1 What is a degree distribution (Olhede and Wolfe, 2012)

That's the model,... that are my asumptions .... $||d||_1 > 0$.

Let us assume that we have a graph $G = (V, E)$ which can be described by an adjacency matrix $(A_{ij})_{i,j \in V}$. Furthermore, lets say that

$$\begin{aligned} A_{ij} &\sim \mathrm{Bern}(p_{ij}) \quad \text{for } i < j, \quad A_{ij} = A_{ji} \quad \text{and} \quad A_{ij} = 0, \\ p_{ij} &= \pi_i \pi_j \quad \text{for} \quad 1 \leq i < j \leq n, \quad \pi_i \in \Pi_n \subseteq [0,1] \,. \end{aligned}$$

I want to generalize the Power law central limit Theorem for π of the paper Olhede and Wolfe (2012), namely

Theorem (1.1) Power law central limit theorem for π
Assume as in Theorem 1 that each $\pi_i = \theta_n i^{-\gamma}$, with $\gamma \in (0,1)$. Define

$$\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}$$

as an estimator of π_i, and assume $\mathbb{E}(d_i|\pi)$ is growing in n. Then as $n \to \infty$, the standardized variate

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}}}$$

converges in distribution to a Normal$(0,1)$ *random variable.*

In Theorem 1 it is in addition assumed that γ is a fixed and that θ_n is a n-depending sequence satisfying $\theta_n \leq 1$ for all n.
For the first approach I left out the power law assumption and analyzed which requirements are then needed to fulfill the Central Limit Theorem (CLT).

Theorem (1.2) Central Limit Theorem for the estimator of π
Assume that we have a graph $G = (V, E)$ which can be described by an adjacency matrix $(A_{ij})_{i,j \in V}$. Furthermore, lets say that

$$\begin{aligned} A_{ij} &\sim \mathit{Bern}(p_{ij}) \quad \textit{for } i < j, \quad A_{ij} = A_{ji} \quad \textit{and} \quad A_{ij} = 0, \\ p_{ij} &= \pi_i \pi_j \quad \textit{for} \quad 1 \leq i < j \leq n, \quad \pi_i \in \Pi_n \subseteq [0,1] \,. \end{aligned}$$

Define an estimator for π_i as

$$\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}.$$

$\hat{\pi}_i$ converges in distribution to Normal$(0,1)$ *if we additionally assume that*

$$\begin{aligned} &1. \quad ||\pi||_1 \overset{n\to\infty}{\rightarrow} \infty, \\ &2. \quad 0 < \pi_i, \\ &3. \quad \frac{\pi_i ||\pi||_2^2}{||\pi||_1} \overset{n\to\infty}{\rightarrow} 0. \end{aligned}$$

The proof for this Theorem works analogously to the proof of the Power law central limit Theorem for π in Olhede and Wolfe (2012).

Unfortunately, an Erdös Rényi graph disagree with the third assumptions that

$$2. \quad \frac{\pi_i ||\pi||_2^2}{||\pi||_1} = \pi_1^2 \overset{n\to\infty}{\nrightarrow} 0.$$

Furthermore, I can proof that this estimator in the Erdös Rényi case does not converge in distribution to a Normal(0,1).

Theorem (1.3) Central limit theorem
Assume we have a simple graph where $A_{ij} \sim Bern(\pi_i \cdot \pi_j)$ with $\pi_i = \sqrt{p}$ for all $i \in \mathbb{N}$. Define an estimator for π_i as

$$\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}.$$

The standardized estimator

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}}}$$

does in general not converges in distribution to a Normal$(0,1)$ *random variable.*

Proof.

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}}} = \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}}} \overset{identical}{=} \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{\pi_i}{n\pi_i}}}$$
$$= \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{1}{n}}} = \left(\frac{d_i}{\sqrt{||d||_1}} - \pi_i\right) \cdot \sqrt{n}$$
$$= \frac{d_i}{\sqrt{\frac{1}{n}||d||_1}} - \pi_i \cdot \sqrt{n}$$

If we now substitute $p = \pi_i^2$, it follows

$$= \frac{d_i}{\sqrt{\bar{d}}} - \sqrt{p \cdot n} = \frac{d_i - \sqrt{\bar{d}} \cdot \sqrt{pn}}{\sqrt{\bar{d}}}$$

This ration can be reformulated as follows

$$= \left[\frac{d_i - \mathbb{E}(d_i|\pi)}{\sqrt{(n-1)p(1-p)}} + \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}}\right] \cdot \frac{\sqrt{(n-1)p(1-p)}}{\sqrt{\bar{d}}} \tag{1.1}$$
$$= \left[\frac{d_i - \mathbb{E}(d_i|\pi)}{\sqrt{\mathrm{Var}(d_i|\pi)}} + \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}}\right] \cdot \frac{\sqrt{\mathbb{E}(d_i|\pi)}\sqrt{(1-p)}}{\sqrt{\bar{d}}} \tag{1.2}$$
$$= [\quad T_1 \quad + \quad T_2 \quad] \quad \cdot \quad T_3 \tag{1.3}$$

Since $\frac{\mathrm{Var}(d_i|\pi)}{\mathbb{E}(d_i|\pi)^2} = \frac{(n-1)p(1-p)}{((n-1)p)^2}$ converges in n to 0 it follows via Chebyshev inequality

$$\mathbb{P}\left(|\frac{d_i}{\mathbb{E}(d_i|\pi)} - 1| \geq \epsilon\sqrt{\frac{\mathrm{Var}(d_i|\pi)}{\mathbb{E}(d_i|\pi)^2}}\right) \leq \frac{1}{\epsilon^2} \quad \text{for any } \epsilon > 0$$

that $\frac{d_i}{\mathbb{E}(d_i|\pi)}$ converges to one. Thus, d_i for sufficiently large n will be close to $\mathbb{E}(d_i|\pi)$. It follows that T_1 converges to a standard Normal(0,1) due to De Moivre Laplace.

We need to show that $T_2 = \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}}$ converges in probability to 0. It follows that

$$\frac{1}{n-1} \cdot \bar{d} = \frac{1}{n-1} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} \sum_{j \neq i} A_{ij} = \frac{1}{n-1} \cdot \frac{1}{n} \cdot 2 \cdot \sum_{i=1}^{n} \sum_{j > i} A_{ij}$$

The A_{ij} are $\binom{n}{2}$ independent Bernoulli distributed random variables. Thus, $\frac{1}{n-1} \cdot \bar{d}$ is the sample mean of independent random variables. It follows

$$\frac{1}{n-1} \cdot \bar{d} \quad \overset{P}{\to} \quad \mathbb{E}(A_{ij}|\pi) = p$$

according to the weak law of large numbers. Furthermore, due to the continuous mapping theorem

$$
\begin{aligned}
&\Rightarrow \quad \sqrt{\frac{1}{n-1}\cdot \bar{d}} \quad \xrightarrow{P} \quad \sqrt{p}.\\
&\Rightarrow \quad \sqrt{\bar{d}}\cdot\sqrt{np} = \sqrt{\frac{n-1}{n-1}}\sqrt{\bar{d}}\cdot\sqrt{np} = \sqrt{\frac{1}{n-1}\bar{d}}\cdot\sqrt{(n-1)np}\\
&\quad = \sqrt{\frac{1}{n-1}\bar{d}}\cdot\sqrt{(n-1)(n-1)p}\cdot\sqrt{\frac{n}{n-1}} \quad \xrightarrow{P} \quad \mathbb{E}(d_i|\pi)\cdot 1\\
&\Rightarrow \quad T_2 = \frac{\mathbb{E}(d_i|\pi)-\sqrt{\bar{d}}\cdot\sqrt{np}}{\sqrt{(n-1)p(1-p)}} \quad \xrightarrow{P} \quad 0.
\end{aligned}
$$

With the same reasoning we conclude

$$
\begin{aligned}
T_3 &= \frac{\sqrt{\mathbb{E}(d_i|\pi)}}{\sqrt{\bar{d}}}\cdot\sqrt{1-p} = \sqrt{\frac{1}{n-1}}\cdot\frac{\sqrt{\mathbb{E}(d_i|\pi)}}{\sqrt{\frac{1}{n-1}\bar{d}}}\cdot\sqrt{1-p}\\
&= \sqrt{\frac{1}{n-1}}\cdot\frac{\sqrt{(n-1)p}}{\sqrt{\frac{1}{n-1}\bar{d}}}\cdot\sqrt{1-p}\\
&\xrightarrow{P} \quad \frac{\sqrt{p}}{\sqrt{p}}\cdot\sqrt{1-p} \quad = \sqrt{1-p}
\end{aligned}
$$

Using the continuous mapping theorem we therefore deduce

$$
\frac{\hat{\pi}_i-\pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}}} \quad \xrightarrow{d} \quad \sqrt{1-p}\cdot \text{Normal}(0,1).
$$

$\square$

That broad up the idea how the estimator would need to look like to work as well in the power law case as for Erdös Rényi graphs. As a first step, we figured that if we change the denominator from $\sqrt{\frac{\pi_i}{||\pi||_1}}$ to $\sqrt{\frac{\pi_i}{||\pi||_1}-\pi_i^2\frac{||\pi||_2^2}{||\pi||_1^2}}$ the CLT holds in the Erdös Rényi case.

Theorem (1.4) Central limit theorem
Assume we have a simple graph where $A_{ij}\sim Bern(\pi_i\cdot\pi_j)$ with $\pi_i=\sqrt{p}$ for all $i\in\mathbb{N}$. Define an estimator for π_i as

$$
\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}.
$$

The standardized estimator

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}}$$

converges in distribution to a Normal$(0, 1)$ *random variable.*

Proof.

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} = \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} \overset{identical}{=} \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{\pi_i}{n\pi_i} - \pi_i^2 \frac{n\pi_i^2}{n^2\pi_i^2}}}$$

$$= \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{1}{n} \cdot (1 - \pi_i^2)}} = \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{1 - \pi_i^2}} \cdot \sqrt{n}$$

$$= \frac{\frac{d_i}{\sqrt{\frac{1}{n}||d||_1}} - \pi_i \cdot \sqrt{n}}{\sqrt{1 - \pi_i^2}}$$

If we now substitute $p = \pi_i^2$, it follows

$$= \frac{\frac{d_i}{\sqrt{\bar{d}}} - \sqrt{p \cdot n}}{\sqrt{1-p}} = \frac{d_i - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{\bar{d}} \cdot \sqrt{1-p}}$$

This ration can be reformulated as follows

$$= \left[\frac{d_i - \mathbb{E}(d_i|\pi)}{\sqrt{(n-1)p(1-p)}} + \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}} \right] \cdot \frac{\sqrt{(n-1)p}}{\sqrt{\bar{d}}} \tag{1.4}$$

$$= \left[\frac{d_i - \mathbb{E}(d_i|\pi)}{\sqrt{\mathrm{Var}(d_i|\pi)}} + \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}} \right] \cdot \frac{\sqrt{\mathbb{E}(d_i|\pi)}}{\sqrt{\bar{d}}} \tag{1.5}$$

$$= [\quad T_1 \quad + \quad T_2 \quad] \quad \cdot \quad T_3 \tag{1.6}$$

Since $\frac{\mathrm{Var}(d_i|\pi)}{\mathbb{E}(d_i|\pi)^2} = \frac{(n-1)p(1-p)}{((n-1)p)^2}$ converges in n to 0 it follows via Chebyshev inequality

$$\mathbb{P}\left(|\frac{d_i}{\mathbb{E}(d_i|\pi)} - 1| \geq \epsilon \sqrt{\frac{\mathrm{Var}(d_i|\pi)}{\mathbb{E}(d_i|\pi)^2}} \right) \leq \frac{1}{\epsilon^2} \quad \text{for any } \epsilon > 0$$

that $\frac{d_i}{\mathbb{E}(d_i|\pi)}$ converges to one. Thus, d_i for sufficiently large n will be close to $\mathbb{E}(d_i|\pi)$. It follows that T_1 converges to a standard Normal(0,1) due to De Moivre

Laplace.

