

Matrix Basics

Shusen Wang

Vector and Matrix

Vector (n -dim) $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Matrix ($m \times n$) $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

Additions

Vector Addition

- Given $n \times 1$ vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector addition: $\mathbf{c} = \mathbf{a} + \mathbf{b} \in \mathbb{R}^n$.

Vector Addition

- Given $n \times 1$ vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector addition: $\mathbf{c} = \mathbf{a} + \mathbf{b} \in \mathbb{R}^n$.

Pseudo Code

- Initialization: $\mathbf{c} \leftarrow [0, 0, \dots, 0]$.
- For $i = 1$ to n :
 $c_i \leftarrow a_i + b_i$.

Time complexity: $O(n)$.

Matrix Addition

- Given $m \times n$ matrices: $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$.
- Matrix addition: $\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{R}^{m \times n}$.

Matrix Addition

- Given $m \times n$ matrices: $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$.
- Matrix addition: $\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{R}^{m \times n}$.

Pseudo Code

- Initialization: $\mathbf{C} \leftarrow m \times n$ all-zero matrix.
- For $i = 1$ to m :
 - For $j = 1$ to n :

$$c_{ij} \leftarrow a_{ij} + b_{ij}.$$

Time complexity: $O(mn)$.

Multiplications

Vector Inner Product

- Given vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $c = \mathbf{a}^T \mathbf{b} = a_1 b_1 + \cdots + a_n b_n$.

Vector Inner Product

- Given vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $c = \mathbf{a}^T \mathbf{b} = a_1 b_1 + \cdots + a_n b_n$.

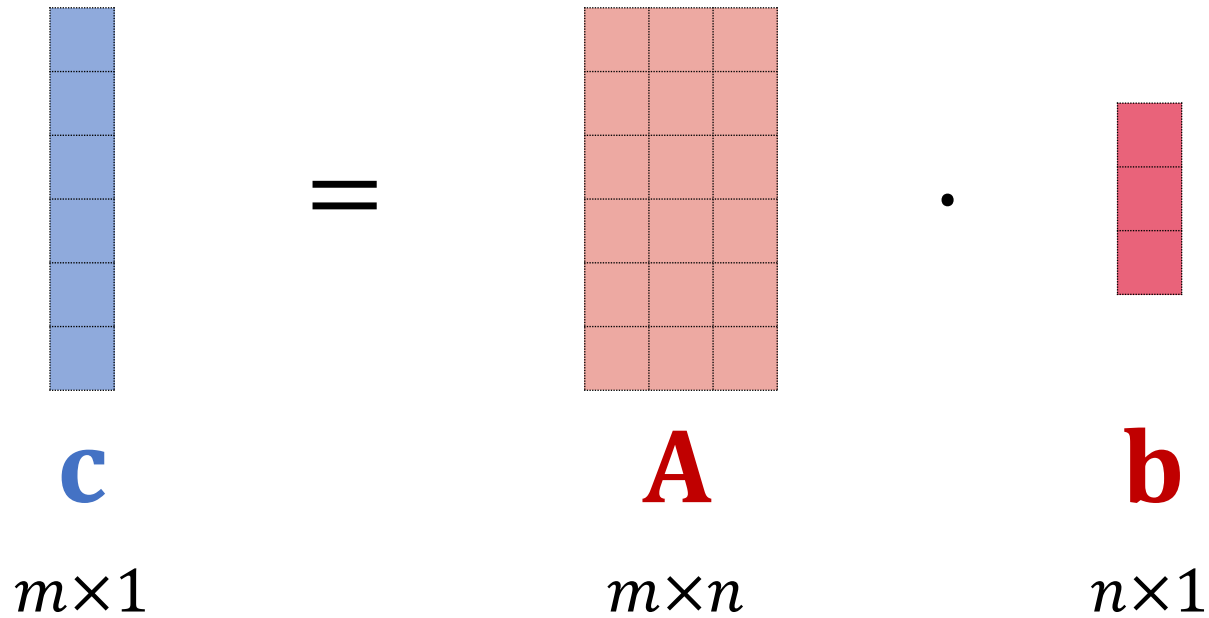
Pseudo Code

- Initialization: $c \leftarrow 0$.
- For $i = 1$ to n :
 $c \leftarrow c + a_i b_i$.

Time complexity: $O(n)$.

Matrix-Vector Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $\mathbf{c} = \mathbf{A}\mathbf{b} \in \mathbb{R}^m$.



Matrix-Vector Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $\mathbf{c} = \mathbf{A}\mathbf{b} \in \mathbb{R}^m$.

