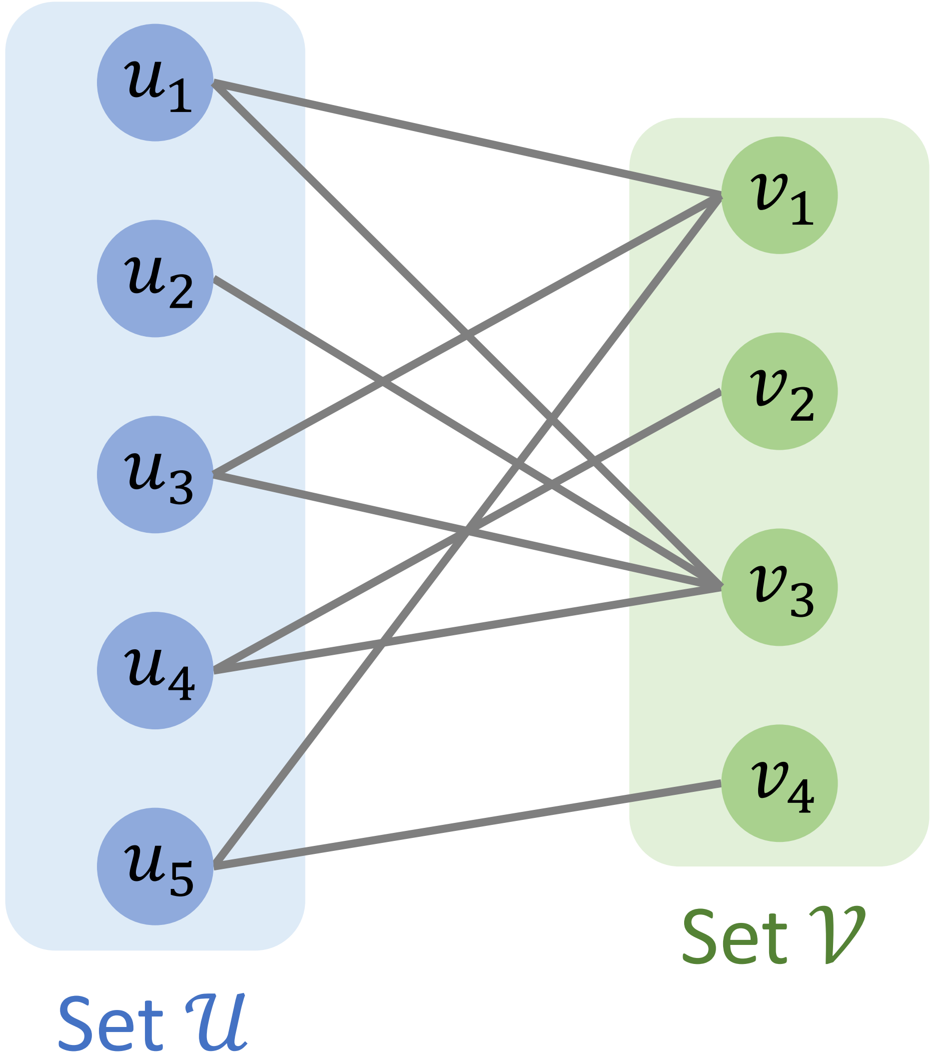
Maximum-Cardinality Bipartite Matching

Shusen Wang

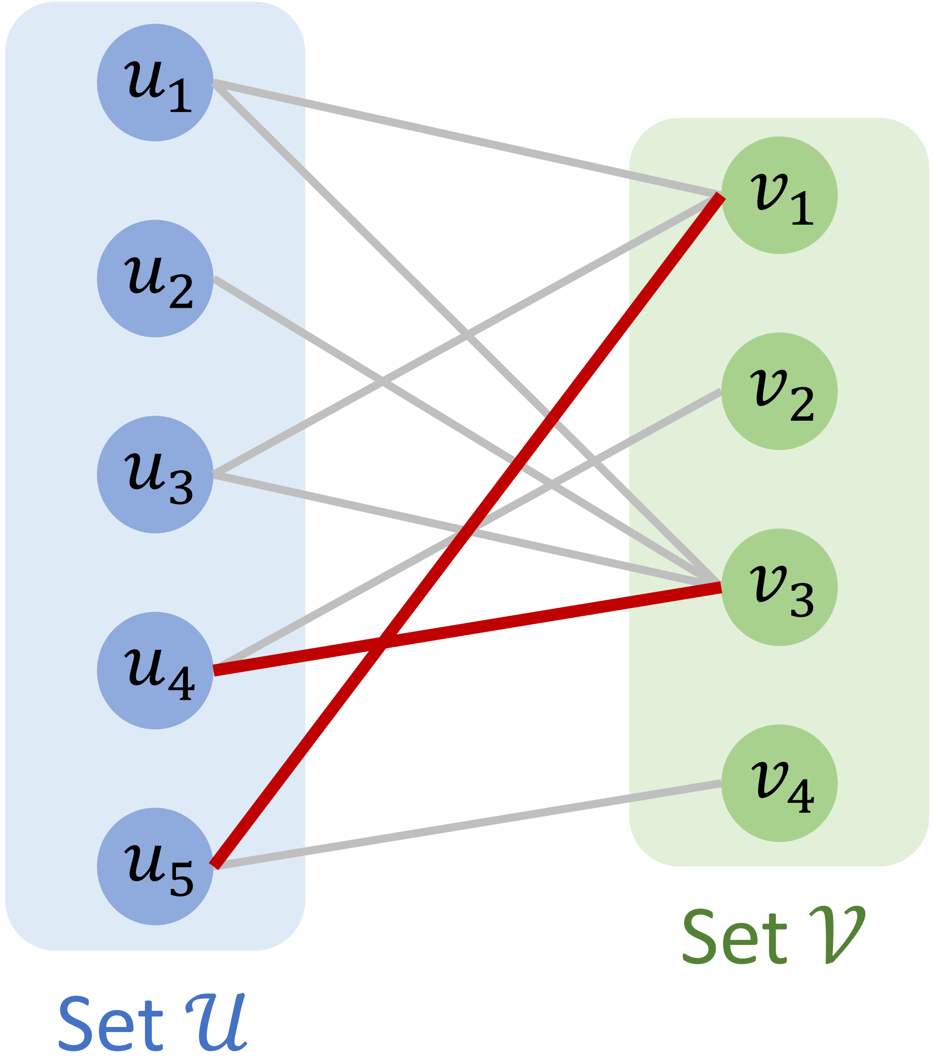
Definition

Bipartite Matching



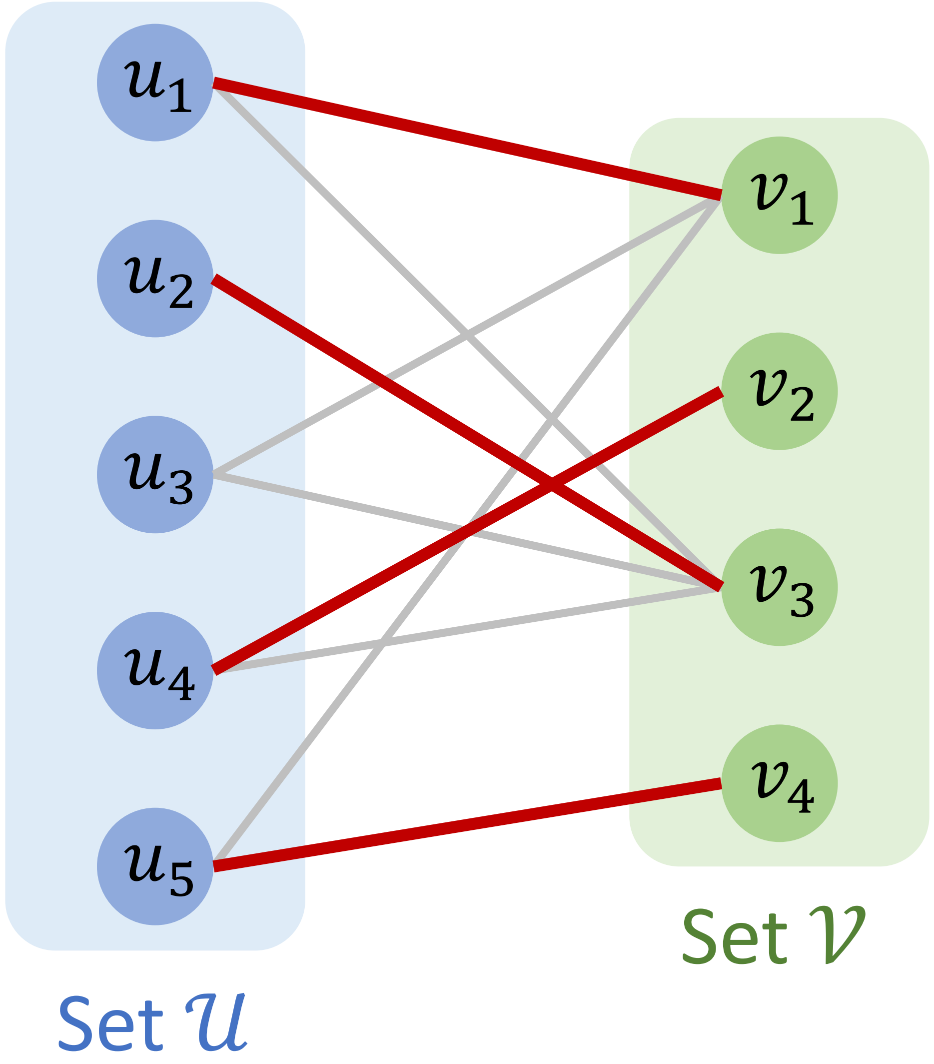
- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Matching is a subset of edges without common vertices.
- Denote the matching by set $\mathcal{S} \subseteq \mathcal{E}$.

