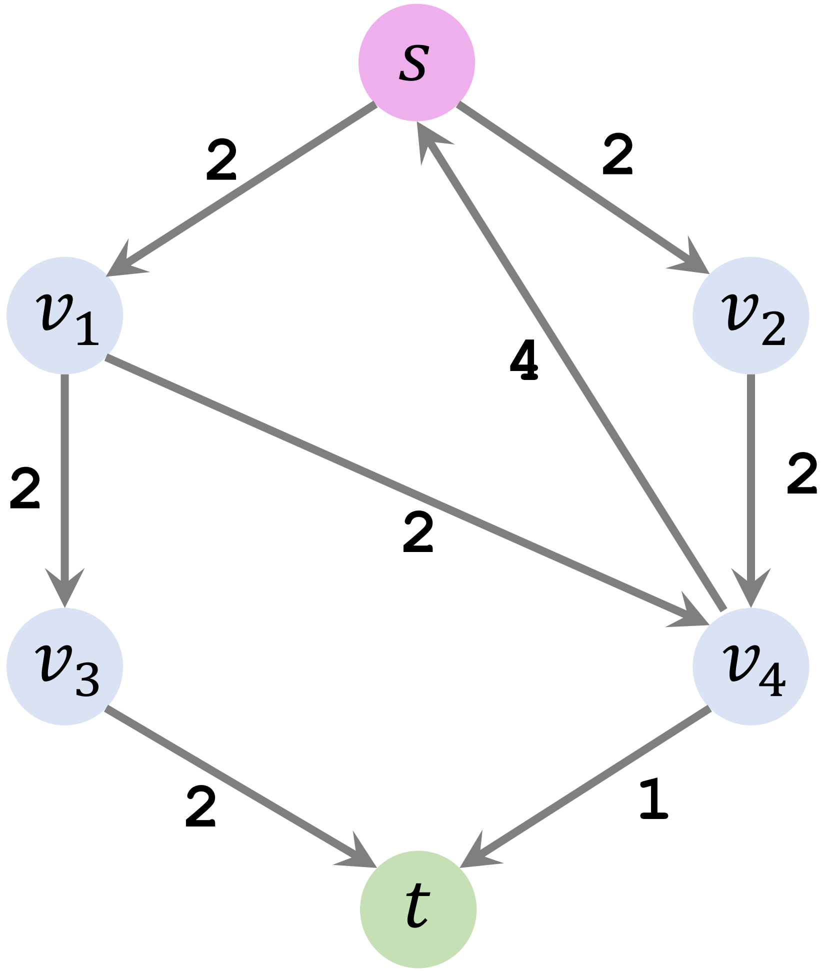
Minimum Cut Problem

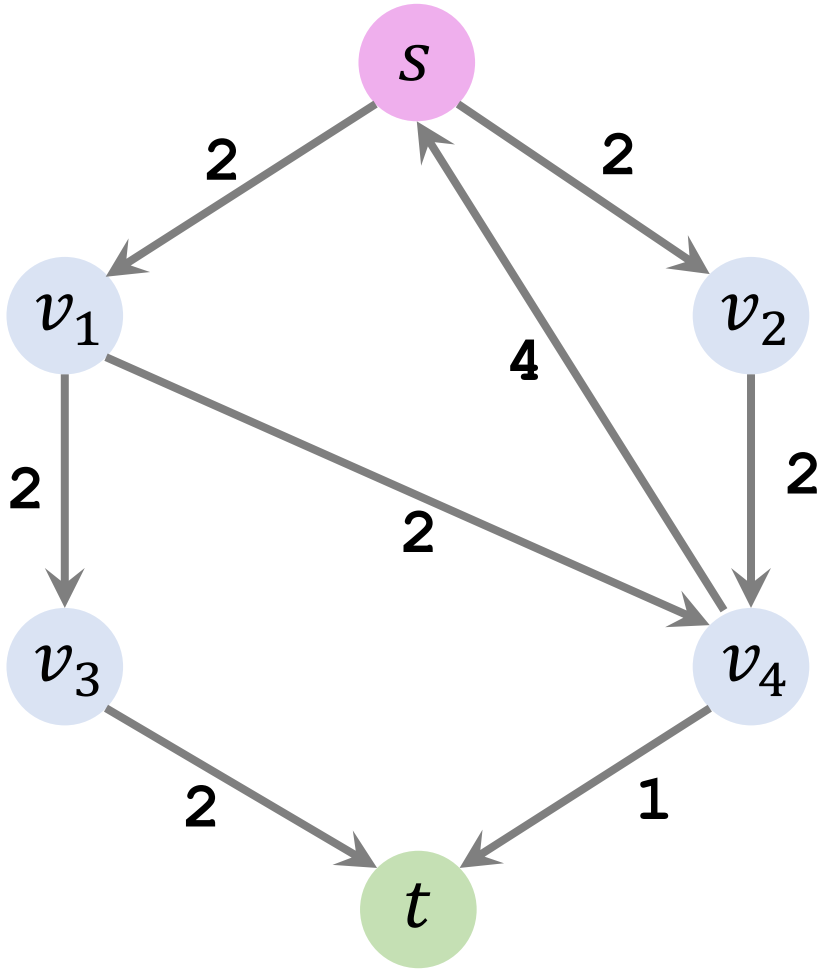
Shusen Wang

Inputs



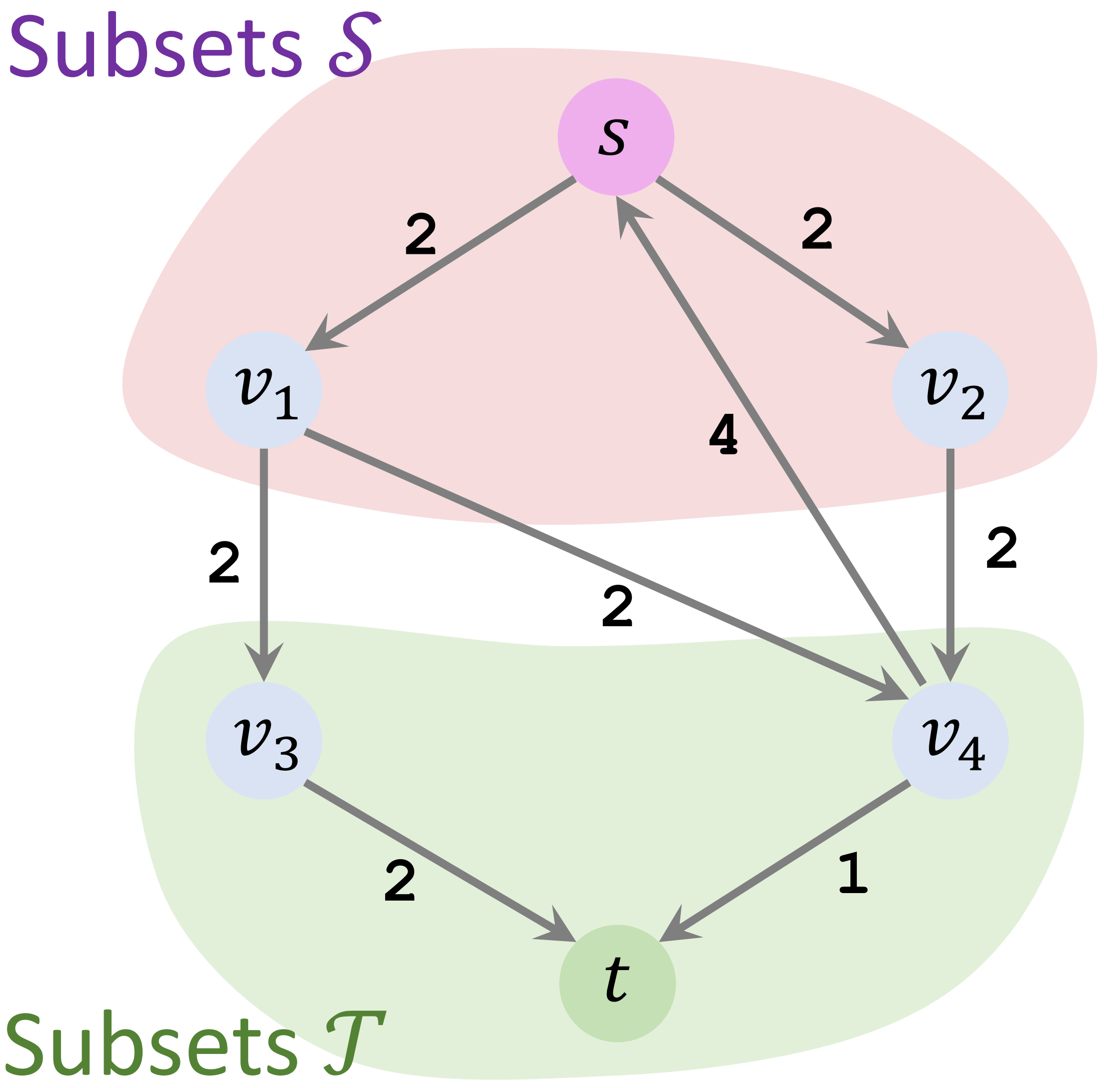
- Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Source: $s \in \mathcal{V}$.
- Sink: $t \in \mathcal{V}$.

S-T Cut



- Split $\mathcal{V}$ into two subsets: $\mathcal{S}$ and $\mathcal{T}$.
  - $\mathcal{S} \cup \mathcal{T} = \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \emptyset$.
  - $s \in \mathcal{S}$ and $t \in \mathcal{T}$.
- The pair $(\mathcal{S}, \mathcal{T})$ is called s-t cut.

