1 Introduction

The Jacobi relaxation method and the Gauss-Seidel method were implemented to solve the discretized Poisson equation in two dimensions for point charges in a grounded box. The two methods were compared with each other in terms of time to solution and number of iterations required. Also, the time to solution was compared to a high performance library Cholesky solver.

2 Algorithm Description

For all cases, the two-dimensional Poisson equation(equation 1) on Ω is discretized using second-order central finite differences in both the x- and the y-direction (equation 2). Both axes share a common grid spacing of $\Delta x = \frac{1}{N+1}$ where N is the number of interior points per axis direction on the grid. Following the established finite difference method procedure to employ natural ordering, the left-hand side of equation 2 then can be written in form of an $N * N \times N * N$ matrix $\mathbf{A}$ while the values of ϕ on the grid get unrolled into a vector $\mathbf{b}$ of size $N * N$ on the right-hand side (equation 3).
The resulting matrix A is both sparse and block tridiagonal.

$$\Delta\Phi = -\phi \quad \text{on } \Omega = (0,1) \times (0,1) \tag{1}$$

$$4x_{i,j} - x_{i-1,j} - x_{x+1,j} - x_{i,j-1} - x_{i,j+1} = -(\Delta x)^2 \cdot \rho(x_{i,j}) \tag{2}$$

$$Ax = b \tag{3}$$

2.1 Jacobi relaxation method

In matrix form, the Jacobi method works by decomposing the matrix $\mathbf{A}$ into a matrix $\mathbf{D}$ consisting only of $\mathbf{A}$'s main diagonal and a remainder matrix $\mathbf{R} = \mathbf{A} - \mathbf{D}$. Starting with an initial guess for $\mathbf{x} = (1, 1, \ldots, 1)^{\mathrm{T}}$, we iterate $\mathbf{x}^{(t+1)} = \mathbf{D}^{-1}(\mathbf{b} - \mathbf{R}\mathbf{x}^{(t)})$ until $\left\|\mathbf{x}^{(t+1)} - \mathbf{x}^{(t)}\right\|$ becomes smaller than a chosen convergence treshold in a chosen norm.

2.2 Gauss-Seidel method

Similar to the Jacobi relaxation method, the matrix $\mathbf{A}$ is decomposed into a lower triagonal matrix $\mathbf{L}$ and a strictly upper triagonal matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{L} + \mathbf{U}$. Completely analogous to the Jacobi relaxation method, the iteration step is $\mathbf{x}^{(t+1)} = \mathbf{L}^{-1}(\mathbf{b} - \mathbf{U}\mathbf{x}^{(t)})$.

3 Results

The program was implemented as described above and submitted with this report.
Both methods reached the set conversion treshold of $\left\|\mathbf{x}^{(t+1)} - \mathbf{x}^{(t)}\right\|_2 \leq 10^{-4}$.
The Jacobi relaxation method took $t = 3478$ iterations but on average only $\sim$ 45ms to do so, while the Gauss-Seidel method only took $t = 1922$ iterations but $\sim$ 6400ms to reach the same treshold. For comparison, Eigen's optimized library Cholesky method solver obtained the reference solution in 9ms.
Both methods' solutions reached similar deviation from the reference Cholesky solution:
$\left\|\mathbf{x}^*_{\text{Jacobi}} - \mathbf{x}^*_{\text{Cholesky}}\right\|_2 \approxeq \left\|\mathbf{x}^*_{\text{Gauss-Seidel}} - \mathbf{x}^*_{\text{Cholesky}}\right\|_2 \approxeq 0.05$. The heat maps for all three solvers are shown in figures 1, 2 and 3.

