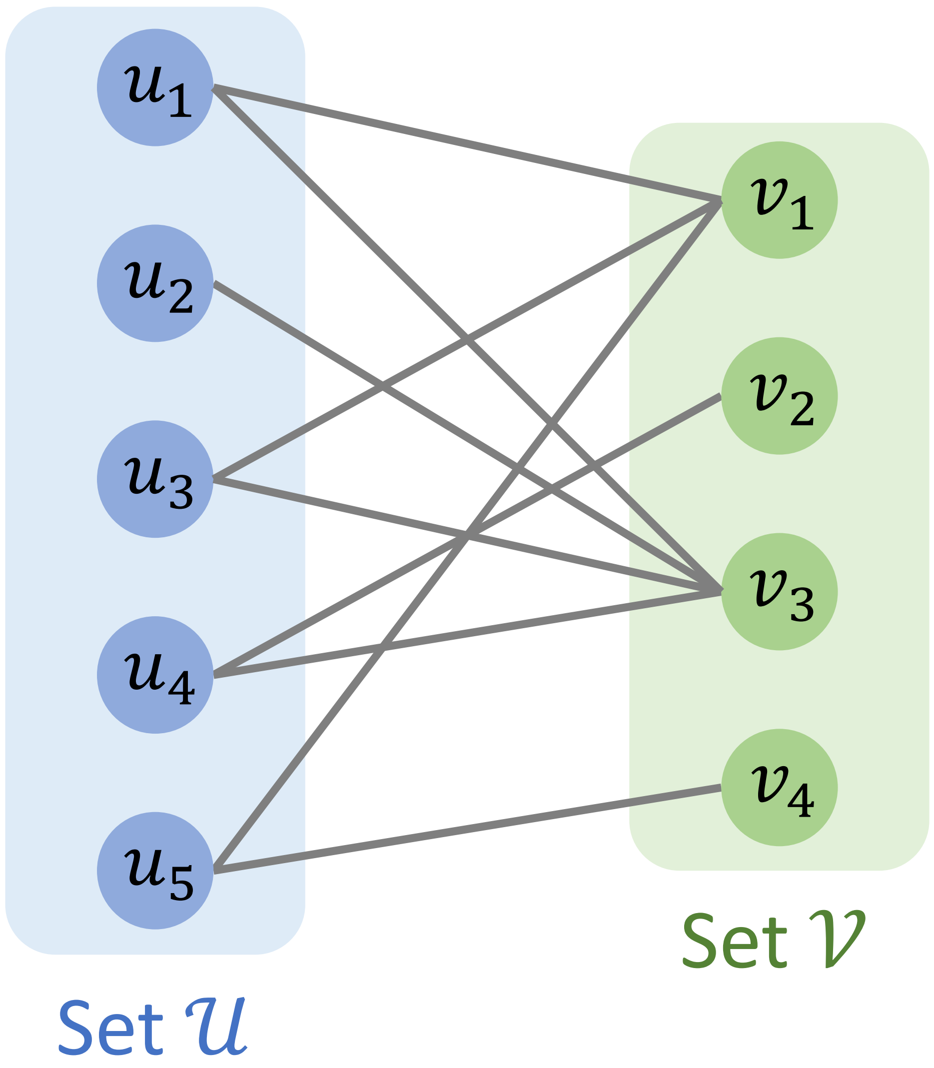
Bipartite Graph

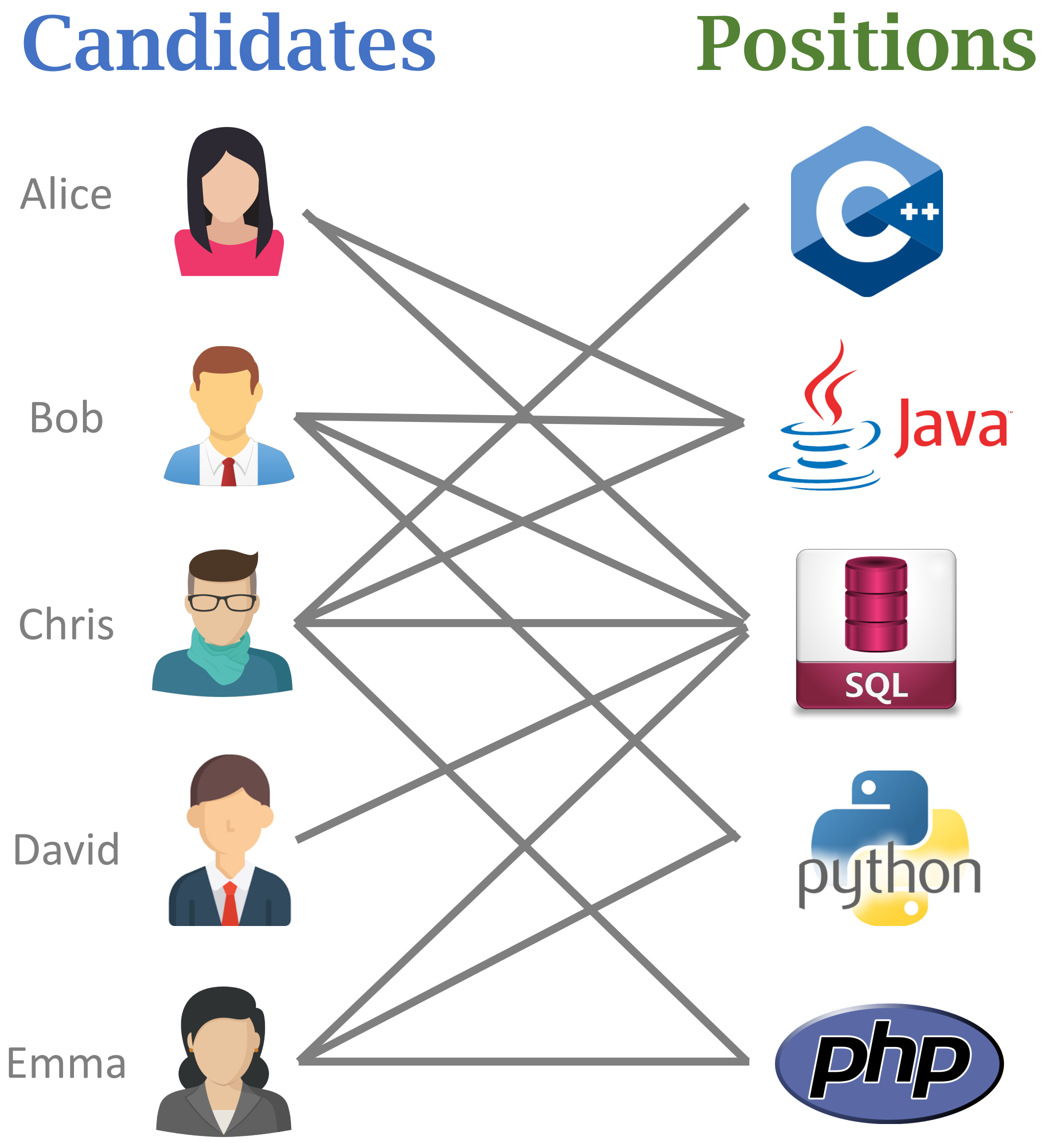
Shusen Wang

Definition

Bipartite Graph

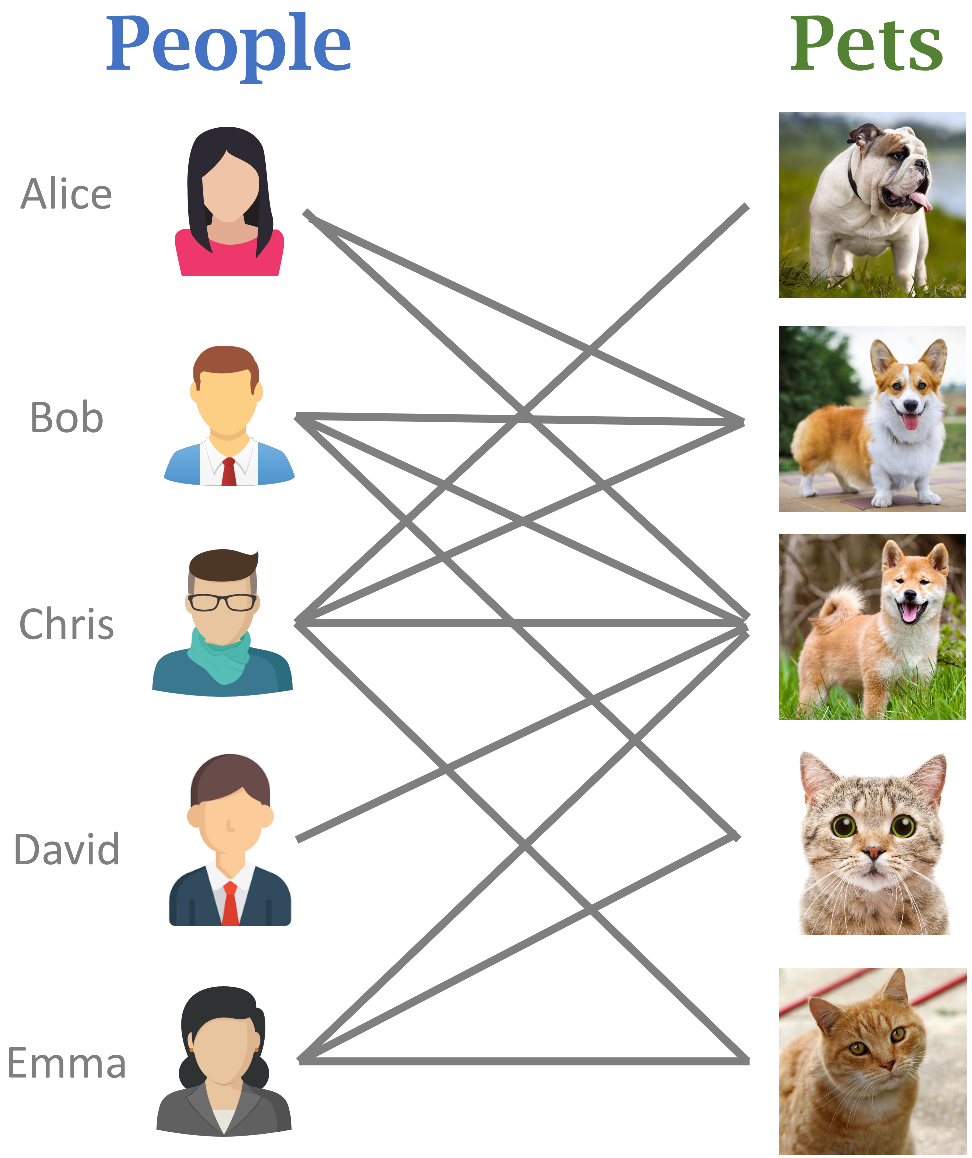


- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- All the edges are between $\mathcal{U}$ and $\mathcal{V}$.
- No edge between two vertices in $\mathcal{U}$.
- No edge between two vertices in $\mathcal{V}$.



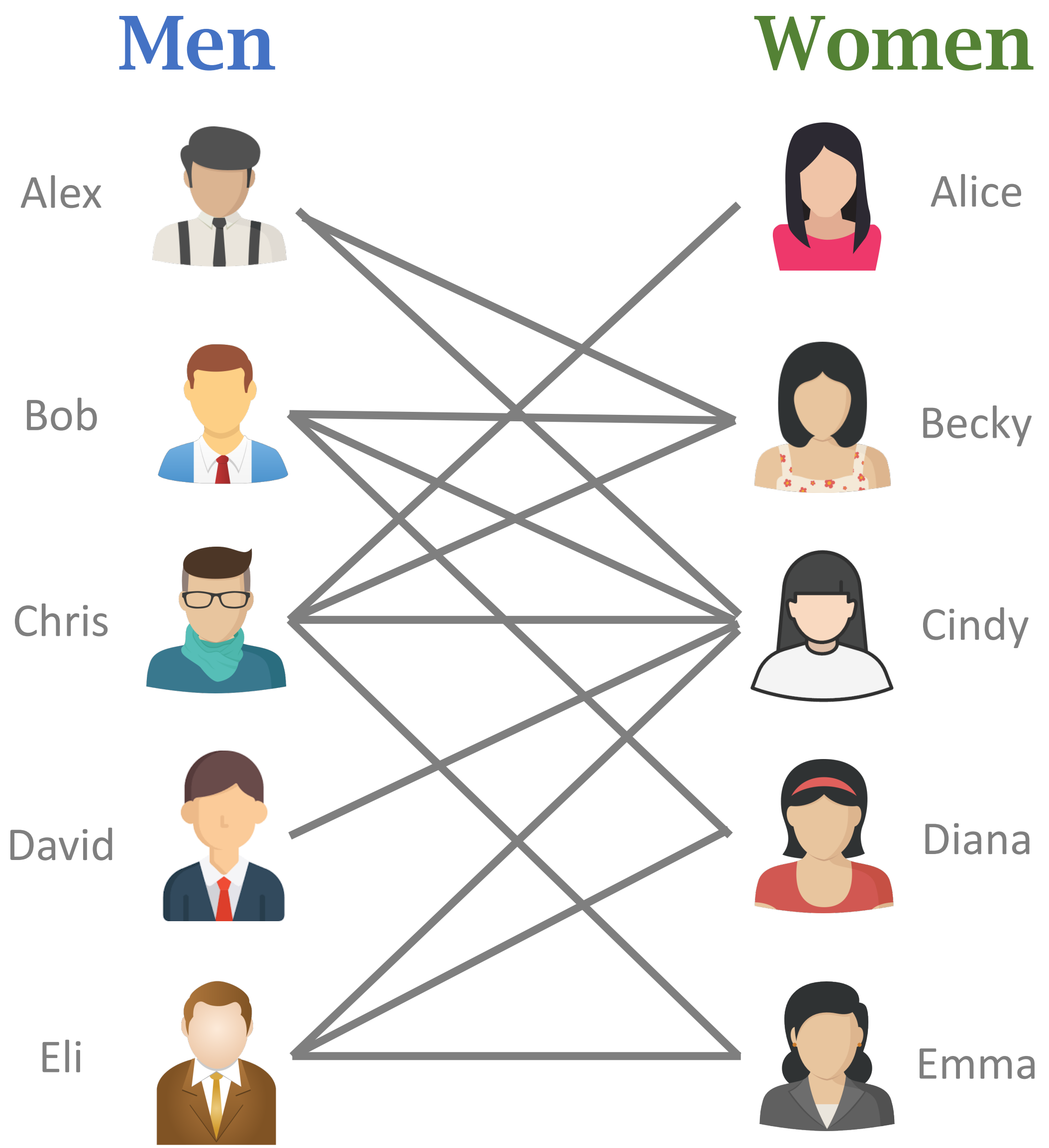
Matching candidates and positions.

- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Set $\mathcal{U}$ contains candidates.
- Set $\mathcal{V}$ contains jobs.
- Edges in $\mathcal{E}$ are candidates' skills.



Pet adoption

- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Set $\mathcal{U}$ contains people.
- Set $\mathcal{V}$ contains pets.
- Edges in $\mathcal{E}$ are people's preference.

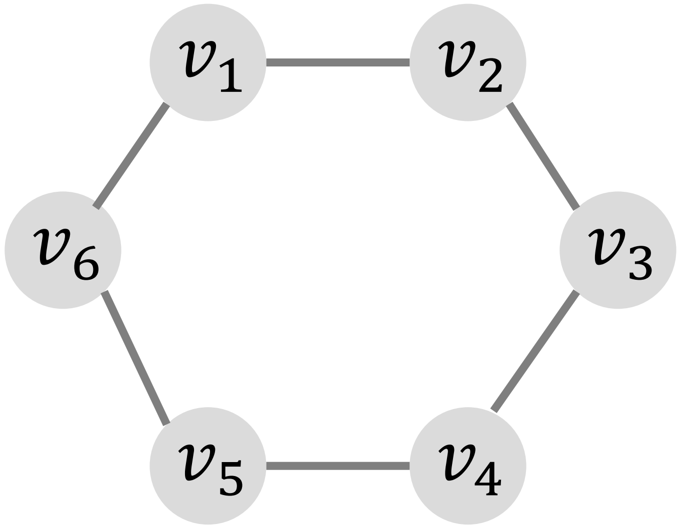


Dating

- Bipartite graph: $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$.
- Set $\mathcal{U}$ contains males.
- Set $\mathcal{V}$ contains females.
- Edges in $\mathcal{E}$ are people’s preference.

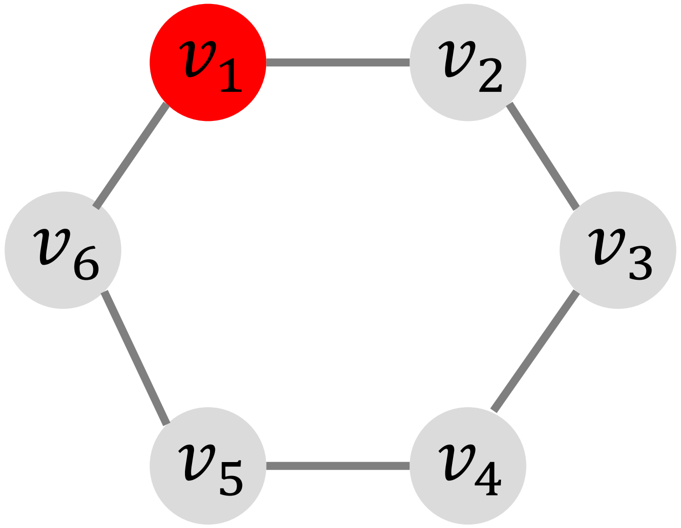
Testing Bipartiteness

Is the graph bipartite?



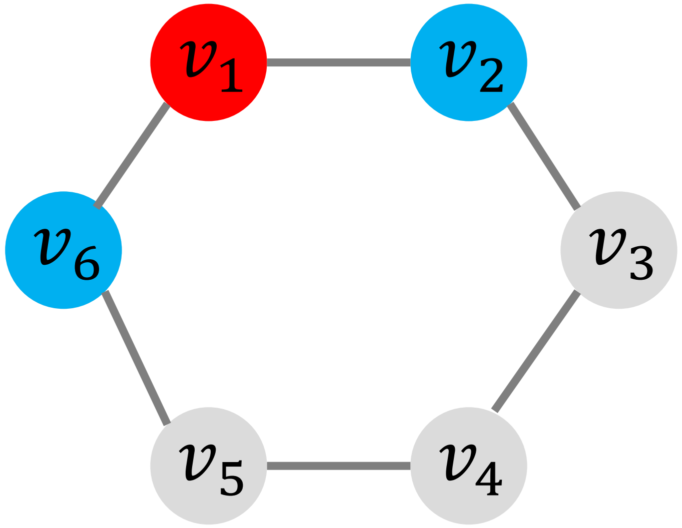
1. Select an arbitrary vertex and assign red color to it.
2. Repeat until all vertices are colored:
   - Color red vertices' neighbors as blue.
   - Color blue vertices' neighbors as red.
   - During the process, if a vertex has the same color as its neighbor, then output FALSE.
3. If no violation is found, return TRUE in the end.

Example 1



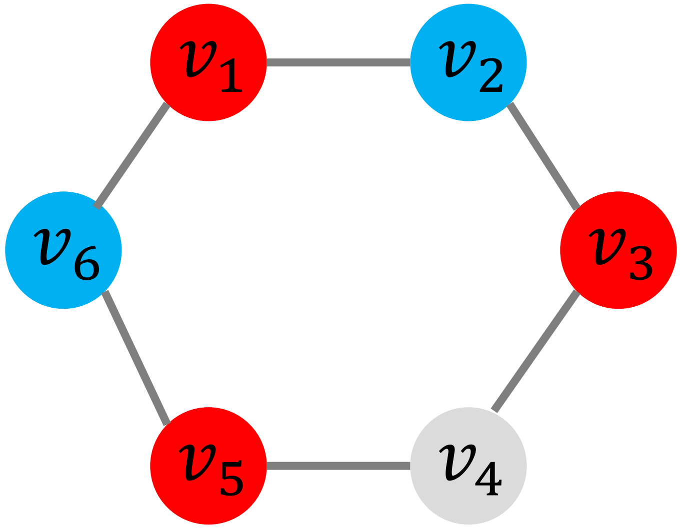
1. Select any vertex and assign red color to it.