The diagram illustrates the matrix-vector product $\mathbf{c} = \mathbf{A}\mathbf{b}$. It shows three vertical structures: a blue vector \mathbf{c} of size $m \times 1$, a red matrix \mathbf{A} of size $m \times n$, and a red vector \mathbf{b} of size $n \times 1$. The vector \mathbf{c} has its i -th element, c_i , highlighted in yellow. The matrix \mathbf{A} is represented by a grid of red squares. The vector \mathbf{b} is represented by a single column of red squares. An equals sign and a dot operator are placed between the structures to indicate the equation $\mathbf{c} = \mathbf{A}\mathbf{b}$.

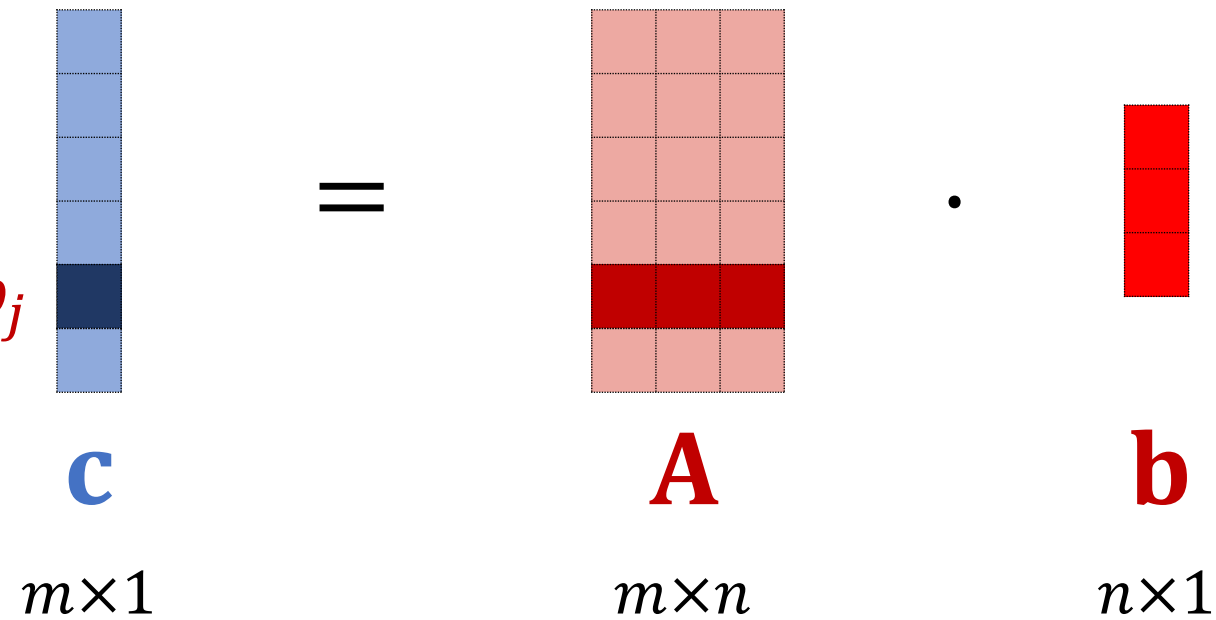
\mathbf{c}
 $m \times 1$

\mathbf{A}
 $m \times n$

\mathbf{b}
 $n \times 1$

Matrix-Vector Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $\mathbf{c} = \mathbf{A}\mathbf{b} \in \mathbb{R}^m$.

$$c_i = \sum_{j=1}^n a_{ij} b_j$$


\mathbf{c}
 $m \times 1$

\mathbf{A}
 $m \times n$

\mathbf{b}
 $n \times 1$

Matrix-Vector Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Vector inner product: $\mathbf{c} = \mathbf{A}\mathbf{b} \in \mathbb{R}^m$.

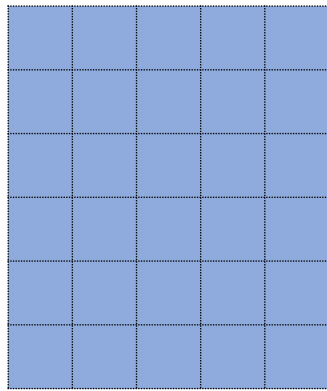
Pseudo Code

- Initialization: $\mathbf{c} \leftarrow$ all-zero vector.
- For $i = 1$ to m :
 - For $j = 1$ to n :
$$c_i \leftarrow c_i + a_{ij} b_j.$$

Time complexity: $O(mn)$.

