A Refresher on Some Mathematical Subjects

or how to stay fresh all the time?

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Topics Covered

Linear Algebra

Multivariate Analysis

Probability Theory

Outline

Linear Algebra

Multivariate Analysis

Probability Theory

Matrix Multiplication

- Let $A \in \mathbb{R}^{m\times n}, x, b \in \mathbb{R}^n$.
  $Ax = b \iff b$ is a linear combination of columns of A.

$$b = \sum_{j=1}^{n} x_j \, a_j$$

- **Outer product.** Let $u \in \mathbb{R}^n, v \in \mathbb{R}^m$. Then we call $uv^\top$ the outer product of u and v:

$$uv^\top = \begin{bmatrix} v_1 u & v_2 u & \cdots & v_m u \end{bmatrix}$$

Range, Kernel and Rank

- **Range** of a matrix A is the span of its columns.
- **Kernel** or *Null Space* of a matrix A is the space of all x such that $Ax = \mathbf{0}$.

$$\ker(A) = \{\mathbf{0}\} \iff A \text{ is injective.}$$

- **Rank** of a matrix A is the dimension of its range. It's equal to dim(col space) = dim(row space).

$$\dim\ker(A) + \operatorname{rank}(A) = \#\text{cols of A}$$

Inverse

- A is *invertible* or *nonsingular* iff it is square and full rank. Equivalently, having $\det(A) \neq 0$, or $\ker(A) = \{\mathbf{0}\}$.
- Multiplication by A^{-1} is a change of basis:

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Coefficients of}\\ \text{expansion of } b\\ \text{in } \{e_1, \ldots, e_n\}\end{array}} \arrow[r, bend left, "\times A^{-1}"] & \boxed{\begin{array}{c}\text{Coefficients of}\\ \text{expansion of } b\\ \text{in } \{a_1, \ldots, a_n\}\end{array}} \arrow[l, bend left, "\times A"] \end{tikzcd}$$

Orthogonality

- **Inner product** of two vectors x and y in $\mathbb{R}^n$ is defined as $\langle x, y \rangle = x^\top y = \sum x_i y_i$.
- Two vectors are *orthogonal* if their inner product is zero.
- Let $\{q_1, \ldots, q_n\}$ be a set of pairwise orthogonal vectors in $\mathbb{R}^n$. Then

$$\forall v \in \mathbb{R}^n : \quad v = \sum_{i=1}^{n} (q_i^\top v) q_i = \sum_{i=1}^{n} (q_i q_i^\top) v$$

Note: $q_i q_i^\top$ is orthogonal projection onto direction q_i, which is a rank-one operator.

Unitary Matrices

- A matrix U is **unitary** or *orthogonal* if $U^\top U = I$, *i.e.* $\langle u_i, u_j \rangle = \delta_{i,j}$.
- If U is unitary, then it preserves angles,

$$\langle Ux, Uy \rangle = \langle x, y \rangle \,,$$

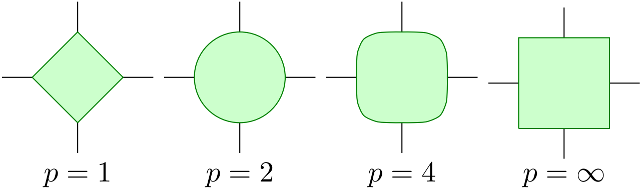
and also lengths,

$$\|Ux\| = \|x\|.$$

If $\det(U) = 1$, then U is a rigid rotation, and if $\det(U) = -1$, then U is a reflection.

Norms

- A function $\|\cdot\|:\mathbb{R}^n\to\mathbb{R}$, satisfying
  - $\forall x\in\mathbb{R}^n:\ \|x\|\geq 0,\ \|x\|=0 \iff x=\mathbf{0}$,
  - $\forall\lambda\in\mathbb{R}:\ \|\lambda x\|=|\lambda|\|x\|$,
  - $\forall x,y\in\mathbb{R}^n:\ \|x+y\|\leq\|x\|+\|y\|$.
- The class of p-norms:
  - $\|x\|_1=\sum_{i=1}^n|x_i|$
  - $\|x\|_p=\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}$ for $p\in(0,1)$
  - $\|x\|_\infty=\max_{1\leq i\leq n}|x_i|$
- Unit balls:



- The *Hölder inequality.* (Case $p=q=2$ is known as *Cauchy-Schwartz* inequality)

$$|\langle x,y\rangle|\leq\|x\|_p\|y\|_q,\ \text{for } 1/p+1/q=1.$$

Matrix Norms

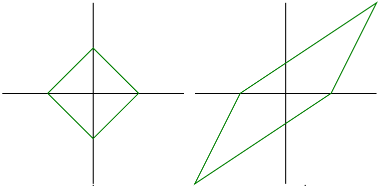
- We can view a matrix as a *linear operator*, and we can define norms on the space of linear operators. A famous norm is the **operator norm** of a matrix A. Let $A : (\mathbb{R}^n, \|\cdot\|_p) \to (\mathbb{R}^m, \|\cdot\|_q)$. Then we define

$$\|A\|_{(p,q)} := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|_q}{\|x\|_p} = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_p = 1}} \|Ax\|_q$$

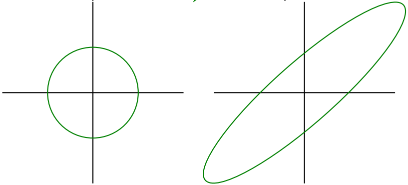
- Defines the maximum *stretch* of the unit ball.
- When $p = q$ we just write $\|A\|_p$.
  e.g. $\|A\|_2$ is the largest *singular value* of A.

Matrix Norms (Example)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$



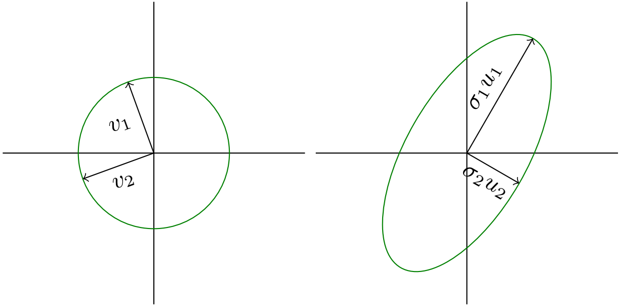
$$\|A\|_1 = 4$$



$$\|A\|_2 \approx 2.92$$

Singular Values and Singular Vectors

- The image of the unit sphere under a linear transform is always a hyperellipse.



- We have $Av_i = \sigma_i u_i$.
- v_1, v_2 are called *right singular vectors* and u_1, u_2 the *left singular vectors*. Also σ_1, σ_2 are *singular values*.

Singular Value Decomposition (SVD)

- We can decompose any matrix A in the form

$$A = U\Sigma V^\top,$$

where U and V are unitary and Σ is a diagonal matrix, *i.e.*

$$U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix},$$

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix},$$

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

Eigenvalues and Eigenvectors

- If for some vector $v \neq \mathbf{0}$ we have $Av = \lambda v$ then v is an **eigenvector** of A associated to the **eigenvalue** λ. In this case we have $(A - \lambda I)v = \mathbf{0}$, and this can only happen when $\det(A - \lambda I) = 0$.
- If A is real and symmetric, then all eigenvalues are real and eigenvectors can be chosen to be orthogonal to each other.
- There is also an **Eigenvalue Decomposition** for a diagonalizable square matrix:

$$A = X^{-1} \Lambda X,$$

which is different from SVD.

Outline

Notion of the Derivative

- As you may recall, for real differentiable functions $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$, the derivative $f'(x) = \frac{df}{dx}(x)$ is the *slope* of the **tangent** line at the point x. This notion is geometrically plausible, but unfortunately hard to generalize.
- A better notion for derivative would be *the best linear approximation* of a function near a point x.
- Formally, let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in \Omega$. We call f to be differentiable at x_0 iff there is a *linear* function $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$, for which we have

$$\lim_{\|h\| \to 0} \frac{|f(x_0 + h) - f(x_0) - Df(x_0)h|}{\|h\|} = 0$$

- We call $Df(x_0)$ the derivative or differential of f at the point x_0.

Derivative

- Take $f(x) = (f_1(x), \dots, f_m(x))$. We call f_i the **components** of f. If the derivative exists, then we have

$$Df(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right]_{1 \le i \le m, 1 \le j \le n},$$

  where $\frac{\partial f_i}{\partial x_j}(x_0)$ is the **partial derivative** of f_i w.r.t. x_j at the point x_0, namely

$$\frac{\partial f_i}{\partial x_j}(x_0) = \lim_{\epsilon \to 0} \frac{f_i(x_0 + \epsilon e_j) - f_i(x_0)}{\epsilon}.$$

- If for a function f, all partial derivatives exist and *are continuous* at the point x_0 then f is continuously differentiable at x_0 and its derivative would be the matrix $Df(x_0)$ above.

