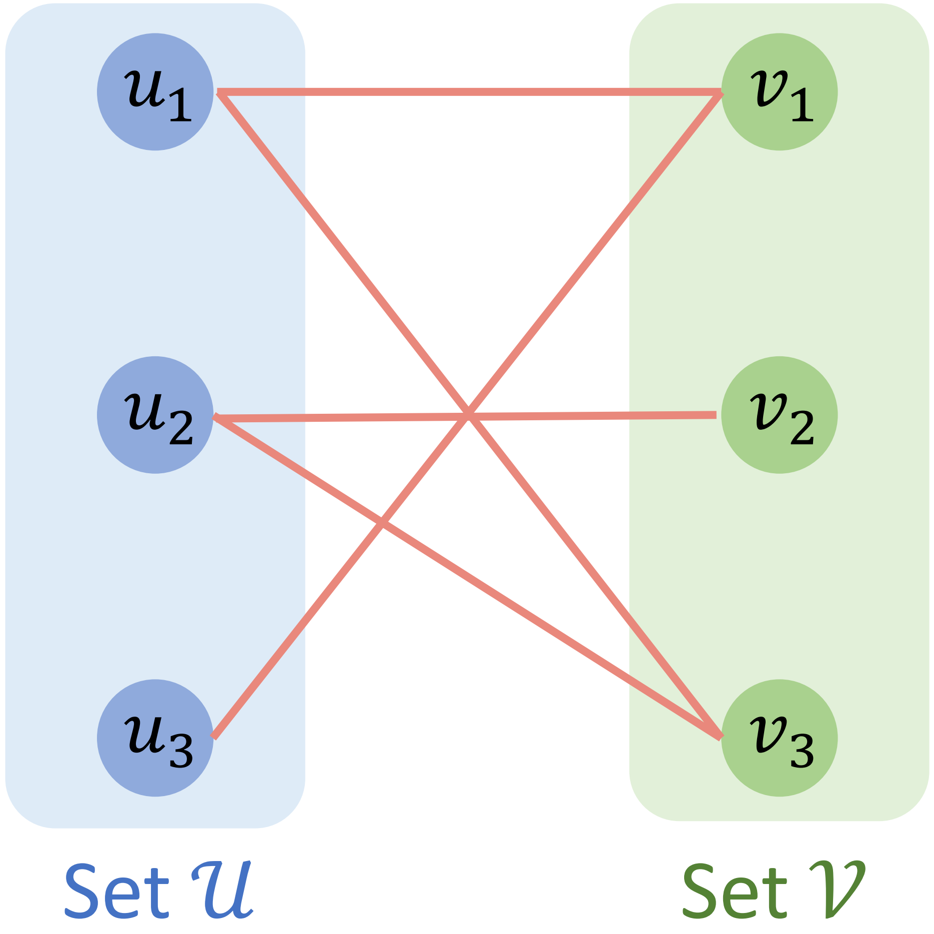
	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

Output the matching

	v_1	v_2	v_3
u_1	**0**	15	**0**
u_2	17	**0**	**0**
u_3	**0**	24	86

- Choose a matching among the zeros.
- Think of the zeros as edges.

Output the matching



	v_1	v_2	v_3
u_1	**0**	15	**0**
u_2	17	**0**	**0**
u_3	**0**	24	86

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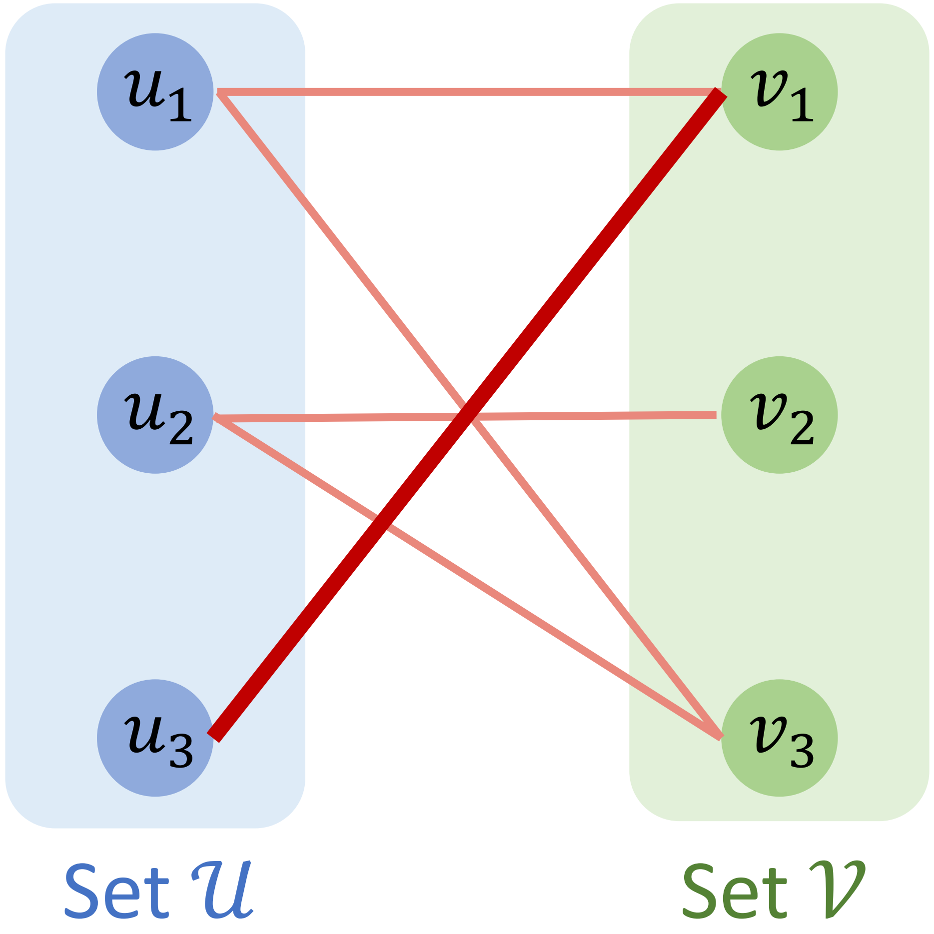
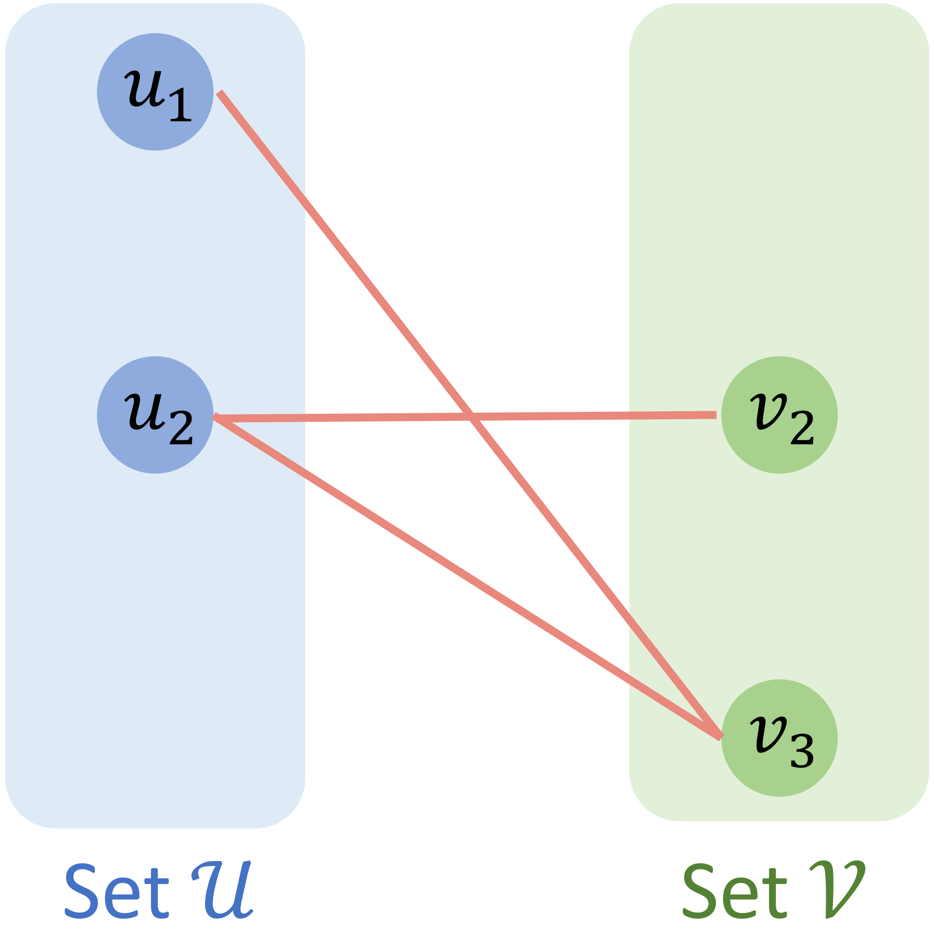
Output the matching