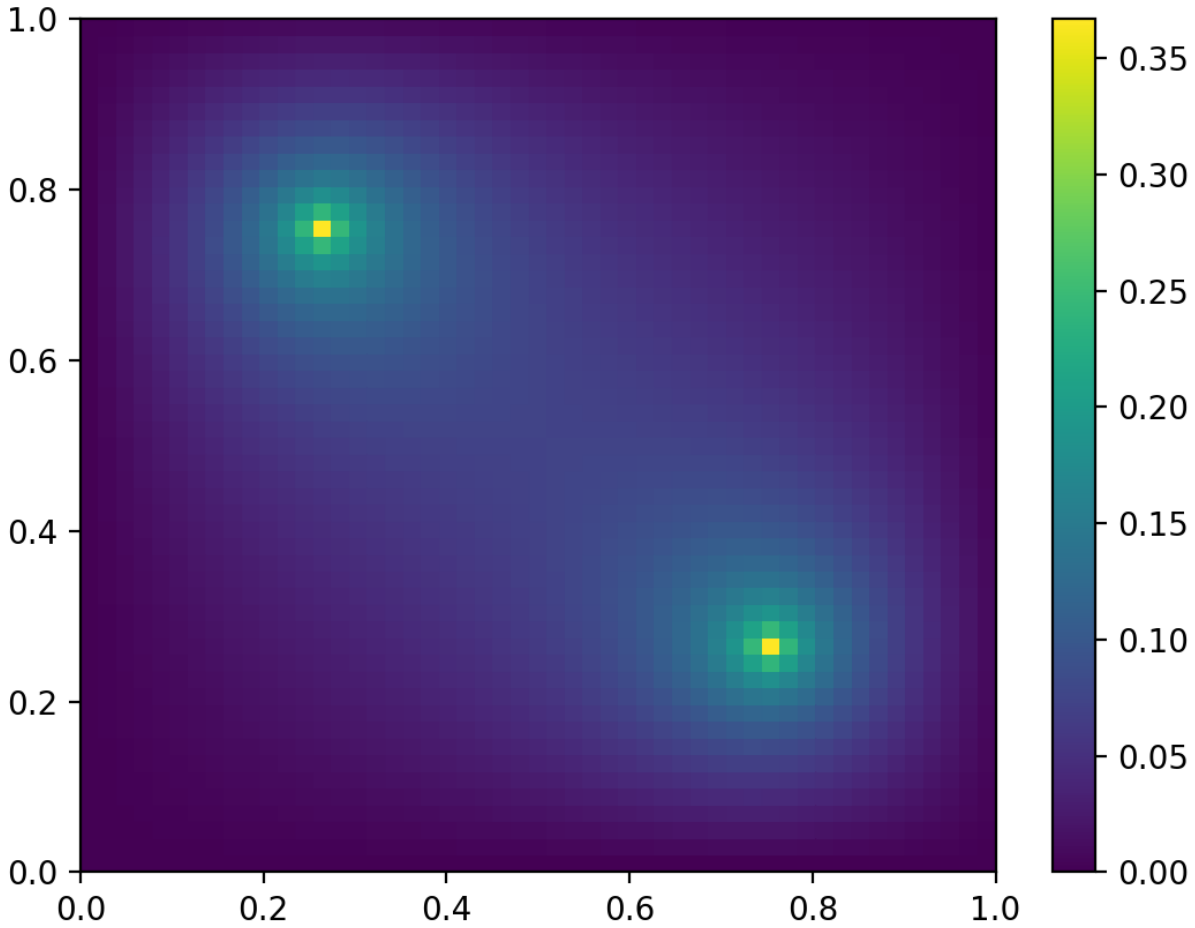

Figure 1: Cholesky solver reference solution for Poisson equation with point charges at $(0.25, 0.75)$, $(0.75, 0.25)$.

4 Discussion

The results are as expected. Further consideration should be given to investigating why the Gauss-Seidel solver in this implementation is an order of magnitude slower than the Jacobi relaxation solver.

