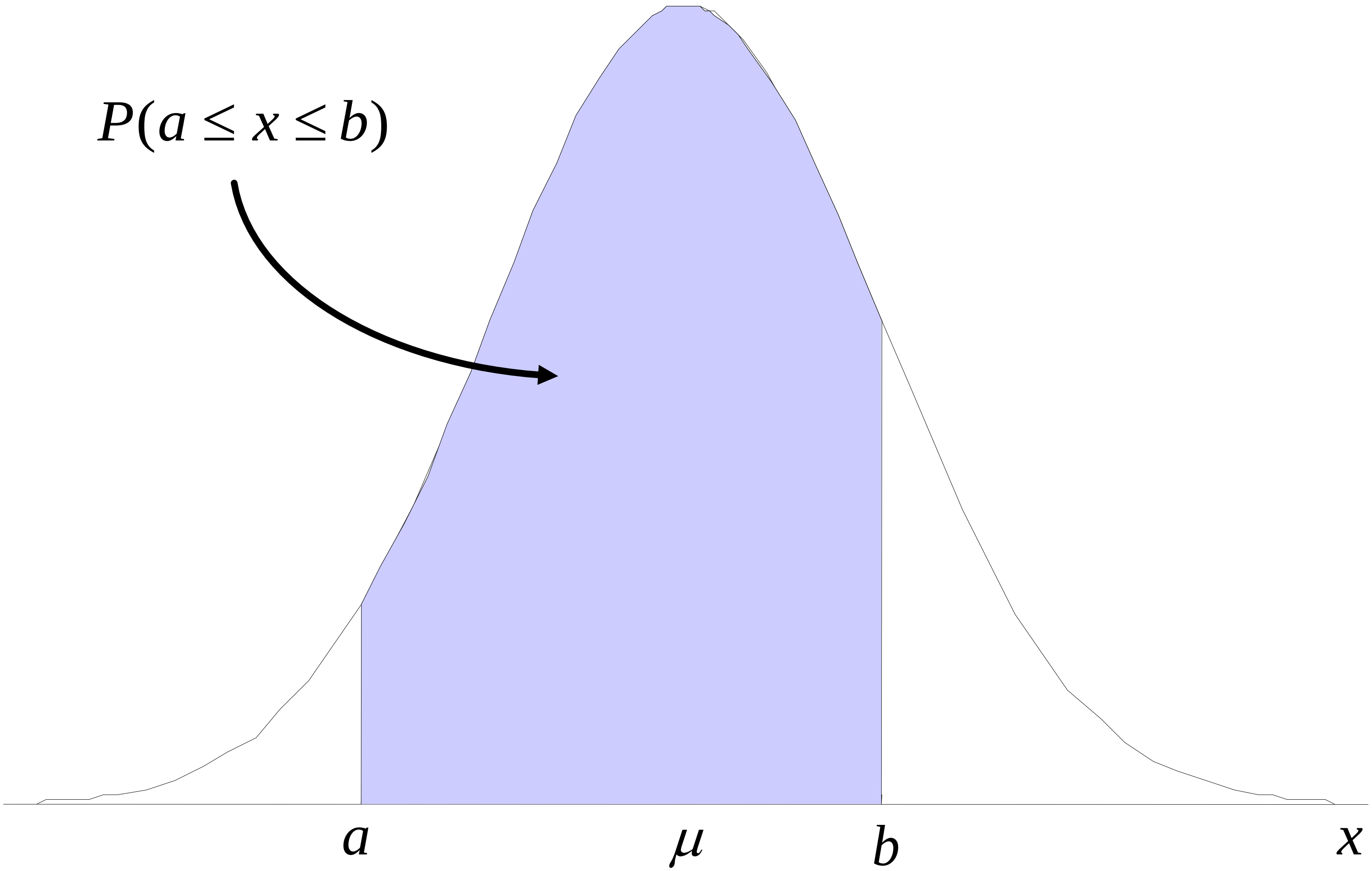
# Chapter 5 ~ Normal Probability Distributions



# Chapter Goals

Learn about the *normal*, *bell-shaped*, or *Gaussian* distribution

- How probabilities are found
- How probabilities are represented
- How normal distributions are used in the real world

# 5.1 ~ Normal Probability Distributions

The normal probability distribution is the most important distribution in all of statistics

- Many continuous random variables have normal or approximately normal distributions

- Need to learn how to describe a normal probability distribution

# Normal Probability Distribution

1. A continuous random variable

2. Description involves two functions:
   a. A function to determine the ordinates of the graph picturing the distribution
   b. A function to determine probabilities

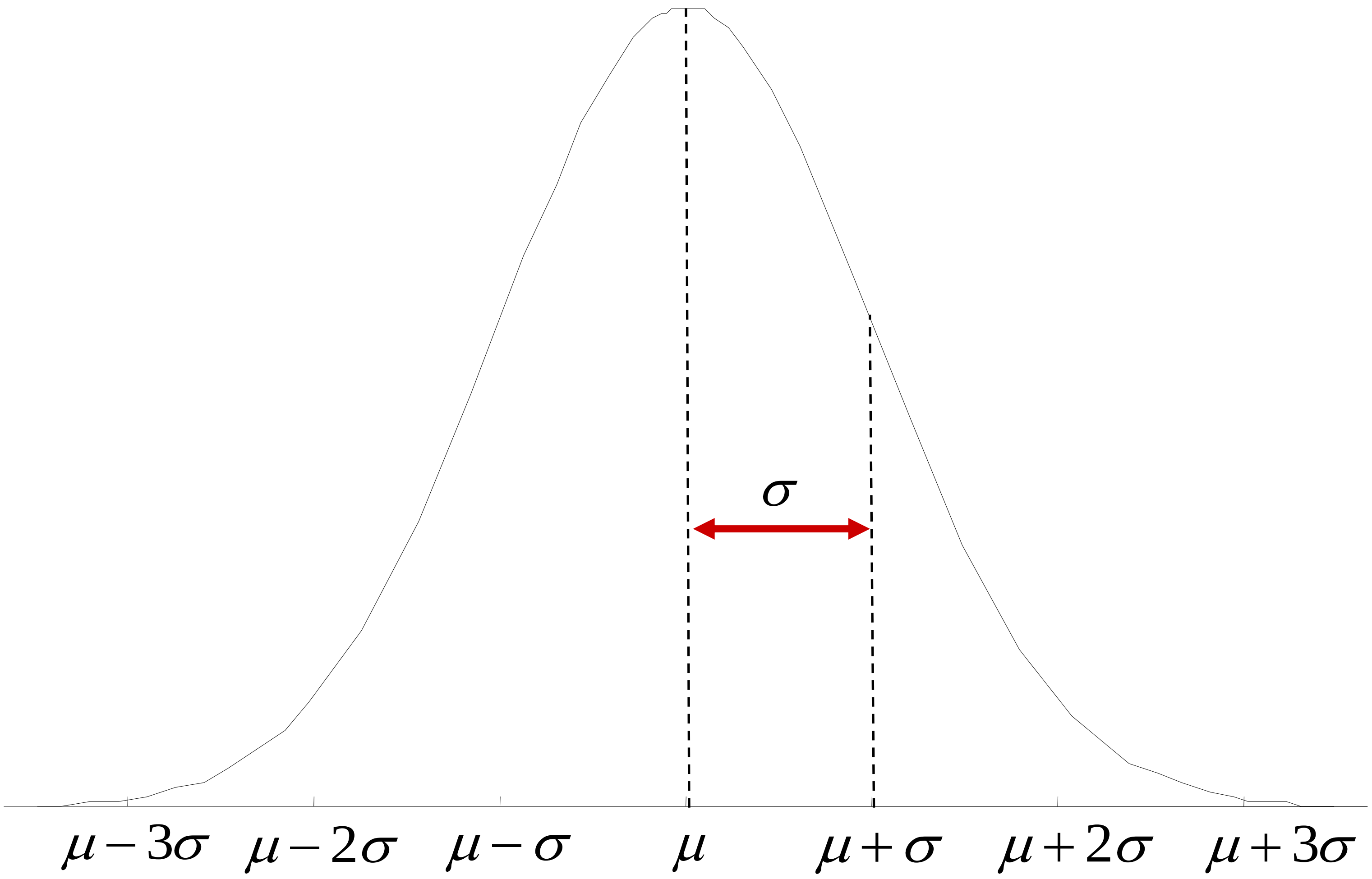
3. Normal probability distribution function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2}$$

   This is the function for the normal (bell-shaped) curve

4. The probability that *x* lies in some interval is the area under the curve

# The Normal Probability Distribution



# 5.2 ~ The Standard Normal Distribution

There are infinitely many normal probability distributions

- They are all related to the *standard normal distribution*

- The standard normal distribution is the normal distribution of the standard variable $z$ (the $z$-score)

# Standard Normal Distribution

## Properties:

The total area under the normal curve is equal to 1

The distribution is mounded and symmetric; it extends indefinitely in both directions, approaching but never touching the horizontal axis

The distribution has a mean of 0 and a standard deviation of 1

The mean divides the area in half, 0.50 on each side

Nearly all the area is between $z$ = -3.00 and $z$ = 3.00

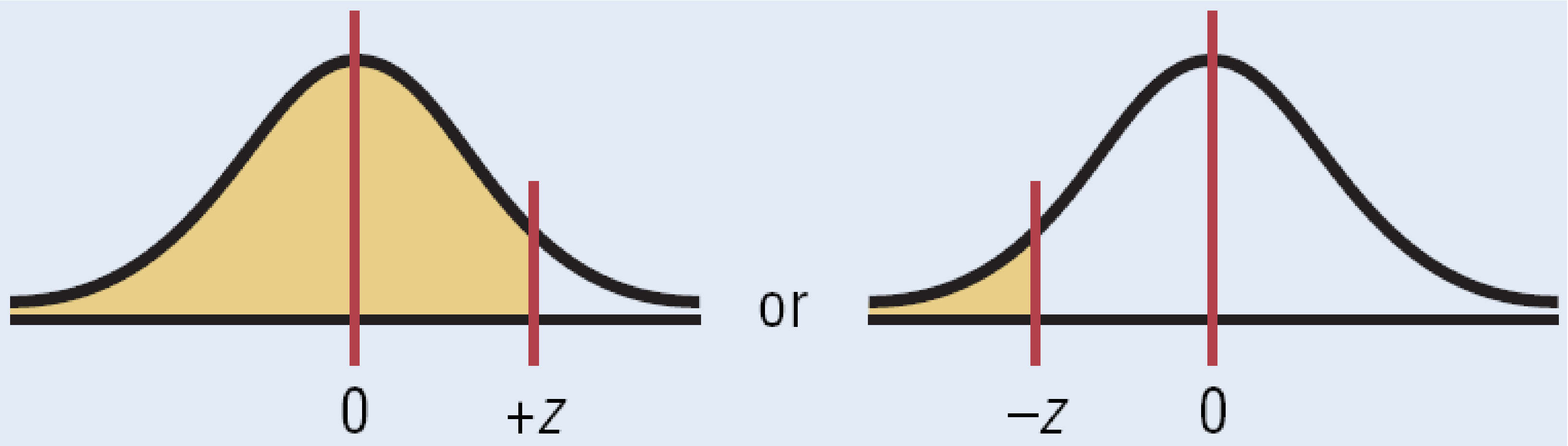
*Notes:*

- Probabilities of other intervals are found using the table entries, addition, subtraction, and the properties above

# *Area under the Standard Normal Distribution Curve*

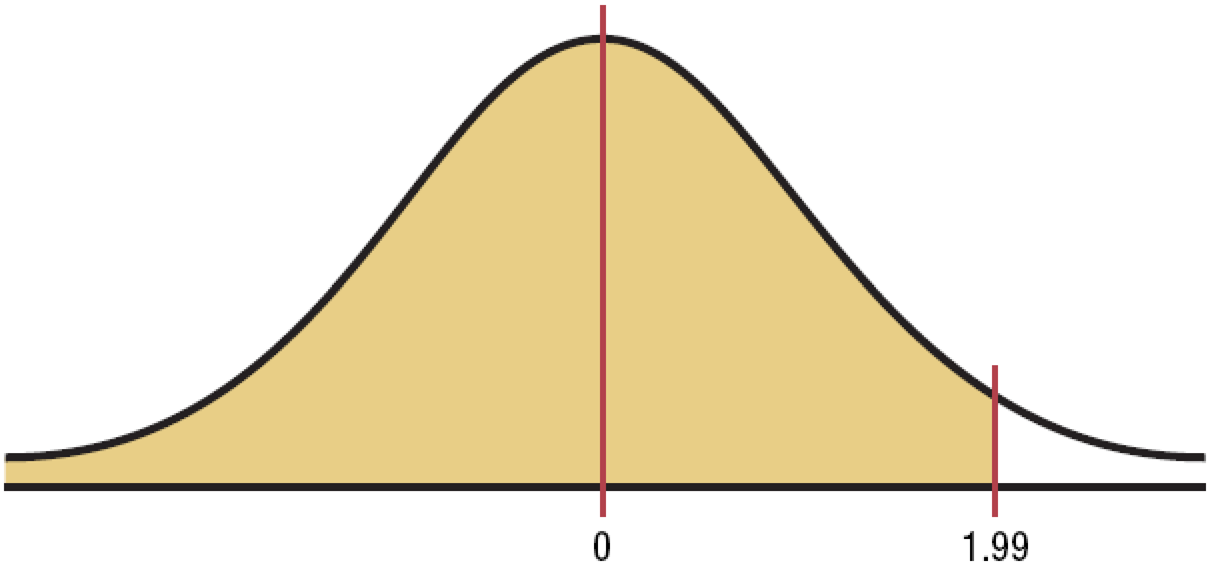
1. To the left of any *z* value:

   Look up the *z* value in the table and use the area given.



# Example

✓**Example:** Find the area under the standard normal curve $Z < 1.99$

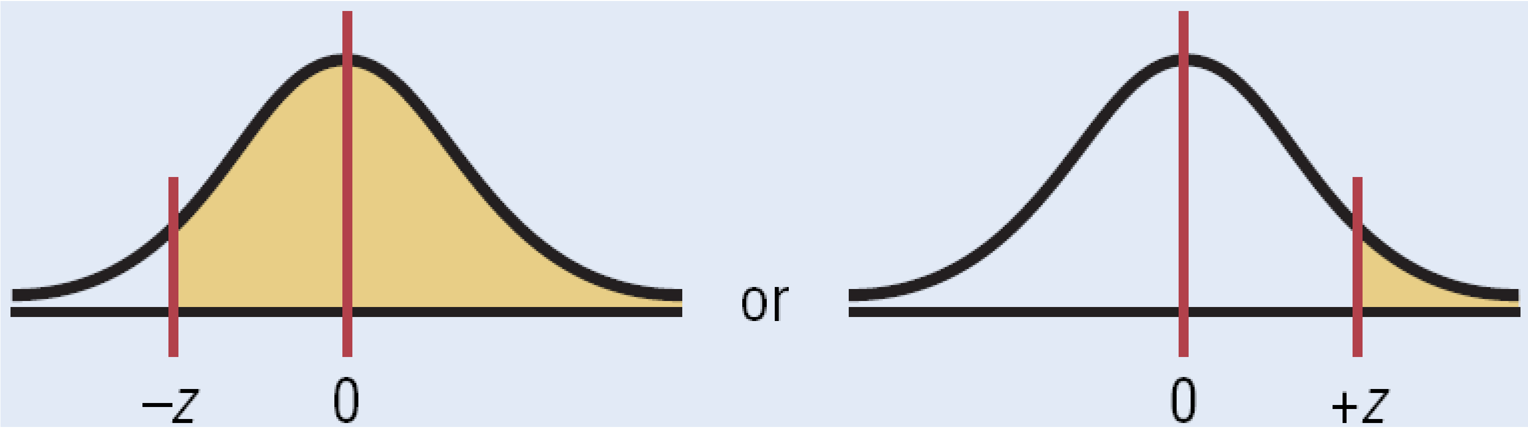



The value in the 1.9 row and the .09 column of Table E is .9767. The area is .9767.

## *Area under the Standard Normal Distribution Curve*

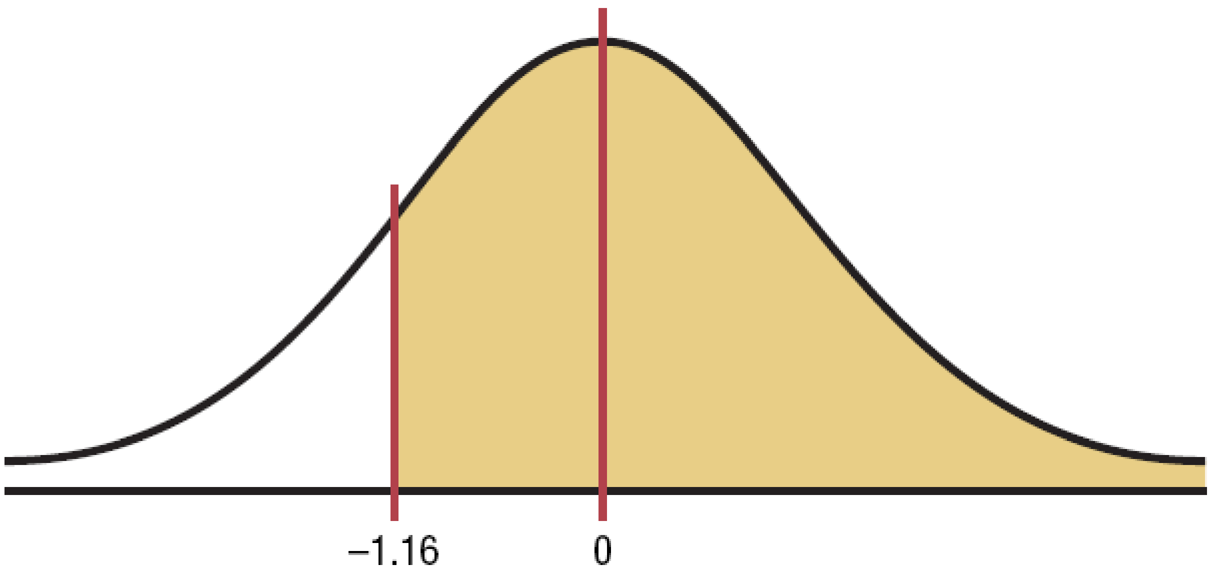
2. To the right of any $z$ value:

   Look up the $z$ value and subtract the area from 1.



# Example

✓Find the area under the standard normal curve $Z > -1.16$.

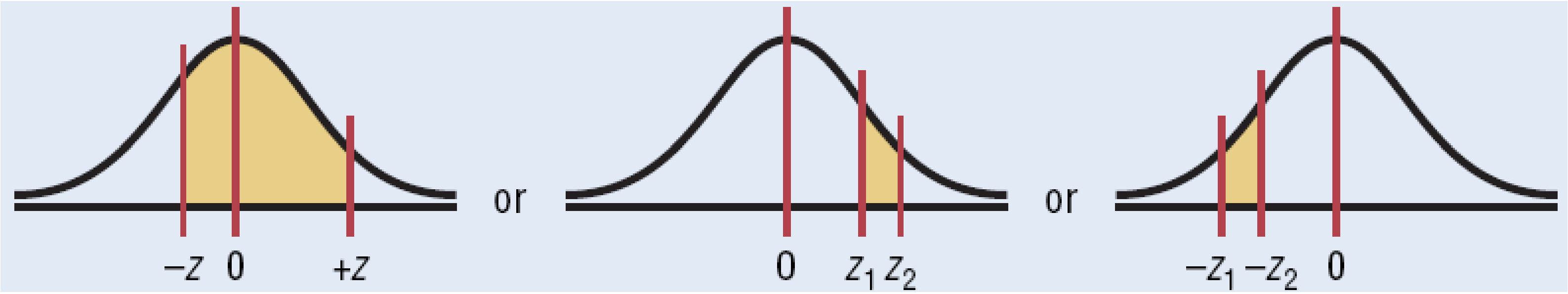



The value in the -1.1 row and the .06 column of Table E is .1230. The area is 1 - .1230 = .8770.

*Area under the Standard Normal Distribution Curve*

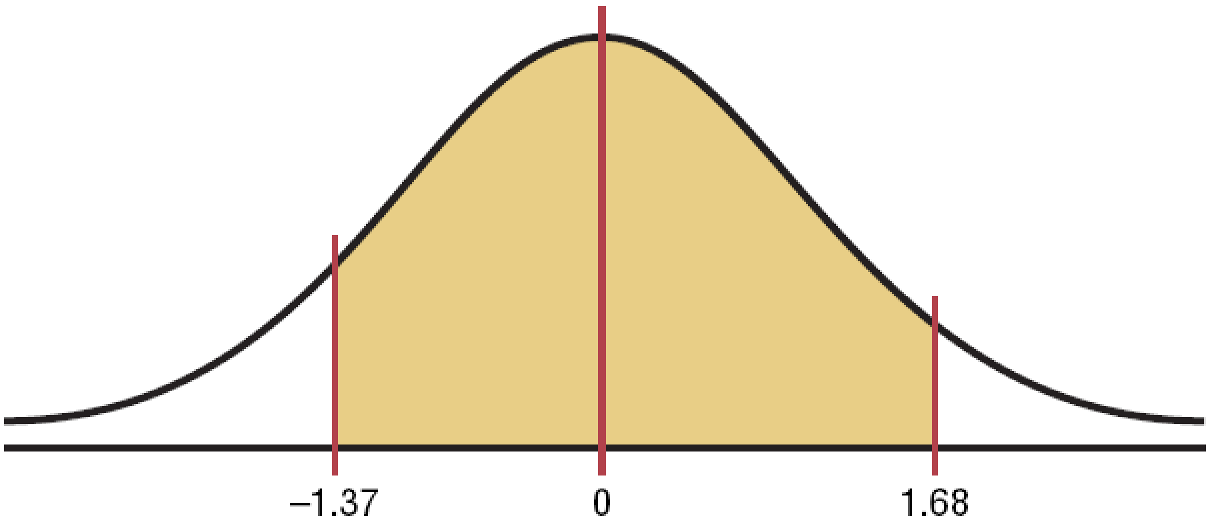
3. Between two *z* values:

   Look up both *z* values and subtract the corresponding areas.



# Example

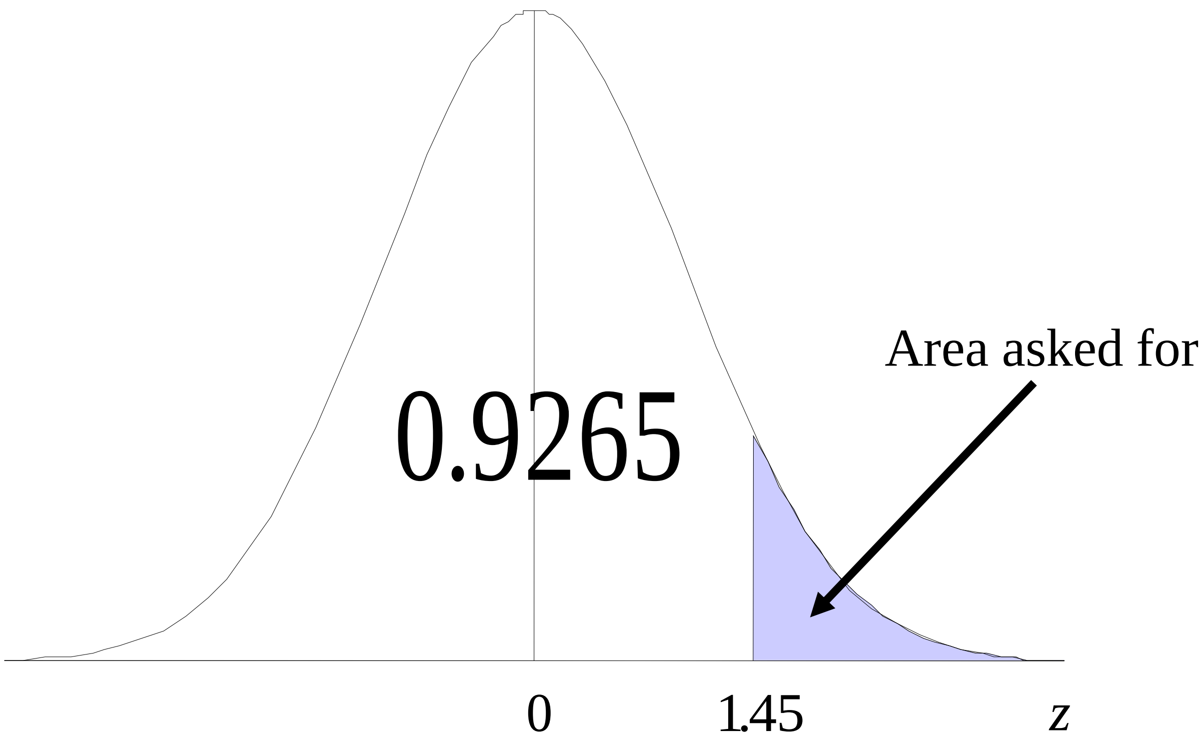
Find the area between $z = 1.68$ and $z = -1.37$.




The values for $z = 1.68$ is .9535 and for $z = -1.37$ is .0853. The area is .9535 - .0853 = .8682.

# Example

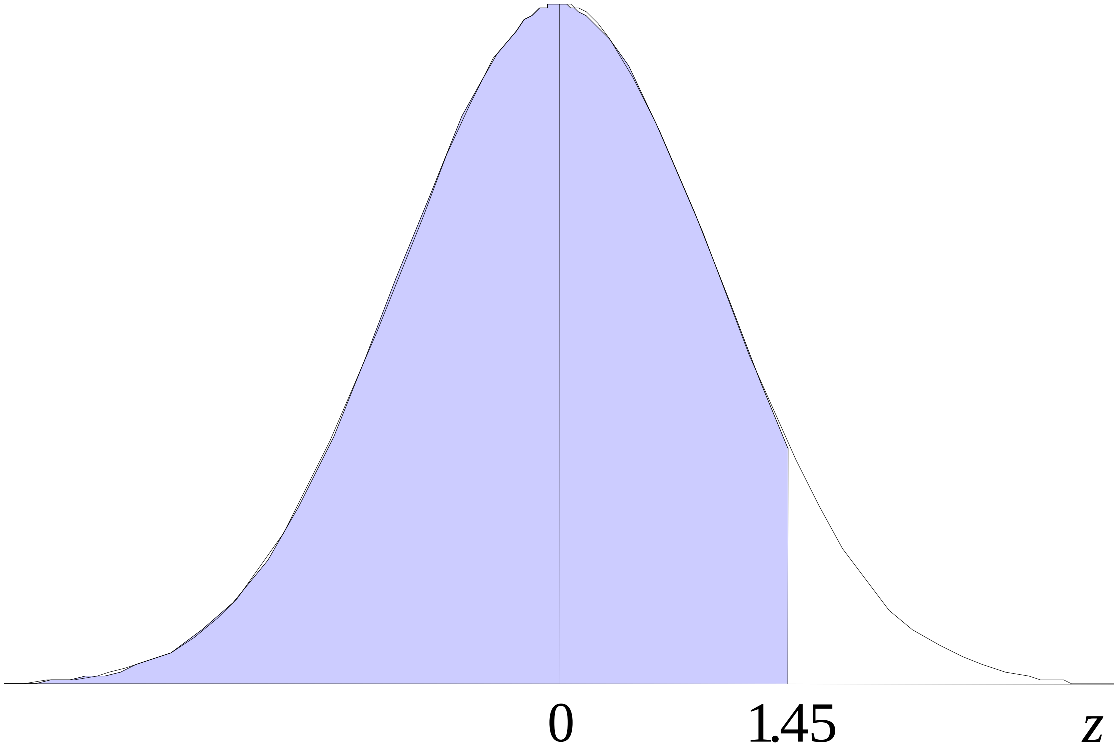
- ✓ **Example:** Find the area under the normal curve to the right of $z = 1.45$; $P(z > 1.45)$




$$P(z > 1.45) = 1 - 0.9265 = 0.0735$$

# Example

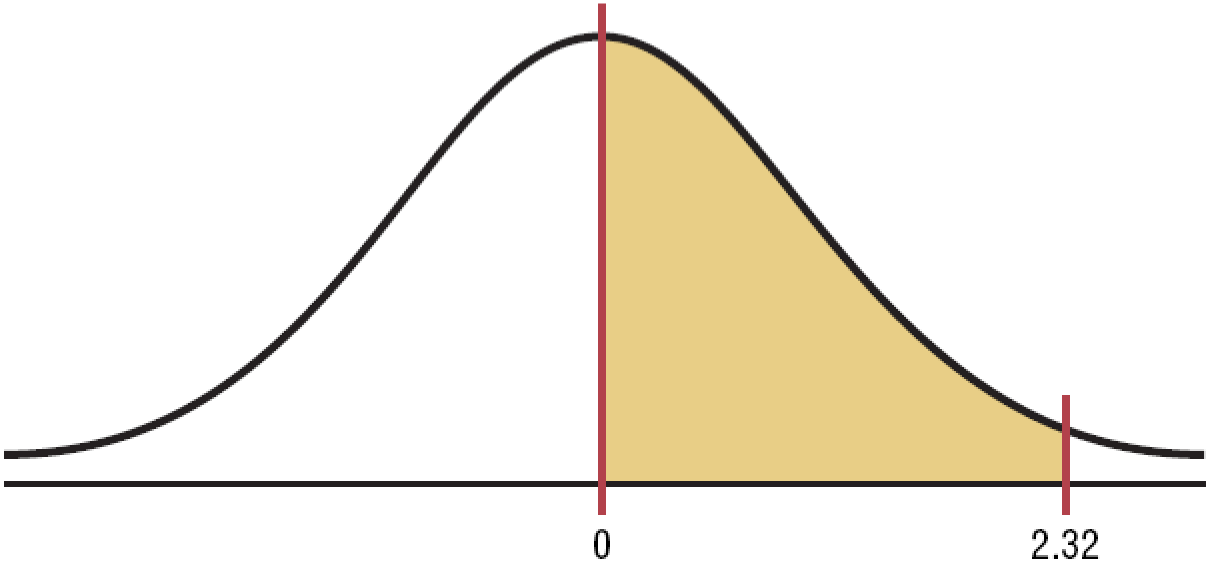
✓ **Example:** Find the area to the left of z = 1.45; P(z < 1.45)




$$P(z < 1.45) = 0.9265$$

# Example

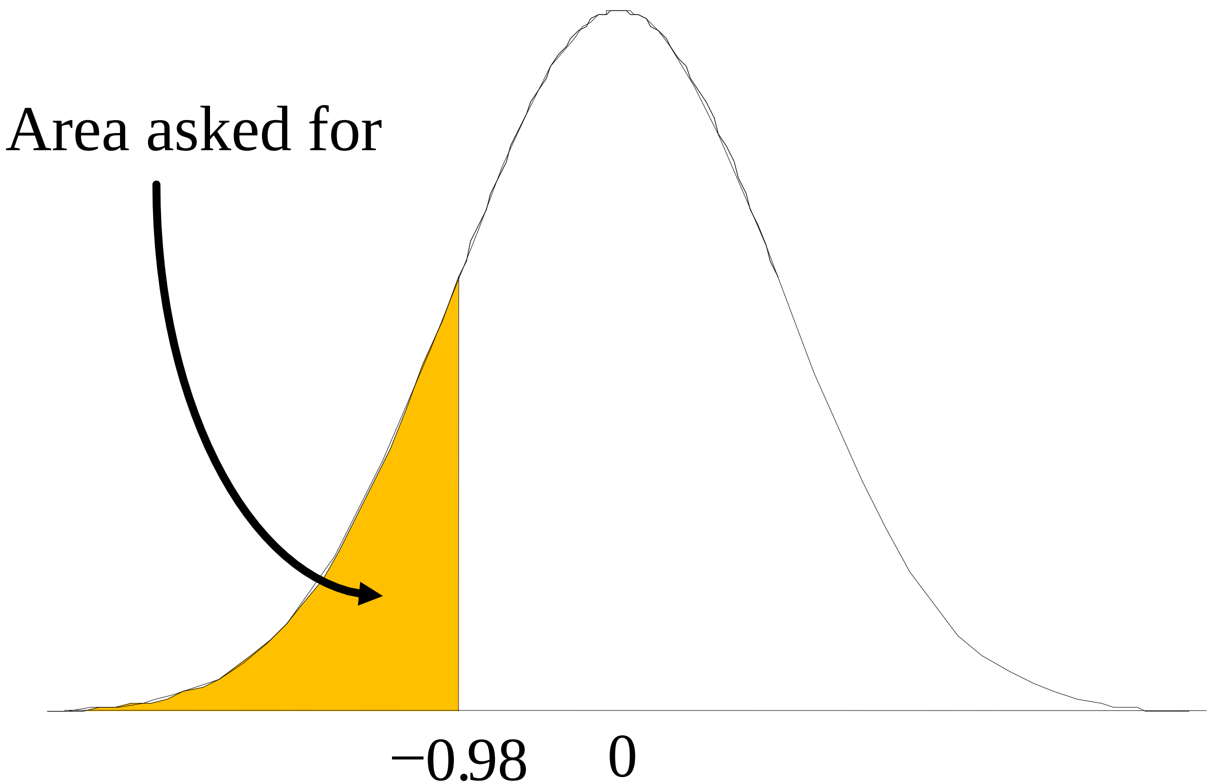
- ✓ **Example:** Find the area between the mean ($z$ = 0) and $z$ = 2.32




The values for $z$ = 2.32 is .9898 and for $z$ = 0 is .5000. The probability is .9898 - .5000 = .4898.

# Example

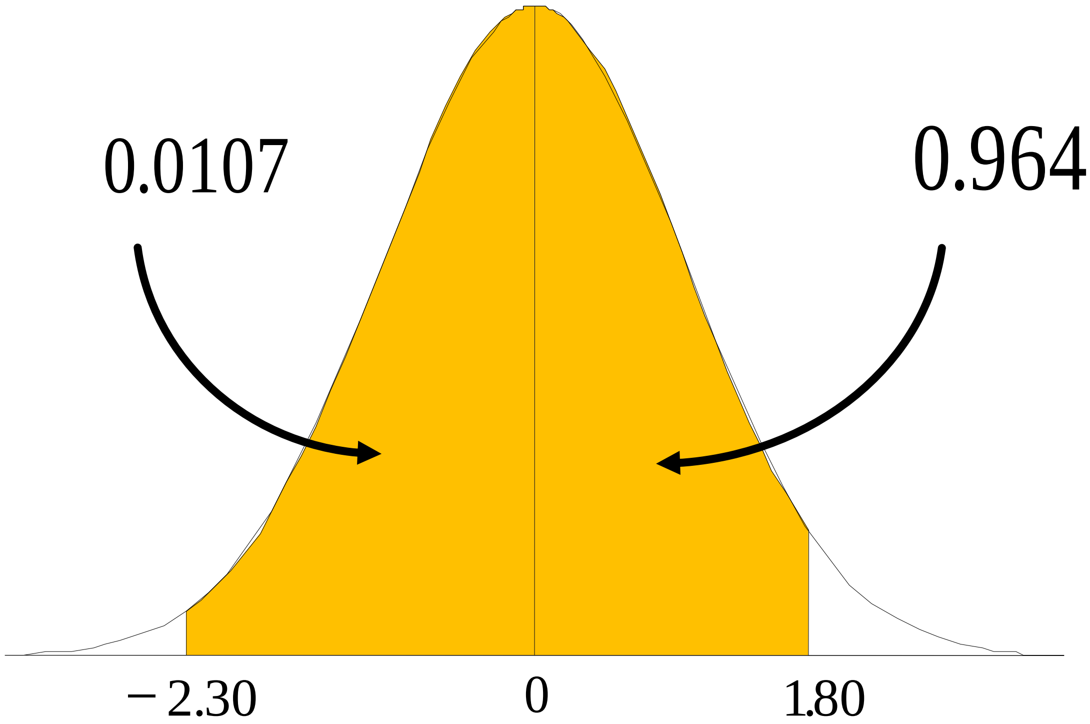
✓ **Example:** Find the area to the left of -0.98; *P*(*z* < -0.98)




$P(z < -0.98) = 0.1635$

# Example

✓ **Example:** Find the area between $z$ = -2.30 and $z$ = 1.80

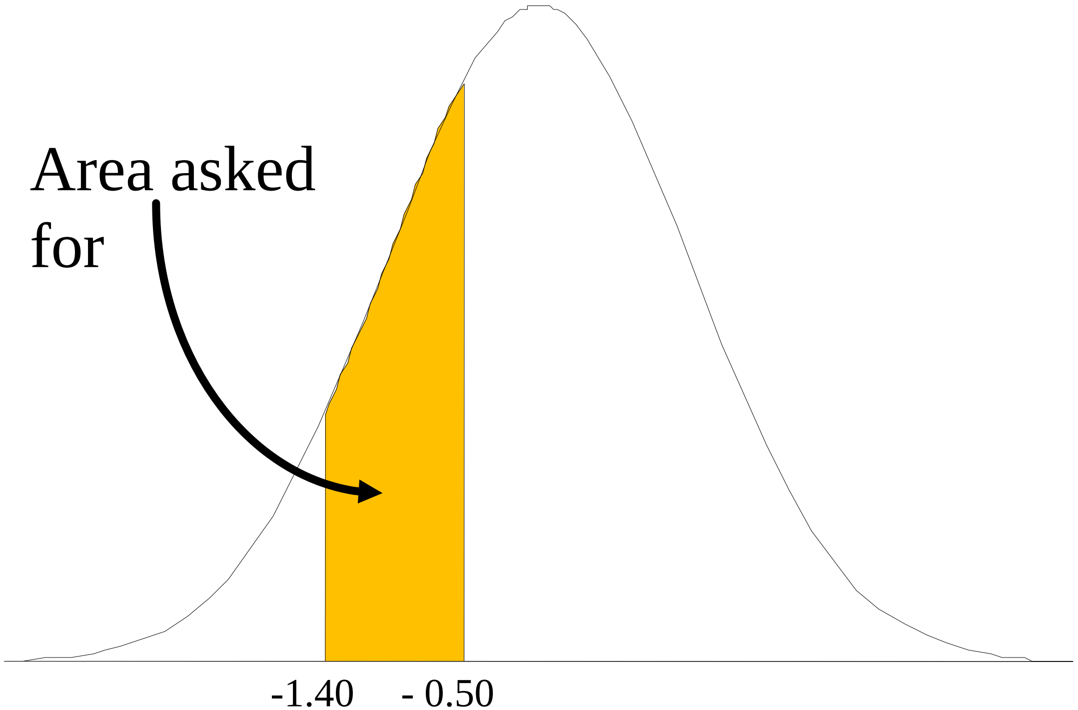



$$P(-2.30 < z < 1.80) = P\ (z{=}1.80){-}P(z{=}{-}2.3){=}\ 0.9641{-}0.0107$$

$$= 0.9534$$

# Example

- ✓ **Example:** Find the area between $z$ = -1.40 and $z$ = -0.50




$$P(-1.40 < z < -0.50) = P(\ Z{=}{-0.5}\ ) - P(Z{=}{-1.40})$$

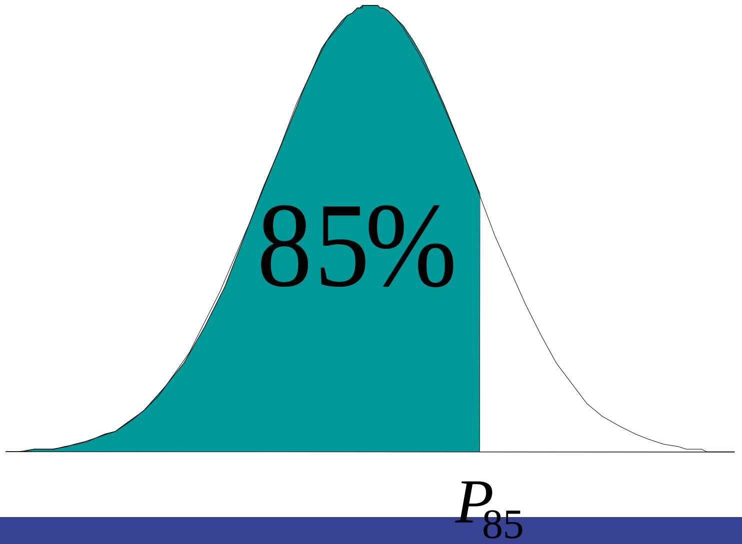
$$= 0.3085 - 0.0808 = 0.2277$$

# Normal Distribution Note

- The normal distribution table may also be used to determine a *z*-score if we are given the area (working backwards)

✓ **Example: What is the z-score associated with the 85th percentile?**



# Solution

In Table 3 Appendix B, find the “area” entry that is closest to 0.8500:

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | | 0.04 | 0.05 |
|---|---|---|---|---|---|---|---|
| ⋮ | | | | | | | |
| 1.0 | | | | 0.8485 | 0.8508 | | |
| ⋮ | | | | | | | |

- The area entry closest to 0.8500 is 0.8485
- The z-score that corresponds to this area is 1.03
- The 85th percentile in a standard normal distribution is 1.03

# 5.3 ~ Applications of Normal Distributions

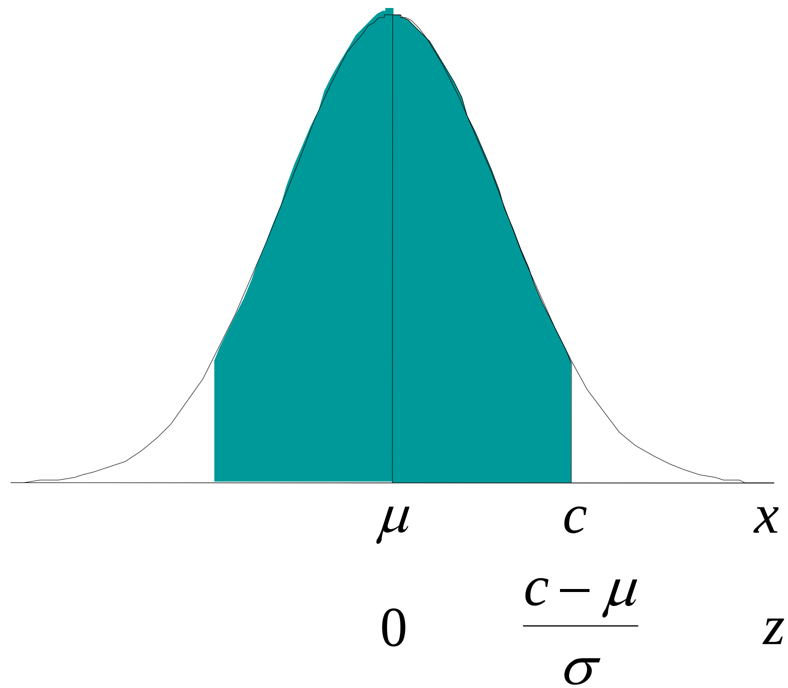
Apply the techniques learned for the *z* distribution to all normal distributions

- Start with a probability question in terms of *x*-values
- Convert, or transform, the question into an equivalent probability statement involving *z*-values

# Standardization

Suppose *x* is a normal random variable with mean $\mu$ and standard deviation $\sigma$

- The random variable $z = \dfrac{x - \mu}{\sigma}$ has a standard normal distribution



# Example

- ✓ **Example:** A bottling machine is adjusted to fill bottles with a mean of 32.0 oz of soda and standard deviation of 0.02. Assume the amount of fill is normally distributed and a bottle is selected at random:

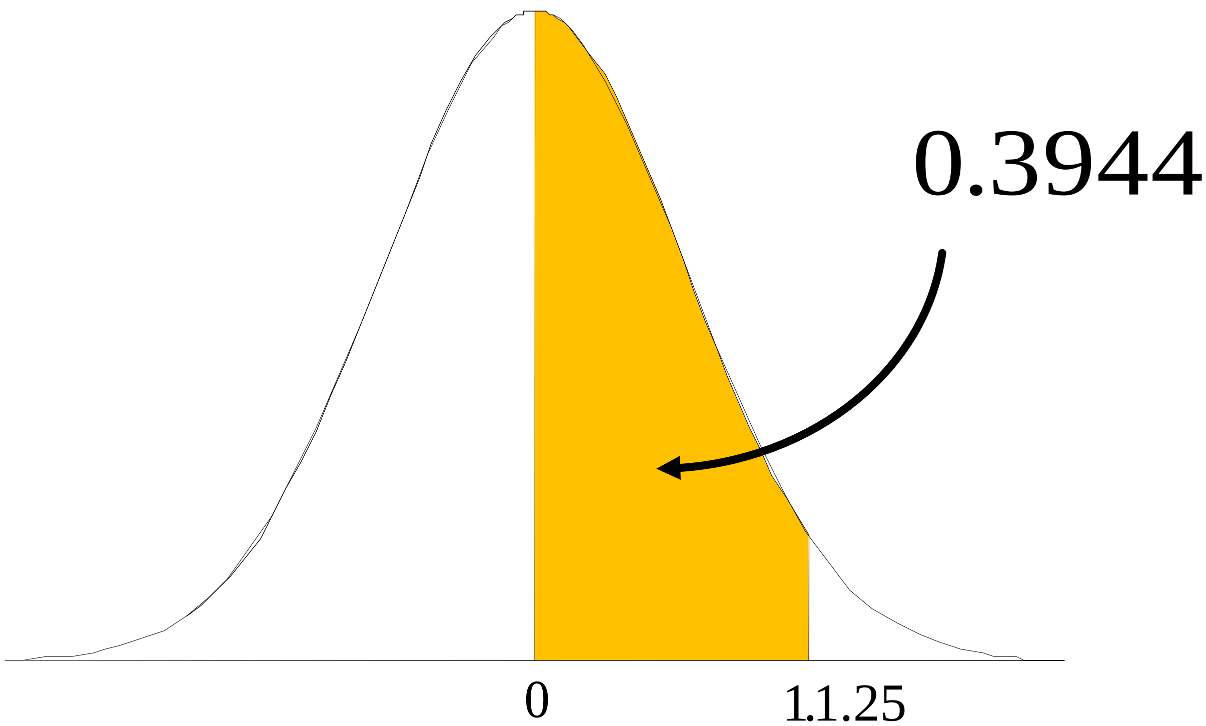
1) Find the probability the bottle contains between 32.00 oz and 32.025 oz
2) Find the probability the bottle contains more than 31.97 oz

**Solutions:**

1) When $x = 32.00;\quad z = \dfrac{32.00 - \mu}{\sigma} = \dfrac{32.00 - 32.0}{0.02} = 0.00$

When $x = 32.025;\quad z = \dfrac{32.025 - \mu}{\sigma} = \dfrac{32.025 - 32.0}{0.02} = 1.25$

# Solution Continued

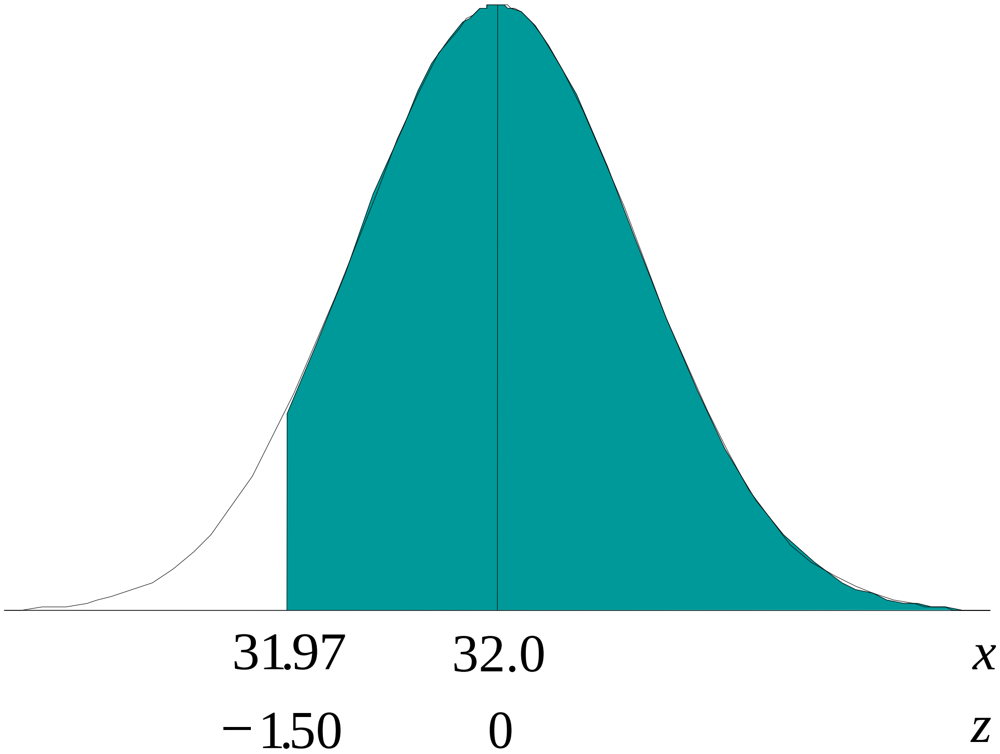



$$P(32.0 < x < 32.025) = P\left(\frac{32.0 - 32.0}{0.02} < \frac{x - 32.0}{0.02} < \frac{32.025 - 32.0}{0.02}\right)$$

$$= P \ (z{=}1.25)\text{-}p(z{=}0) = 0.8944\text{-}0.5 = 0.3944$$

# Example, Part 2

2)




$$P(x > 31.97) = P\left(\frac{x - 32.0}{0.02} > \frac{31.97 - 32.0}{0.02}\right) = P\ (z > -1.50)$$

$1\text{-}\ p(z\!=\!-1.50) = 1 - 0.0668\ = 0.9332$

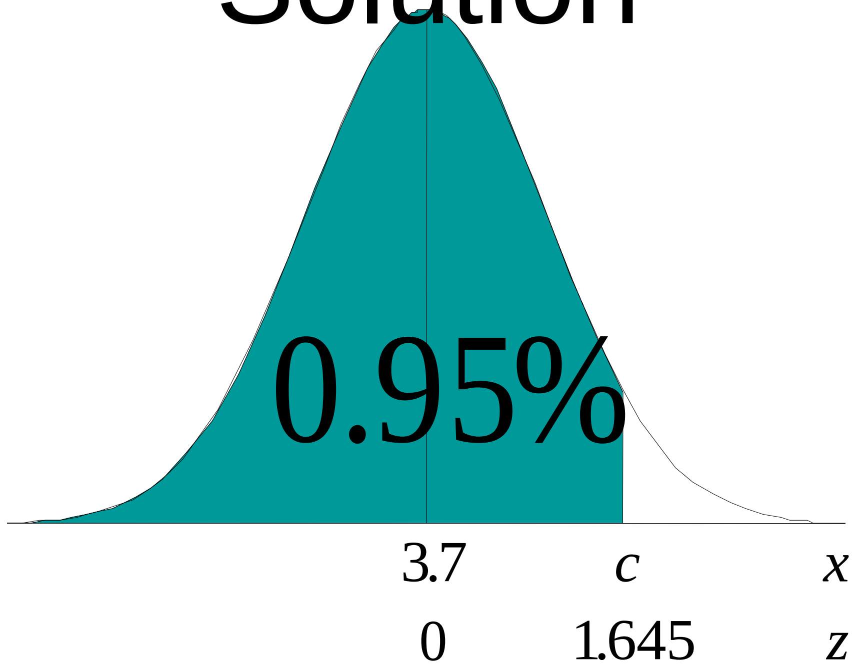
# Notes

The normal table may be used to answer many kinds of questions involving a normal distribution

- Often we need to find a *cutoff point*: a value of $x$ such that there is a certain probability in a specified interval defined by $x$

✓ **Example:** The waiting time $x$ at a certain bank is approximately normally distributed with a mean of 3.7 minutes and a standard deviation of 1.4 minutes. The bank would like to claim that 95% of all customers are waited on by a teller within c minutes. Find the value of $c$ that makes this statement true.

# Solution




$$P(x \le c) = 0.95$$

$$P\left(\frac{x-3.7}{1.4} \le \frac{c-3.7}{1.4}\right) = 0.95$$

$$P\left(z \le \frac{c-3.7}{1.4}\right) = 0.95$$

$$\frac{c-3.7}{1.4} = 1.645$$

$$c = (1.645)(1.4) + 3.7 = 6.003$$

$$c \approx 6 \text{ minutes}$$