HPCSE II

HW 3 - PART 1

BEAT HUBMANN

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1 / INVERSION METHOD FOR GAUSSIAN SAMPLING

GIVEN: RV $X = \sqrt{2} \cdot \mathrm{erf}^{-1}(2U-1)$, $U \sim \mathrm{Unif}(0,1)$,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

/a SHOW THAT: $X \sim \mathcal{N}(0,1)$

NOTING THAT $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ AND WRITING

$t^2 = \frac{s^2}{2} \Rightarrow t = \frac{s}{\sqrt{2}}$ AS $t \geq 0$ AND $dt = \frac{1}{\sqrt{2}} ds$ WE CAN WRITE:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-\frac{s^2}{2}} ds = 2\Bigg[\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}x} e^{-\frac{s^2}{2}} ds}_{\Phi(\sqrt{2}x)} - \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{s^2}{2}} ds}_{\Phi(0)} \Bigg]$$

WHERE $\Phi(x)$ IS THE CDF OF THE STANDARD NORMAL DISTRIBUTION:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

AND THUS

$$\mathrm{erf}(x) = 2\left(\Phi(\sqrt{2}x) - \frac{1}{2}\right) = 2\Phi(\sqrt{2}x) - 1$$

$$\Leftrightarrow \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = 2\Phi(x) - 1 \Leftrightarrow \underline{\Phi(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)}$$

NOW INVERTING $x = F^{-1}(u) := \sqrt{2}\, \mathrm{erf}^{-1}(2u-1)$ [THE NAMING F^{-1} WILL BE OBVIOUS]

WE SEE THAT $\underline{F(x) := u = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)} = \Phi(x)$ AS JUST SHOWN

WHERE $F^{-1}(u) = \inf\{x : F(x) \geq u,\ 0 \leq u \leq 1\}$. IN OTHER WORDS,

AS $\Phi(x)$ IS THE CDF OF THE STANDARD NORMAL DISTRIBUTION,

FEEDING $u \sim \mathrm{Unif}(0,1)$ INTO $F^{-1}(u)$ YIELDS NORMALLY DISTRIBUTED $x \sim \mathcal{N}(0,1)$.

More formally: $P\left(F^{-1}(U) \le x\right) \overset{F(\cdot)}{=} P\left(F(F^{-1}(U)) \le F(x)\right) \overset{F(F^{-1})=id}{=} P\left(U \le F(x)\right)$

$$\overset{U \sim Unif(0,1)}{=} F(x) = \Phi(x) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow X \sim \mathcal{N}(0,1) \quad \blacksquare$$

```python
def F(u):
    return sqrt(2.0) * erfinv(2.0 * u - 1)

u = np.random.uniform(0,1, 100000)
hist = plt.hist(F(u), bins=100, density=True)
plt.show()
```

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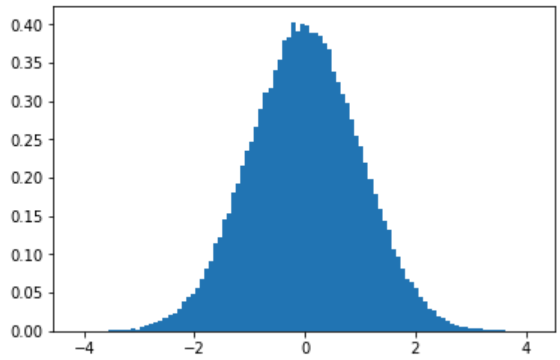

1b In practice, other methods are used as the evaluation of erf^{-1} isn't computationally cheap due to the lack of an analytical expression for erf. Being given $u \sim Unif(0,1)$, e.g. the Box-Muller transform is more efficient. $\blacksquare$

2/ IMPORTANCE SAMPLING

GIVEN:

[MONTE CARLO METHOD]

$$\mathbb{E}_f(h(X)) = \int h(x) f(x)\, dx$$

$$\Downarrow$$

$$\hat{I}_f = \frac{1}{N} \sum_{i=1}^{N} h(Y_i), \quad Y_i \sim f$$

[IMPORTANCE SAMPLING]

$$\mathbb{E}_g\left(h(X) \frac{f(X)}{g(X)}\right) = \int h(x) \frac{f(x)}{g(x)} g(x)\, dx$$

$$\Downarrow$$

$$\hat{I}_g = \frac{1}{N} \sum_{i=1}^{N} h(X_i) \frac{f(X_i)}{g(X_i)}; \quad X_i \sim g$$

<u>SHOW</u> THAT $g^*(x) = \dfrac{h(x) f(x)}{\int |h(x)| f(x)\, dx}$ MINIMIZES VARIANCE OF $\hat{I}_g$.

THE VARIANCE OF $\dfrac{h(x) f(x)}{g(x)}$ 'S, GIVEN THE ABOVE:

$$\mathbb{E}_g\left(\left(\frac{h(x) f(x)}{g(x)}\right)^2\right) - \left(\mathbb{E}\left(\frac{h(x) f(x)}{g(x)}\right)\right)^2 =$$

$$= \int \frac{h^2(x) f^2(x)}{g^2(x)} g(x)\, dx - \left(\int \frac{h(x) f(x)}{g(x)} g(x)\, dx\right)^2$$

$$= \int \frac{h^2(x) f^2(x)}{g^2(x)} g(x)\, dx - \underbrace{\left(\int h(x) f(x)\, dx\right)^2}_{\text{INDEPENDENT OF } g(x),\ \text{NO MINIMIZATION TARGET}}$$

AIMING TO MINIMIZE THE FIRST INTEGRAL, WE USE JENSEN'S INEQUALITY AS SUGGESTED:

$$\mathbb{E}_g\left(\frac{h^2(x) f^2(x)}{g^2(x)}\right) \geq \left(\mathbb{E}_g\left(\left|\frac{h(x) f(x)}{g(x)}\right|\right)\right)^2$$

HENCE $\left(\mathbb{E}_g\left(\left|\frac{h(x)f(x)}{g(x)}\right|\right)\right)^2 = \left(\int |h(x)|f(x)\,dx\right)^2$ AS PREVIOUSLY DEFINED ESTABLISHES A LOWER BOUND ON $\mathbb{E}_g\left(\frac{h^2(x)f^2(x)}{g^2(x)}\right)$.

