

# Workshop 6: General equilibrium

FIE463: Numerical Methods in Macroeconomics and Finance using Python

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See GitHub repository for notebooks and data:

<https://github.com/richardfoltyn/FIE463-V25>

## Exercise 1: Labor supply without capital

Recall the consumption & labor choice problem studied in the lecture. In this exercise, we revisit this setting but assume that there is no capital in the economy.

### Household problem

Households choose  $c$  and  $h$  to maximize utility

$$u(c, h) = \frac{c^{1-\gamma} - 1}{1-\gamma} - \psi \frac{h^{1+1/\theta}}{1+1/\theta}$$

subject to the budget constraint

$$c = w \cdot h + \pi$$

where  $\pi$  are firm profits which are distributed to all households equally. Since all households are identical, we assume that the economy is populated by a *single* representative household.

### Firm problem

We assume that firms have the decreasing-returns-to-scale production function

$$Y = zL^{1-\alpha}$$

where  $z$  is productivity (TFP) and labor  $L$  is the only input factor. Firms maximize profits  $\Pi$ ,

$$\max_L \Pi = zL^{1-\alpha} - wL$$

which gives rise to the first-order condition

$$\frac{\partial \Pi}{\partial L} = (1-\alpha)zL^{-\alpha} - w = 0 \implies w = (1-\alpha)zL^{-\alpha} \implies L = \left( \frac{(1-\alpha)z}{w} \right)^{\frac{1}{\alpha}}$$

This equation characterizes the firm's labor demand for a given wage rate  $w$ . For simplicity, we assume there is a *single* firm which takes wages and the price of output as given, where the latter is normalized to one.

## Equilibrium

The general equilibrium in this economy is a set of quantities  $(L, Y, \Pi, c, h, \pi)$  and the wage rate  $w$  which solve the household's and firm's problem, and the following conditions are satisfied:

- Labor market:  $L = h$  (hours  $h$  supplied by households equal labor  $L$  demanded by firms).
- Goods market:  $Y = c$  (the amount of goods  $c$  consumed by households equals aggregate output).
- Profits:  $\Pi = \pi$  (profits distributed by firms equal profits received by households).

## Analytical solution

By combining the household and firm first-order conditions, the problem can be reduced to a single equation in a single unknown,  $L$  (or  $h$ ):

$$L = \left( \frac{(1-\alpha)z^{1-\gamma}}{\psi} \right)^{\frac{1}{1/\theta + \alpha + \gamma(1-\alpha)}}$$

We will use this expression later to compare the numerical to the exact solution.

## Numerical solution

In the following, you are asked to adapt the code from the lecture to solve this problem. You can use the template file `workshop06_ex1.py` provided for this exercise to implement your solution.

1. Adapt the Parameters data class

```
@dataclass
class Parameters:
    pass
```

so that it contains the following parameters as attributes:  $\alpha = 0.36, z = 1, \gamma = 2, \psi = 1, \theta = 0.5$ .

2. Write the function `solve_hh(w, pi, par)` to solve the household problem using a minimizer for a given  $w$  and  $\pi$ .

Use the utility function `util(c, h, par)` defined in the template file for this purpose (this is the same function we used in the lecture).

3. Write the function `solve_firm(w, par)` which returns the firm's labor demand  $L$ , output  $Y$ , and profits  $\Pi$  for a given wage  $w$ .
4. Write the function `compute_labor_ex_demand(w, par)` which returns the excess labor demand for a given wage  $w$ .
5. Write the function `compute_equilibrium(par)` which uses a root-finder to locate the equilibrium, computes the equilibrium quantities  $(L, Y, \Pi, c, h, \pi)$  and prices  $(w, r)$  and stores these using an `Equilibrium` instance defined as

```
@dataclass
class Equilibrium:
    c: float = None
    h: float = None
    w: float = None
    L: float = None
    Y: float = None
    Pi: float = None
```

6. Compute the equilibrium using the function you just implemented and print the quantities and prices using the `print_equilibrium()` function implemented in the template file (you don't need to write this function yourself).

7. Write a function to compute the analytical solution derived above and compare it to your numerical result.

*Note:* Include the following cell magic to automatically reload any changes you make to the template file:

```
[1]: %load_ext autoreload
      %autoreload 2
```

## Exercise 2: Unequal distribution of profits

We now extend the setting from Exercise 1 and assume that a fraction of households solely live on their labor income (type 1), while profits are only distributed to a subset of households (type 2). We can think of these households as workers and entrepreneurs, respectively. We assume the economy is populated by  $N_1$  households of type 1 and  $N_2$  households of type 2.

### Household problem

All households have identical preferences which are unchanged from the previous exercise, but their budget constraints differ. For type-1 households, it is given by

$$c_1 = w \cdot h_1$$

whereas for type-2 households it's

$$c_2 = w \cdot h_2 + \pi_2$$

The subscripts in  $c_1$ ,  $c_2$ ,  $h_1$ ,  $h_2$ , and  $\pi_2$  index the household type since different households will choose different levels of consumption and labor supply.

