Botlhale Village

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ARE YOU READY?

NONLINEAR PROGRAMMING

A Nonlinear program (NLP) can have a linear or nonlinear objective function with linear and/or nonlinear constraints (If both the objective function and the constraints are linear then it is a linear programming model)

THE DIFFERENCE BETWEEN NLP'S AND LP'S

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- For any LP, the coefficients of the variable are fixed. For example: a company wants to sell shirts for R150 and shorts for R300. We will define our objective function as $15x_1 + 30x_2$.
- For any NLP the variables can be dependant on each other. For example: Q & H Company advertises on soap operas and football games. if *S* soap opera ads are bought, they will be seen by $5\sqrt{S}$ men. If *F* football ads are bought, they will be seen by $17\sqrt{F}$ men. We will define our constraint as $5S^{1/2} + 17F^{1/2} \geq 40 \quad (\text{Men})$.
- For an NLP we need to use some form of numerical search.

NLP EXAMPLE AND SOLUTION

- If K units of capital and L units of labour are used, a company can produce KL units of a manufactured good. Capital can be purchased at \$4/unit and labour can be purchased at \$1/unit. A total of \$8 is available to purchase capital and labour. How can the firm maximize the quantity of the goods that can be manufactured?
- Let $K =$ units of capital purchased and $L =$ units of labour purchased. Then K and L must satisfy $4K + L \leq 8, K \geq 0$, and $L \geq 0$. Thus, the firm wants to solve the following constrained maximization problem:

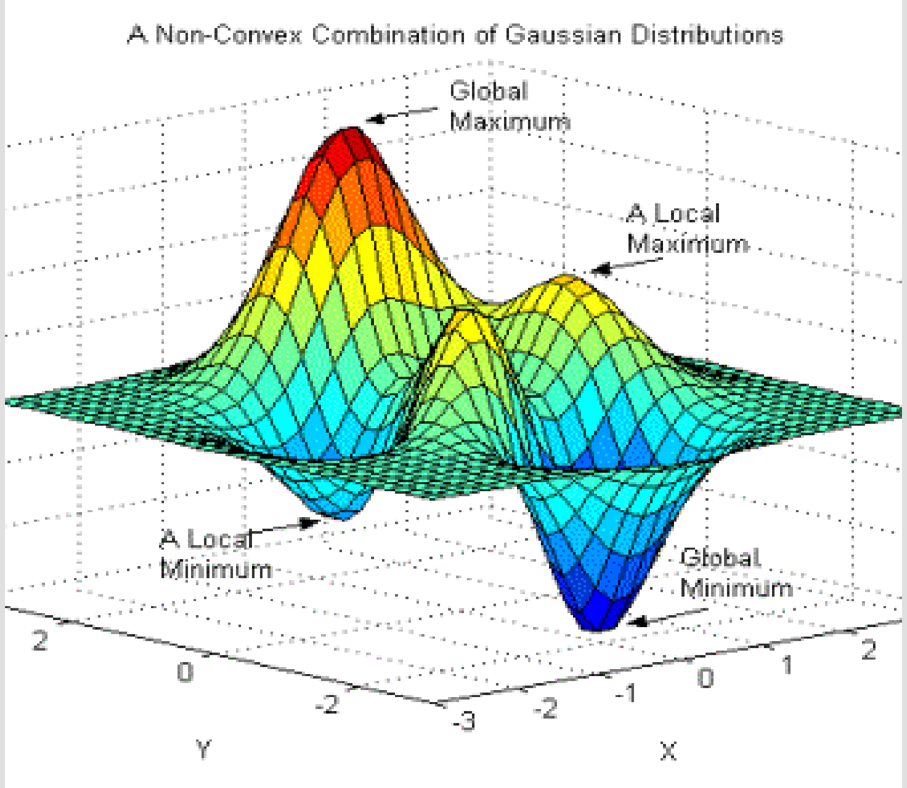
$$\max z = KL$$

$$\text{s.t. } 4K + L \leq 8$$

$$K, L \geq 0$$

INTRODUCTION TO NLP

- An NLP with no constraints is an **unconstrained NLP**
- Non-convexity can be an issue in an NLP model, because solver may not find the global minimum and may instead find a local minimum.
- The objective function or the constraints can be non-convex.
- A local maximum/minimum may not be the optimal solution to an NLP, like shown in the picture.