	v_1	v_2	v_3
u_1	**0**	15	**0**
u_2	17	**0**	**0**
u_3	**0**	24	86

- The edge (u_3, v_1) must be chosen.
- Because it is the only zero in the row.

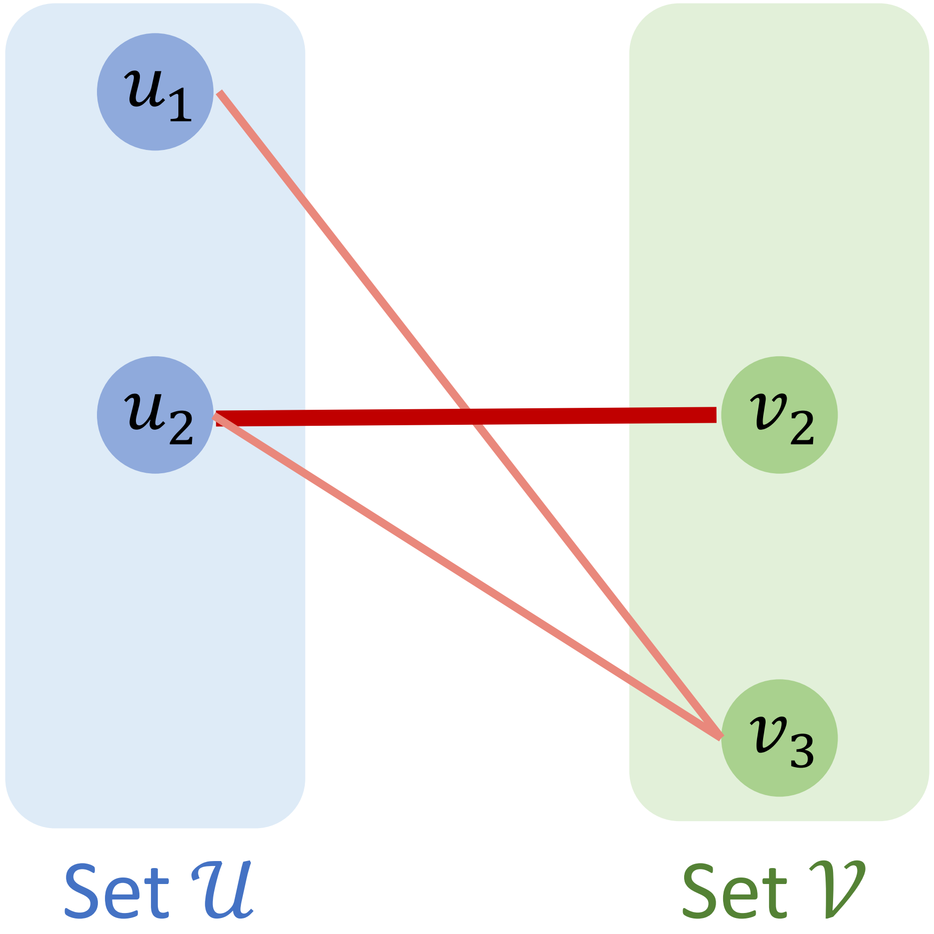
Output the matching



	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

- Cover the row of u_3.
- Cover the column of v_1.

