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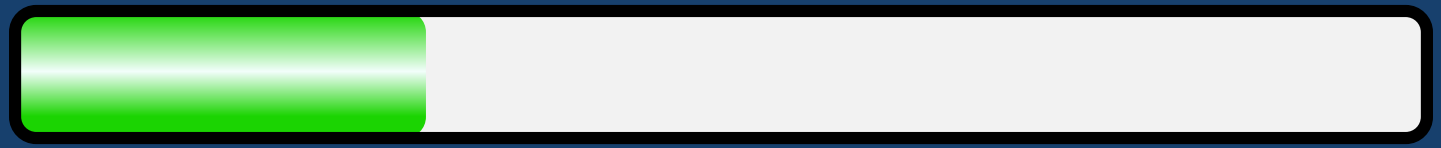
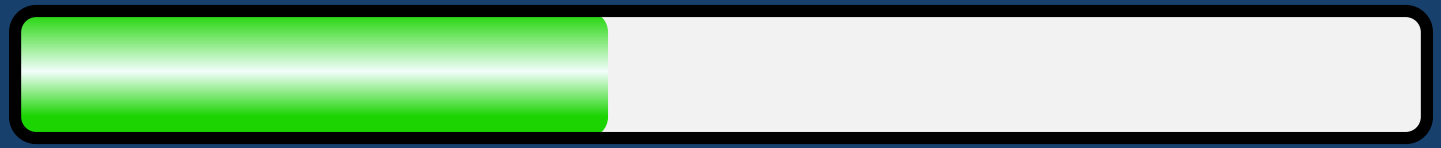
Python programming Introduction Content