IF WE THEN INSERT $g^*(x) = \dfrac{|h(x)|f(x)}{\int |h(x)|f(x)\,dx}$ INTO THE EXPRESSION ON THE LEFT SIDE OF THE ABOVE INEQUALITY,

WE SEE THAT $\mathbb{E}_{g^*}\left(\frac{h^2(x)f^2(x)}{g^{*2}(x)}\right) = \mathbb{E}_{g^*}\left(\left(\cancel{\frac{h^2(x)f^2(x)}{|h^2(x)|f^2(x)}}\int |h(x)|f(x)\,dx\right)^2\right) =$

$$= \left(\int |h(x)|f(x)\,dx\right)^2$$

THUS g^* ACTUALLY REALIZES THE LOWER BOUND. $\blacksquare$

IN PRACTICE, THIS MOST LIKELY IS USELESS, AS WE EMPLOY IMPORTANCE SAMPLING IF WE'RE UNABLE TO SAMPLE FROM f. HOWEVER, IF WE CAN'T SAMPLE FROM f, WE MOST PROBABLY WON'T BE ABLE TO SAMPLE FROM $\dfrac{|h(x)|f(x)}{\int |h(x)|f(x)\,dx}$ EITHER. $\blacksquare$

[SEE FILE FOR CODE]

1b

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import truncexpon
from scipy.stats import norm

%matplotlib inline

def f_rvs(N):
    return norm.rvs(size=N)

def f(x):
    return norm.pdf(x)

def g_rvs(N, limit=4.5):
    return truncexpon.rvs(loc=limit, b=np.inf, size=N)

def g(x, limit=4.5):
    return truncexpon.pdf(x, loc=limit, b=np.inf)

def h(x, limit=4.5):
    return np.where(x > limit, 1.0, 0.0)

def I_1(rvs, limit=4.5):
    return np.mean(h(rvs, limit))

def I_2(rvs, limit=4.5):
    return np.mean(h(rvs, limit) * f(rvs) / g(rvs, limit))

N = 10000
cut_off = 4.5

norm_rvs = f_rvs(N)
result_1 = I_1(norm_rvs, cut_off)

expon_rvs = g_rvs(N, cut_off)
result_2 = I_2(expon_rvs, cut_off)

result_true = 1.0 - norm.cdf(cut_off)

print('Monte Carlo integration result = {:.5f}'.format(result_1))
print('Importance sampling integration result = {:.5f}'.format(result_2))
print('True result = {:.5f}'.format(result_true))
print(80 * '-')
print('Monte Carlo integration error = {:.5f}'.format(np.abs(result_1-result_true)))
print('Importance sampling integration error = {:.5f}'.format(np.abs(result_2-result_true)))