S-T Cut



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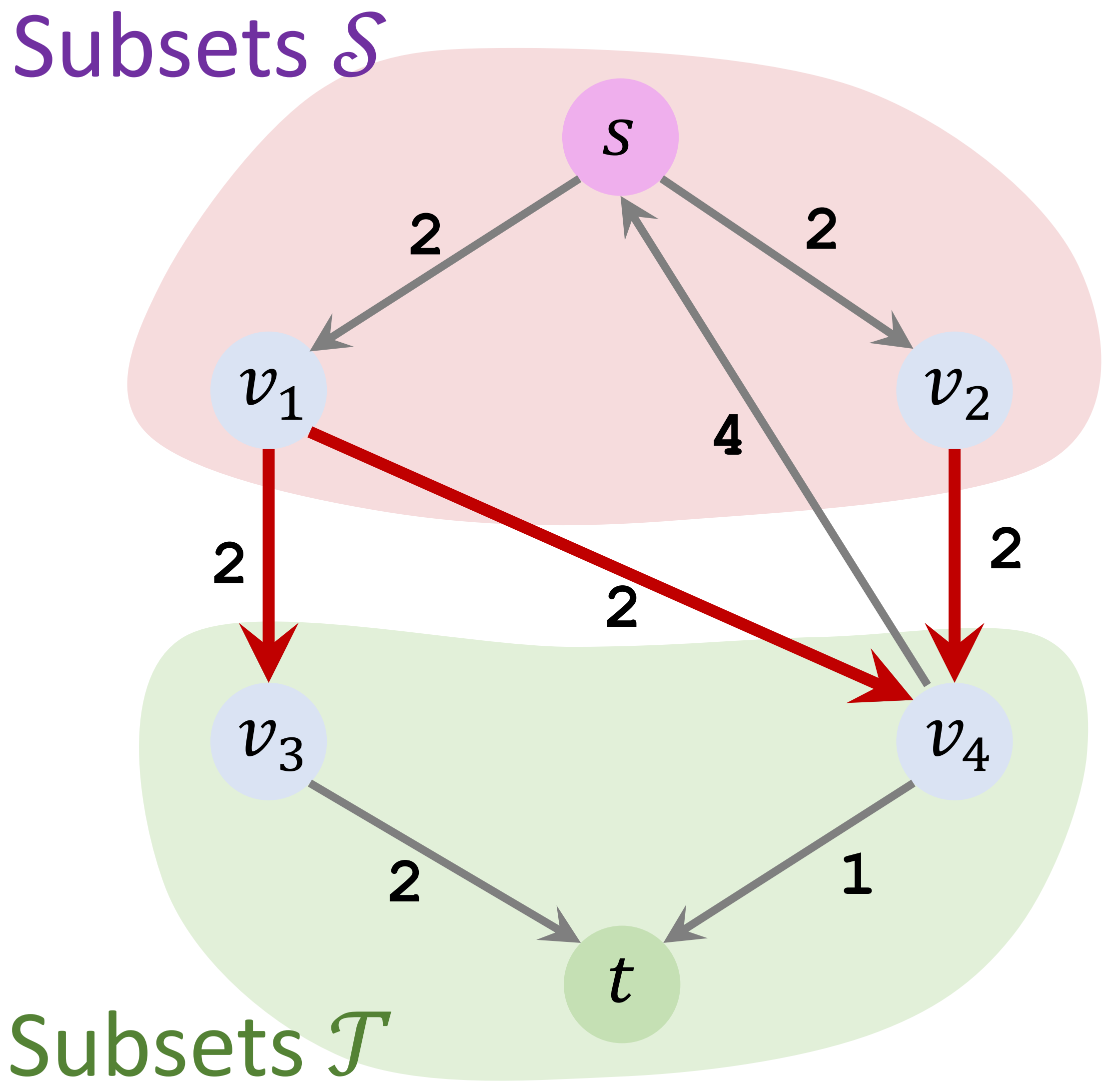
S-T Cut



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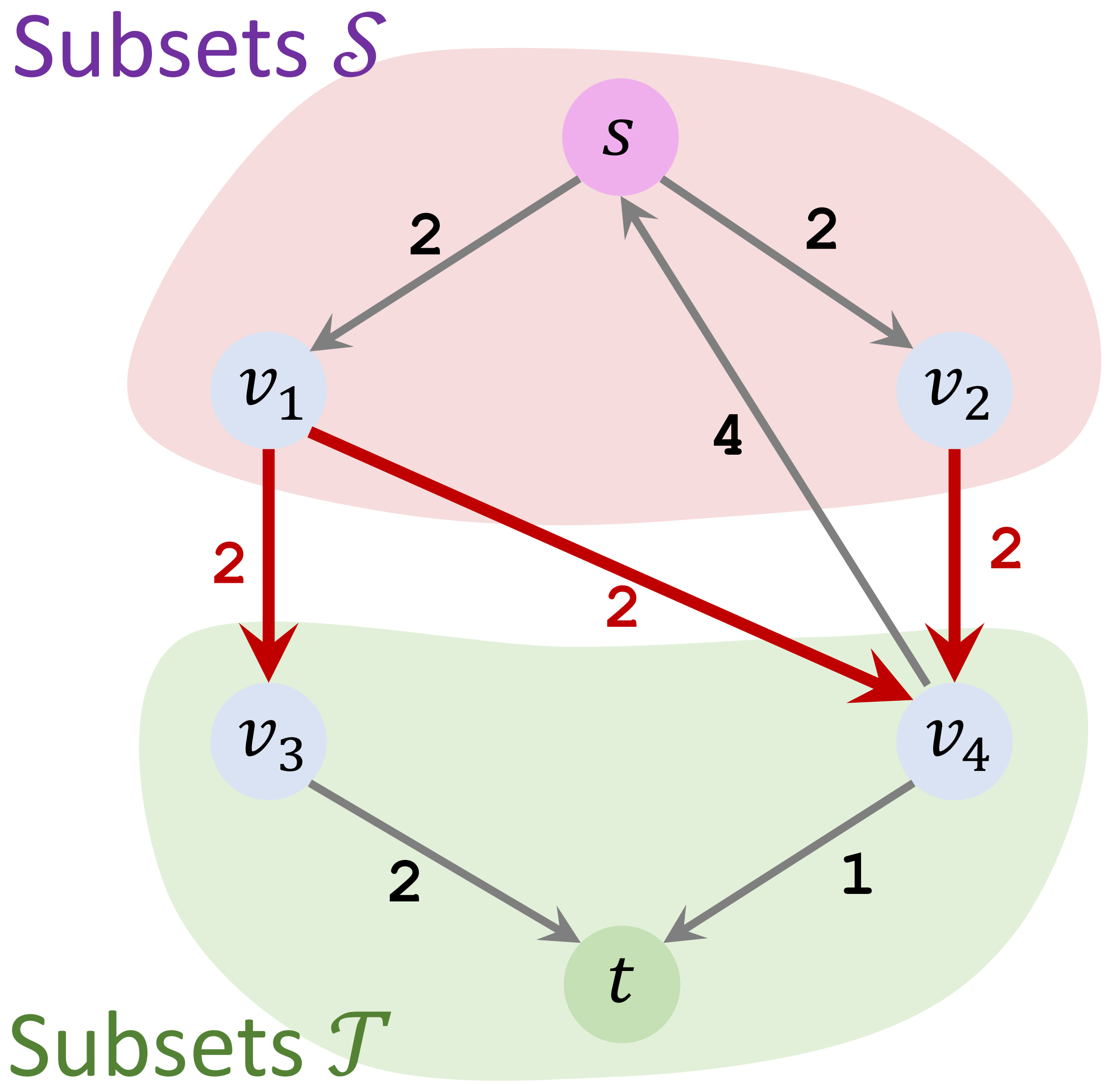
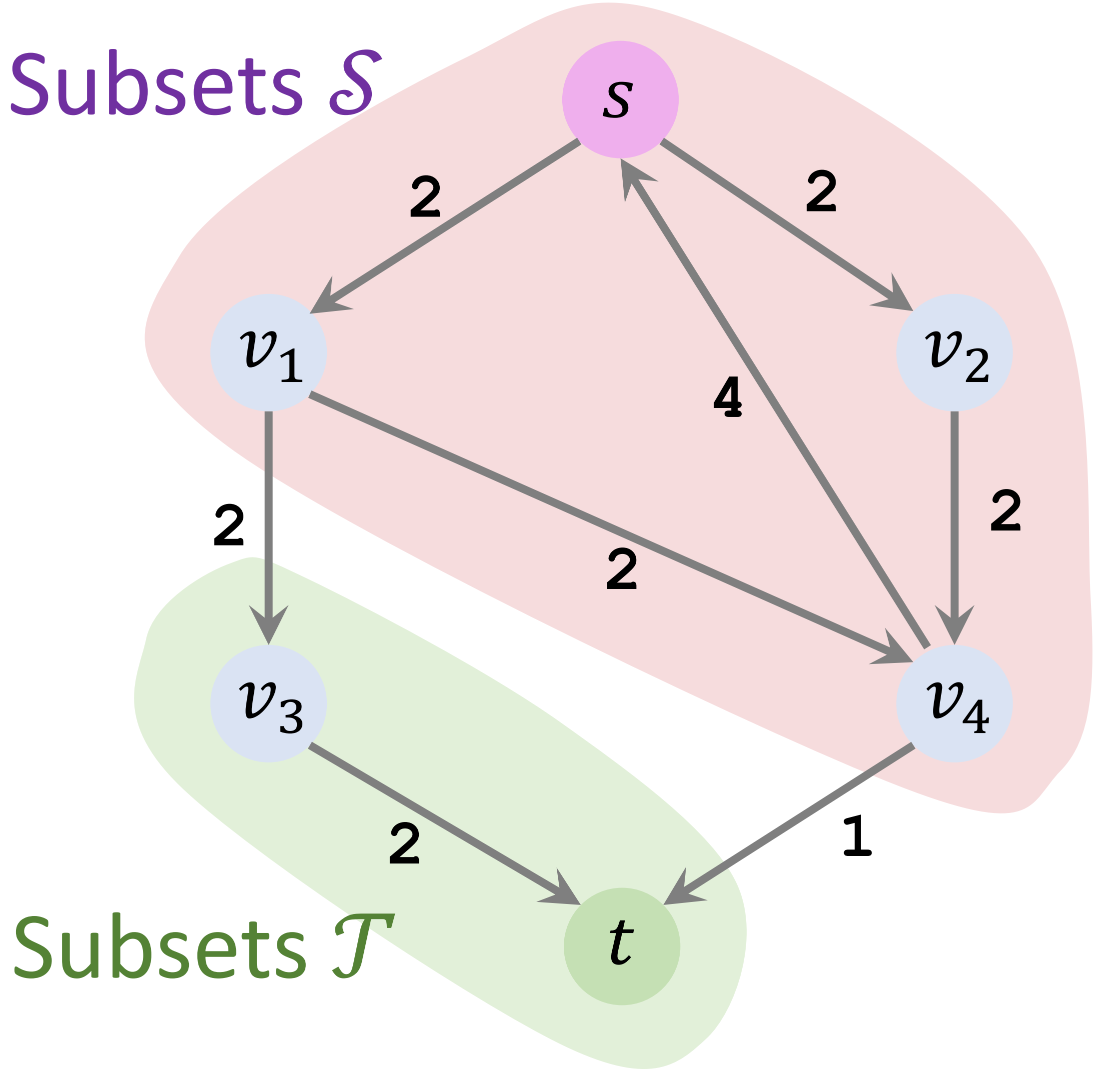
S-T Cut