2. Data types/outputs/inputs
3. Operators
4. Functions and Modules



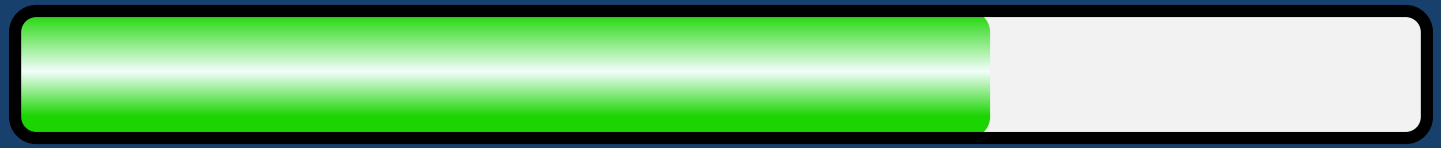
Python programming Introduction Content

5. Conditional statements and expression
6. Loops
7. Work with standard Library and Modules



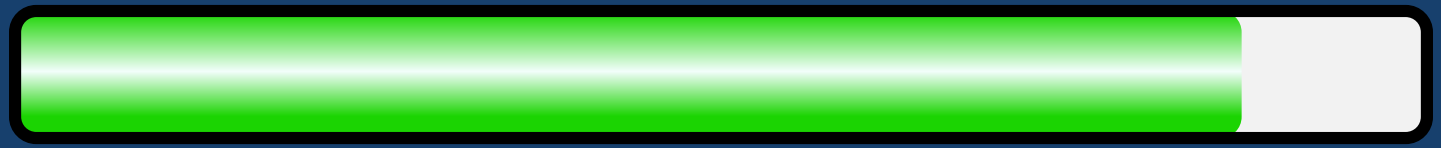
Python programming Introduction Content

8. Data structure in python
9. List
10. Tuple
11. Dictionaries
12. Set



Python programming Introduction Content

13. Files
14. Functions and Modules
15. Classes
16. Introduction to Numpy
17. Introduction to Pandas