Notion of the Gradient

- Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real-valued differentiable function. Then we have

$$Df(x_0) = \left[\frac{\partial f}{\partial x_1}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0)\right] =: \nabla f(x_0)^\top$$

- The **gradient** of f has the following properties:
  - It points to the direction in which f has the maximum rate of increase. (likewise, $-\nabla f$ points to the direction of maximum decrease)
  - It is always orthogonal to the contour line $\{x \ : \ f(x) = f(x_0)\}$.
  - If f attains a local minimum (or maximum) at some point x_0, then $\nabla f(x_0) = \mathbf{0}$. (The **first derivative test**)
  - So we have the following (first-order) approximation for x sufficiently close to x_0:

$$f(x) \approx f(x_0) + \nabla f(x_0)^\top (x - x_0)$$

Chain Rule

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : \Omega' \subseteq \mathbb{R}^p \to \mathbb{R}^n$. Assume g is differentiable at x_0 and $g(x_0) \in \Omega$ and f is differentiable at $g(x_0)$. Then $f \circ g : \Omega' \to \mathbb{R}^m$ is differntiable at x_0 and we have

$$D(f \circ g)(x_0) = Df(g(x_0)) \circ Dg(x_0)$$

A good example is the directional derivatives. Assume $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Let $u \in \mathbb{R}^n$. We want to find the rate of change of f in the direction of u, *i.e.* $\frac{d}{dt} f(x_0 + tu)$ for $t = 0$. Define $g(t) = x_0 + tu$. We have f

$$D(f \circ g)(0) = Df(g(0)) \circ Dg(0) = \nabla f(x_0)^\top u.$$

Second-Order Approximation

- The error of the first-order approximation is sub-linear, *i.e.* $\lim_{\|h\|\to 0} R(h)/\|h\| = 0$. Can we do better?
- In the single-variable regime, we can use quadratic polynomials to approximate a function in some neighborhood.
- We need to understand what are quadratic functions in multi-dimensional case and try to approximate our function.
- We expect our new approximation's error has a faster convergence to 0 than $\|h\|^2$.

Multidimensional Quadratic Functions

- Let A be an $n \times n$ symmetric matrix. We define the **quadratic form** induced by A to be

$$f \ : \ x \in \mathbb{R}^n \mapsto x^\top A x.$$

- Note that the quadratic form is a weighted sum of all possible second degree terms, *e.g.* $x_i x_j$ or x_i^2.
- We call A a **positive (negative) definite** matrix, iff all eigenvalues of A are positive (negative). We say A is a **positive semi-definite** or p.s.d., if all eigenvalues are nonnegative.
- If A is p.s.d., then the contour levels of f are concentric ellipsoids.
- One can prove that $\nabla f(x) = 2Ax$.

The Hessian

- Assume $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice-differentiable at x_0. Then there exists some symmetric matrix $D^2 f(x_0)$ which we call the **Hessian** of f at x_0, with

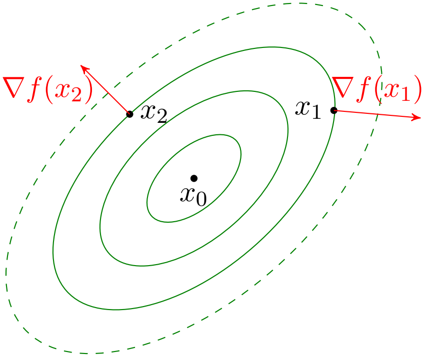
  $$D^2 f(x_0) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right]_{1 \leq i,j \leq n},$$

  where $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right)(x_0)$.
- We have the following (second-order) approximation for x sufficiently close to x_0:

  $$f(x) \approx f(x_0) + \nabla f(x_0)^\top (x - x_0) + \frac{1}{2}(x - x_0)^\top D^2 f(x_0)(x - x_0)$$

All-in-one Picture

In this example, we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is twice-differentiable, having a local minimum at x_0. This picture demonstrates a neighborhood of x_0:



Outline

Linear Algebra

Multivariate Analysis

Probability Theory

Basic Notations

- Let Ω be a set, which we call the **sample space**. This set contains all possible outcomes of an experiment. Note that this set depends on how we model the problem, *e.g.* in the problem of a single dart throw to a circular dartboard, we have the following possibilities:
  - $\Omega_1 = \mathbb{R}^2$ (exact place of landing),
  - $\Omega_2 = \{\text{Hit}, \text{Miss}\}$ (indicator),
  - $\Omega_3 = \{0, 10, 20, \ldots, 100\}$ (score of the throw).
- A family $\mathcal{F}$ of subsets of Ω, called the **events**, are also interesting for us. Rather than asking whether a certain outcome has happen, we want to ask harder questions. For example if we want to ask whether the score is higher than 60 or not, we are asking about the event $\{80, 100\}$.
- We also want to apply rules of logic. Taking "and" is translated to intersection of events, "or" is union, and "not" is complements. So we desire our family of events to be closed under these operations.

Basic Notations

- We also assign a belief (which we may obtain through experiments, or just arbitrary) to each of these events. However, this assignment should be consistent. It is agreed that the following rules suffice to model our philosophy about beliefs and probabilities. If we assign to each event A, a probability $\mathbb{P}(A)$, we should have:
  - $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$,
  - $\mathbb{P}(\Omega) = 1$,
  - If $A \cap B = \emptyset$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.
    - If $A_1, A_2, \ldots$ are pairwise disjoint, $\mathbb{P}(\bigcup A_i) = \sum \mathbb{P}(A_i)$.
- We call $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.
- Examples:
  - $\Omega = \{1, \ldots, 6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\{i\}) = \frac{1}{6}$ for all $i \in \Omega$.
  - $\Omega = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(\mathbb{N})$, $\mathbb{P}(\{i\}) = \frac{1}{2^i}$ for all $i \in \mathbb{N}$.
  - $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, $\mathbb{P}((a, b)) = b - a$.

Random Variables

- RVs are different points-of-view to the same probability space. We call a function $X : \Omega \to \mathbb{R}$ a **random variable**. We now ask our questions through the lens of X.
- In the dart throwing problem, let's suppose $\Omega = \mathbb{D}^2$, $\mathcal{F} = \mathcal{B}(\mathbb{D}^2)$ and $\mathbb{P}$ be the uniform measure, meaning that $\mathbb{P}(A) = \text{area}(A)/\text{area}(\mathbb{D}^2)$. Take X to be

  $$X(\omega) = \text{distance of } \omega \text{ to the center of dartboard.}$$

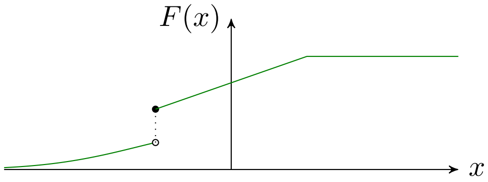
  Now we can ask, whether the throw was further than 0.3 cm, via asking about the event $\{\omega : X(\omega) \geq 0.3\}$. For brevity we write this event as $\{X \geq 0.3\}$.
- Link between X and Ω is the **inverse image** X^{-1}.
- We can only ask a question A from X if the inverse image of A is already inside $\mathcal{F}$, *i.e.* $X^{-1}(A) \in \mathcal{F}$.

Distributions

- Any random variable induces a probability space on $\mathbb{R}$, *i.e.* for each interval I we assign the probability $\mathbb{P}(X^{-1}(I))$.
- This defines a right-continuous nondecreasing function $F : \mathbb{R} \to [0, 1]$,

$$F(x) := \mathbb{P}(\{X \leq x\}) = \mathbb{P}(X \leq x) = \mathbb{P}(X^{-1}((-\infty, x]))),$$

  which we call the **cumulative distribution function** (or CDF).
  - We have $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$.
  - $\mathbb{P}(a < X \leq b) = F(b) - F(a)$.



Densities

- Exactly like physical concept of density, we can define density for a random variable (if it is regular enough). For a random variable X we define

$$f(x) := \lim_{\substack{|I|\to 0 \\ x\in I}} \frac{\mathbb{P}(X \in I)}{|I|},$$

where $\mathbb{P}(\cdots)$ replaces "mass" and $|I|$ is in place of "volume".