Example 1



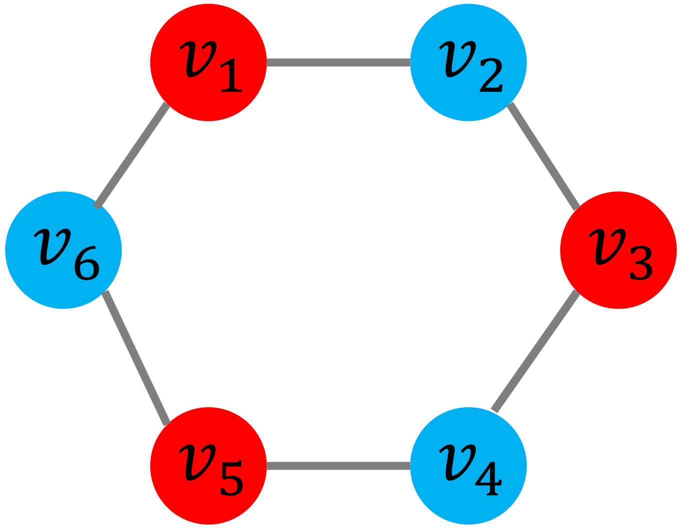
1. Select any vertex and assign red color to it.
2. Color red vertices' neighbors as blue.

Example 1



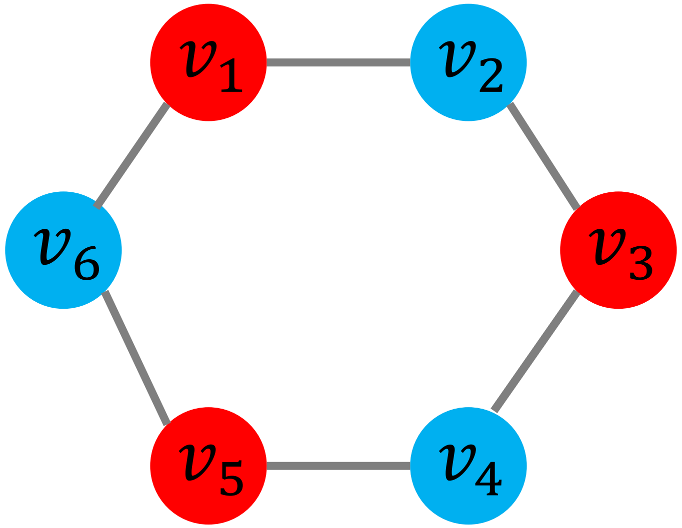
1. Select any vertex and assign red color to it.
2. Color red vertices' neighbors as blue.
3. Color blue vertices' neighbors as red.

Example 1



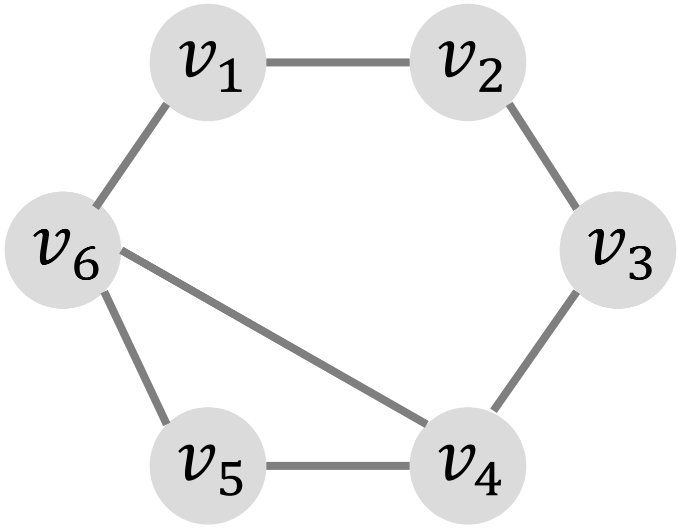
1. Select any vertex and assign red color to it.
2. Color red vertices’ neighbors as blue.
3. Color blue vertices’ neighbors as red.
4. Color red vertices’ neighbors as blue.

Example 1

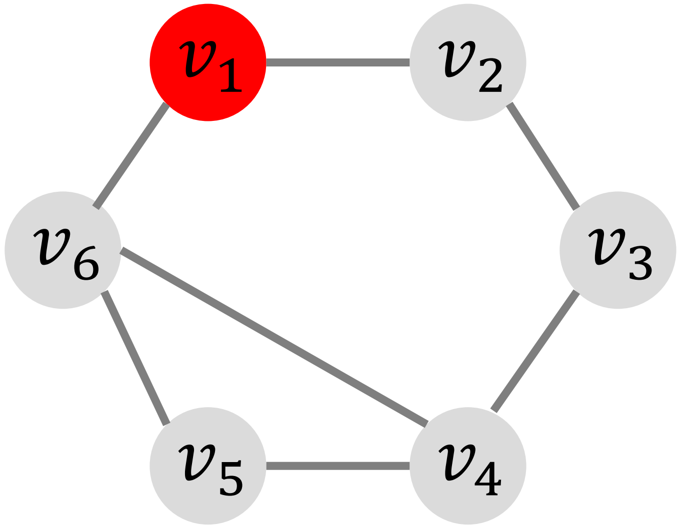


- No violation has been found!
- It is bipartite graph.

Example 2

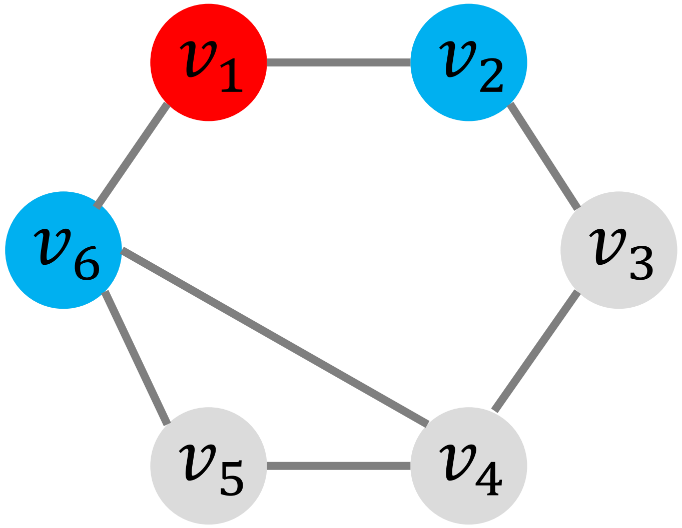


Example 2



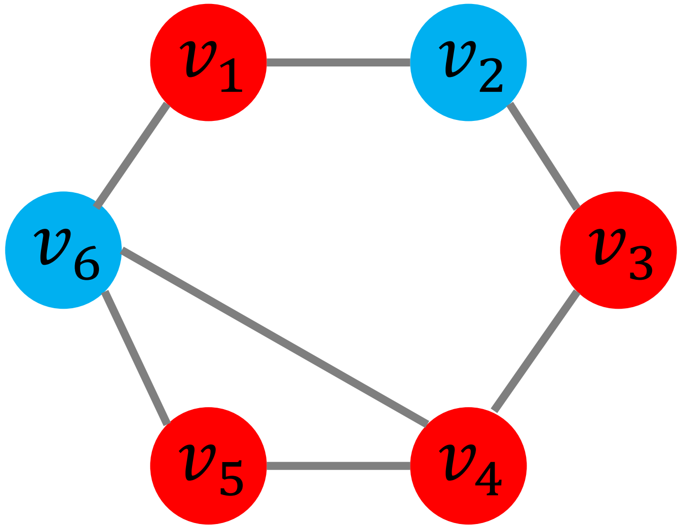
1. Select any vertex and assign red color to it.

Example 2



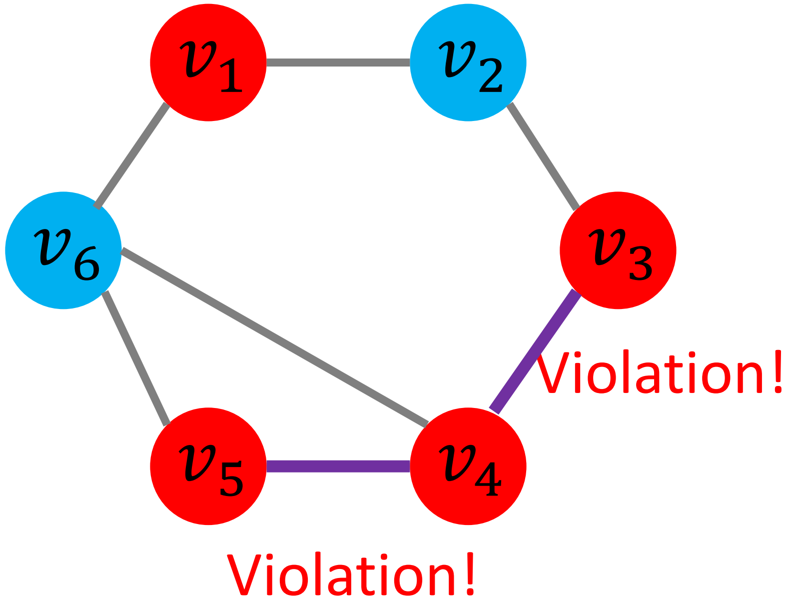
1. Select any vertex and assign red color to it.
2. Color red vertices’ neighbors as blue.

Example 2



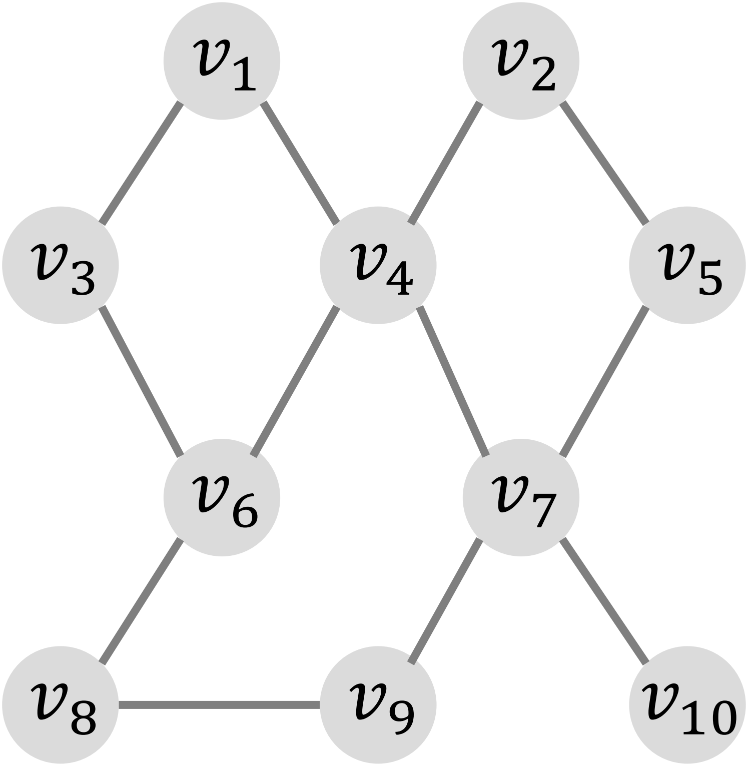
1. Select any vertex and assign red color to it.
2. Color red vertices' neighbors as blue.
3. Color blue vertices' neighbors as red.

Example 2

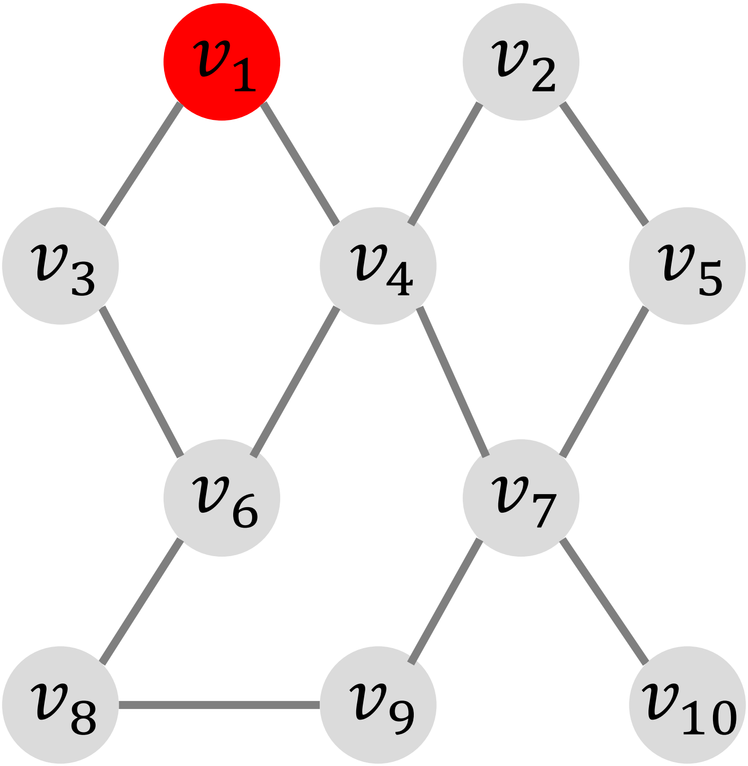


- Violation found!
- It is not bipartite graph.

Example 3

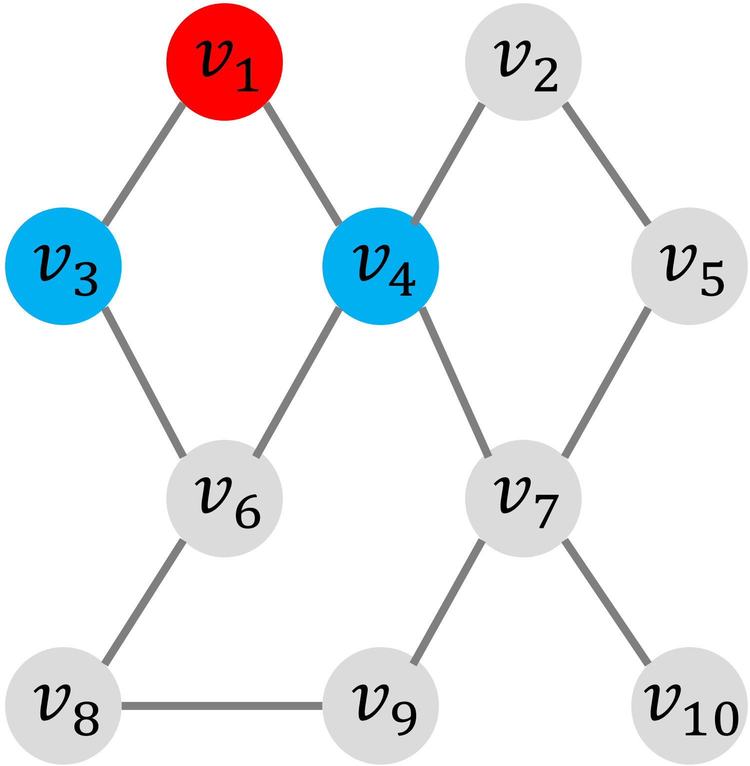


Example 3



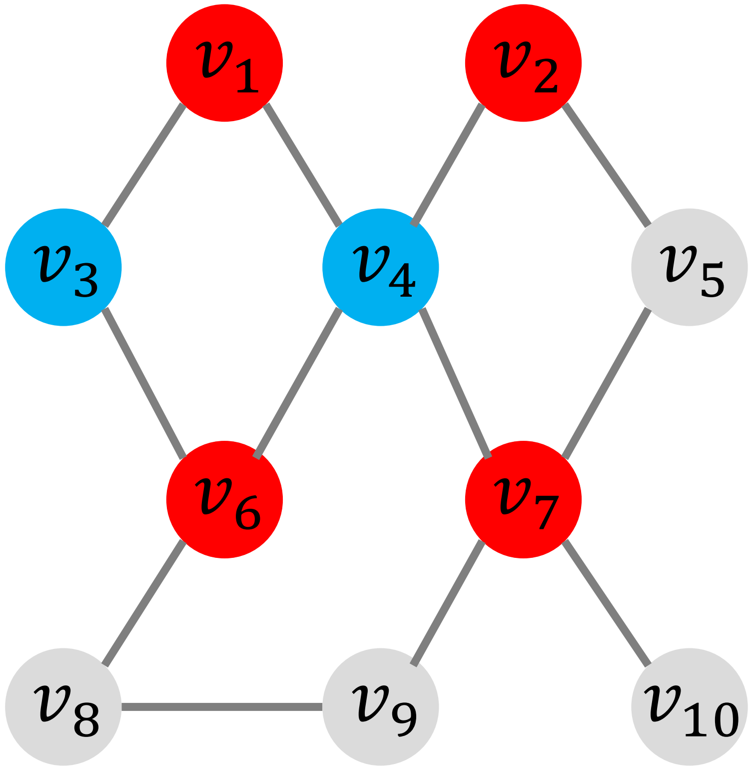
1. Select any vertex and assign red color to it.

Example 3



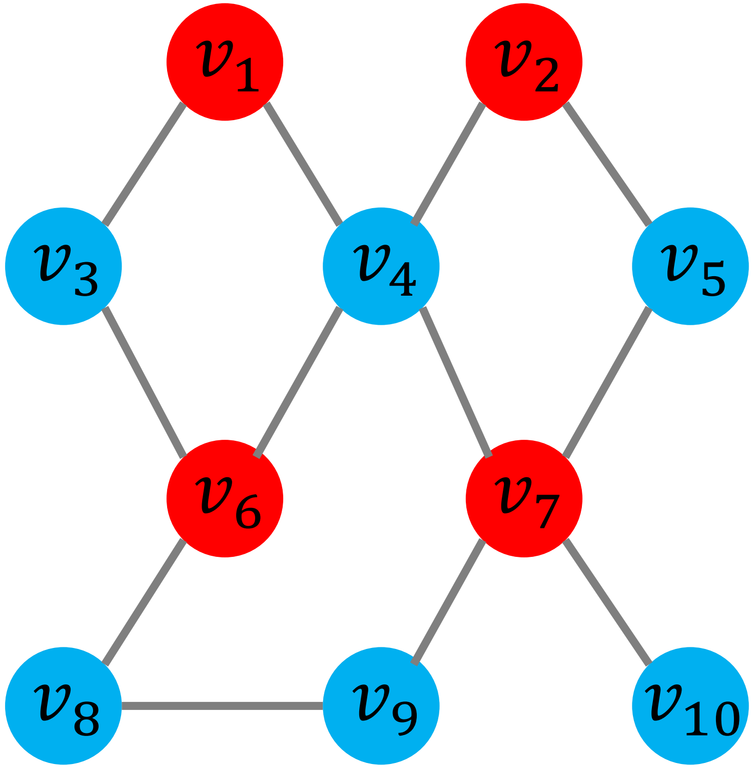
1. Select any vertex and assign red color to it.
2. Color red vertices' neighbors as blue.

Example 3



1. Select any vertex and assign red color to it.
2. Color red vertices’ neighbors as blue.
3. Color blue vertices’ neighbors as red.