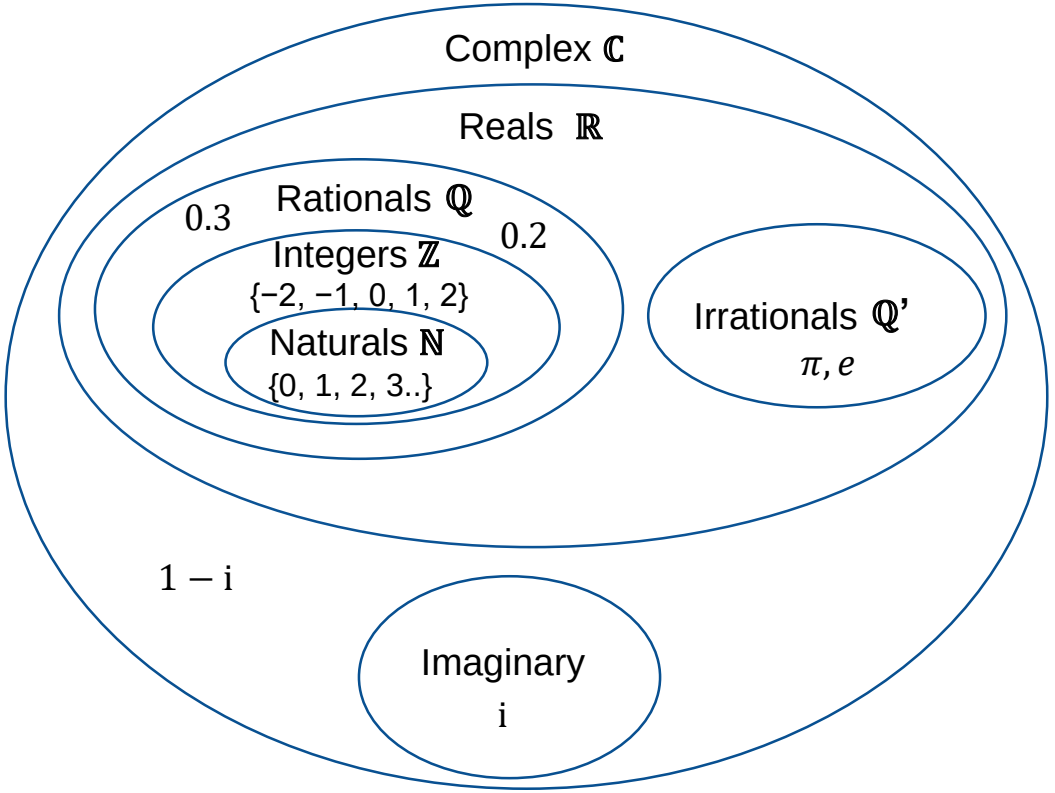
Python programming Introduction Content

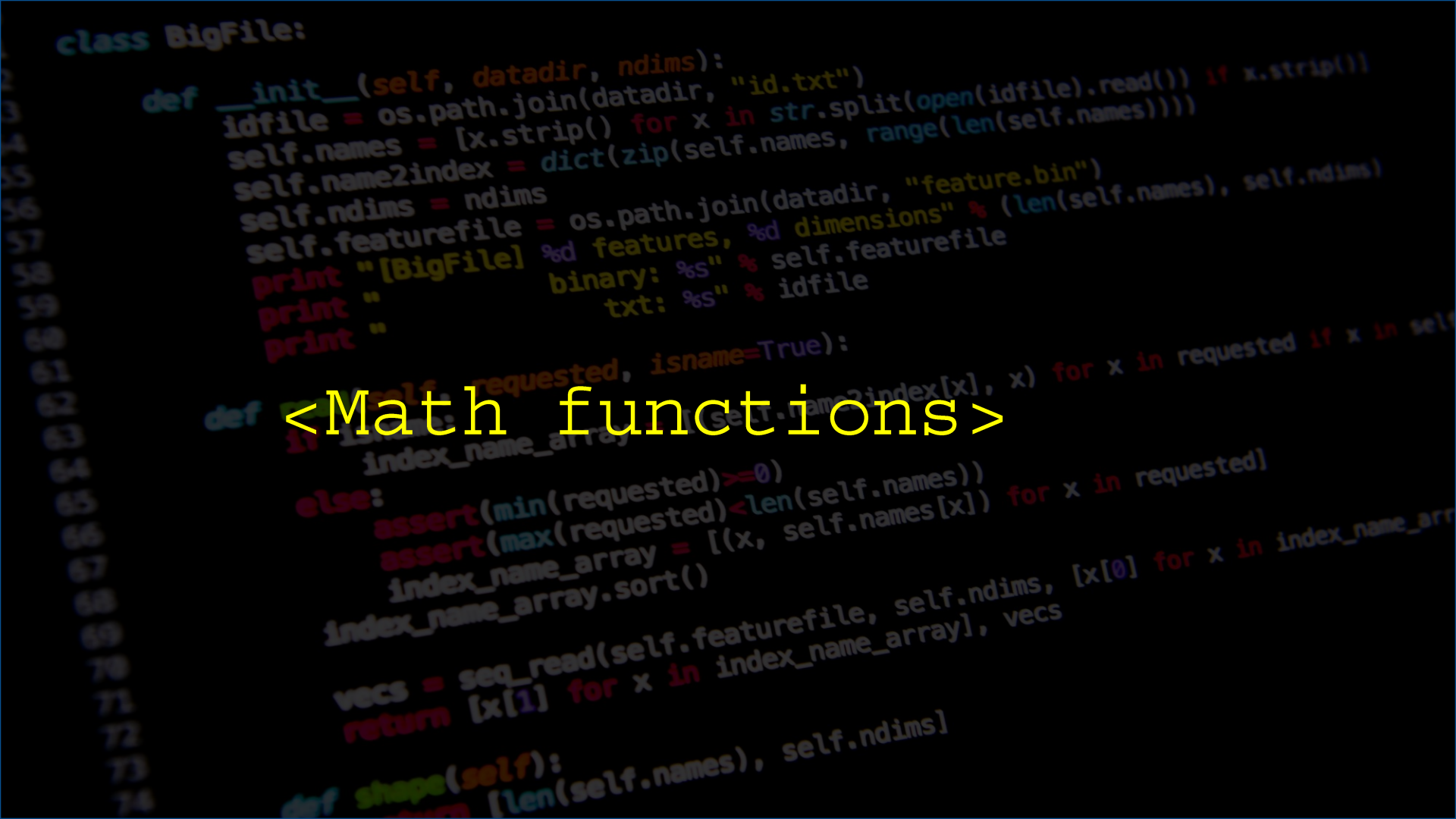
18. Introduction to matplotlib for data visualization
19. Data Preprocessing

Math Revision Content

1. **Functions**
2. **Exponents, logarithms and radicals**
3. **Trigonometry**
4. **Sequences**
5. **Statistics**
6. **Random numbers distributions**
7. **Matrices operations**

Numbers


Complex ℂ
Reals ℝ
Rationals ℚ
0.3
0.2
Integers ℤ
{−2, −1, 0, 1, 2}
Naturals ℕ
{0, 1, 2, 3..}
Irrationals ℚ'
π, e
1 − i
Imaginary
i


<Math functions>

Math Functions

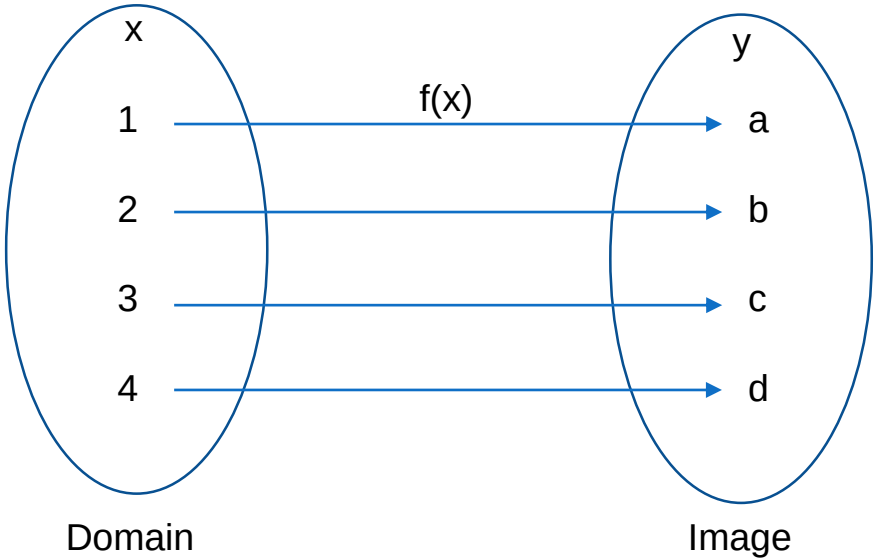
Math functions definition originates from equations that at most one output for any input. Usually, **y** is the variable that we want to find and **x** is the variable that we have in order to evaluate y.

$$y = 2x + 4$$

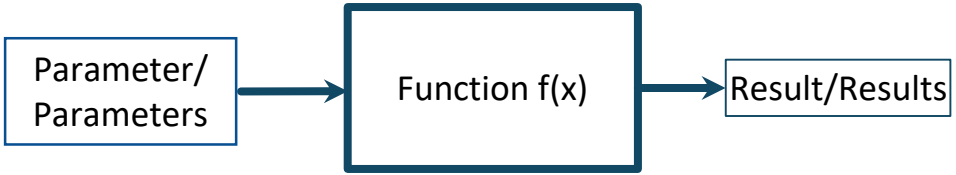
$$y = f(x) = \begin{cases} 1 \text{ , } x = 0 \\ x + 2 \text{ , } x > 0 \\ x - 3 \text{ , } x < 0 \end{cases}$$

f(4) = 6
f(-3) = -6
f(0) = 1

Math Functions

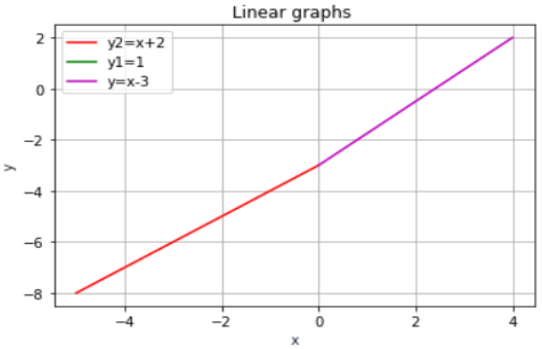


Math Functions



$$y = f(x) = \begin{cases} 1, & x = 0 \\ x + 2, & x > 0 \\ x - 3, & x < 0 \end{cases} \qquad \begin{array}{l} f(4) = 6 \\ f(-3) = -6 \\ f(0) = 1 \end{array}$$