Bipartite Matching



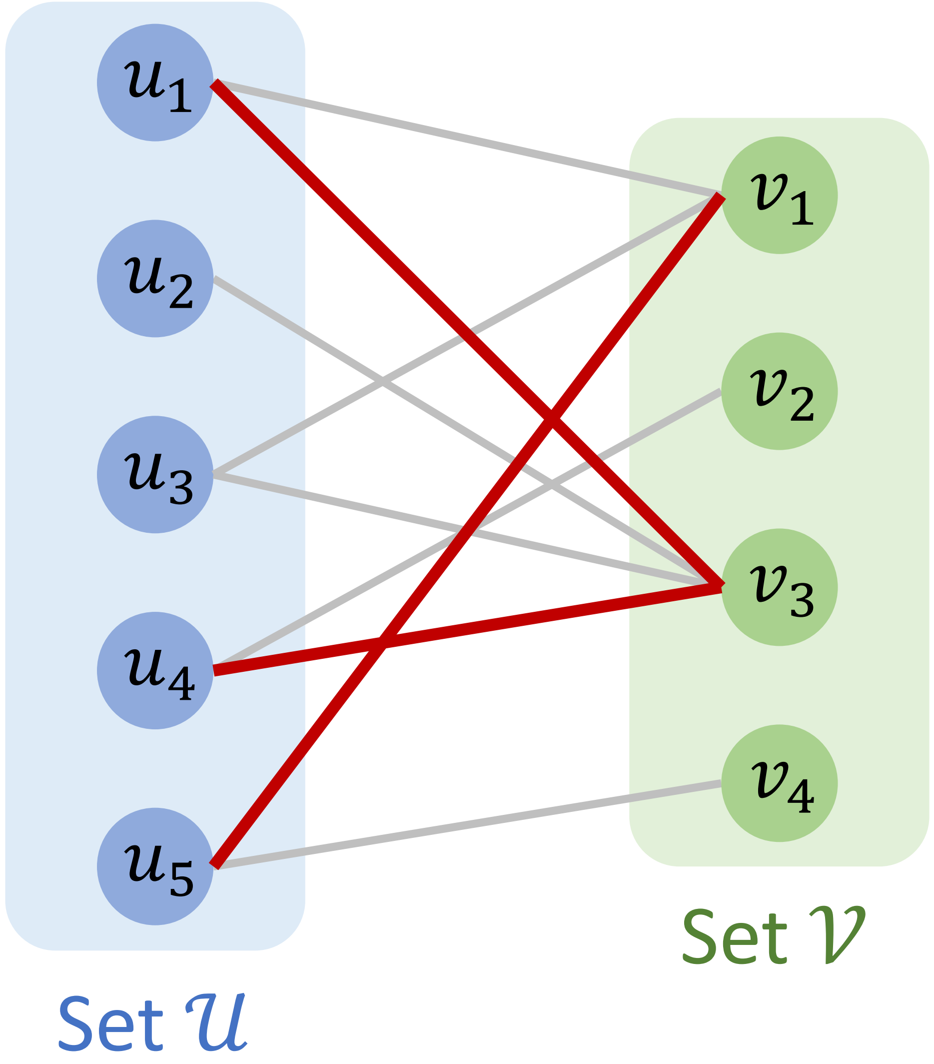
- $\mathcal{S} = \{e_{51}, e_{43}\}$ is a matching. (But not a maximum matching.)

Bipartite Matching



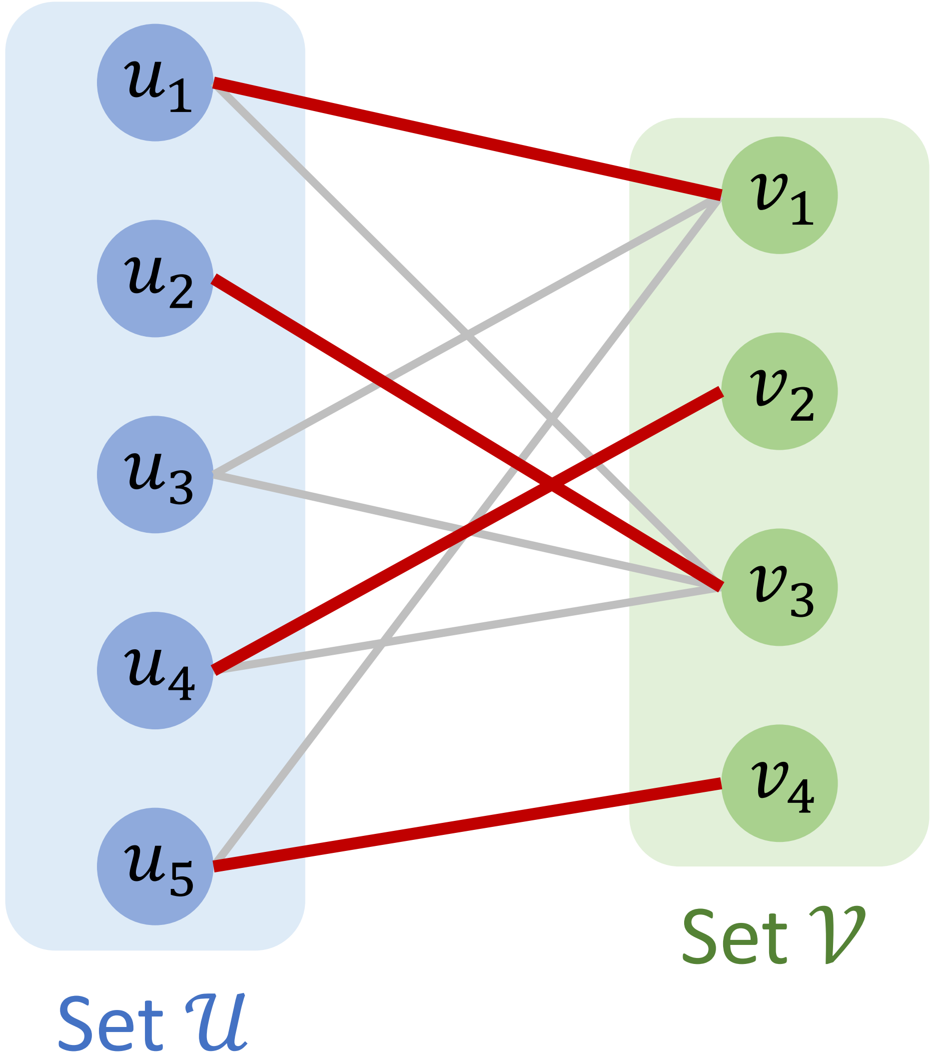
- $\mathcal{S} = \{e_{11}, e_{23}, e_{42}, e_{54}\}$ is a matching.

Bipartite Matching



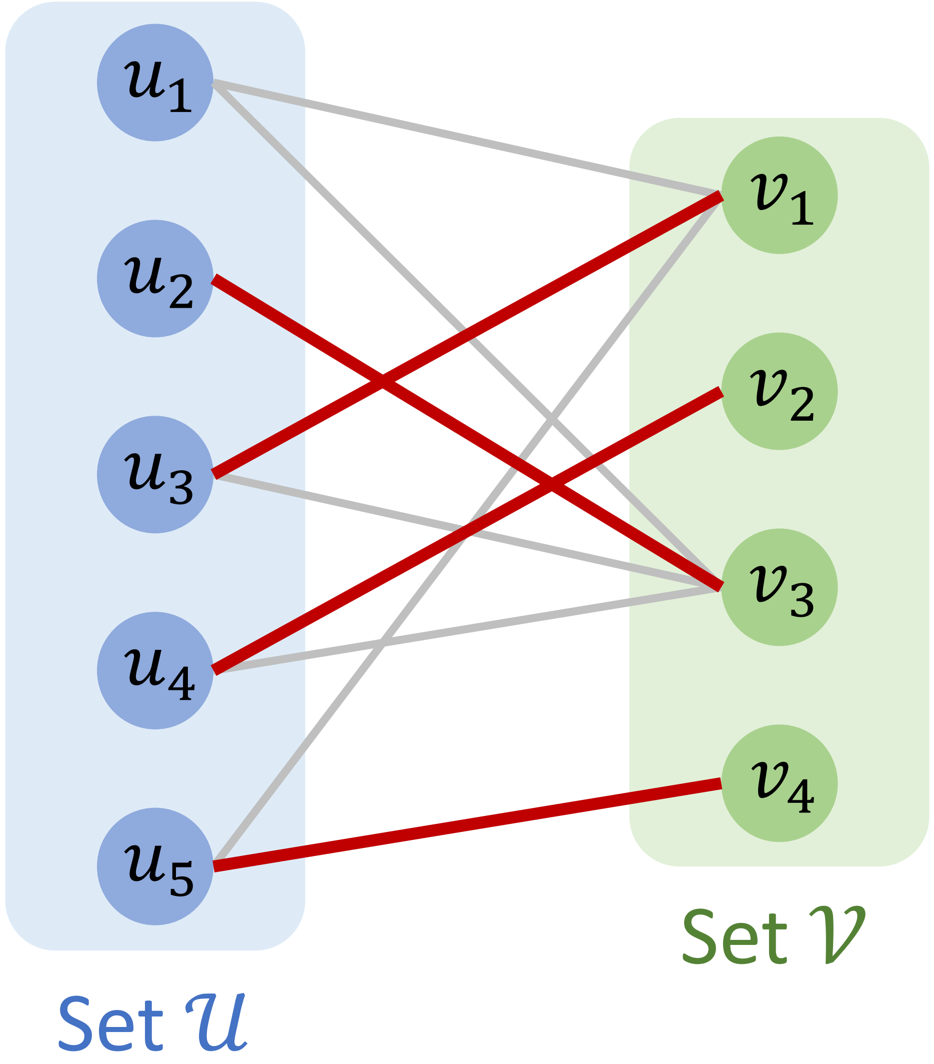
- $\{e_{13}, e_{51}, e_{43}\}$ is not a matching.
- Two edges have common vertex.