plt.hist(norm_rvs, bins=100, label='$Yi$ from standard normal distribution', alpha=0.5, density=True)
plt.hist(expon_rvs, bins=100, label='$X_i$ from exponential distribution truncated at {}'.format(cut_off),
         alpha=0.5, density=True)
plt.plot(np.linspace(-5,5,10000), norm.pdf(np.linspace(-5,5,10000)), label='Standard normal pdf')
plt.legend()
plt.show()
```

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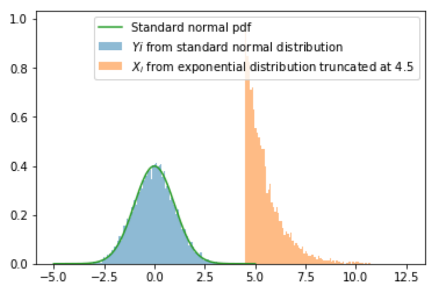
```
Monte Carlo integration result = 0.00000
Importance sampling integration result = 0.00000
True result = 0.00000
--------------------------------------------------------------------------------
Monte Carlo integration error = 0.00000
Importance sampling integration error = 0.00000
```



THE ERROR OBVIOUSLY IS ZERO FOR BOTH ESTIMATORS AS EVIDENT FROM $\frac{1}{\sqrt{2\pi}} e^{-\frac{(4.5)^2}{2}} \approx 1.5 \cdot 10^{-5}$. HOWEVER, IT IS CLEAR THAT THE IMPORTANCE SAMPLING ESTIMATOR IS MUCH MORE EFFICIENT, AS ALMOST NO Y_i WILL BE IN THE RELEVANT INTERVAL $(4.5, \infty)$ FOR THE STANDARD MONTE CARLO ESTIMATOR, WHILE THE TRUNCATED EXPONENTIAL DISTRIBUTION HAS ITS PEAK IN THE DESIRED AREA.

3/ MCMC

GIVEN: HASTINGS' ACCEPTANCE PROBABILITY

$$\alpha(x|y) = \frac{s(x|y)}{1 + \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}}$$

WITH $q \equiv$ PROPOSAL DISTRIBUTION

$p \equiv$ STATIONARY DISTRIBUTION

$s(x|y) = s(y|x)$ S.T. $\alpha(x|y) < 1 \quad \forall x, y \in \mathbb{R}$

/a SHOW $t(x|y) = \alpha(x|y)\,q(x|y)$ FULFILLS DETAILED BALANCE:

A SUFFICIENT CONDITION FOR DETAILED BALANCE IS :

$$p(y)\,t(x|y) \overset{!}{=} p(x)\,t(y|x)$$

THUS

$$p(y)\,\alpha(x|y)\,q(x|y) \overset{!}{=} p(x)\,\alpha(y|x)\,q(y|x)$$

BY SYMMETRY OF S

$$\Rightarrow p(y) \cdot \frac{\cancel{s(x|y)}}{1 + \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}} \cdot q(x|y) = p(x) \cdot \frac{\cancel{s(y|x)}}{1 + \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}} \cdot q(y|x) =$$

$$= p(y) \cdot \frac{q(x|y)}{1 + \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}} \qquad = p(x) \cdot \frac{q(y|x)}{1 + \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}} =$$

$$= p(y) \qquad = p(x) \cdot \frac{q(y|x)}{1 + \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}} \cdot \frac{1 + \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}}{q(x|y)} =$$

$$= p(y) = p(x) \cdot \frac{q(y|x)}{q(x|y)} \cdot \frac{1 + \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}}{1 + \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}} =$$

$$= p(y) = p(x) \cdot \frac{q(y|x)}{q(x|y)} \cdot \frac{\frac{q(y|x)\,p(x) + q(x|y)\,p(y)}{q(y|x)\,p(x)}}{\frac{q(x|y)\,p(y) + q(y|x)\,p(x)}{q(x|y)\,p(y)}} =$$

$$= p(y) = \cancel{p(x)} \cdot \frac{\cancel{q(y|x)}}{\cancel{q(x|y)}} \cdot \frac{\cancel{q(x|y)\,p(y) + q(y|x)\,p(x)}}{\cancel{q(x|y)\,p(y) + q(y|x)\,p(x)}} \cdot \frac{\cancel{q(x|y)}\,p(y)}{\cancel{q(y|x)}\,\cancel{p(x)}} =$$

$$= p(y) = p(y) \qquad \blacksquare$$

/b **Show** Metropolis' choice $\alpha_M(x|y) = \min\left\{1, \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}\right\}$ is special case of Hastings's α. Find $s(x|y)$.

1st case: $\alpha_M(x|y) = 1 \iff q(y|x)\,p(x) > q(x|y)\,p(y)$

$$\Rightarrow \alpha_M(y|x) = \min\left\{1, \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}\right\} = \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}$$

Detailed balance:

$$p(y)\,\underbrace{\alpha_M(x|y)}_{=1} \cdot q(x|y) = p(x) \cdot \underbrace{\alpha_M(y|x)}_{\frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}} \cdot q(y|x)$$

$$\iff p(y)\,q(x|y) = \cancel{p(x)} \cdot \frac{q(x|y)\,p(y)}{\cancel{q(y|x)}\,\cancel{p(x)}} \cdot \cancel{q(y|x)} = q(x|y)\,p(y)$$

2nd case: $\alpha_M(x|y) = \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)} \iff q(y|x)\,p(x) < q(x|y)\,p(y)$

$$\Rightarrow \alpha_M(y|x) = \min\left\{1, \frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}\right\} = 1$$

Detailed balance:

$$p(y)\,\alpha_M(x|y) \cdot q(x|y) = p(x) \cdot \alpha_M(y|x) \cdot q(y|x)$$

$$\iff \cancel{p(y)} \cdot \frac{q(y|x)\,p(x)}{\cancel{q(x|y)}\,\cancel{p(y)}} \cdot \cancel{q(x|y)} = p(x)\,q(y|x)$$ $\blacksquare$ for detailed balance.

TO FIND $s(x|y)$:

COMPARE
$$\alpha_H(x|y) = \frac{s(x|y)}{1+\frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}}$$

AND
$$\alpha_M(x|y) = \min\left\{1, \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}\right\}$$

1ST CASE: $\alpha_M(x|y) = 1 \iff q(y|x)\,p(x) > q(x|y)\,p(y)$

$$\Rightarrow \quad s(x|y) \overset{!}{=} \frac{q(x|y)\,p(y) + q(y|x)\,p(x)}{q(y|x)\,p(x)} =$$

$$= s(y|x) = \frac{q(x|y)\,p(y) + q(y|x)\,p(x)}{q(x|y)\,p(y)} \qquad \text{IF } q(y|x) = q(x|y)$$

2ND CASE: $\alpha_M(x|y) = \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)} \iff q(y|x)\,p(x) < q(x|y)\,p(y)$

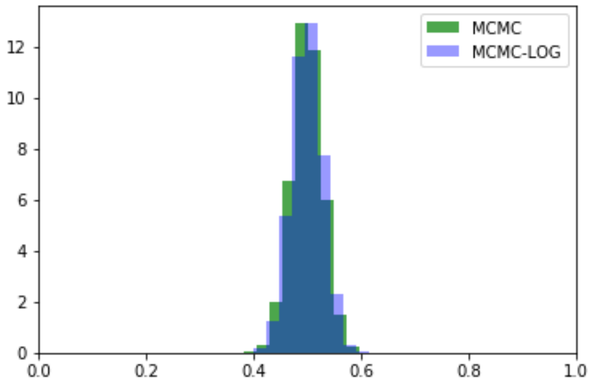
$$\frac{s(x|y)}{1+\frac{q(x|y)\,p(y)}{q(y|x)\,p(x)}} \overset{!}{=} \frac{q(y|x)\,p(x)}{q(x|y)\,p(y)}$$

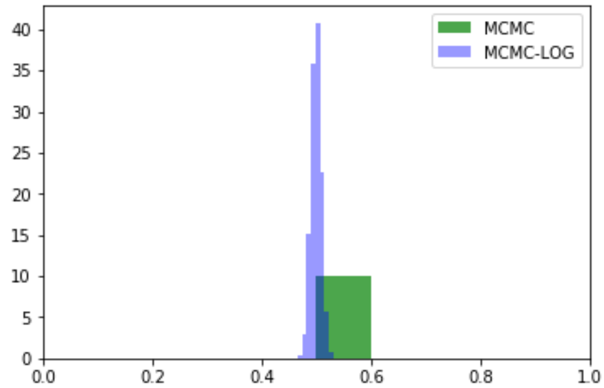
$$\iff \underline{\underline{s(x|y)}} = \frac{\cancel{q(y|x)\,p(x)}}{q(x|y)\,p(y)} \cdot \frac{q(x|y)\,p(y) + q(y|x)\,p(x)}{\cancel{q(y|x)\,p(x)}} =$$

$$= \underline{\underline{\frac{q(x|y)\,p(y) + q(y|x)\,p(x)}{q(x|y)\,p(y)} = s(y|x)}}$$

HENCE $s(x|y) = s(y|x)$ ARE CONFIRMED SHOWING METROPOLIS' CHOICE FOR A SYMMETRIC PROPOSAL FUNCTION: $q(x|y) = q(y|x)$ ∎

1c+d [See file for code]





Plot with NN=300, NH=150:

Results are identical (taking stochastic character into account) with a large-ish error due to low N.

Plot with NN=3000, NH=1500:

Numerical errors due to calculating H^{1500} for $0 \le H \le 1$

=> (all) samples taken by MCMC are copies of the starting value 0.5

OTOH, MCMC-LOG works fine due to the numerical stability gained from using the logarithm and now shows a narrower posterior due to N larger / decreased error by factor of $\sqrt{1000} \approx 30$.

HPCSE II

HW3 - Part 2

Beat Hubmann

1/ The Farm

1a

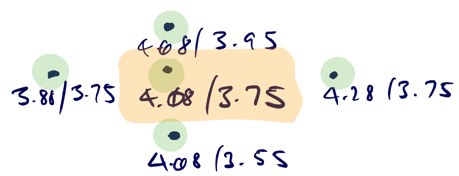
Koehli output: $x/y = 4.0852/3.7476$