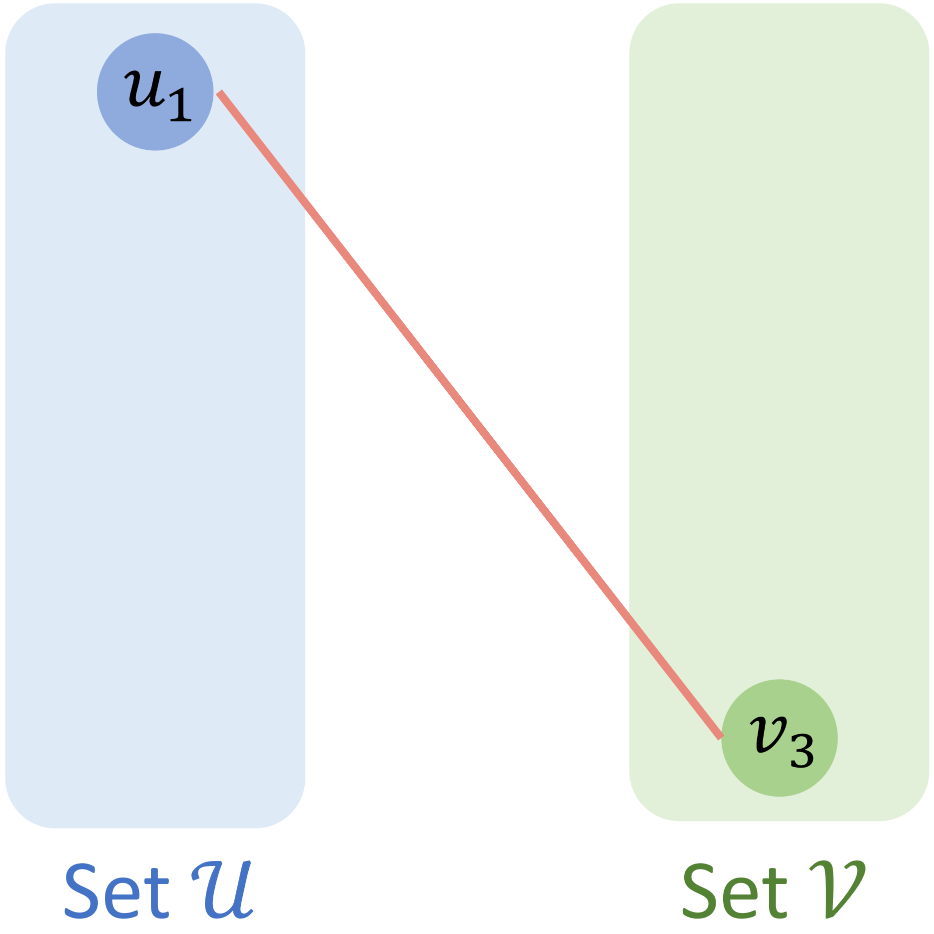
Output the matching



	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

- The edge (u_2, v_2) must be chosen.
- Because it is the only zero in the column.

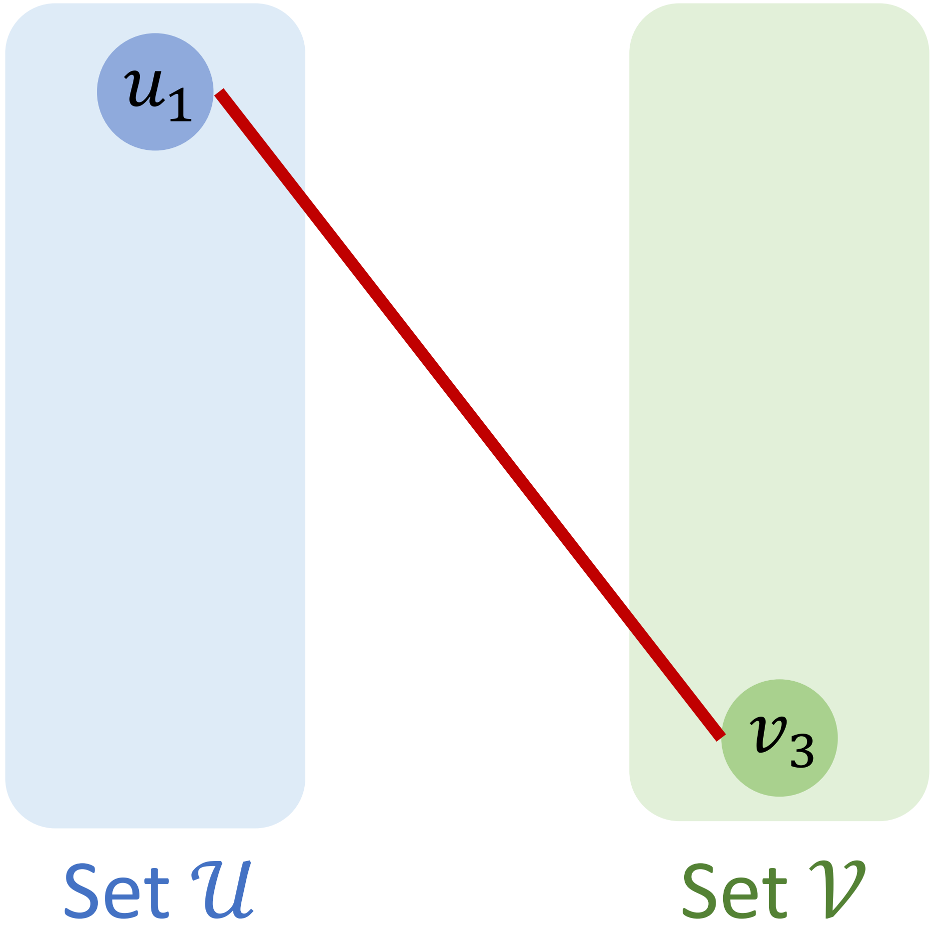
Output the matching



	v_1	v_2	v_3
u_1	0	15	0
u_2	17	0	0
u_3	0	24	86

- Cover the row of u_2.
- Cover the column of v_2.

