Fibonacci Numbers

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Compute Fibonacci Numbers by Recursion

Fibonacci Numbers

Definition

- $F(n)$: the n-th Fibonacci number.
- $F(1) = 1.$
- $F(2) = 1.$
- $F(n) = F(n-1) + F(n-2),$ for $n \geq 3.$

Fibonacci Numbers

C/C++ Implementation

```
long F(int n) {
    if (n <= 2)
        return 1;
    else
        return F(n-1) + F(n-2);
}
```

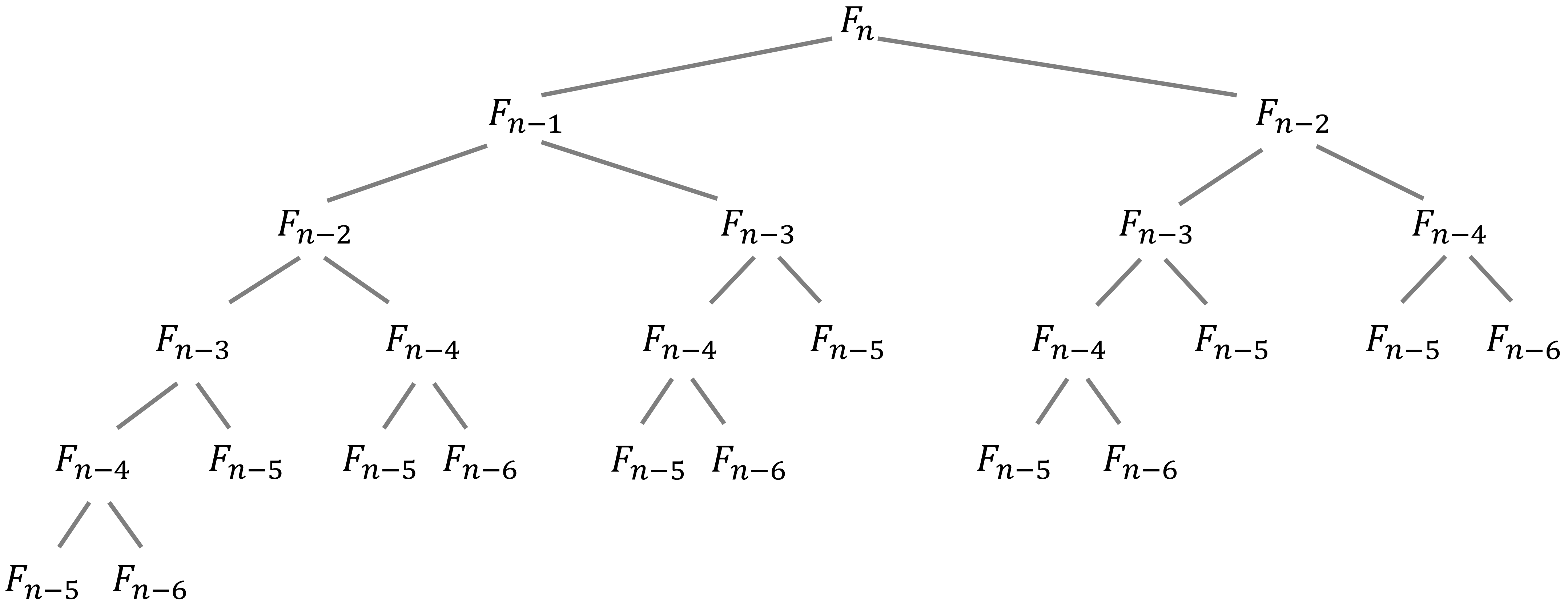
Fibonacci Numbers

- Recursion is slow.
- Time complexity:

$$T(n) = a^n, \quad \text{where } a = \frac{1+\sqrt{5}}{2}.$$

Why is recursion slow?