```
Searching for cows near (4.085200, 3.747600)...
New Cow Found! Id:#95, Position: (4.153557, 3.972554), Distance: 0.235111
New Cow Found! Id:#130, Position: (4.219451, 3.755086), Distance: 0.134460
New Cow Found! Id:#267, Position: (3.901680, 3.571877), Distance: 0.254083
New Cow Found! Id:#279, Position: (4.127642, 3.933766), Distance: 0.190943
New Cow Found! Id:#298, Position: (4.049541, 3.729459), Distance: 0.040008
New Cow Found! Id:#378, Position: (3.899571, 3.599574), Distance: 0.237423
New Cow Found! Id:#386, Position: (4.098821, 3.938975), Distance: 0.191859
New Cow Found! Id:#407, Position: (4.144751, 4.004599), Distance: 0.263809
New Cow Found! Id:#449, Position: (4.097446, 3.971806), Distance: 0.224540
New cows found: 9
Total cows found so far: 9
Herr Kueheli says: "I knew we should look around the spot with the tallest grass!."
Herr Kueheli says: "The rest of the cows should be around here. Lets try these nearby points:"
[4.08, 3.95]
[4.28, 3.75]
[3.88, 3.75]
[4.08, 3.55]
>>> Time until deadline: 55 minutes. <<<
```

1b

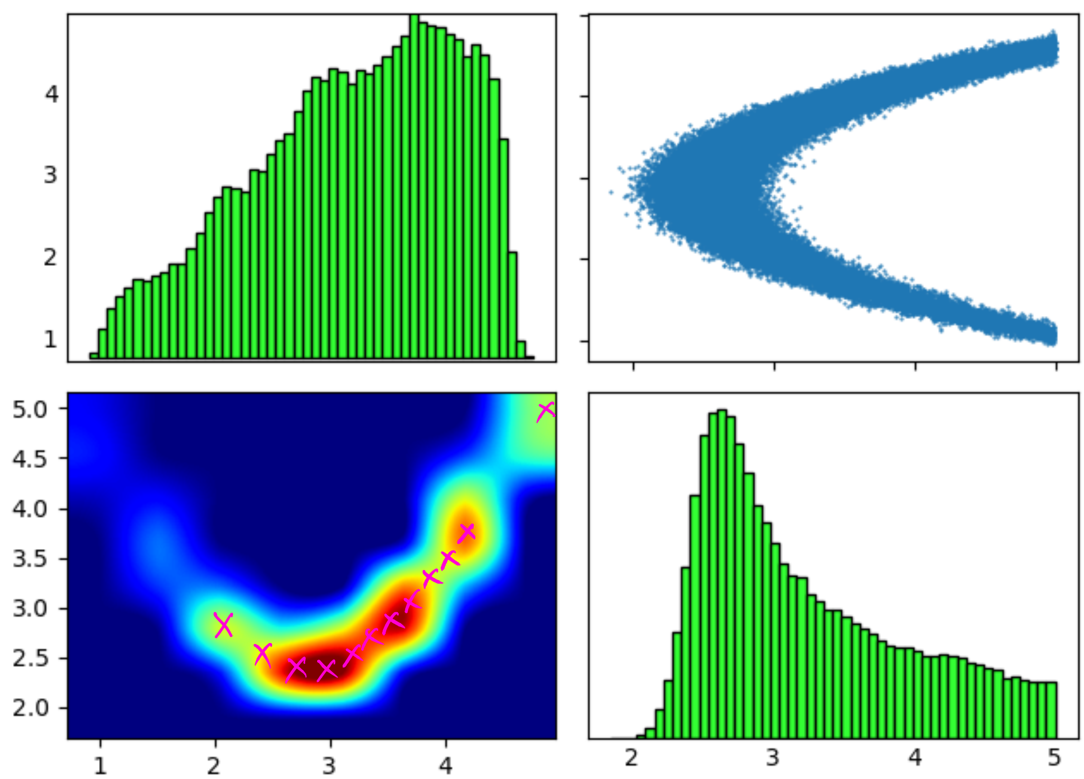
```
Searching for cows near (4.080000, 3.950000)...
New Cow Found! Id:#20, Position: (4.274126, 4.013635), Distance: 0.204289
New Cow Found! Id:#24, Position: (4.312318, 4.051145), Distance: 0.253381
New Cow Found! Id:#135, Position: (4.300406, 4.138333), Distance: 0.289911
New Cow Found! Id:#159, Position: (4.257531, 4.011793), Distance: 0.187977
New Cow Found! Id:#190, Position: (4.242909, 4.156680), Distance: 0.263165
New Cow Found! Id:#235, Position: (4.269124, 4.096134), Distance: 0.239004
New Cow Found! Id:#238, Position: (4.272254, 4.042875), Distance: 0.213512
New Cow Found! Id:#265, Position: (4.256778, 4.150874), Distance: 0.267583
New Cow Found! Id:#317, Position: (4.257531, 4.011793), Distance: 0.187977
New Cow Found! Id:#336, Position: (4.240627, 4.143513), Distance: 0.251492
New Cow Found! Id:#361, Position: (4.307999, 4.099270), Distance: 0.272516
New Cow Found! Id:#420, Position: (4.282964, 4.083839), Distance: 0.243120
New Cow Found! Id:#470, Position: (4.224628, 4.105519), Distance: 0.212376
New Cow Found! Id:#487, Position: (4.224628, 4.105519), Distance: 0.212376
New Cow Found! Id:#495, Position: (4.171655, 4.122416), Distance: 0.195264
New cows found: 15
Total cows found so far: 24
Herr Kueheli says: "That was good, but we need to find them faster!."
>>> Time until deadline: 50 minutes. <<<
bhubmann@eu-login-20-ng:/cluster/home/bhubmann/project-3/part_2/task1 > ./check_cows 4.28 3.75