Matrix-Matrix Product

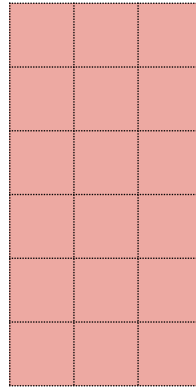
- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times p}$.
- Vector inner product: $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.



C

$m \times p$

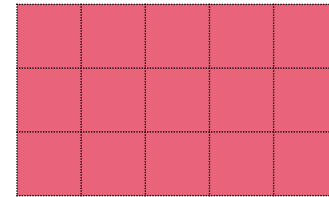
=



A

$m \times n$

•

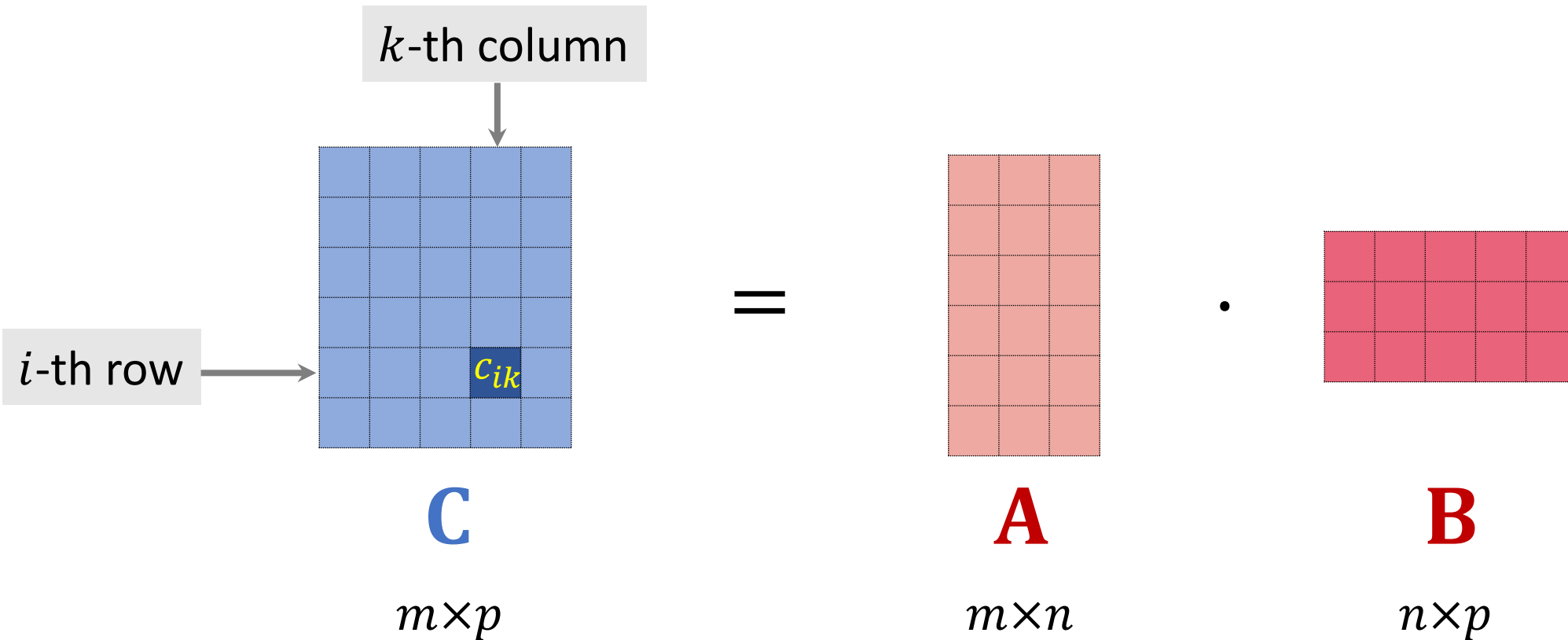


B

$n \times p$

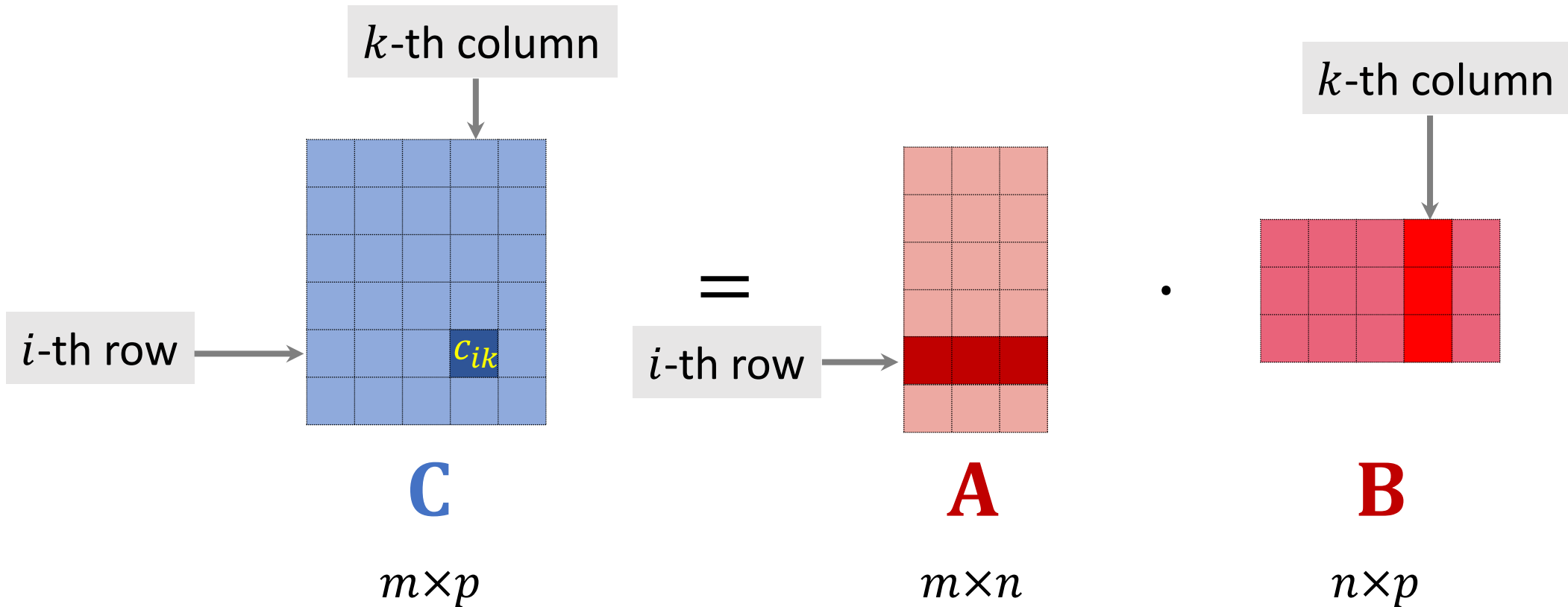
Matrix-Matrix Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times p}$.
- Vector inner product: $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.



Matrix-Matrix Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times p}$.
- Vector inner product: $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.



Matrix-Matrix Product

- Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times p}$.
- Vector inner product: $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}$.

Pseudo Code

- Initialization: $\mathbf{C} \leftarrow$ all-zero matrix.
- For $i = 1$ to m :
 - For $k = 1$ to p :
 - For $j = 1$ to n :
$$c_{ik} \leftarrow c_{ik} + a_{ij} b_{jk}.$$

Time complexity: $O(mnp)$.

Summary

Addition

- Given vector **a** $\in \mathbb{R}^n$ and vector **b** $\in \mathbb{R}^n$.