Maximum-Cardinality Bipartite Matching (MCBM)



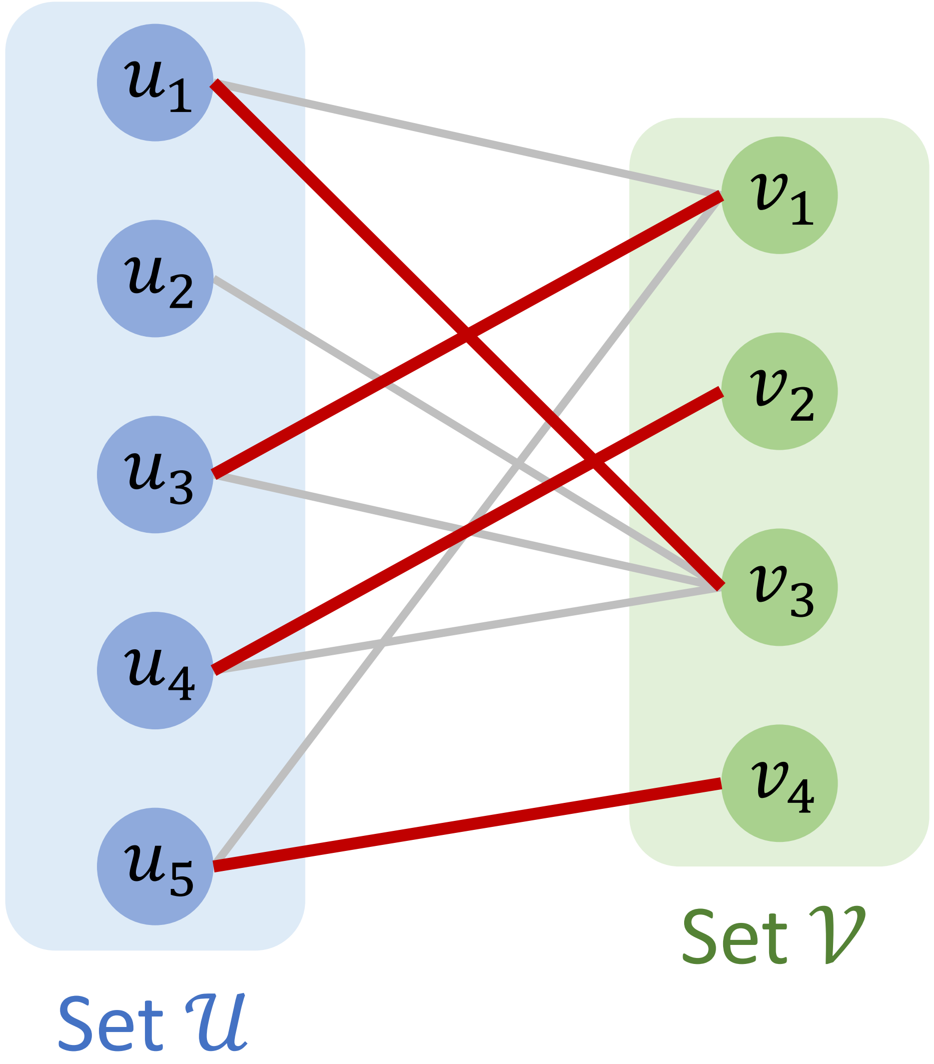
- **MCBM:** The matching $\mathcal{S}$ that has the maximum cardinality $|\mathcal{S}|$.

Maximum-Cardinality Bipartite Matching (MCBM)



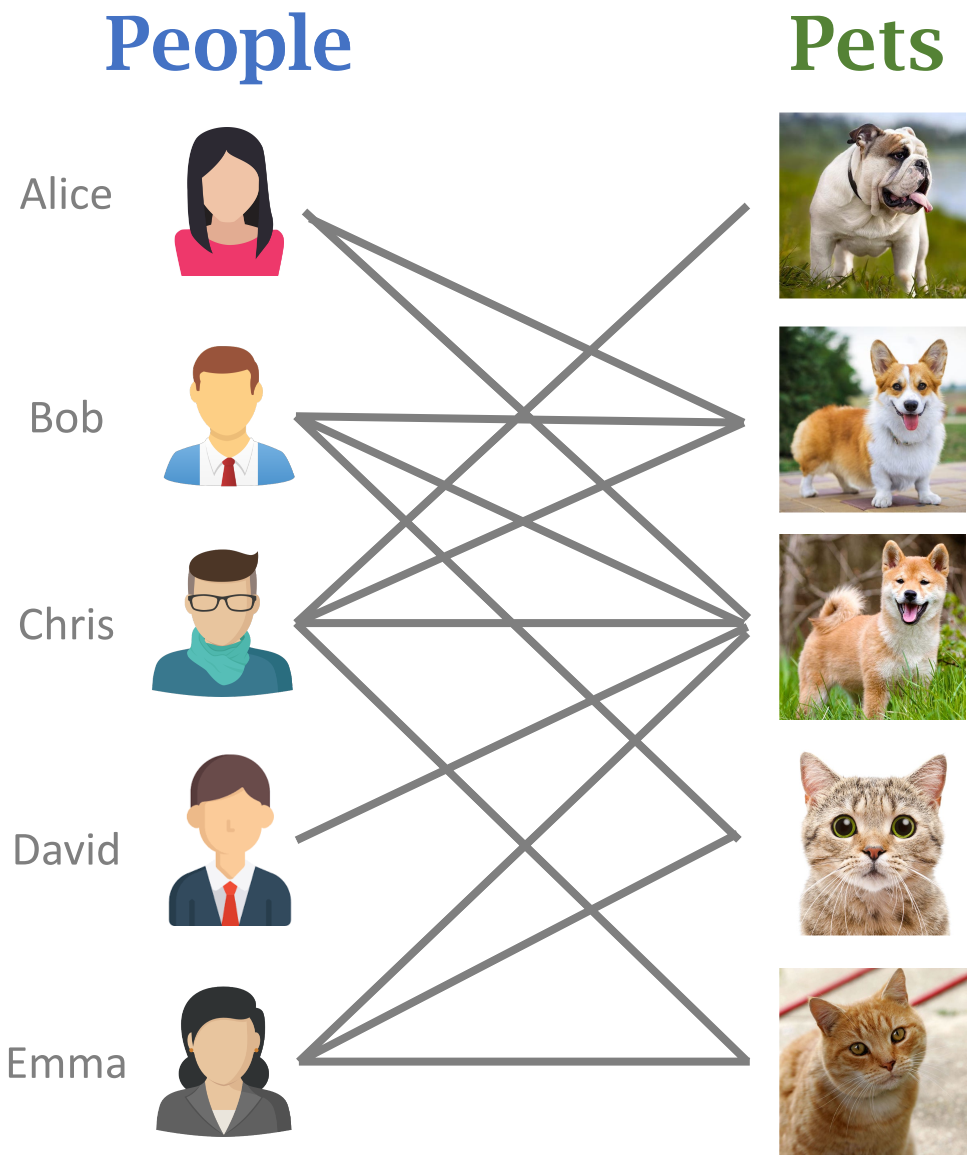
- **MCBM:** The matching $\mathcal{S}$ that has the maximum cardinality $|\mathcal{S}|$.
- MCBM may not be unique.

Maximum-Cardinality Bipartite Matching (MCBM)



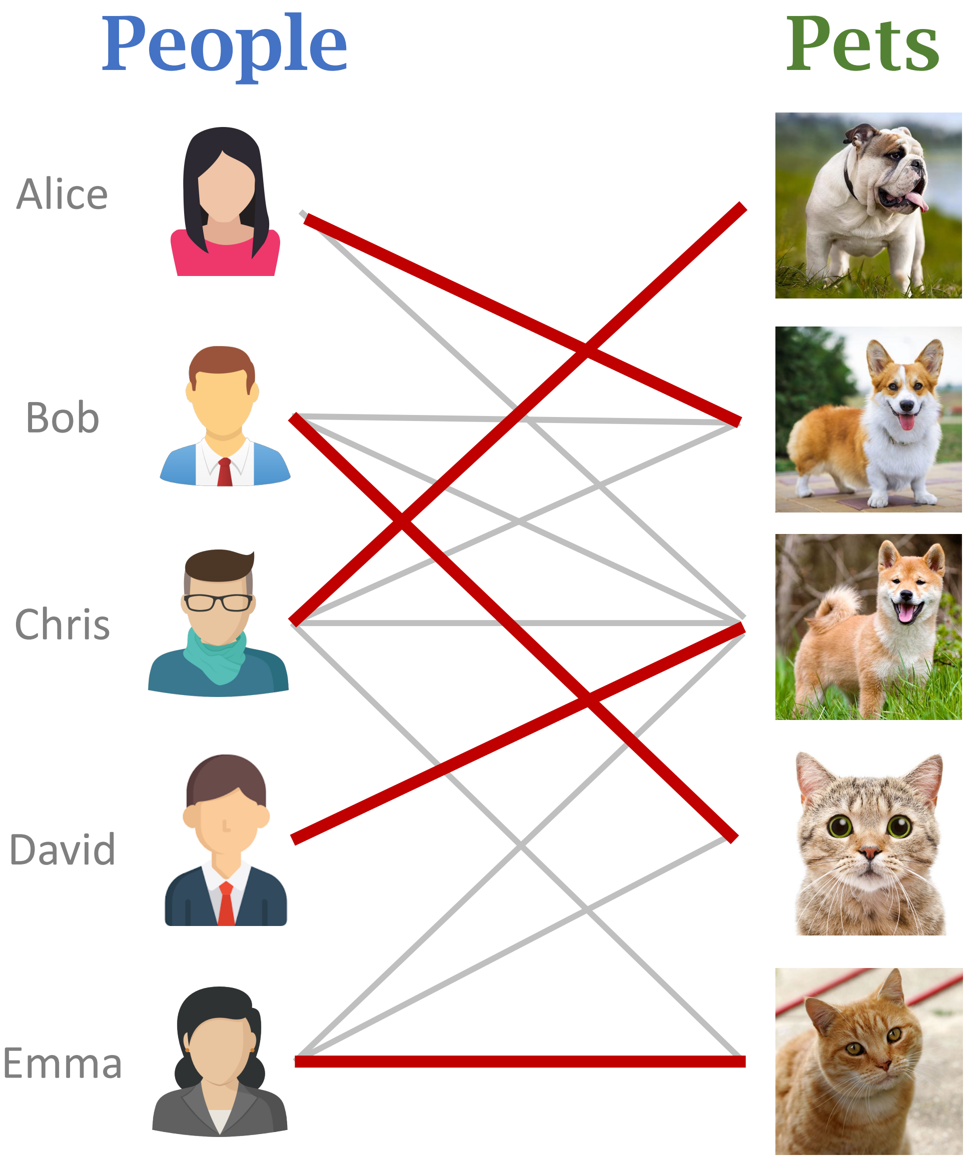
- **MCBM:** The matching $\mathcal{S}$ that has the maximum cardinality $|\mathcal{S}|$.
- MCBM may not be unique.

Greedy Algorithm



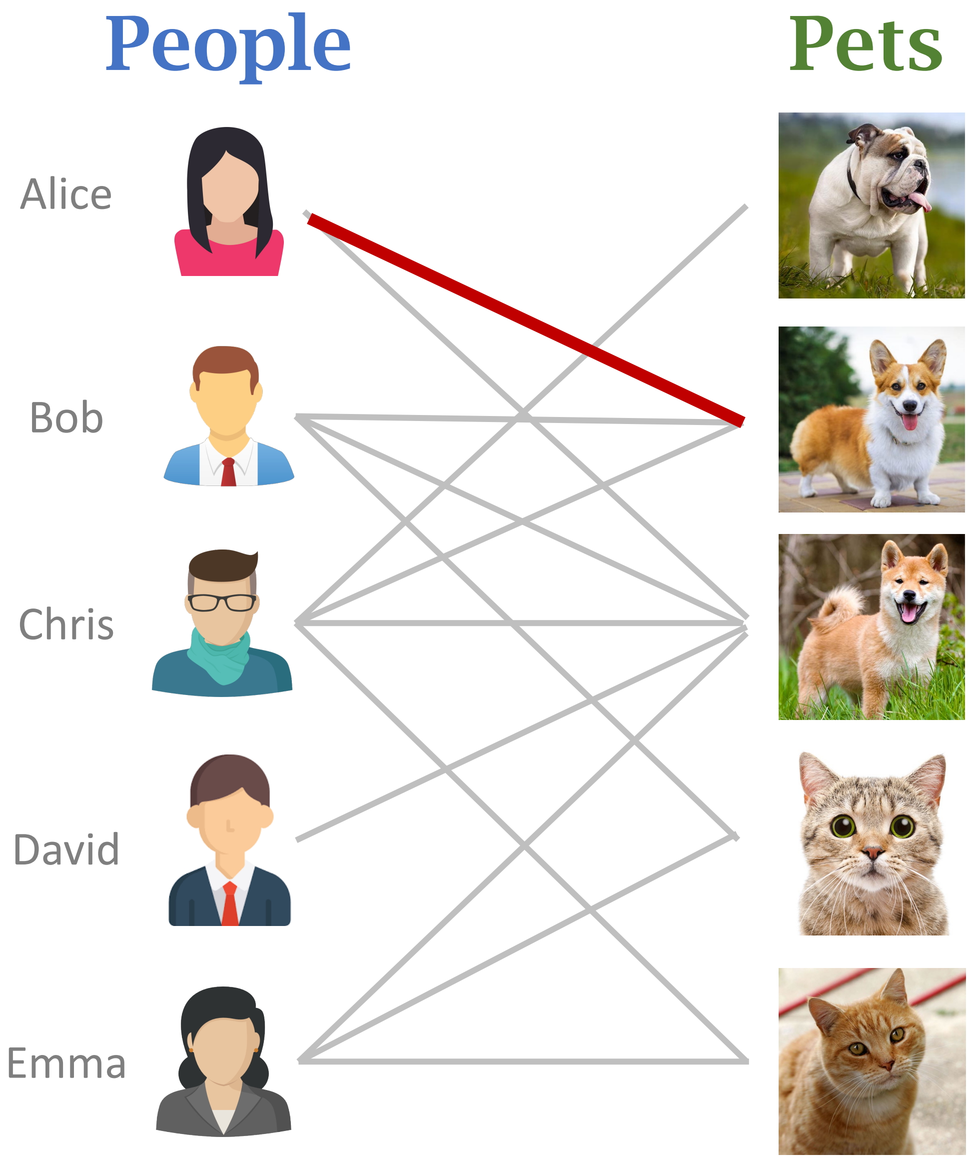
Pet adoption

- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Set $\mathcal{U}$ contains candidates.
- Set $\mathcal{V}$ contains jobs.
- Edges in $\mathcal{E}$ are candidates' skills.
- **Goal:** Maximizing the cardinality of matching.

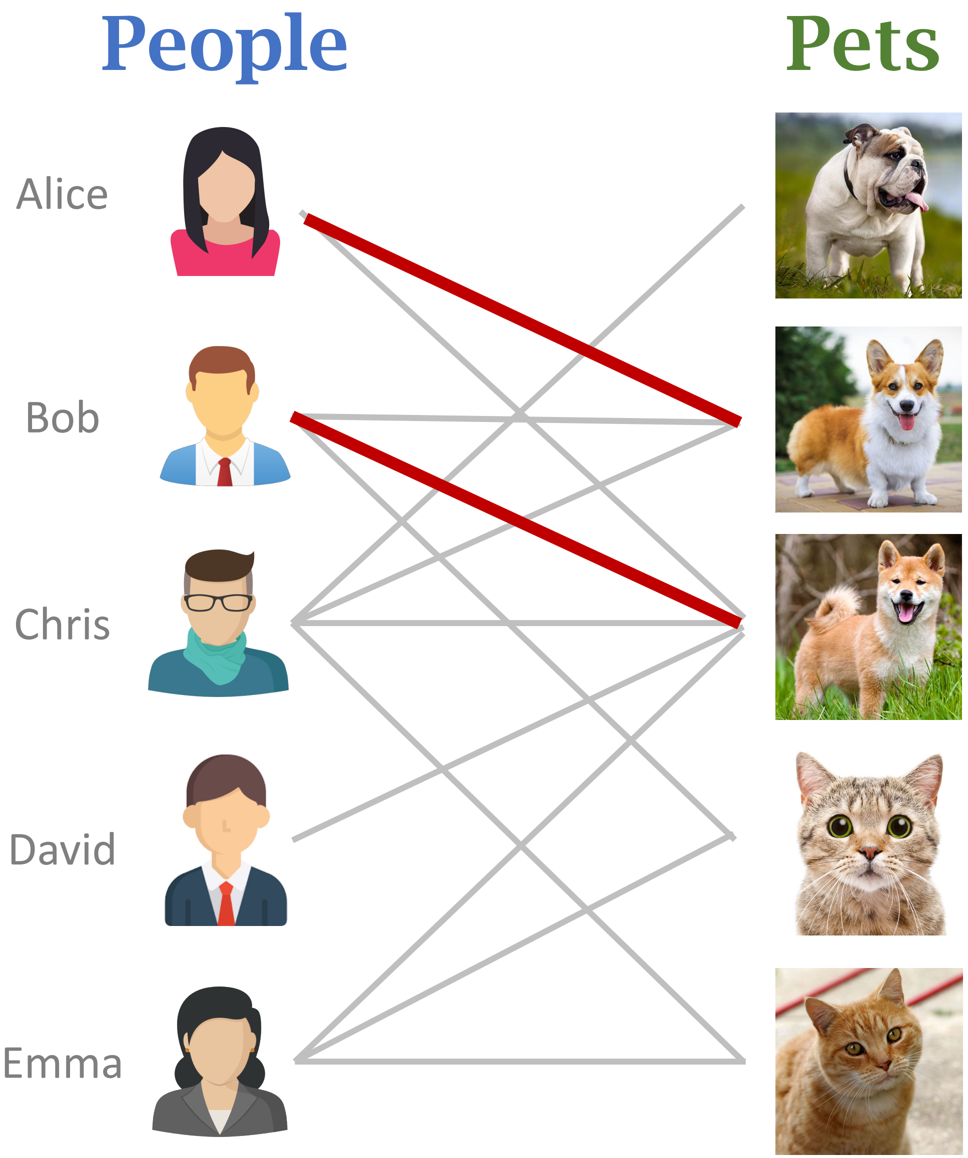


Pet adoption

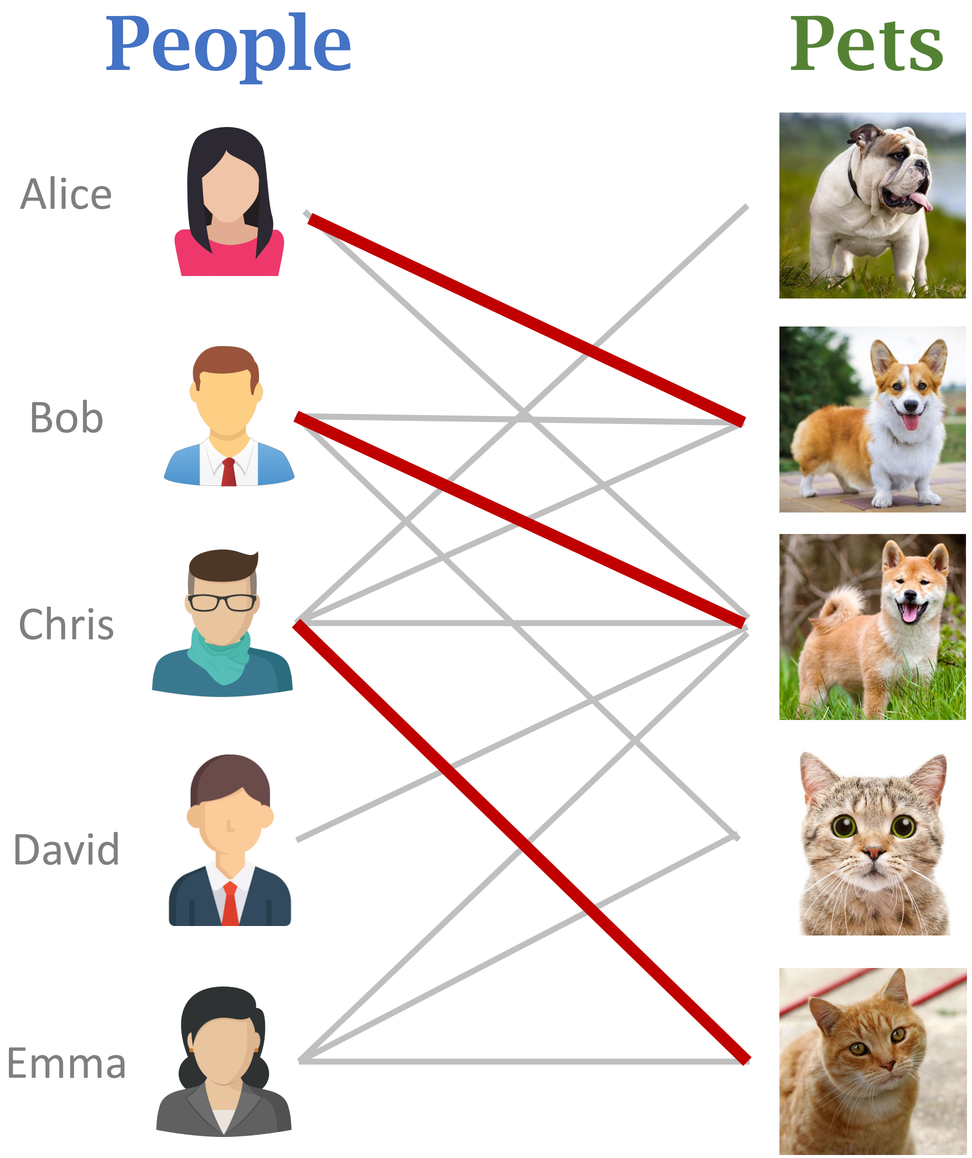
- The cardinality of the maximum matching is 5.



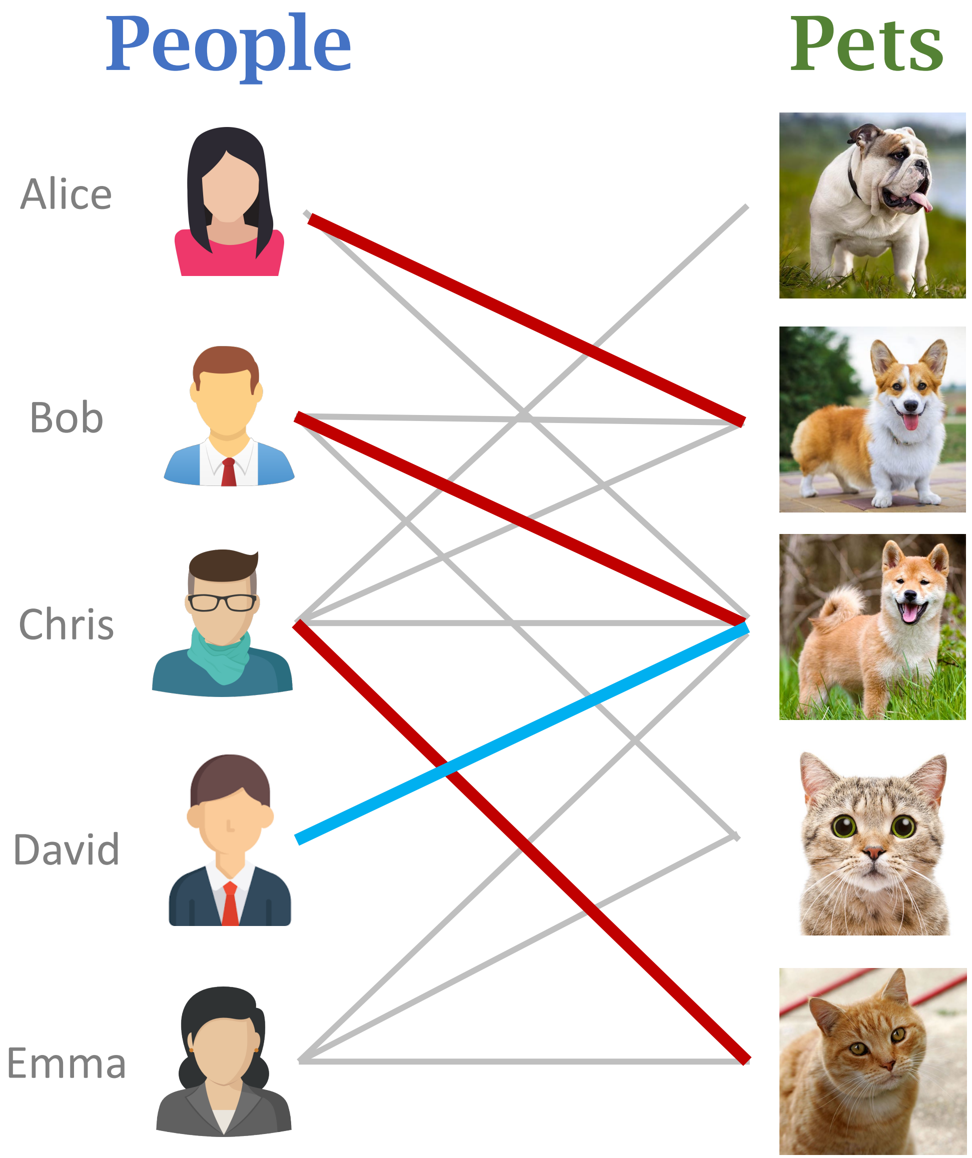
Greedy Algorithm



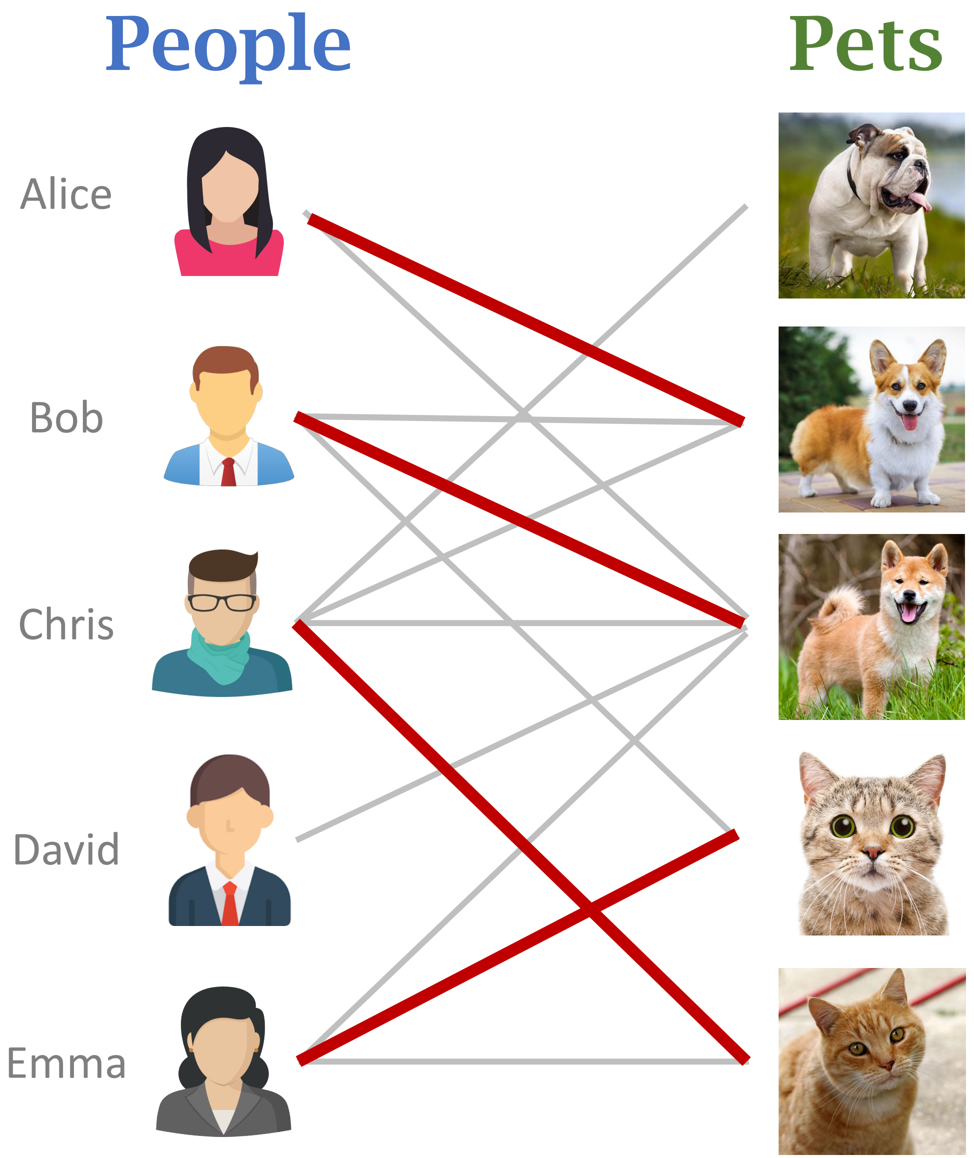
Greedy Algorithm



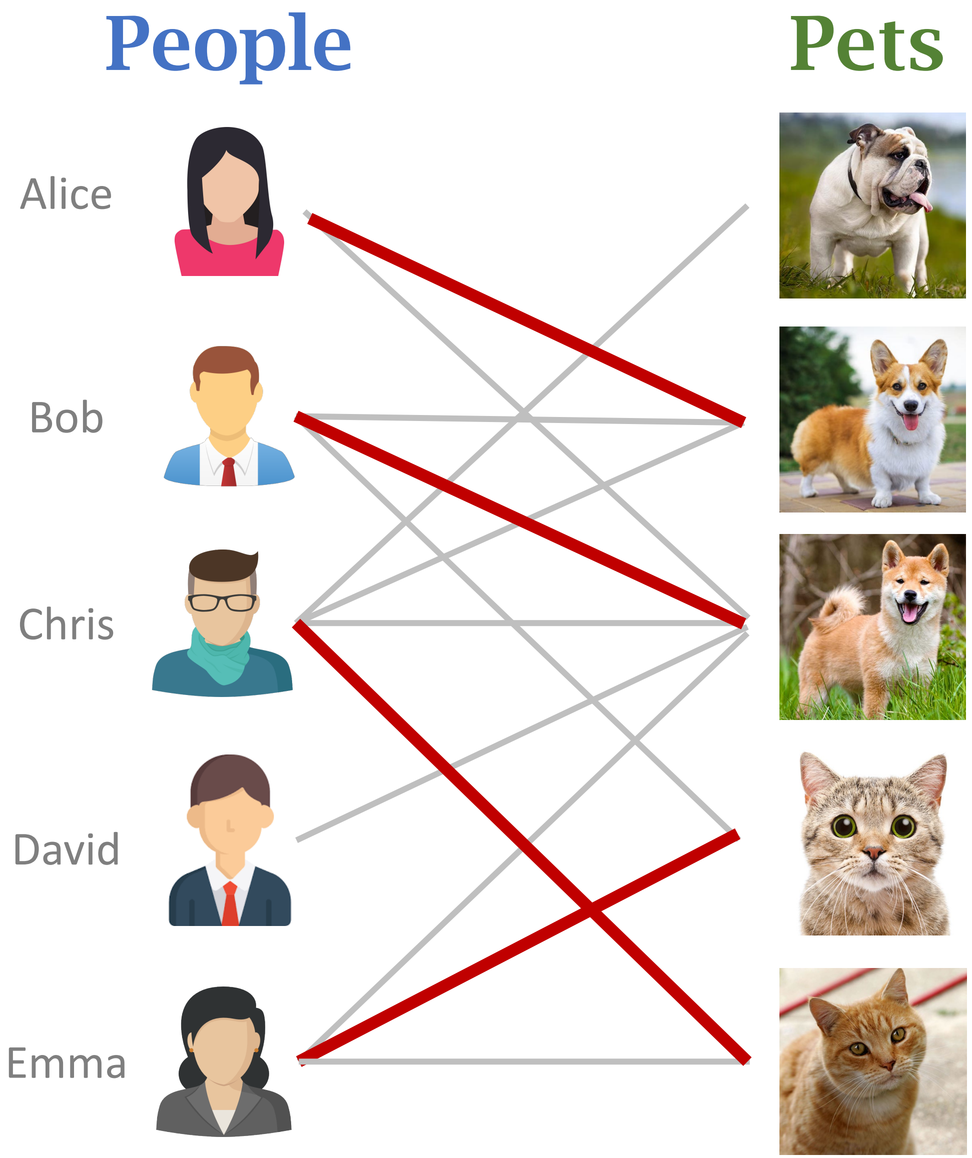
Greedy Algorithm



Greedy Algorithm