### Firm problem

The firm problem remains unchanged from the previous exercise.

### Equilibrium

The general equilibrium in this economy is a set of quantities  $(L, Y, \Pi, c_1, c_2, h_1, h_2, \pi_2)$  and the wage rate  $w$  which solve the household's and firm's problem, and the following conditions are satisfied:

- Labor market clearing:  $L = N_1 h_1 + N_2 h_2$  (hours supplied by households equal labor  $L$  demanded by firms).
- Goods market clearing:  $Y = N_1 c_1 + N_2 c_2$  (the amount of goods consumed by households equals aggregate output).
- Profits:  $\Pi = N_2 \pi_2$  (profits distributed by firms equal profits received by type-2 households).

## Numerical solution

In the following, you are asked to adapt the code you wrote for exercise 1 to solve the modified problem. The new solution only requires changes at a few selected points to take into account the unequal distribution of profits. You can use the template file `workshop06_ex2.py` provided for this exercise.

1. Adapt the `Parameters` class to include the two new parameters  $N_1$  and  $N_2$  which represent the number of type-1 and type-2 households, respectively. Set  $N_1 = 5$  and  $N_2 = 5$ .

For the remaining parameters, use the same values as in exercise 1.

2. Write the function `compute_labor_ex_demand(w, par)` which returns the excess labor demand for given  $w$ . Use the function `solve_hh()` and `solve_firm()` you wrote for exercise 1 to solve this task.

*Hint:* Don't copy the implementations for `solve_hh()` and `solve_firm()` but directly import them from the module which contains the solution for exercise 1.

3. Write the function `compute_equilibrium(par)` which uses a root-finder to locate the equilibrium, computes the equilibrium quantities  $(L, Y, \Pi, c_1, h_1, c_2, h_2, \pi_2)$  and the wage rate  $w$ , and stores these using an `Equilibrium` instance defined as

```
@dataclass
class Equilibrium:
    par: Parameters = None
    c1: float = None
    h1: float = None
    c2: float = None
    h2: float = None
    pi2: float = None
    w: float = None
    L: float = None
    Y: float = None
    Pi: float = None
```

4. Compute the equilibrium using the function you just implemented and print the quantities and prices using the `print_equilibrium()` function implemented in the template file.

How does the unequal distribution of profits affect consumption and labor supply of type-1 vs type-2 households?

5. You are interested to see if and how the allocation and prices in the economy change as we vary the number of type-1 and type-2 households. Assume that there are a total of  $N = N_1 + N_2 = 10$  households in the economy

- Using the function `compute_equilibrium()` you wrote earlier, compute the equilibrium when  $N_1$  takes on the integer values from  $0, \dots, 9$  and  $N_2 = N - N_1$ .
- Create a graph with four panels ( $2 \times 2$ ) which shows the aggregates  $Y$ ,  $L$ ,  $\Pi$ , and  $w$  as a function of  $N_1$ .
- Create a graph with three columns which shows  $(c_1, c_2)$  in the first,  $(h_1, h_2)$  in the second, and  $\pi_2$  in the third column. Use different colors and line styles to distinguish household types and include a legend.

What do you conclude about the effects of inequality on the equilibrium allocation and prices?

## Bonus: using analytical results and root-finding

Unlike the previous exercise, this economy no longer has a closed-form solution for the equilibrium quantities. From the households' first-order conditions, we can derive that the equilibrium is characterized by the two non-linear equations

$$\begin{aligned}(wh_1)^{-\gamma} &= \psi \frac{h_1^{1/\theta}}{w} \\ (wh_1 + \pi_2)^{-\gamma} &= \psi \frac{h_2^{1/\theta}}{w}\end{aligned}$$

Note that  $w$  and  $\pi_2 = \frac{\Pi}{N_2}$  itself are functions of  $(h_1, h_2)$  via the labor market clearing  $L = N_1h_1 + N_2h_2$ :

$$\begin{aligned}w &= (1 - \alpha)zL^{-\alpha} = (1 - \alpha)z(N_1h_1 + N_2h_2)^{-\alpha} \\ \Pi &= \alpha zL^{1-\alpha} = \alpha z(N_1h_1 + N_2h_2)^{1-\alpha}\end{aligned}$$

We can substitute these two equations into the non-linear equation system above and numerically find a solution  $(h_1, h_2)$  that satisfies these conditions.

1. Use the multivariate root-finder `root()` from `scipy.optimize` with `method='hybr'` to solve the above equation system.

To do this, you need to write a function which takes as argument a vector  $x$  which contains the values  $(h_1, h_2)$ , and return the errors in the two first-order conditions, i.e., a vector that contains the left-hand minus the right-hand side for each of the two equations.

2. Make sure the results obtained from this approach are the same as in the main exercise.