NLP WITH MULTIPLE LOCAL OPTIMAL SOLUTIONS

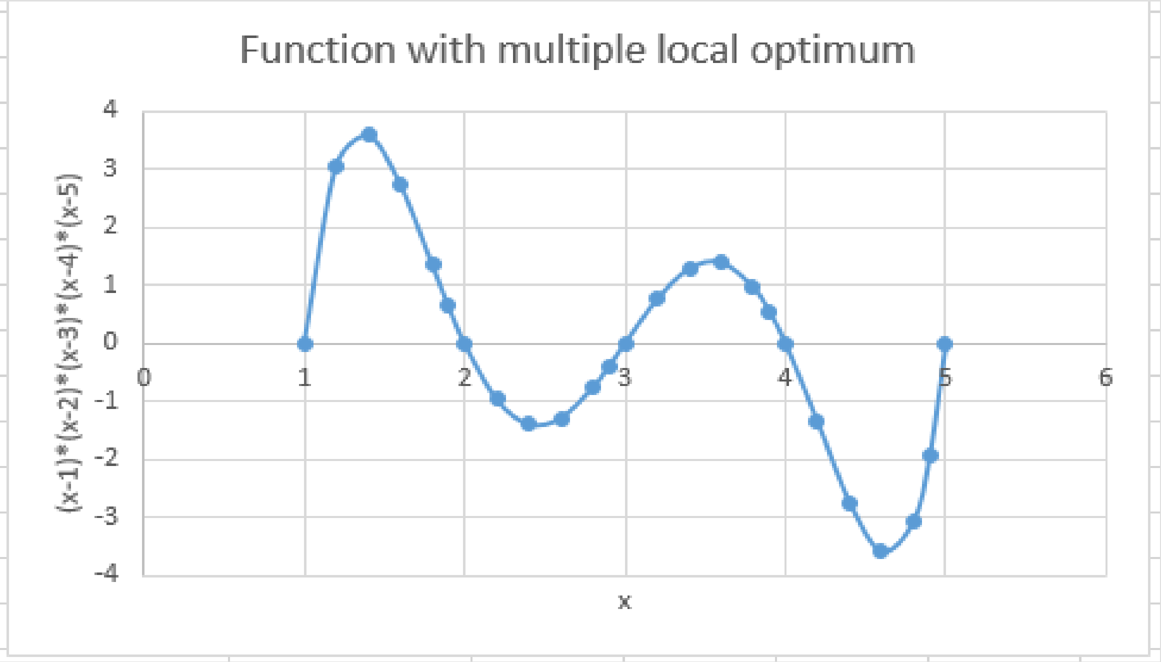
- For NLPs having multiple local optimal solutions, the Excel Solver may fail to find the optimal solution, because it may pick a local extremum that is not a global extremum. To illustrate, consider the following NLP:

$$\max z = (x-1)(x-2)(x-3)(x-4)(x-5)$$

$$s.t.\ x \geq 1$$

$$x \leq 5$$

Note there are two local maxima for this problem.



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USING SOLVER FOR AN NLP WITH MULTIPLE LOCAL OPTIMAL SOLUTIONS

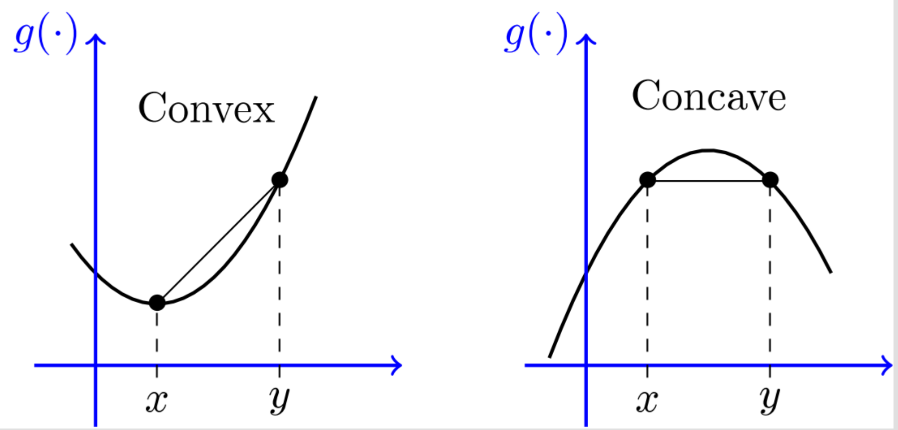
- The first time we find the optimal solution, which is x =3.5 and $z = 1.4$ and we can see on the graph in the previous slide that this point is not the best maximum point.
- The second time, we begin with changing the second constraint to $x \leq 2$ and we find the other local maximum, $x = 1.4$ and $z = 3.5$.
- The reason for this is that Solver use calculus-based methods (to be described later) to solve NLPs. Any calculus-based approach to solving NLPs runs the risk of finding a local extremum that is not a global extremum.