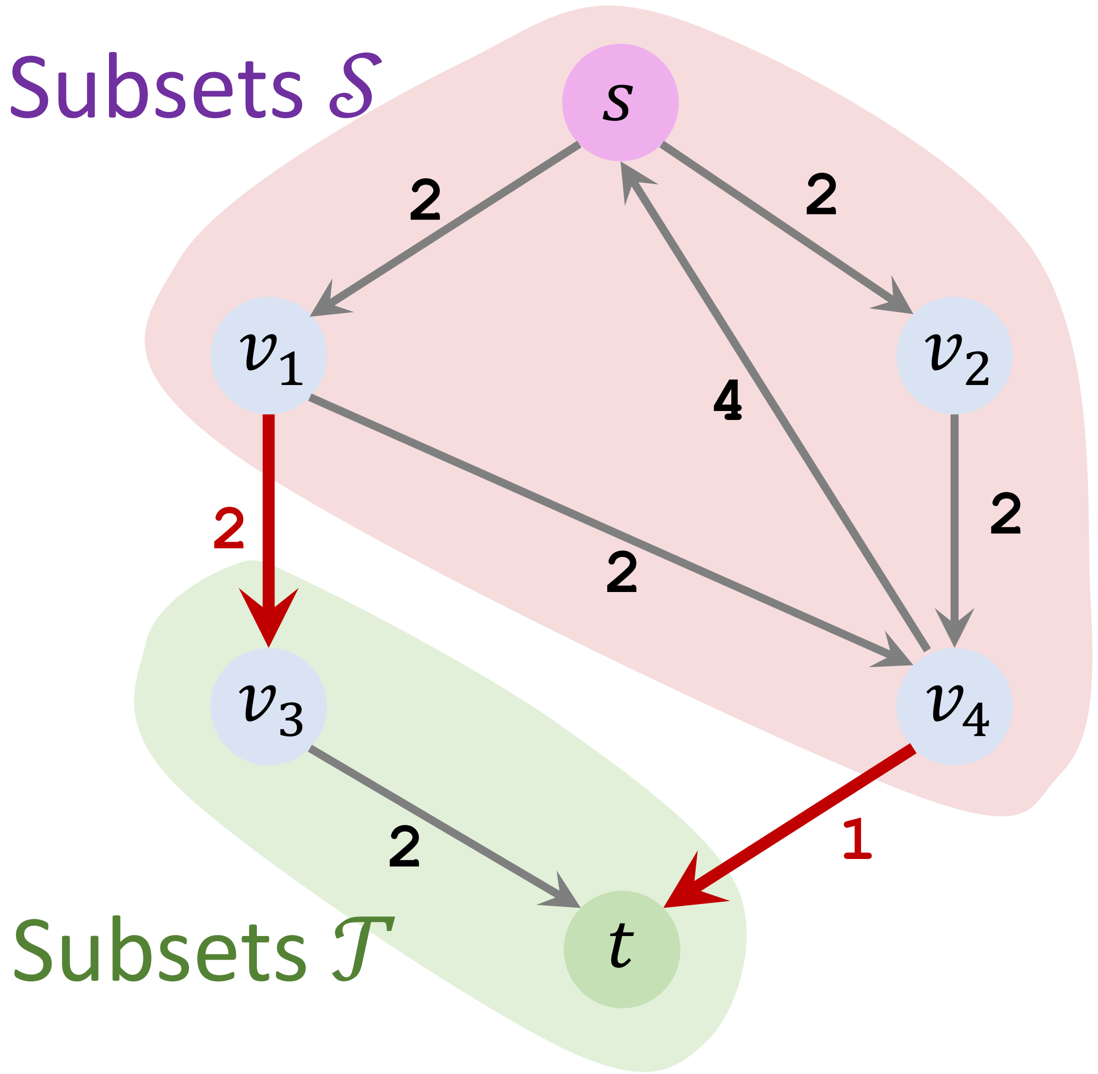
- Capacity$(\mathcal{S}, \mathcal{T})$ = sum of weights of edges leaving $\mathcal{S}$.
- In the figure, three edges leave $\mathcal{S}$.
- Capacity$(\mathcal{S}, \mathcal{T}) = 2 + 2 + 2 = 6$.

S-T Cut: Another Example



- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.
- Subset $\mathcal{T} = \{t, v_3\}$.

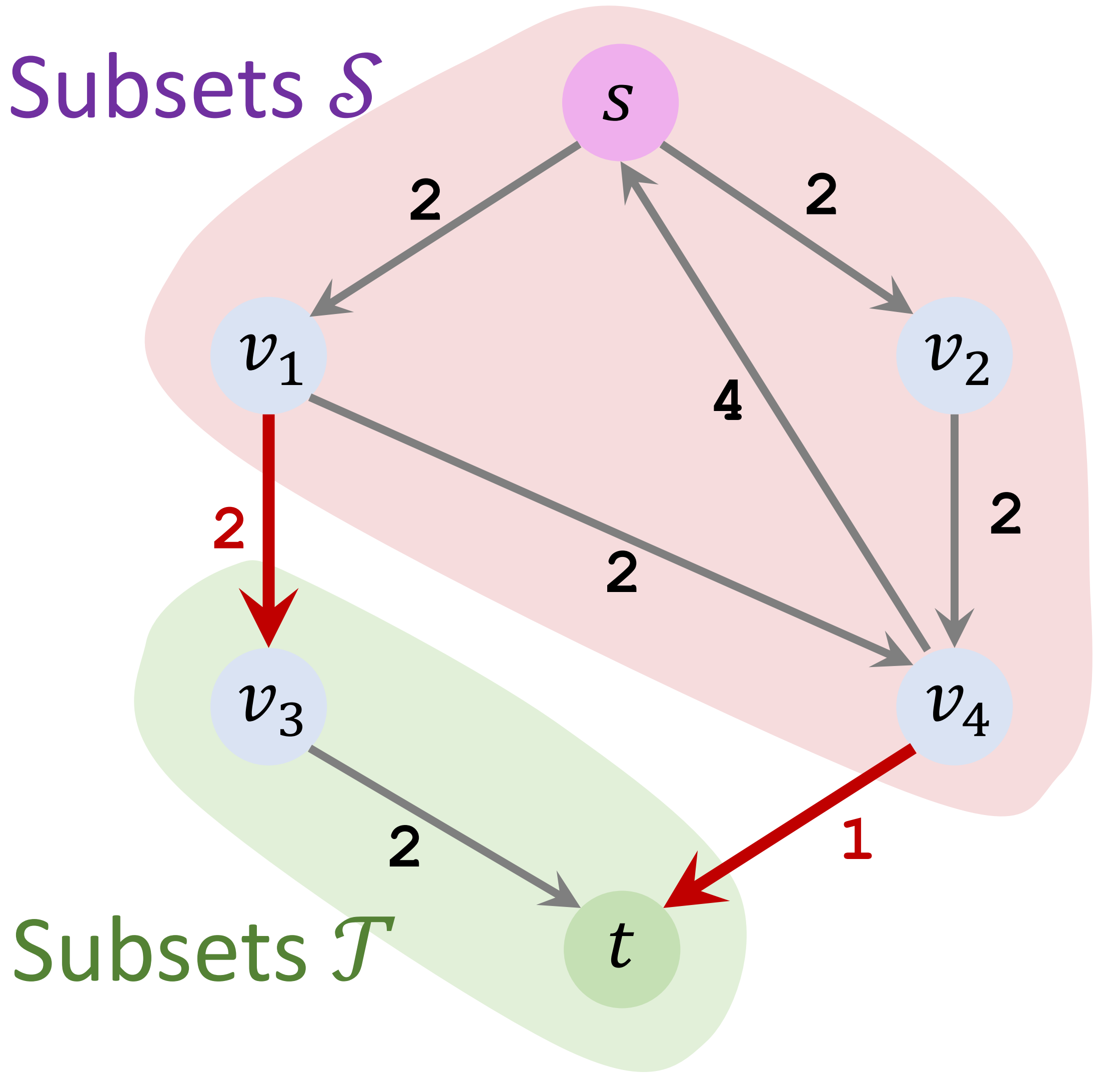
S-T Cut: Another Example



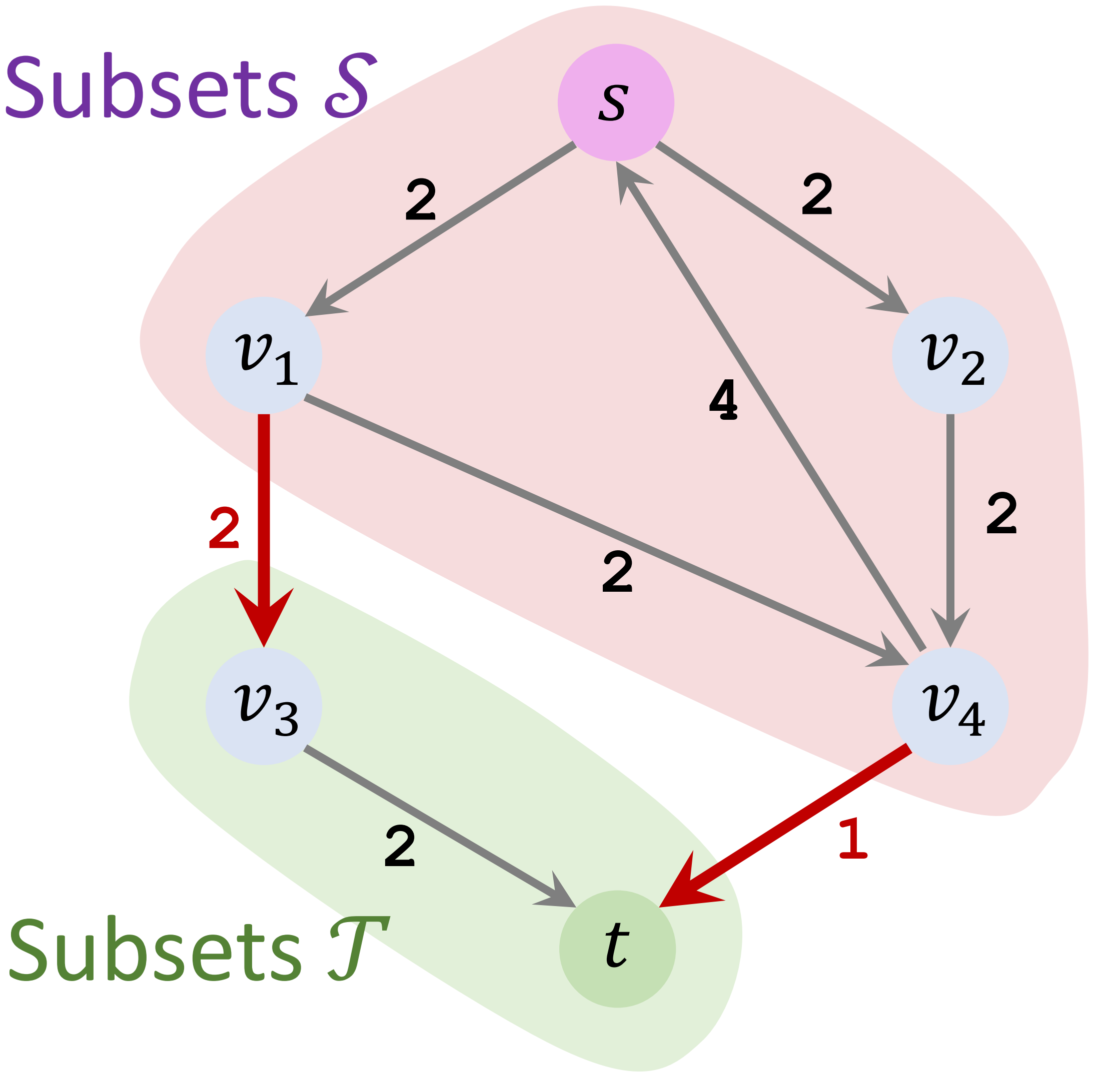
- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.
- Subset $\mathcal{T} = \{t, v_3\}$.

- In the figure, two edges leave $\mathcal{S}$.
- Capacity$(\mathcal{S}, \mathcal{T}) = 2 + 1 = 3$.

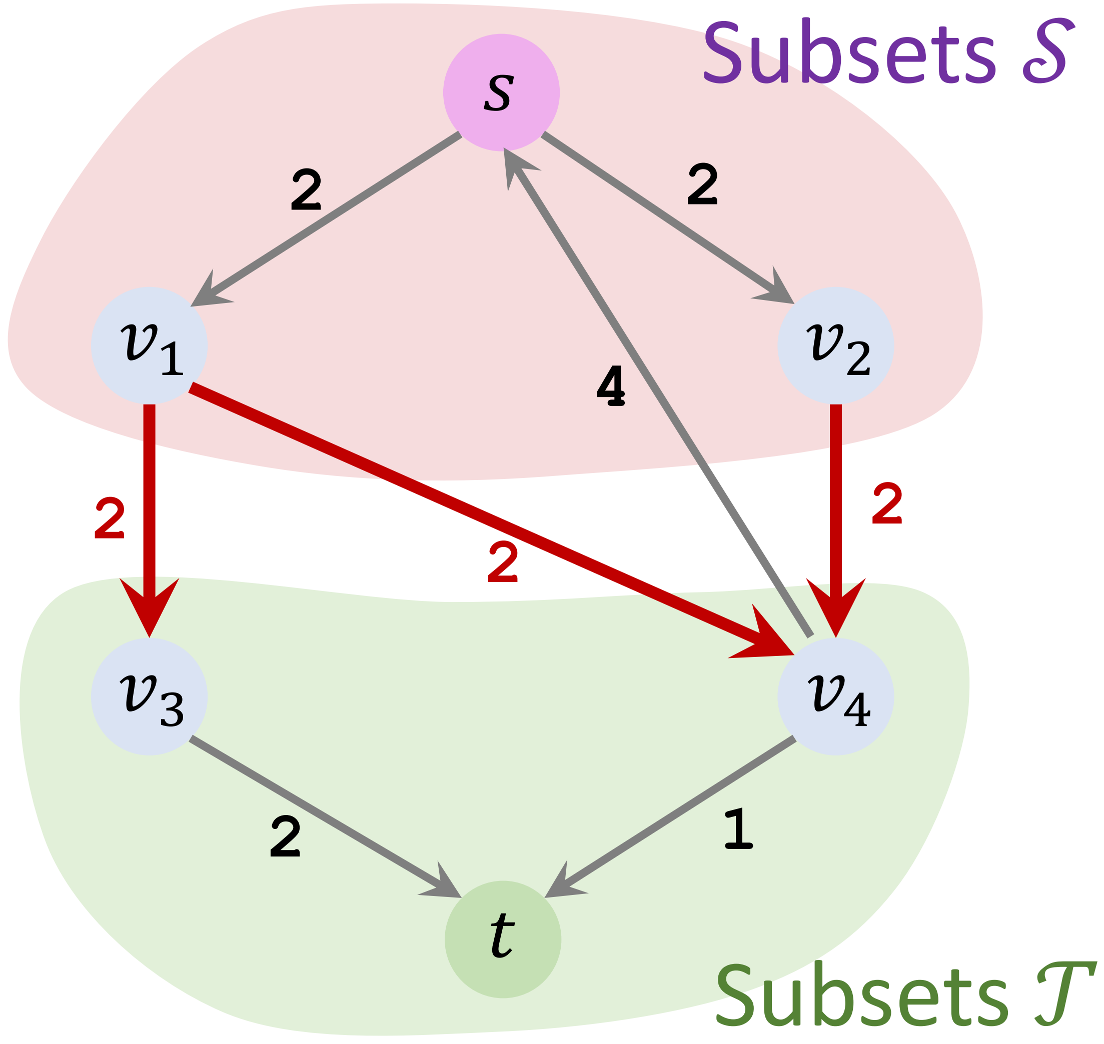
Minimum S-T Cut Problem



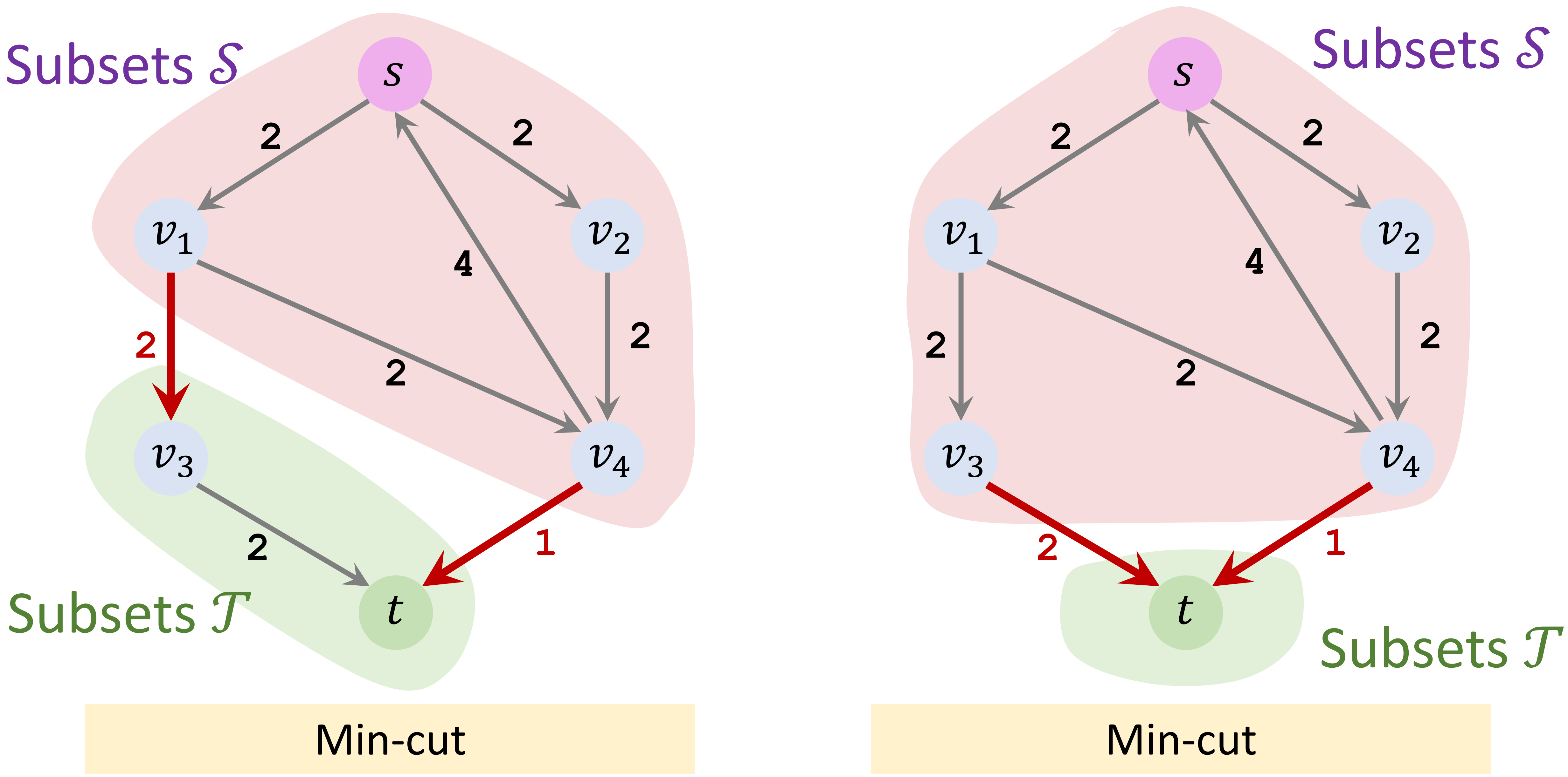
- Inputs:
  - Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,
  - Source $s \in \mathcal{V}$,
  - Sink $t \in \mathcal{V}$.
- Problem: Find s-t cut $(\mathcal{S}, \mathcal{T})$ that minimizes Capacity$(\mathcal{S}, \mathcal{T})$.
- Such a cut is called minimum s-t cut (min-cut).



Min-cut



Not min-cut



Subsets $\mathcal{S}$
s
v_1
v_2
v_3
v_4
t
2
2
4
2
2
2
2
1
Subsets $\mathcal{T}$
Min-cut
Subsets $\mathcal{S}$
s
v_1
v_2
v_3
v_4
t
2
2
4
2
2
2
2
1
Subsets $\mathcal{T}$
Min-cut

Max-Flow Min-Cut Theorem

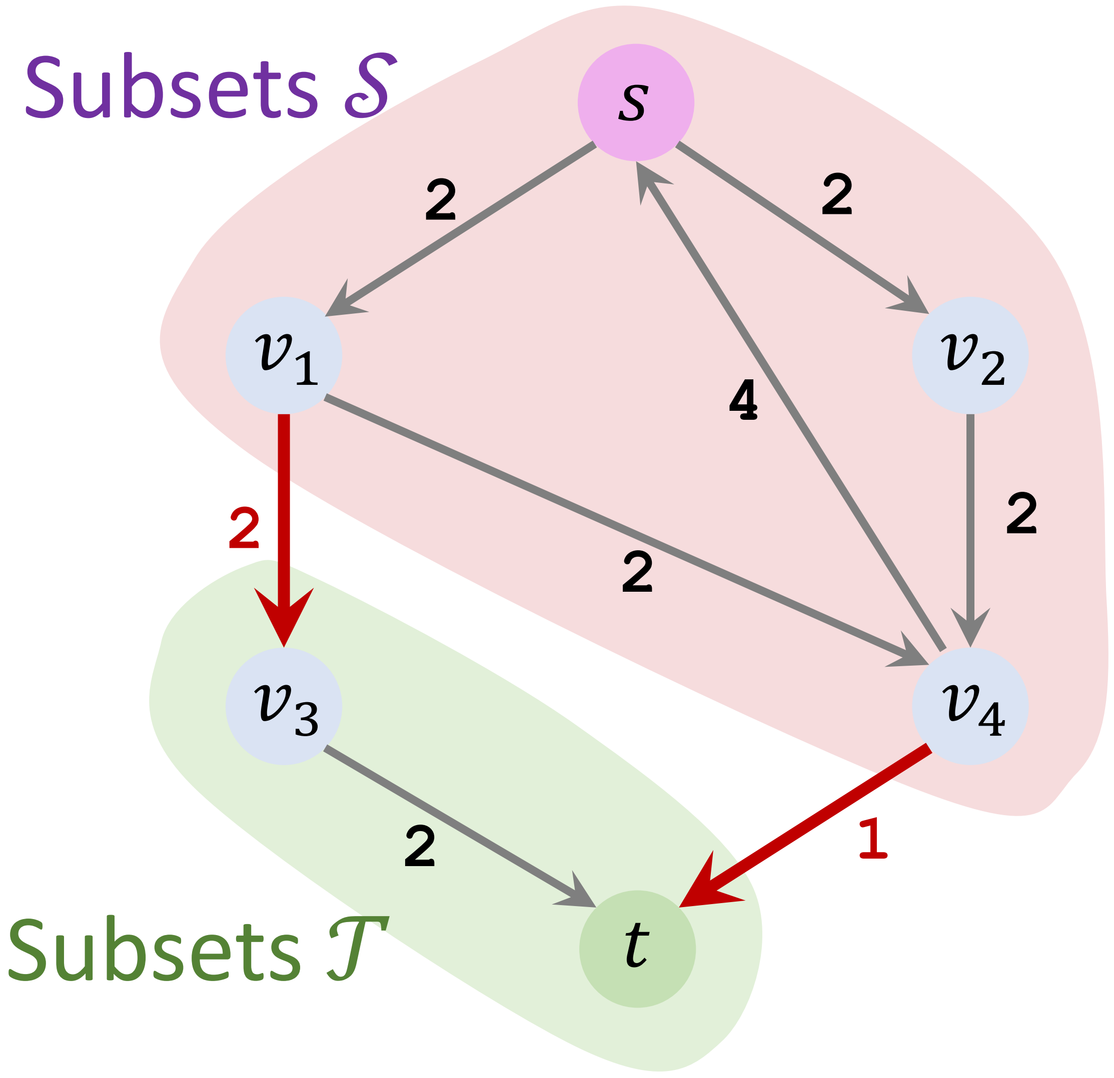
Max-Flow Min-Cut Theorem

- In a flow network, the maximum amount of flow from s to t is equal to the capacity of the minimum s-t cut.
- In short, amount of max-flow $=$ capacity of min-cut.

Reference

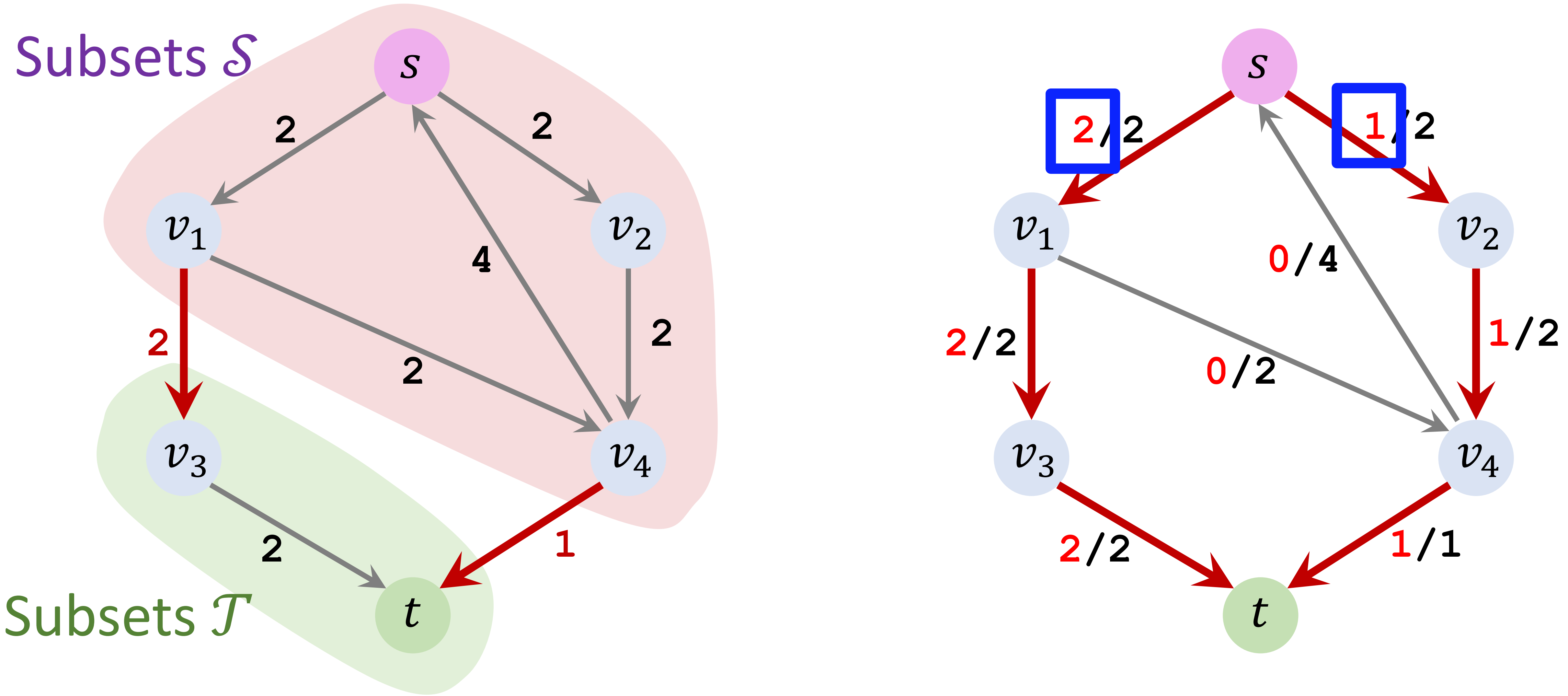
- L. R. Ford and D. R. Fulkerson. Flows in Networks. *Princeton University Press*, 1962.

Max-Flow Min-Cut Theorem



Capacity of min-cut $= 3$

Max-Flow Min-Cut Theorem



Capacity of min-cut = 3

Amount of max-flow = 3

Find Min-Cut

Algorithm

1. Run a max-flow algorithm to obtain the final residual graph.
    - E.g., using Edmonds–Karp algorithm or Dinic’s algorithm.
    - Ignore the backward edges in the final residual graph.

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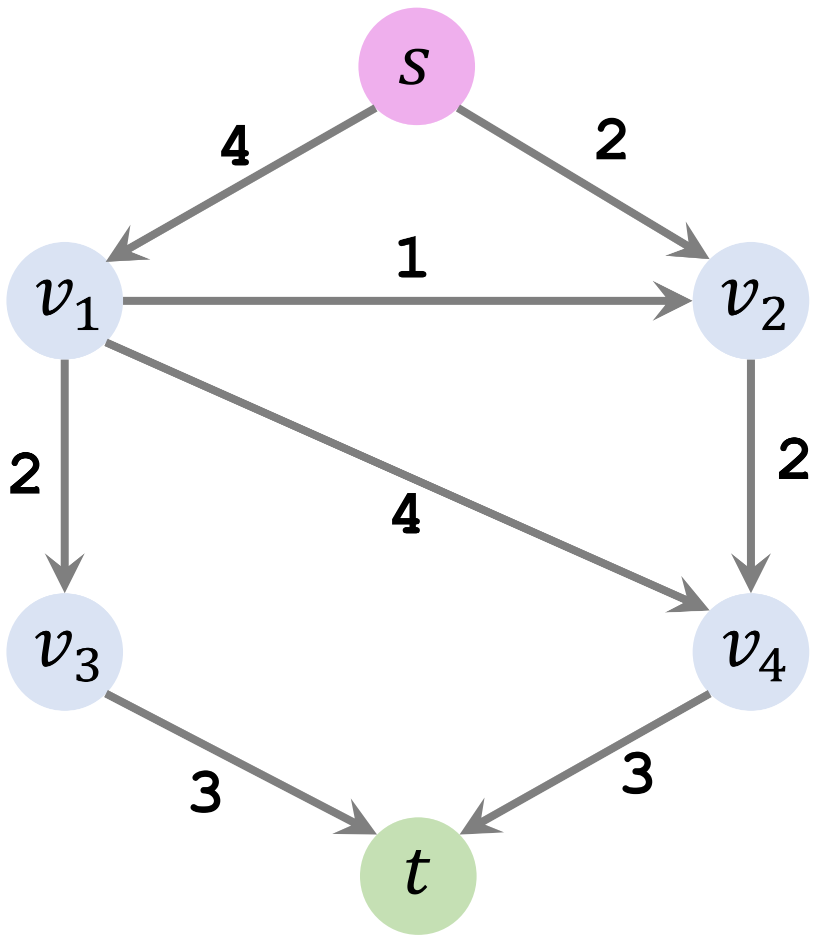
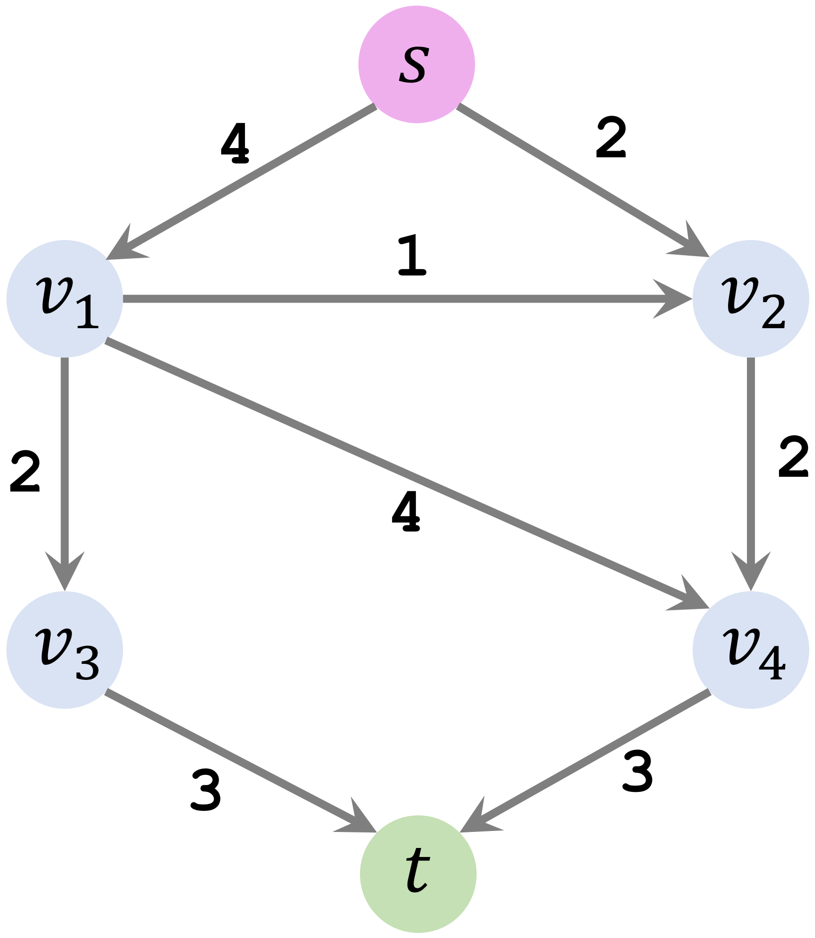
1. Run a max-flow algorithm to obtain the final residual graph.
   - E.g., using Edmonds–Karp algorithm or Dinic’s algorithm.
   - Ignore the backward edges in the final residual graph.
2. Find the minimum s-t cut $(\mathcal{S}, \mathcal{T})$:
   a. On the residual graph, find paths from source s to all the other vertices.
   b. $\mathcal{S} \leftarrow$ all the vertices that has finite distance. (Reachable from s.)
   c. $\mathcal{T} \leftarrow$ all the remaining vertices. (Not reachable from s.)

Example 1

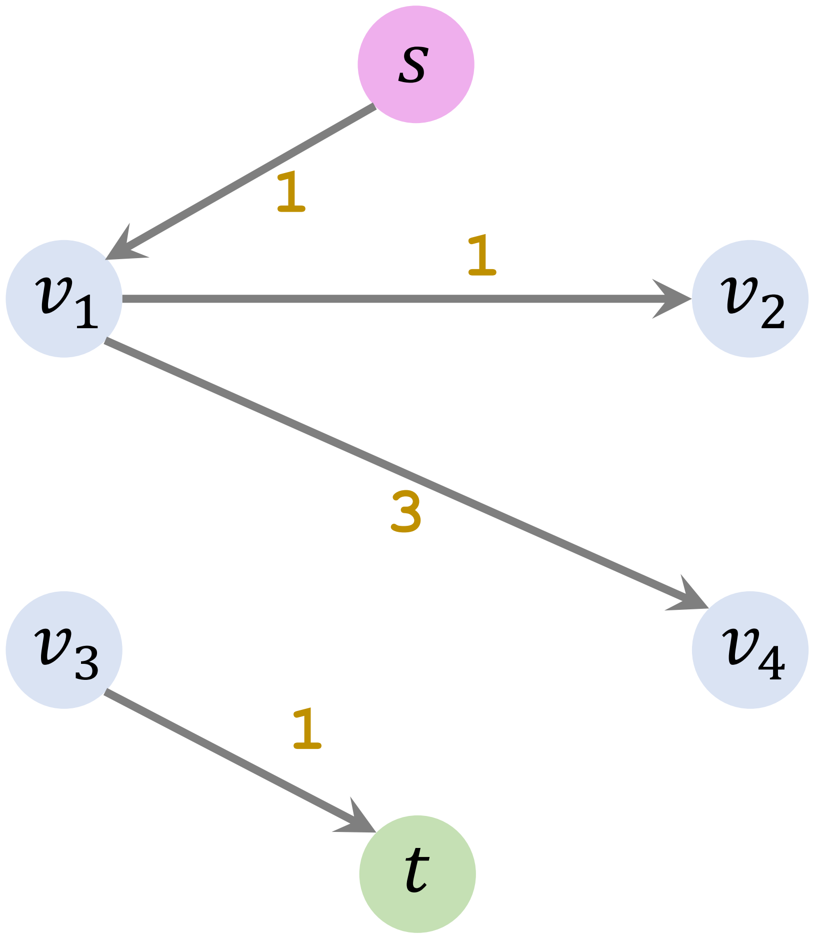


Original Graph

Example 1: Run a max-flow algorithm



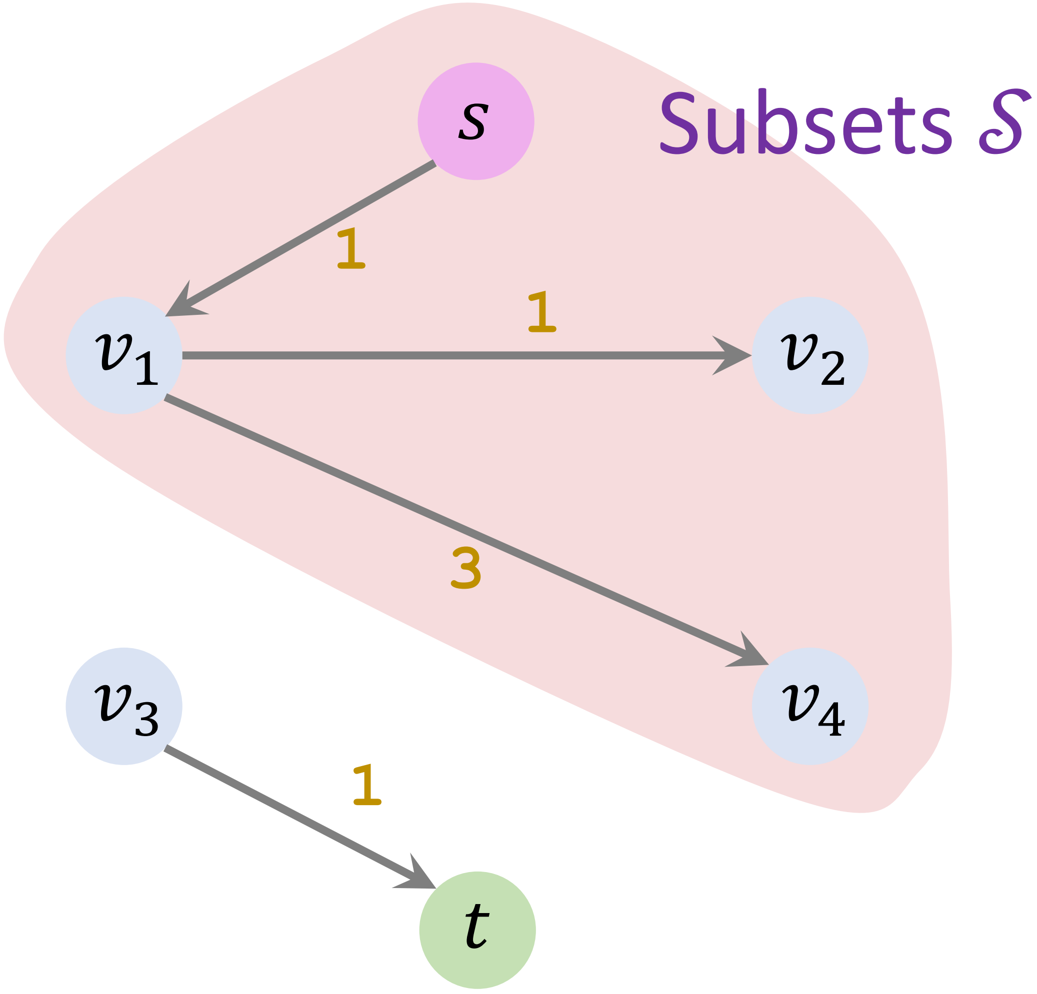
Original Graph



Residual Graph

Example 1: Find the min-cut

- Find the vertices reachable from s.
- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.

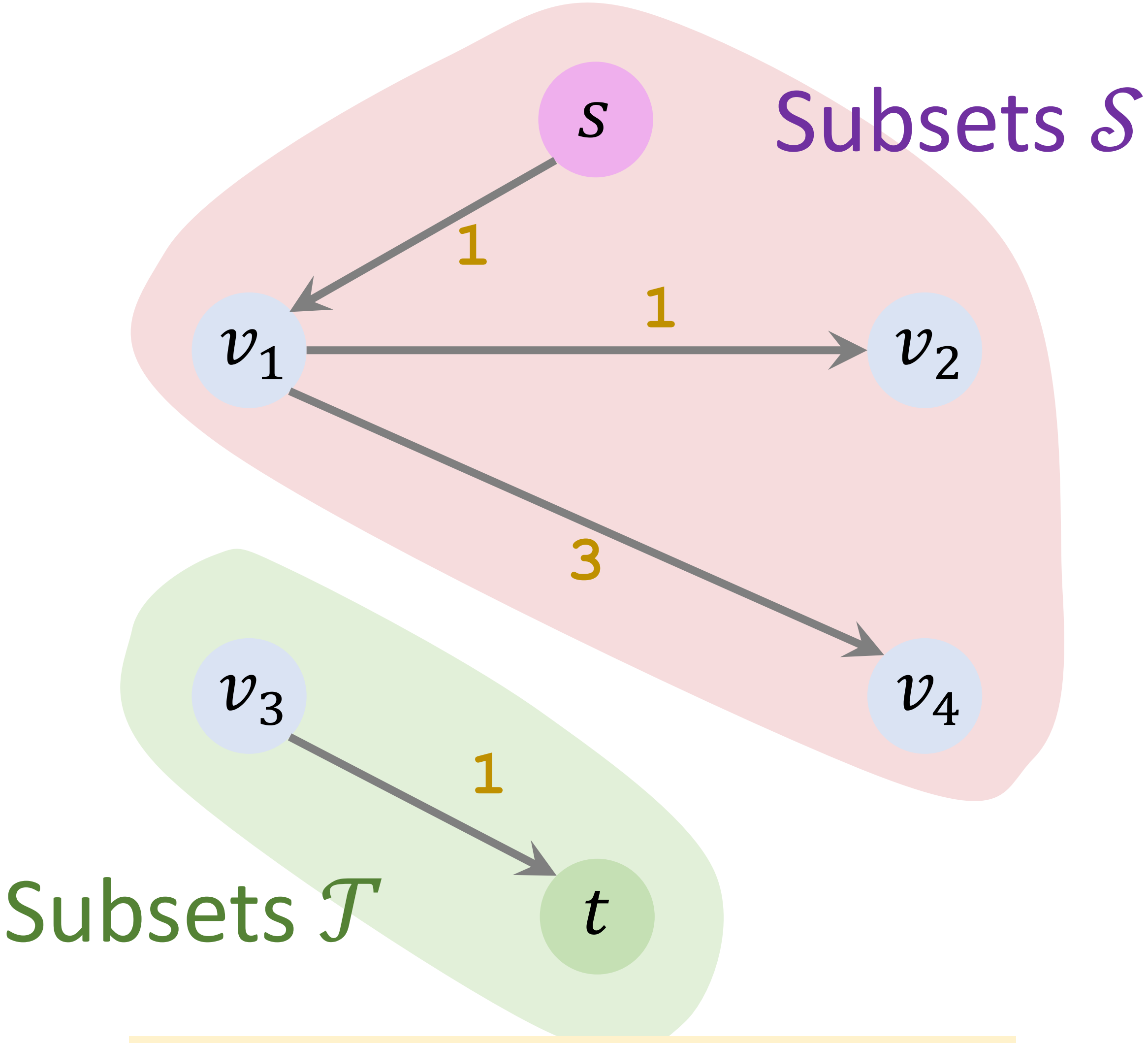


Residual Graph

Example 1: Find the min-cut

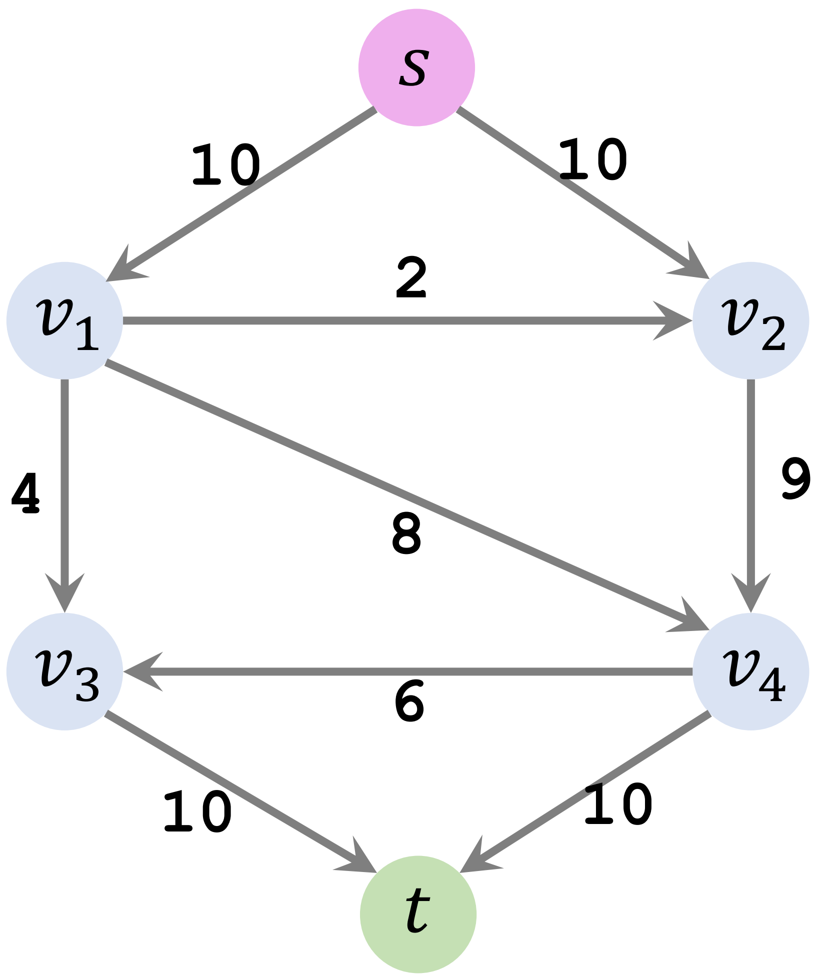
- Find the vertices reachable from s.
- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.

- The remaining vertices: t and v_3.
- Subset $\mathcal{T} = \{t, v_3\}$.



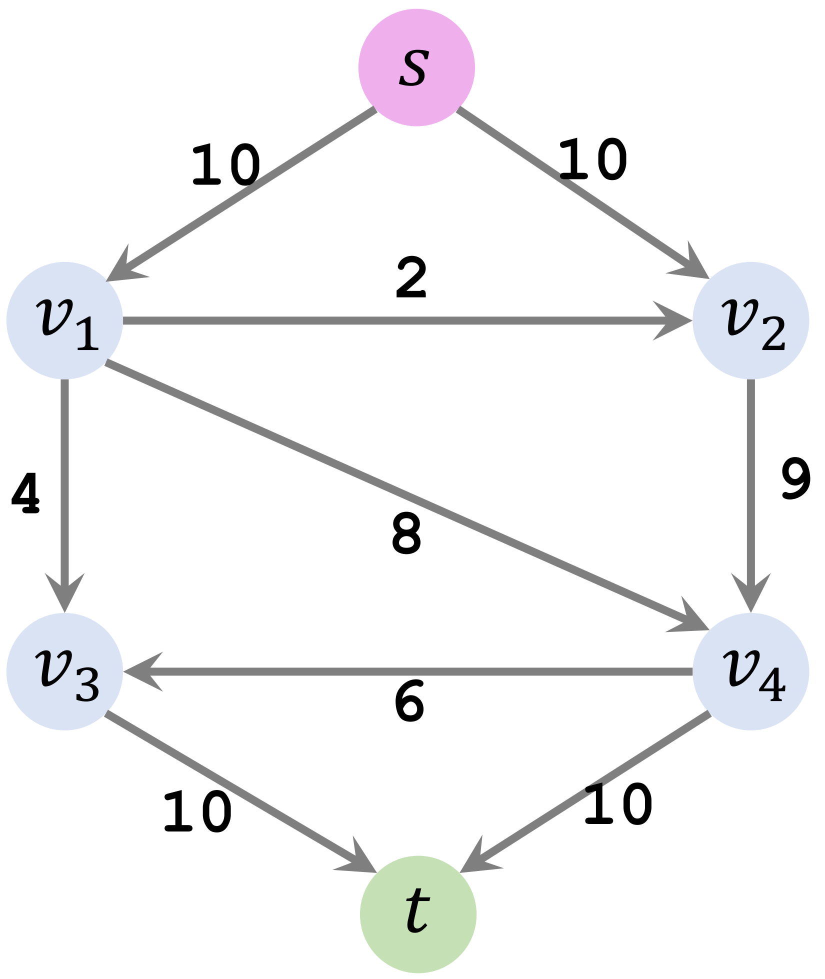
Residual Graph

Example 2

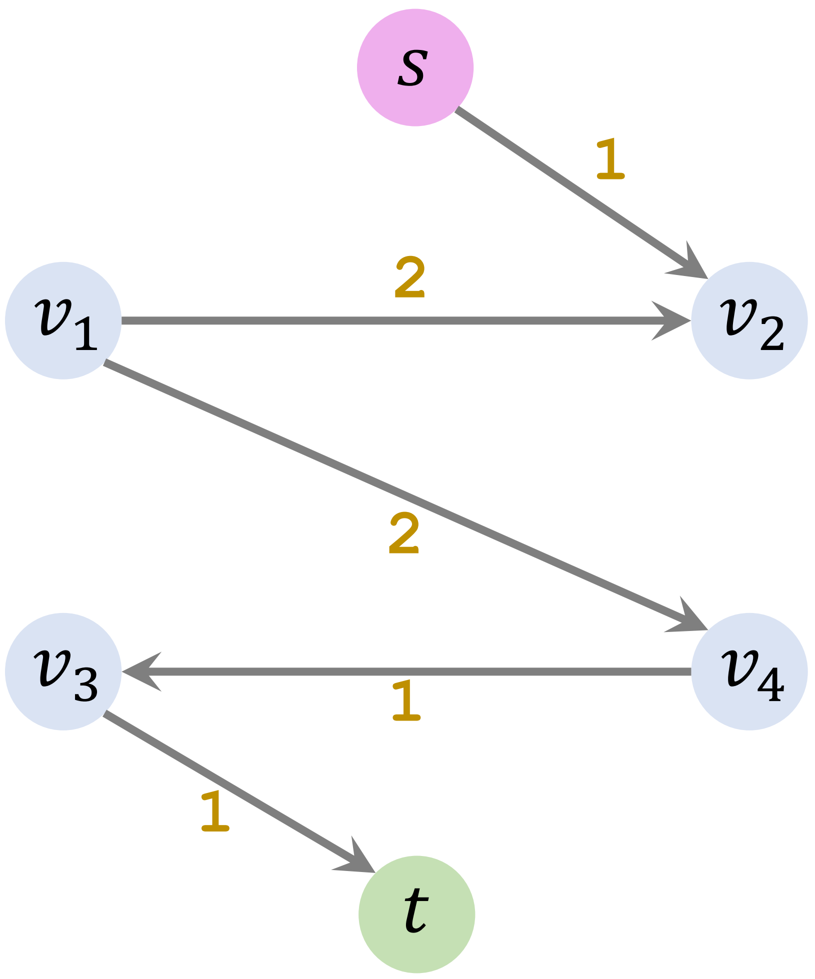


Original Graph

Example 2: Run a max-flow algorithm



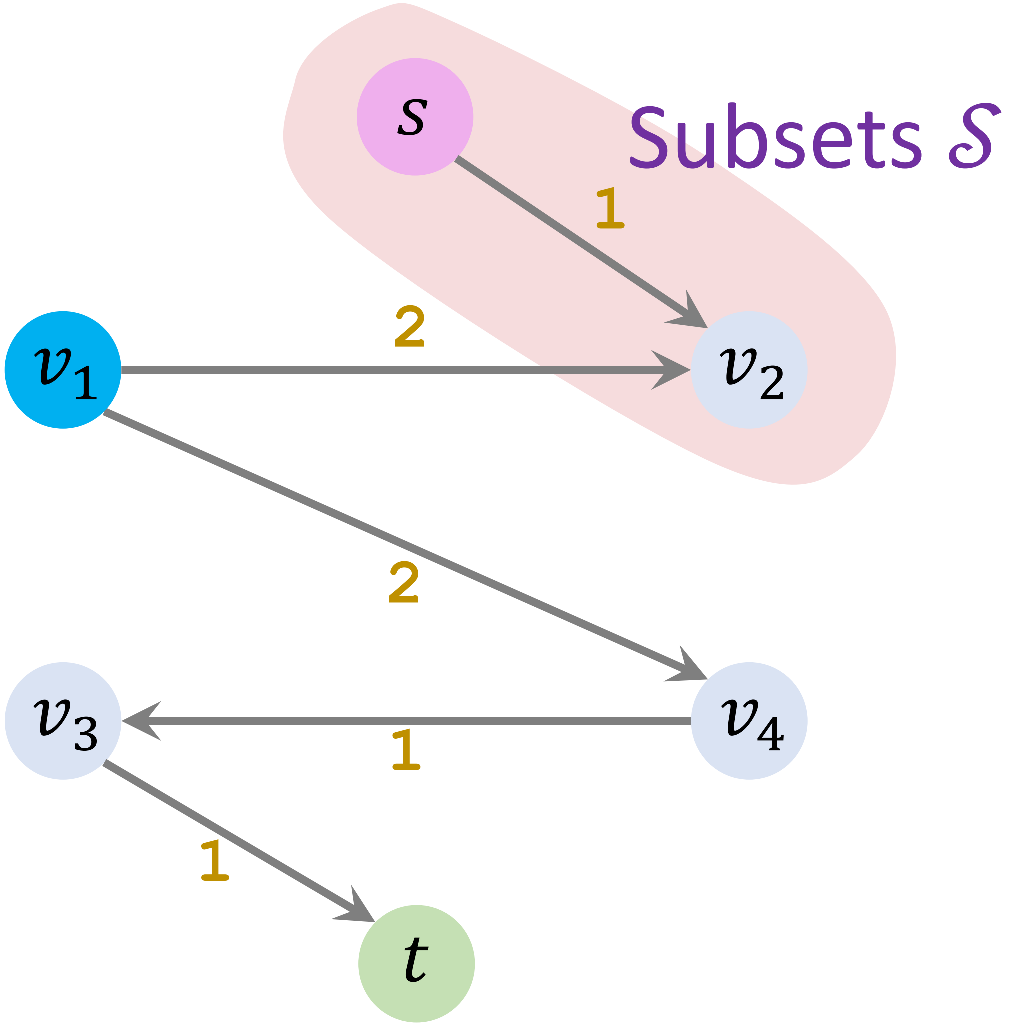
Original Graph



Residual Graph

Example 2: Find the min-cut

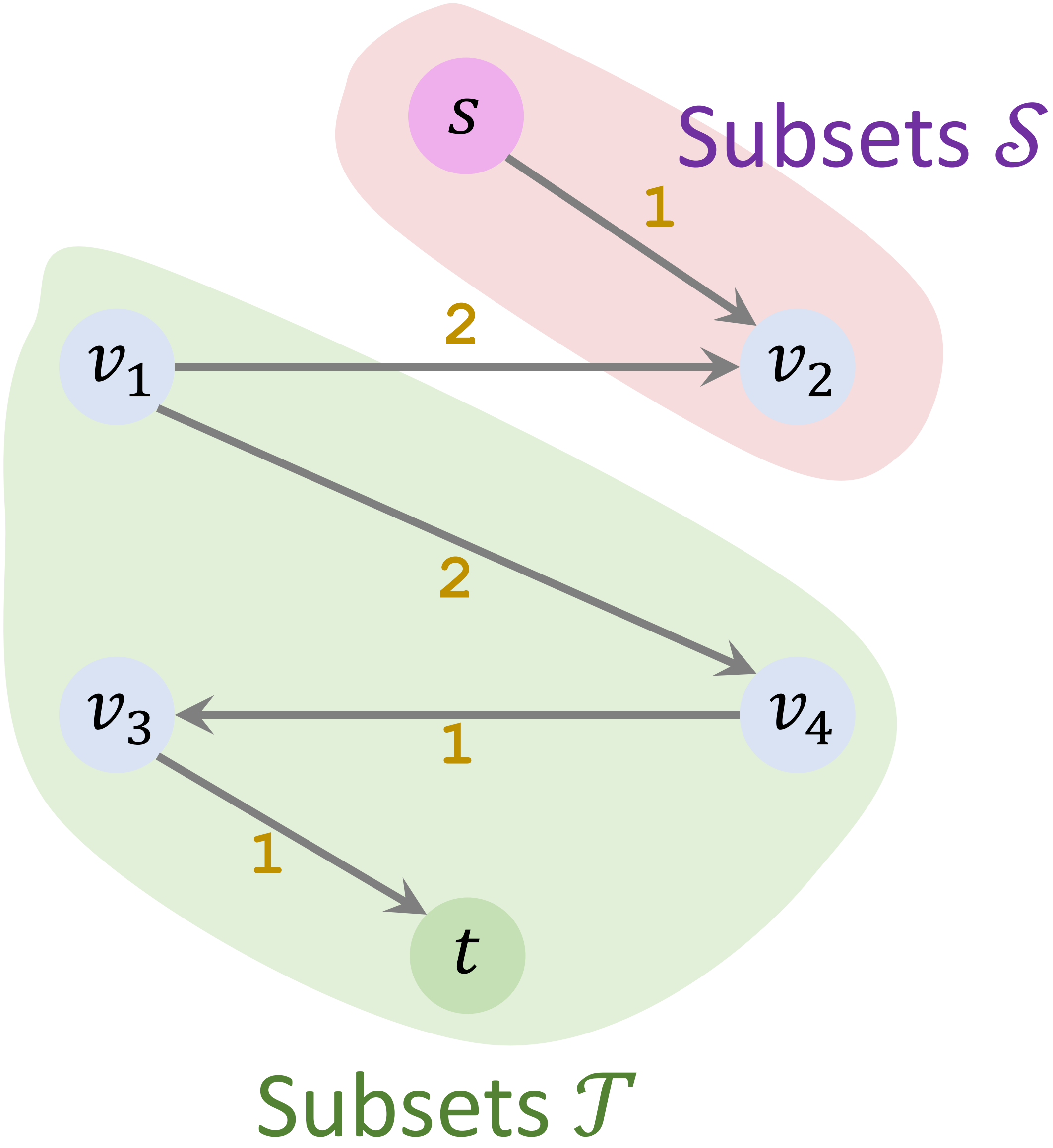
- Find the vertices reachable from s.
- Subset $\mathcal{S} = \{s, v_2\}$.



Example 2: Find the min-cut

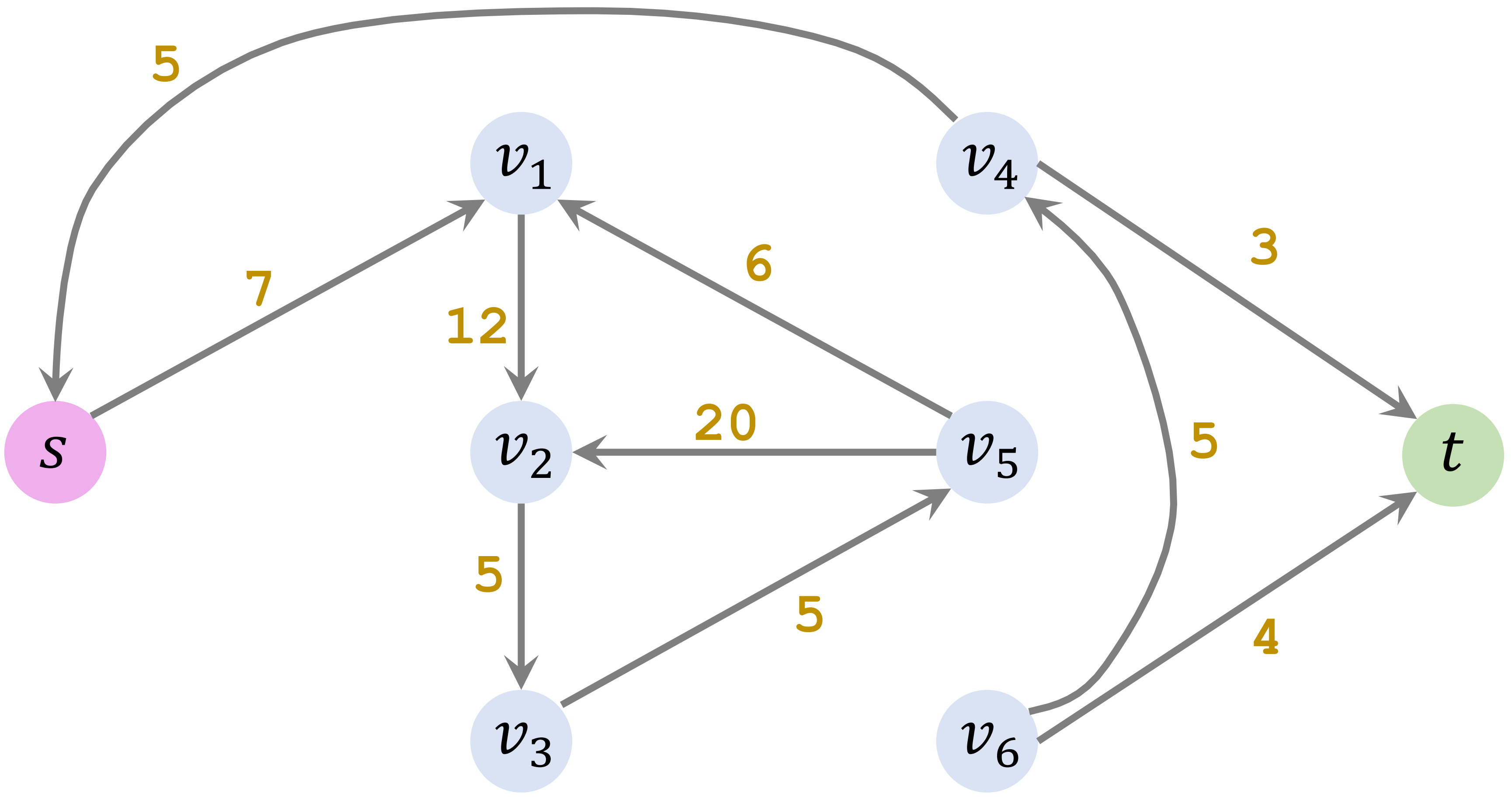
- Find the vertices reachable from s.
- Subset $\mathcal{S} = \{s, v_2\}$.

- The remaining vertices: t, v_1, v_3, v_4.
- Subset $\mathcal{T} = \{t, v_1, v_3, v_4\}$.



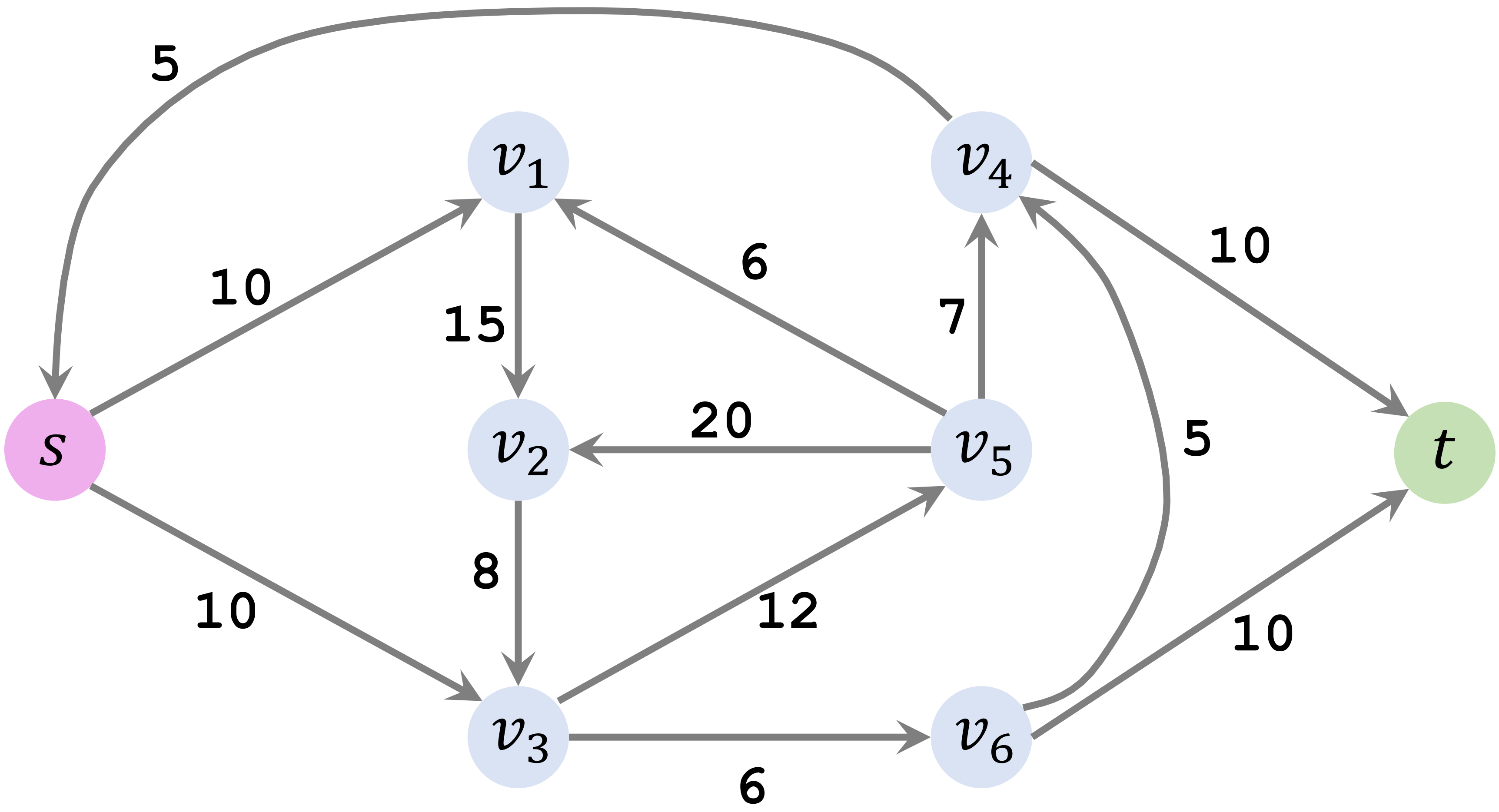
Questions

Q1: What is the min-cut?



This is the residual graph found by Dinic’s algorithm.

Q2: What is the capacity of the min-cut?



Hint: The min-cut is the same to the previous page.

Thank You!