Example 3



1. Select any vertex and assign red color to it.
2. Color red vertices' neighbors as blue.
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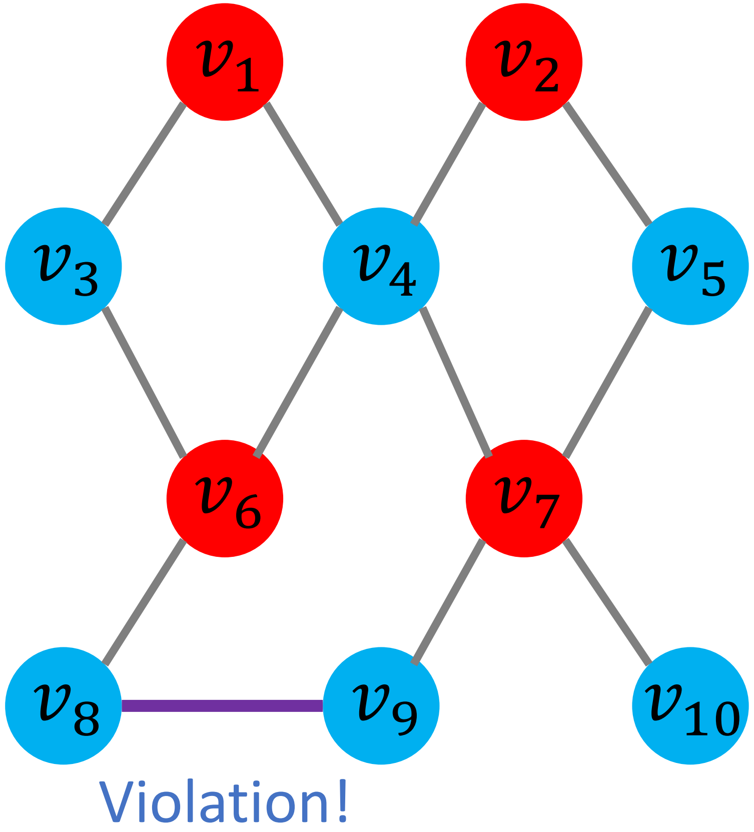
4. Color red vertices' neighbors as blue.

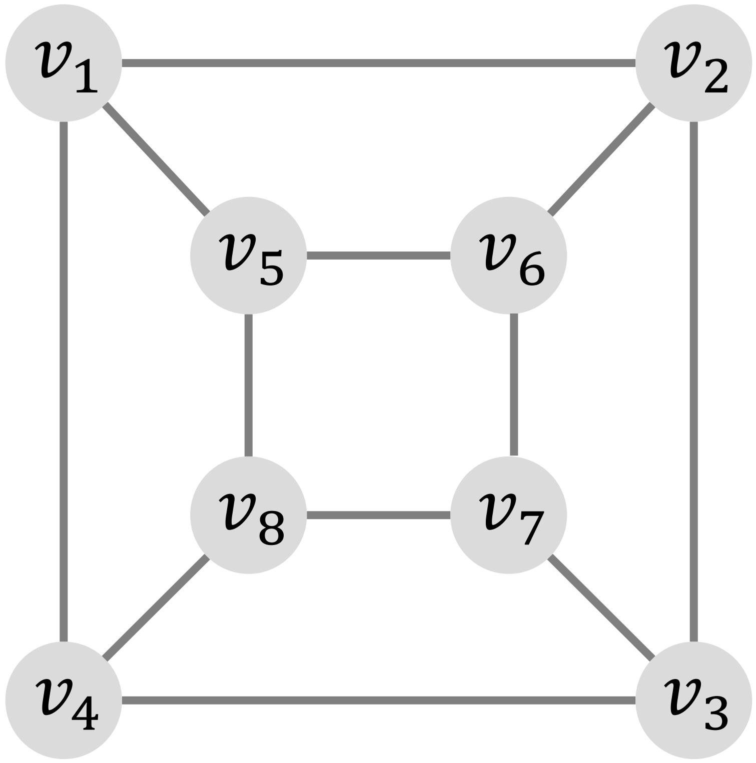
Example 3



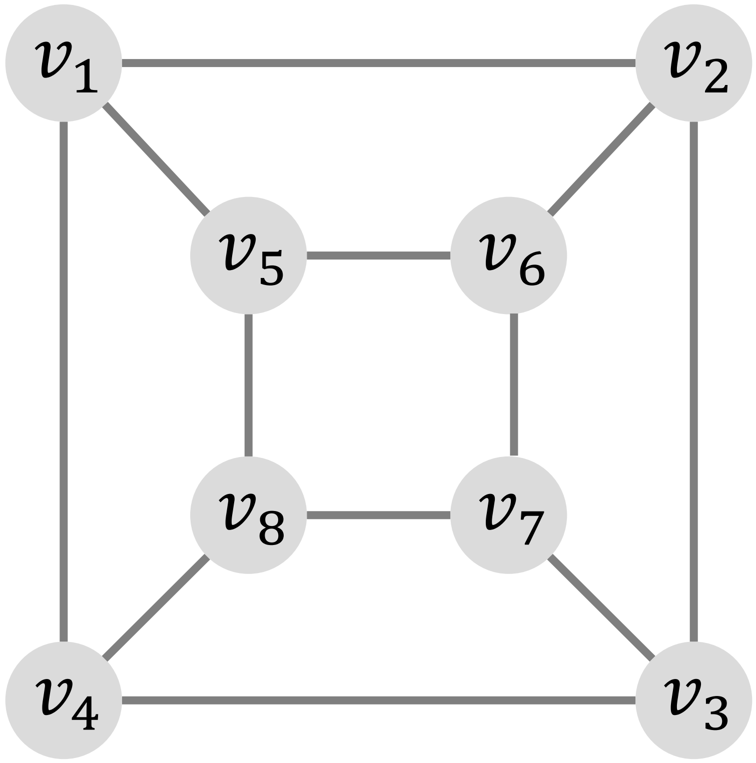
- Violation found!
- It is not bipartite graph.

Algorithm Details

Is the graph bipartite?

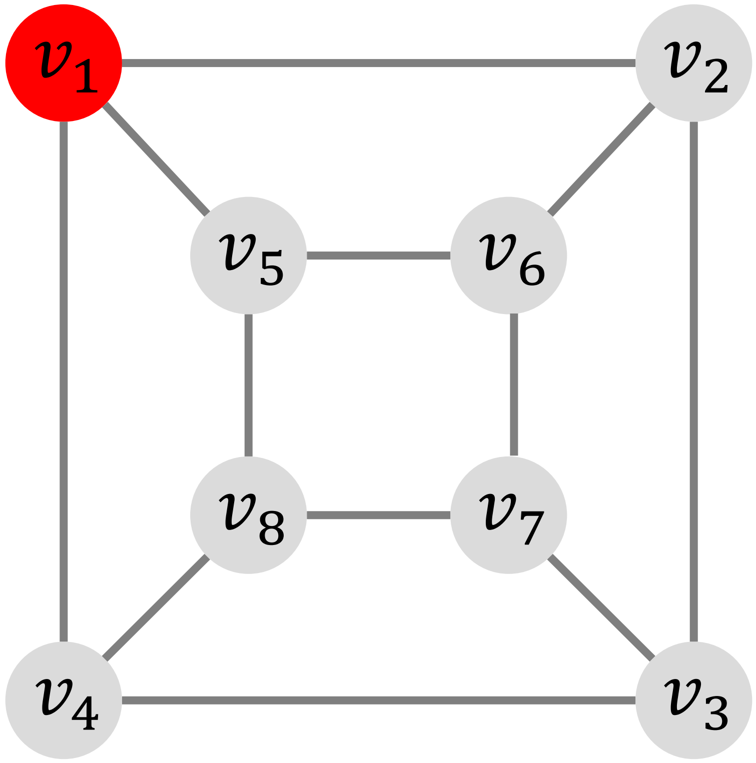


Initial State



Queue:

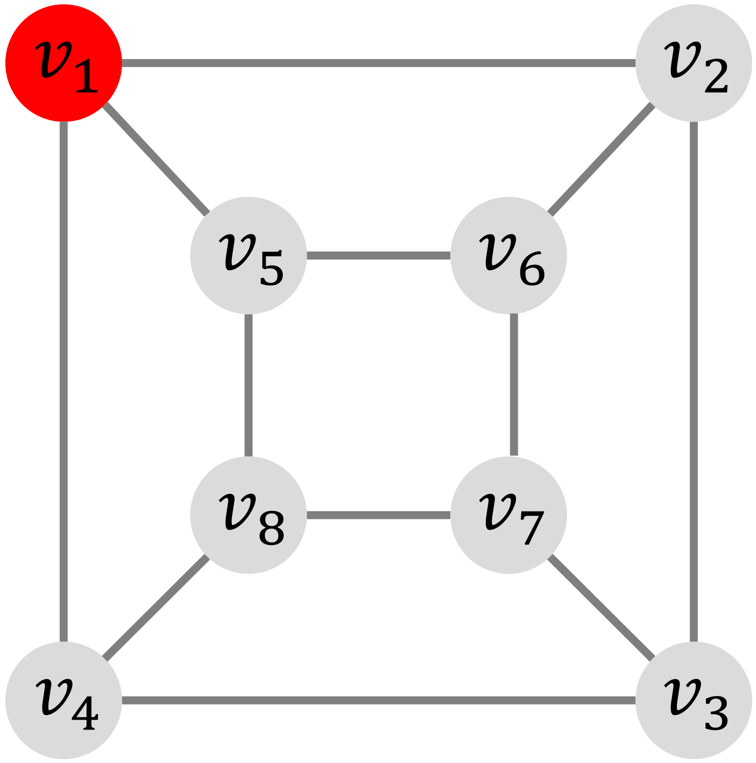
Initial State



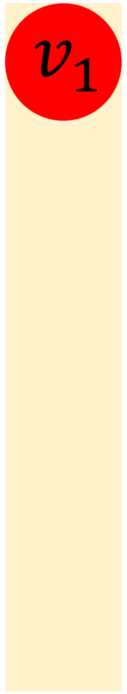
Queue:

- Assign red color to v_1.

Initial State

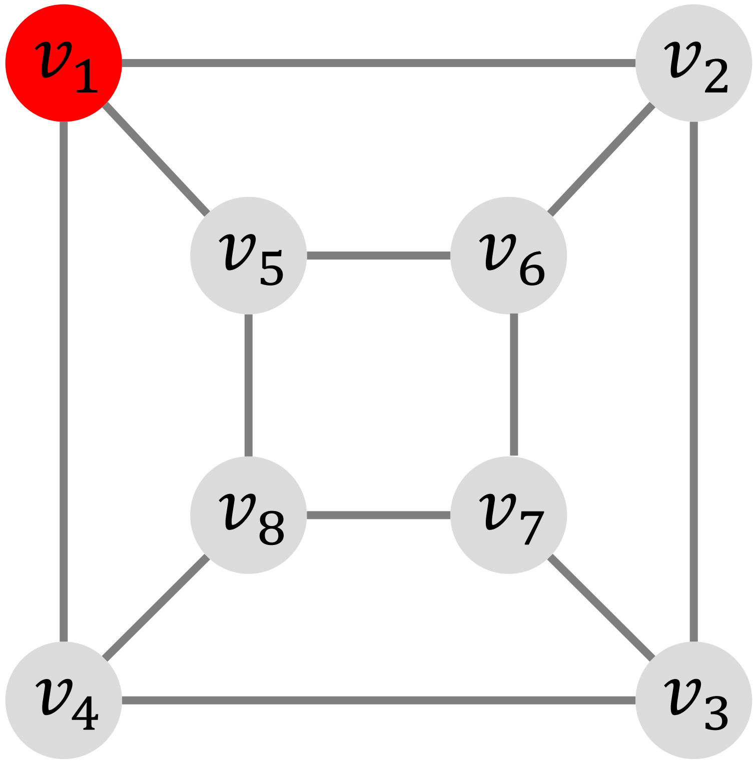


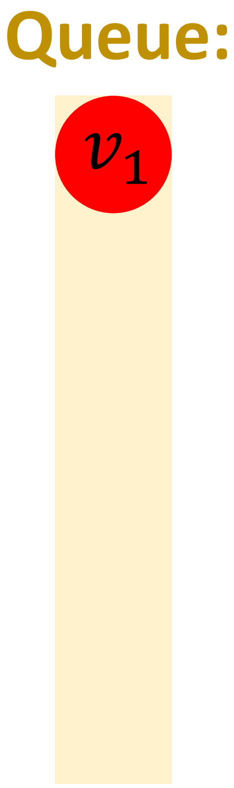
Queue:



- Assign red color to v_1.
- enqueue(v_1).

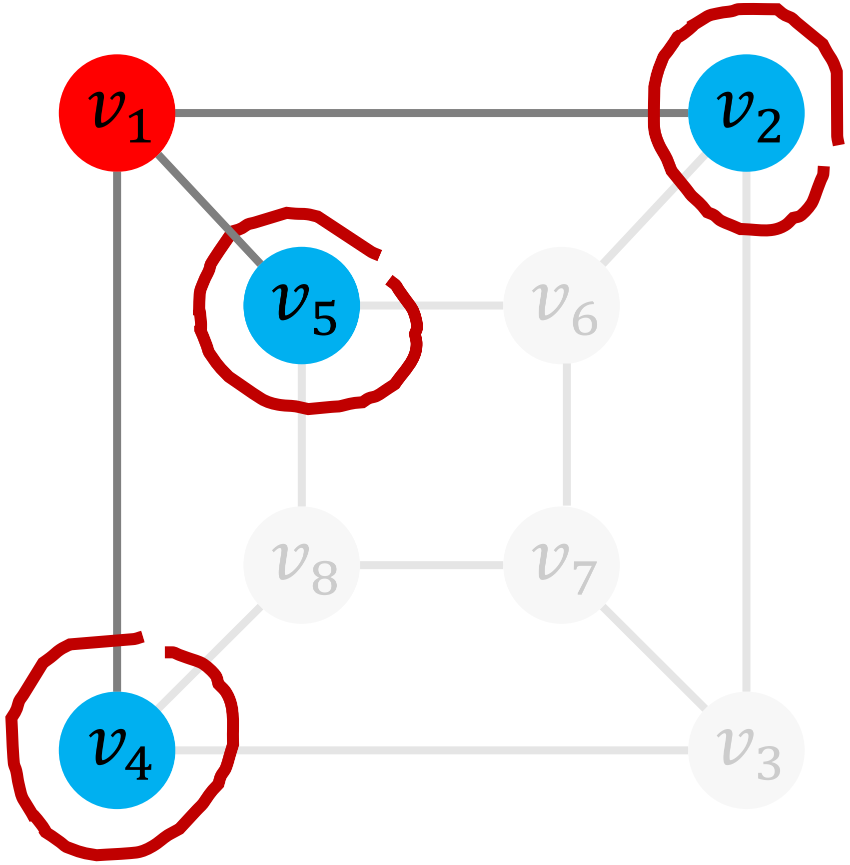
Iteration 1





- $v_1 \leftarrow$ dequeue( ).

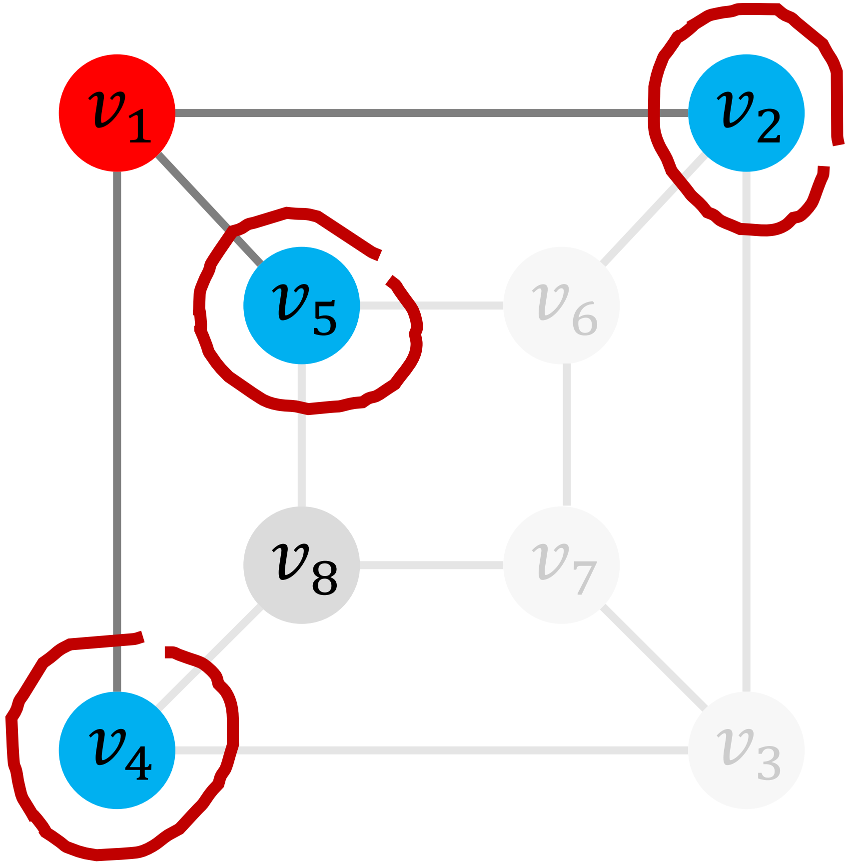
Iteration 1

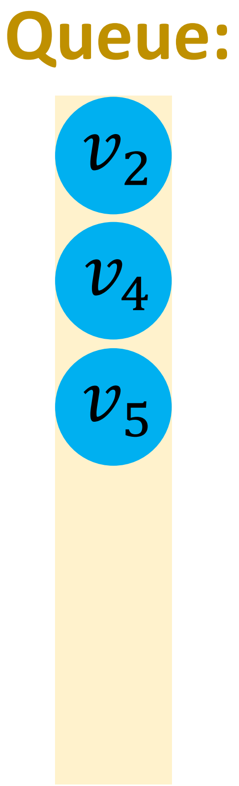


Queue:

- $v_1 \leftarrow$ dequeue( ).
- Assign blue color to its unvisited neighbors, v_2, v_4, and v_5.

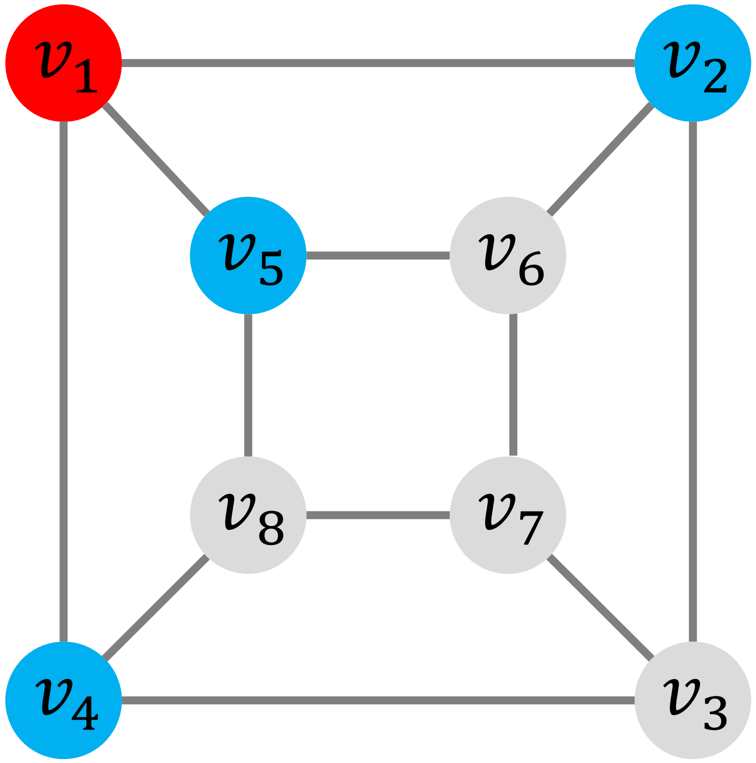
Iteration 1

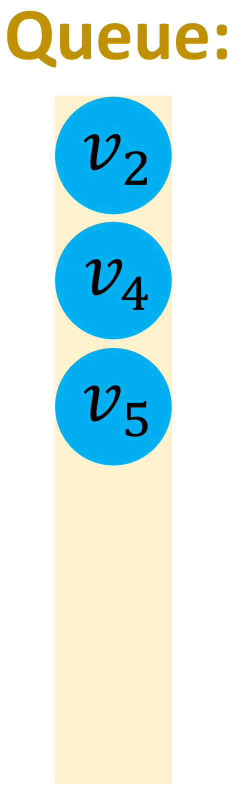




- $v_1 \leftarrow \mathrm{dequeue}(\,)$.
- Assign blue color to its unvisited neighbors, v_2, v_4, and v_5.
- Put the unvisited neighbors, v_2, v_4, and v_5, in the queue.

