Matrix Basics

Shusen Wang

Vector and Matrix

Vector (n-dim) $\quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Matrix ($m \times n$) $\quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

Additions

Vector Addition

- Given $n \times 1$ vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector addition: $\mathbf{c} = \mathbf{a} + \mathbf{b} \in \mathbb{R}^n$.

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- Vector addition: $\mathbf{c} = \mathbf{a} + \mathbf{b} \in \mathbb{R}^n$.

Pseudo Code

- Initialization: $\mathbf{c} \leftarrow [0, 0, \cdots, 0]$.
- For $i = 1$ to n:

$$c_i \leftarrow a_i + b_i.$$

Time complexity: $O(n)$.

Matrix Addition

- Given $m \times n$ matrices: $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$.
- Matrix addition: $\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{R}^{m \times n}$.

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- Given $m \times n$ matrices: $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$.
- Matrix addition: $\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{R}^{m \times n}$.

Pseudo Code

- Initialization: $\mathbf{C} \leftarrow m \times n$ all-zero matrix.
- For $i = 1$ to m:
  - For $j = 1$ to n:

$$c_{ij} \leftarrow a_{ij} + b_{ij}.$$

Time complexity: $O(mn)$.

Multiplications

Vector Inner Product

- Given vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $c = \mathbf{a}^T\mathbf{b} = a_1 b_1 + \cdots + a_n b_n$.

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- Vector inner product: $c = \mathbf{a}^T\mathbf{b} = a_1 b_1 + \cdots + a_n b_n$.

Pseudo Code

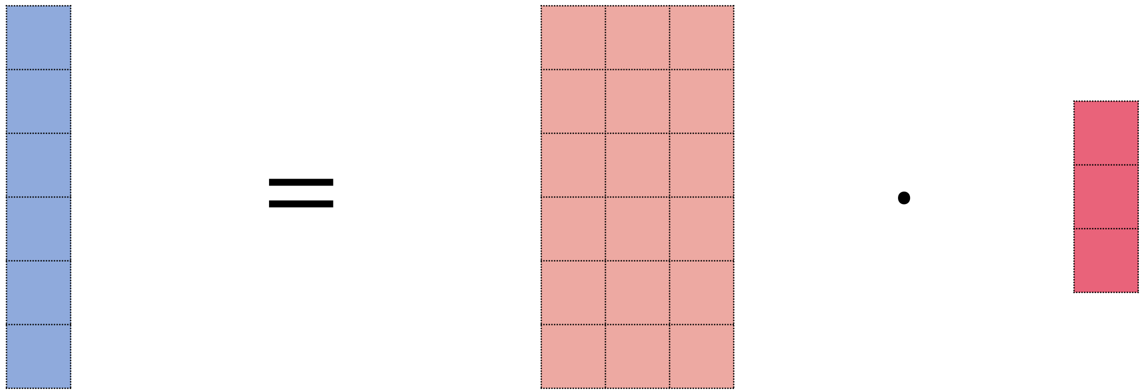
- Initialization: $c \leftarrow 0$.
- For $i = 1$ to n:

  $c \leftarrow c + a_i b_i$.

Time complexity: $O(n)$.

Matrix-Vector Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^{n}$.
- Vector inner product: $\mathbf{c} = \mathbf{A}\mathbf{b} \in \mathbb{R}^{m}$.