Trace of the recursive calculation of Fibonacci numbers:



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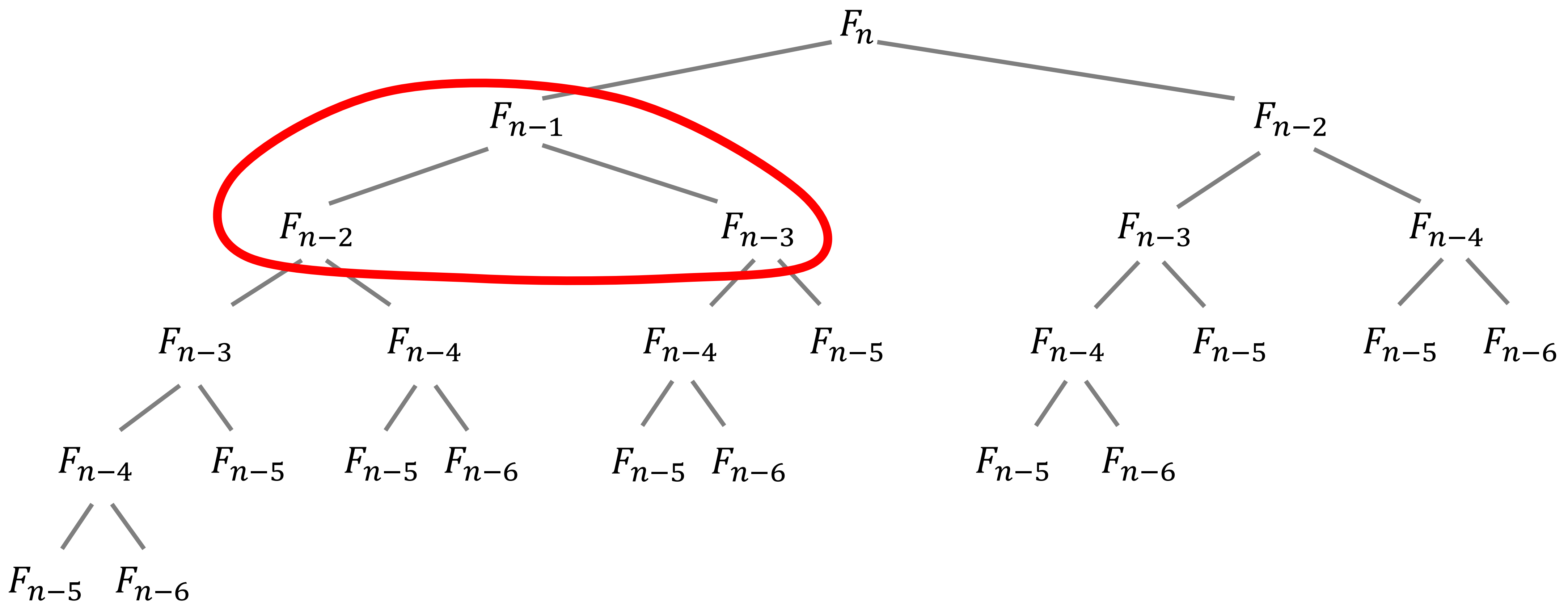
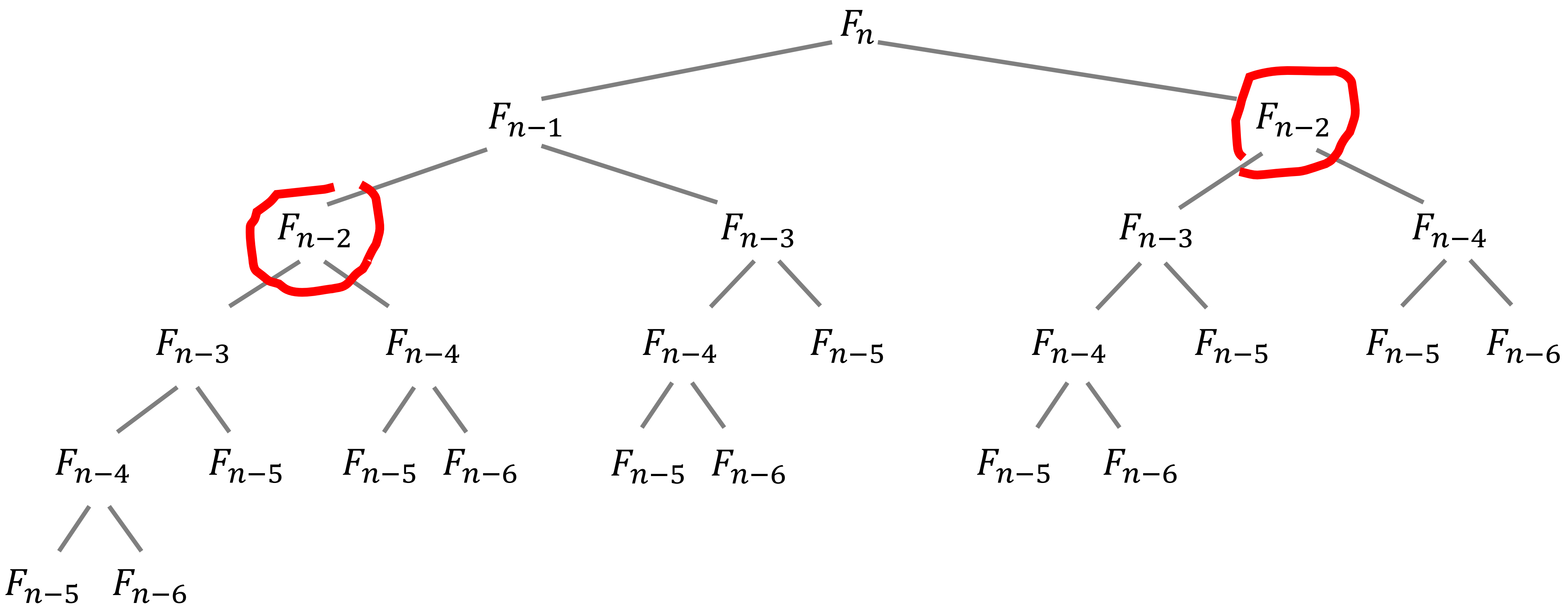