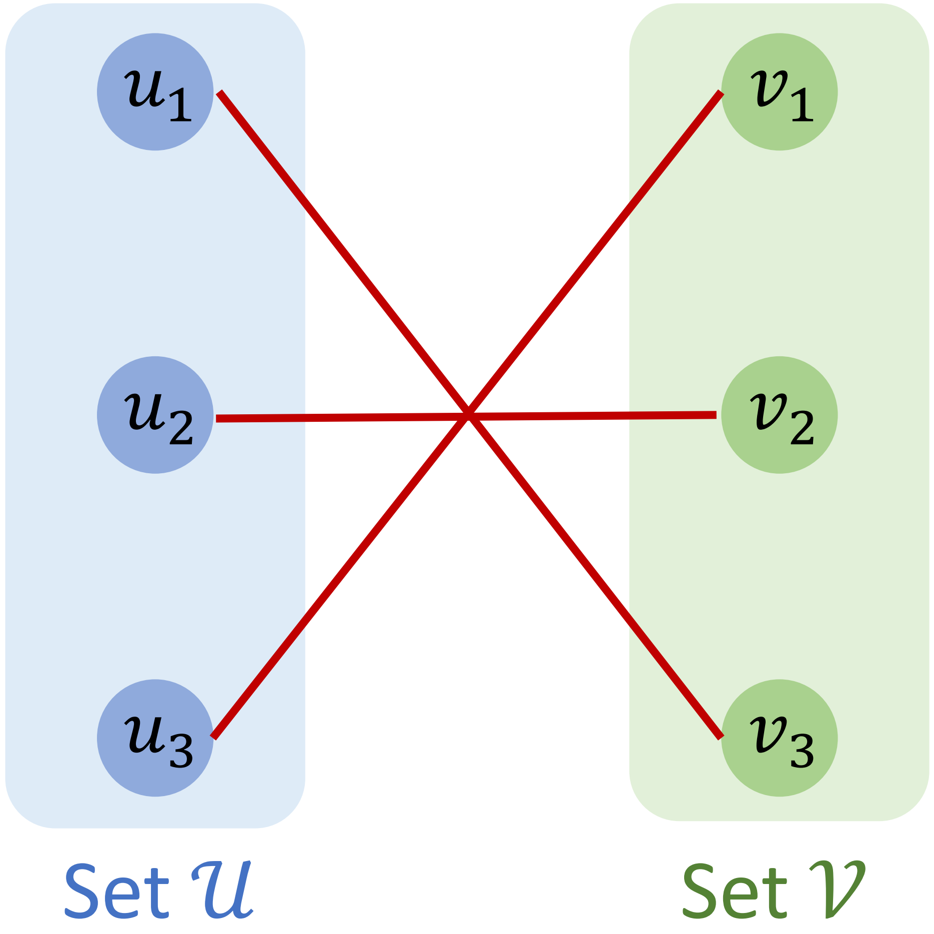
Output the matching



	v_1	v_2	v_3
u_1	0	15	**0**
u_2	17	0	0
u_3	0	24	86

- The edge (u_1, v_3) must be chosen.
- Because it is the only zero.

Output the matching



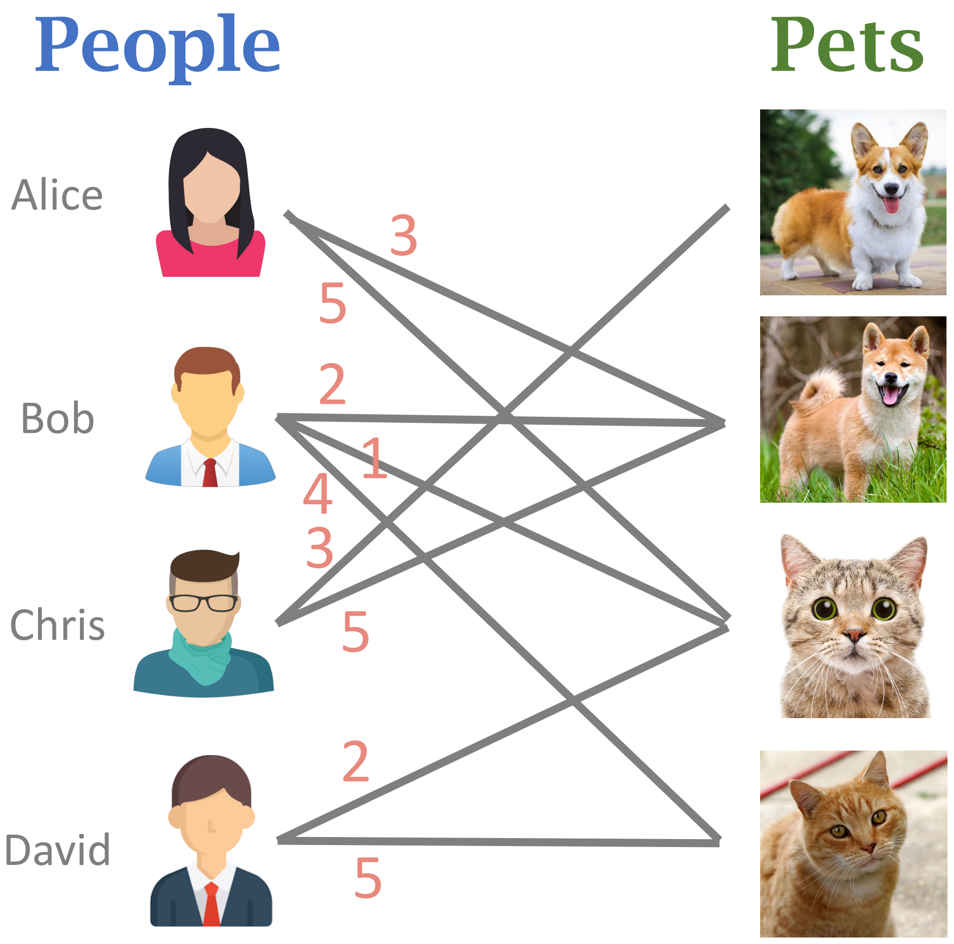
	v_1	v_2	v_3
u_1	**0**	15	**0**
u_2	17	**0**	**0**
u_3	**0**	24	86