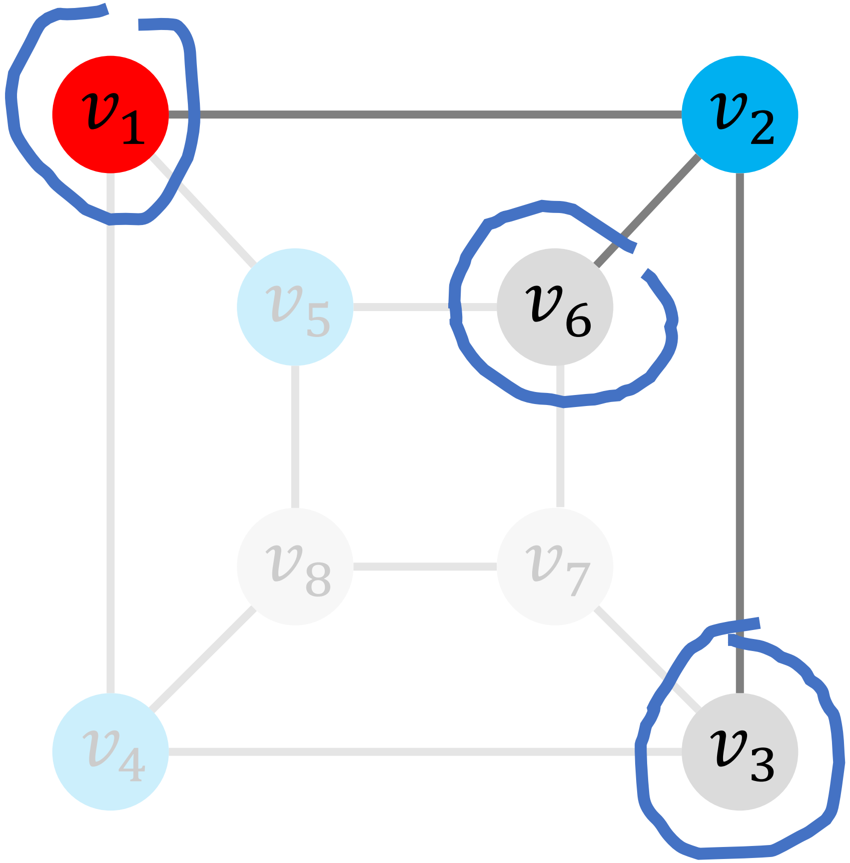
Iteration 2

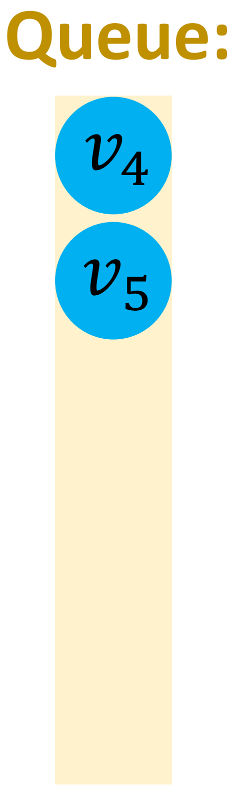




- $v_2 \leftarrow$ dequeue( ).

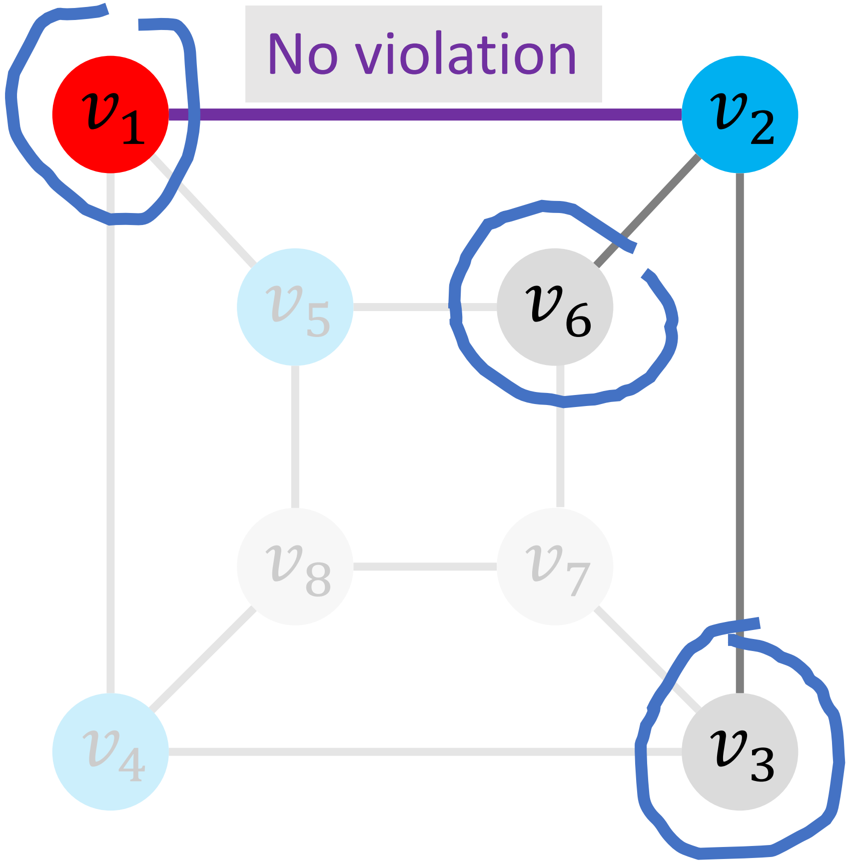
Iteration 2

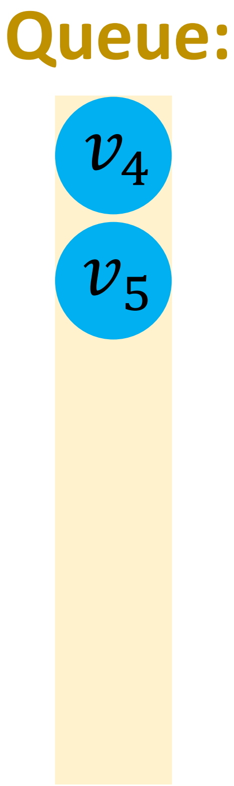




- $v_2 \leftarrow$ dequeue( ).

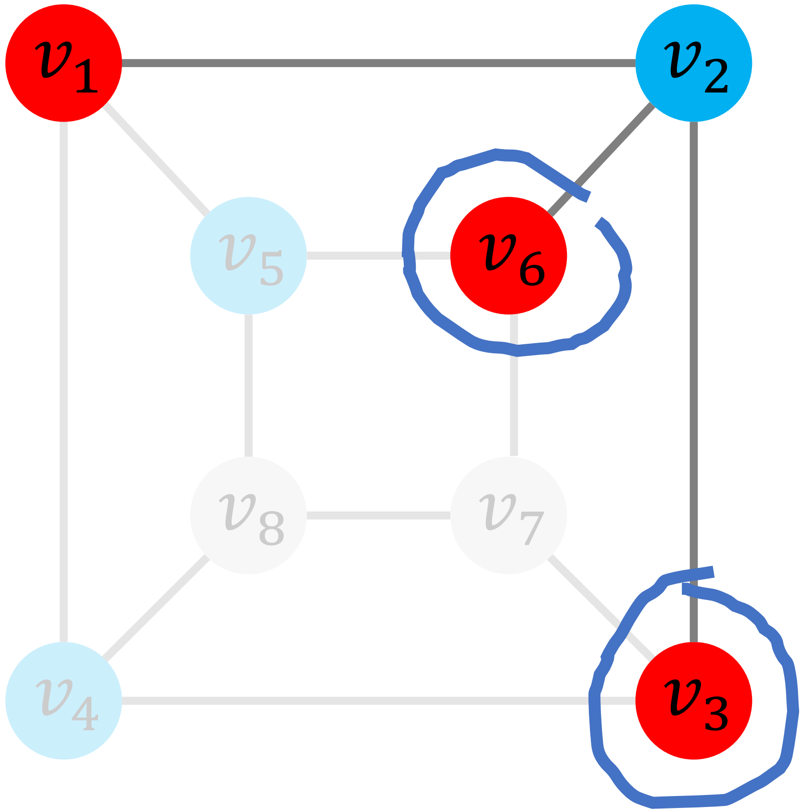
Iteration 2

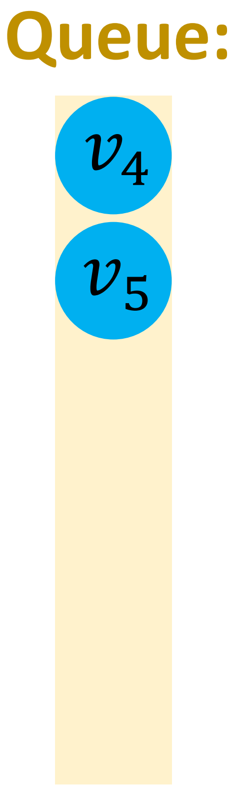




- $v_2 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.

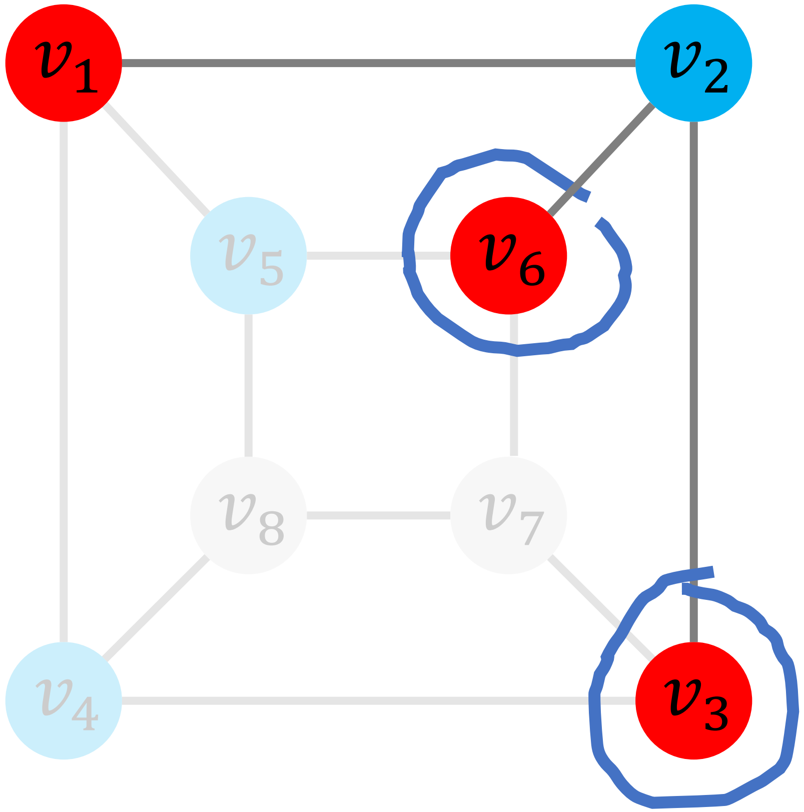
Iteration 2



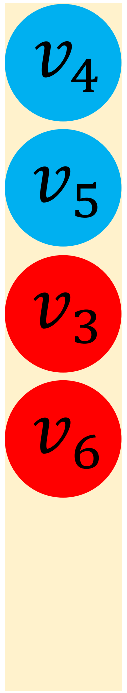


- $v_2 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Assign red color to its neighbors, v_3 and v_6.

Iteration 2

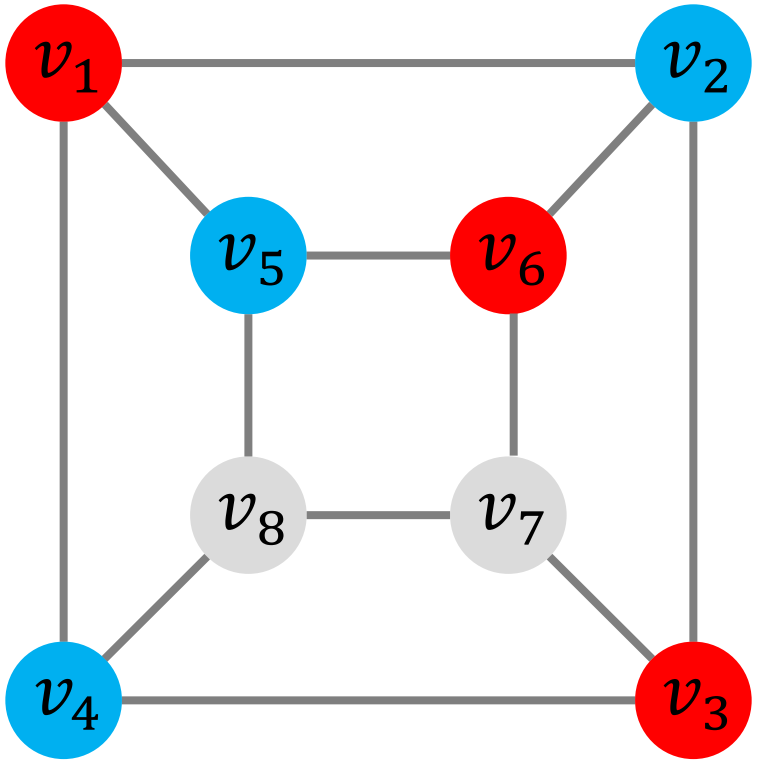


Queue:

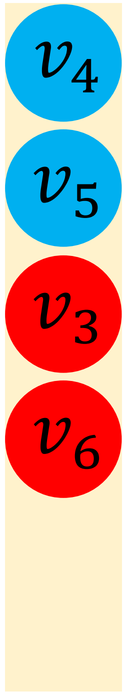


- $v_2 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Assign red color to its neighbors, v_3 and v_6.
- Put the unvisited neighbors, v_3 and v_6, in the queue.

Iteration 3

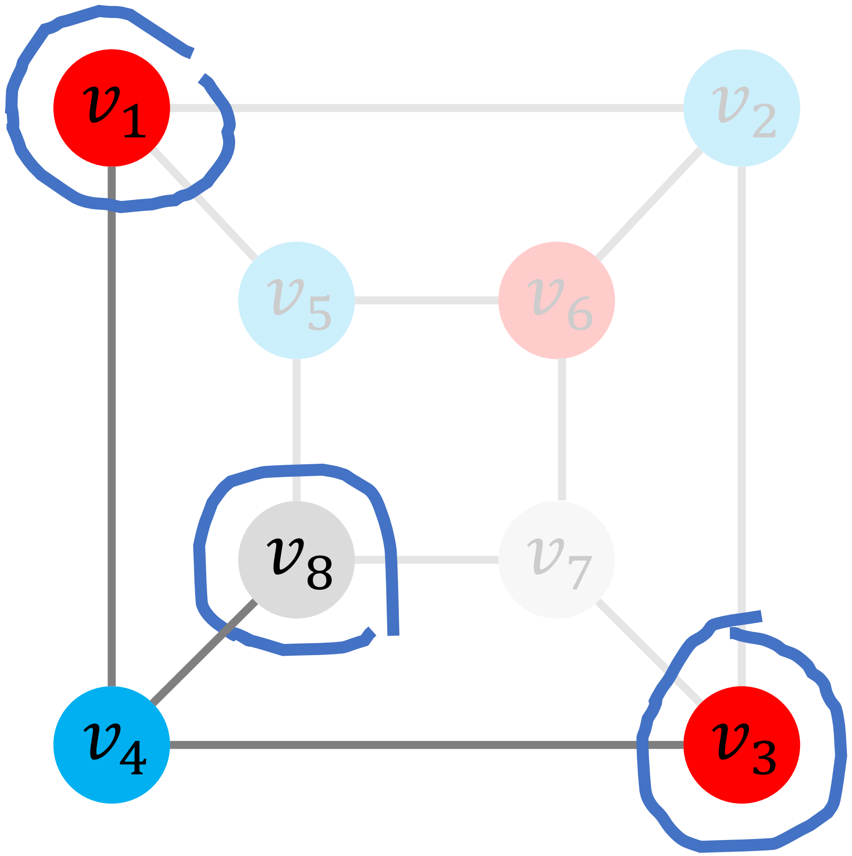


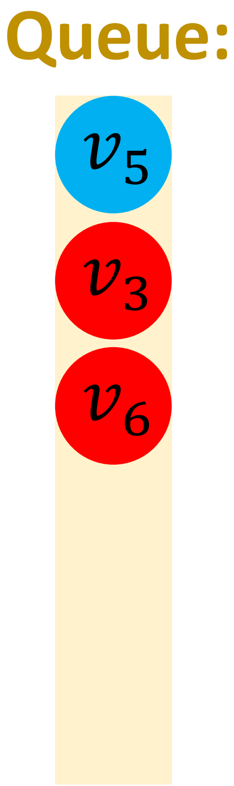
Queue:



- $v_4 \leftarrow$ dequeue( ).

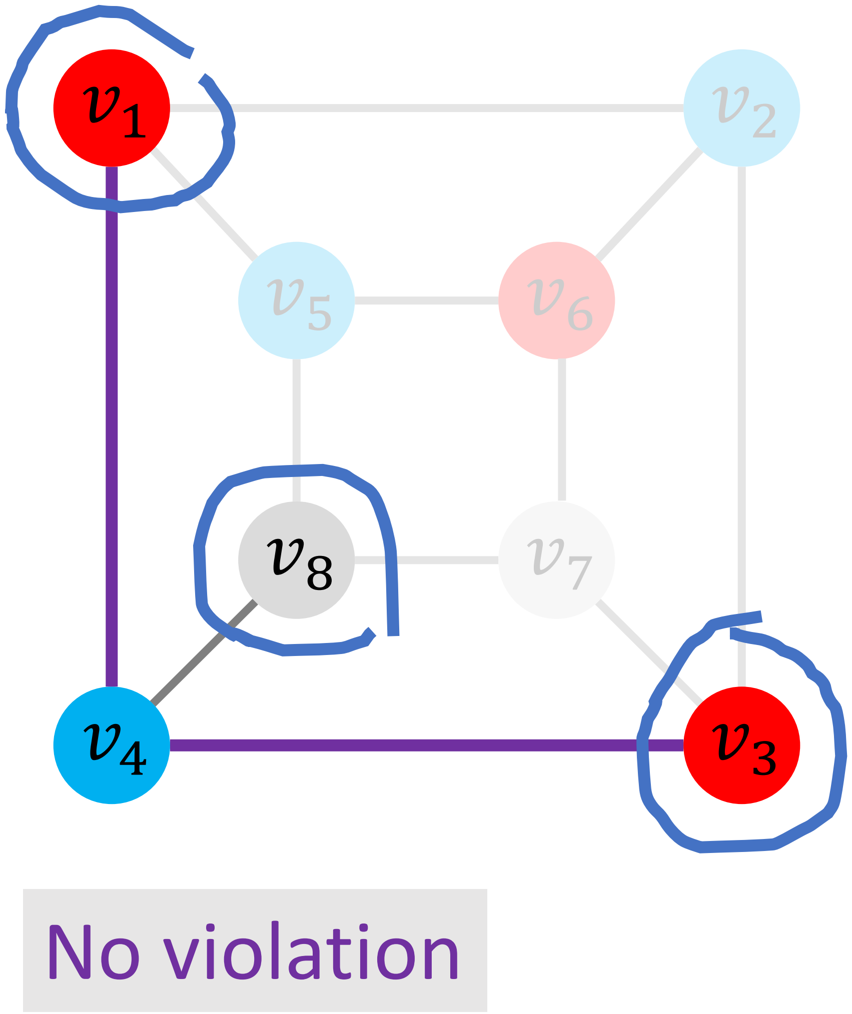
Iteration 3



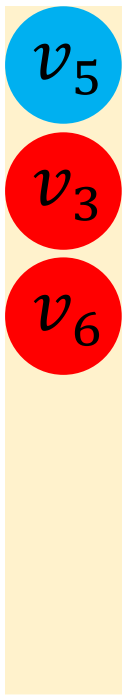


- $v_4 \leftarrow$ dequeue( ).

Iteration 3

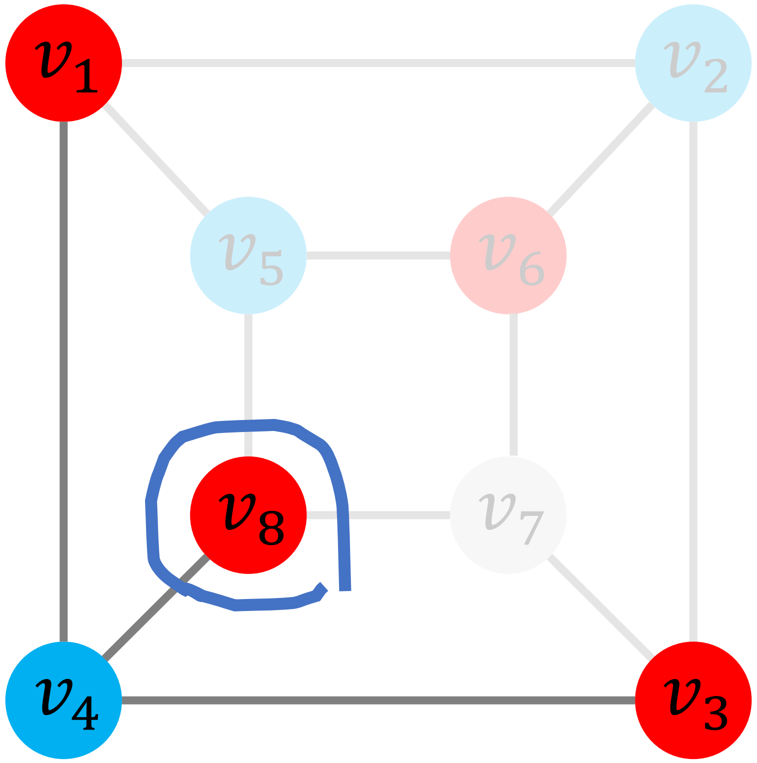


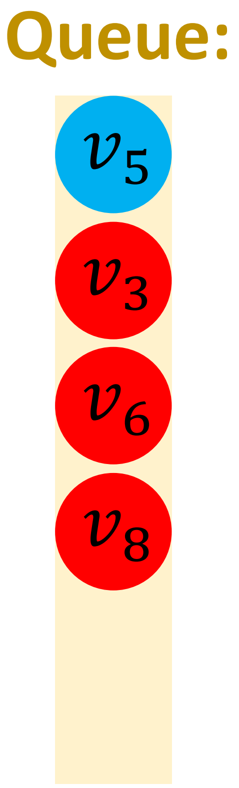
Queue:



- $v_4 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.

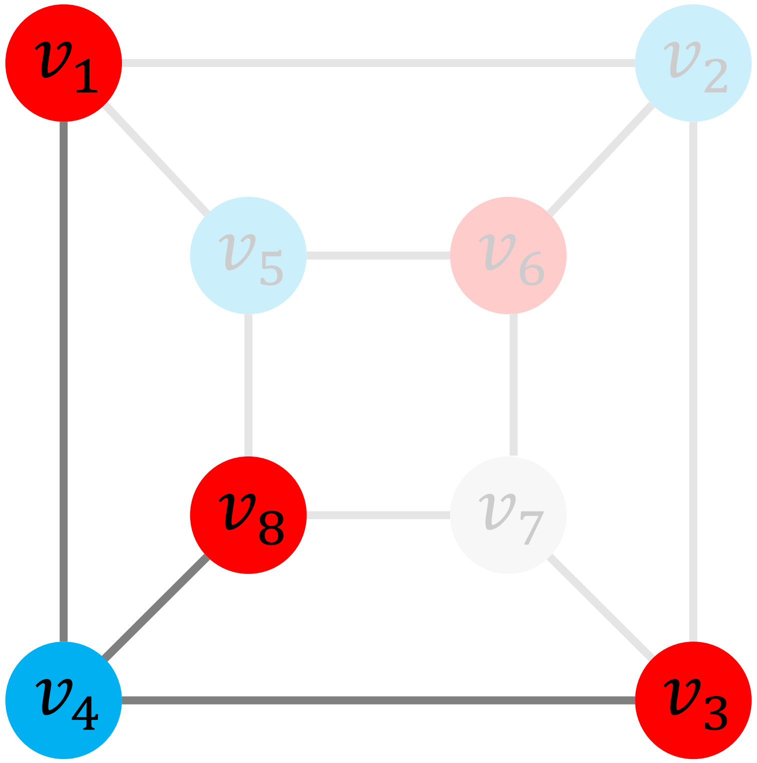
Iteration 3



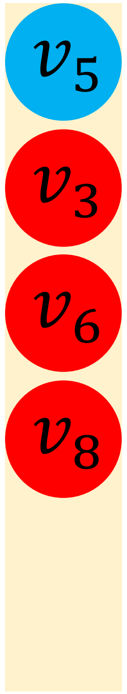


- $v_4 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Assign red color to its neighbor, v_8.

Iteration 3

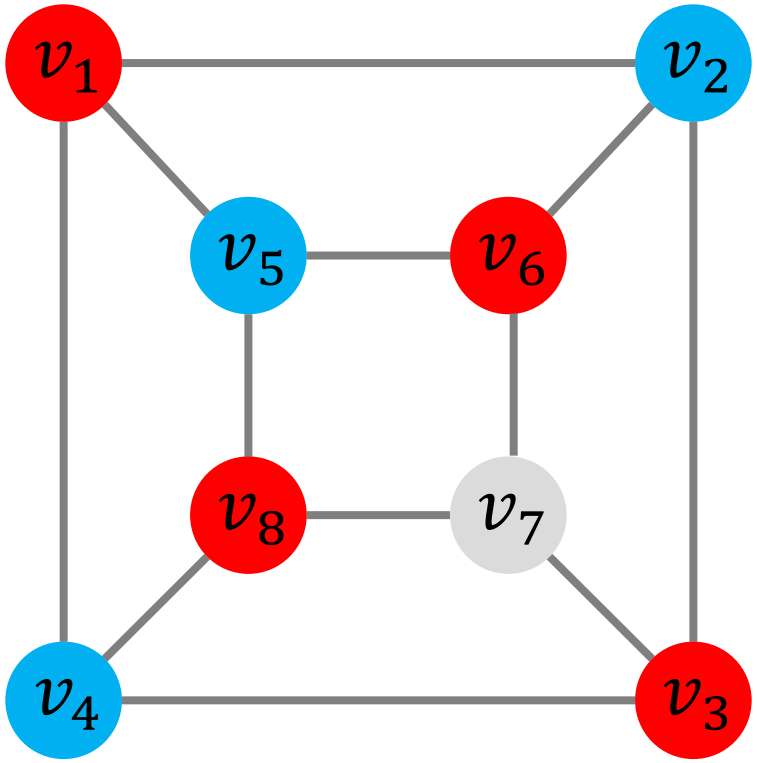


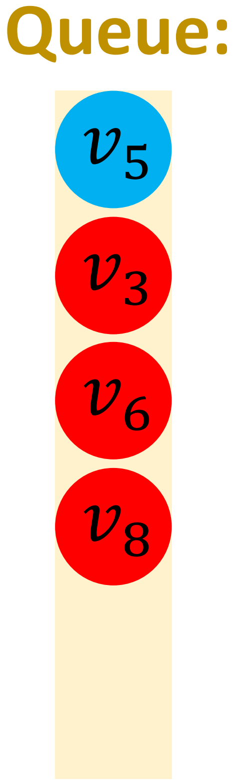
Queue:



- $v_4 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.
- Assign red color to its neighbor, v_8.
- Put the unvisited neighbor, v_8, in the queue.

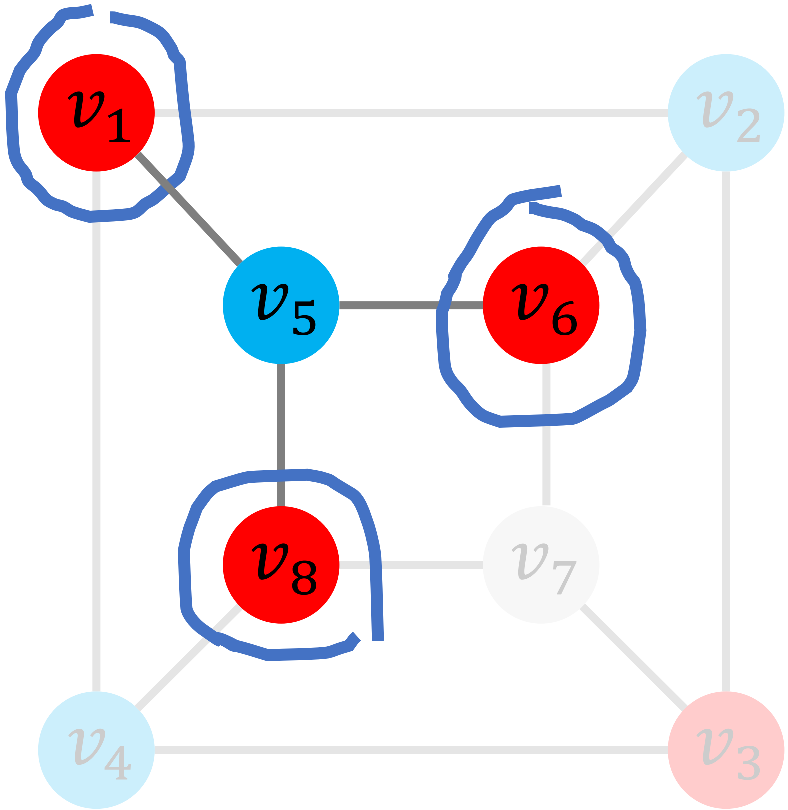
Iteration 4





- $v_5 \leftarrow$ dequeue( ).

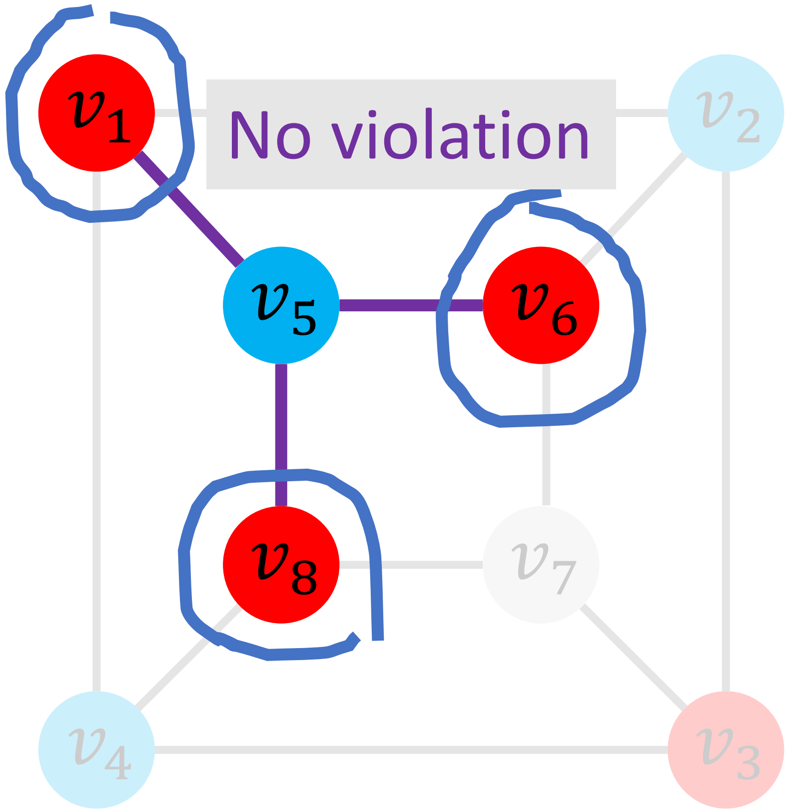
Iteration 4



Queue:

- $v_5 \leftarrow$ dequeue( ).

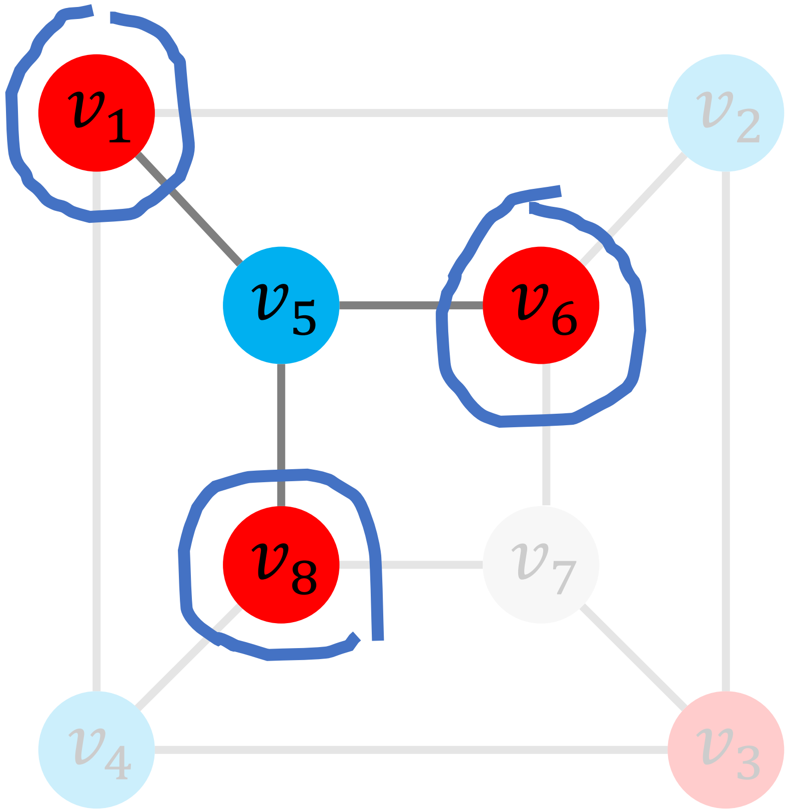
Iteration 4

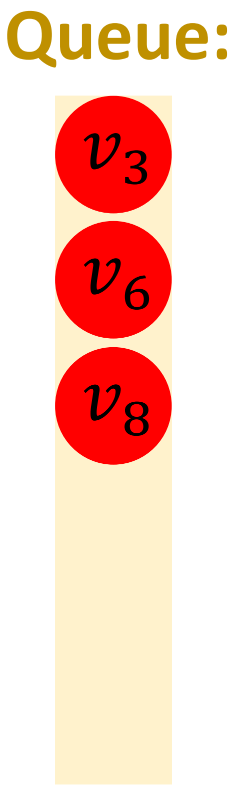


Queue:

- $v_5 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.

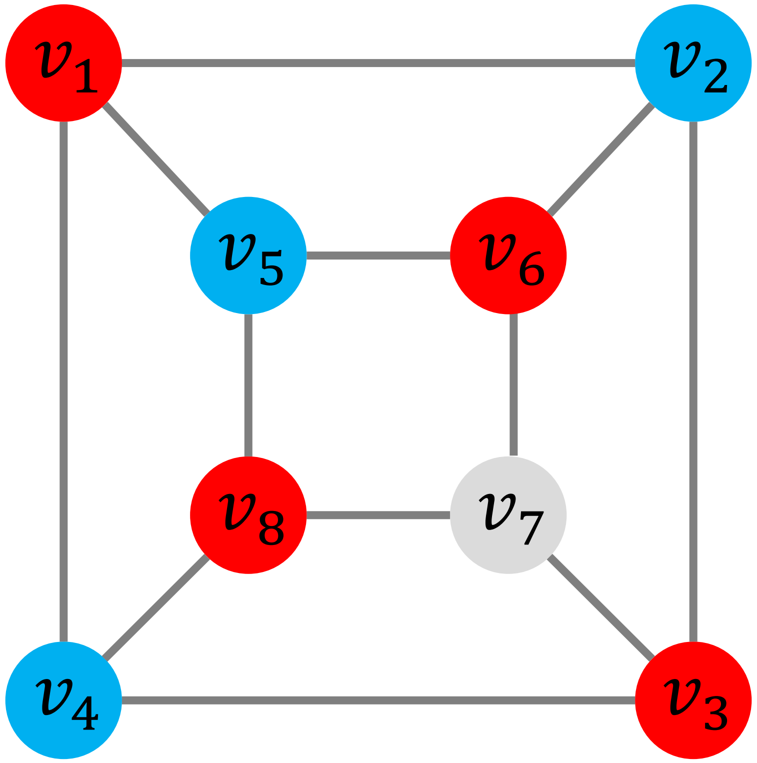
Iteration 4

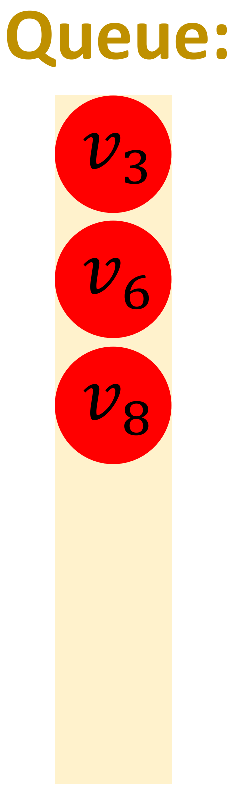




- $v_5 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

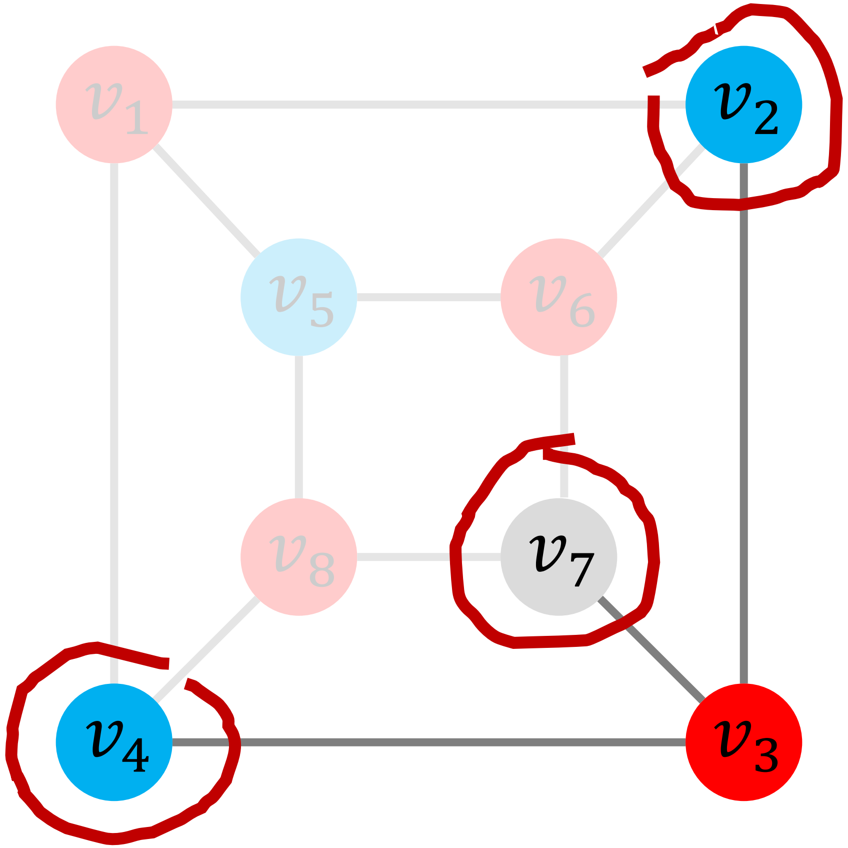
Iteration 5



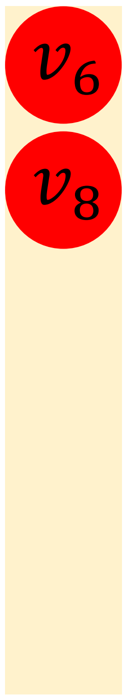


- $v_3 \leftarrow$ dequeue( ).

Iteration 5

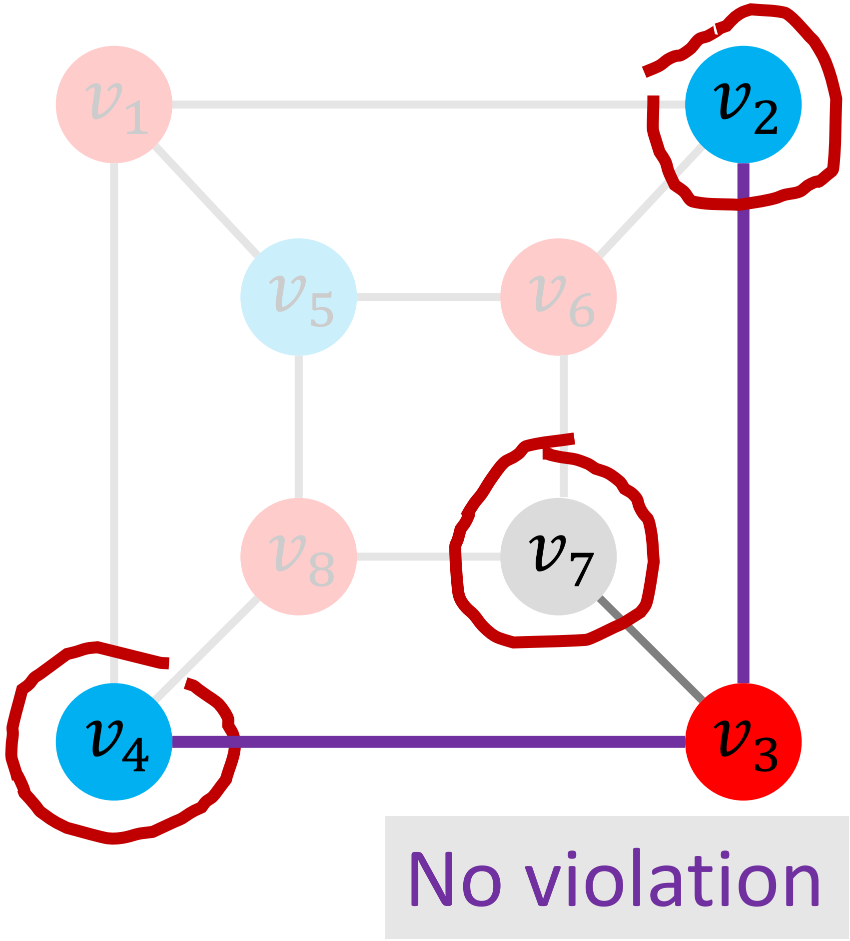


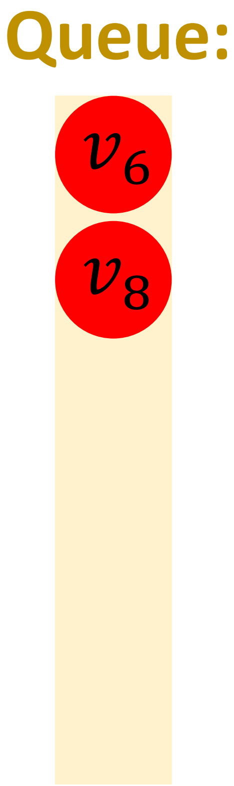
Queue:



- $v_3 \leftarrow$ dequeue( ).

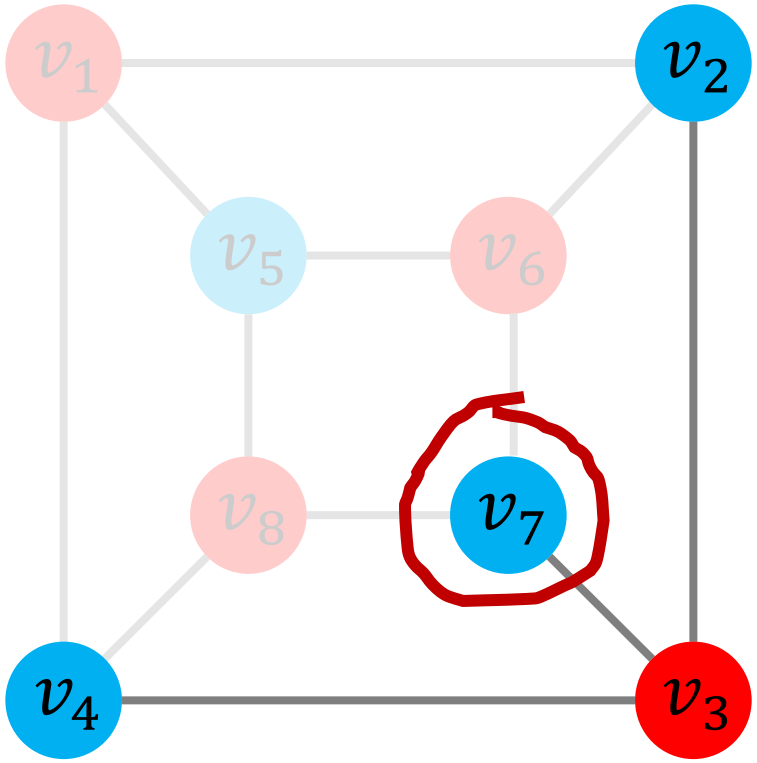
Iteration 5

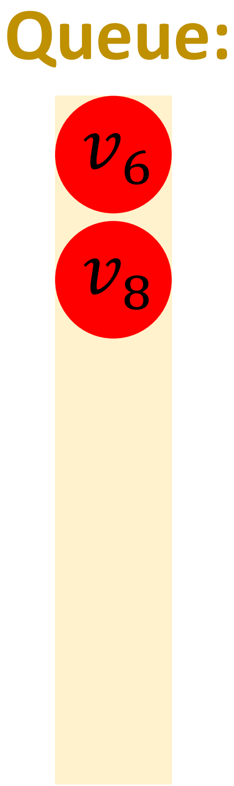




- $v_3 \leftarrow \text{dequeue}()$.
- Check the visited neighbors to see if there is any violation.

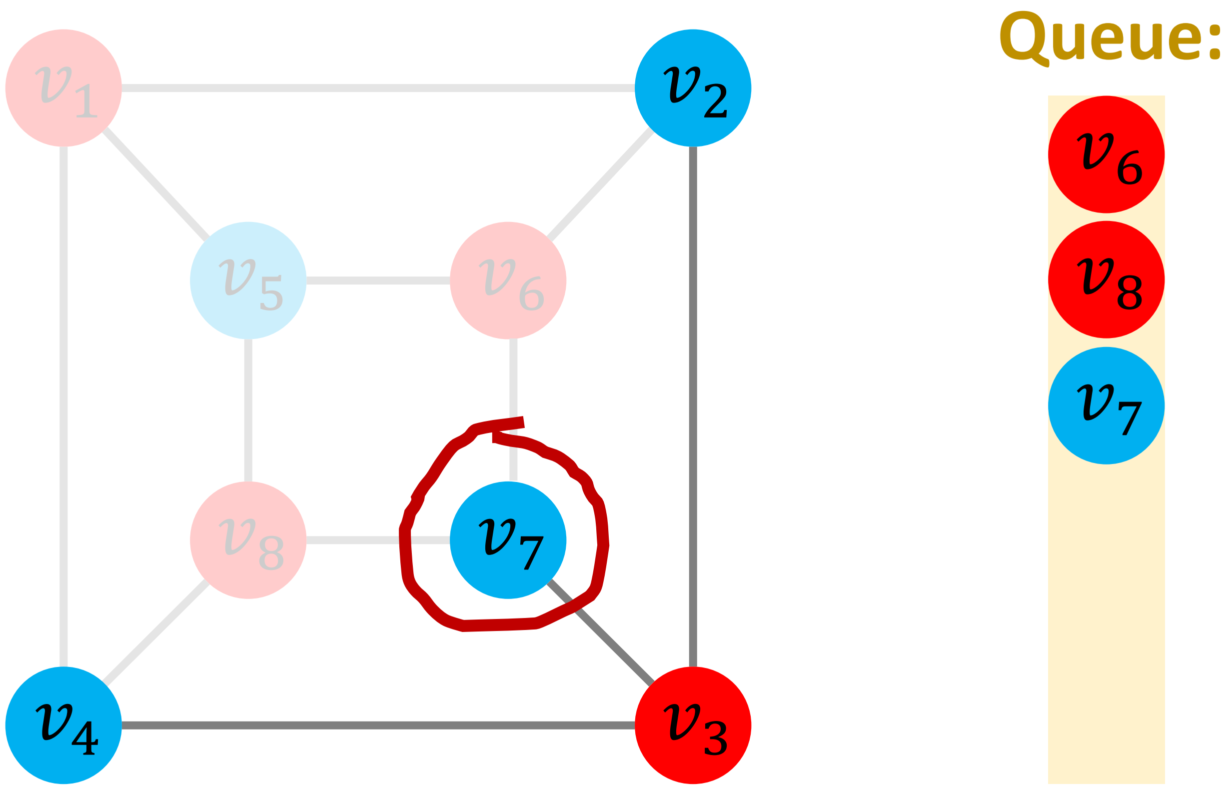
Iteration 5





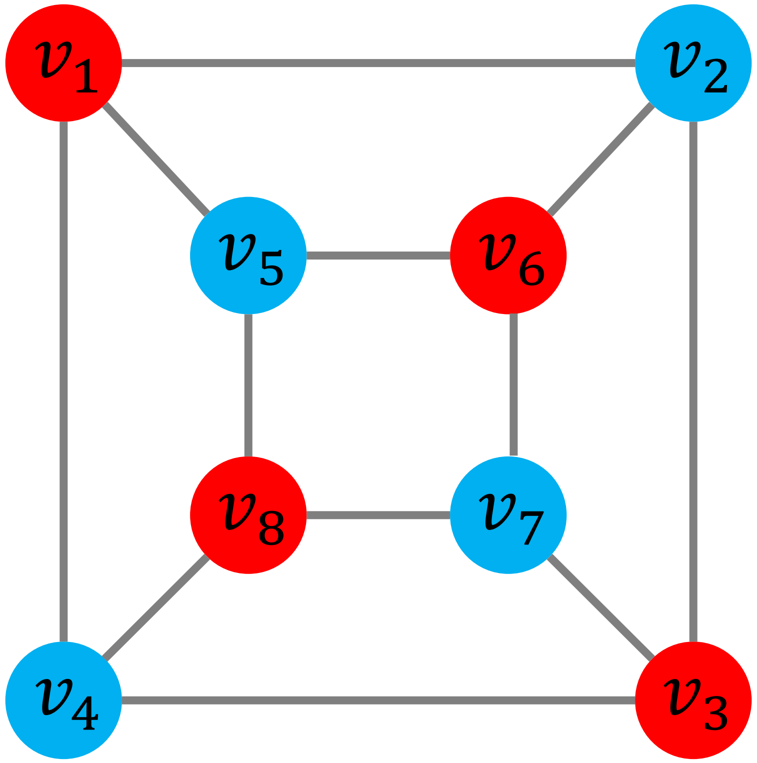
- $v_3 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.
- Assign blue color to its neighbor v_7.

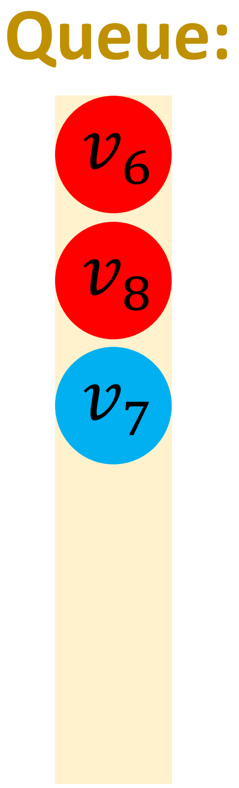
Iteration 5



- $v_3 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Assign blue color to its neighbor v_7.
- Put the unvisited neighbor v_7 in the queue.

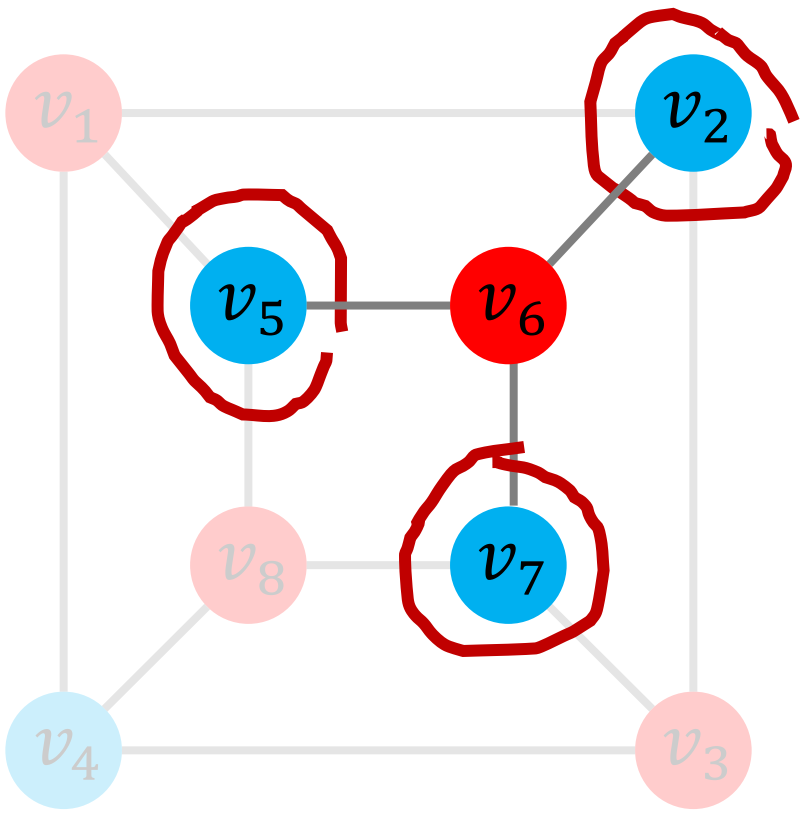
Iteration 6

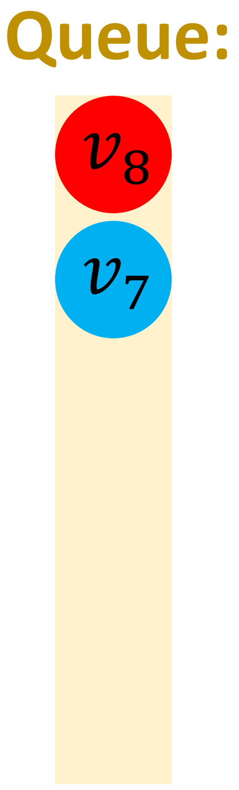




- $v_6 \leftarrow$ dequeue( ).

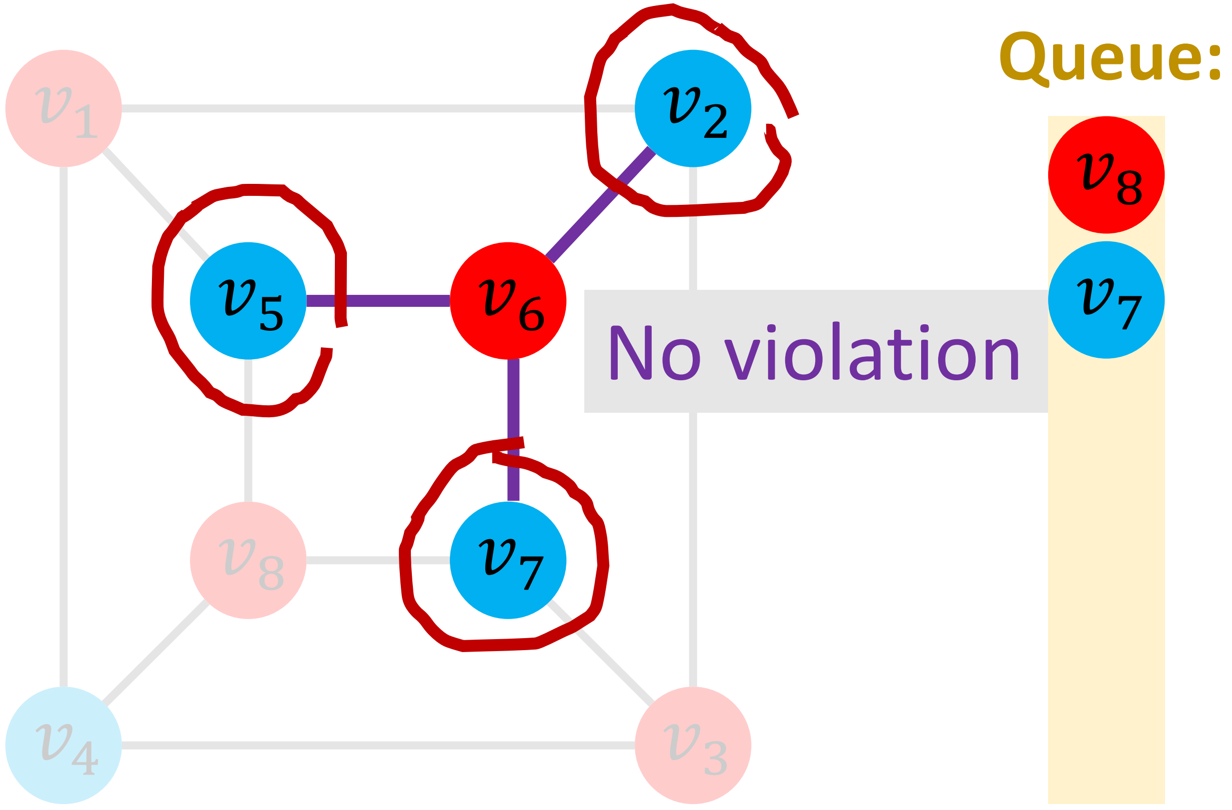
Iteration 6





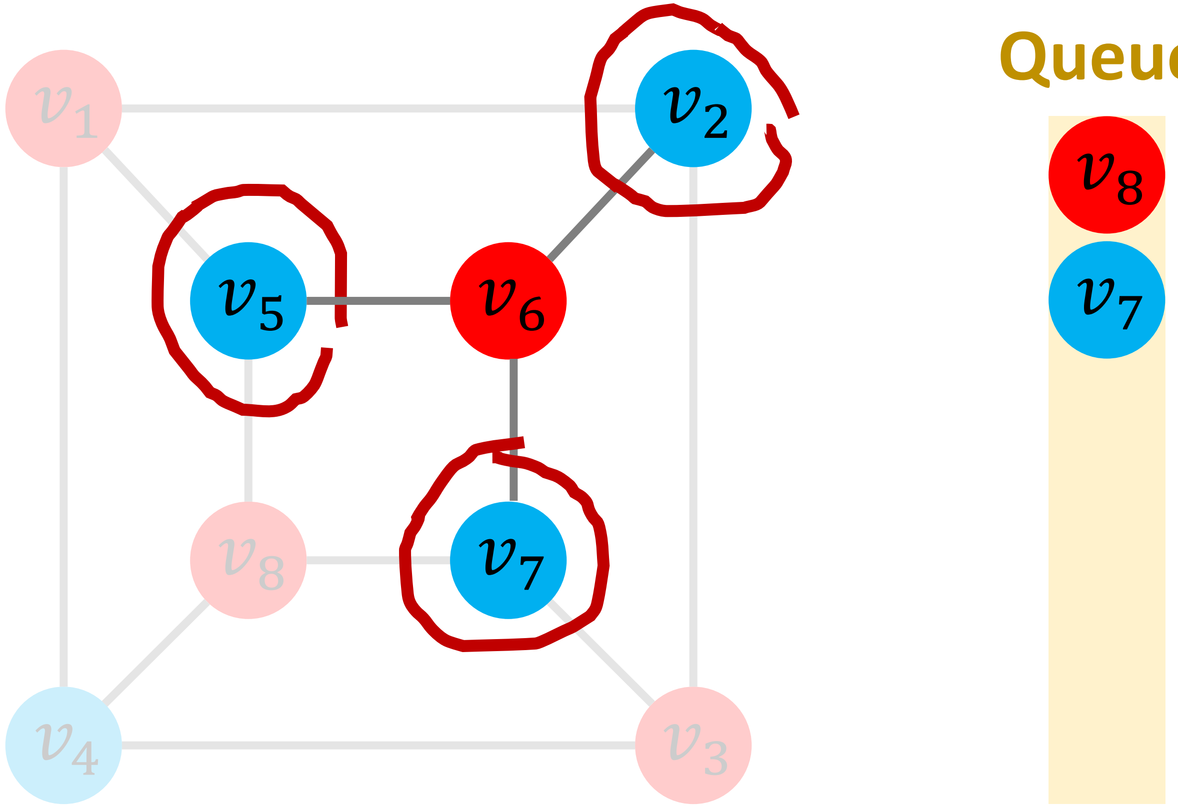
- $v_6 \leftarrow$ dequeue( ).