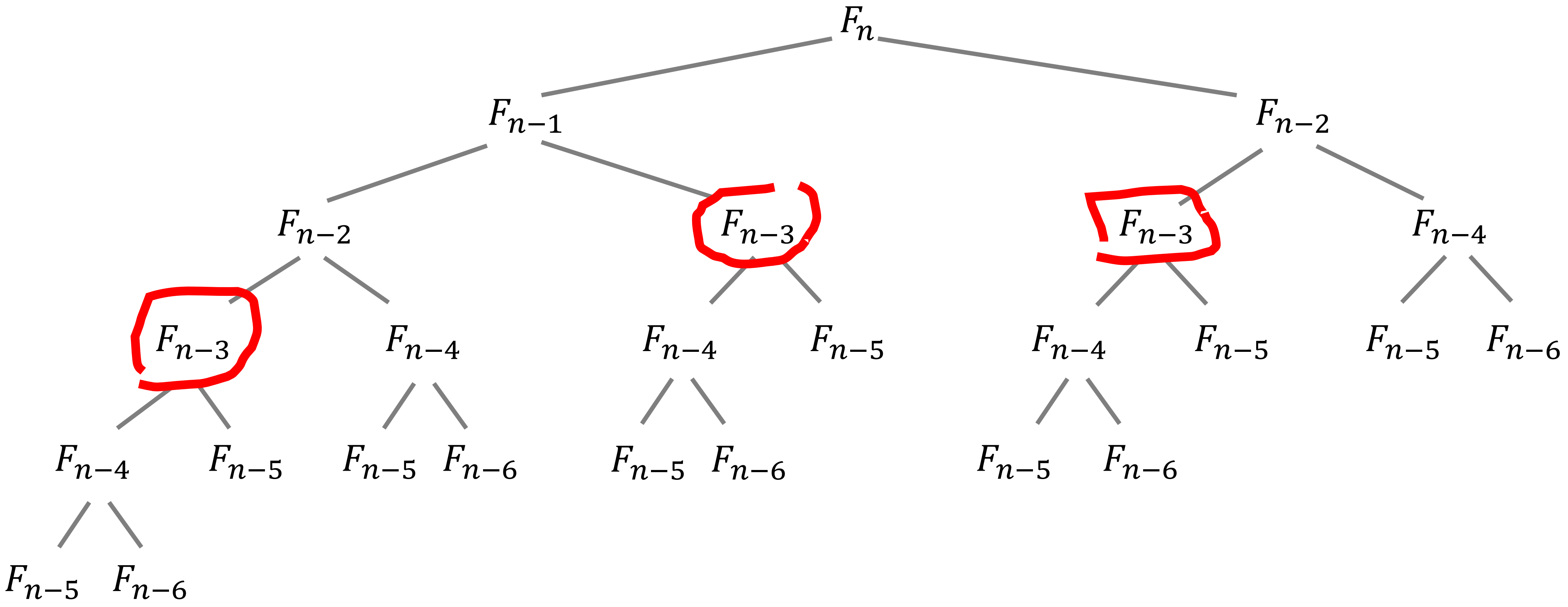
Why is recursion slow?

2 separate calls to compute $F(n-2)$.