The matching is

$$\mathcal{S} = \{(u_3, v_1),\ (u_1, v_3),\ (u_2, v_2)\}.$$

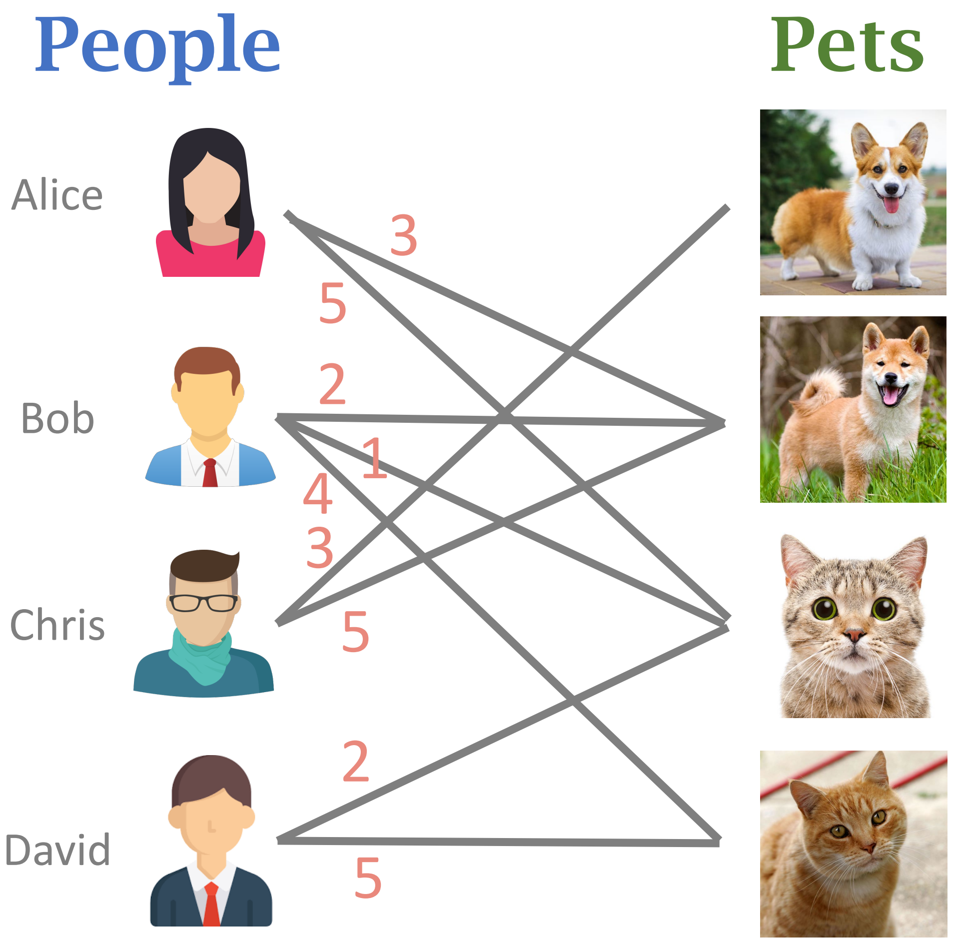
Hungarian Algorithm for Maximum Matching

Maximum Matching



- Pet adoption is a max matching problem.
- A weight quantifies how much a person loves a pet.
- Maximize the weights of matching. (Maximize people's happiness.)

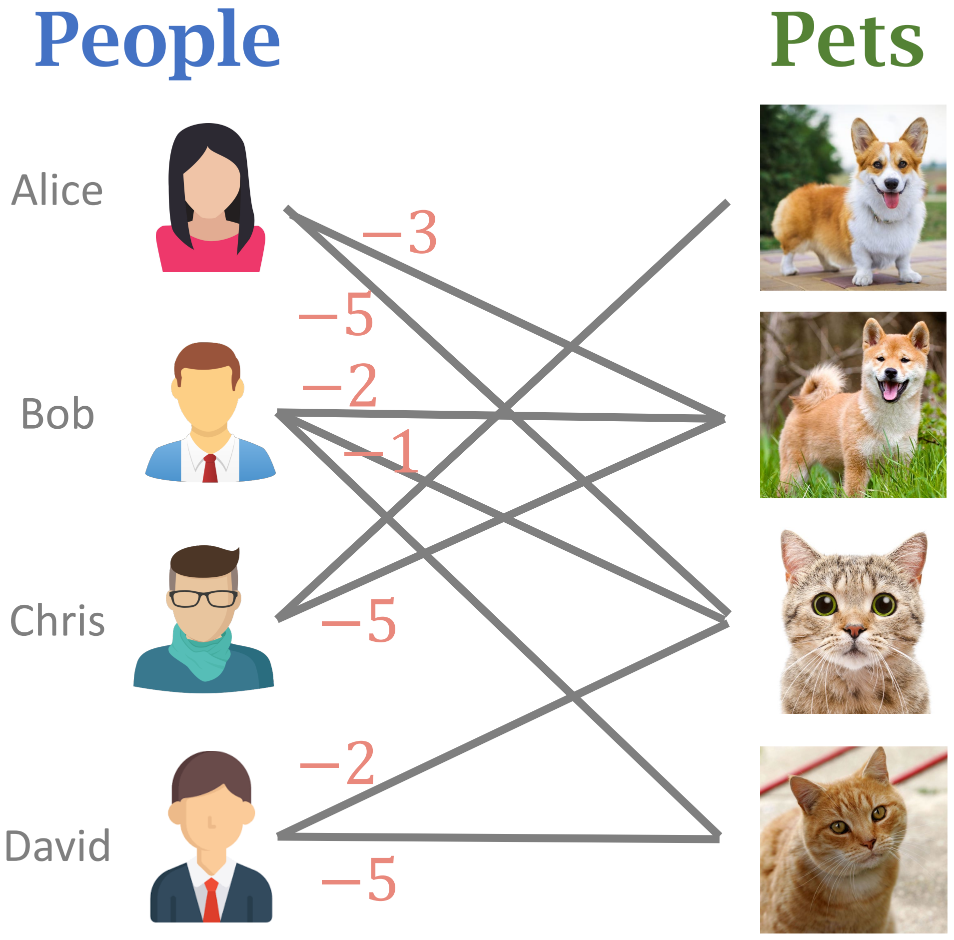
Hungarian Algorithm for Maximum Matching



Idea: Max Matching ➡ Min Matching

- Flip the signs of all the weights.
- It is equivalent to the minimum matching.
- Run the Hungarian algorithm.

Hungarian Algorithm for Maximum Matching



Idea: Max Matching ➡ Min Matching

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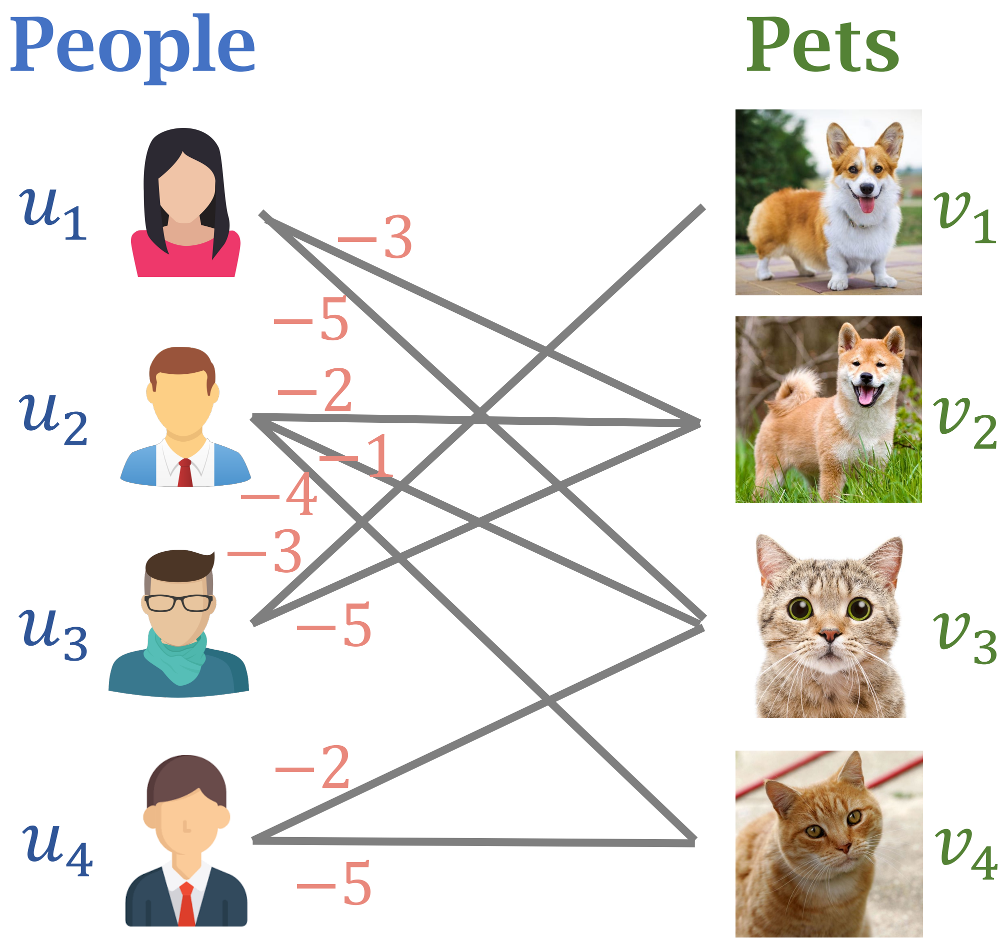
Hungarian Algorithm for Maximum Matching



	v_1	v_2	v_3	v_4
u_1	0	-3	-5	0
u_2	0	-2	-1	-4
u_3	-3	-5	0	0
u_4	0	0	-2	-5

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.

	v_1	v_2	v_3	v_4
u_1	0 −(−5)	−3 −(−5)	−5 −(−5)	0 −(−5)
u_2	0 −(−4)	−2 −(−4)	−1 −(−4)	−4 −(−4)
u_3	−3 −(−5)	−5 −(−5)	0 −(−5)	0 −(−5)
u_4	0 −(−5)	0 −(−5)	−2 −(−5)	−5 −(−5)

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.
  - The minimum of the row is equal to 0.

	v_1	v_2	v_3	v_4
u_1	5	2	0	5
u_2	4	2	3	0
u_3	2	0	5	5
u_4	5	5	3	0

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.
  - The minimum of the row is equal to 0.
- Step 2: Subtract column minima.
  - Subtract the smallest entry of each column from all the entries in the column.

	v_1	v_2	v_3	v_4
u_1	5 −2	2 −0	0 −0	5 −0
u_2	4 −2	2 −0	3 −0	0 −0
u_3	2 −2	0 −0	5 −0	5 −0
u_4	5 −2	5 −0	3 −0	0 −0

Hungarian Algorithm

- Step 1: Subtract row minima.
  - Subtract the smallest entry of each row from all the entries in the row.
  - The minimum of the row is equal to 0.
- Step 2: Subtract column minima.
  - Subtract the smallest entry of each column from all the entries in the column.
  - The minimum of the column is equal to 0.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	2	2	3	0
u_3	0	0	5	5
u_4	3	5	3	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
  - Use either horizontal or vertical lines.
  - Minimize the total number of lines.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	2	2	3	0
u_3	0	0	5	5
u_4	3	5	3	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
  - If n lines are required, the algorithm stops.
  - If less than n lines are required, continue with Step 3C.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	2	2	3	0
u_3	0	0	5	5
u_4	3	5	3	0

- The number of line is 3.
- Number of vertices is $n = 4$.
- Thus continue to Step 3C.

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	2	2	3	0
u_3	0	0	5	5
u_4	3	5	3	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
    - Find the smallest element (call it k) that is not covered by a line.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	2	2 =k	3	0
u_3	0	0	5	5
u_4	3	5	3	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	2 −2	2 −2	3 −2	0
u_3	0	0	5	5
u_4	3 −2	5 −2	3 −2	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5
u_2	0	0	1	0
u_3	0	0	5	5
u_4	1	3	1	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.
  - Add k to all elements that are covered twice.

	v_1	v_2	v_3	v_4
u_1	3	2	0	5 +2
u_2	0	0	1	0
u_3	0	0	5	5 +2
u_4	1	3	1	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
- **Step 3C:** Create additional zeros.
  - Find the smallest element (call it k) that is not covered by a line.
  - Subtract k from all uncovered elements.
  - Add k to all elements that are covered twice.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Hungarian Algorithm

Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
  - Use either horizontal or vertical lines.
  - Minimize the total number of lines.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Hungarian Algorithm