Searching for cows near (4.280000, 3.750000)...
New cows found: 0
Total cows found so far: 24
Herr Kueheli says: "That was not good enough, perhaps this strategy does not really work."
>>> Time until deadline: 45 minutes. <<<
bhubmann@eu-login-20-ng:/cluster/home/bhubmann/project-3/part_2/task1 > ./check_cows 3.88 3.75
Searching for cows near (3.880000, 3.750000)...
New Cow Found! Id:#73, Position: (3.890320, 3.468693), Distance: 0.281496
New cows found: 1
Total cows found so far: 25
Herr Kueheli says: "That was not good enough, perhaps this strategy does not really work."
>>> Time until deadline: 40 minutes. <<<
bhubmann@eu-login-20-ng:/cluster/home/bhubmann/project-3/part_2/task1 > ./check_cows 4.08 3.55
Searching for cows near (4.080000, 3.550000)...
New cows found: 0
Total cows found so far: 25
Herr Kueheli says: "That was not good enough, perhaps this strategy does not really work."
>>> Time until deadline: 35 minutes. <<<
```

▷ CLEARLY, THE OTHER SPOTS WERE SUGGESTED IN A GRID PATTERN AROUND THE MAXIMUM FIRST FOUND:



▷ NO, WE DIDN'T ACHIEVE THE DESIRED RESULTS WITH ONLY 16 ADDITIONAL COWS FOUND AROUND THE INITIAL POINT.

1c THE RESULTING OUTPUT FROM THE TMCMC SAMPLER:



▷ USING THIS INFORMATION, WE SEARCH AS FOLLOWS: EVERY CHECK TAKES 5min, HENCE WE CAN DO 12 CHECKS. SOMEWHAT HEURISTICALLY CHOOSE 12 SPOTS FROM THE ABOVE HEAT MAP WHICH SEEM PROMISING AND CHECK:

2.05/2.9 : 45 COWS, 45 TOTAL
2.3/2.55 : 42 COWS, 87 TOTAL

2.75 / 2.4 : 58 cows , 145 total

2.95 / 2.4 : 41 cows , 186 total

3.2 / 2.65 : 104 cows , 249 (?) total

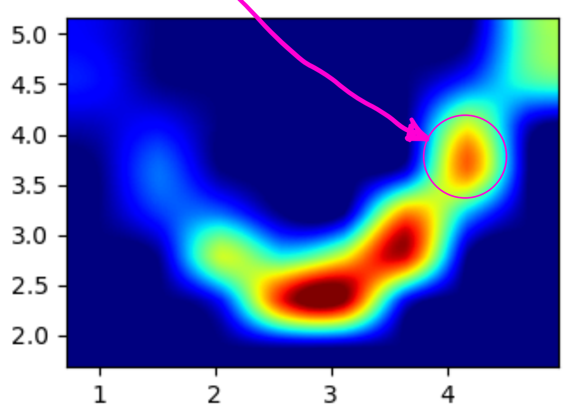
3.3 / 2.75 : 101 cows , 246 (??) total

3.6 / 2.9 : 65 cows , 210 (???) total

. . .

- While the check_cows script obviously has a sinning bug, this strategy clearly allows us to find 100 animals in one hour.

- The initial strategy was not effective because we were searching the grid around a local maximum and thus missed all the other maxima.



Performing the importance sampling allowed us to find other promising spots efficiently without getting stuck in local optima or having to sample the whole space.

1d
- We use Koruzi to maximize the posterior as seen in class for the Ackley example.

We derive two priors from expert opinion:

$$pH \sim \text{Unif}(4.0, 9.0)$$

$$\text{rain\_mm} \sim \mathcal{N}(90, 20)$$

We also limit rain_mm to be between zero for obvious reasons and $\mu + 5\sigma = 190$ mm because if it's good enough for CERN it's good enough for pompeins.

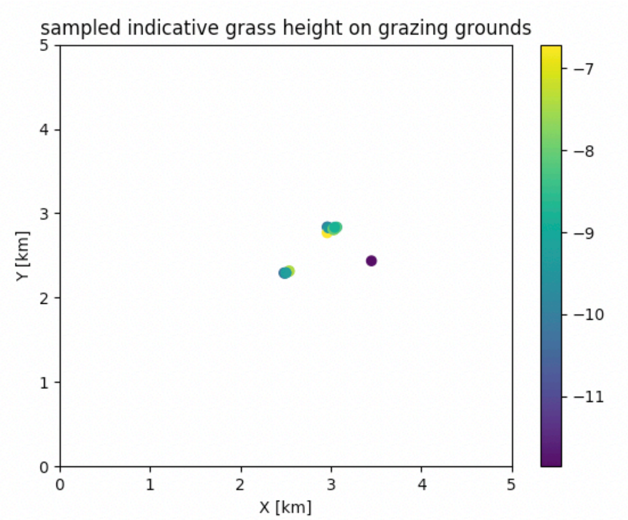
THIS GIVES THE FOLLOWING OUTPUT:

```
[Korali] Finished - Reason: Function value differences (4.74e-13) < (1.00e-12)
[Korali] Parameter 'Sigma' Value: 0.144359
[Korali] Parameter 'pH' Value: 7.446896
[Korali] Parameter 'rain_mm' Value: 115.583404
[Korali] Total Elapsed Time: 0.005675s
```

▷ AT FIRST GLANCE, IT COULD SEEM PROMISING TO PLANT PUMPKINS NEXT TO THE GRAZING GROUNDS AS $5.5 \leq pH \leq 7.5$ AND $rain\_mm > 100$ ARE BOTH PREDICTED TO BE FULFILLED.

THERE ARE HOWEVER SEVERAL CAVEATS:

- pH POSTERIOR IS DANGEROUSLY CLOSE TO THE LIMIT WHILE THE SIGMA IS ONLY 1-2 ORDERS OF MAGNITUDE SMALLER THAN THE INFERRED HEIGHTS (I.E. NOT "ZERO" FOR PRACTICAL PURPOSES)
- NOT ONLY ARE THERE ONLY 20 SAMPLES TAKEN AS REFERENCE DATA, THEY'RE ALSO:
  - ☐ NOT CLOSE TO THE BORDER WHERE THE PUMPKIN GARDEN WOULD BE
  - ☐ CLOSELY SPACED AROUND A SPOT WHERE WE KNOW FROM ABOVE THE GRASS GROWS WELL:



AND THUS NOT TRULY REPRESENTATIVE OF THE FARM AS A WHOLE.

- FINALLY AND PROBABLY MOST RELEVANT FOR ANYONE WHO KNOWS THE SLIGHTEST BIT ABOUT GARDENING, INFERRING GROWTH BEHAVIOUR FOR ONE PLANT SPECIES (PUMPKIN) BASED ON DATA COLLECTED FROM A COMPLETELY DIFFERENT PLANT SPECIES (GRASS) CAN GO WRONG IN MANY WAYS ONE IS NOT WELL EDUCATED ABOUT THE SPECIES' REQUIREMENTS AND GROWTH BEHAVIOUR.

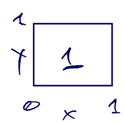
BASED ON ALL THE ABOVE ALONE, PLANTING PUMPKINS ON THE SUGGESTED SPOT CAN'T BE FULLY RECOMMENDED WITHOUT FURTHER INVESTIGATIONS.

2/ THE CANDLES

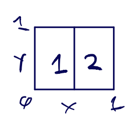
[SEE FILE FOR CODE]

2/a ▷ ACCORDING TO THE WORKERS' TESTIMONIES, WE DEFINE THREE MODELS FOR 1, 2 AND 3 CANDLES:

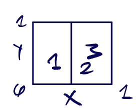
- model_1: 2 PARAMETERS: $x\,Pos \sim Unif(0,1)$, $y\,Pos \sim Unif(0,1)$



- model_2: 4 PARAMETERS: $x\,Pos\,1 \sim Unif(0, 0.5)$, $y\,Pos\,1 \sim Unif(0, 1)$, $x\,Pos\,2 \sim Unif(0.5, 1)$, $y\,Pos\,2 \sim Unif(0, 1)$



- model_3: 6 PARAMETERS: $x\,Pos\,1 \sim Unif(0, 0.5)$, $y\,Pos\,1 \sim Unif(0, 1)$, $x\,Pos\,2 \sim Unif(0.5, 1)$, $y\,Pos\,2 \sim Unif(0, 1)$, $x\,Pos\,3 \sim Unif(0.5, 1)$, $y\,Pos\,3 \sim Unif(0, 1)$



AND SOLVE A LIKELIHOOD PROBLEM GIVEN THE REFERENCE DATA.

▷ RUNNING THE TMCMC SAMPLER, WE DISCOVER THAT THE MODEL WITH 3 CANDLES HAS THE HIGHEST EVIDENCE:

MODEL:

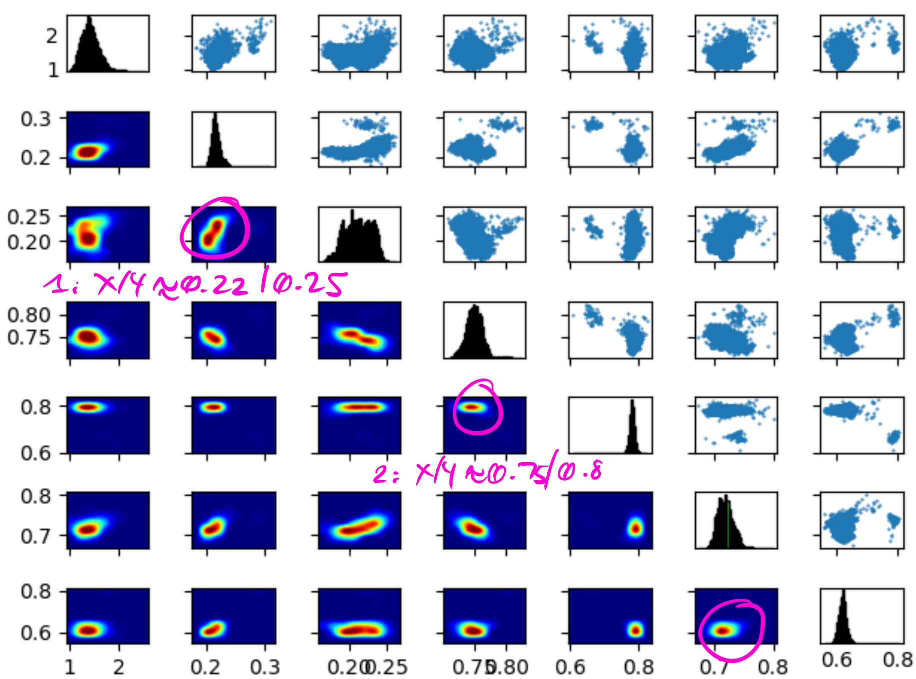
1 CANDLE →

2 CANDLES →

3 CANDLES →

