



香港電腦奧林匹克競賽  
Hong Kong Olympiad in Informatics

# Mathematics in OI (II)

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## Motivation

- In Mathematics in OI (I), we mainly dealt with topics in **number theory**, with an emphasis on how to compute things **correctly** and **efficiently**.
- In today's session, we will shift our focus to **combinatorics** – the mathematics of counting and arranging.

## Motivation

- As you can see, knowing how to code things is not enough. You can never be certain whether a statement is true or false.
- That's why **proving** is an important aspect in mathematics.
- Today, we'll focus on proving. Hopefully it helps in developing your critical reasoning skills, and develop better intuition.

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- 1 Basics of Counting
- 2 Fibonacci and Catalan Sequence
- 3 Inclusion-Exclusion Principle
- 4 Sums and Expected Value
- 5 Problem Set  
(This section is not archived to open up possibilities of reuse in the future)

## Addition and Multiplication Principle

### Addition Principle

If we want to choose 1 element out of  $k$  given sets with sizes  $n_1, n_2, \dots, n_k$ , and no two sets share a common element, then there are in total  $n_1 + n_2 + \dots + n_k$  ways to choose an element.

### Multiplication Principle

If we want to choose 1 element out of **each** of the  $k$  given sets with sizes  $n_1, n_2, \dots, n_k$ , then there are in total  $n_1 \times n_2 \times \dots \times n_k$  ways to choose the elements.

## Factorial

$n!$  ( $n$  factorial)

Recursive definition:  $n! = \begin{cases} 1 & \text{if } n = 0 \\ n \times (n - 1)! & \text{otherwise} \end{cases}$

Combinatorial definition:  $n!$  is the **number of permutations** of  $1 \dots n$

**⚠ Are these two definitions equivalent?**

## Factorial

 **Aim:** To show that the two definitions are equivalent.

Let  $n!$  be the number of permutations of  $1 \dots n$ .  *combinatorial definition*

- If  $n = 0$ , there is 1 permutation of 0 objects.


$$\therefore n! = 1$$

- Otherwise, consider any permutation of  $1 \dots n - 1$ :



There are  $n$  locations to insert the number  $n \Rightarrow$  permutation of  $1 \dots n$ .

$$\therefore n! = n \times (n - 1)!$$

Combining the two cases,  $n! = \begin{cases} 1 & \text{if } n = 0 \\ n \times (n - 1)! & \text{otherwise} \end{cases}$ .  *recursive definition*

# Combinations

## Definition

Let  $n$  and  $r$  be nonnegative integers. Define  $\binom{n}{r}$  ( $n$  choose  $r$ , also  $C_r^n$ ) to be the number of ways to choose  $r$  elements from  $1 \dots n$ .

Example:  $\binom{4}{2} = 6$



Combinations:





## Combinations

### Definition

Let  $n$  and  $r$  be nonnegative integers. Define  $\binom{n}{r}$  ( $n$  choose  $r$ , also  $C_r^n$ ) to be the number of ways to choose  $r$  elements from  $1 \dots n$ .

### Boundary Cases:

- $\binom{n}{0} = 1$
- $\binom{n}{n} = 1$
- $\binom{n}{r} = 0$  for  $r > n$ .
- Sometimes we extend the definition and say  $\binom{n}{r} = 0$  for negative  $r$ .

## Explicit Formula for $\binom{n}{r}$

### Theorem

For  $0 \leq r \leq n$ , the following holds:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

## Explicit Formula for $\binom{n}{r}$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad (1)$$

We show that  $n! = \binom{n}{r} r!(n-r)!$ .

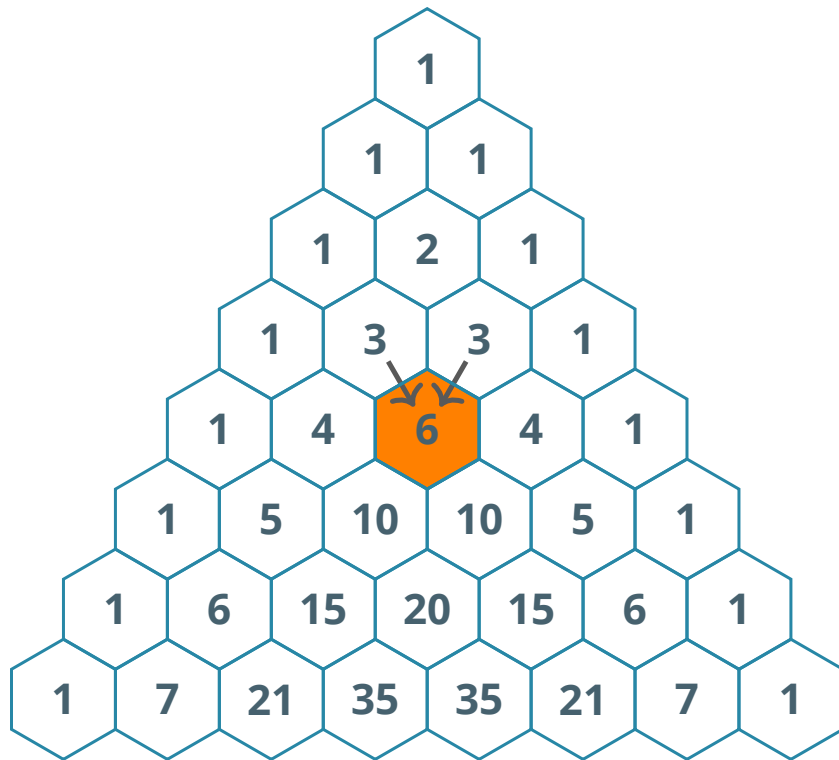
### What are we counting?

The number of permutations of  $1 \dots n$ .

- LHS: By definition, it is  $n!$ .
- RHS: Choose  $r$  elements out of  $n$  elements to be the first  $r$  elements.
  - For the first  $r$  elements, there are  $r!$  ways to permute them.
  - For the next  $n-r$  elements, there are  $(n-r)!$  ways to permute them.

By multiplication principle, there are  $\binom{n}{r} r!(n-r)!$  permutations.

## Combinations and Pascal's Triangle



**Pascal Triangle:** The  $r$ -th column on the  $n$ -th row is  $\binom{n}{r}$ .

**Pascal's Identity:**

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Let's use **two different methods** to prove this.

## Equation (2) – Pascal's Identity

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \quad (2)$$

### Method 1:

### What are we counting?

The number of ways to choose  $r$  elements from  $1 \dots n$ .

- LHS: By definition, it is  $\binom{n}{r}$ .
- RHS: Condition on whether  $n$  is chosen.
  - If no, we need to choose  $r$  elements from  $1 \dots (n-1)$ .  
By definition, there are  $\binom{n-1}{r}$  ways.
  - If yes, we need to choose  $r-1$  elements from  $1 \dots (n-1)$ .  
By definition, there are  $\binom{n-1}{r-1}$  ways.

## Equation (2) – Pascal's Identity

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \quad (2)$$

**Method 2:** Use the explicit formula for  $\binom{n}{r}$ .

$$\begin{aligned} \binom{n-1}{r} + \binom{n-1}{r-1} &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!} \cdot \left( \frac{1}{r} + \frac{1}{n-r} \right) \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!} \cdot \frac{n}{r(n-r)} \\ &= \frac{n!}{r!(n-r)!} \end{aligned}$$

## Combinatorial Proof and Algebraic Proof

Notice that Method 1 and Method 2 are very different approaches.

- **Method 1 (Combinatorial Proof):** “Tells a story” and through the metaphor of the story proves the result.
  - Question asked: What are we counting?
- **Method 2 (Algebraic Proof):** Manipulate formulas – Simplifications, substitutions, applying other equations.
  - Question asked: How can we make LHS equal to RHS?

## Equation (3) – Sum of a row

$$\sum_{r=0}^n \binom{n}{r} = 2^n \quad (3)$$

Let's try both combinatorial proof and algebraic proof.