We need to show that $T_2 = \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}}$ converges in probability to 0. It follows that

$$\frac{1}{n-1} \cdot \bar{d} = \frac{1}{n-1} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} \sum_{j \neq i} A_{ij} = \frac{1}{n-1} \cdot \frac{1}{n} \cdot 2 \cdot \sum_{i=1}^{n} \sum_{j > i} A_{ij}$$

The A_{ij} are $\binom{n}{2}$ independent Bernoulli distributed random variables. Thus, $\frac{1}{n-1} \cdot \bar{d}$ is the sample mean of independent random variables. It follows

$$\frac{1}{n-1} \cdot \bar{d} \quad \overset{P}{\to} \quad \mathbb{E}(A_{ij}|\pi) = p$$

according to the weak law of large numbers. Furthermore, due to the continuous mapping theorem

$$\begin{aligned}
\Rightarrow \quad & \sqrt{\frac{1}{n-1} \cdot \bar{d}} \quad \overset{P}{\to} \quad \sqrt{p}. \\
\Rightarrow \quad & \sqrt{\bar{d}} \cdot \sqrt{np} = \sqrt{\frac{n-1}{n-1}} \sqrt{\bar{d}} \cdot \sqrt{np} = \sqrt{\frac{1}{n-1} \bar{d}} \cdot \sqrt{(n-1)np} \\
& = \sqrt{\frac{1}{n-1} \bar{d}} \cdot \sqrt{(n-1)(n-1)p} \cdot \sqrt{\frac{n}{n-1}} \quad \overset{P}{\to} \quad \mathbb{E}(d_i|\pi) \cdot 1 \\
\Rightarrow \quad & T_2 = \frac{\mathbb{E}(d_i|\pi) - \sqrt{\bar{d}} \cdot \sqrt{np}}{\sqrt{(n-1)p(1-p)}} \quad \overset{P}{\to} \quad 0.
\end{aligned}$$

With the same reasoning we conclude

$$\begin{aligned}
T_3 = \frac{\sqrt{\mathbb{E}(d_i|\pi)}}{\sqrt{\bar{d}}} &= \sqrt{\frac{1}{n-1}} \cdot \frac{\sqrt{\mathbb{E}(d_i|\pi)}}{\sqrt{\frac{1}{n-1}\bar{d}}} \\
&= \sqrt{\frac{1}{n-1}} \cdot \frac{\sqrt{(n-1)p}}{\sqrt{\frac{1}{n-1}\bar{d}}} \\
&\overset{P}{\to} \quad \frac{\sqrt{p}}{\sqrt{p}} \quad = 1
\end{aligned}$$

Using the continuous mapping theorem we therefore deduce

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} \quad \overset{d}{\to} \quad \text{Normal}(0, 1).$$

$\square$

In a second step, I proved the convergence in distribution of the estimator with new standardization also for the power law case.

Theorem (1.5) Central limit theorem
Assume we have a simple graph where $A_{ij} \sim Bern(\pi_i \cdot \pi_j)$ *with* $\pi_i = \theta_n i^{-\gamma}$*, for all* $i \in \mathbb{N}$ *and with* $\gamma \in (0,1)$*. Furthermore, let* θ_n *be a* n*- depending sequence satisfying* $\theta_n \leq 1$ *for all* n *and assume* $\mathbb{E}(d_i|\pi)$ *is growing in* n*. Define an estimator for* π_i *as*

$$\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}.$$

The standardized estimator

$$\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}}$$

converges in distribution to a Normal$(0,1)$ *random variable.*

Proof.

$$\begin{aligned}
\frac{\hat{\pi}_i - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} &= \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} \\
&= \frac{\frac{d_i}{\sqrt{||d||_1}} - \pi_i}{\sqrt{\pi_i \cdot ||\pi||_1 - \pi_i^2 ||\pi||_2^2}} \cdot ||\pi||_1 \\
&= \frac{d_i - \pi_i \sqrt{||d||_1}}{\sqrt{\pi_i ||\pi||_1 - \pi_i^2 ||\pi||_2^2}} \cdot \frac{||\pi||_1}{\sqrt{||d||_1}} \\
&= \left[\frac{d_i - \mathbb{E}(d_i|\pi)}{\sqrt{\mathbb{E}(d_i|\pi)}} + \frac{\mathbb{E}(d_i|\pi) - \pi_i \sqrt{||d||_1}}{\sqrt{\mathbb{E}(d_i|\pi)}} \right] \cdot \frac{||\pi||_1}{\sqrt{||d||_1}} \cdot \sqrt{\frac{\mathbb{E}(d_i|\pi)}{\pi_i ||\pi||_1}} \\
&\quad \cdot \sqrt{\frac{\pi_i ||\pi||_1}{\pi_i ||\pi||_1 - \pi_i^2 ||\pi||_2^2}}
\end{aligned}$$

The first row of the last equation is equivalent to the "$[T_1 + T_2]\, T_3$" step in the supplementary material to the paper Olhede and Wolfe (2012). It needs to be shown that the additional factor

$$\sqrt{\frac{\pi_i ||\pi||_1}{\pi_i ||\pi||_1 - \pi_i^2 ||\pi||_2^2}} = \sqrt{\frac{1}{1 - \pi_i \frac{||\pi||_2^2}{||\pi||_1}}}$$

converges to 1. Due to Lemma 1 in the supplementary material, we know that

$$\begin{aligned}
||\pi||_1 =& \theta_n \sum_{i=1}^{n} i^{-\gamma} = \theta_n \left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right] \\
||\pi||_2^2 =& \theta_n^2 \sum_{i=1}^{n} i^{-2\gamma} \\
=& \theta_n^2 \begin{cases} \frac{n^{1-2\gamma}}{1-2\gamma} + \mathcal{O}(1), & \text{if } 0 < \gamma < 1/2; \\ \ln(n) + \gamma_E + \mathcal{O}(n^{-1}), & \text{if } \gamma = 1/2; \\ \zeta(2\gamma) + \mathcal{O}(n^{-(2\gamma-1)}), & \text{if } 1/2 < \gamma < 1 \end{cases}
\end{aligned}$$

where γ_E is the Euler-Mascheroni constant and $\zeta(.)$ is the Riemann zeta function.
Case 1: $0 < \gamma < 1/2$

$$\begin{aligned}
\pi_i \frac{||\pi||_2^2}{||\pi||_1} =& \theta_n i^{-\gamma} \frac{\theta_n^2 \left[\frac{n^{1-2\gamma}}{1-2\gamma} + \mathcal{O}(1)\right]}{\theta_n \left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right]} \\
=& \frac{i^{-\gamma}\theta_n^2 \left[\frac{1}{(1-2\gamma)n^{\gamma}} + \mathcal{O}(\frac{1}{n^{1-\gamma}})\right]}{\left[\frac{1}{1-\gamma} + \mathcal{O}(\frac{1}{n^{1-\gamma}})\right]}
\end{aligned}$$

Since $\theta_n \leq 1$ as well as $i^{-\gamma} \leq 1$ this limit goes to zero in n.
Case 2: $\gamma = 1/2$

$$\begin{aligned}
\pi_i \frac{||\pi||_2^2}{||\pi||_1} =& \theta_n i^{-\gamma} \frac{\theta_n^2 \left[\ln(n) + \gamma_E + \mathcal{O}(n^{-1})\right]}{\theta_n \left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right]} \\
=& \frac{i^{-\gamma}\theta_n^2 \left[\ln(n) + \gamma_E + \mathcal{O}(n^{-1})\right]}{\left[2n^{1/2} + \mathcal{O}(1)\right]}
\end{aligned}$$

As $\theta_n, i^{-\gamma} \leq 1$ and γ_E is a constant, it follows again that this limit goes to zero in n.
Case 3: $1/2 < \gamma < 1$

$$\begin{aligned}
\pi_i \frac{||\pi||_2^2}{||\pi||_1} =& \theta_n i^{-\gamma} \frac{\theta_n^2 \left[\zeta(2\gamma) + \mathcal{O}(n^{-(2\gamma-1)})\right]}{\theta_n \left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right]} \\
=& \frac{i^{-\gamma}\theta_n^2 \left[\zeta(2\gamma) + \mathcal{O}(n^{-(2\gamma-1)})\right]}{\left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right]}
\end{aligned}$$

$\theta_n, i^{-\gamma} \leq 1$, $(2\gamma - 1) > 0$ and the zeta function ζ converges for 2γ. It results that as well for this case the limit goes to zero in n. Thus, the additional factor among which the estimator stated in Olhede and Wolfe (2012) differs from the estimator stated here

$$\sqrt{\frac{1}{1-\pi_i\frac{||\pi||_2^2}{||\pi||_1}}}$$

converges to 1 in n for all $0<\gamma<1$. $\square$

Theorem (1.6) Central limit theorem
Assume we have a n-node simple graph whose edges are independent Bern($\pi_i\pi_j$) random variables. Consider d to be the degree vector. Define $\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}$ as an estimator for π_i. In addition, let us consider that the Var($d_i|\pi$) is growing to ∞ in n. Then as $n\to\infty$, the standardized estimator $\frac{\hat{\pi}_i-\pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}-\pi_i^2\frac{||\pi||_2^2}{||\pi||_1^2}}}$ converges in distribution to a Normal$(0,1)$ *random variable.*

Proof.

$$\begin{aligned}
\frac{\hat{\pi}_i-\pi_i}{\sqrt{\frac{\pi_i}{||\pi||_1}-\pi_i^2\frac{||\pi||_2^2}{||\pi||_1^2}}} &= \frac{\frac{d_i}{\sqrt{||d||_1}}-\frac{\pi_i\sqrt{||d||_1}}{\sqrt{||d||_1}}}{\sqrt{\pi_i||\pi||_1-\pi_i^2||\pi||_2^2}}\sqrt{||\pi||_1^2} \\
&= \left[\frac{d_i-\mathbb{E}(d_i|\pi)}{\sqrt{\mathrm{Var}(d_i|\pi)}}+\frac{\mathbb{E}(d_i|\pi)-\pi_i\sqrt{||d||_1}}{\sqrt{\mathrm{Var}(d_i|\pi)}}\right]\frac{||\pi||_1}{\sqrt{||d||_1}}\frac{\sqrt{\mathrm{Var}(d_i|\pi)}}{\sqrt{\pi_i||\pi||_1-\pi_i^2||\pi||_2^2}} \\
&= [T_1+T_2]\,T_3
\end{aligned}$$

...

<u>Term 1</u>

The Lindeberg-Feller Central Limit Theorem states for the sum of independent but not identical distributed random variables converence in distribution to a *Normal*$(0,1)$. The assumption that the Var$(d_i|\pi) \overset{n\to\infty}{\to} \infty$ is a sufficient condition for this result to hold (the Lyapunov condition for $\delta=1$ (Lehmann, 1998, p. 572)). Thus, we conclude that $T_1 \overset{d}{\to}$ *Normal*$(0,1)$.

<u>Term 2</u>

$$\begin{aligned}
T_2 &= \frac{\mathbb{E}(d_i|\pi)-\pi_i\sqrt{||d||_1}}{\sqrt{\mathrm{Var}(d_i|\pi)}} \\
&= \frac{\mathbb{E}(d_i|\pi)-\pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\mathrm{Var}(d_i|\pi)}} - \frac{\pi_i\sqrt{||d||_1}-\pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\mathrm{Var}(d_i|\pi)}}
\end{aligned}$$

The first ratio ...

$$\frac{\mathbb{E}(d_i|\pi) - \pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\text{Var}(d_i|\pi)}} = \frac{\pi_i||\pi||_1 - \pi_i^2 - \pi_i\sqrt{||\pi||_1^2 - ||\pi||_2^2}}{\sqrt{\text{Var}(d_i|\pi)}}$$
$$= \frac{\pi_i||\pi||_1\left[1 - \sqrt{1 - ||\pi||_2^2/||\pi||_1^2}\right] - \pi_i^2}{\sqrt{\text{Var}(d_i|\pi)}}$$

If we now Taylor expand $\sqrt{1-x}$ at 0, we will get

$$\frac{\mathbb{E}(d_i|\pi) - \pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\text{Var}(d_i|\pi)}}$$
$$= \frac{\pi_i||\pi||_1\left[1 - \left(1 - (1-0)^{-1/2}(||\pi||_2^2/||\pi||_1^2 - 0) - \frac{1}{2}(1-0)^{-3/2}\frac{(||\pi||_2^2/||\pi||_1^2-0)^2}{2} + \dots\right)\right] - \pi_i^2}{\sqrt{\text{Var}(d_i|\pi)}}$$
$$= \frac{\pi_i||\pi||_1\left[1 - \left(1 - ||\pi||_2^2/||\pi||_1^2 + o\left(||\pi||_2^2/||\pi||_1^2\right)\right)\right] - \pi_i^2}{\sqrt{\text{Var}(d_i|\pi)}}$$

The last reformulation is true because $\left(||\pi||_2^2/||\pi||_1^2\right)^k = \left(\frac{\sum \pi_i^2}{(\sum \pi_i)^2}\right)^k \leq \left(\frac{\sum \pi_i}{(\sum \pi_i)^2}\right)^k = \left(\frac{1}{||\pi||_1}\right)^k$. Since $||\pi||_1 \to \infty$ it follows that $\left(||\pi||_2^2/||\pi||_1^2\right)^k = o\left(||\pi||_2^2/||\pi||_1^2\right)$.

$$\frac{\mathbb{E}(d_i|\pi) - \pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\text{Var}(d_i|\pi)}} = \frac{\pi_i||\pi||_1 \cdot o\left(||\pi||_2^2/||\pi||_1^2\right) - \pi_i^2}{\sqrt{\text{Var}(d_i|\pi)}}$$
$$= \frac{\pi_i \cdot o\left(||\pi||_2^2/||\pi||_1\right) - \pi_i^2}{\sqrt{\text{Var}(d_i|\pi)}} \qquad = \frac{\pi_i \cdot \mathcal{O}\left(1\right) - \pi_i^2}{\sqrt{\text{Var}(d_i|\pi)}}$$

Since, the variance is growing to ∞ it follows that $\frac{\mathbb{E}(d_i|\pi) - \pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\text{Var}(d_i|\pi)}}$ converges to 0 in n. Next, we want to prove that the second ratio $\frac{\pi_i\sqrt{||d||_1} - \pi_i\sqrt{\mathbb{E}(||d||_1|\pi)}}{\sqrt{\text{Var}(d_i|\pi)}}$ converges in probability to 0. Therefore, we first need to prove the following lemma.

Lemma (1.7)
Assumptions as above. Then,

$$\frac{||d||_1 - \mathbb{E}(||d||_1|\pi)}{\sqrt{Var(d_i|\pi)}} \xrightarrow{d} \text{Normal}(0,1)$$

and $\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}$ *converges in probability to 1.*

Proof. For the convergence in distribution due to the Lindeberg Feller Central Limit theorem it would be sufficient if the variance $\mathrm{Var}(||d||_1|\pi)$ is growing in n to ∞ (the Lyapunov condition $\delta = 1$(Lehmann, 1998, p. 572)). Note that A_{ij} are independent for $i = 1, \dots n$ and $j > i$.

$$\begin{aligned}
\mathrm{Var}(||d||_1|\pi) &= \mathrm{Var}(2\sum_{i=1}^{n}\sum_{j>i} A_{ij}|\pi) = 4\sum_{i=1}^{n}\sum_{j>i}\mathrm{Var}(A_{ij}|\pi)\\
&= 2\sum_{i=1}^{n}\sum_{j\neq i}\mathrm{Var}(A_{ij}|\pi) = 2\sum_{i=1}^{n}\mathrm{Var}(\sum_{j\neq i} A_{ij}|\pi)\\
&= 2\sum_{i=1}^{n}\mathrm{Var}(d_i|\pi)
\end{aligned}$$

We assumed $\mathrm{Var}(d_i|\pi)$ to be growing to ∞. Thus, $\mathrm{Var}(||d||_1|\pi)$ is growing to ∞ as well and we therefore meet the Lyapunov condition.
Due to Chebyshev's inequality, as long as $\mathrm{Var}\left(\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}\right)$ is decreasing to 0, $\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}$ is converging in probability to 1. Assumptions as above. Then,

$$\begin{aligned}
\mathrm{Var}\left(\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}\right) &= \frac{\mathrm{Var}\left(||d||_1\right)}{(\mathbb{E}(||d||_1|\pi))^2} = \frac{2}{\mathbb{E}(||d||_1|\pi)}\left[1+\frac{||\pi||_4^4-||\pi||_2^4}{||\pi||_1^2-||\pi||_2^2}\right]\\
&\leq \frac{2}{\mathbb{E}(||d||_1|\pi)}\left[1+\frac{||\pi||_4^4/||\pi||_1}{||\pi||_1-\frac{||\pi||_2^2}{||\pi||_1}}\right]\\
&\leq \frac{2}{\mathbb{E}(||d||_1|\pi)}\left[1+\frac{1}{||\pi||_1-1}\right]
\end{aligned}$$

We made use of the fact that $\frac{||\pi||_4^4}{||\pi||_1}$ as well as $\frac{||\pi||_2^2}{||\pi||_1}$ are upper bounded by 1. We assumed the variance to grow. Since, $\mathrm{Var}(d_i|\pi) = \mathbb{E}(d_i|\pi) - \pi_i^2\sum_{j\neq i}\pi_j^2$ it implies that $\mathbb{E}(d_i|\pi)$ grows to ∞ which in turn implies that $||\pi||_1$ grows to ∞. Thus, the RHS goes to 0 in n and the second result of the lemma is proved. □

Lemma (1.8)

Assumptions as above. Then, $\frac{\pi_i\left(\sqrt{||d||_1}-\sqrt{\mathbb{E}(||d||_1|\pi)}\right)}{\sqrt{Var(d_i|\pi)}}$ *converges in probability to 0.*

Proof. First, we Taylor expand $\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}$ about 1. Lemma 1.7 verifies that $\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}$ converges in probability to 1. Furthermore,

1. it follows that divided by its standard derivation $\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}$ is bounded in prob-

ability; i.e.,

$$
\begin{aligned}
&\mathbb{P}\left(Var\left(\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}|\pi\right)^{-1/2}\left|\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}-1\right|\geq\epsilon\right)\leq\epsilon^{-2}\\
\Leftrightarrow\quad &\mathbb{P}\left(\left|\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}-1\right|\geq\epsilon\, Var\left(\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}|\pi\right)^{1/2}\right)\leq\epsilon^{-2}\\
\Leftrightarrow\quad &\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}=1+\mathcal{O}_p\left(Var\left(\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}|\pi\right)^{1/2}\right)\\
&\qquad\qquad=1+\mathcal{O}_p\left(\frac{\sqrt{Var\left(||d||_1|\pi\right)}}{\mathbb{E}(||d||_1|\pi)}\right).
\end{aligned}
$$

2. and the ratio $\frac{\sqrt{Var(||d||_1|\pi)}}{\mathbb{E}(||d||_1|\pi)}$ goes to zero in n as long as the expectation is growing because

$$
0<\frac{\sqrt{Var\left(||d||_1|\pi\right)}}{\mathbb{E}(||d||_1|\pi)}\leq\frac{2\sqrt{\mathbb{E}(||d||_1|\pi)}}{\mathbb{E}(||d||_1|\pi)}=\frac{2}{\sqrt{\mathbb{E}(||d||_1|\pi)}}
$$

This two consequences from Lemma 1.7 are necessary for a Taylor expansion in probability (Brockwell and Davis, 1991, p. 201). In addition, it is needed that the square root function has continous derivatives at 1. Due to this three conditions, we can Taylor expand $\sqrt{\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}}$ as

$$
\sqrt{\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}}=1+\frac{1}{2}\left(\frac{||d||_1}{\mathbb{E}(||d||_1|\pi)}-1\right)+o_p\left(\frac{\sqrt{Var\left(||d||_1|\pi\right)}}{\mathbb{E}(||d||_1|\pi)}\right).
$$

□

□

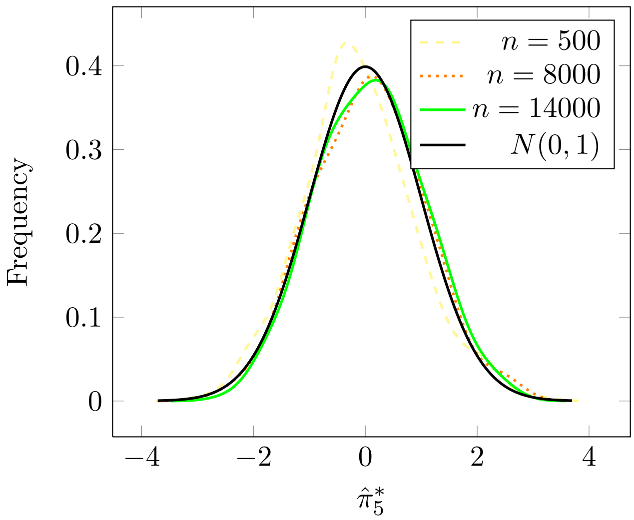

Figure 1.1: Comparison between empirical densities of estimators $\hat{\pi}_5^*$ for different number of edges n and $N(0, 1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $\gamma = 0.01$, $\theta_n = 1$)

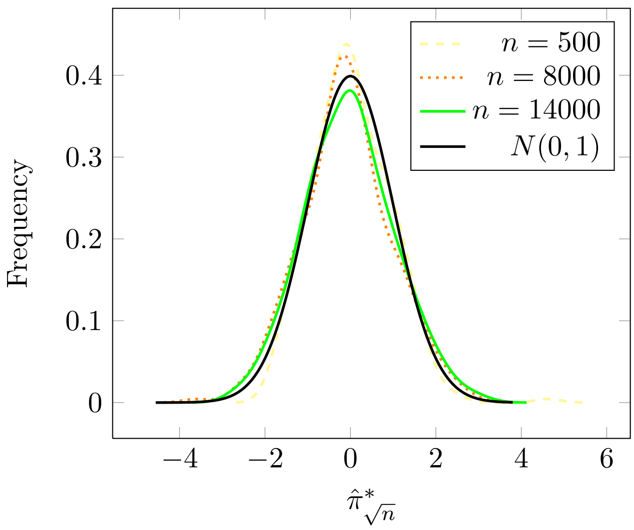

Figure 1.2: Comparison between empirical densities of estimators $\hat{\pi}_{\sqrt{n}}^*$ for different number of edges n and $N(0, 1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $\gamma = 0.01$, $\theta_n = 1$)

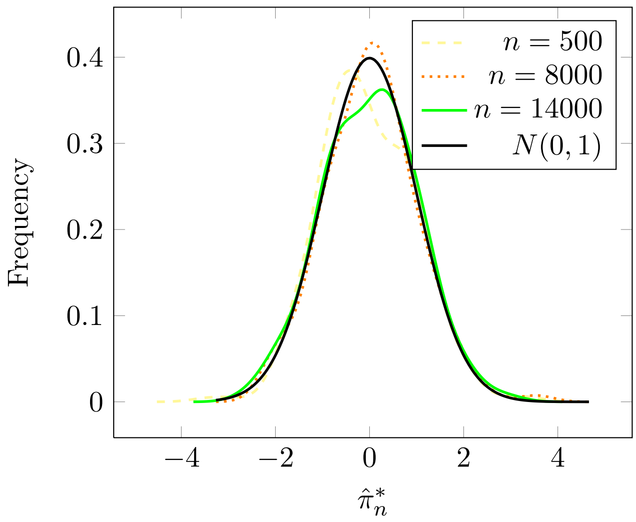

Figure 1.3: Comparison between empirical densities of estimators $\hat{\pi}^*_n$ for different number of edges n and $N(0, 1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $\gamma = 0.01$, $\theta_n = 1$)

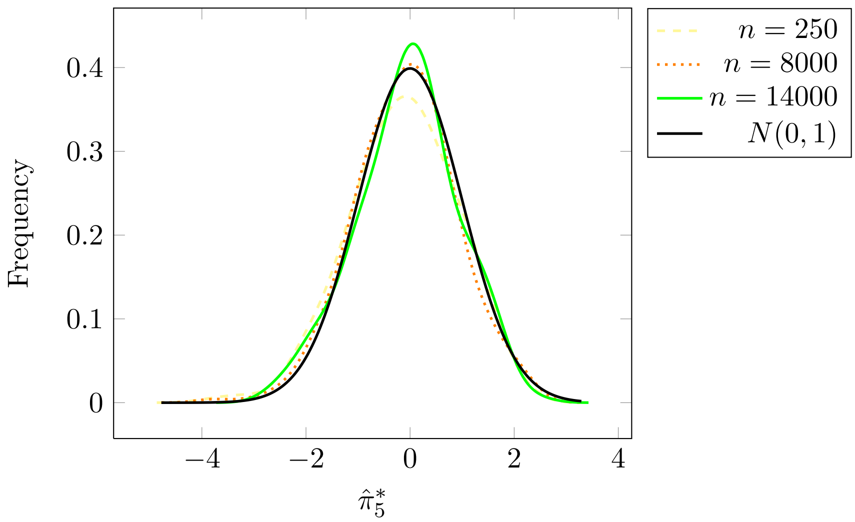

Figure 1.4: Comparison between empirical densities of estimators $\hat{\pi}^*_5$ for different number of edges n and $N(0, 1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $\gamma = 0.5$, $\theta_n = 1$)

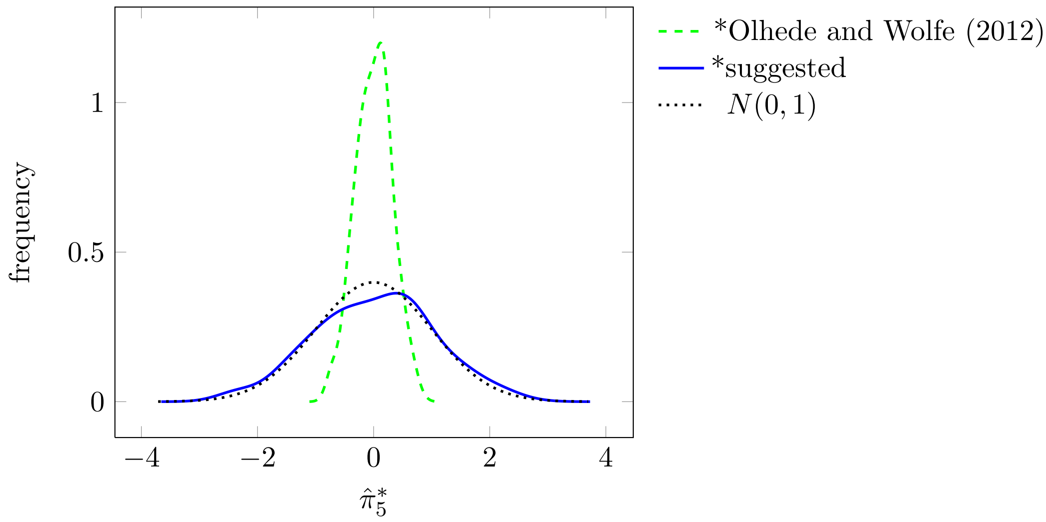

Figure 1.5: Comparison between densities of estimators $\hat{\pi}_5^*$ with different standardizations * and $N(0,1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $n = 8000$, $\gamma = 0.01$, $\theta_n = 1$)

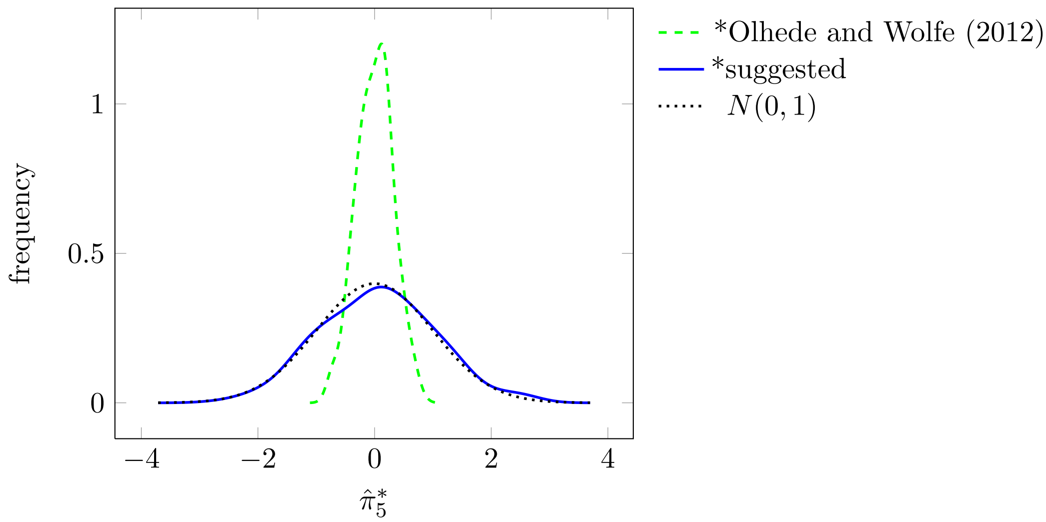

Figure 1.6: Comparison between empirical densities of estimators $\hat{\pi}_5^*$ with different standardizations * and $N(0,1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $n = 8000$, $\gamma = 0.01$, $\theta_n = 1$)

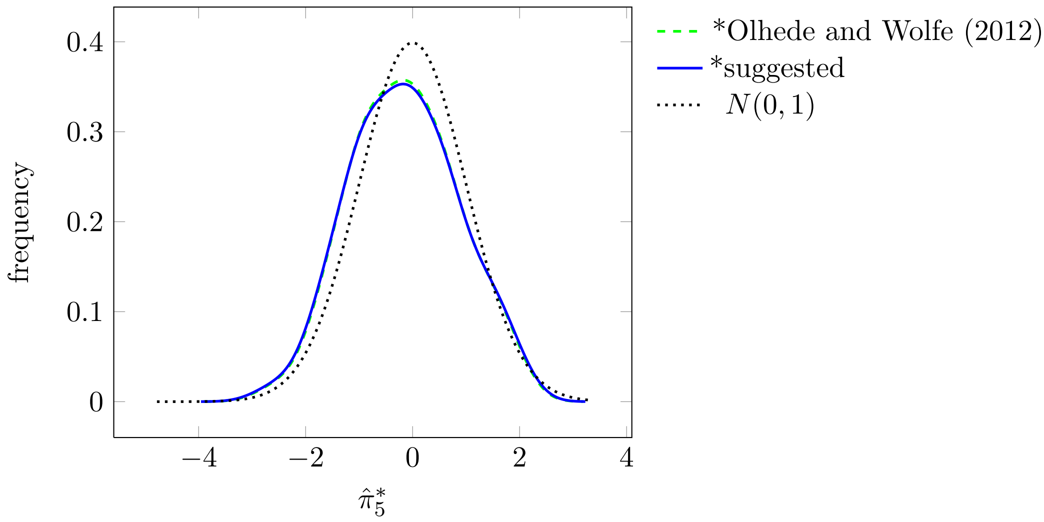

Figure 1.7: Comparison between densities of estimators $\hat{\pi}_5^*$ with different standardizations * and $N(0,1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $n = 8000$, $\gamma = 0.5$, $\theta_n = 1$)

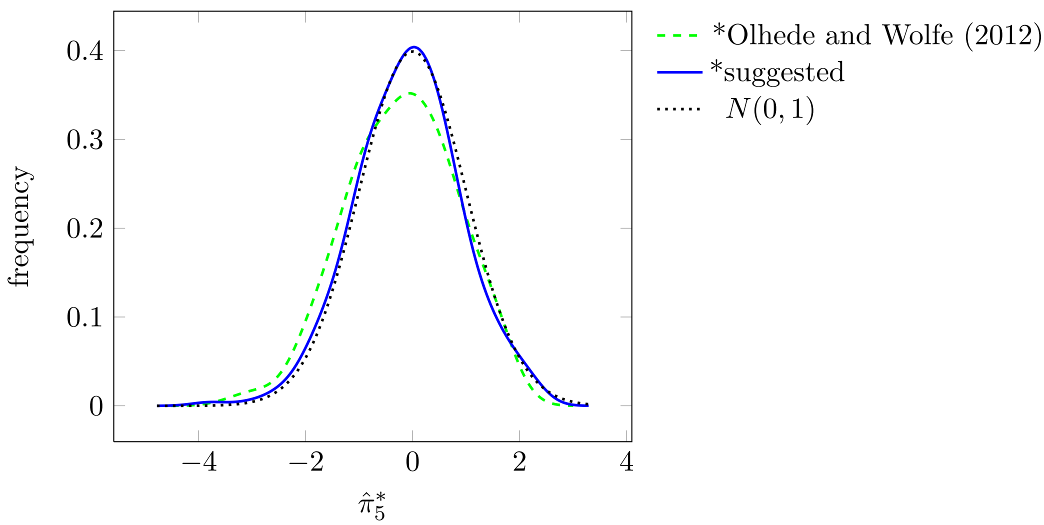

Figure 1.8: Comparison between empirical densities of estimators $\hat{\pi}_5^*$ with different standardizations * and $N(0,1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $n = 8000$, $\gamma = 0.5$, $\theta_n = 1$)

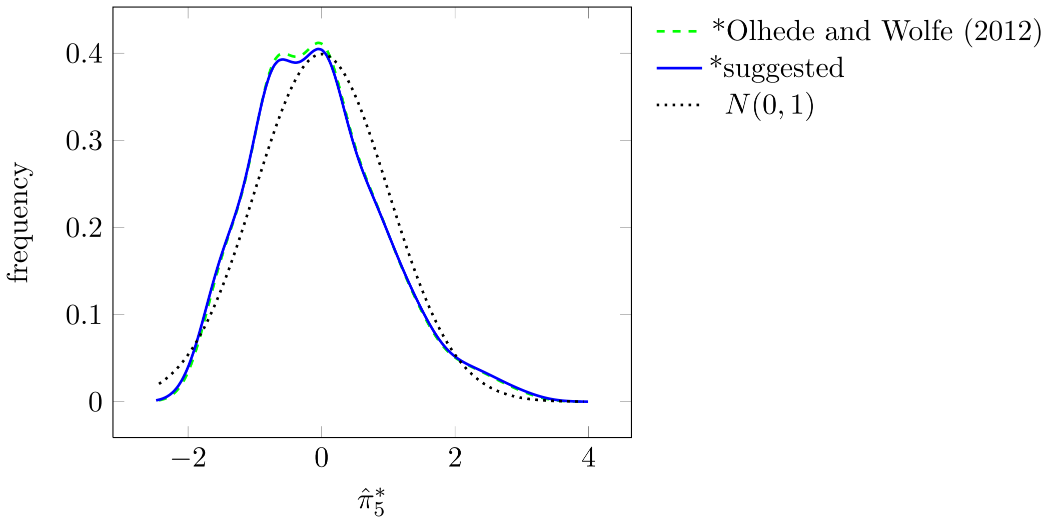

Figure 1.9: Comparison between densities of estimators $\hat{\pi}_5^*$ with different standardizations * and $N(0,1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $n = 8000$, $\gamma = 0.99$, $\theta_n = 1$)

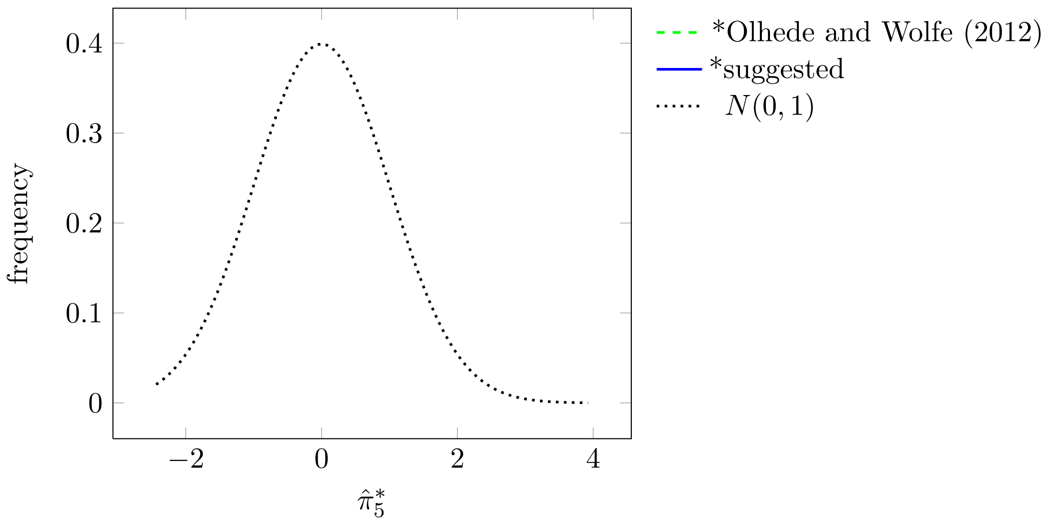

Figure 1.10: Comparison between empirical densities of estimators $\hat{\pi}_5^*$ with different standardizations * and $N(0,1)$ (deterministic power law setting of Olhede and Wolfe (2012) with $n = 8000$, $\gamma = 0.99$, $\theta_n = 1$)

2 consistency

Theorem (2.1) Consistent estimator for the denominator for the Power law case
Assume the deterministic power law setting of Olhede and Wolfe (2012) , so that $\pi_i = \theta_n i^{-\gamma}, i = 1, \ldots, n$*, with* d *the degree vector of an* n*-node simple graph whose edges are independent* $\text{Bernoulli}(\pi_i \pi_j)$ *trials where* $\gamma \in (0,1)$ *and* θ_n *is a* n*- depending sequence satisfying* $0 \leq \theta_n \leq 1$ *for all* n*. Consider degree* d_i *under the assumption* $\mathbb{E}(d_i \,|\, \pi) \underset{n}{\longrightarrow} \infty$*. The denominator* $\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}$ *can be estimated as* $\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}$*, where* $\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}$*. This estimator is consistent; i.e., for any* $\epsilon > 0$ *it holds that*

$$\lim_{n \to \infty} \mathbb{P} \left(\left| \frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} - 1 \right| > \epsilon \right) = 0.$$

Proof. We divide the proof into four steps where for step 3 and 4 it is necessary to distinguish between three cases, $0 < \gamma < 1/2$, $\gamma = 1/2$ and $1/2 < \gamma < 1$.

1. Rearrange the estimator $\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}} = \sqrt{\frac{d_i}{||d||_1}} \cdot \sqrt{1 - \frac{d_i}{||d||_1} \cdot \frac{||d||_2^2}{||d||_1}}$.

2. Lemma 2.2:
$$\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{\|d\|_1} \xrightarrow{P} 1.$$

3. Lemma 2.3:
1.case: $0 < \gamma < 1/2$
$$\frac{1-2\gamma}{\theta_n^2 n^{1-2\gamma}} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\to} \quad 1 \quad \text{as } n \to \infty$$
2.case: $\gamma = 1/2$
$$\frac{1}{\theta_n^2 \ln(n)} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\to} 1 \quad \text{as } n \to \infty$$

3.case: $1/2 < \gamma < 1$

$$\frac{1}{\theta_n^2 n^{1-2\gamma}} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\to} \quad 1 \quad \text{as } n \to \infty$$

4. Lemma 2.4:

$$\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} \overset{P}{\to} 1 \quad \text{as } n \to \infty$$

Compute that $\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} = \sqrt{\frac{\frac{d_i}{||d||_1}}{\frac{\pi}{||\pi||_1}[1-d_n]}} \sqrt{1 - \frac{d_i}{||d||_1} \cdot \frac{||d||_2^2}{||d||_1}}$ where d_n is a null sequence, and $\frac{d_i}{||d||_1} \cdot \frac{||d||_2^2}{||d||_1}$ and the ratio $\frac{\frac{d_i}{||d||_1}}{\frac{\pi}{||\pi||_1}}$ converges in probability to 0 and 1, respectively.

The rearrangements in step 1 can be computed directly. In the second step we consider the ratio $\frac{d_i}{\|d\|_1}$.

Lemma (2.2)
Consider degree d_i under the assumption $\mathbb{E}(d_i \,|\, \pi) \underset{n}{\longrightarrow} \infty$. Then as $n \to \infty$, the standardized variate $\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{\|d\|_1}$ converges in probability to 1.

Proof. By Chebyshev's inequality, $d_i/\mathbb{E}(d_i \,|\, \pi) \xrightarrow{P} 1$ and $\|d\|_1/\|\pi\|_1^2 \xrightarrow{P} 1$ under the hypothesis $\mathbb{E}(d_i \,|\, \pi) \underset{n}{\longrightarrow} \infty$ of the lemma. Write the ratio of interest in terms of these known results as

$$\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{\|d\|_1} = \left(\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{\mathbb{E}(d_i \,|\, \pi)}{\|\pi\|_1^2} \right) \left(\frac{\|d\|_1}{\|\pi\|_1^2} \right)^{-1} \left(\frac{d_i}{\mathbb{E}(d_i \,|\, \pi)} \right) = c_n Y_n^{-1} X_n,$$

where $\{c_n\}$ is a non-random sequence converging to a constant, and $\{X_n\}, \{Y_n\}$ are sequences of random variables converging in probability as stated. We calculate directly that $\lim_{n\to\infty} c_n = c = 1$, and note that $Y_n \xrightarrow{P} 1 \neq 0$. Therefore $c_n Y_n^{-1} X_n \xrightarrow{P} c1^{-1}1 = 1$ (Lehmann, 1998, p.50 Thm 2.1.3) and the result is proved. □

Now we analyze the asymptotic behavior of the ratio $\frac{||d||_2^2}{||d||_1}$.

Lemma (2.3)
Assume $\mathbb{E}(d_i|\pi)$ *is growing to* ∞ *in* n. *Then* $a_{ni}\frac{||d||_2^2}{||d||_1}$ *converges in probability to 1 where* a_{ni} *is equal to* $\frac{1-2\gamma}{\theta_n^2 n^{1-2\gamma}}$, $\frac{1}{\theta_n^2 \ln(n)}$ *and* $\frac{1}{\theta_n^2 n^{1-2\gamma}}$, *respectively.*

Proof. Note that since $\mathbb{E}(d_i|\pi)$ is assumed to be growing the factor a_{ni} is well defined in all three cases.
Due to Chebyshev's inequality, $\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \xrightarrow{P} 1$. A sufficient condition therefore is that the $\mathrm{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \,|\, \pi\right)$ is converging to 0.

$$\mathrm{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \,|\, \pi\right) = \frac{\mathrm{Var}\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} A_{ij}A_{il} \,|\, \pi\right)}{\left(\mathbb{E}(||d||_2^2|\pi)\right)^2}$$

Since the product $A_{ij}A_{il}$ is again a Bernoulli random variable it follows that $(A_{ij1}A_{il1})$ and $(A_{ij2}A_{il2})$ are uncorrelated for all $i, j1, l1, j2$ and $l2$ and that its variance arises as $\mathrm{Var}\left(A_{ij}A_{il} \,|\, \pi\right) = \pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2$.

To compute the variance it is necessary to ensure that $(A_{ij1}A_{il1})$ $(A_{ij2}A_{il2})$ are uncorrelated for all $i, j1, l1, j2$ and $l2$.

$$\mathrm{Cov}\left(A_{ij1}A_{il1}, A_{ij2}A_{il2} \,|\, \pi\right) = \mathbb{E}(A_{ij1}A_{il1} \cdot A_{ij2}A_{il2} \,|\, \pi) - \mathbb{E}(A_{ij1}A_{il1} \,|\, \pi) \cdot \mathbb{E}(A_{ij2}A_{il2} \,|\, \pi)$$

Since the product of Bernoulli random variables is a Bernoulli-random variable it follows that

$$\begin{aligned}\mathrm{Cov}\left(A_{ij1}A_{il1}, A_{ij2}A_{il2} \,|\, \pi\right) &= \pi_i^4\pi_{j1}\pi_{l1}\pi_{j2}\pi_{l2} - \pi_i^4\pi_{j1}\pi_{l1} \cdot \pi_{j2}\pi_{l2}.\\ &= 0\end{aligned}$$

It implies that $\mathrm{Var}\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} A_{ij}A_{il} \,|\, \pi\right) = \sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \mathrm{Var}\left(A_{ij}A_{il} \,|\, \pi\right)$.
The product $A_{ij}A_{il}$ is again a Bernoulli random variable and therefore its variance

arises as $\pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2$. Thus,

$$\begin{aligned}
\mathrm{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)}\,|\,\pi\right) =& \frac{\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\mathbb{E}(||d||_2^2|\pi)\right)^2} \\
=& \frac{\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\mathbb{E}(\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}A_{ij}A_{il}\,|\,\pi)\right)^2} \\
=& \frac{\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\mathbb{E}(A_{ij}A_{il}\,|\,\pi)\right)^2} \\
=& \frac{\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l\right)^2} \\
\leq& \frac{\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l}{\left(\sum_{i=1}^n\sum_{j\neq i}\sum_{l\neq i}\pi_i^2\pi_j\pi_l\right)^2} \\
=& \frac{1}{\mathbb{E}(||d||_2^2|\pi)} \leq \frac{1}{\mathbb{E}(||d||_1|\pi)}
\end{aligned}$$

in addition, $\|d\|_1/\|\pi\|_1^2 \xrightarrow{P} 1$ (see above) under the hypothesis $\mathbb{E}(d_i\,|\,\pi) \underset{n}{\longrightarrow} \infty$ of the lemma. Write the ratio of interest in terms of these known results as

$$a_{ni}\frac{||d||_2^2}{||d||_1} = a_{ni}||\pi||_2^2\cdot\left(\frac{||d||_1}{||\pi||_1^2}\right)^{-1}\cdot\frac{||d||_2^2}{\mathbb{E}(||d||_2^2\,|\,\pi)}\cdot\frac{\mathbb{E}(||d||_2^2\,|\,\pi)}{||\pi||_2^2||\pi||_1^2} = a_nY_n^{-1}X_nc_n.$$

We conclude directly that the non-random sequence $\{a_n\}$ is converging to 1 using that

$$||\pi||_2^2 = \theta_n^2\begin{cases}\frac{n^{1-2\gamma}}{1-2\gamma} + \mathcal{O}(1) & \text{if } 0<\gamma<1/2; \\ \ln n + \gamma_E + \mathcal{O}(n^{-1}) & \text{if } \gamma = 1/2; \\ \zeta(2\gamma) + \mathcal{O}(n^{-(2\gamma-1)}) & \text{if } 1/2<\gamma<1.\end{cases}$$

$\{c_n\}$ is as well a converging non-random sequence. Since,

$$c_n = \frac{\mathbb{E}(||d||_2^2\,|\,\pi)}{||\pi||_2^2||\pi||_1^2} = 1 - \frac{\sum_{i=1}^n\pi_i^4}{||\pi||_2^2||\pi||_1^2}$$

it fallows that

$$1 \geq \frac{\mathbb{E}(||d||_2^2\,|\,\pi)}{||\pi||_2^2||\pi||_1^2} \geq 1 - \frac{1}{||\pi||_1^2}.$$

Since, we assume that $\mathbb{E}(d_i|\pi)$ is growing to ∞ it holds that $||\pi||_1$ is growing as well. Taking the limit results in $\lim_{n\to\infty} c_n = 1$. Note that $Y_n \xrightarrow{P} 1 \neq 0$. Therefore, $a_n Y_n^{-1} X_n c_n \xrightarrow{P} 1$ (Lehmann, 1998, p.50 Thm 2.1.3) and the result is proved.

$\square$

Lemma (2.4)

The ratio $\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}}$ *converges to 1 in probability.*

Proof. Rearranging the ratio of estimator and true values in terms of the results in step 1, 2 and 3 leads us to

$$\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}}$$

$$= \sqrt{\frac{\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{||d||_1}}{\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{\pi_i}{||\pi||_1} \left[1 - \frac{\pi_i}{||\pi||_1} ||\pi||_2^2\right]}} \sqrt{1 - \frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{||d||_1} \cdot a_{ni} \frac{||d||_2^2}{||d||_1} \cdot \frac{i^{-\gamma}(1-\gamma)}{n^{1-\gamma}} a_{ni}^{-1}}.$$

$$= \sqrt{\frac{X_n}{x_n \left[1 - d_n\right]}} \sqrt{1 - X_n Y_n c_n}$$

As stated above, $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables converging in probability to 1, and we calculate directly that $\lim_{n\to\infty} c_n = 0$. Furthermore, due to $||\pi||_1 = \theta_n \left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right]$ we compute the limit of the non-random sequence $\{x_n\}$ to be 1. The sequence $\{d_n\}$is non-random as well and converges to 0 for all γ because $||\pi||_2^2 = o\left(||\pi||_1\right)$. Note that $\lim_{n\to\infty} x_n \left[1 - d_n\right] = 1 \neq 0$. Using Lehmann (1998, p.50 Thm 2.1.3) and the continuous mapping theorem, we deduce the required result of the theorem, namely $\sqrt{\frac{X_n}{x_n[1-d_n]}} \sqrt{1 - X_n Y_n c_n} \xrightarrow{P} 1$.

$\square$

$\square$

Theorem (2.5) Consistent estimator for the Erdös Rényi graphs
Assume we have a simple graph where $A_{ij} \sim Bern(\pi_i \cdot \pi_j)$ *with* $\pi_i = \sqrt{p}$, *for all* $i \in \mathbb{N}$. *The denominator* $\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}$ *can be estimated as* $\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}$, *where* $\hat{\pi}_i = \frac{d_i}{\sqrt{||d||_1}}$. *This estimator is consistent; i.e., for any* $\epsilon > 0$ *it holds that*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} - 1\right| > \epsilon\right) = 0.$$

Proof. **Step 1**
First we rearrange the estimator such that

$$\sqrt{n}\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}} = \sqrt{n} \cdot \sqrt{\frac{d_i}{||d||_1}} \cdot \sqrt{1 - n\frac{d_i}{||d||_1} \cdot \frac{1}{n-1}\frac{||d||_2^2}{||d||_1} \cdot \frac{n-1}{n}}.$$

Secondly, we consider the ratio $\frac{d_i}{||d||_1}$.

Lemma (2.6)
The standardized variate $n\,\frac{d_i}{||d||_1}$ *converges in probability to 1 as* $n \to \infty$.

Proof. $\frac{1}{n-1}d_i = \frac{1}{n-1}\sum_{j\neq i} A_{ij}$ is a sample mean of IID Bernoulli-random veraibles with $\mathbb{E}(d_i|\pi) = p$. Thus, due to the WLLN $\frac{1}{n-1}d_i$ is converging in probability to the expectation p. Furthermore, $\frac{1}{n(n-1)}||d||_1 = 2\frac{1}{n(n-1)}\sum_{i=1}^n \sum_{j>i} A_{ij}$ is as well a sample mean of IID Bern(p)-variables and it therefore holds that $\frac{1}{n(n-1)}||d||_1 \xrightarrow{P} p$. Accordingt to Lehmann (1998, p.50 Thm 2.1.3) we conclude that

$$n\,\frac{d_i}{||d||_1} = \frac{\frac{1}{n-1}d_i}{\frac{1}{n(n-1)}||d||_1} \quad \xrightarrow{P} 1.$$

□

Third, the asymptotic behavior of the ration $\frac{||d||_2^2}{||d||_1}$ is analyzed.

Lemma (2.7)
For $n \to \infty$, *the ratio* $\frac{1}{n-1}\frac{||d||_2^2}{||d||_1}$ *converges in probability to* p.

$$\frac{1}{n-1}\frac{||d||_2^2}{||d||_1} \quad \xrightarrow{P} p \quad \textit{as } n \to \infty$$

Proof. The term of interest can be splitted up into four components as

$$\frac{1}{n-1}\frac{||d||_2^2}{||d||_1} = \frac{1}{n-1}||\pi||_2^2 \cdot \left(\frac{||d||_1}{||\pi||_1^2}\right)^{-1} \cdot \frac{||d||_2^2}{\mathbb{E}(||d||_2^2 \,|\, \pi)} \cdot \frac{\mathbb{E}(||d||_2^2 \,|\, \pi)}{||\pi||_2^2||\pi||_1^2} = a_n Y_n^{-1} X_n c_n.$$

We can directly compute that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n-1}p = p$. $\{Y_n\}$ is a sequence of random variables for which holds that

$$Y_n = \frac{||d||_1}{||\pi||_1^2} = \frac{2\sum_{i=1}^n \sum_{j>i} A_{ij}}{\left(\sum_{i=1}^n \sqrt{p}\right)^2} = \frac{\frac{2}{n(n-1)}\sum_{i=1}^n \sum_{j>i} A_{ij}}{\frac{n^2}{n(n-1)}p}.$$

WLLN tells us that Y_n therefore converges to 1 in probability. Due to Chebyshev's inequality, it is suffcient to show that $\mathrm{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2 \,|\, \pi)}\right)$ converges to 0 to proof that $X_n = \frac{||d||_2^2}{\mathbb{E}(||d||_2^2 \,|\, \pi)}$ is converging to 1 in probability.

$$\mathrm{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2 \,|\, \pi)}\right) = \frac{1}{(\mathbb{E}(||d||_2^2 \,|\, \pi))^2}\mathrm{Var}(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} A_{ij}A_{il})$$

Since $\{A_{ij}A_{il}\}$ are uncorrelated it follows that

$$\mathrm{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2 \,|\, \pi)}\right) = \frac{n(n-1)^2p^2(1-p^2)}{n^2(n-1)^4p^4} = \frac{1-p^2}{n(n-1)^2p^2},$$

which converges to 0 in n. In addition,

$$\frac{\mathbb{E}(||d||_2^2 \,|\, \pi)}{||\pi||_2^2||\pi||_1^2} = \frac{n(n-1)^2p^2}{n^3p^2} \overset{n\to\infty}{\to} 1.$$

Note that $Y_n \xrightarrow{P} 1 \neq 0$. Therefore, $a_n Y_n^{-1} X_n c_n \xrightarrow{P} 1$ (Lehmann, 1998, p.50 Thm 2.1.3) and the result is proved.

$\square$

In the last step, we compare the estimator with the true value.

$$\begin{aligned}
&\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2\frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2\frac{||\pi||_2^2}{||\pi||_1^2}}} \\
&= \sqrt{\frac{n\frac{d_i}{||d||_1}}{n\frac{\pi_i}{||\pi||_1}}}\sqrt{\frac{1 - n\frac{d_i}{||d||_1}\cdot\frac{1}{n-1}\frac{||d||_2^2}{||d||_1}\cdot\frac{n-1}{n}}{1-\frac{\pi_i}{||\pi||_1}||\pi||_2^2}}. \\
&= \sqrt{\frac{X_n}{x_n}}\sqrt{\frac{1-X_nY_nc_n}{1-d_n}}
\end{aligned}$$

$\{X_n\}$ and $\{Y_n\}$ are sequences of random varaibles converging to 1 and p, respectively. We calculate directly that $\lim_{n\to\infty} x_n = 1$, $\lim_{n\to\infty} c_n = 1$ and $\lim_{n\to\infty} d_n = p$. Note that neither the limit of x_n nor $[1 - d_n]$ is equal to 0. Using Lehmann (1998, p.50 Thm 2.1.3) and the continuous mapping theorem, the first ratio $\sqrt{\frac{X_n}{x_n}}$ and the second ration $\sqrt{\frac{1-X_nY_nc_n}{1-d_n}}$ both converge in probability to 1 and we deduce the required result of the theorem.

□

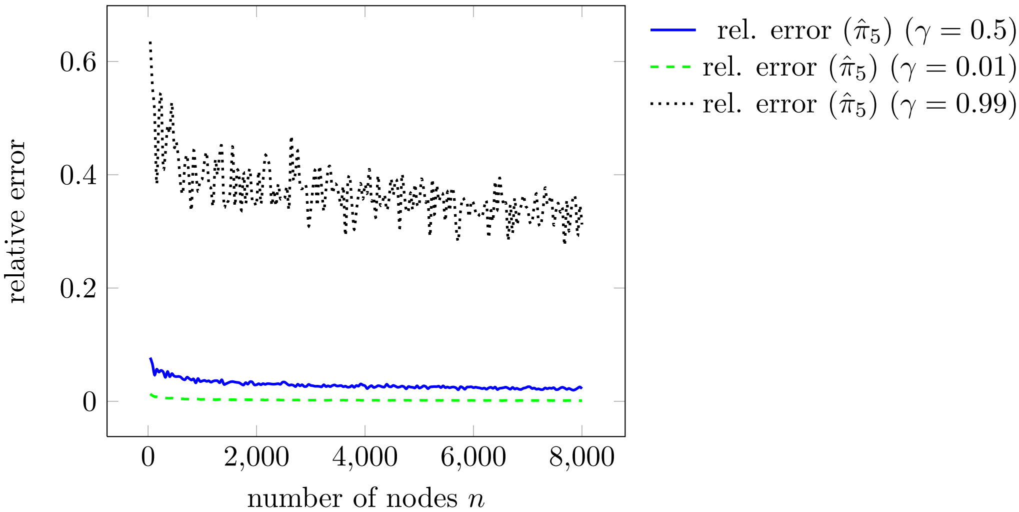

Figure 2.1: Relative error of the estimator of the standardization factor (averaged over 100 repetitions, $\theta_n = 1$)

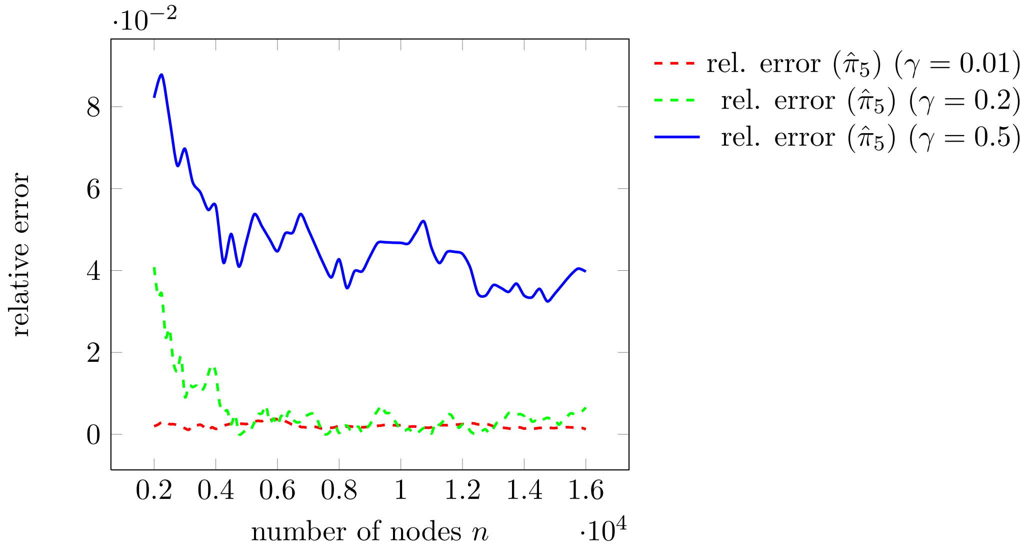

Figure 2.2: Relative error of the estimator of the standardization factor ($\theta_n = 1$)

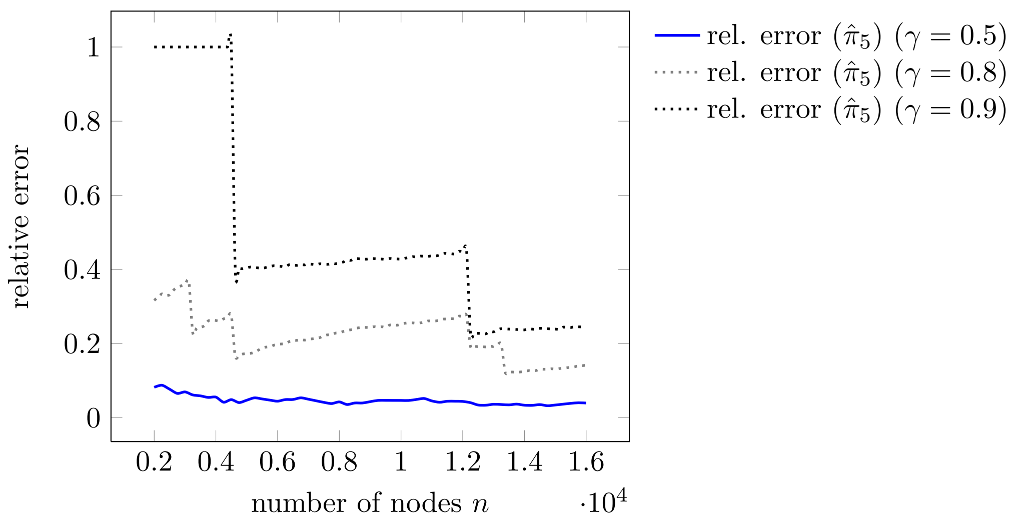

Figure 2.3: Relative error of the estimator of the standardization factor ($\theta_n = 1$)

A Appendix

A.1 Details to proof Theorem (2.1)

Proof. We divide the proof into four steps where for step 3 and 4 it is necessary to distinguish between three cases, $0 < \gamma < 1/2$, $\gamma = 1/2$ and $1/2 < \gamma < 1$.

1. Rearrange the estimator $\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}} = \sqrt{\frac{d_i}{||d||_1}} \cdot \sqrt{1 - \frac{d_i}{||d||_1} \cdot \frac{||d||_2^2}{||d||_1}}$.

2. Lemma A.1:
$$\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{\|d\|_1} \xrightarrow{P} 1.$$

3. Lemma A.2:
   1.case: $0 < \gamma < 1/2$
$$\frac{1-2\gamma}{\theta_n^2 n^{1-2\gamma}} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\to} \quad 1 \quad \text{as } n \to \infty$$
   2.case: $\gamma = 1/2$
$$\frac{1}{\theta_n^2 \ln(n)} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\to} 1 \quad \text{as } n \to \infty$$
   3.case: $1/2 < \gamma < 1$
$$\frac{1}{\theta_n^2 n^{1-2\gamma}} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\to} \quad 1 \quad \text{as } n \to \infty$$

4. Compute that $\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} = \sqrt{\frac{\frac{d_i}{||d||_1}}{\frac{\pi}{||\pi||_1}}} \sqrt{\frac{1-k_n}{1-d_n}}$ where d_n is a null sequence, and k_n and the ratio $\frac{\frac{d_i}{||d||_1}}{\frac{\pi}{||\pi||_1}}$ converges in probability to 0 and 1, respectively.

Step 1

$$\begin{aligned}
&\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}} \\
=&\sqrt{\frac{\hat{\pi}_i ||\hat{\pi}||_1 - \hat{\pi}_i^2 ||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}} \\
=&\sqrt{\frac{\frac{d_i}{\sqrt{||d||_1}} \sum_j \frac{d_j}{\sqrt{||d||_1}} - \left(\frac{d_i}{\sqrt{||d||_1}}\right)^2 \sum_j \left(\frac{d_j}{\sqrt{||d||_1}}\right)^2}{\left(\sum_j \frac{d_j}{\sqrt{||d||_1}}\right)^2}} \\
=&\sqrt{\frac{\frac{1}{||d||_1} \cdot d_i \cdot \sum_j d_j - \frac{1}{||d||_1^2} \cdot d_i^2 \cdot \sum_j d_j^2}{\frac{\left(\sum_j d_j\right)^2}{||d||_1}}}
\end{aligned}$$

$$\begin{aligned}
=&\sqrt{\frac{d_i \cdot \sum_j d_j - \frac{1}{||d||_1} \cdot d_i^2 \cdot \sum_j d_j^2}{\left(\sum_j d_j\right)^2}} \\
=&\sqrt{\frac{d_i \cdot ||d||_1 - \frac{1}{||d||_1} \cdot d_i^2 \cdot ||d||_2^2}{||d||_1^2}} \\
=&\sqrt{\frac{d_i}{||d||_1} - d_i^2 \cdot \frac{||d||_2^2}{||d||_1^3}} \\
=&\sqrt{\frac{d_i}{||d||_1}} \cdot \sqrt{1 - \frac{d_i}{||d||_1} \cdot \frac{||d||_2^2}{||d||_1}}
\end{aligned} \tag{A.1}$$

$$=\sqrt{\frac{d_i}{||d||_1}} \cdot \sqrt{1 - n^{1-\gamma} \frac{d_i}{||d||_1} \cdot \frac{1}{n^{1-2\gamma}} \frac{||d||_2^2}{||d||_1} \cdot \frac{1}{n^\gamma}}$$

Step 2

Lemma (A.1)
Under the same assumptions as for the theorem it holds that

$$\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{\|d\|_1} \xrightarrow{P} 1.$$

Proof. By Chebyshev's inequality, $d_i / \mathbb{E}(d_i \mid \pi) \xrightarrow{P} 1$ and $\|d\|_1 / \|\pi\|_1^2 \xrightarrow{P} 1$ under the hypothesis $\mathbb{E}(d_i \mid \pi) \underset{n}{\longrightarrow} \infty$ of the lemma (see supplementary material proof Thm 2, Term 3). Write the ratio of interest in terms of these known results as

$$\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{d_i}{\|d\|_1} = \left(\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{\mathbb{E}(d_i \mid \pi)}{\|\pi\|_1^2} \right) \left(\frac{\|d\|_1}{\|\pi\|_1^2} \right)^{-1} \left(\frac{d_i}{\mathbb{E}(d_i \mid \pi)} \right) = c_n Y_n^{-1} X_n,$$

where $\{c_n\}$ is a non-random sequence converging to a constant, and $\{X_n\}, \{Y_n\}$ are sequences of random variables converging in probability as stated. We calculate directly that $\lim_{n\to\infty} c_n = c = 1$, and note that $Y_n \xrightarrow{P} 1 \neq 0$. Therefore $c_n Y_n^{-1} X_n \xrightarrow{P} c 1^{-1} 1 = 1$ (Lehmann, 1998, p.50 Thm 2.1.3) and the result is proved. □

Step 3

In this step we analyze the asymptotic behavior of the ratio $\frac{||d||_2^2}{||d||_1}$.

Lemma (A.2)

1.case: $0 < \gamma < 1/2$

$$\frac{1-2\gamma}{\theta_n^2 n^{1-2\gamma}} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\rightarrow} \quad 1 \quad \text{as } n \to \infty$$

2.case: $\gamma = 1/2$

$$\frac{1}{\theta_n^2 \ln(n)} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\rightarrow} 1 \quad \text{as } n \to \infty$$

3.case: $1/2 < \gamma < 1$

$$\frac{1}{\theta_n^2 \zeta(2\gamma)} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\rightarrow} \quad 1 \quad \text{as } n \to \infty$$

Proof. Note that the lemma can be rephrased by $a_{ni} \frac{||d||_2^2}{||d||_1} \quad \overset{P}{\rightarrow} \quad 1$ as $n \to \infty$ where a_{ni} is equal to $\frac{1-2\gamma}{\theta_n^2 n^{1-2\gamma}}$, $\frac{1}{\theta_n^2 \ln(n)}$ and $\frac{1}{\theta_n^2 \zeta(2\gamma)}$, respectively. Due to the equation of $\mathbb{E}(d_i|\pi)$ in Theorem 1 in the paper Olhede and Wolfe (2012) we conclude that if $\mathbb{E}(d_i|\pi)$ is growing then $\theta_n^2 = \omega\left(\frac{1}{i^{-\gamma} n^{1-\gamma}}\right)$. The sequence of scaling constants $\{\theta_n\}$ is the same for all i. Thus, $\theta_n^2 = \omega\left(\frac{1}{n^{1-2\gamma}}\right)$. Consequently,

$$\begin{array}{ll} \text{1.case: } 0 < \gamma < 1/2 & \theta_n^2 \cdot n^{1-2\gamma} = \omega(1) \\ \text{2.case: } \gamma = 1/2 & \theta_n^2 \cdot \ln(n) = \omega(\ln(n)) \\ \text{3.case: } 1/2 < \gamma < 1 & \theta_n^2 \cdot \zeta(2\gamma) = \omega\left(n^{2\gamma-1}\right) \end{array}$$

and the factor a_{ni} is well defined in all three cases.

By Chebyshev's inequality, we can first show that $\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \xrightarrow{P} 1$. Therefore, we compute the variance as

$$\operatorname{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \,|\, \pi\right) = \frac{\operatorname{Var}\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} A_{ij}A_{il} \,|\, \pi\right)}{\left(\mathbb{E}(||d||_2^2|\pi)\right)^2}$$

To compute the variance it is necessary to ensure that $(A_{ij1}A_{il1})$ $(A_{ij2}A_{il2})$ are uncorrelated for all $i, j1, l1, j2$ and $l2$.

$$\operatorname{Cov}(A_{ij1}A_{il1}, A_{ij2}A_{il2} \,|\, \pi) = \mathbb{E}(A_{ij1}A_{il1} \cdot A_{ij2}A_{il2} \,|\, \pi) - \mathbb{E}(A_{ij1}A_{il1} \,|\, \pi) \cdot \mathbb{E}(A_{ij2}A_{il2} \,|\, \pi)$$

Since the product of Bernoulli random variables is a Bernoulli-random variable it follows that

$$\begin{aligned}\operatorname{Cov}(A_{ij1}A_{il1}, A_{ij2}A_{il2} \,|\, \pi) =& \pi_i^4 \pi_{j1}\pi_{l1}\pi_{j2}\pi_{l2} - \pi_i^4 \pi_{j1}\pi_{l1} \cdot \pi_{j2}\pi_{l2}. \\ =& 0\end{aligned}$$

It implies that $\operatorname{Var}\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} A_{ij}A_{il} \,|\, \pi\right) = \sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \operatorname{Var}(A_{ij}A_{il} \,|\, \pi)$. The product $A_{ij}A_{il}$ is again a Bernoulli random variable and therefore its variance arises as $\pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2$. Thus,

$$\begin{aligned}\operatorname{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \,|\, \pi\right) =& \frac{\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\mathbb{E}(||d||_2^2|\pi)\right)^2} \\ =& \frac{\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\mathbb{E}(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} A_{ij}A_{il} \,|\, \pi)\right)^2} \\ =& \frac{\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \mathbb{E}(A_{ij}A_{il} \,|\, \pi)\right)^2} \\ =& \frac{\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l - \pi_i^4\pi_j^2\pi_l^2}{\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l\right)^2} \\ \leq& \frac{\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l}{\left(\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l\right)^2} \\ =& \frac{1}{\mathbb{E}(||d||_2^2|\pi)} \leq \frac{1}{\mathbb{E}(||d||_1|\pi)}\end{aligned}$$

Since we assume $\mathbb{E}(d_i \,|\, \pi) \underset{n}{\longrightarrow} \infty$ we know that $\mathbb{E}(||d||_1|\pi)$ is growing in n to ∞ implying that $\lim_{n\to\infty} \operatorname{Var}\left(\frac{||d||_2^2}{\mathbb{E}(||d||_2^2|\pi)} \,|\, \pi\right) = 0$.

In addition, $\|d\|_1/\|\pi\|_1^2 \xrightarrow{P} 1$ (see above) under the hypothesis $\mathbb{E}(d_i \,|\, \pi) \underset{n}{\longrightarrow} \infty$ of the lemma. Write the ratio of interest in terms of these known results as

$$a_{ni}\frac{||d||_2^2}{||d||_1} = a_{ni}||\pi||_2^2 \cdot \left(\frac{||d||_1}{||\pi||_1^2}\right)^{-1} \cdot \frac{||d||_2^2}{\mathbb{E}(||d||_2^2 \,|\, \pi)} \cdot \frac{\mathbb{E}(||d||_2^2 \,|\, \pi)}{||\pi||_2^2||\pi||_1^2} = a_n Y_n^{-1} X_n c_n.$$

$\{a_n\}$ is a non-random sequence converging to 1 in all three cases because
1.case: $0 < \gamma < 1/2$

$$\begin{aligned} a_n = \frac{1-2\gamma}{\theta_n^2 n^{1-2\gamma}}||\pi||_2^2 &= \frac{(1-2\gamma)\theta_n^2}{\theta_n^2 n^{1-2\gamma}}\left[\frac{n^{1-2\gamma}}{1-2\gamma} + \mathcal{O}(1)\right] \\ &= 1 + \mathcal{O}\left(\frac{1}{n^{1-2\gamma}}\right); \end{aligned}$$

2.case: $\gamma = 1/2$

$$\begin{aligned} a_n = \frac{||\pi||_2^2}{\theta_n^2 \ln(n)} &= \frac{\theta_n^2}{\theta_n^2 \ln(n)}\left[\ln n + \gamma_E + \mathcal{O}(n^{-1})\right]. \\ &= \left[1 + \frac{\gamma_E}{\ln(n)} + \mathcal{O}\left(\frac{1}{\ln(n) n}\right)\right]; \end{aligned}$$

and 3.case: $1/2 < \gamma < 1$

$$\frac{||\pi||_2^2}{\theta_n^2\zeta(2\gamma)} = \frac{\theta_n^2}{\theta_n^2\zeta(2\gamma)}\left[\zeta(2\gamma) + \mathcal{O}(n^{-(2\gamma-1)})\right] = 1 + \mathcal{O}(n^{-(2\gamma-1)}).$$

$\zeta(x)$ is the Riemann zeta function which for $x = 2\gamma$ converges.
$\{c_n\}$ is as well a converging non-random sequence.

$$\begin{aligned} \frac{\mathbb{E}(||d||_2^2 \,|\, \pi)}{||\pi||_2^2||\pi||_1^2} =& \frac{\mathbb{E}(\sum_{i=1}^n(\sum_{j\neq i} A_{ij})^2 \,|\, \pi)}{\sum_{i=1}^n \pi_i^2 \cdot (\sum_{j=1}^n \pi_j \cdot \sum_{l=1}^n \pi_l)} \\ =& \frac{\sum_{i=1}^n \sum_{j\neq i} \sum_{l\neq i} \pi_i^2\pi_j\pi_l}{\sum_{i=1}^n \pi_i^2 \cdot (\sum_{j=1}^n \pi_j \cdot \sum_{l=1}^n \pi_l)} \\ =& 1 - \frac{\sum_{i=1}^n \pi_i^4}{\sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n \pi_i^2\pi_j\pi_l} \end{aligned}$$

From the former equation and due to $0 < \pi_i < 1$ it fallows that

$$\begin{aligned} 1 \geq \frac{\mathbb{E}(||d||_2^2 \,|\, \pi)}{||\pi||_2^2||\pi||_1^2} \geq& 1 - \frac{\sum_{i=1}^n \pi_i^2}{||\pi||_2^2||\pi||_1^2}. \\ =& 1 - \frac{1}{||\pi||_1^2} \\ =& 1 - \frac{1}{\theta_n^2[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)]^2} \\ =& 1 - \frac{1}{\theta_n^2\frac{n^{2-2\gamma}}{(1-1\gamma)^2} + \theta_n^2\frac{n^{1-\gamma}}{1-\gamma}\mathcal{O}(1) + \theta_n^2\mathcal{O}(1)} \end{aligned} \tag{A.2}$$

Since $\mathbb{E}(d_i|\pi)$ is growing it holds that

$$\theta_n^2 = \omega\left(\frac{1}{n^{1-2\gamma}}\right). \quad \Rightarrow \quad \theta_n^2 \cdot n^{2-2\gamma} = \omega\left(n\right)$$

Thus, taking the limit in equation (A.2) results in

$$1 \geq \lim_{n\to\infty} \frac{\mathbb{E}(||d||_2^2 \,|\, \pi)}{||\pi||_2^2||\pi||_1^2} \geq \lim_{n\to\infty} \left(1 - \frac{1}{\theta_n^2 \frac{n^{2-2\gamma}}{(1-1\gamma)^2} + \theta_n^2 \frac{n^{1-\gamma}}{1-\gamma}\mathcal{O}(1) + \theta_n^2\mathcal{O}(1)}\right) = 1$$

Therefore, $\lim_{n\to\infty} c_n = 1$ holds. Note that $Y_n \xrightarrow{P} 1 \neq 0$. Therefore, $a_n Y_n^{-1} X_n c_n \xrightarrow{P}$ 1 (Lehmann, 1998, p.50 Thm 2.1.3) and the result is proved.

□

Step 4

Rearranging the ratio of estimator and true values in terms of the results in step 1, 2 and 3 leads us to

$$\begin{aligned}
&\frac{\sqrt{\frac{\hat{\pi}_i}{||\hat{\pi}||_1} - \hat{\pi}_i^2 \frac{||\hat{\pi}||_2^2}{||\hat{\pi}||_1^2}}}{\sqrt{\frac{\pi_i}{||\pi||_1} - \pi_i^2 \frac{||\pi||_2^2}{||\pi||_1^2}}} \\
&= \sqrt{\frac{\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)}\frac{d_i}{||d||_1}\left[1 - \frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)}\frac{d_i}{||d||_1} \cdot a_{ni}\frac{||d||_2^2}{||d||_1} \cdot \frac{i^{-\gamma}(1-\gamma)}{n^{1-\gamma}} a_{ni}^{-1}\right]}{\frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)}\frac{\pi_i}{||\pi||_1}\left[1 - \frac{\pi_i}{||\pi||_1}||\pi||_2^2\right]}}. \\
&= \sqrt{\frac{X_n\left[1 - X_n Y_n c_n\right]}{x_n\left[1 - d_n\right]}} \\
&= \sqrt{\frac{X_n}{x_n}}\sqrt{\frac{1 - X_n Y_n c_n}{1 - d_n}}
\end{aligned}$$

As stated above, $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables converging in probability to 1, and we calculate that $\lim_{n\to\infty} c_n = 0$ because

$$\begin{aligned}
&\text{1.case: } 0 < \gamma < 1/2 && c_n = \frac{\theta_n^2\, n^{1-2\gamma}\, i^{-\gamma}\,(1-\gamma)}{(1-2\gamma)n^{1-\gamma}} = \mathcal{O}\left(n^{-\gamma}\right) \\
&\text{2.case: } \gamma = 1/2 && c_n = \frac{\theta_n^2\, \ln n\, i^{-\gamma}\,(1-\gamma)}{n^{1-\gamma}} = \mathcal{O}\left(\frac{\ln n}{\sqrt{n}}\right) \\
&\text{3.case: } 1/2 < \gamma < 1 && c_n = \frac{\theta_n^2\, \zeta(2\gamma)\, i^{-\gamma}\,(1-\gamma)}{n^{1-\gamma}} = \mathcal{O}\left(n^{-(1-\gamma)}\right)
\end{aligned}$$

Furthermore, due to $||\pi||_1 = \theta_n \left[\frac{n^{1-\gamma}}{1-\gamma} + \mathcal{O}(1)\right]$ we compute the limit of the non-random sequence $\{x_n\}$ as

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^{1-\gamma}}{i^{-\gamma}(1-\gamma)} \frac{\pi_i}{||\pi||_1} = \lim_{n\to\infty} \frac{1}{1+\mathcal{O}(\frac{1}{n^{1-\gamma}})} = 1.$$

The sequence $\{d_n\}$is non-random as well and converges to 0 for all γ because $||\pi||_2^2 = o\left(||\pi||_1\right)$; i.e.,

1.case: $0 < \gamma < 1/2$

$$d_n = \frac{\pi_i}{||\pi||_1}||\pi||_2^2 = n^{-\gamma} \frac{i^{-\gamma}\theta_n^2\left[\frac{1}{1-2\gamma} + \mathcal{O}\left(\frac{1}{n^{1-2\gamma}}\right)\right]}{\frac{1}{1-\gamma} + \mathcal{O}(n^{\gamma-1})};$$

2.case: $\gamma = 1/2$

$$d_n = \frac{\pi_i}{||\pi||_1}||\pi||_2^2 = \frac{\ln n}{\sqrt{n}} \frac{i^{-\gamma}\theta_n^2\left[1 + \frac{\gamma_E}{\ln(n)} + \mathcal{O}(\frac{1}{\ln(n)n})\right]}{\left[\frac{1}{1-\gamma} + \mathcal{O}(n^{\gamma-1})\right]};$$

and 3.case: $1/2 < \gamma < 1$

$$d_n = \frac{\pi_i}{||\pi||_1}||\pi||_2^2 = \frac{1}{n^{1-\gamma}} \frac{i^{-\gamma}\theta_n^2\left[\zeta(2\gamma) + \mathcal{O}(n^{-(2\gamma-1)})\right]}{\frac{1}{1-\gamma} + \mathcal{O}(n^{\gamma-1})}.$$

Note that $\lim_{n\to\infty} x_n\left[1-d_n\right] = 1 \neq 0$. Using Lehmann (1998, p.50 Thm 2.1.3) and the continuous mapping theorem, we deduce the required result of the theorem, namely $\sqrt{\frac{X_n}{x_n}}\sqrt{\frac{1-X_nY_nc_n}{1-d_n}} \xrightarrow{P} 1$.

□

Note:
But the summand $\frac{\gamma_E}{\ln(n)}$ should not be underestimated.

n	200	2000	8000	10000	12000	16000	2E+25
$\frac{\gamma_E}{\ln(n)}$	0.1089	0.0759	0.0642	0.0627	0.0615	0.0596	0.0099

Table A.1: Euler Mascheroni constant = 0.57721

Bibliography

P.J. Brockwell and R.A. Davis. *Time Series: Teory and Methods*. Springer Series in Statistics Series. Springer-Verlag, 1991. ISBN 9780387974293. URL `http://books.google.co.uk/books?id=ZW_ThhYQiXIC`.

E.L. Lehmann. *Elements of Large-Sample Theory*. Springer, 1998.

Sofia C Olhede and Patrick J Wolfe. What is a Degree Distribution? *arXiv:1211.6537v1*, pages 1–18, 2012.