```
[Korali] Starting TMCMC. Parameters: 3, Seed: 0x5CA0D776
[Korali] Generation 0 - Time: 33.032195s, Annealing: 0.00%, Acceptance: 100.00%
[Korali] Generation 1 - Time: 3.328683s, Annealing: 7.61%, Acceptance: 49.45%
[Korali] Generation 2 - Time: 3.357518s, Annealing: 19.45%, Acceptance: 50.93%
[Korali] Generation 3 - Time: 3.368001s, Annealing: 37.10%, Acceptance: 48.71%
[Korali] Generation 4 - Time: 3.387735s, Annealing: 70.33%, Acceptance: 49.24%
[Korali] Finished. Evidence: -122.158929.
[Korali] Total Time: 46.598142s - Sampling Time: 46.474131s - Engine Time: 0.123969s.
[Korali] Saving results to file: tmcmc.txt.
[Korali] Starting TMCMC. Parameters: 5, Seed: 0x5CA0D776
[Korali] Generation 0 - Time: 33.608048s, Annealing: 0.00%, Acceptance: 100.00%
[Korali] Generation 1 - Time: 3.787736s, Annealing: 6.16%, Acceptance: 50.21%
[Korali] Generation 2 - Time: 3.877246s, Annealing: 12.95%, Acceptance: 53.76%
[Korali] Generation 3 - Time: 3.958197s, Annealing: 19.39%, Acceptance: 51.50%
[Korali] Generation 4 - Time: 4.009036s, Annealing: 27.46%, Acceptance: 49.01%
[Korali] Generation 5 - Time: 4.026064s, Annealing: 43.55%, Acceptance: 48.01%
[Korali] Generation 6 - Time: 4.034069s, Annealing: 77.43%, Acceptance: 48.89%
[Korali] Finished. Evidence: -115.557590.
[Korali] Total Time: 57.508220s - Sampling Time: 57.300395s - Engine Time: 0.207774s.
[Korali] Saving results to file: tmcmc.txt.
[Korali] Starting TMCMC. Parameters: 7, Seed: 0x5CA0D776
[Korali] Generation 0 - Time: 34.265250s, Annealing: 0.00%, Acceptance: 100.00%
[Korali] Generation 1 - Time: 4.281534s, Annealing: 5.31%, Acceptance: 50.63%
[Korali] Generation 2 - Time: 4.417991s, Annealing: 11.12%, Acceptance: 53.65%
[Korali] Generation 3 - Time: 4.545829s, Annealing: 16.46%, Acceptance: 53.17%
[Korali] Generation 4 - Time: 4.637275s, Annealing: 21.50%, Acceptance: 53.11%
[Korali] Generation 5 - Time: 4.679911s, Annealing: 26.92%, Acceptance: 51.71%
[Korali] Generation 6 - Time: 4.678683s, Annealing: 33.18%, Acceptance: 51.63%
[Korali] Generation 7 - Time: 4.676100s, Annealing: 41.03%, Acceptance: 50.62%
[Korali] Generation 8 - Time: 4.654426s, Annealing: 53.87%, Acceptance: 48.23%
[Korali] Generation 9 - Time: 4.647539s, Annealing: 81.68%, Acceptance: 48.87%
[Korali] Finished. Evidence: -94.663675.
[Korali] Total Time: 75.763137s - Sampling Time: 75.484538s - Engine Time: 0.278532s.
[Korali] Saving results to file: tmcmc.txt.
```

ALSO, THE CANDLES' POSITIONS ARE FAIRLY WELL ESTABLISHED IN THE 3 CANDLE MODEL:



1: X/Y ≈ 0.22 / 0.25

2: X/Y ≈ 0.75/0.8

3: X/Y 0.7/0.65

▷ Switching to the CMAES solver to confirm the result and obtain a numerical solution, we get for the 3-candle model:

```
[Korali] Finished - Reason: Function value differences (3.98e-13) < (1.00e-12)
[Korali] Parameter 'Sigma' Value: 1.187578
[Korali] Parameter 'xpos_1' Value: 0.284123
[Korali] Parameter 'ypos_1' Value: 0.225208
[Korali] Parameter 'xpos_2' Value: 0.702163
[Korali] Parameter 'ypos_2' Value: 0.704550
[Korali] Parameter 'xpos_3' Value: 0.800293
[Korali] Parameter 'ypos_3' Value: 0.789581
```

While the 1-candle model shows a higher sigma:

```
[Korali] Finished - Reason: Function value differences (3.98e-13) < (1.00e-12)
[Korali] Parameter 'Sigma' Value: 6.112970
[Korali] Parameter 'xpos_single' Value: 0.721352
[Korali] Parameter 'ypos_single' Value: 0.754621
```

And the 2-candle model doesn't converge : (also higher sigma):

```
[Korali] Finished - Reason: Maximal condition number 9.01e+12 reached.
[Korali] Parameter 'Sigma' Value: 3.387104
[Korali] Parameter 'xpos_1' Value: 0.286146
[Korali] Parameter 'ypos_1' Value: 0.246752
[Korali] Parameter 'xpos_2' Value: 0.735151
[Korali] Parameter 'ypos_2' Value: 0.766711
```

▷ We thus conclude that 3 workers at the positions indicated above : 1 : $x/y \approx 0.28 / 0.23$
2 : $x/y \approx 0.7 / 0.7$
3 : $x/y \approx 0.8 / 0.79$

are the most likely case.

/b ▷ Using the new information, we adapt the respective xPos parameters to Gaussians around 0.25 respectively 0.75 with $\sigma = 0.05$ and limits $(0, 0.5)$ respectively $(0.5, 1)$ while leaving the yPos uniform parameters.

CMAES gives similar values but with lower sigma:

```
[Korali] Finished - Reason: Function value differences (9.52e-13) < (1.00e-12)
[Korali] Parameter 'Sigma' Value: 1.039126
[Korali] Parameter 'xpos_1' Value: 0.208636
[Korali] Parameter 'ypos_1' Value: 0.219382
[Korali] Parameter 'xpos_2' Value: 0.804045
[Korali] Parameter 'ypos_2' Value: 0.802827
[Korali] Parameter 'xpos_3' Value: 0.705800
[Korali] Parameter 'ypos_3' Value: 0.709230
```

- Hence we would correct our recommendations for the torch positions:

  1 : $x/y \approx 0.21 / 0.22$
  2 : $x/y \approx 0.8 / 0.8$
  3 : $x/y \approx 0.71 / 0.71$

- Running the updated models for 2 and 3 torches with TMCMC again gives a higher evidence for 3 candles which, as expected from the informative prior, is even higher now (evidence was − 94.7 best value for 3 candles; uninformed):

```
[Korali] Starting TMCMC. Parameters: 5, Seed: 0x5CA0EF49
[Korali] Generation 0 - Time: 34.556678s, Annealing: 0.00%, Acceptance: 100.00%
[Korali] Generation 1 - Time: 4.062543s, Annealing: 5.40%, Acceptance: 51.42%
[Korali] Generation 2 - Time: 4.111483s, Annealing: 11.71%, Acceptance: 52.17%
[Korali] Generation 3 - Time: 4.142668s, Annealing: 19.08%, Acceptance: 50.07%
[Korali] Generation 4 - Time: 4.159910s, Annealing: 30.60%, Acceptance: 49.64%
[Korali] Generation 5 - Time: 4.165978s, Annealing: 50.04%, Acceptance: 49.55%
[Korali] Generation 6 - Time: 4.159693s, Annealing: 83.15%, Acceptance: 48.31%
[Korali] Finished. Evidence: -122.025112.
```

2 candles →

```
[Korali] Starting TMCMC. Parameters: 7, Seed: 0x5CA0EF49
[Korali] Generation 0 - Time: 35.163790s, Annealing: 0.00%, Acceptance: 100.00%
[Korali] Generation 1 - Time: 4.557577s, Annealing: 4.10%, Acceptance: 52.76%
[Korali] Generation 2 - Time: 4.623621s, Annealing: 7.89%, Acceptance: 54.12%
[Korali] Generation 3 - Time: 4.761215s, Annealing: 11.25%, Acceptance: 53.22%
[Korali] Generation 4 - Time: 4.756775s, Annealing: 14.95%, Acceptance: 50.13%
[Korali] Generation 5 - Time: 4.760949s, Annealing: 21.47%, Acceptance: 48.15%
[Korali] Generation 6 - Time: 4.747416s, Annealing: 37.63%, Acceptance: 49.15%
[Korali] Generation 7 - Time: 4.752566s, Annealing: 67.33%, Acceptance: 49.07%
[Korali] Finished. Evidence: -70.039198.
```

3 candles →

- Finally, plotting shows the 3 candle positions are more distinct now with narrower distributions while still fitting well with the CMAES solver's numerical results: