Shortest Path

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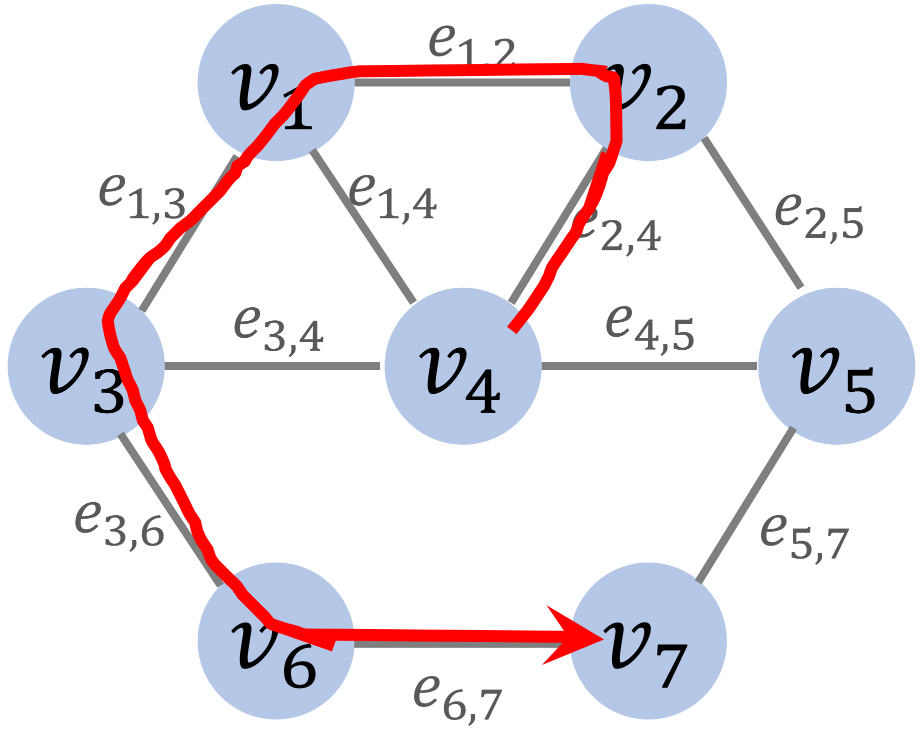
Path in Undirected Graphs

Path in undirected unweighted graph



Definition: Path

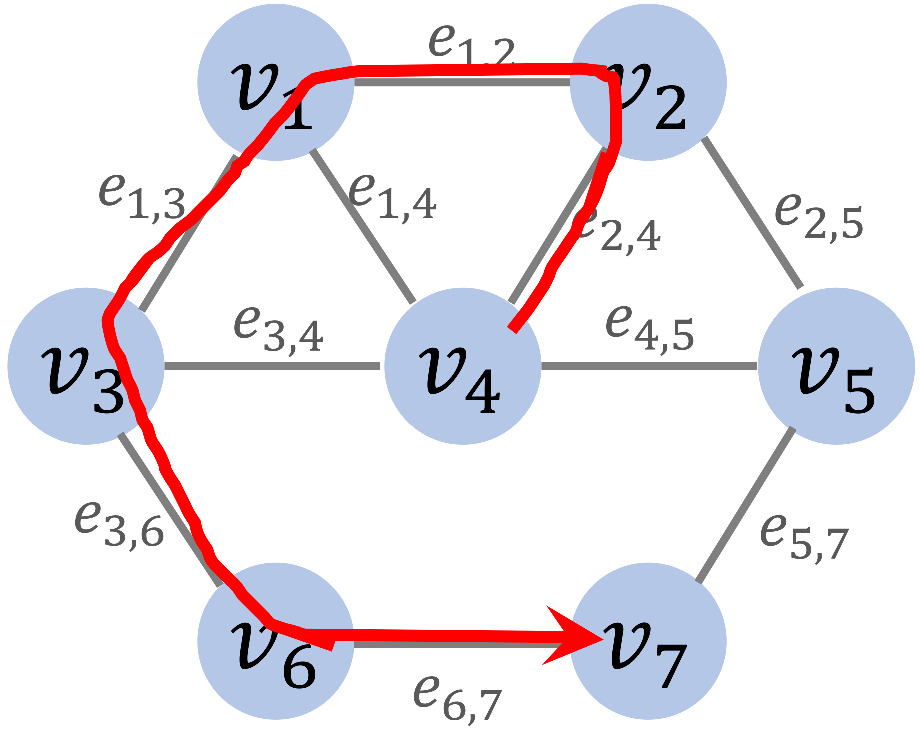
- A sequence of vertices

$$\boxed{(v_4, v_2, v_1, v_3, v_6, v_7)}$$

- Or a sequence of edges

$$\boxed{(e_{4,2}, e_{2,1}, e_{1,3}, e_{3,6}, e_{6,7}).}$$

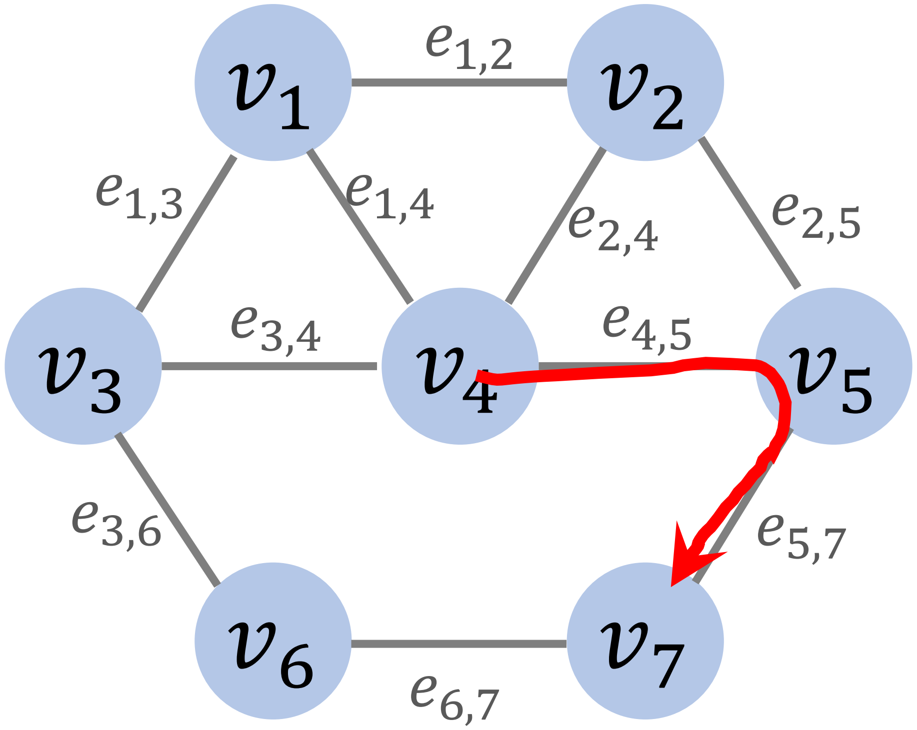
Path in undirected unweighted graph



Definition: Length of path

- Length of a path is the number of edges on the path.
- In this example, the length is 5.

Path in undirected unweighted graph



Definition: Path

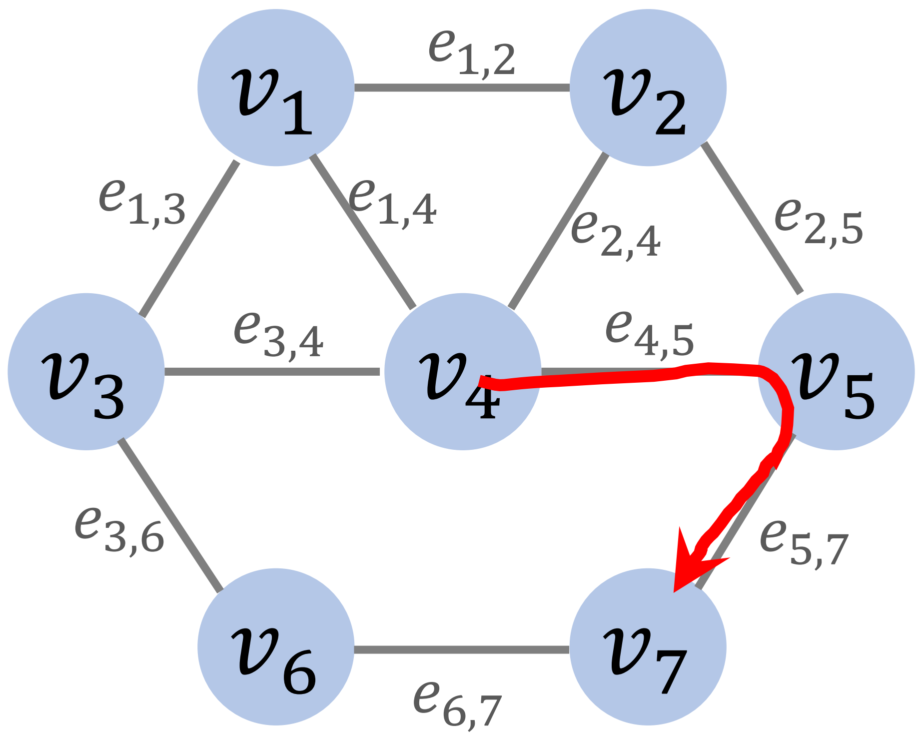
- A sequence of vertices

$$\boxed{(v_4, v_5, v_7)}$$

- Or a sequence of edges

$$\boxed{(e_{4,5}, e_{5,7}).}$$

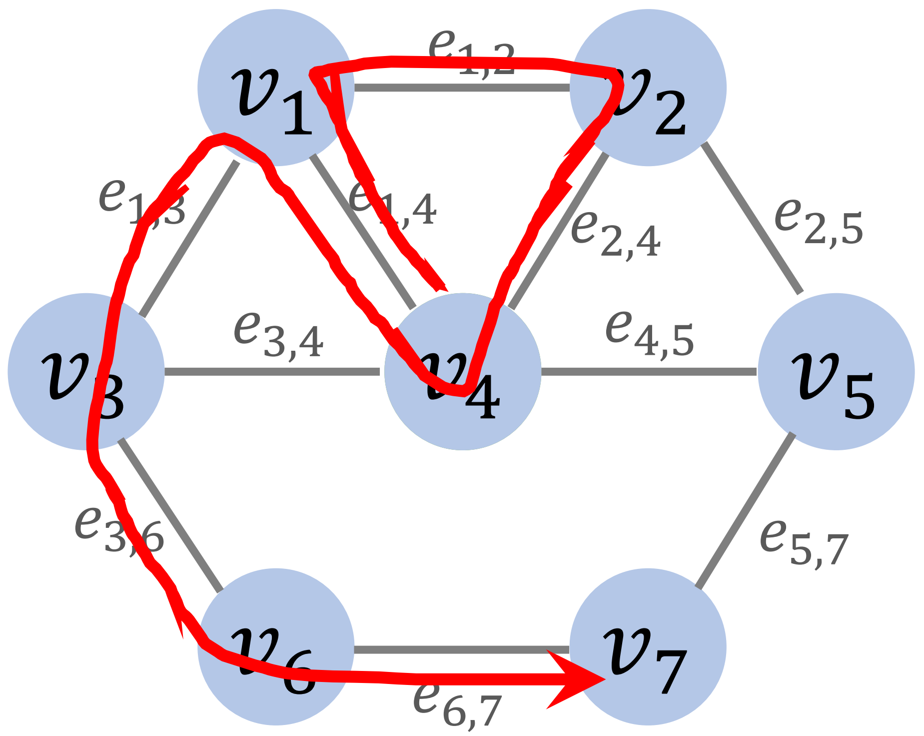
Path in undirected unweighted graph



Definition: Length of path

- Length of a path is the number of edges on the path.
- In this example, the length is 2.

Simple Path



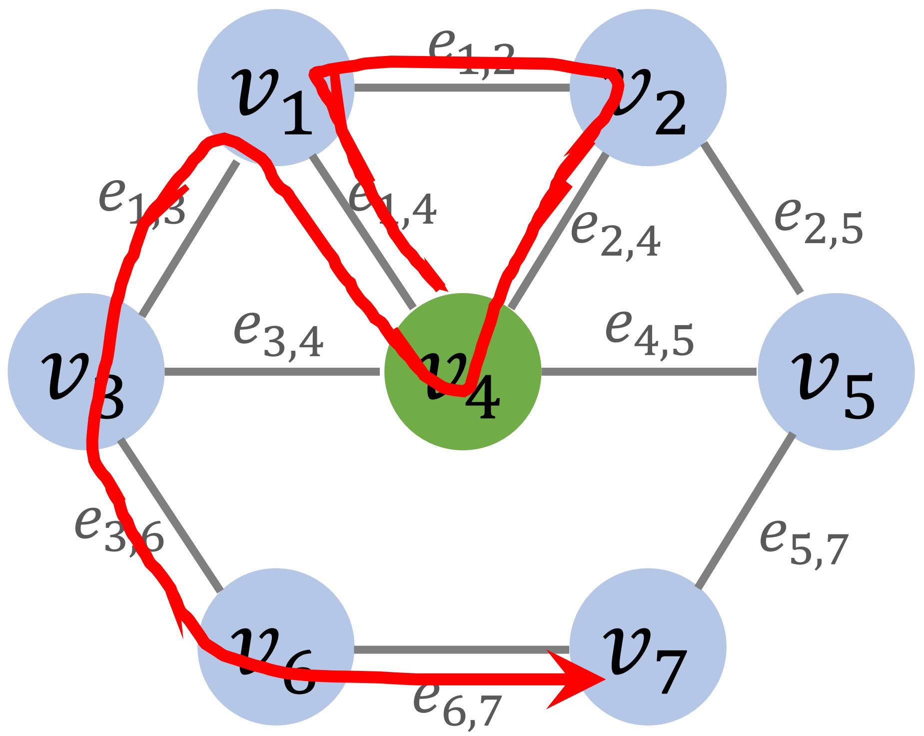
Definition: Path

A sequence of vertices

$$(v_4, v_1, v_2, v_4, v_1, v_3, v_6, v_7) \, .$$

Simple Path

Not a simple path!



Definition: Path

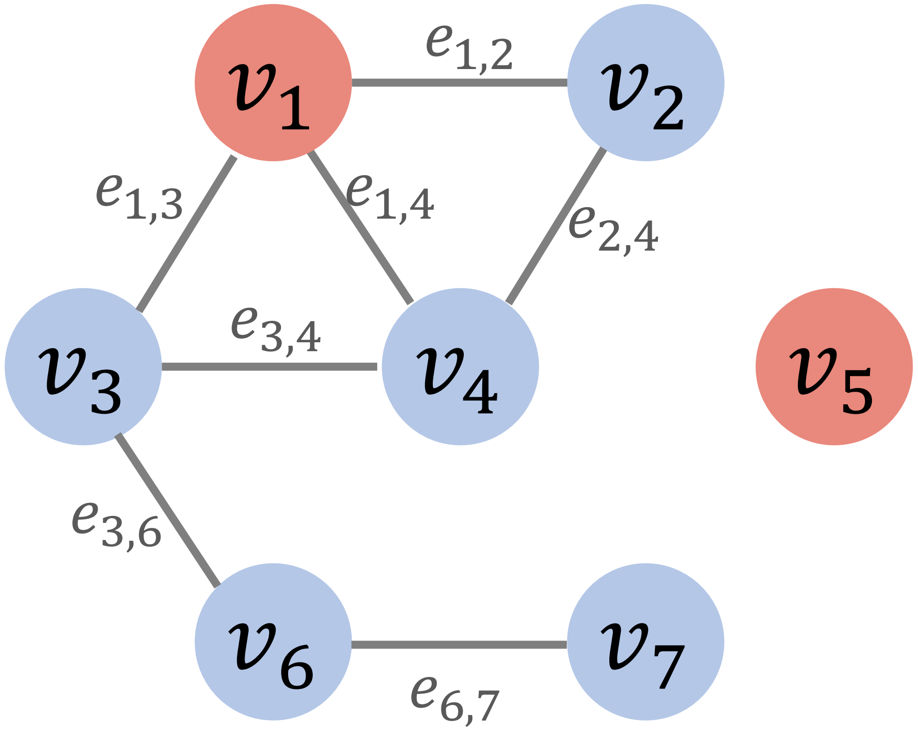
A sequence of vertices

$$(v_4, v_1, v_2, v_4, v_1, v_3, v_6, v_7) .$$

Definition: simple path

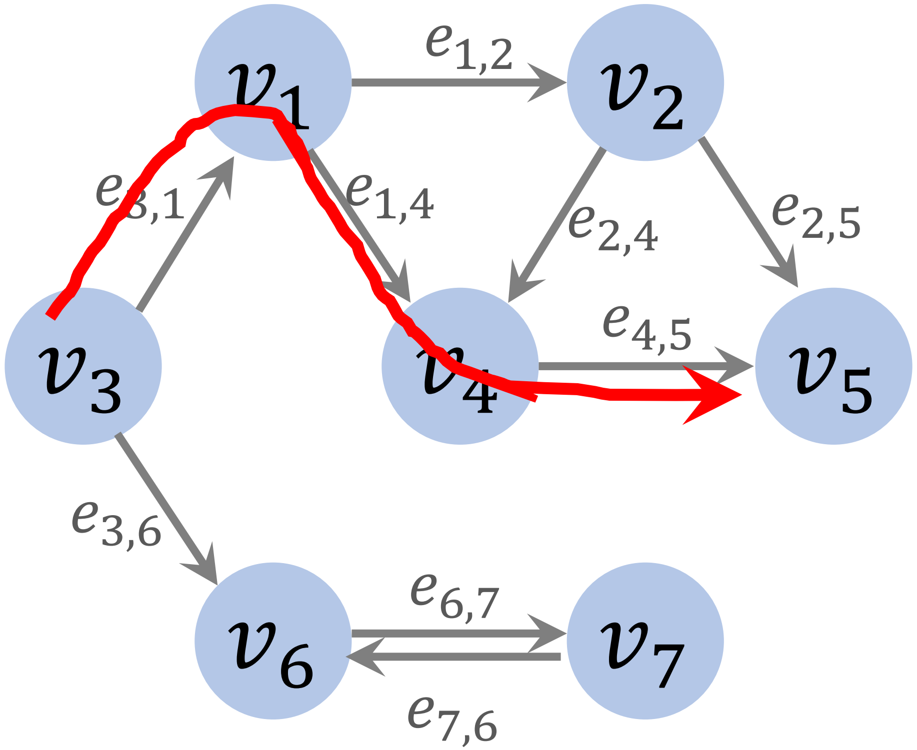
- A path that does not repeat vertices is called a simple path.

Path may not always exist



- There is no path between v_1 and v_5.
- The path between v_1 and v_5 has a length of ∞.

Path in directed unweighted graph



Definition: Path

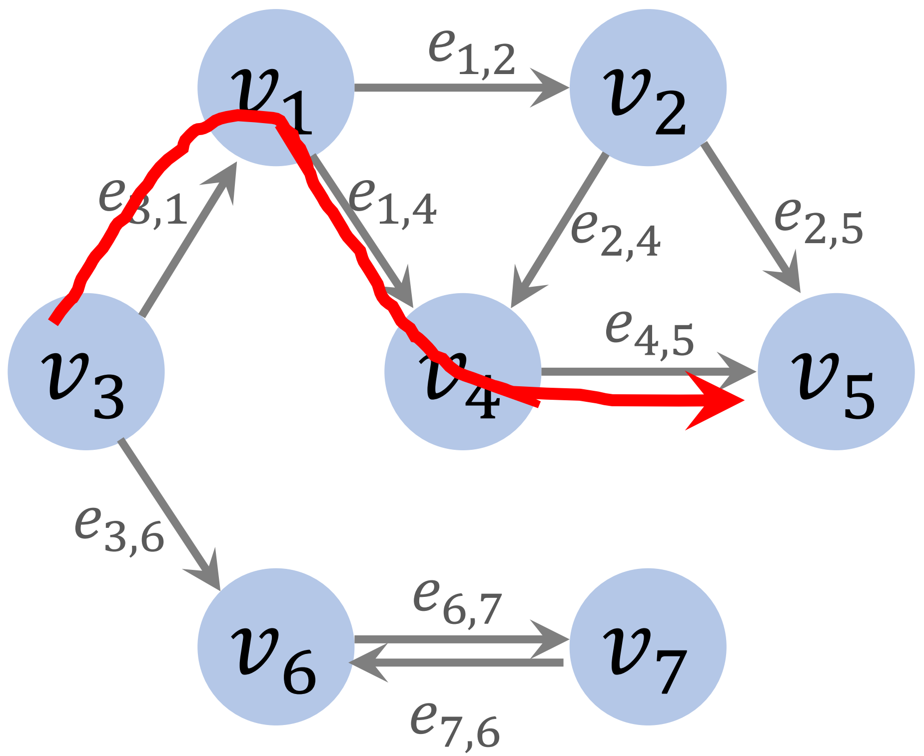
- A sequence of vertices

$$(v_3, v_1, v_4, v_5)$$

- Or a sequence of edges

$$(e_{3,1}, e_{1,4}, e_{4,5}).$$

Path in directed unweighted graph

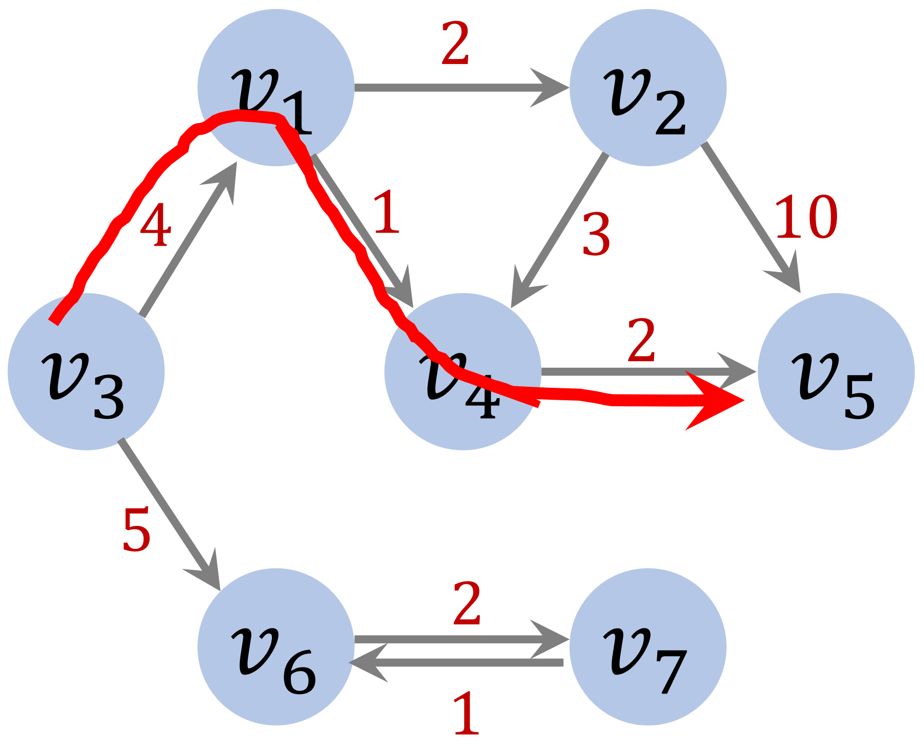


Definition: Length of path

- Length of a path is the number of edges on the path.
- In this example, the length is 3.

Path in Directed Graphs

Path in directed weighted graph

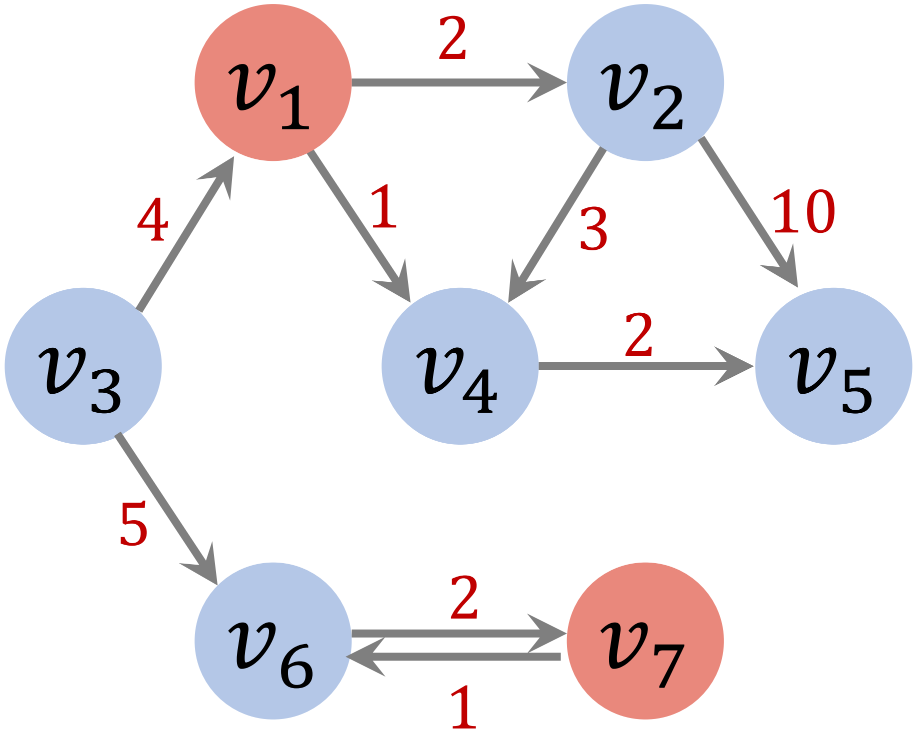


Definition: Length of path

- Length of a path is the sum of weights.
- In this example, the length is

$$\boxed{4 + 1 + 2 = 7.}$$

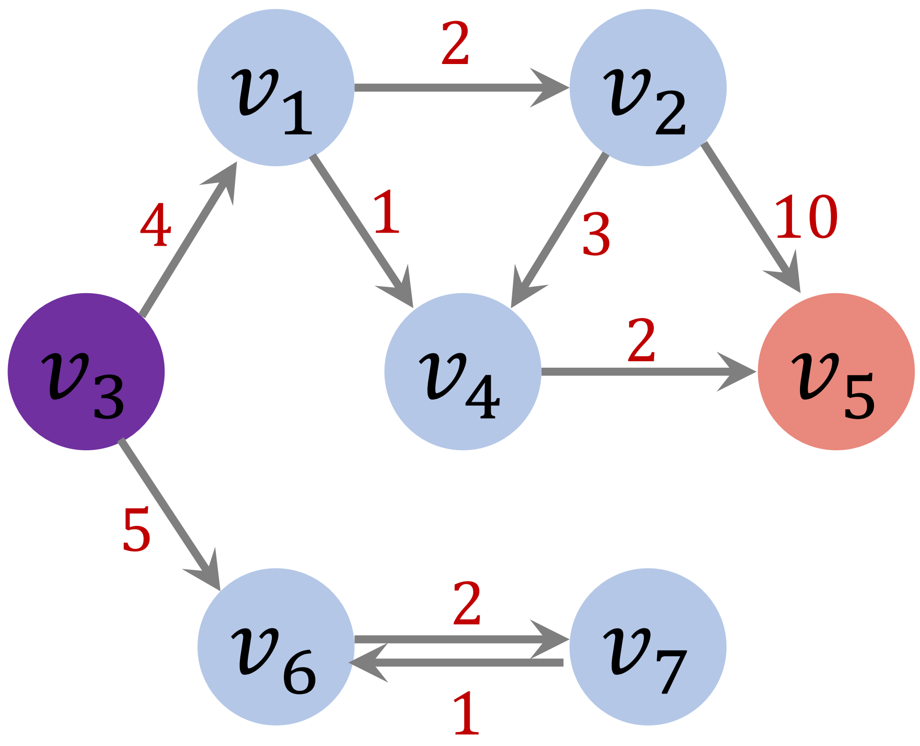
Path may not always exist



- There is no path from v_1 to v_7.
- The path from v_1 to v_7 has a length of ∞.

Shortest Path

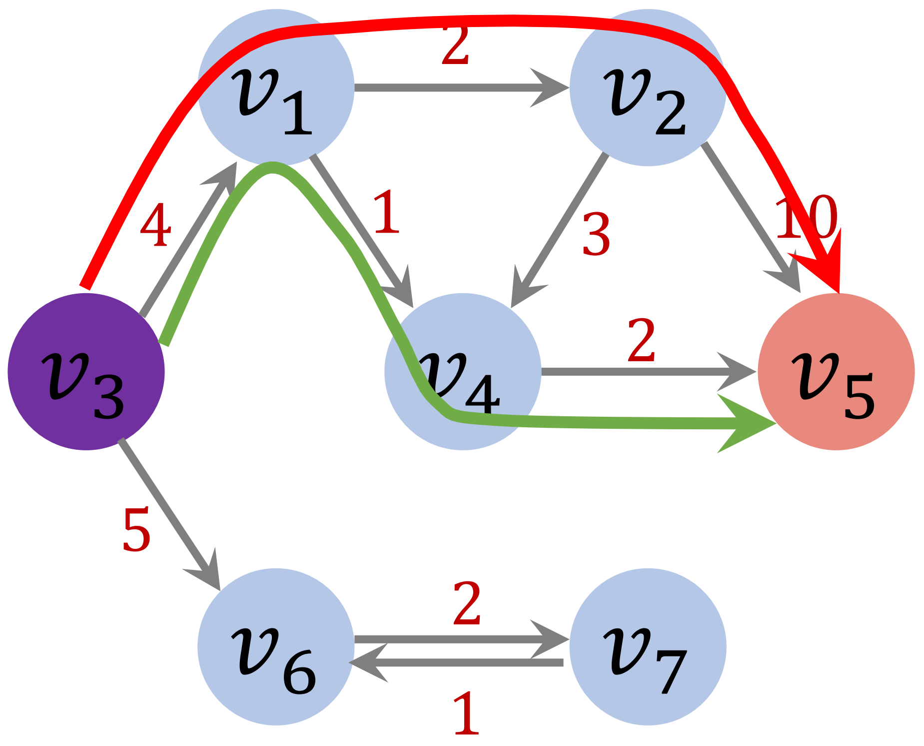
Shortest Path



Definition

- **Inputs:** graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the source vertex, s, and the destination, d.

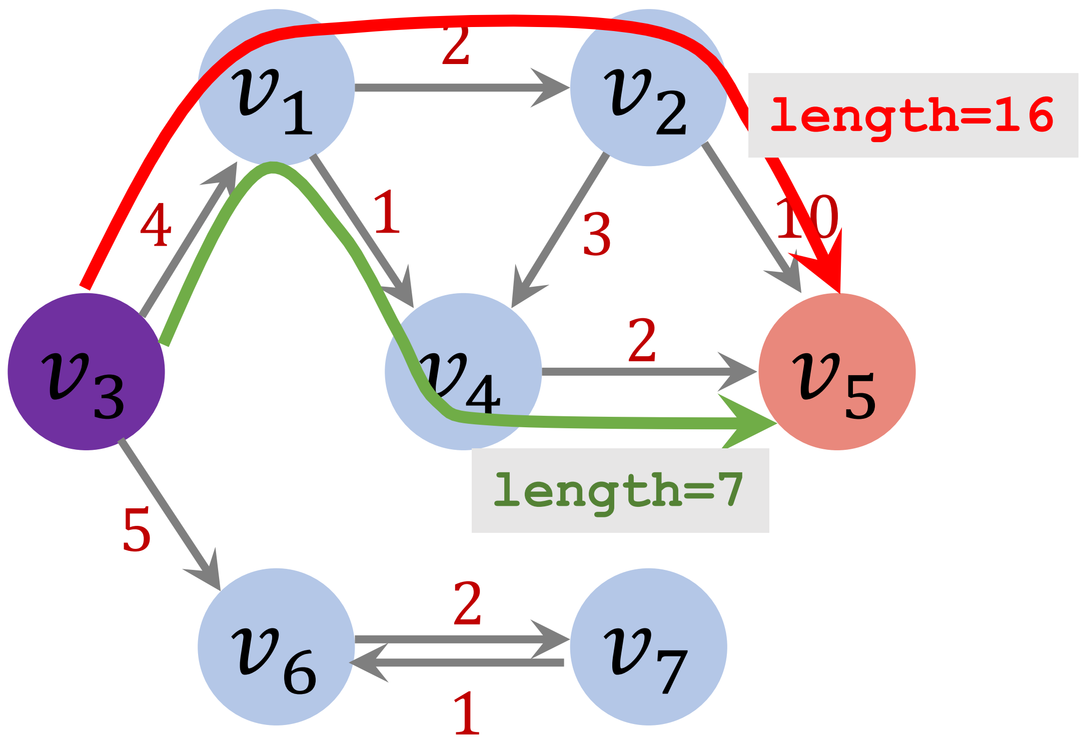
Shortest Path



Definition

- **Inputs:** graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the source vertex, s, and the destination, d.
- There can be multiple paths from s to d.

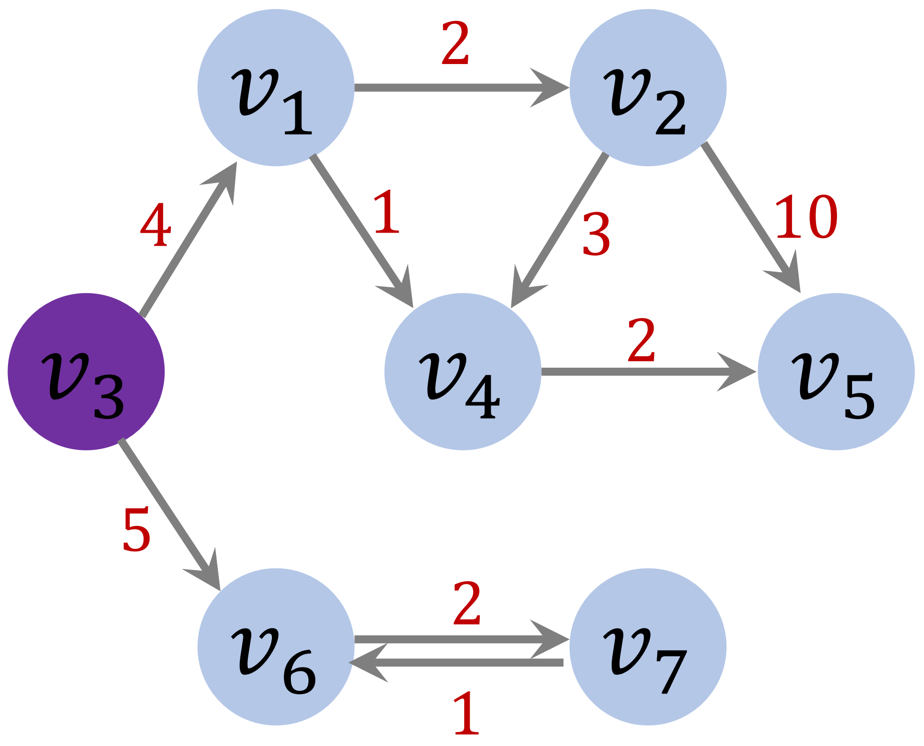
Shortest Path



Definition

- **Inputs:** graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the source vertex, s, and the destination, d.
- There can be multiple paths from s to d.
- Among all the paths, the one with the smallest length is called the shortest path.

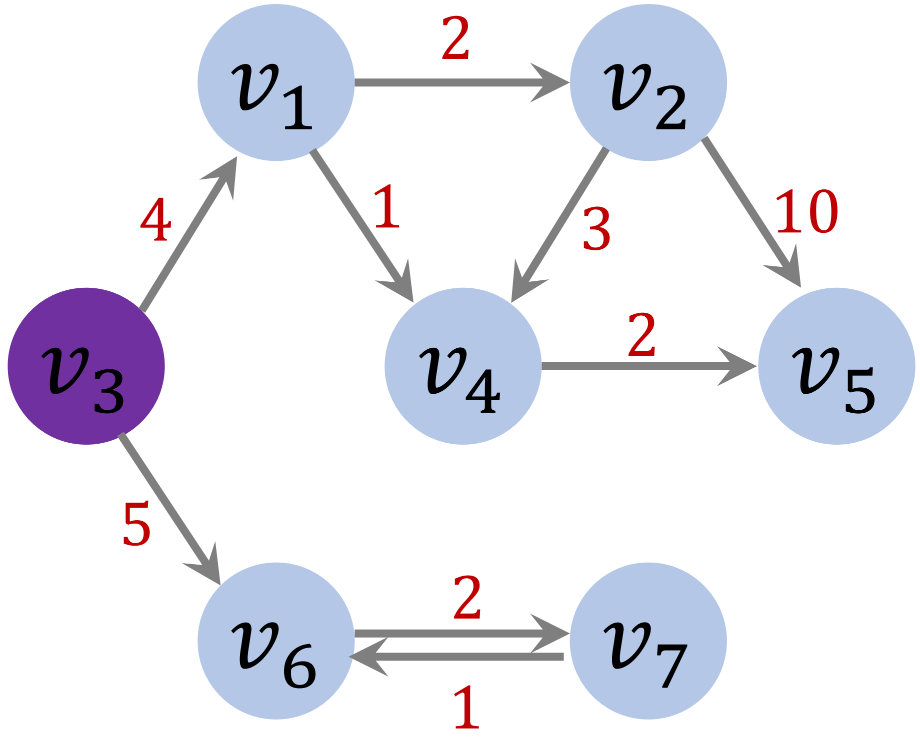
Single-Source Shortest Path Problem



Definition

- **Inputs:** graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the source vertex, s.
- **Goal:** find the shortest path from s to every other vertex in $\mathcal{G}$.

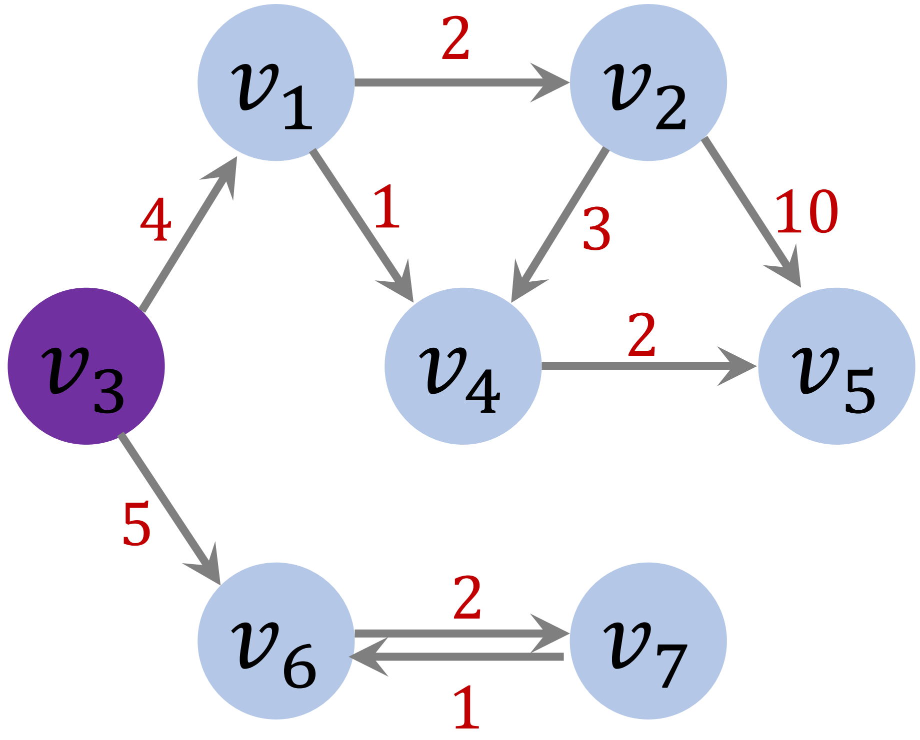
Single-Source Shortest Path Problem



Example

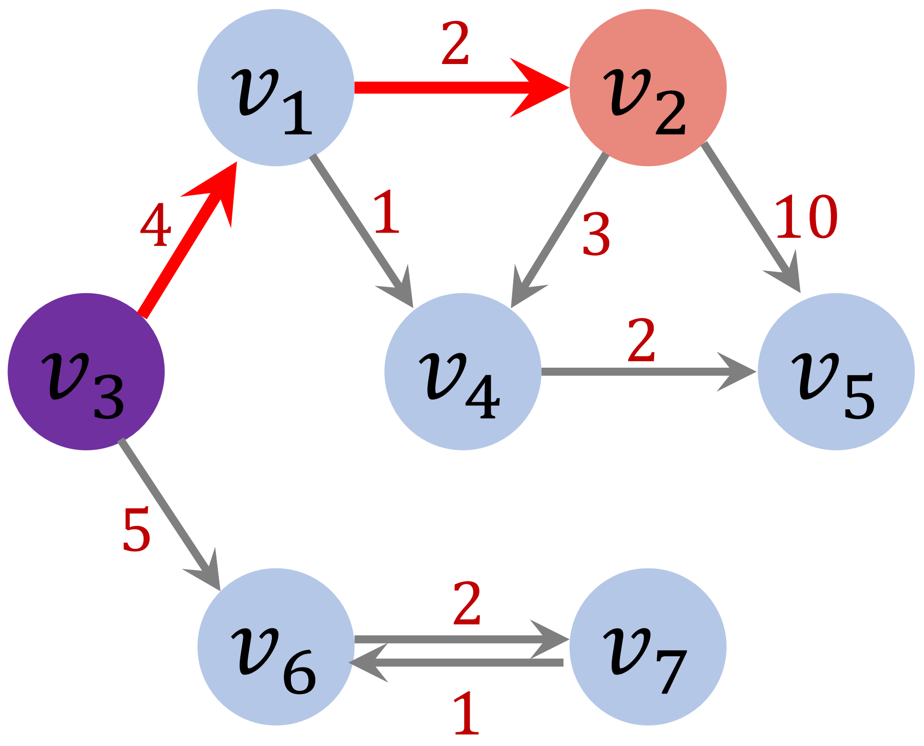
- Source: $s = v_3$.
- Find the shortest path from v_3 to all the other vertices.

Single-Source Shortest Path Problem



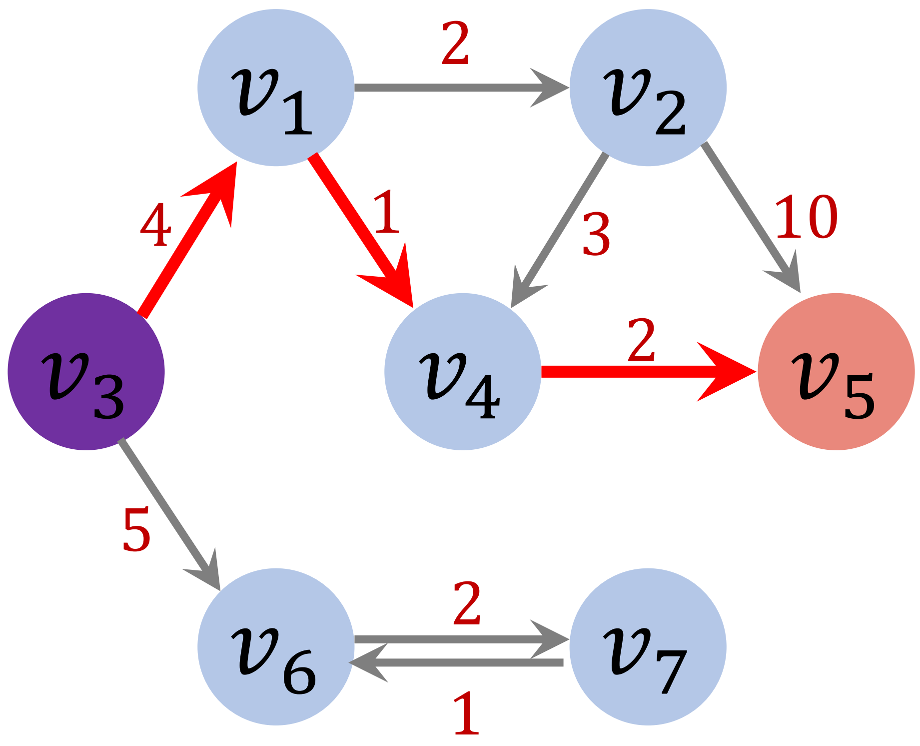
vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Lengths of the shortest paths



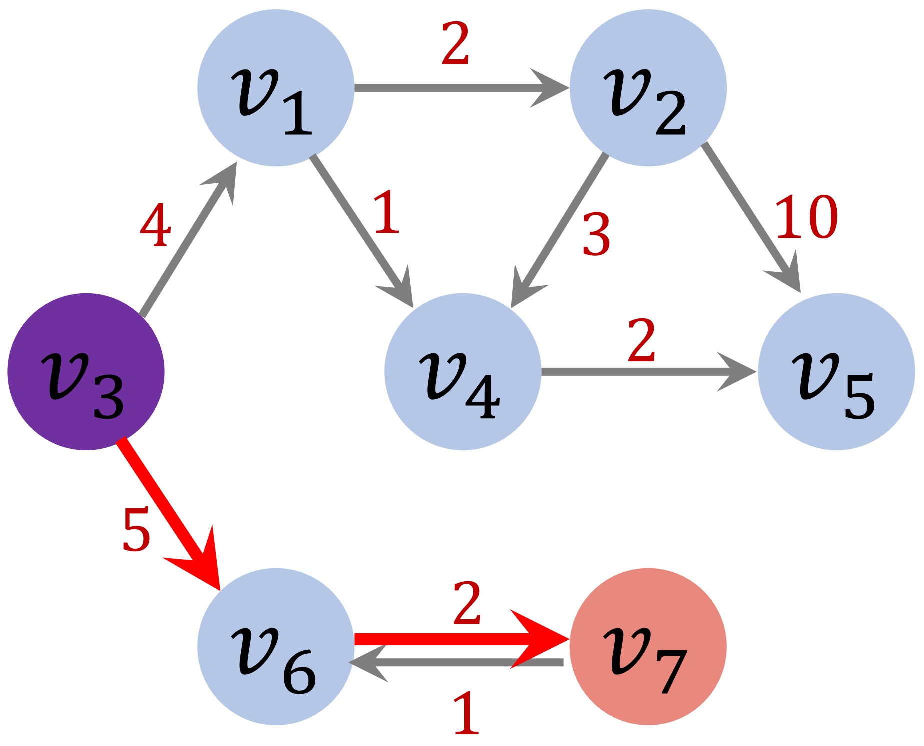
vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Lengths of the shortest paths



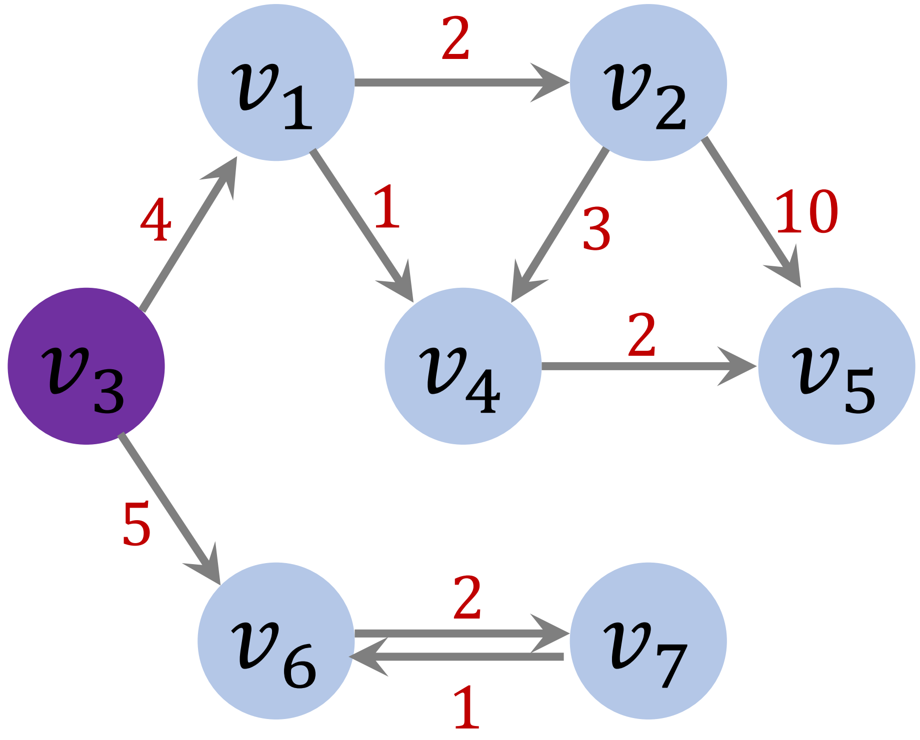
vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Lengths of the shortest paths



vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

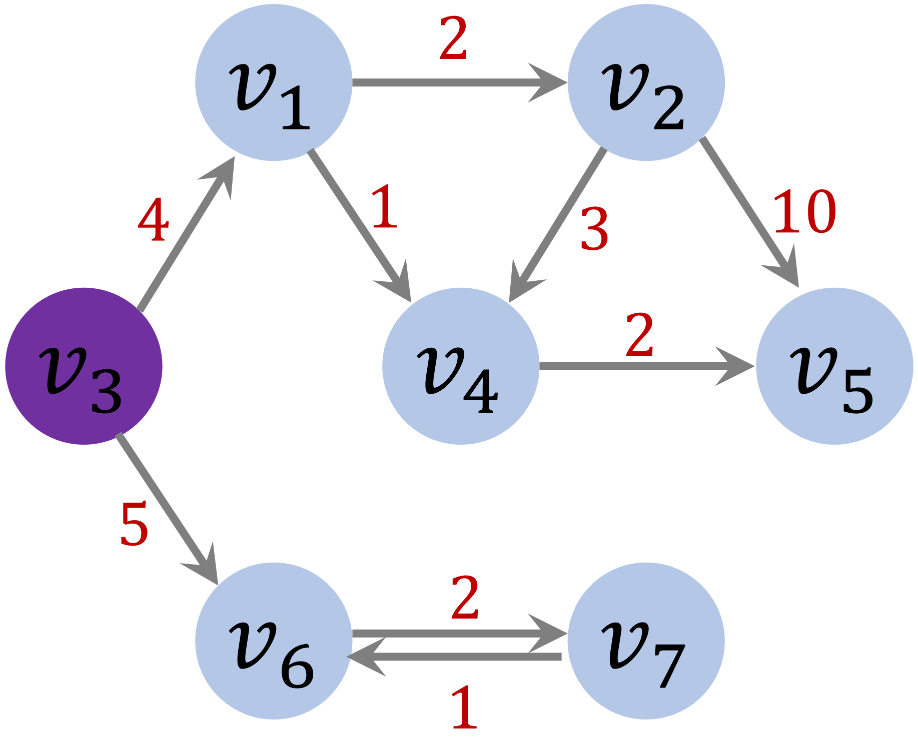
Lengths of the shortest paths



Lengths of the shortest paths.

vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Recover the shortest paths



The previous vertex along the path.

vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Single-Source Shortest Path Problem

What is the shortest path from v_3 to v_2?

- The 2nd row in the table: $v_1 \rightarrow v_2$.

vertex	dist	path
v_1	4	v_3
v_2	6	$\boxed{v_1}$
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Single-Source Shortest Path Problem

What is the shortest path from v_3 to v_2?

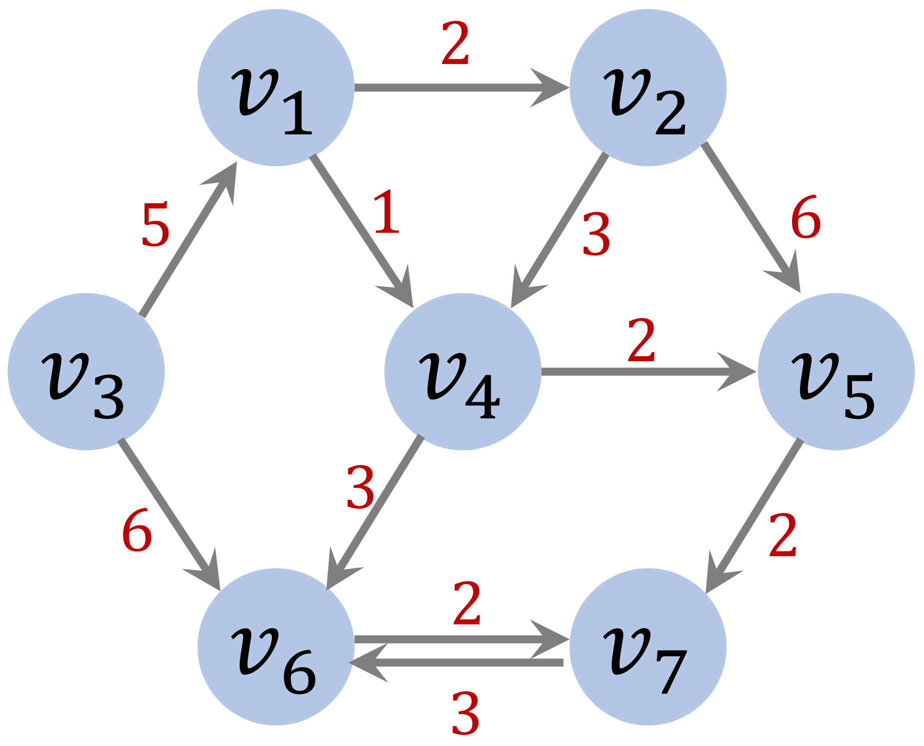
- The 2nd row in the table: $v_1 \rightarrow v_2$.
- The 1st row in the table: $v_3 \rightarrow v_1$.
- Thus, the shortest path is

$$v_3 \rightarrow v_1 \rightarrow v_2.$$

vertex	dist	path
v_1	4	$\boxed{v_3}$
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Questions

Question 1



- **Q:** What is the shortest paths?
  - From v_1 to v_1.
  - From v_1 to v_3.
  - From v_1 to v_7.
- **Q:** What are the lengths of these shortest paths?

Question 2

- The graph is directed and weighted.
- Let v_3 be the source vertex.
- The table reflects the single-source shortest paths.
- **Q:** What is the shortest path from v_3 to v_7?
- **Q:** What is the length of this shortest path?

vertex	dist	path
v_1	4	v_3
v_2	6	v_1
v_3	0	0
v_4	5	v_1
v_5	7	v_4
v_6	5	v_3
v_7	7	v_6

Thank You!