- If F is differentiable at x, then $f(x) = F'(x)$ and by FTC

$$F(x) = \int_{-\infty}^{x} f(y)dy.$$

- The value of $f(x)$ can be used to estimate probabilities, *e.g.* if $f(x) = 2$, then for a small interval I of size ϵ around x, we know that $\mathbb{P}(X \in I) \approx 2\epsilon$.

Motto!

With random variables and their densities (or distributions) we can even forget about Ω and just look at the probability space that is defined on $\mathbb{R}$ via X. So the following holds:

> *If* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* X *are known, we can find the distribution of* X *(and its density, if it exists).*
>
> *If we know* $F(x)$ *or* $f(x)$*, we can build a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a random variable* X*, such that the distribution of* X *is exactly* $F(x)$*.*

Joint Distribution and Marginals

- Let X, Y be two random variables over the same probability space. Then we can define the **joint distribution** as

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y).$$

- The **joint density** can also be defined as

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

- Given the joint distribution, one can find the distribution of each of variables by **marginalizing**:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad F_X(x) = F_{X,Y}(x, \infty)$$

Independence

- Two "events" A and B are said to be independent if we have
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$
- Two "RVs" X and Y are independent if their joint distribution function factorizes, *i.e.*
$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$
- A sequence of n RVs $X_1, \ldots, X_n$ are said to be independent, iff their joint distribution factorizes. Note that if X_i are pairwise independent, it does *not* follow that they are independent.

Conditional Probability

- How does information affect our belief?
- "knowing" that an event B has occured, what is the probability of A happening?
- Denote by $\mathbb{P}(A|B)$ by conditional probability of A given B.
- This can be defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad (\mathbb{P}(B) \neq 0)$$

- **Law of total probability.** Let $A_1, \ldots, A_n$ be a partition of Ω. We have

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i).$$

- **Bayes Rule.**

$$\mathbb{P}(A|B) = \mathbb{P}(B|A)\frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Bayes Rule and the Chain Rule

- Let $A_1, \ldots, A_n$ be a partition of Ω. We have

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n} \mathbb{P}(B|A_j)\mathbb{P}(A_j)}.$$

- **Chain Rule.** Let $A_1, \ldots, A_n$ be arbitrary events. We have

$$\mathbb{P}(A_1, \ldots, A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1, A_2) \cdots \mathbb{P}(A_n|A_1, \ldots, A_{n-}$$

Expected Value

- If I do an experiments multiple times and look at my RV's value, what does the average look like?
- This average converges (as I make more experiments) to a certain number, called **expected value** or **mean** of X.
- By definition,

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\mathbb{P}(d\omega) = \int_{\mathbb{R}} x f(x) dx$$

- Expected value is linear! $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, even when X and Y are not independent. For a distribution, we usually use μ to represent its mean.

Variance

- Variance is a measure of scattering.
- "10^6 pockets, that only one of them has a golden coin with value 10^6" vs. "1 pocket with a coin of value 1". Which one do you choose?
- Expected value is the same, the second one has lower variance. . .
- Can be defined as

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \geq 0$$

- For a distribution, we show its variance by σ^2.

Law of Large Numbers

- Let's say, you did an experiment infinitely many times. The outcomes are listed as $X_1, X_2, \ldots$. We assume that each time we did the experiment *fresh*! Meaning that X_i does not depend on each other. Usually we say X_i are **iid RVs**; meaning that they have the same distribution and are independent of eachother.
- Up to time n, we take the average of what we saw,

$$\bar{X}_n := (X_1 + \cdots + X_n)/n.$$

- WLLN states that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) = 0$$

- SLLN states that

$$\mathbb{P}\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$$

Central Limit Theorem(s)

- If $X_1, X_2, \ldots$ is an iid seq. of RVs, having mean μ and variance σ^2, we have the following:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

- $\xrightarrow{d}$ means **convergence in distribution**, *i.e.* pointwise convergence of distribution functions.
- $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2.
- *Basically*, for large n, every scaled distribution is essentially normal distribution!
- Good for creating *approximate confidence intervals*
- Caveat! Speed of convergence, Uniform convergence, regularity conditions…

What was not covered!

Please check these on the web! If you had troubles, you can contact me:

- Covariance, Covariance Matrix of n RVs
- Multivariate Normal Distribution

All the best!