First Solution				
	x			
Dec. variables	3.573700358			
Objective function	1.41283319	5.049042476		
s. t.	1	3.573700358	>=	1
	1	3.573700358	<=	5
		wrong answer		

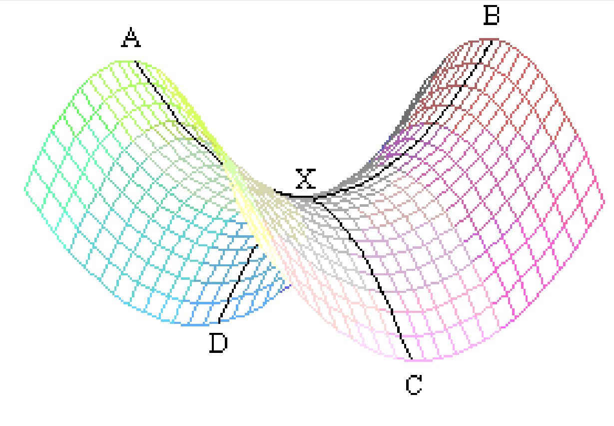
Second Solution				
	x			
Dec. variables	1.426299971			
Objective function	3.539958339	5.049042476		
s. t.	1	1.426299971	>=	1
	1	1.426299971	<=	2
		right answer		

CONVEX AND CONCAVE FUNCTIONS

- Convex and concave functions play an extremely important role in the study of nonlinear programming problems.
- We need to check whether the function is a convex or concave function.
- With a Single variable NLP, we can determine whether it is convex or concave with second derivatives.



Saddle point



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SINGLE VARIABLE CONVEX AND CONCAVE FUNCTIONS

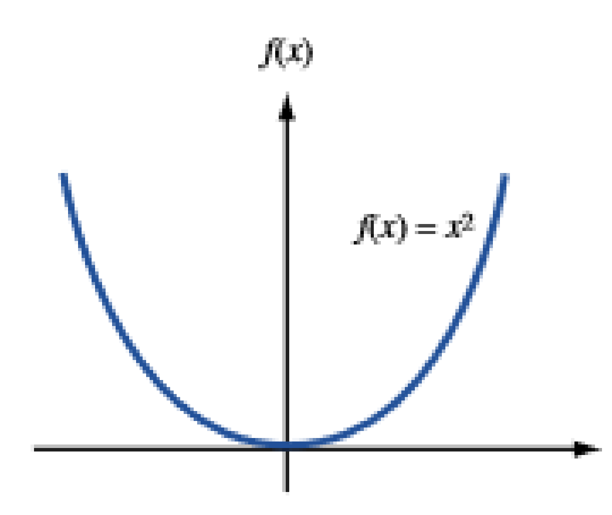
- For a single variable function, we can use second partial derivative to see whether a function is concave or convex:
  - If $f''(x) \geq 0$ then the function is convex (the function has a local minimum)
  - If $f''(x) \leq 0$ then the function is concave (the function has a local maximum)
  - For example:

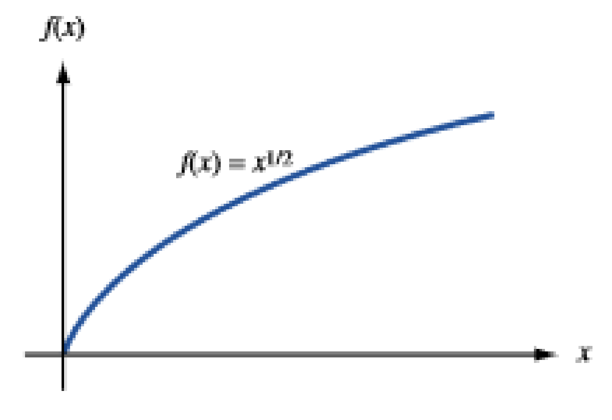
$$f(x) = x^2$$

$f''(x) = 2$ which is ≥ 0 showing that this function is convex.

$$f(x) = x^{\frac{1}{2}}$$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ which is ≤ 0 showing that this function is concave.





MULTI-VARIABLE CONVEX AND CONCAVE FUNCTIONS

- To determine whether a multi-variable function is convex or concave, we need to use the Hessian matrix with partial second derivatives.
- The determinant of the Hessian matrix denoted by $|H|$ can have three cases:
  - If $|H| > 0$ and $\frac{\partial^2 f}{\partial^2 x^2} > 0$ then $f(x, y)$ has a local minimum (Convex).
  - If $|H| > 0$ and $\frac{\partial^2 f}{\partial^2 x^2} < 0$ then $f(x, y)$ has a local maximum (Concave).
  - If $|H| < 0$ and then $f(x, y)$ has a saddle point.

FINDING THE HESSIAN MATRIX

- For example Let's calculated the hessian matrix at point (2; 1) for the function $f(x,y) = x^2y^2$
- $\frac{\partial f}{\partial x} = 2xy^2, \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^2 f}{\partial^2 x^2} = 2y^2$
- $\frac{\partial f}{\partial y} = x^2 2y, \frac{\partial^2 f}{\partial^2 y^2} = 2x^2$
- $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x} = 4xy$
- $H = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y^2} \end{bmatrix} = \begin{bmatrix} 2y^2 & 4xy \\ 4xy & 2x^2 \end{bmatrix}$

At point (2; 1):

$$\begin{bmatrix} 2(1)^2 & 4(2)(1) \\ 4(2)(1) & 2(2)^2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 8 & 8 \end{bmatrix}$$

First principal 2 and 8 are positive and the determinant = -48 which is negative.

$|H| < 0$ -> saddle point

Using the Hessian matrix to identify whether a function is convex or concave Example 1

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Show that $f(x_1, x_2) = -x_1^2 - x_1x_2 - 2x_2^2$ is a concave function on R^2

$$H(x_1, x_2) = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$$

- The first principal minors are the diagonal entries of the Hessian $(-2 \text{ and } -4)$. These are both nonpositive.
- The second principal minor is the determinant of $H(x_1, x_2)$ and equals $-2(-4) - (-1)(-1) = 7 > 0$. Thus, $f(x_1, x_2)$ is a concave function on R^2.

$|H| > 0$ and $\frac{\partial^2 f}{\partial^2 x^2} < 0$ -> concave

Using the Hessian matrix to identify whether a function is convex or concave Example 2

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Show that $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ is a convex function on $S = R^2$

$$H(x_1, x_2) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

- The first principal minors of the Hessian are the diagonal entries (both equal $2 \geq 0$).
- The second principal minor is $2(2) - 2(2) = 0 \geq 0$. For any point, all principal minors of H are nonnegative, so Thus, $f(x_1, x_2)$ is a convex function on R^2

$|H| > 0$ and $\frac{\partial^2 f}{\partial^2 x^2} > 0$ -> convex

Using the Hessian matrix to identify whether a function is convex or concave Example 3

Show that for $S = R^2, f(x_1, x_2) = x_1^2 - 3x_1x_2 + 2x_2^2$ is not a convex or a concave function.

$$H(x_1, x_2) = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$$

- The first principal minors of the Hessian are 2 and 4. Because both the first principal minors are positive, $f(x_1, x_2)$ cannot be concave.
- The second principal minor is $2(4) - (-3)(-3) = -1 < 0$. Thus, $f(x_1, x_2)$ cannot be convex. Together, these facts show that $f(x_1, x_2)$ cannot be a convex or a concave function.

$|H| < 0$ -> saddle point

Using the Hessian matrix to identify whether a function is convex or concave Example 4

Show that for $S = R^3, f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 - x_1x_2 - x_2x_3 - x_1x_3$ is a convex function.

$$H(x_1, x_2, x_3) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

- The first principal minors of the Hessian are 2, 2 and 4. Because the first principal minors are positive, $f\left(x_1, x_{2,} x_3\right)$ cannot be concave.
- We can find the second order principal minors doing the following:
  - By deleting row 1 and column 1 of Hessian, we find the second-order principal minor

$$det \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = 7 > 0$$

Using the Hessian matrix to identify whether a function is convex or concave Example 4

- By deleting row 2 and column 2 of Hessian, we find the second-order principal minor

$$det\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = 7 > 0$$

- By deleting row 3 and column 3 of Hessian, we find the second-order principal minor

$$det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 > 0$$

- The third-order principal minor is simply the determinant of the Hessian itself. Expanding by row 1 cofactors we find the third-order principal minor

$$2[(2)(4) - (-1)(-1)] - (-1)[(-1)(4) - (-1)(-1)] + (-1)[(-1)(-1) - (-1)(2)]$$
$$= 14 - 5 - 3 = 6 > 0$$

Because for all (x_1, x_2, x_3) all principal minors of the Hessian are non-negative, we have shown that $f(x_1, x_2, x_3)$ is a convex function on R^3.

SOLVING A NLP USING A NUMERICAL SEARCH METHOD

- Once we identified whether the unconstrained function is convex, concave or neither, we can use a Numerical search method.
- For solving an NLP with one variable, we can use the Golden Section Search algorithm
- For solving an NLP with *n* number of variable, we can use the Steepest ascent/descent algorithm.

Exercises

Using the Hessian matrix, show whether the following functions are convex, concave or if they have a saddle point.

- $f(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2$
- $f(x_1, x_2) = -x_1^2 - 2x_1x_2 - 2x_2^2$

Solutions:

- $H(x_1, x_2) = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ -> det=-1<0 -> Saddle point
- $H(x_1, x_2) = \begin{bmatrix} -2 & -2 \\ -2 & -4 \end{bmatrix}$ -> det=4>0 and -2 & -4 are both negative <0-> Concave

END

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