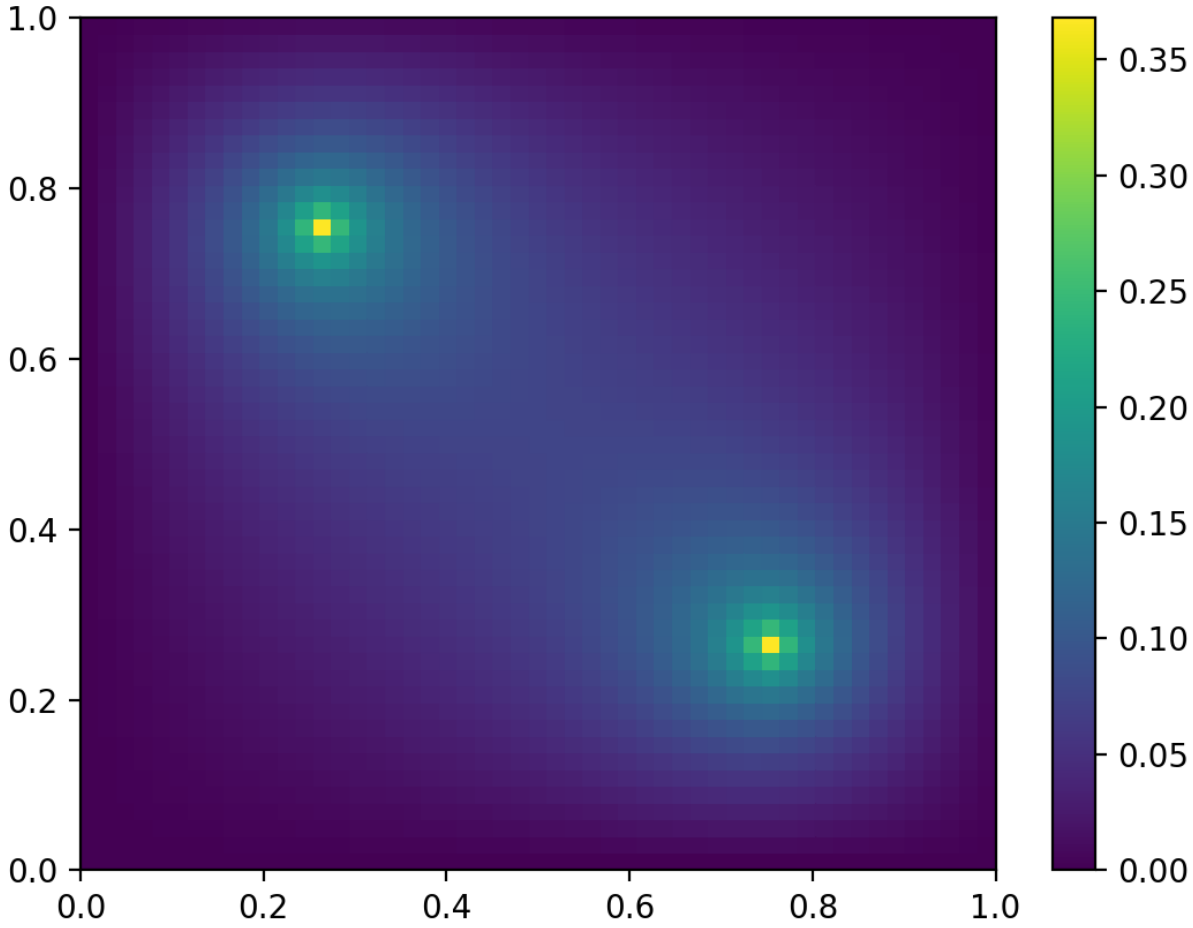

Figure 2: Jacobi relaxation solver solution for Poisson equation with point charges at $(0.25, 0.75)$, $(0.75, 0.25)$.

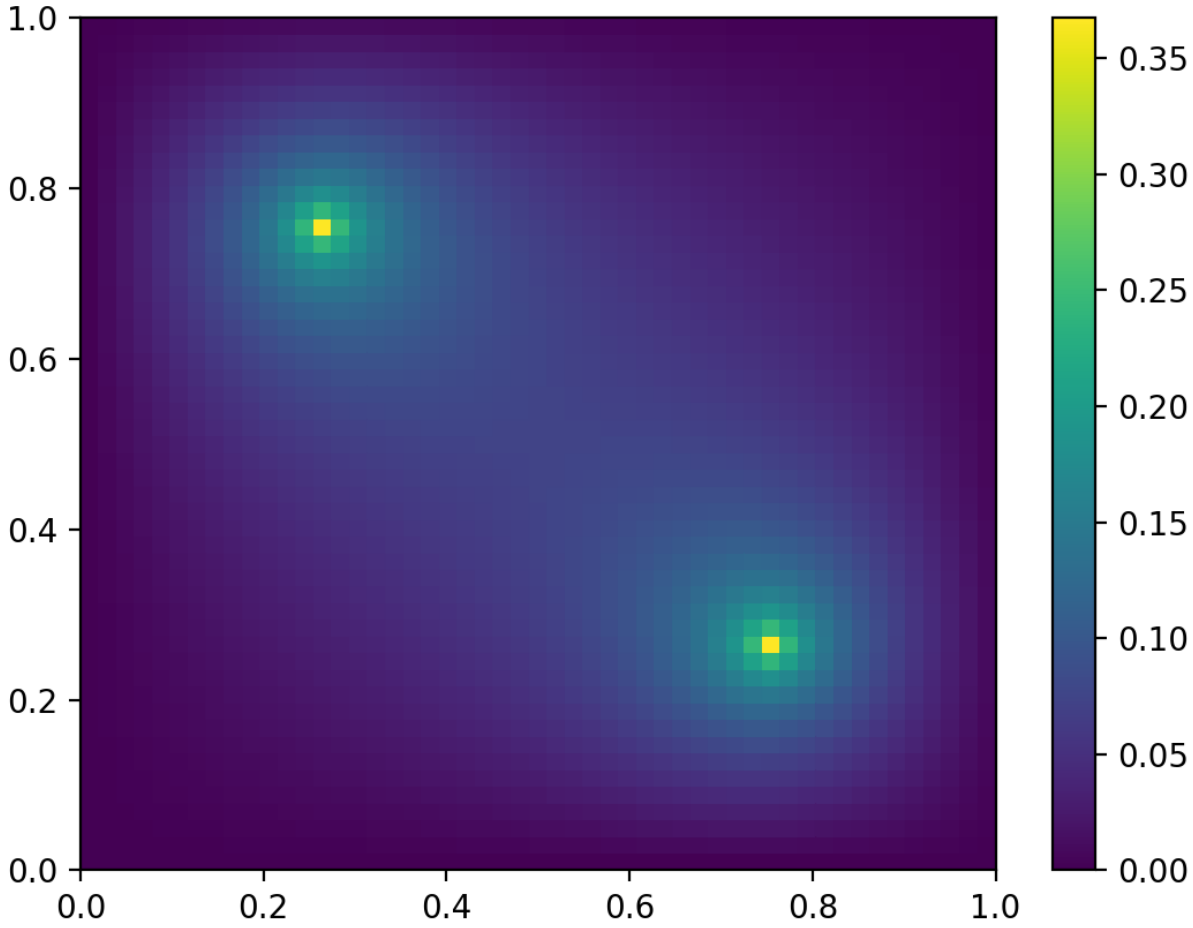

Figure 3: Gauss-Seidel solver solution for Poisson equation with point charges at $(0.25, 0.75)$, $(0.75, 0.25)$.