- Given matrix **A** $\in \mathbb{R}^{n \times n}$ and vector **B** $\in \mathbb{R}^{n \times n}$.

- Vector addition: **c** = **a** + **b**.
- Time complexity: $O(n)$.

- Matrix addition: **C** = **A** + **B**.
- Time complexity: $O(n^2)$.

Multiplication

- Given vector $\mathbf{a} \in \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^n$.
- Given matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and vector $\mathbf{B} \in \mathbb{R}^{n \times n}$.

- Vector-vector product: $\mathbf{c} = \mathbf{a}^T \mathbf{b}$.
- Time complexity: $O(n)$.

- Matrix-vector product: $\mathbf{c} = \mathbf{A} \mathbf{b}$.
- Time complexity: $O(n^2)$.

- Matrix-matrix product: $\mathbf{C} = \mathbf{A} \mathbf{B}$.
- Time complexity: $O(n^3)$.

Questions

Vector and Matrix Norms

- Given $n \times 1$ vector \mathbf{a} and $m \times n$ matrix \mathbf{B} .

Question: What are the costs of computing the following norms?

- Vector ℓ_1 -norm: $\|\mathbf{a}\|_1 = |a_1| + |a_2| + \cdots + |a_n|.$
- Vector ℓ_2 -norm: $\|\mathbf{a}\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$
- Matrix Frobenius norm: $\|\mathbf{B}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n b_{ij}^2}.$

Thank You!