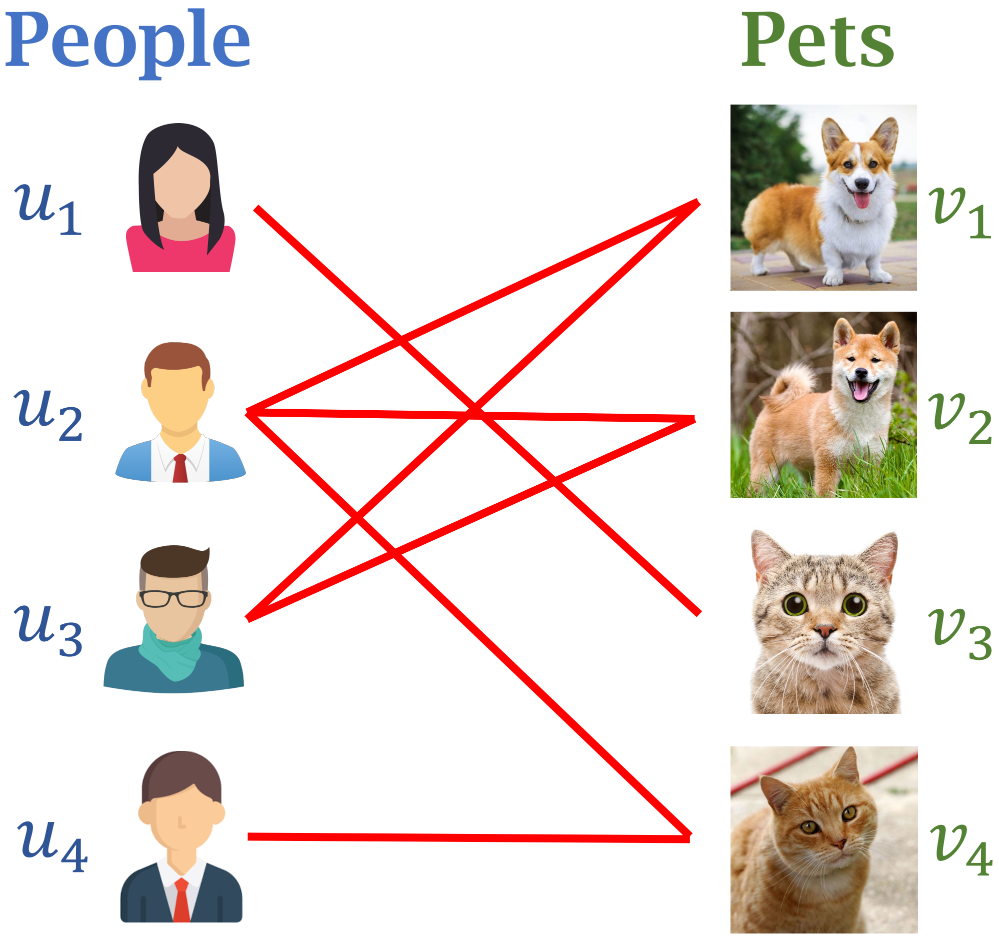
Step 3: Repeat the following:

- **Step 3A:** Cover all zeros with a minimum number of lines.
- **Step 3B:** Decide whether to stop.
  - If n lines are required, the algorithm stops.

The algorithm stops.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching



	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- The edge (u_1, v_3) must be selected.
  - Otherwise, u_1 would have no matching.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- The edge (u_1, v_3) must be selected.
  - Otherwise, u_1 would have no matching.
  - Cover the row of u_1.
  - Cover the column of v_3.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- The edge (u_4, v_4) must be selected.
  - Otherwise, u_4 would have no matching.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- The edge (u_4, v_4) must be selected.
  - Otherwise, u_4 would have no matching.
  - Cover the row of u_4.
  - Cover the column of v_4.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- Select edges (u_2, v_1) and (u_3, v_2).

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- Select edges (u_2, v_1) and (u_3, v_2).
- Or select edges (u_3, v_1) and (u_2, v_2).

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- Return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_2, v_1), (u_3, v_2)\}$.

- Or return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_3, v_1), (u_2, v_2)\}$.

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- Return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_2, v_1), (u_3, v_2)\}.$

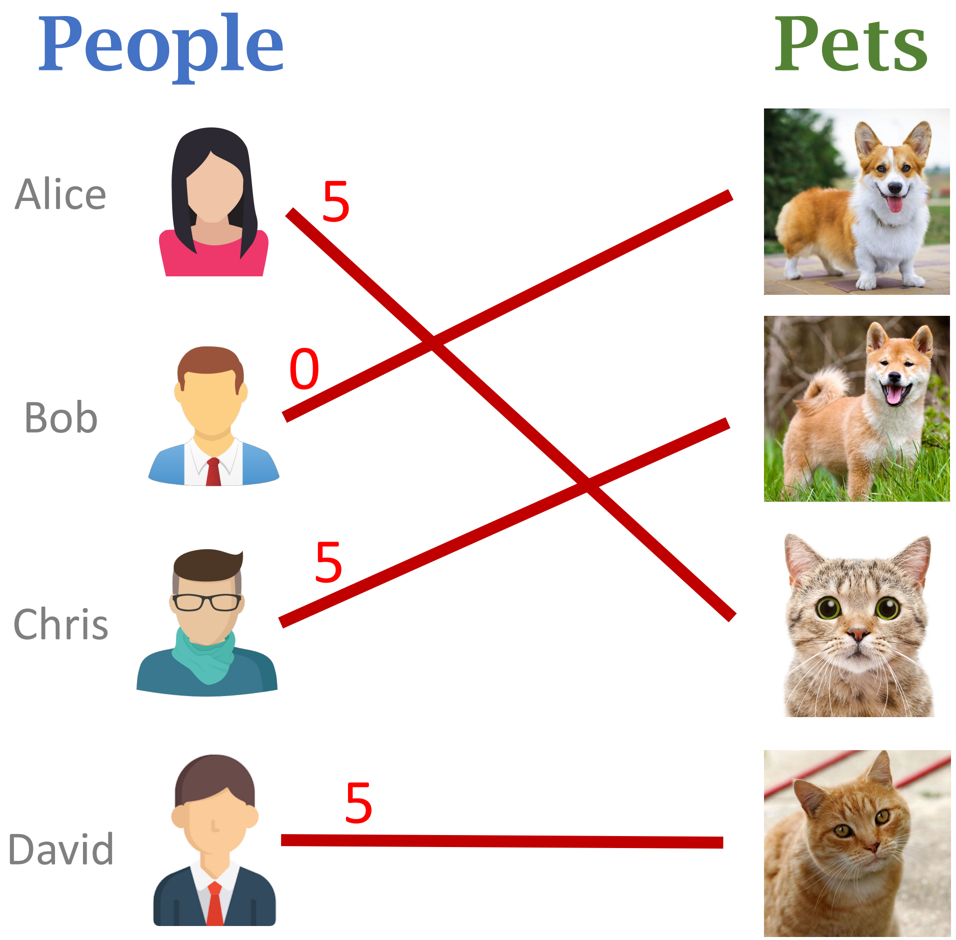
- Or return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_3, v_1), (u_2, v_2)\}.$

	v_1	v_2	v_3	v_4
u_1	3	2	0	7
u_2	0	0	1	0
u_3	0	0	5	7
u_4	1	3	1	0

Output the matching

- Return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_2, v_1), (u_3, v_2)\}.$
- The matching is equal to 15.

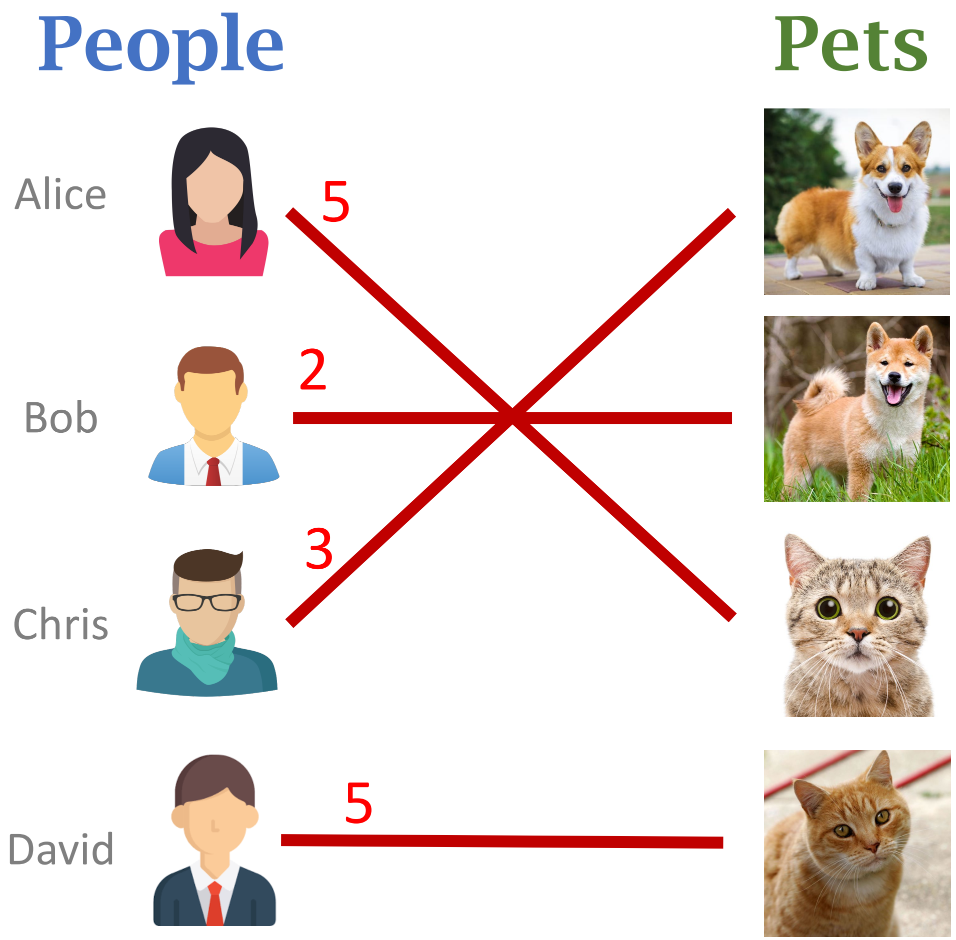
- Or return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_3, v_1), (u_2, v_2)\}.$



Output the matching

- Return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_2, v_1), (u_3, v_2)\}.$
- The matching is equal to 15.

- Or return the matching:
  $\mathcal{S} = \{(u_1, v_3), (u_4, v_4), (u_3, v_1), (u_2, v_2)\}.$
- The matching is equal to 15.



Summary

Maximum-Weight Bipartite Matching

- Weighted bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$. (Edges have weights: w_{uv}.)
- Matching is a subset of edges without common vertices.
- Denote the matching by set $\mathcal{S} \subseteq \mathcal{E}$.
- Sum of weights in matching $\mathcal{S}$:

$$f(\mathcal{S}) = \sum_{(u,v)\in\mathcal{S}} w_{uv}.$$

- Find matching $\mathcal{S}$ that has the maximum weight:

$$\max_{\mathcal{S}} f(\mathcal{S}).$$

Maximum Matching $\Longleftrightarrow$ Minimum Matching

- Maximum matching: $\max_{\mathcal{S}} f(\mathcal{S})$.
- Minimum matching: $\min_{\mathcal{S}} f(\mathcal{S})$.
- The maximum matching problem can be reduced to minimum matching problem by flipping the signs of weights.
- Algorithms that solve minimum matching can also solve the maximum matching problem.

Hungarian Algorithm

- Hungarian algorithm finds minimum-weight bipartite matching.
- It requires $|\mathcal{U}| = |\mathcal{V}| = n$.
- Time complexity: $O(n^3)$.

Questions

Question 1

- The right is the adjacency matrix of a bipartite graph.
- Find the minimum matching in the graph.

	v_1	v_2	v_3	v_4	v_5
u_1	20	15	18	24	25
u_2	18	20	12	14	15
u_3	21	23	25	27	26
u_4	17	18	21	23	22
u_5	19	22	16	21	20

Question 2

- The right is the adjacency matrix of a bipartite graph.
- Find the maximum matching in the graph.

	v_1	v_2	v_3	v_4	v_5
u_1	20	15	18	24	25
u_2	18	20	12	14	15
u_3	21	23	25	27	26
u_4	17	18	21	23	22
u_5	19	22	16	21	20

Thank You!