$\mathbf{c}$ $\qquad$ $\mathbf{A}$ $\qquad$ $\mathbf{b}$

$m \times 1$ $\qquad$ $m \times n$ $\qquad$ $n \times 1$

Matrix-Vector Product

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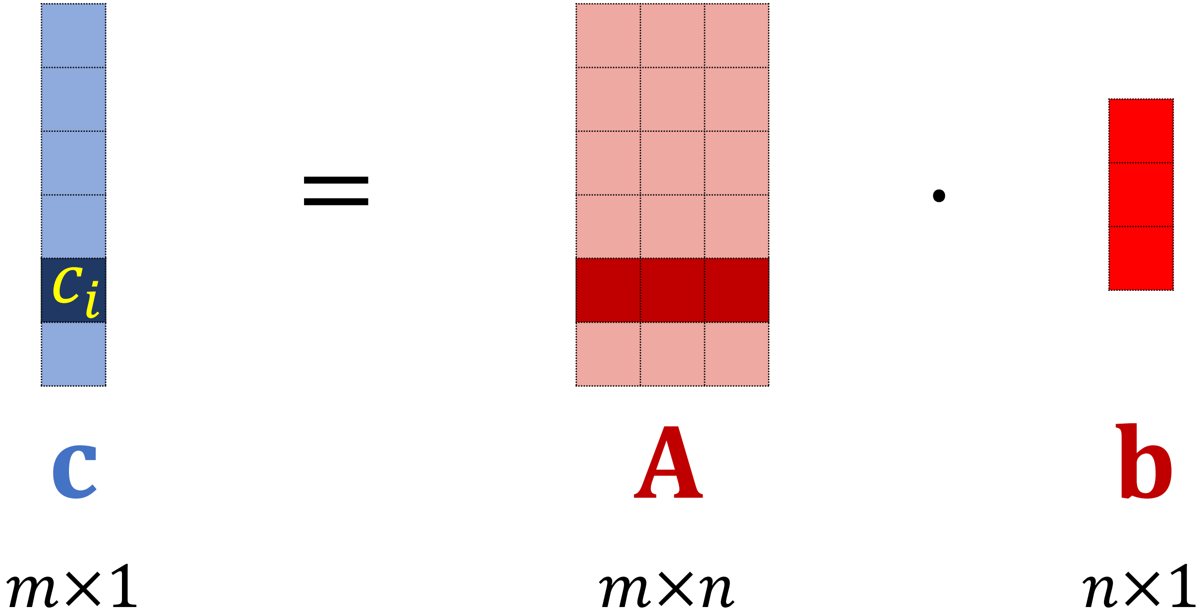

Matrix-Vector Product

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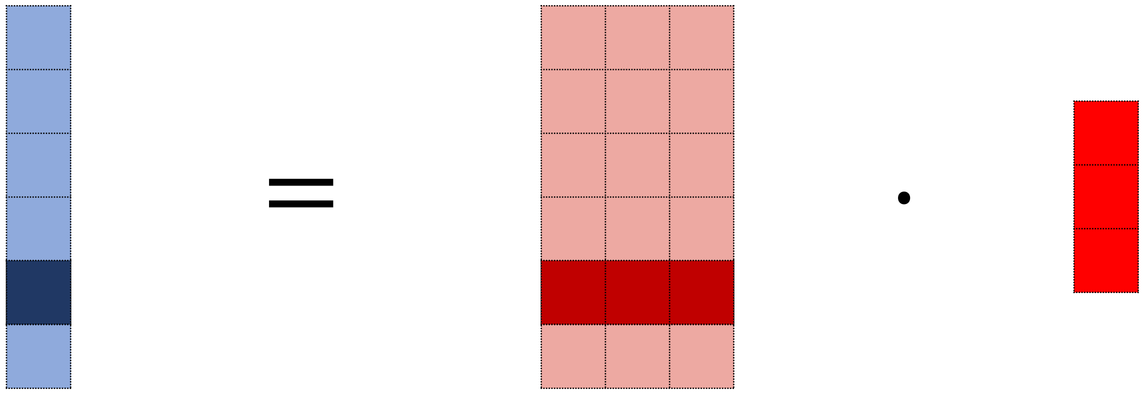
$$c_i = \sum_{j=1}^{n} a_{ij}\, b_j$$



$\mathbf{c}$ $m \times 1$ = $\mathbf{A}$ $m \times n$ · $\mathbf{b}$ $n \times 1$

Matrix-Vector Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^{n}$.
- Vector inner product: $\mathbf{c} = \mathbf{A}\mathbf{b} \in \mathbb{R}^{m}$.

Pseudo Code

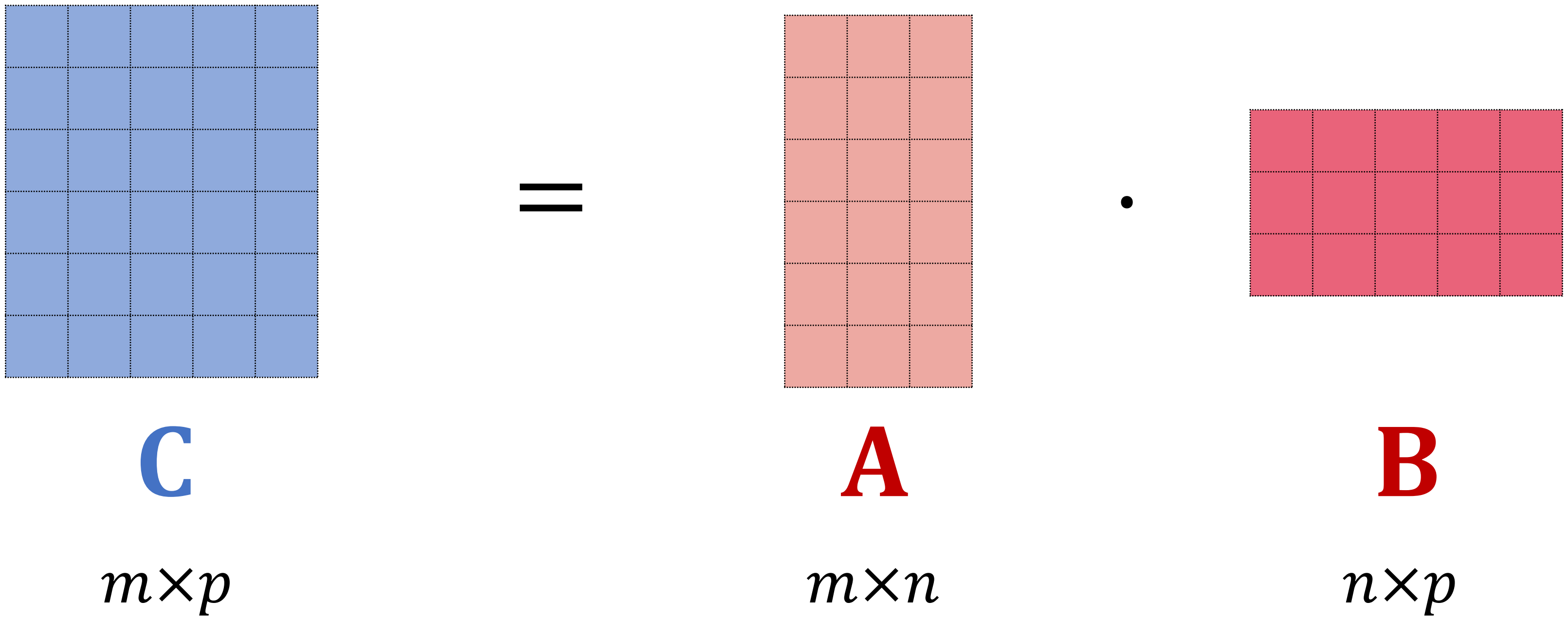
- Initialization: $\mathbf{c} \leftarrow$ all-zero vector.
- For $i = 1$ to m:
  - For $j = 1$ to n:

$$c_i \leftarrow c_i + a_{ij}\, b_j.$$

Time complexity: $O(mn)$.

Matrix-Matrix Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times p}$.
- Vector inner product: $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.



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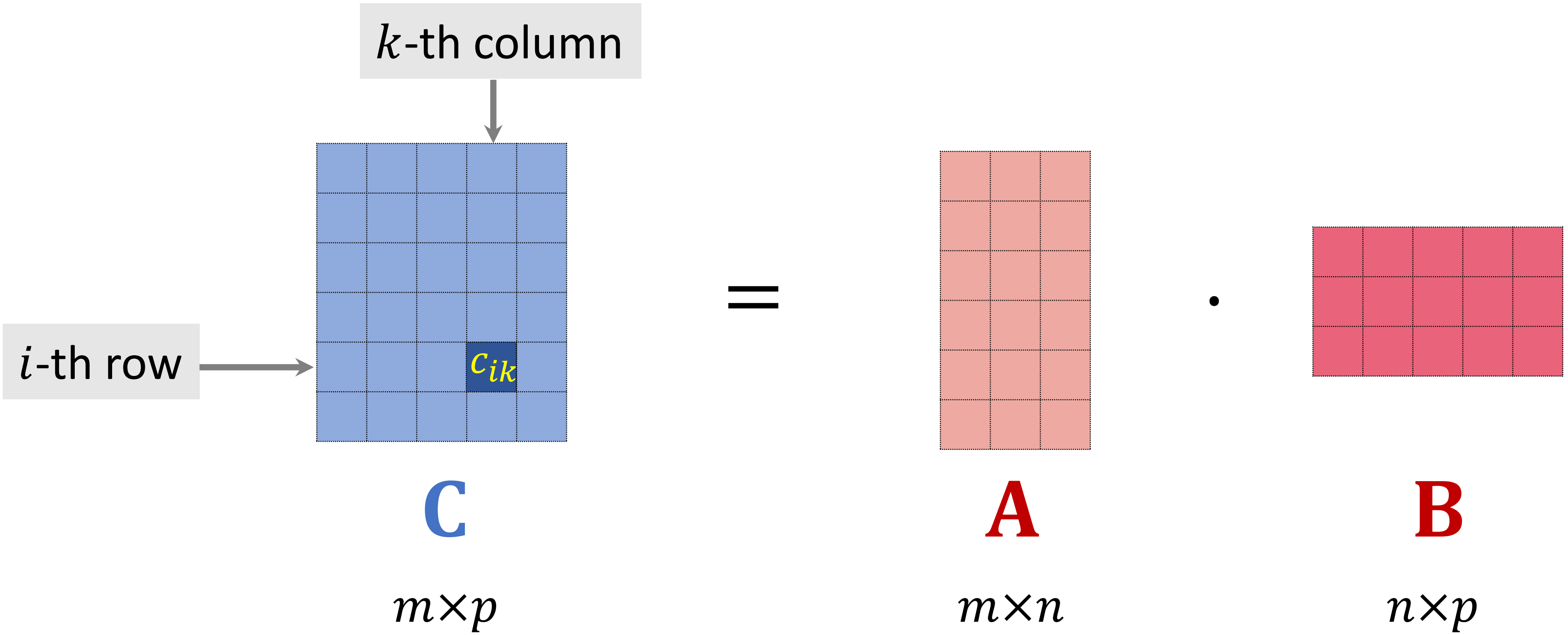

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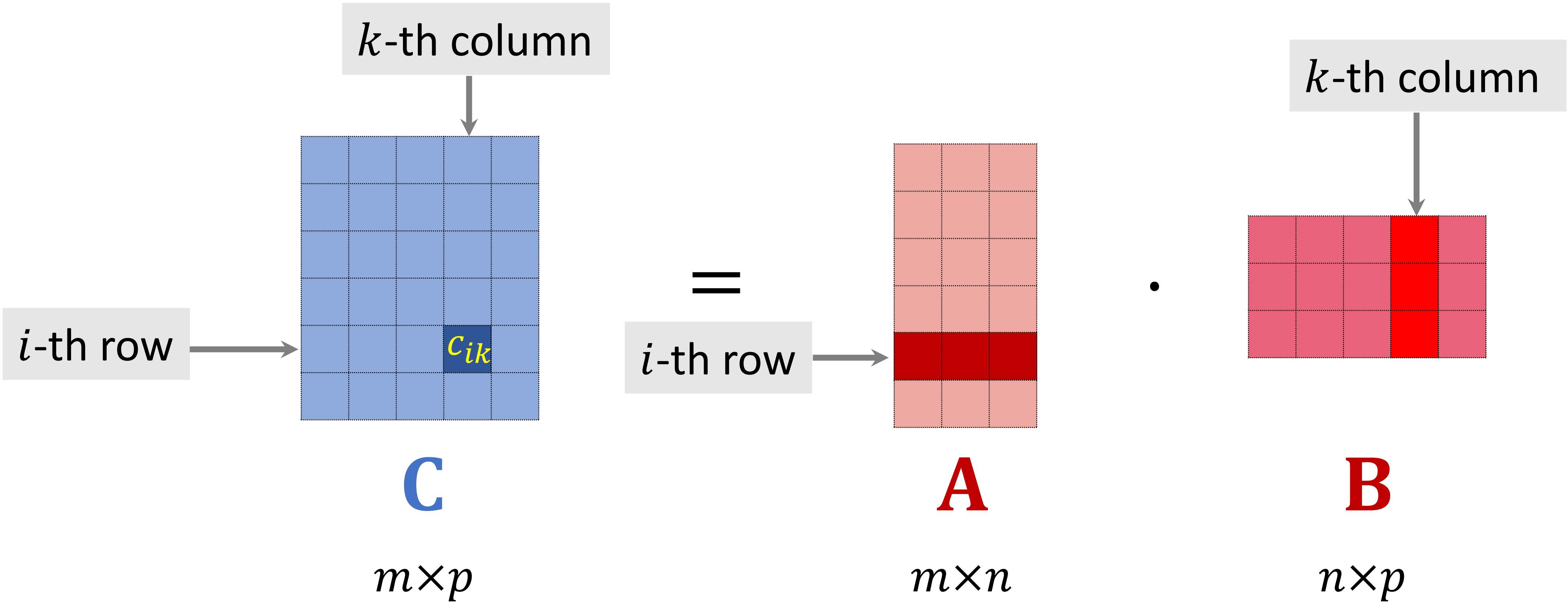

Matrix-Matrix Product

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- Vector inner product: $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.

Pseudo Code

- Initialization: $\mathbf{C} \leftarrow$ all-zero matrix.
- For $i = 1$ to m:
  - For $k = 1$ to p:
    - For $j = 1$ to n:

      $c_{ik} \leftarrow c_{ik} + a_{ij}\, b_{jk}$.

Time complexity: $O(mnp)$.

Summary

Addition

- Given vector $\mathbf{a} \in \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Given matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times n}$.

- **Vector addition:** $\mathbf{c} = \mathbf{a} + \mathbf{b}$.
- **Time complexity:** $O(n)$.

- **Matrix addition:** $\mathbf{C} = \mathbf{A} + \mathbf{B}$.
- **Time complexity:** $O(n^2)$.

Multiplication

- Given vector $\mathbf{a} \in \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Given matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n\times n}$.

- **Vector-vector product:** $c = \mathbf{a}^T \mathbf{b}$.
- **Time complexity:** $O(n)$.

- **Matrix-vector product:** $\mathbf{c} = \mathbf{A}\,\mathbf{b}$.
- **Time complexity:** $O(n^2)$.

- **Matrix-matrix product:** $\mathbf{C} = \mathbf{A}\,\mathbf{B}$.
- **Time complexity:** $O(n^3)$.

Questions

Vector and Matrix Norms

- Given $n \times 1$ vector $\mathbf{a}$ and $m \times n$ matrix $\mathbf{B}$.

Question: What are the costs of computing the following norms?

- Vector ℓ_1-norm: $||\mathbf{a}||_1 = |a_1| + |a_2| + \cdots + |a_n|$.
- Vector ℓ_2-norm: $||\mathbf{a}||_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.
- Matrix Frobenius norm: $||\mathbf{B}||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}^2}$.

Thank You!