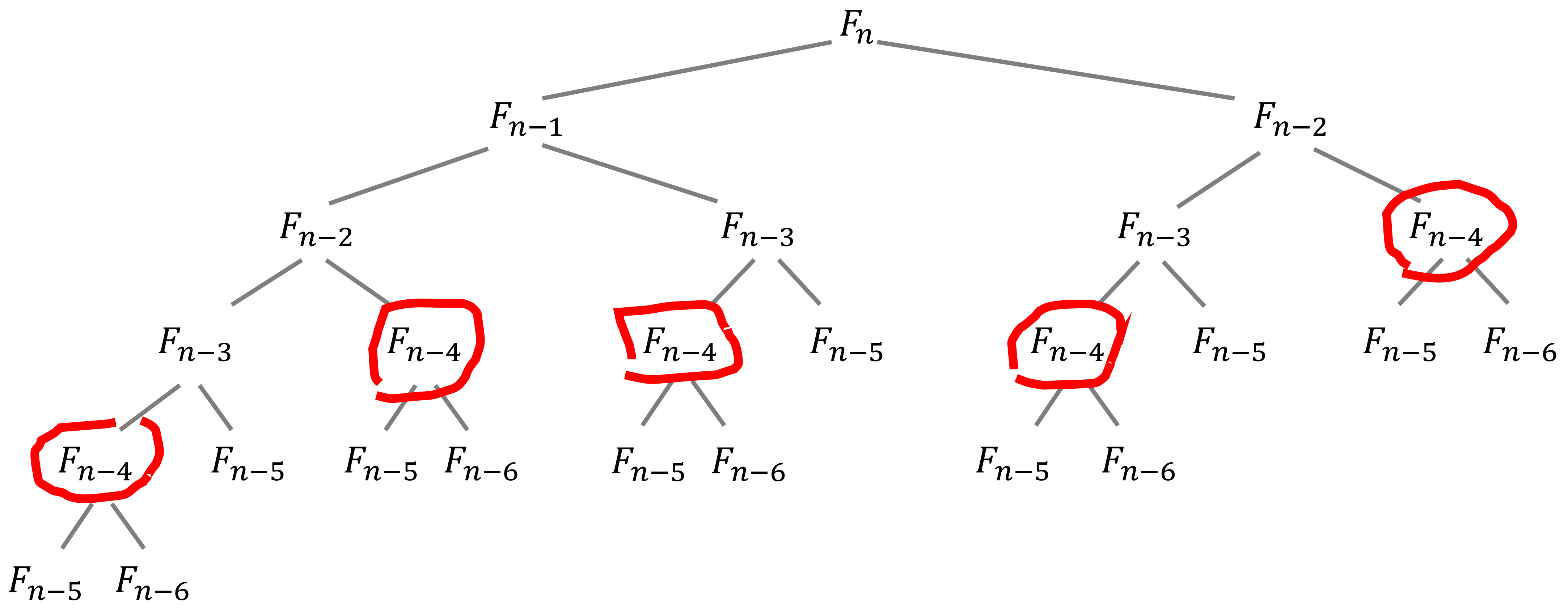
Why is recursion slow?

3 separate calls to compute $F(n-3)$.



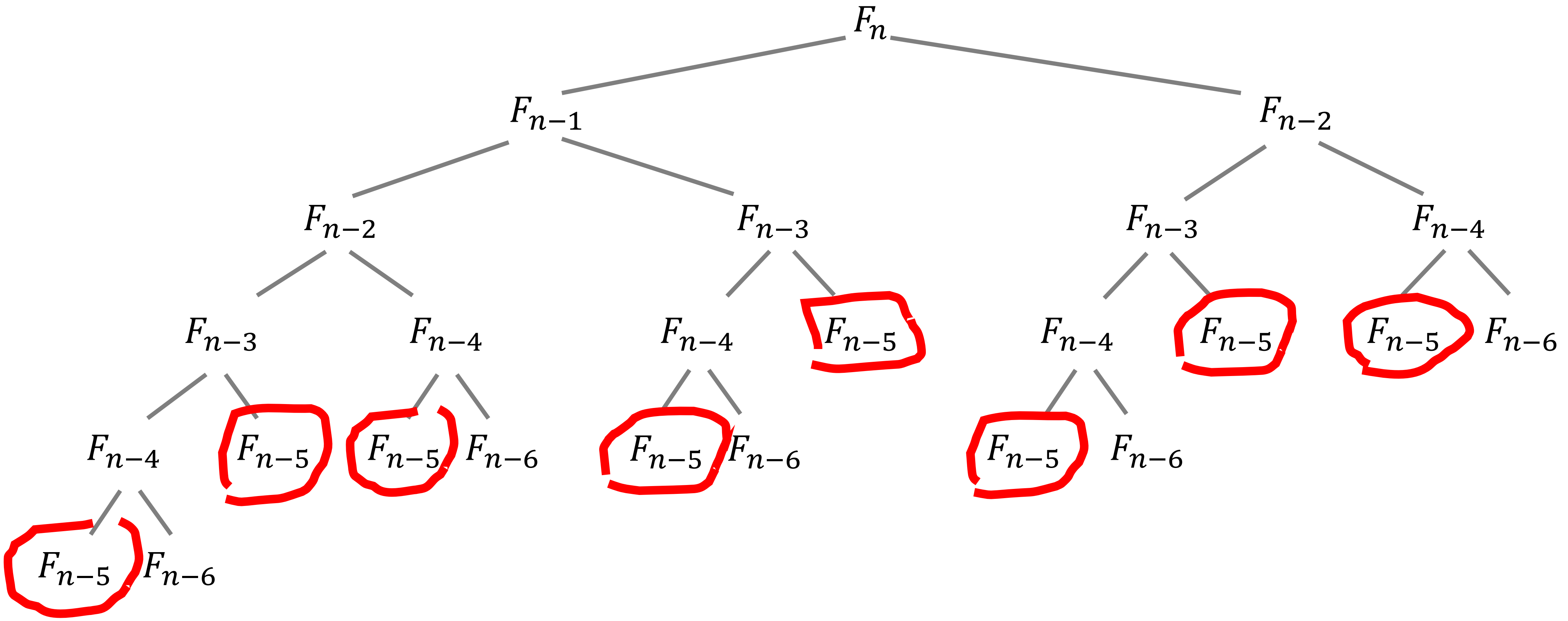
Why is recursion slow?

5 separate calls to compute $F(n-4)$.



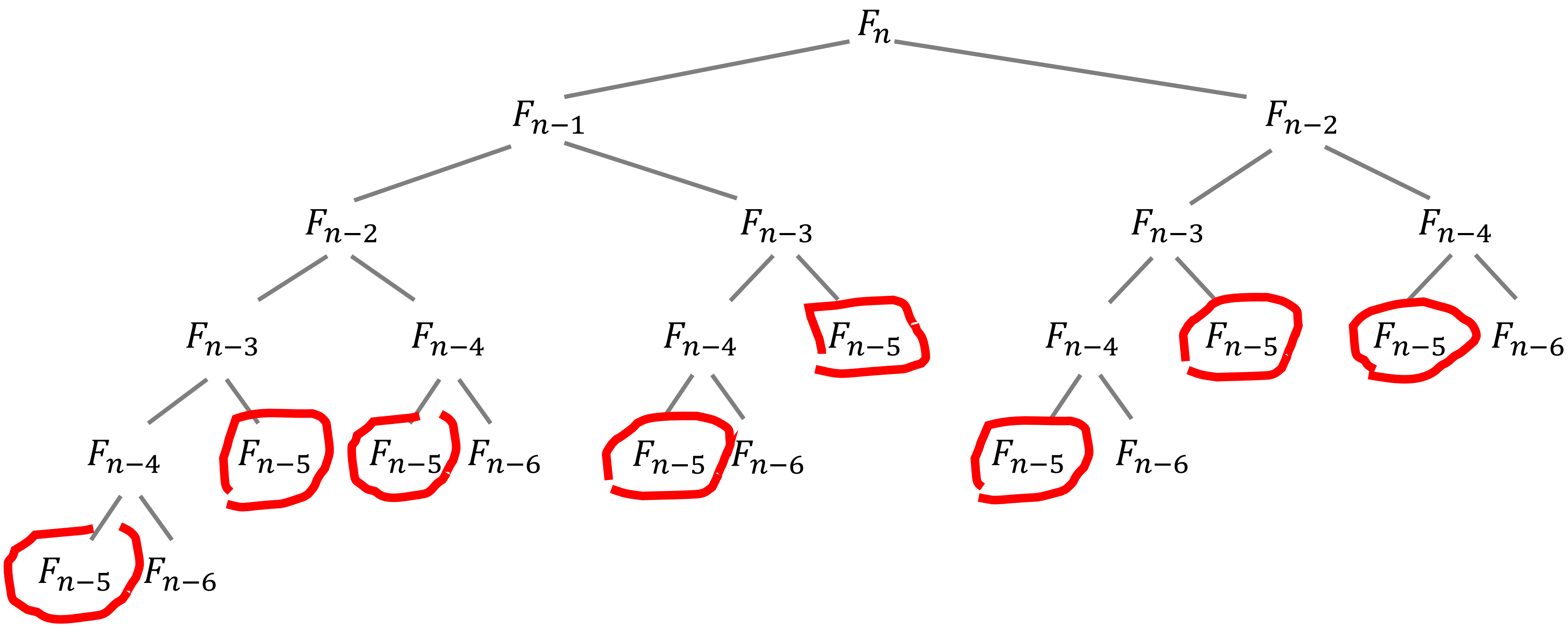
Why is recursion slow?

8 separate calls to compute $F(n-5)$.



Why is recursion slow?

Repeatedly computing the same function is a waste of computation.



Dynamic Programming

Dynamic Programming (DP)

- Basic idea of DP: Using a table instead of recursion.

k	1	2	3	4	5	6	7	8	9	10
$f(k)$										

Dynamic Programming (DP)

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- DP for Fibonacci numbers:
  - Compute $F(k)$ only once, for $k = 1, 2, \cdots, n-1$.
  - Record it in a table.
  - Use $F(k-2)$ and $F(k-1)$ for computing $F(k)$.

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k	1	2	3	4	5	6	7	8	9	10
$F(k)$	1	1	2	3	5	?				

DP for Fibonacci Numbers

```
long fib(int n) {
    if (n == 1 || n == 2) return 1;
    long F[n + 1];
    F[1] = 1;
    F[2] = 1;
    for (int k=3; k<=n; k++)
        F[k] = F[k-1] + F[k-2];
    return F[n];
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$F(k)$	1	1	2	3						

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$F(k)$	1	1	2	3	**5**	**8**	13			

DP for Fibonacci Numbers

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k	1	2	3	4	5	6	7	8	9	…
$F(k)$	1	1	2	3	5	8	**13**	**21**	34	

DP for Fibonacci Numbers

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```

Time complexity: $O(n)$

k	1	2	3	4	5	6	7	8	9	$\cdots$
$F(k)$	1	1	2	3	5	8	13	21	34	$\cdots$

Dynamic Programming

Step 1: Break a big problem into smaller sub-problems.

- Write down the recursive formula.
- E.g., $F(k) = F(k-2) + F(k-1)$.

Step 2: Use a table instead of recursion.

- Record the solutions of the small sub-problems.
- Use the recorded solutions for solving bigger sub-problems.

Time Complexity

Input Size

Input size: the number of bits for encoding the inputs.

- Find the n-th Fibonacci number.
- Input size: $b = \log_2 n$ bits.

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- **Example 1:** $b = 13$ bits for encoding $n = 5678$.

$$5678 = \underbrace{\boxed{1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0}}_{\text{13 bits}}$$

Input Size

Input size: the number of bits for encoding the inputs.

- Find the n-th Fibonacci number.
- Input size: $b = \log_2 n$ bits.
- **Example 1:** $b = 13$ bits for encoding $n = 5678$.
- **Example 2:** $b = 20$ bits for encoding $n = 1$ million.
- **Example 3:** $b = 30$ bits for encoding $n = 1$ billion.

Time Complexities

Input size: the number of bits for encoding the inputs.

- Find the n-th Fibonacci number.
- Input size: $b = \log_2 n$ bits.

Time complexity of dynamic programming is exponential:

$$O(n) = O\left(2^b\right).$$

Time Complexities

Input size: the number of bits for encoding the inputs.

- Find the n-th Fibonacci number.
- Input size: $b = \log_2 n$ bits.

Time complexity of recursion is double exponential:

$$O(a^n) = O\left(a^{2^b}\right).$$

Thank You!

Time Complexity of Recursion

- **Time complexity:** $T(n) = a^n$, where $a = \frac{1+\sqrt{5}}{2}$.
- **Proof by induction:**
  - Recursion: $T(n) = T(n-1) + T(n-2)$.
  - Assume $T(n-2) = a^{n-2}$.
  - Assume $T(n-1) = a^{n-1} = a \cdot a^{n-2}$.
  - Then $T(n) = T(n-1) + T(n-2)$

$$= a^{n-2} \cdot (a+1).$$

$$= a^{n-2} \cdot a^2$$

$$= a^n.$$

Why? Since $a = \frac{1+\sqrt{5}}{2}$, we have $a^2 = a + 1$.