Iteration 6



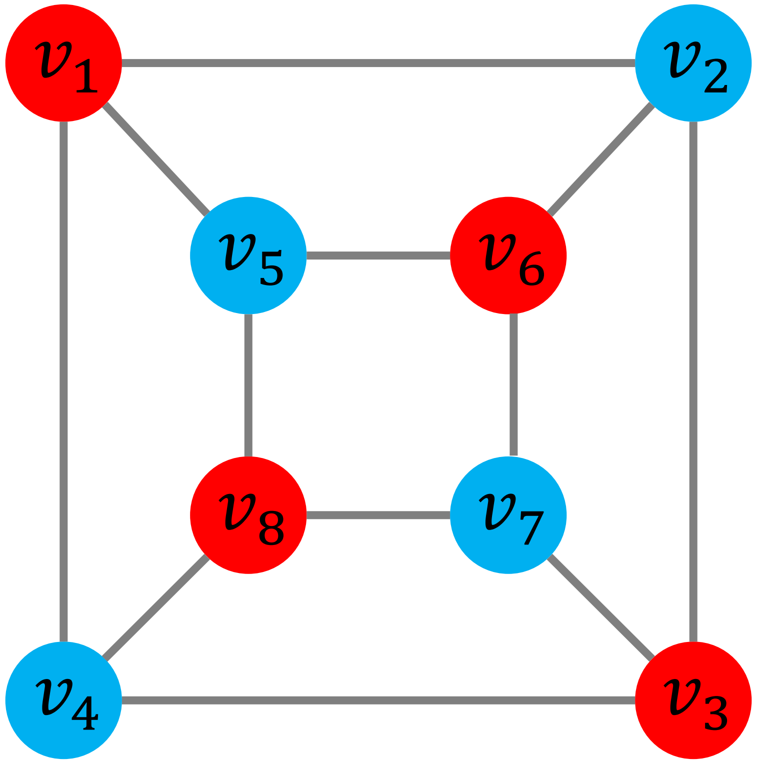
- $v_6 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.

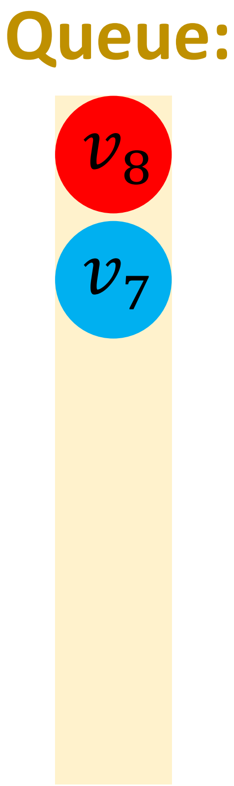
Iteration 6



- $v_6 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

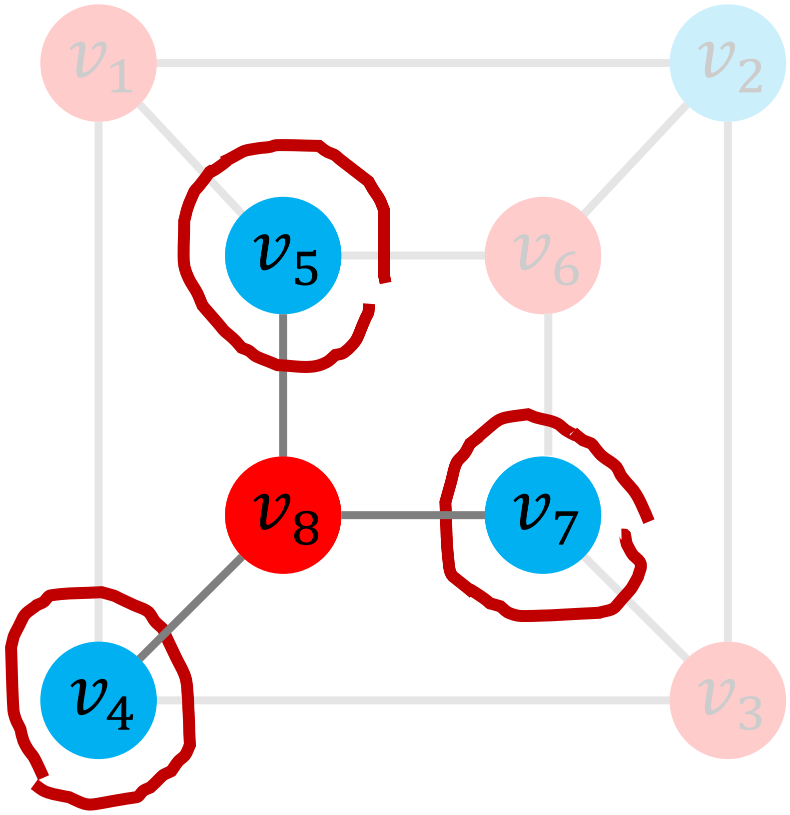
Iteration 7



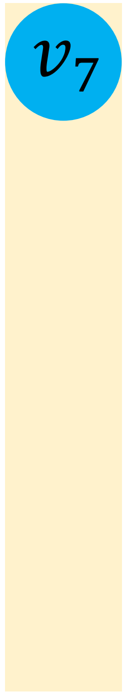


- $v_8 \leftarrow$ dequeue( ).

Iteration 7

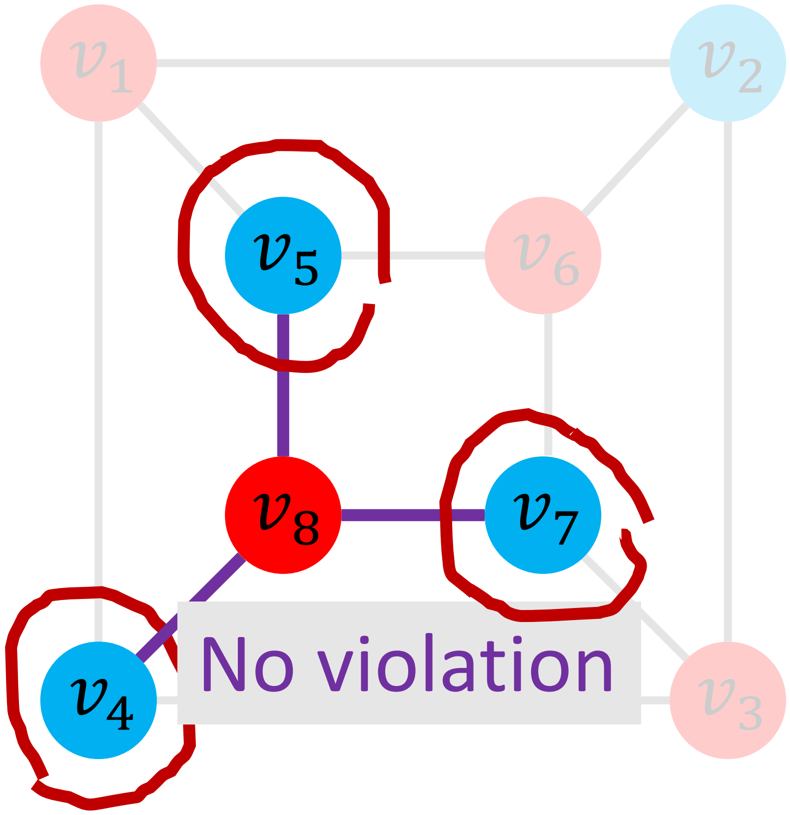


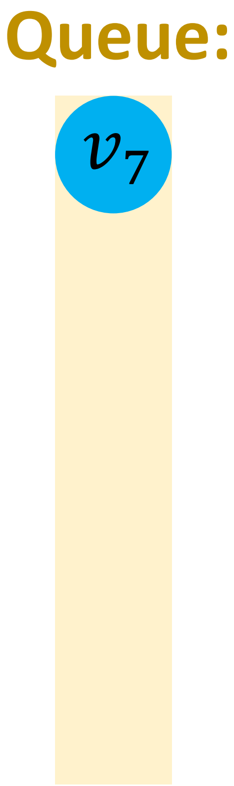
Queue:



- $v_8 \leftarrow$ dequeue( ).

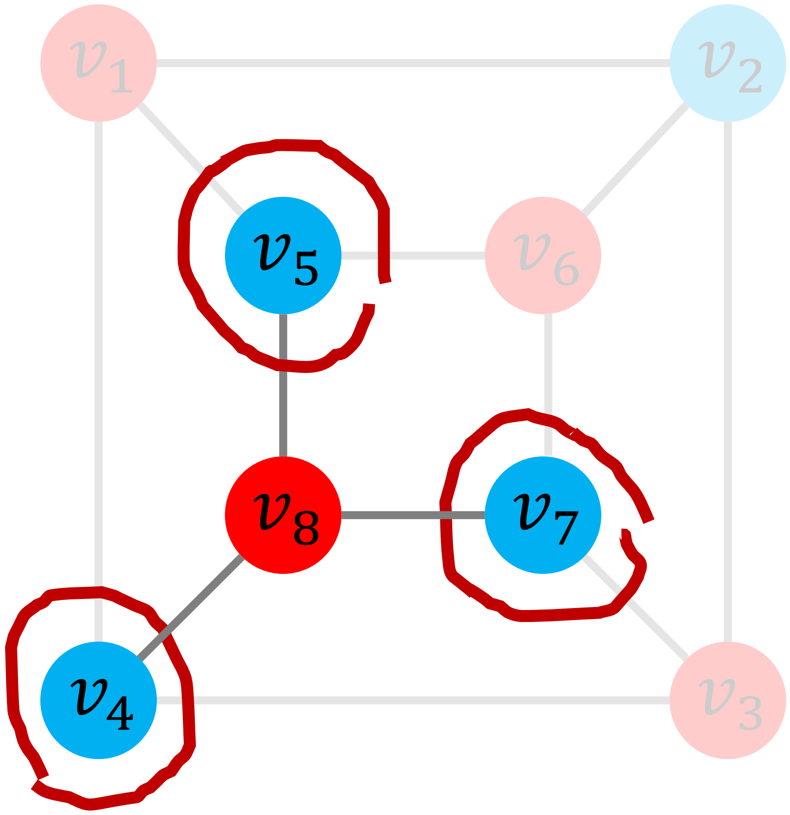
Iteration 7

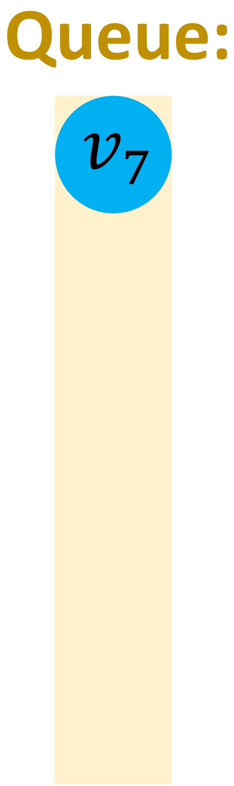




- $v_8 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.

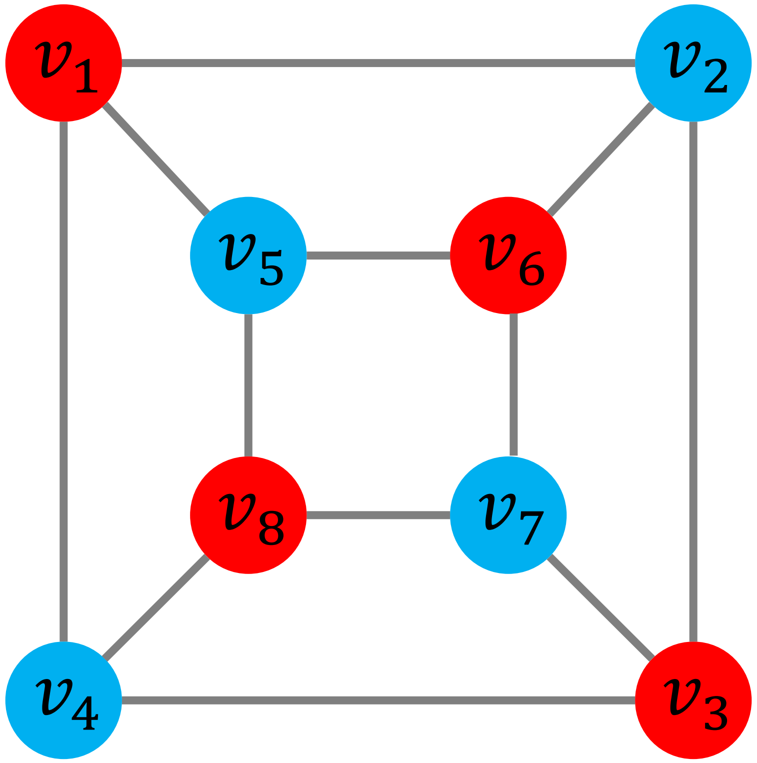
Iteration 7



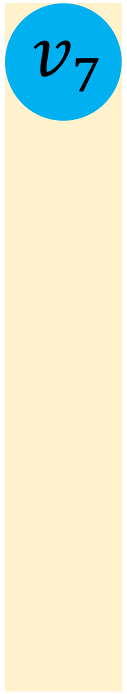


- $v_8 \leftarrow$ dequeue( ).
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

Iteration 8

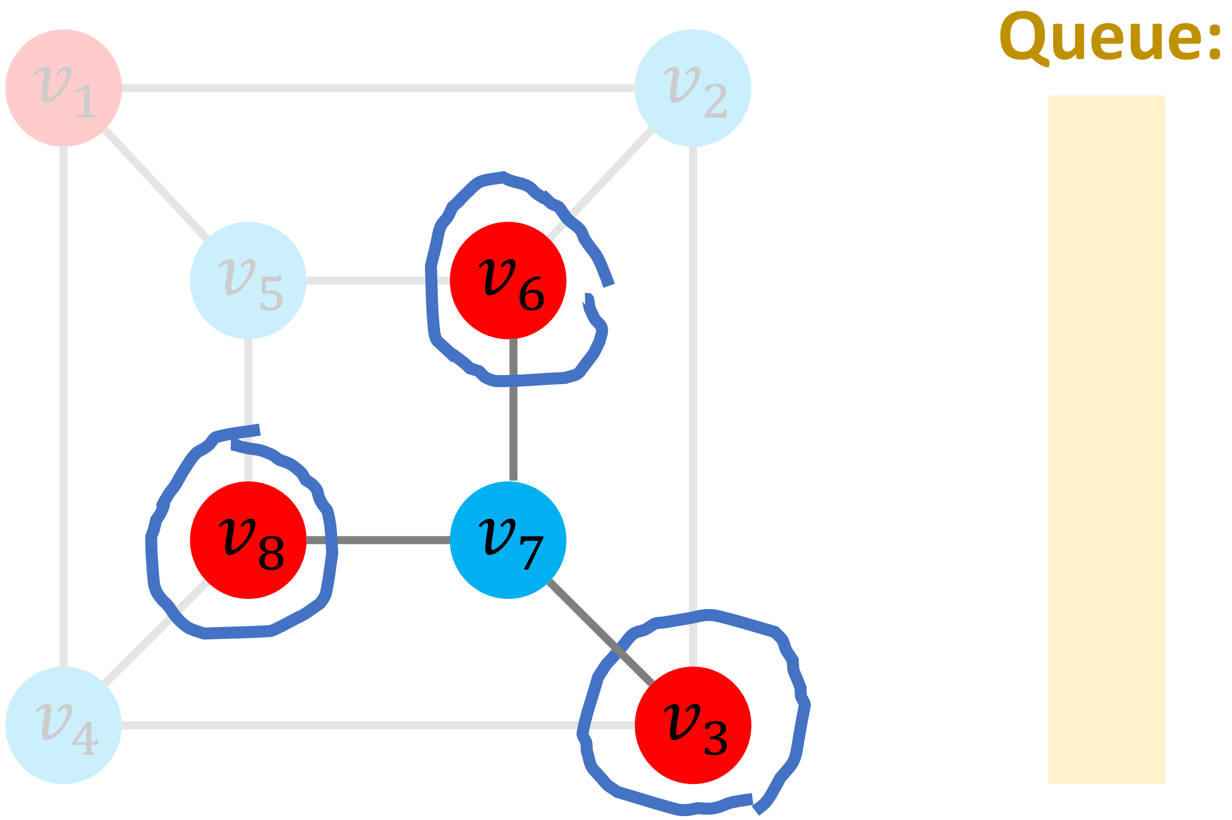


Queue:



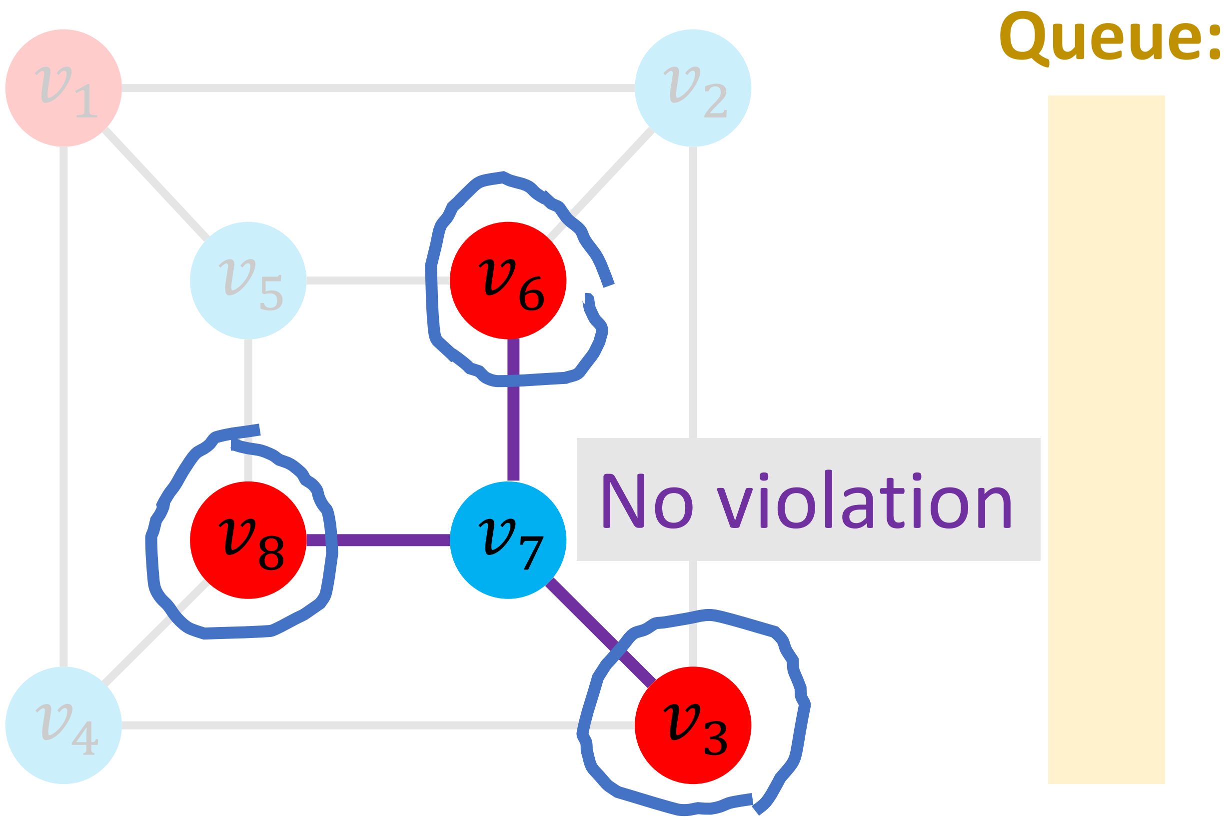
- $v_7 \leftarrow$ dequeue( ).