## Equation (3) – Sum of a row

$$\sum_{r=0}^n \binom{n}{r} = 2^n \quad (3)$$

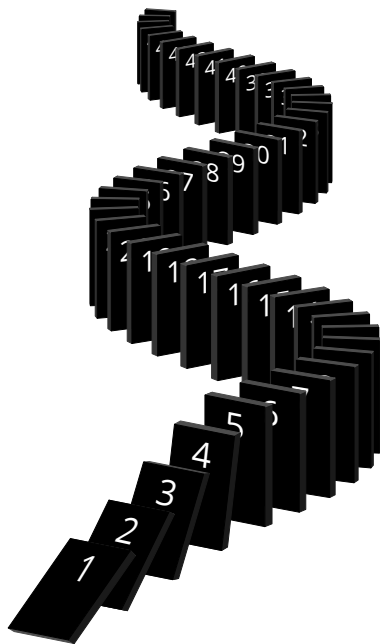
### Combinatorial Proof:

## What are we counting?

The number of subsets of  $1 \dots n$ .

- LHS: Condition on the size of the subset. There are  $\binom{n}{r}$  subsets of size  $r$ .
- RHS: By definition, it is  $2^n$ .

## Sidetrack: What is Induction?

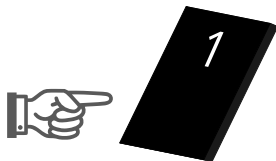


*“Mathematical induction proves that we can knock down as many dominoes as we like, by proving that we can knock down the first domino (the basis) and each domino knocked down causes the next one to fall (the induction).”*

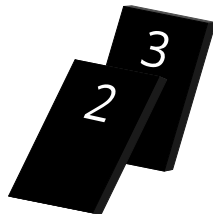
## Sidetrack: What is Induction?

**Goal:** Prove that some statement about the integer  $n$  is true for all  $n \geq n_0$ .

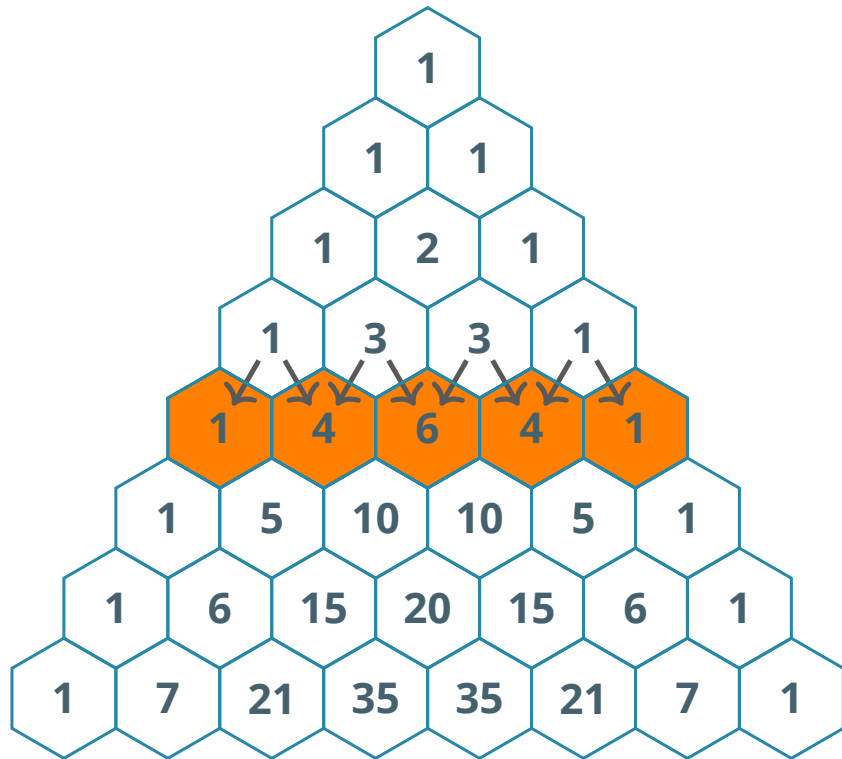
**Basis step:** Prove the statement when  $n$  has its smallest value  $n_0$ .



**Induction step:** Prove that if the statement is true for the value  $n - 1$ , then it is true for the value  $n$ .



## Equation (3) – Sum of a row



What is the relationship between  $\sum_{r=0}^k \binom{k}{r}$  and  $\sum_{r=0}^{k+1} \binom{k+1}{r}$ ?

**Observation:**

$$\begin{aligned} \sum_{r=0}^{k+1} \binom{k+1}{r} &= \sum_{r=0}^{k+1} \left( \binom{k}{r} + \binom{k}{r-1} \right) \\ &= 2 \sum_{r=0}^k \binom{k}{r} \end{aligned}$$

## Equation (3) – Sum of a row

**Algebraic Proof:** Induct on  $n$ .

- Let  $P(n)$  be the proposition  $\sum_{r=0}^n \binom{n}{r} = 2^n$ .
- **Basis step:** When  $n = 0$ ,  $\text{LHS} = \text{RHS} = 1$ .  $\therefore P(0)$  is true.
- **Induction step:** Assume  $P(k)$  is true for some  $k \geq 0$ . Then

$$\sum_{r=0}^{k+1} \binom{k+1}{r} = \sum_{r=0}^{k+1} \left( \binom{k}{r} + \binom{k}{r-1} \right) = 2 \sum_{r=0}^k \binom{k}{r} = 2 \times 2^k = 2^{k+1}$$

$\therefore P(k+1)$  is true.

- By mathematical induction,  $P(n)$  is true for all  $n \geq 0$ .

## Equation (4) – Symmetry

$$\binom{n}{r} = \binom{n}{n-r} \quad (4)$$

### Combinatorial Proof:

### What are we counting?

The number of ways to choose  $r$  elements in  $1 \dots n$ .

- LHS: By definition, it is  $\binom{n}{r}$ .
- RHS: Pick  $(n - r)$  elements to **exclude**. There are  $\binom{n}{n-r}$  ways to do so.

## Equation (4) – Symmetry

$$\binom{n}{r} = \binom{n}{n-r} \quad (4)$$

**Algebraic Proof:** Use the explicit formula for  $\binom{n}{r}$ .

$$\begin{aligned} \binom{n}{r} &= \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \binom{n}{n-r} \end{aligned}$$

## Equation (5) – Weighted sum of a row

$$\sum_{r=0}^n r \times \binom{n}{r} = n \times 2^{n-1} \quad (5)$$

### Combinatorial Proof:

## What are we counting?

From  $n$  people, the number of ways to form a committee with a chairperson.  
(Yes, you can use everyday language to reason about mathematics!)

- LHS: Form a committee first, then elect chairperson.
- RHS: Choose chairperson first, then pick the rest of the committee.



## Equation (5) – Weighted sum of a row

**Algebraic Proof:** Let  $P(n)$  be the proposition  $\sum_{r=0}^n r \times \binom{n}{r} = n \times 2^{n-1}$ .

**Basis Step:** When  $n = 0$ ,  $\text{LHS} = \text{RHS} = 0$ .  $\therefore P(0)$  is true.

**Induction Step:** Assume  $P(k)$  is true for some  $k \geq 0$ . Then

$$\begin{aligned} \sum_{r=0}^{k+1} r \times \binom{k+1}{r} &= \sum_{r=0}^{k+1} r \times \left( \binom{k}{r-1} + \binom{k}{r} \right) \\ &= \sum_{r=0}^{k+1} (r-1) \times \binom{k}{r-1} + \sum_{r=0}^{k+1} \binom{k}{r-1} + \sum_{r=0}^{k+1} r \times \binom{k}{r} \\ &= k \times 2^{k-1} + 2^k + k \times 2^{k-1} = k \times 2^k + 2^k = (k+1) \times 2^k \end{aligned}$$

$\therefore P(k+1)$  is true.

By mathematical induction,  $P(n)$  is true for all  $n \geq 0$ .

## Equation (6) – Binomial Theorem

$$\sum_{r=0}^n \binom{n}{r} a^r b^{n-r} = (a+b)^n \quad (6)$$

- Think about the number of ways to form the term  $a^r b^{n-r}$  from the expansion of  $(a+b)(a+b)\dots(a+b)$ .

- $a = b = 1$ :

$$\sum_{r=0}^n \binom{n}{r} = 2^n$$

- $a = -1, b = 1$ :

$$\sum_{r=0}^n (-1)^r \binom{n}{r} = 0$$

## Counting Techniques – Stars and Bars

- What is the number of solutions to  $x_1 + x_2 + x_3 = 10$ , where  $x_i \geq 1$  for  $i = 1, 2, 3$ ?



Answer:  $\binom{9}{2}$

- What is the number of solutions to  $x_1 + x_2 + \dots + x_r = n$ , where  $x_i \geq 1$  for  $i = 1, 2, \dots, r$ ?

Answer:  $\binom{n-1}{r-1}$

## Counting Techniques – Stars and Bars

- What is the number of solutions to  $x_1 + x_2 + x_3 = 10$ , where  $x_i \geq 0$  for  $i = 1, 2, 3$ ?



Let  $y_i = x_i + 1$ , then  $y_1 + y_2 + y_3 = 13$ ,  $y_i \geq 1$ . Answer:  $\binom{12}{2}$

- What is the number of solutions to  $x_1 + x_2 + \dots + x_r = n$ , where  $x_i \geq 0$  for  $i = 1, 2, \dots, r$ ?

Answer:  $\binom{r+n-1}{r}$

## Counting Techniques – Stars and Bars

Four ways to choose  $r$  elements from  $n$  elements:

	Order Matters	Order Does Not Matter
<b>Repetition Is Allowed</b>	$n^r$	$\binom{r+n-1}{r}$
<b>Repetition Is Not Allowed</b>	$P(n, r) = \frac{n!}{(n-r)!}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

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## Fibonacci Numbers (A000045)

(0, )1, 1, 2, 3, 5, 8, 13, 21, ...

Recursive Definition:

- $F_0 = 0, F_1 = 1$
- $F_{n+2} = F_n + F_{n+1}$

Combinatorial Definition:

- $F_n$  counts the number of ways to tile a  $1 \times (n - 1)$  board with squares and dominoes.
- $F_0 = 0$  because, well, a  $1 \times (-1)$  board does not exist.

**⚠ Are these two definitions equivalent?**

## Fibonacci Numbers (A000045)

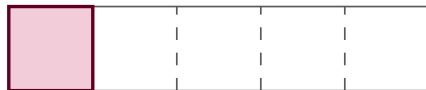
**Aim:** To show that the two definitions are equivalent.

$F_n$  = number of ways to tile a  $1 \times (n - 1)$  board with squares and dominoes.

- $F_1 = 1$  since a  $1 \times 0$  board can only be tiled with nothing.
- $F_{n+2}$ : We tile a  $1 \times (n + 1)$  board with squares and dominoes.
  - Start with a domino: We're left with a  $1 \times (n - 1)$  board –  $F_n$  ways.



- Start with a square: We're left with a  $1 \times n$  board –  $F_{n+1}$  ways.



$$\therefore F_0 = 0, F_1 = 1 \text{ and } F_{n+2} = F_n + F_{n+1}.$$

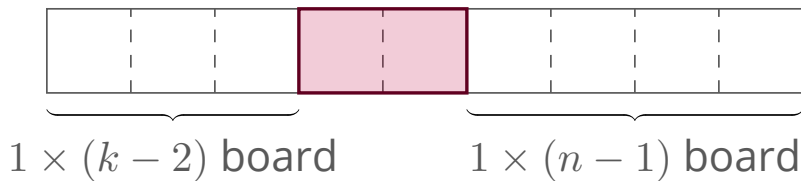


## Fibonacci Numbers – Equation (7)

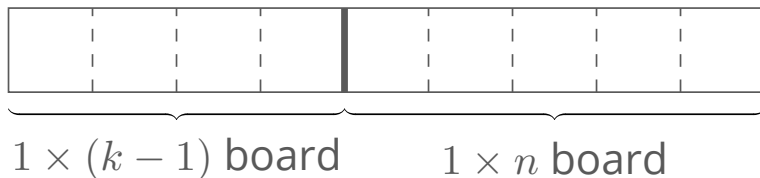
$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad (k \geq 1) \quad (7)$$

The number of square-domino tilings of a  $1 \times (n + k - 1)$  board.

- LHS: By definition, it is  $F_{n+k}$ .
- RHS: Condition on whether a domino covers cells  $(k - 1)$  and  $k$ .
  - If yes, there are  $F_{k-1} \times F_n$  ways to tile the rest.



- If no, there are  $F_k \times F_{n+1}$  ways to tile the rest.



## Fibonacci Numbers – Equation (7)

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad (k \geq 1) \quad (7)$$

Putting  $k = n$  and  $k = n + 1$ , respectively, we get

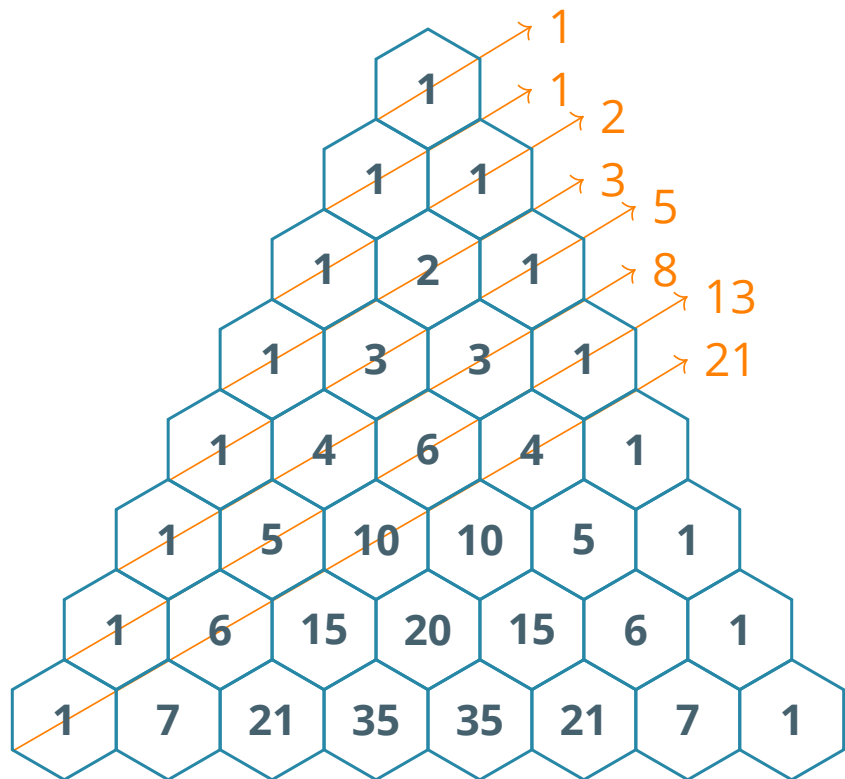
$$\begin{aligned} F_{2n} &= F_n F_{n+1} + F_{n-1} F_n \\ F_{2n} &= F_n F_{n+1} + (F_{n+1} - F_n) F_n \\ \mathbf{F_{2n}} &= \mathbf{2F_n F_{n+1} - F_n F_n} \end{aligned}$$

and

$$\mathbf{F_{2n+1}} = \mathbf{F_{n+1} F_{n+1} + F_n F_n},$$

which give an  $O(\log n)$  algorithm to compute  $F_n \bmod M$ .

## Relating Fibonacci Numbers and $\binom{n}{r}$



$$F_{n+1} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r}$$

**Exercise:** Let's prove this using a combinatorial proof and an algebraic proof.

## Relating Fibonacci Numbers and $\binom{n}{r}$

$$F_{n+1} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} \quad (8)$$

### Combinatorial Proof:

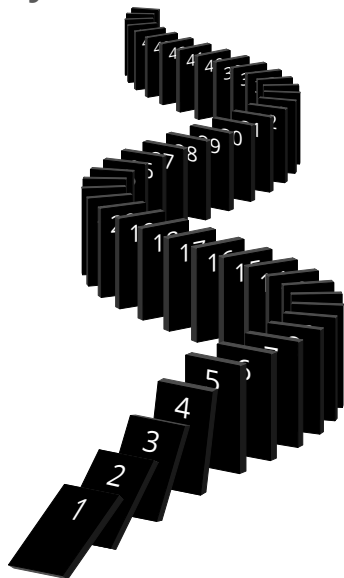
## What are we counting?

The number of square-domino tilings of a  $1 \times n$  board.

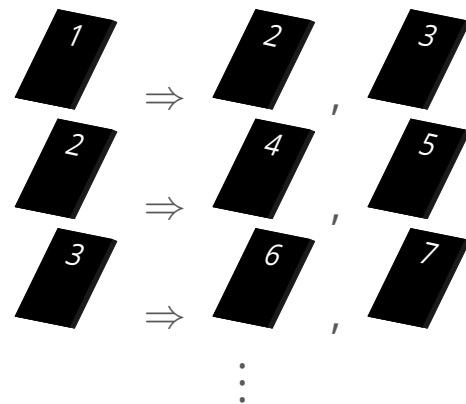
- LHS: By definition, it is  $F_{n+1}$ .
- RHS: Condition on the number of dominoes. For tilings with  $k$  dominoes, a total of  $n - k$  pieces will be used, and so there are  $\binom{n-k}{k}$  ways to tile the board.

## Sidetrack: Weak vs Strong Mathematical Induction

**Weak MI:** Knock the dominoes one by one.  $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$ .



**Strong MI:** Knock the dominoes in **any creative order**, e.g.  
 $P(n) \rightarrow P(2n) \text{ \& } P(2n + 1)$ .



## Relating Fibonacci Numbers and $\binom{n}{r}$

**Algebraic Proof:** Let  $P(n)$  be the proposition  $F_{n+1} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r}$ .

**Basis step:** When  $n = 0$  and  $n = 1$ , LHS = RHS = 1.  $\therefore P(0)$  and  $P(1)$  are true.

**Induction step:** Assume  $P(k)$  and  $P(k+1)$  are true for some  $k \geq 0$ . Then

$$\begin{aligned} F_{k+3} &= F_{k+1} + F_{k+2} \\ &= \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{k-r}{r} + \sum_{r=0}^{\lfloor (k+1)/2 \rfloor} \binom{k+1-r}{r} \\ &= \sum_{r=0}^{\lfloor (k+2)/2 \rfloor} \binom{k+1-r}{r-1} + \sum_{r=0}^{\lfloor (k+2)/2 \rfloor} \binom{k+1-r}{r} = \sum_{r=0}^{\lfloor (k+2)/2 \rfloor} \binom{k+2-r}{r} \end{aligned}$$

$\therefore P(k+2)$  is true.

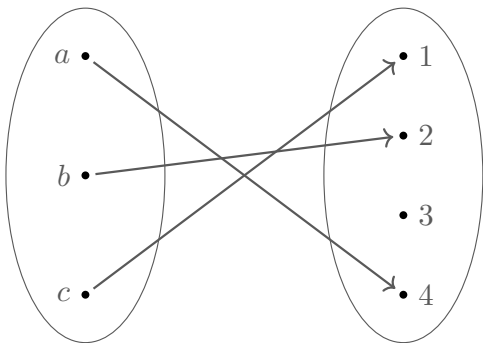
By **strong** mathematical induction,  $P(n)$  is true for all  $n \geq 0$ .

## A Brief Summary: Proof Techniques

- **Combinatorial Proof:** “Tells a story” and through the metaphor of the story proves the result.
  - Question asked: What are we counting?
- **Algebraic Proof:** Manipulate formulas – Simplifications, substitutions, applying other equations.
  - Question asked: How can we make LHS equal to RHS?
- **Induction:** A tool for us to prove a statement for all  $n \geq n_0$ . It can be used in **BOTH** combinatorial and algebraic proofs.
  - Weak Mathematical Induction: Prove that  $P(n_0)$  is true and  $P(k) \rightarrow P(k+1)$  for all  $k \geq n_0$ .
  - Strong Mathematical Induction: Prove in any creative order!

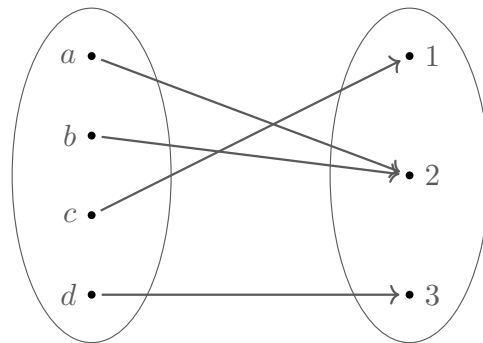
## Injectons and Surjections

Given two sets  $X, Y$  and a function  $f : X \rightarrow Y$ .



**Injection** (One-to-one)

Each element in  $Y$  is mapped to  
**at most** one element from  $X$



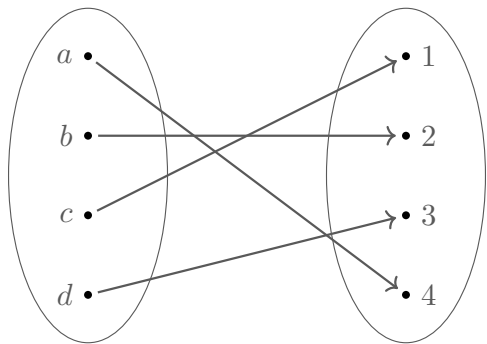
**Surjection** (Onto)

Each element in  $Y$  is mapped to  
**at least** one element from  $X$



## Bijections

What if a function  $f : X \rightarrow Y$  is **both** an injection and a surjection?



### Bijection

Each element in  $Y$  is mapped to **exactly** one element from  $X$

**Important property:** If we can find a bijection between two finite sets  $X$  and  $Y$ , then  $|X| = |Y|$  (i.e.  $X$  and  $Y$  has the same cardinality).

## Bijjective Proofs

**Bijjective Proof** can be used to show that two sets have the same cardinality.

- 1 Define a (bijjective) function between the sets  $X$  and  $Y$ .
- 2 Prove that your function is a bijection.
  - ❗ A function is a bijection if and only if an **inverse function** exists.  
Therefore, we can prove that the function is a bijection by providing the inverse function.
- 3 Conclude that  $|X| = |Y|$ .

## Bijjective Proof – An example

**Example:** Prove that the number of subsets of  $\{1, 2, \dots, n\}$  without two consecutive integers is  $F_{n+2}$ .

Consider the following sets:

- $X$ : The number of subsets of  $\{1, 2, \dots, n\}$  without two consecutive integers.
- $Y$ : The number of ways to tile a  $1 \times (n + 1)$  board with squares and dominoes.

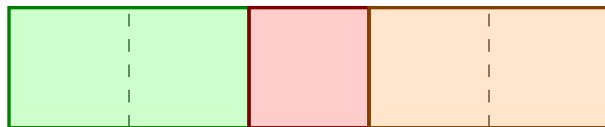
Since we want to prove that  $|X| = |Y|$ , it's a good idea to try establishing a bijection between the two sets.

## Bijjective Proof – An example

Given a way to tile a  $1 \times (n + 1)$  board with squares and dominoes:

- Consider the  $n$  gaps between the cells.
- If there is a gap between cell  $i$  and cell  $i + 1$ , exclude  $i$  from the set.
- Otherwise, include  $i$  in the set.
- This forms a subset of  $\{1, 2, \dots, n\}$  without two consecutive integers.

$\{ 1 \quad \cancel{2} \quad , \quad \cancel{3} \quad 4 \}$

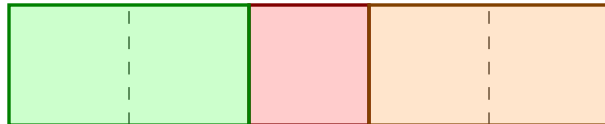


## Bijjective Proof – An example

**Inverse function:** Given a subset of  $\{1, 2, \dots, n\}$  without two consecutive integers.

- For each  $1 \leq i \leq n$ , if  $i$  is in the set, tile cells  $i$  and  $i + 1$  with a domino.
- For all the other cells, tile them with squares.
- This forms a unique square-domino tiling for a  $1 \times (n + 1)$  board.

$\{ 1, 4 \}$



## Catalan Numbers (A000108)

(1, )1, 2, 5, 14, 42, ...

What does this sequence  $C_n$  describe?

### Many Things!

- $C_n$  is the number of valid bracket sequences of length  $2n$ .
- $C_n$  is the number of triangulations of a convex  $(n + 2)$ -gon.
- $C_n$  is the number of “ordered” rooted binary trees with  $n$  nodes.
- The list goes on...

## Let's Take A Look At Small $n$ 's

### Definition

Define  $C_n$  to be the number of valid bracket sequences of length  $2n$ .

Valid means the ( and ) brackets form matching pairs.

- $n = 0$ : (empty sequence)
- $n = 1$ : ( )
- $n = 2$ : ( ) ( ), ( ( ) )
- $n = 3$ : ( ) ( ) ( ), ( ) ( ( ) ), ( ( ) ) ( ), ( ( ) ( ) ), ( ( ( ) ) )
- $n = 4$ : ( ) ( ) ( ) ( ), ( ) ( ) ( ( ) ), ( ) ( ( ) ) ( ), ( ) ( ( ) ( ) ),  
( ) ( ( ( ) ) ), ( ( ) ) ( ) ( ), ( ( ) ) ( ( ) ), ( ( ) ( ) ) ( ), ( ( ( ) ) ) ( ),  
( ( ) ) ( ) ( ), ( ( ) ( ( ) ) ), ( ( ( ) ) ) ( ), ( ( ( ) ( ) ) ), ( ( ( ( ) ) ) )

## Recurrence Formula

$$C_n = \sum_{k=1}^n C_{k-1} \times C_{n-k} \quad (9)$$

### What are we counting?

The number of valid bracket sequences of length  $2n$ .

- LHS: By definition, it is  $C_n$ .
- RHS: Condition on the position of  $)$  which closes the first bracket  $($ . For  $1 \leq k \leq n$ , there are  $C_{k-1} \times C_{n-k}$  valid bracket sequences with  $)$  positioned at  $2k$  (note that the position must be even).

This gives an  $O(n^2)$  algorithm to compute  $C_n \bmod M$ .



## Explicit Formula For $C_n$

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (10)$$

Reformulate the Catalan model as follows:

- You start from  $(0, 0)$  and want to reach  $(2n, 0)$ .
- From  $(x, y)$  you can move “up” to  $(x + 1, y + 1)$  or “down” to  $(x + 1, y - 1)$ .
- $C_n$  is the number of “good” paths – paths that do not go below the x-axis.

Ideas:

- We count, instead, the number of bad paths.
- Then  $C_n = \binom{2n}{n} - (\text{number of bad paths})$ .
- Therefore, for the formula to hold, we expect there to be  $\binom{2n}{n+1}$  bad paths.

## Explicit Formula For $C_n$

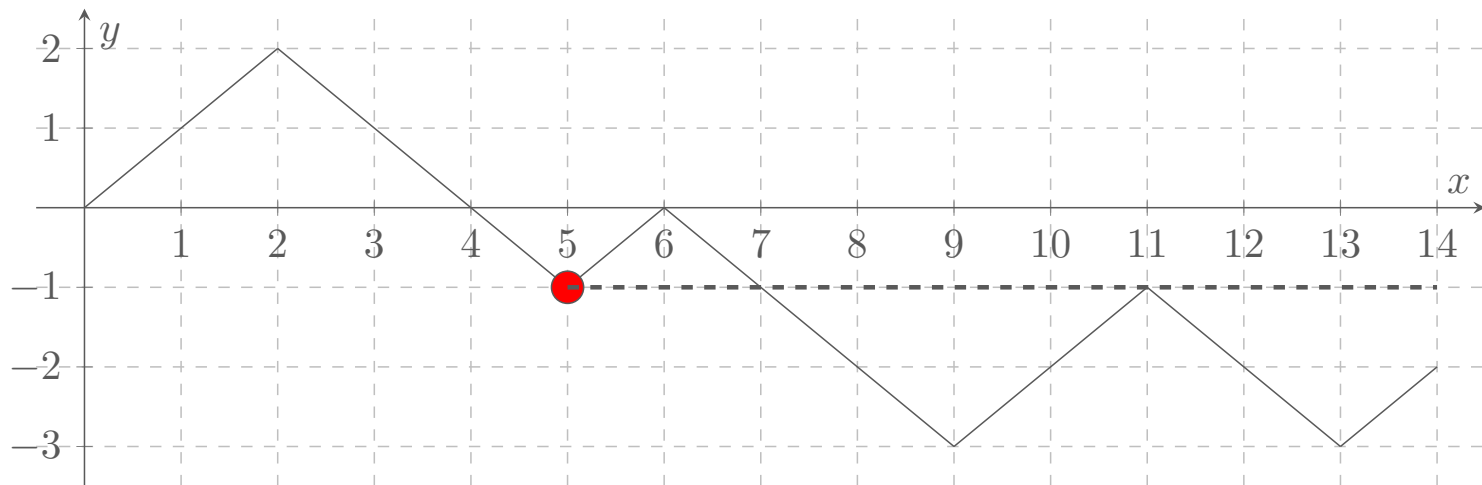
Establish a bijection between:

- Bad paths from  $(0, 0)$  to  $(2n, 0)$ , and
- Paths from  $(0, 0)$  to  $(2n, -2)$

## Explicit Formula For $C_n$

Given a bad path from  $(0, 0)$  to  $(2n, 0)$ :

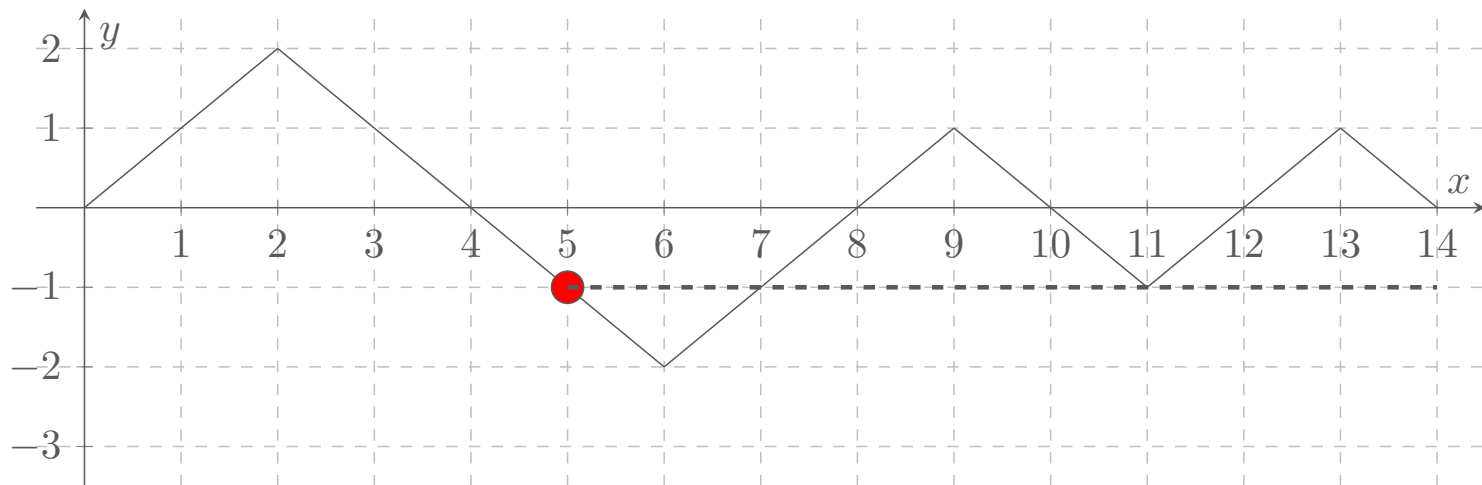
- We take the portion of the path **after** the first visit below  $x$ -axis.
- Reflect it about  $y = -1 \Rightarrow$  a path from  $(0, 0)$  to  $(2n, -2)$ .



## Explicit Formula For $C_n$

**Inverse function:** Given a path from  $(0, 0)$  to  $(2n, -2)$ :

- We take the portion of the path **after** the first visit below  $x$ -axis.
- Reflect it about  $y = -1 \Rightarrow$  a bad path from  $(0, 0)$  to  $(2n, 0)$ .



## Reflection Trick – Exercise

Exercise: Codeforces 26D, La Salle-Pui Ching Programming Challenge 2017

Problem L (Let Me Count The Ways)

Further Reading: Young Tableaux

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- 1 Basics of Counting
- 2 Fibonacci and Catalan Sequence
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- 4 Sums and Expected Value
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(This section is not archived to open up possibilities of reuse in the future)

## Set Union, Set Intersection

### Definition

Given two sets  $A$  and  $B$ .

Their union  $A \cup B$  is the set of elements belonging to *at least one* of  $A$  and  $B$ .

Their intersection  $A \cap B$  is the set of elements belonging to *both* of  $A$  and  $B$ .

### Example 1

$A := \{\text{Alex, Anson, Jason}\}, B := \{\text{Alex, Sampson, RB}\}.$

Then  $A \cup B = \{\text{Alex, Anson, Jason, Sampson, RB}\}; A \cap B = \{\text{Alex}\}.$

### Example 2

$A := \{\text{prime numbers}\}, B := \{\text{even numbers}\}$

Then  $A \cap B = \{2\}.$

Mental Shortcut: intersection = AND, union = OR

## Subset, Complement

### Definition

Given two sets  $X$  and  $Y$ .

$Y$  is said to be a subset of  $X$  if all elements of  $Y$  can be found in  $X$ .

We write  $Y \subseteq X$ .

If  $Y \subseteq X$ , define its complement (with respect to  $X$ )  $X \setminus Y$  to be the set of elements in  $X$  but not in  $Y$ .

We also use  $Y^c$  to denote  $X \setminus Y$  when it is clear what  $X$  is.

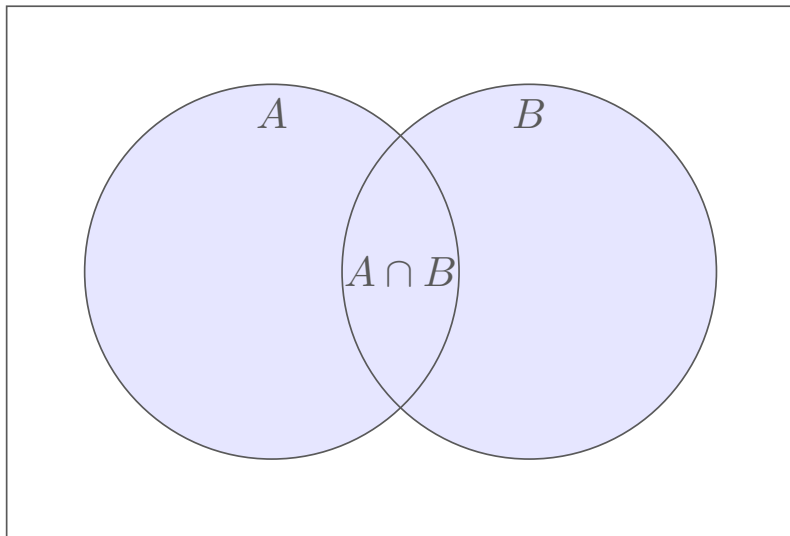
### Example

Let  $X := \{\text{all positive integers}\}$ ,  $Y := \{1, 4, 9, 16, \dots\}$ .

Then  $Y \subseteq X$  and  $X \setminus Y = \{\text{positive integers that are not perfect squares}\}$ .



## Inclusion-Exclusion (for two sets)



**Generic form:**

$$|A \cup B| = |A| + |B| - |A \cap B|$$

**Alternative form:**

$$|A \cap B| = |X| - |A^c| - |B^c| + |A^c \cap B^c|$$

## Inclusion-Exclusion (for two sets)

Generic form:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

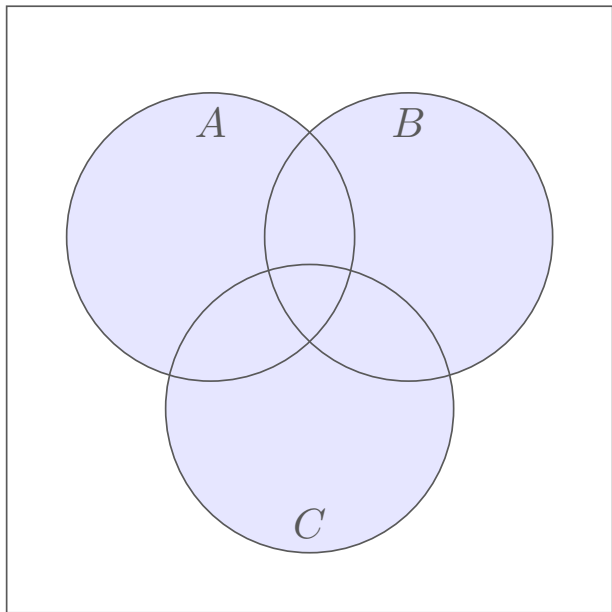
- Answers “how many elements satisfy at least one of the conditions?”
- Useful when **satisfaction of specific conditions** is easier to count

Alternative form:

$$|A \cap B| = |X| - |A^c| - |B^c| + |A^c \cap B^c|$$

- Answers “how many elements satisfy both conditions?”
- Useful when **violation of specific conditions** is easier to count

## Inclusion-Exclusion (for three sets)



**Generic form:**

$$|A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

**Alternative form:**

$$|A \cap B| = |X| - |A^c| - |B^c| - |C^c| \\ + |A^c \cap B^c| + |A^c \cap C^c| + |B^c \cap C^c| \\ + |A^c \cap B^c \cap C^c|$$

## Inclusion-Exclusion (for three sets)

Generic form:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - (|A \cap B| + |A \cap C| + |B \cap C|) \\ &\quad + |A \cap B \cap C| \end{aligned}$$

Alternative form:

$$\begin{aligned} |A \cap B \cap C| &= |X| \\ &\quad - (|A^c| + |B^c| + |C^c|) \\ &\quad + (|A^c \cap B^c| + |A^c \cap C^c| + |B^c \cap C^c|) \\ &\quad - |A^c \cap B^c \cap C^c| \end{aligned}$$

Can you see a pattern?

## Inclusion-Exclusion (for $n$ sets)

Given  $S_1, \dots, S_n \subseteq X$ .

Generic form:

$$|S_1 \cup \dots \cup S_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}|$$

Alternative form:

$$|S_1 \cap \dots \cap S_n| = \sum_{k=0}^n (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1}^c \cap \dots \cap S_{i_k}^c|$$

## Inclusion-Exclusion – Bijective Proof

Let's try a **bijective proof** for the generic form:

$$|S_1 \cup \dots \cup S_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}|$$

Rearrange the negative terms to the left hand side. Then we have

- LHS:  $|S_1 \cup \dots \cup S_n| + \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}|$  for all even  $k$ .
- RHS:  $\sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}|$  for all odd  $k$ .

We want to establish a bijection between LHS and RHS.

## Inclusion-Exclusion – Bijective Proof

(From Previous Slide) We have:

- LHS:  $|S_1 \cup \dots \cup S_n| + \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}|$  for all even  $k$ .
- RHS:  $\sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}|$  for all odd  $k$ .

Let's now consider **one** element  $x$ . Let  $S_{c_1}, S_{c_2}, \dots, S_{c_m}$  be all the sets that  $x$  belong to.

- LHS:  $x$  is definitely in  $|S_1 \cup \dots \cup S_n|$ . +1 contribution.  
For each set  $\{i_1, i_2, \dots, i_k\} \subseteq \{c_1, c_2, \dots, c_m\}$  such that  $k$  is even, there will be another +1 contribution.
- RHS: For each set  $\{i_1, i_2, \dots, i_k\} \subseteq \{c_1, c_2, \dots, c_m\}$  such that  $k$  is odd, there will be another +1 contribution.

## Inclusion-Exclusion – Bijective Proof

Consider a set  $\{i_1, i_2, \dots, i_k\} \subseteq \{c_1, c_2, \dots, c_m\}$  such that  $k$  is even, or the “special element”  $|S_1 \cup \dots \cup S_n|$ . (LHS)

- If the element is the “special element”, turn it into the odd set  $\{c_1\}$ .  $\{c_1\} \subseteq \{c_1, c_2, \dots, c_m\}$ , and 1 is odd. (RHS)
- Otherwise, “toggle” whether  $c_1$  is in the set, i.e. if  $c_1$  is not in  $\{i_1, i_2, \dots, i_k\}$ , add  $c_1$  to the set; if  $c_1$  is already in the set, remove it from the set. It becomes a set  $\{i_1, i_2, \dots, i_{k \pm 1}\} \subseteq \{c_1, c_2, \dots, c_m\}$ , where  $k \pm 1$  is odd. (RHS)



## Inclusion-Exclusion – Bijective Proof

**Inverse Function:** Consider a set  $\{i_1, i_2, \dots, i_k\} \subseteq \{c_1, c_2, \dots, c_m\}$  such that  $k$  is odd. (RHS)

- If  $\{i_1, i_2, \dots, i_k\} = \{c_1\}$ , turn it into the “special element”  $|S_1 \cup \dots \cup S_n|$ . (LHS)
- Otherwise, “toggle” whether  $c_1$  is in the set, i.e. if  $c_1$  is not in  $\{i_1, i_2, \dots, i_k\}$ , add  $c_1$  to the set; if  $c_1$  is already in the set, remove it from the set. It becomes a set  $\{i_1, i_2, \dots, i_{k \pm 1}\} \subseteq \{c_1, c_2, \dots, c_m\}$ , where  $k \pm 1$  is even. (LHS)

## Inclusion-Exclusion – Variants

Another question: “How many elements satisfy **exactly**  $k$  conditions?”

$$\begin{aligned}
 X = & \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_k}| \\
 & - (k+1) \sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} |S_{i_1} \cap \dots \cap S_{i_{k+1}}| \quad \blacktriangleleft \binom{k+1}{k} \\
 & + \binom{k+2}{2} \sum_{1 \leq i_1 < \dots < i_{k+2} \leq n} |S_{i_1} \cap \dots \cap S_{i_{k+2}}| \quad \blacktriangleleft \binom{k+2}{k} - (k+1)\binom{k+2}{k+1} \\
 & + \binom{k+3}{3} \sum_{1 \leq i_1 < \dots < i_{k+3} \leq n} |S_{i_1} \cap \dots \cap S_{i_{k+3}}| \quad \blacktriangleleft \binom{k+3}{k} - (k+1)\binom{k+3}{k+1} + \binom{k+2}{2}\binom{k+3}{k+2} \\
 & + \dots
 \end{aligned}$$

## Inclusion-Exclusion – Variants

General formula (for **exactly**  $k$  conditions):

$$X = \sum_{j=k}^n (-1)^{j-k} \binom{j}{j-k} \sum_{1 \leq i_1 < \dots < i_k \leq n} |S_{i_1} \cap \dots \cap S_{i_j}|$$

Exercise: What about **at least**  $k$  conditions?

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## Introduction to Probability Space

A (finite) probability space consists of two parts:

- 1 A set  $\Omega = \{a_1, \dots, a_n\}$  of outcomes
- 2 Associated probabilities  $p_1, \dots, p_n$  of each outcome ( $\sum_i p_i = 1$ )

Probabilities can be abstract (based on idealised model) or empirical (based on statistical evidence).

Usually we consider uniform probability spaces, i.e.  $p_1 = \dots = p_n = \frac{1}{n}$ .

Examples:

- Dice roll
- Coin toss
- A sequence of  $N$  coin tosses
- Drawing a ball from a bag of colored balls

## Event Probability

An event is (a short description of) a subset of  $\Omega$ .

For example:

- $\Omega_1 :=$  sequences of 3 coin tosses,  $A_1 :=$  seqs with two consecutive heads.  
Then,  $A_1 = \{HHH, HHT, THH\}$ .  
 $P(A_1) = \frac{3}{8}$ .
- $\Omega_2 :=$  dice roll,  $A_2 :=$  odd outcomes.  
Then,  $A_2 = \{1, 3, 5\}$ .  
 $P(A_2) = \frac{3}{6} = \frac{1}{2}$ .

For uniform probability spaces,

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of possible outcomes}},$$

so we just need to count :)

## Conditional Probability

For two events  $A$  and  $B$  with  $P(B) > 0$ , **define** conditional probability as

$$P(A|B) := \frac{P(A \cap B)}{P(B)}.$$

For DP on probabilities, we often use the fact that

$$P(A \cap B) = P(A|B) \cdot P(B)$$

new state    transition    previous  
state

Exercise: Suppose we pick an integer  $x$  in  $[1, 20]$  uniformly at random. Calculate  $P(x \text{ is odd} \mid x \text{ is prime})$  and  $P(x \text{ is prime} \mid x \text{ is odd})$ .



## Random Variable, Expectation, Variance

A random variable is a function  $f : \Omega \longrightarrow \mathbb{R}$ .

It assigns a value of each of  $a_1, \dots, a_n$ .

Examples: slot machine, dice roll

### More Definitions

$$\text{Expectation of } f : \quad \mathbb{E}[f] := \sum_{i=1}^n f(a_i)p_i.$$

$$\text{Variance of } f : \quad \text{Var}(f) := \mathbb{E}[(f - \mathbb{E}[f])^2].$$

$\mathbb{E}[f]$  is the average value of  $f$ , weighted by probability (think lottery).

$\text{Var}(f)$  measures how  $f$  deviates from its average.

## A Remark

Event probability is just the expected value of certain **indicator functions**.

Just choose  $f(x) := 1$  if  $x \in A$  and  $f(x) := 0$  otherwise.

Then  $\mathbb{E}[f] = P(A)$ .

We denote this function by  $1_A$ .

## Linearity of Expectation

### Theorem (Linearity of Expectation)

Suppose random variable  $f$  is written as sum  $f_1 + \cdots + f_k$ . Then,

$$\mathbb{E}[f] = \sum_{i=1}^k \mathbb{E}[f_i].$$

(Think about: do we have “linearity of variance”?) NO.

This simple theorem is very powerful!

Core Idea: Break down a quantity into small,  
easy-to-calculate parts.

## Example 1

Suppose we pick a permutation  $p = [p_1, \dots, p_n]$  of  $1..n$  uniformly at random. What is the expected number of fixed points of  $p$ ?

Let  $X$  be the number of fixed points of  $p$ . Then  $X = X_1 + \dots + X_n$  where  $X_i$  is indicator function for  $p_i = i$ .

By linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^n \mathbb{E}[X_i] \\ &= \sum_{i=1}^n P(p_i = i) \\ &= \sum_{i=1}^n \frac{1}{n} \\ &= 1.\end{aligned}$$

## Example 2 (HKOJ S232)

Dr Jones partitions  $N$  students into  $K$  teams randomly.  $M$  pairs of students are enemies. What is the expected number of pairs of enemies that are placed in the same team?

Let  $X$  be the number of pairs of enemies that are placed in the same team. Then  $X = X_1 + \cdots + X_M$  where  $X_i$  is the indicator function for “the  $i$ -th pair of enemy is placed in the same team”.

By linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^M \mathbb{E}[X_i] \\ &= \sum_{i=1}^M P(i\text{-th pair of enemy is placed in the same team}) \\ &= \sum_{i=1}^M \frac{1}{K} \\ &= \frac{M}{K}\end{aligned}$$

## Example 3 (c.f. CF453A)

You roll a 6-sided die  $N$  times.

What is the expected value of the biggest number obtained?

Let  $X$  be the biggest number obtained. Then  $X = X_1 + \cdots + X_6$  where  $X_i$  is indicator function for “exists a number  $\geq i$ ”.

By linearity of expectation,

$$\begin{aligned}
 \mathbb{E}[X] &= \sum_{i=1}^6 \mathbb{E}[X_i] \\
 &= \sum_{i=1}^6 P(\text{exists a number} \geq i) \\
 &= \sum_{i=1}^6 (1 - P(\text{all numbers smaller than } i)) \\
 &= \sum_{i=1}^6 (1 - (\frac{i-1}{6})^n) \\
 &= 6 - \frac{1^n + 2^n + 3^n + 4^n + 5^n}{6^n}
 \end{aligned}$$

## Example 4

Generate a sequence of length  $n$ , where each element is chosen independently and uniformly from  $1..k$ .

What is the expected number of distinct elements?

Let  $X$  be the number of distinct elements. Then  $X = X_1 + \cdots + X_k$  where  $X_i$  is indicator function for “number  $i$  appears in the sequence”.

By linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^k \mathbb{E}[X_i] \\ &= \sum_{i=1}^k P(\text{number } i \text{ appears in the sequence}) \\ &= \sum_{i=1}^k (1 - P(\text{number } i \text{ does not appear in the sequence})) \\ &= \sum_{i=1}^k (1 - (\frac{k-1}{k})^n) \\ &= k(1 - (\frac{k-1}{k})^n)\end{aligned}$$

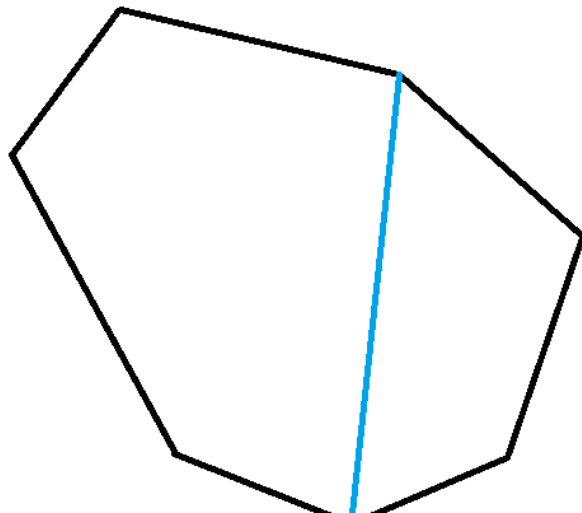
## Example 5

Given a convex polygon with  $N$  vertices. Choose a random subset of vertices, each vertex chosen with probability  $\frac{1}{2}$  and independent of other vertices. Form a new polygon by connecting adjacent chosen vertices. What is the expected perimeter of the new polygon?

A diagonal is a part of the perimeter if and only if:

- Its two endpoints are chosen
- All points on at least one side are not chosen

Notice that if the endpoints are too far away (in terms of number of vertices in between), the probability is negligible.





## Expectation and Counting

Same Core Idea:

Break down a quantity into small, easy-to-calculate parts.

Question on expectation:

Suppose we pick a permutation  $p = [p_1, \dots, p_n]$  of  $1..n$  uniformly at random.  
What is the **expected number** of fixed points of  $p$ ?

Can be easily translated into

Question on counting:

Consider all permutations  $p = [p_1, \dots, p_n]$  of  $1..n$ . What is the **total number** of fixed points?

## Counting Fixed Points – $n = 4$

p[1]	p[2]	p[3]	p[4]	fixed pts		p[1]	p[2]	p[3]	p[4]	fixed pts
1	2	3	4	4		3	1	2	4	1
1	2	4	3	2		3	1	4	2	0
1	3	2	4	2		3	2	1	4	2
1	3	4	2	1		3	2	4	1	1
1	4	2	3	1		3	4	1	2	0
1	4	3	2	2		3	4	2	1	0
2	1	3	4	2		4	1	2	3	0
2	1	4	3	0		4	1	3	2	1
2	3	1	4	1		4	2	1	3	1
2	3	4	1	0		4	2	3	1	2
2	4	1	3	0		4	3	1	2	0
2	4	3	1	1		4	3	2	1	0
					SUM	6	6	6	6	24

Summing “vertically” down the four columns is much easier!

## Example 6

Given an integer set  $S = \{b_1, b_2, \dots, b_n\}$ . For each subset  $T$  of  $S$ , let  $sum(T)$  denote the sum of elements of  $T$ . Calculate

$$\sum_{T \subseteq S} sum(T) \mod M.$$

Let

$$contain_i(T) := \begin{cases} 1, & \text{if } T \text{ contains } b_i; \\ 0, & \text{otherwise.} \end{cases}$$

Then  $sum(T) = \sum_{i=1}^n b_i \times contain_i(T)$ , and

$$\begin{aligned} \sum_{T \subseteq S} sum(T) &= \sum_{T \subseteq S} \sum_{i=1}^n b_i \times contain_i(T) \\ &= \sum_{i=1}^n b_i \left( \sum_{T \subseteq S} contain_i(T) \right) \\ &= \sum_{i=1}^n b_i 2^{n-1} \end{aligned}$$

## Example 7

Given an array  $a[1..n]$ . For  $1 \leq l \leq r \leq n$ , let  $sum(l, r)$  denote  $a[l] + a[l + 1] + \cdots + a[r]$ . Calculate

$$\sum_{1 \leq l \leq r \leq n} sum(l, r).$$

Let

$$contain_p(l, r) := \begin{cases} 1, & \text{if } l \leq p \leq r; \\ 0, & \text{otherwise.} \end{cases}$$

Then  $sum(l, r) = \sum_{p=1}^n a[p] \times contain_p(l, r)$ , and

$$\begin{aligned} \sum_{1 \leq l \leq r \leq n} sum(l, r) &= \sum_{1 \leq l \leq r \leq n} \sum_{p=1}^n a[p] \times contain_p(l, r) \\ &= \sum_{p=1}^n a[p] \times p \times (n + 1 - p) \end{aligned}$$

## Example 8

Given an integer set  $S = \{s_1, s_2, \dots, s_n\}$ . For each nonempty subset  $T$  of  $S$ , let  $\text{diff}(T)$  denote the difference between the largest and the smallest elements of  $T$ . Calculate

$$\sum_{\emptyset \neq T \subseteq S} \text{diff}(T) \pmod{M}.$$

Sort the numbers. For  $1 \leq p < n$ , let

$$\text{contain}_p(T) := \begin{cases} 1, & \text{if } \min\_index(T) \leq p < \max\_index(T) \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\text{diff}(T) = \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times \text{contain}_p(T)$ , and

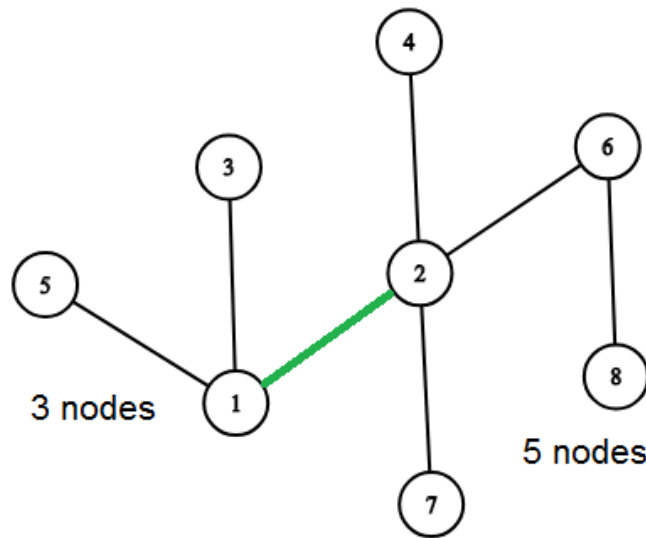
$$\begin{aligned} \sum_{\emptyset \neq T \subseteq S} \text{diff}(T) &= \sum_{\emptyset \neq T \subseteq S} \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times \text{contain}_p(T) \\ &= \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times \sum_{\emptyset \neq T \subseteq S} \text{contain}_p(T) \\ &= \sum_{p=1}^{n-1} (s_{p+1} - s_p) \times (2^p - 1) \times (2^{n-p} - 1) \end{aligned}$$

## Example 9

Given a tree with  $N$  nodes. Let  $l(x, y)$  be the length (number of edges) of the simple path from  $x$  to  $y$ . Calculate

$$\sum_{x,y} l(x, y).$$

For each edge, count the number of paths that pass through it.  
It is (number of nodes on one side)  $\times$  (number of nodes on the other side).  
This can be done using a DFS on tree.



## Further Reading

Proofs That Really Count (The Art of Combinatorial Proof)  
by Arthur T. Benjamin and Jennifer J. Quinn

Sums and Expected Value – Part 1 (<https://codeforces.com/blog/entry/62690>)  
Sums and Expected Value – Part 2 (<https://codeforces.com/blog/entry/62792>)  
by Errichto

Concrete Mathematics: A Foundation for Computer Science  
by Ronald L. Graham, Donald E. Knuth and Oren Patashnik

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