Greedy Algorithm



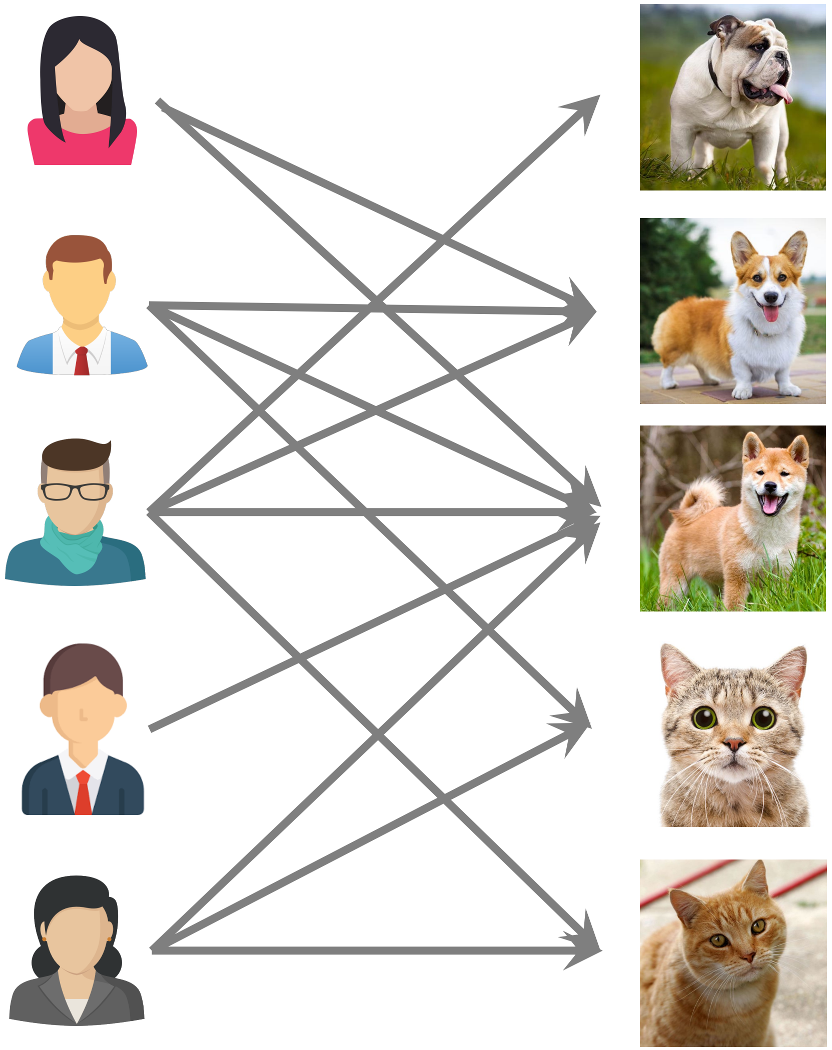
Greedy Algorithm

- The cardinality of matching is 4.
(Not the maximum.)

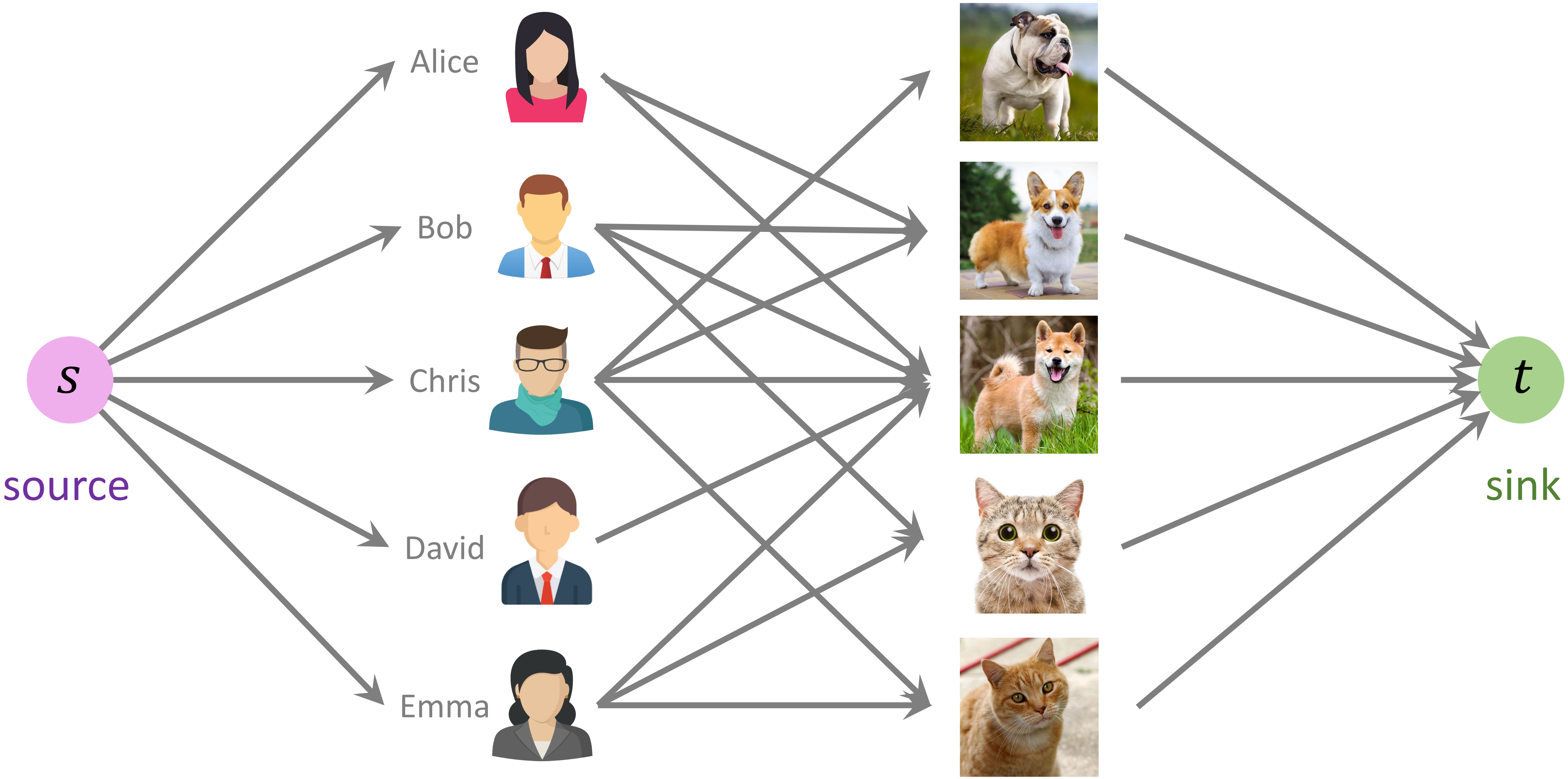
Greedy algorithm can fail!

Reducing Bipartite Matching to Network Flow

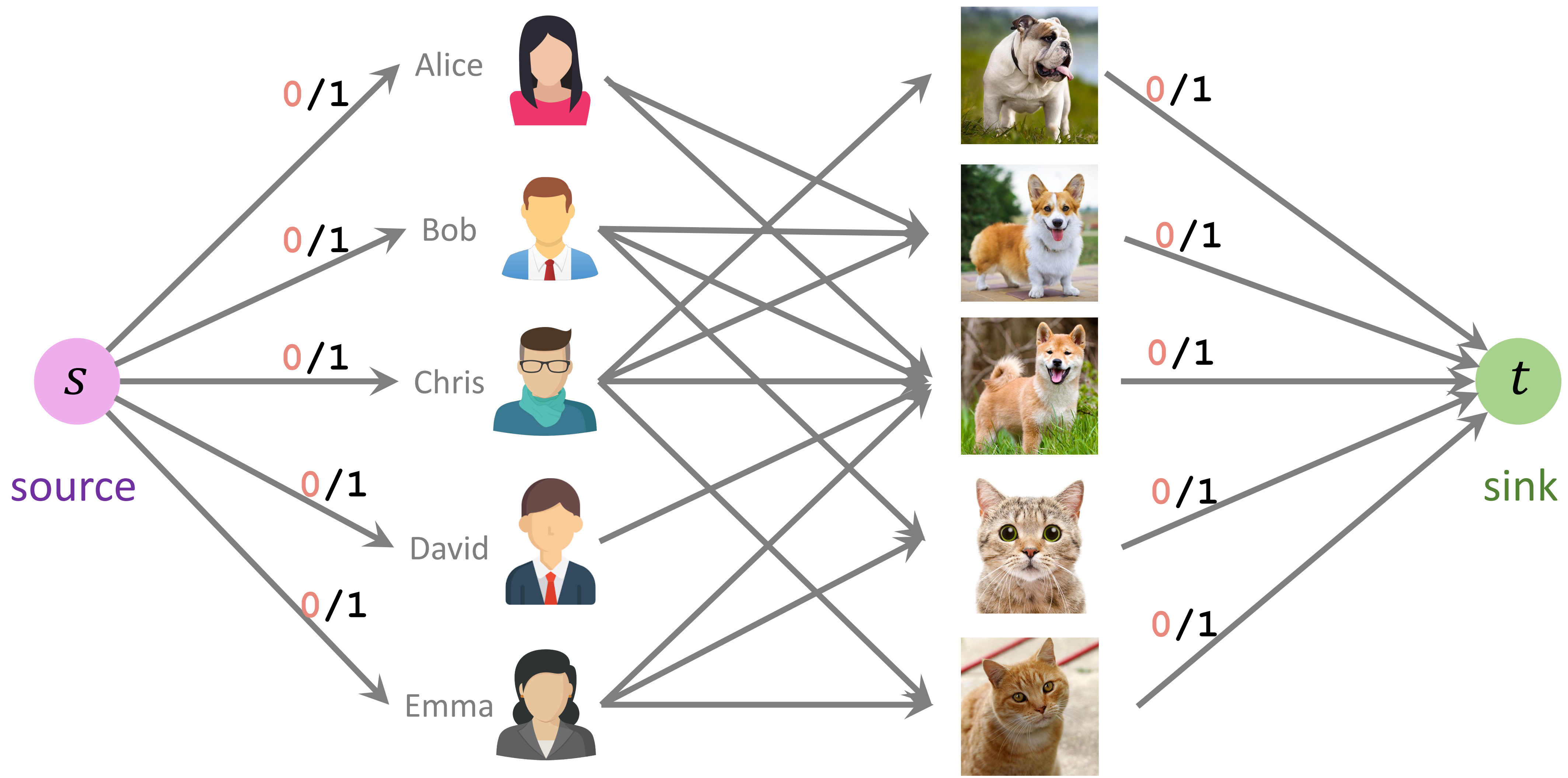
Make the edges directed



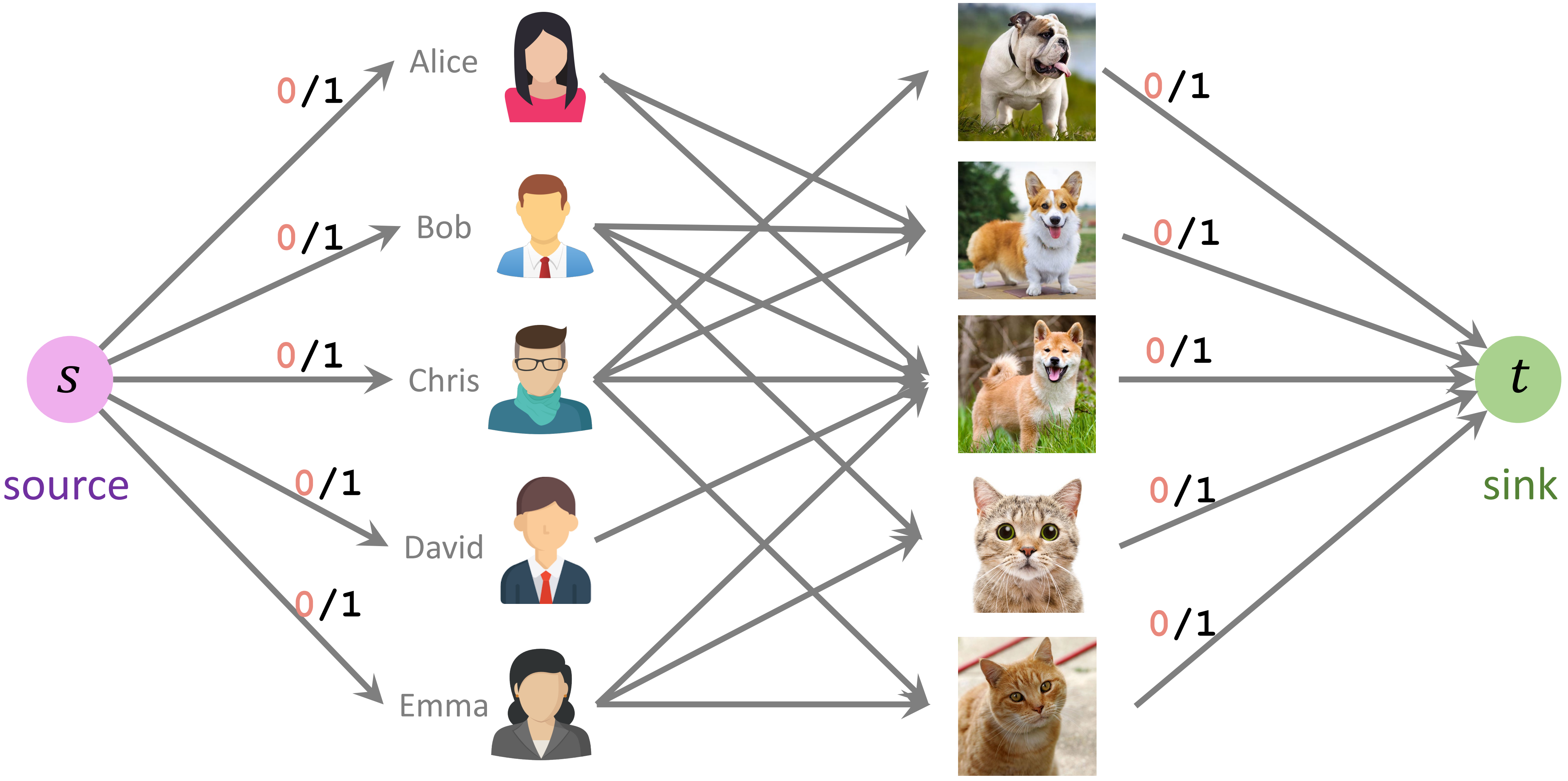
Add source and sink



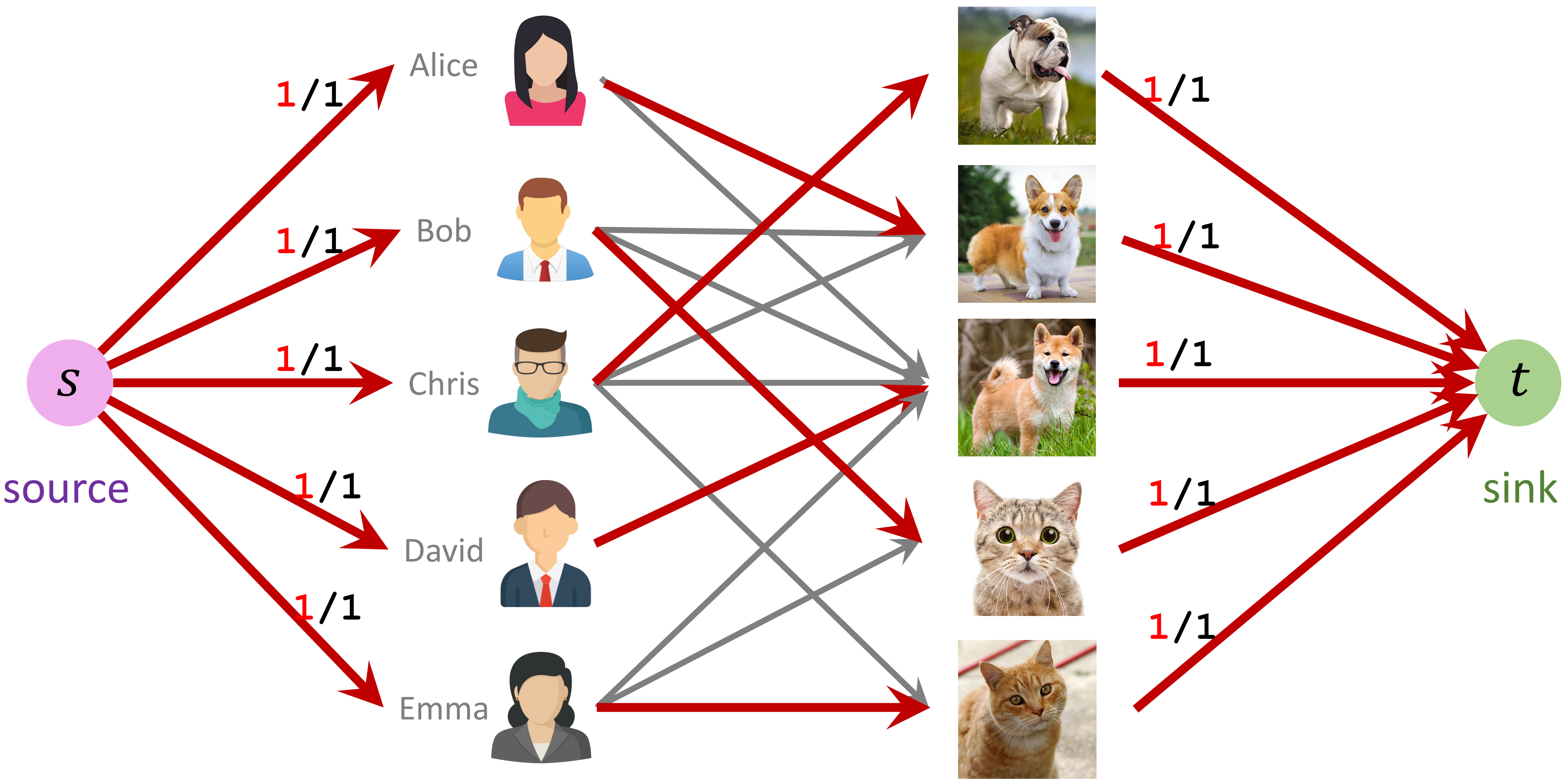
All the edge weights (capacities) are ones