Iteration 8



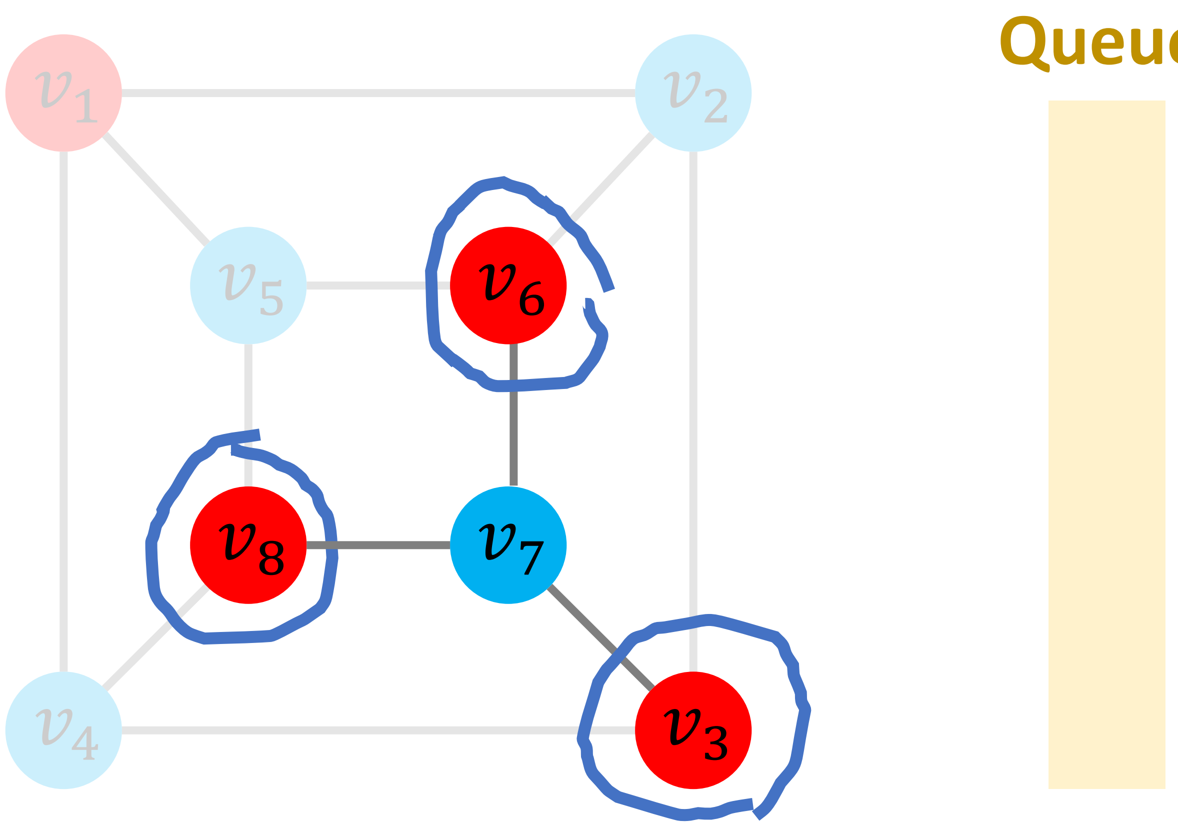
- $v_7 \leftarrow$ dequeue( ).

Iteration 8



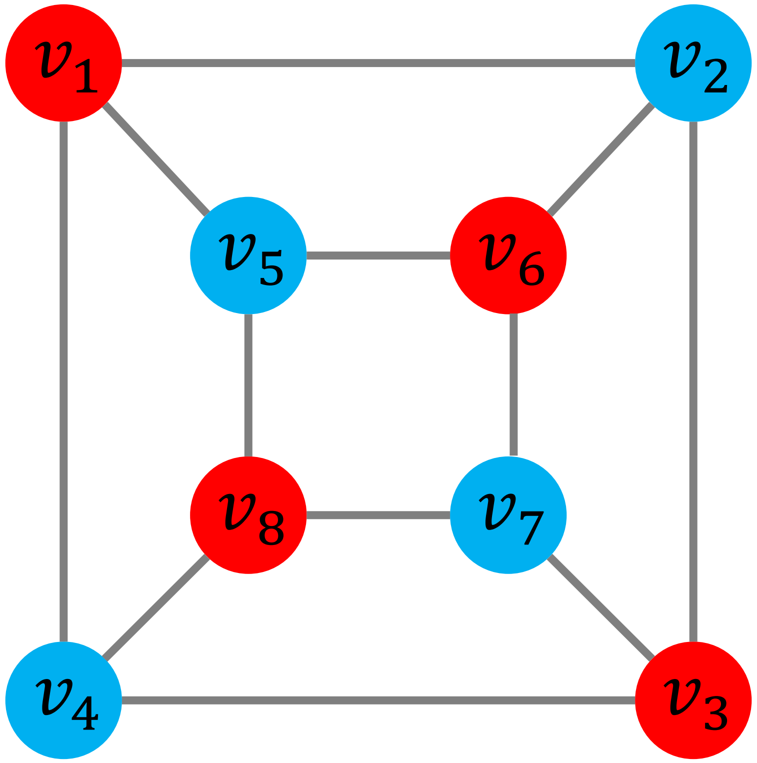
- $v_7 \leftarrow \text{dequeue}(\,)$.
- Check the visited neighbors to see if there is any violation.

Iteration 8



- $v_7 \leftarrow \text{dequeue( )}$.
- Check the visited neighbors to see if there is any violation.
- Do not put visited neighbors in the queue.

End of Procedure



Queue:

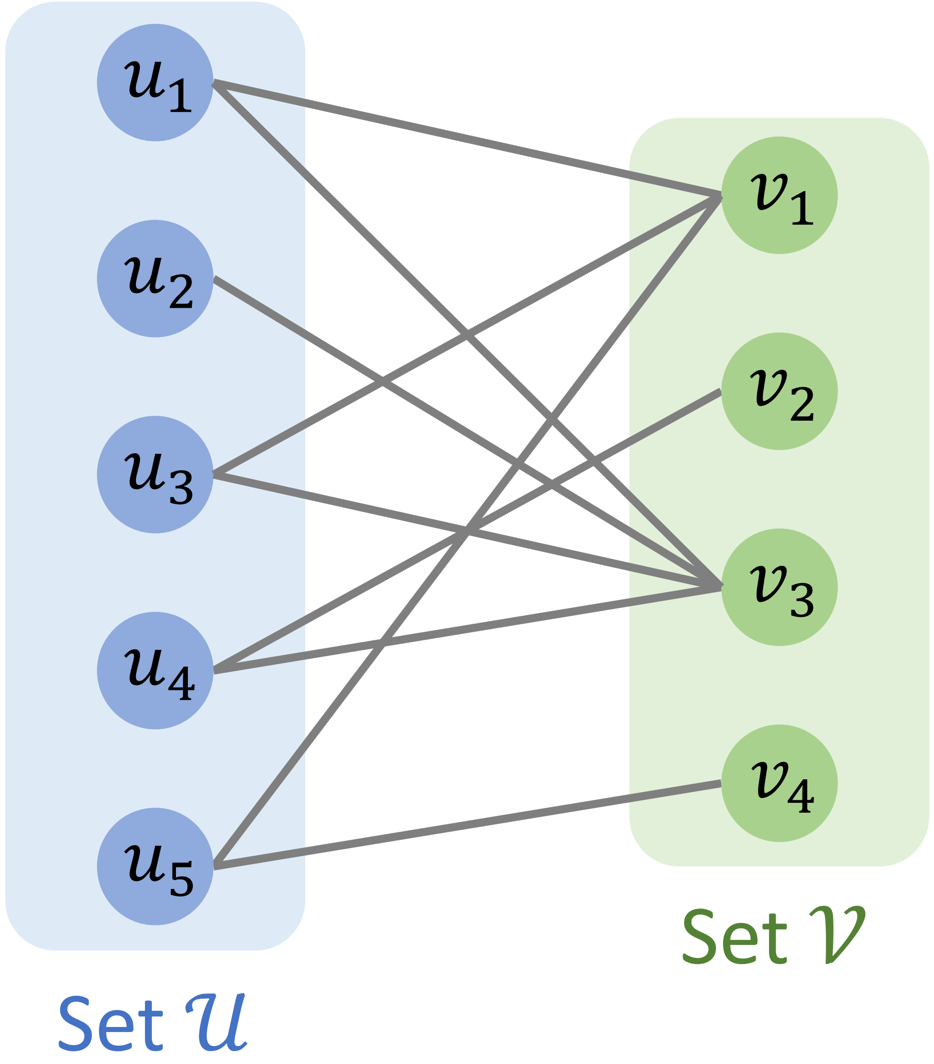
- All the vertices have been visited.
- The queue is empty.
- No violation has been found.
- Thus, the graph is bipartite.

Testing Bipartiteness

1. Select a vertex, assign red color to it, and add it to the queue.
2. While the queue is not empty:
   a. $v \leftarrow \text{dequeue}(\,);$
   b. $c \leftarrow$ the opposite color of v;
   c. For each $u \in \text{Neighbor}(v)$:
      i. If u has been visited, check whether there is a violation;
      ii. Otherwise, assign color c to u, and add u to the queue;
3. If violation is found in step 2c(i), return FALSE (not bipartite); otherwise, return TRUE.

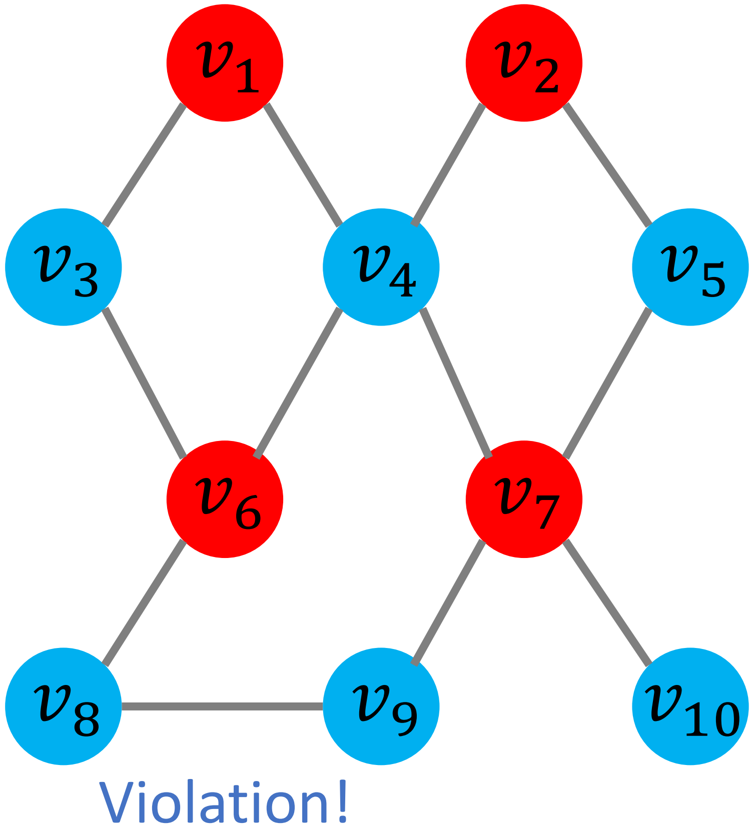
Summary

Bipartite Graph



- The vertices can be partitioned into two subsets, $\mathcal{U}$ and $\mathcal{V}$.
- No edge between two vertices in $\mathcal{U}$.
- No edge between two vertices in $\mathcal{V}$.
- Application: matching.
  - Matching candidates and positions.
  - Pet adoption.
  - Dating.

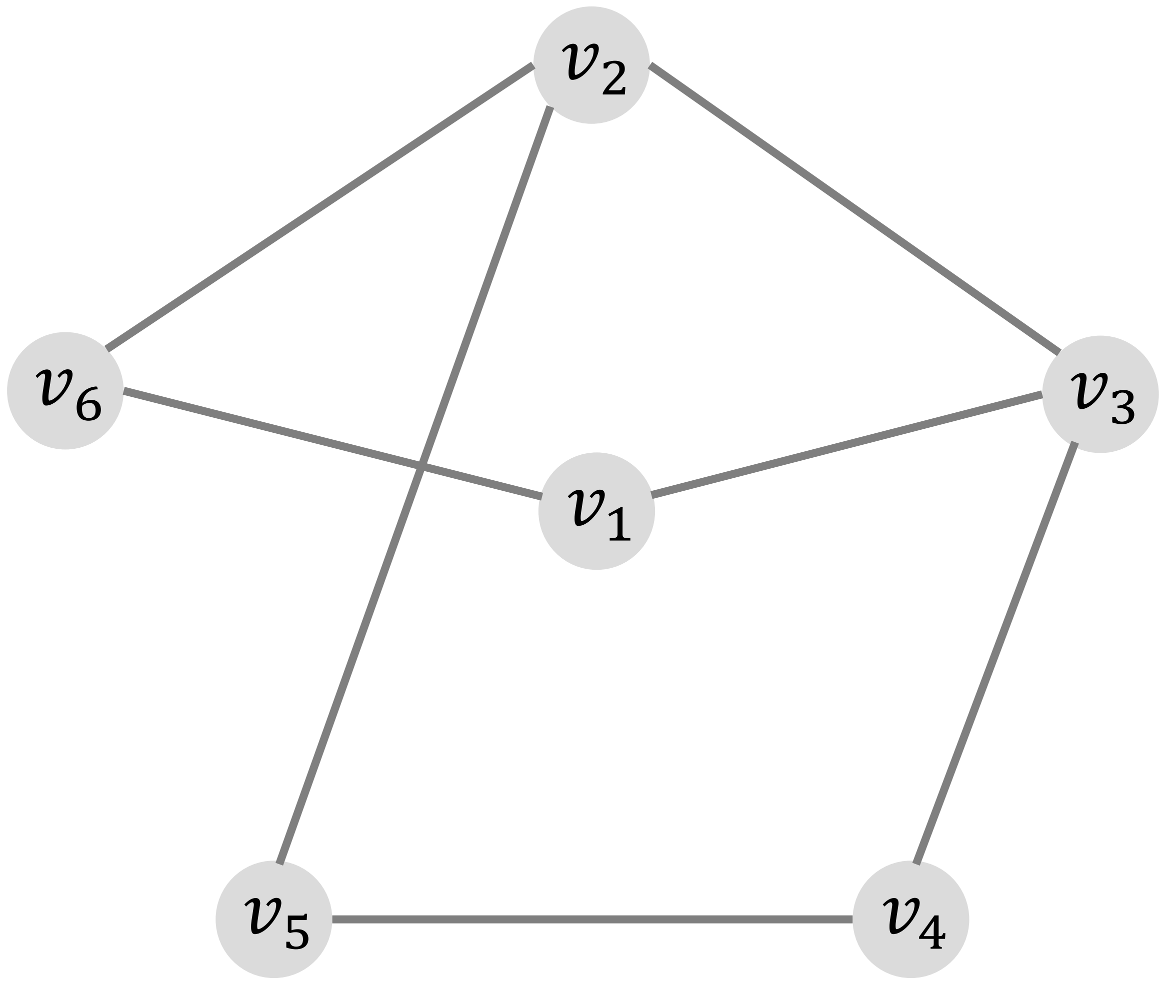
Testing Bipartiteness



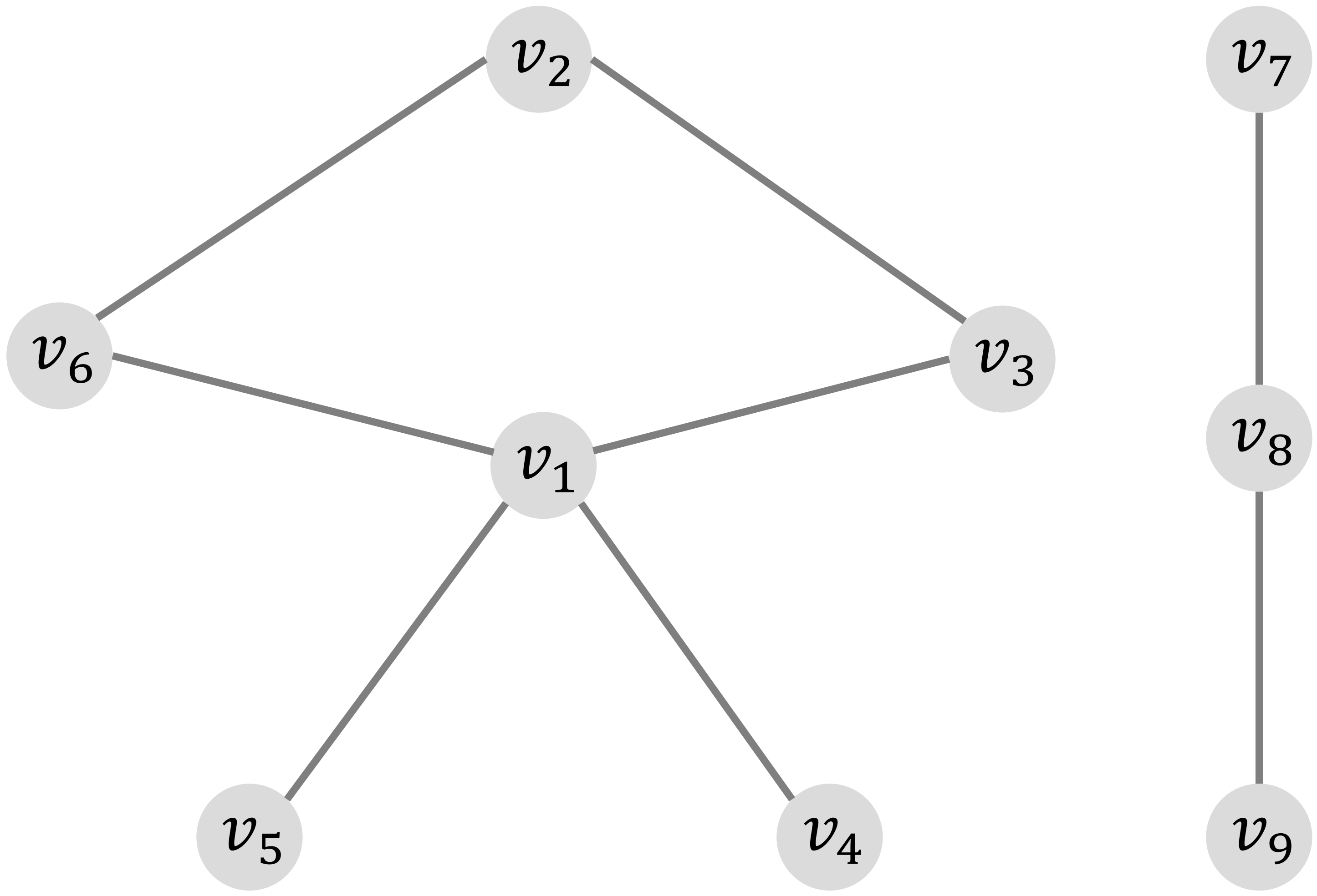
- Basic idea:
  - Coloring the nodes using red and blue.
  - Find whether there is any violation.
- Algorithm: breadth-first search (BFS).
- Time complexity: $O(|\mathcal{E}| + |\mathcal{V}|)$.

Questions

Q1: Is it a bipartite graph?



Q2: Is it a bipartite graph?



Thank You!