Find the max-flow using any algorithm

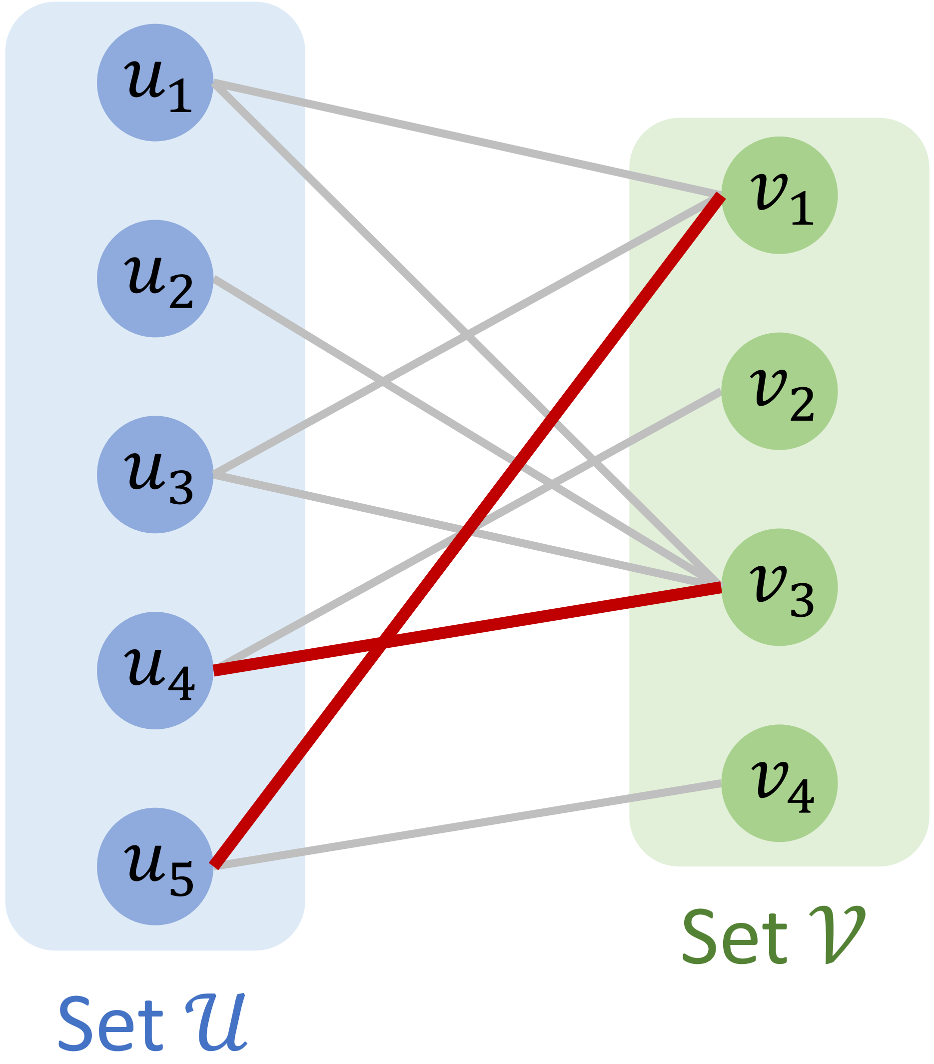


capacity of max-flow = cardinality of max-matching



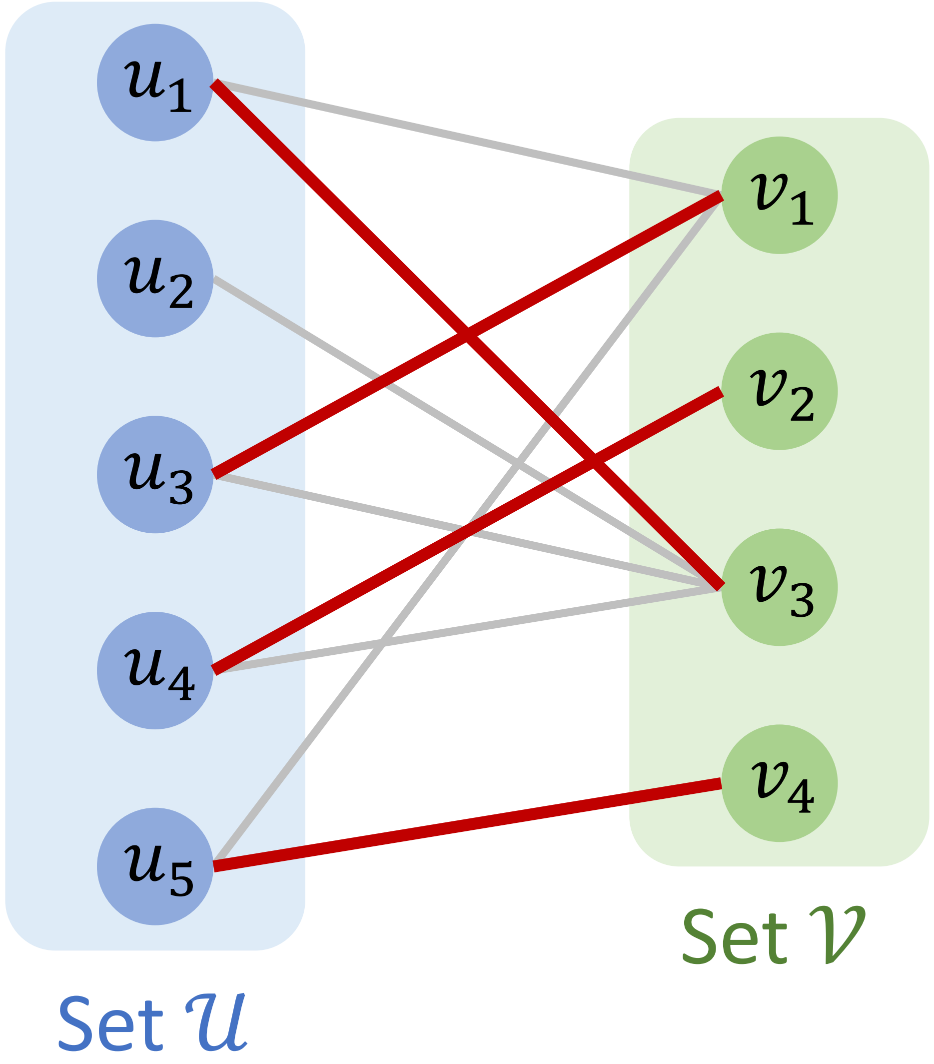
Summary

Bipartite Matching



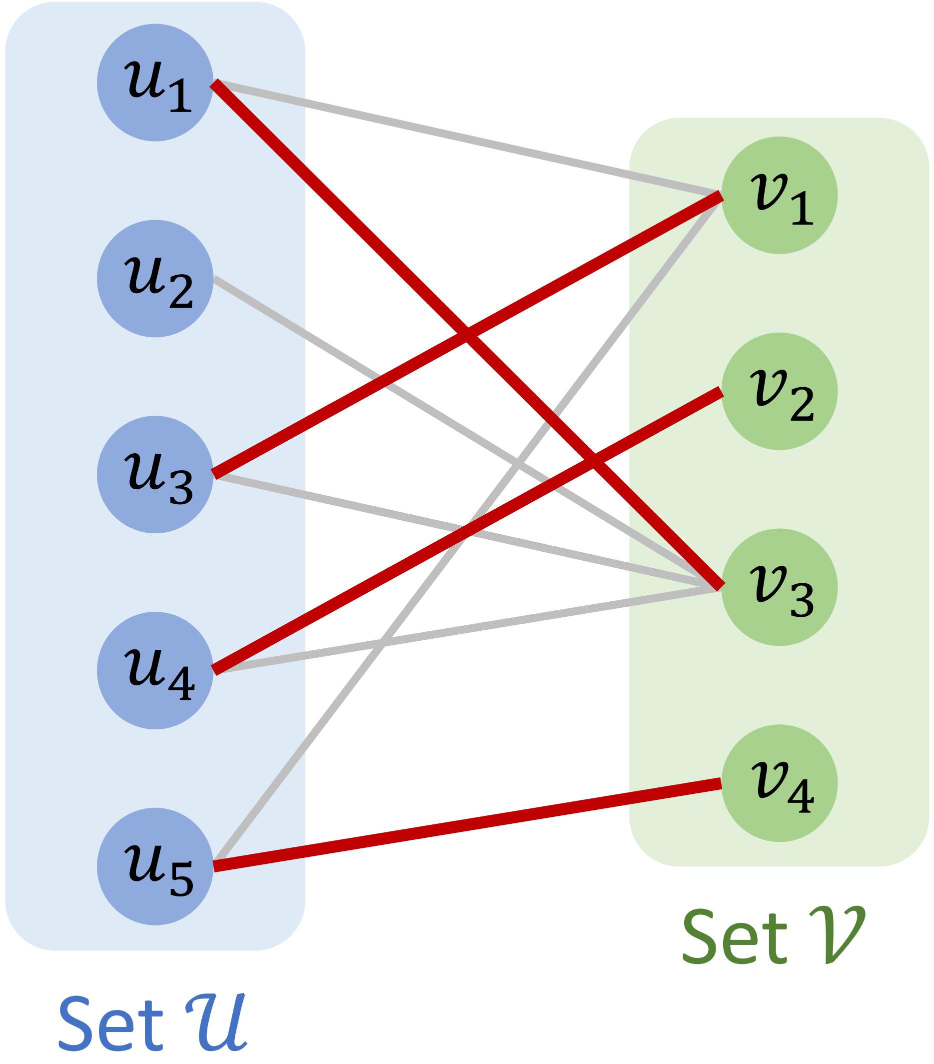
- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Matching is a subset of edges without common vertices.
- Denote the matching by set $\mathcal{S} \subseteq \mathcal{E}$.

Maximum Cardinality Bipartite Matching (MCBM)



- Given an unweighted bipartite graph.
- MCBM: Find matching $\mathcal{S}$ that has the maximum cardinality $|\mathcal{S}|$.
- Bipartite matching in unweighted graphs is easier than in weighted bipartite graph.

Algorithms for finding MCBM



- Greedy algorithm can fail.
- Reduce MCBM to the max-flow problem and solve it using any max-flow algorithm.

Thank You!