Linear function →




```
import matplotlib.pyplot as plt
import numpy as np
x = np.linspace(-5,0,5)
x1 = 0
x2 = np.linspace(0,4,5)
y = x-3
y1 = 1
y2 = x+2
plt.plot(x, y, '-r',
label='y=x+2')
plt.plot(x1, y1, '-g',
label='y=1')
plt.plot(x2, y2, '-m',
label='y=x-3')
plt.title('Linear graphs ')
plt.xlabel('x')
plt.ylabel('y')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

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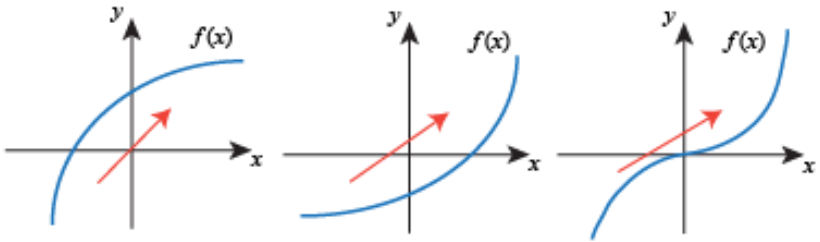
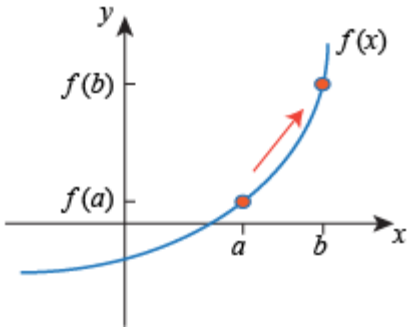
Increasing Functions



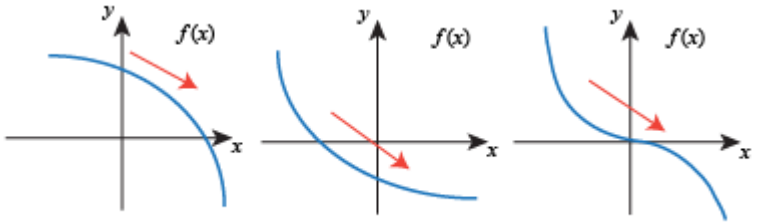
As x moves right, y gets bigger

If a < b; Then f(a) < f(b)



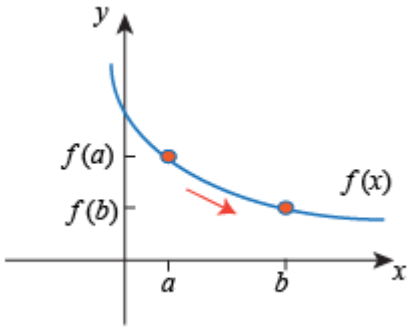
https://www.shmoop.com/points-vectors-functions/increasing-decreasing-functions.html

Decreasing Functions



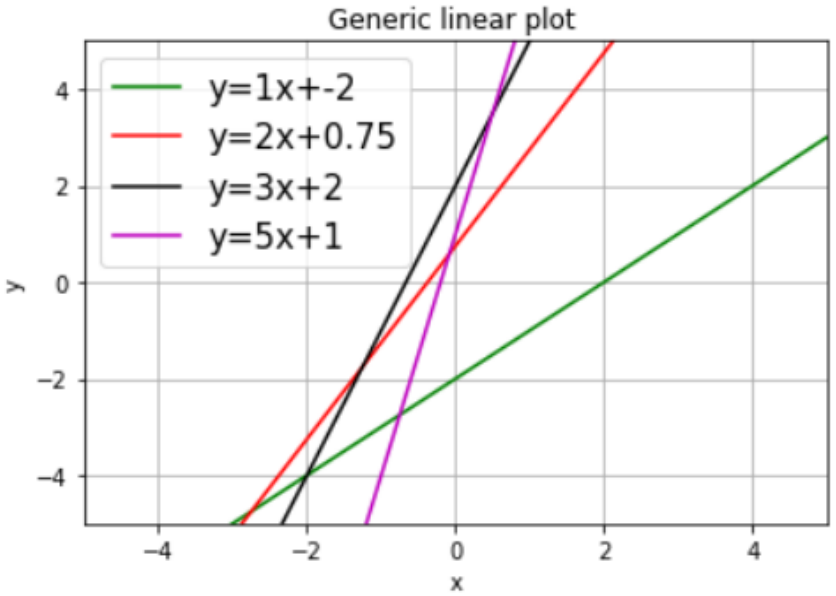
As x moves right, y goes down

If a > b; Then f(a) > f(b)



https://www.shmoop.com/points-vectors-functions/increasing-decreasing-functions.html

- Parallel linear functions m1 = m2
- Perpendicularity m1*m2 = -1




```
import matplotlib.pyplot as plt
import numpy as np

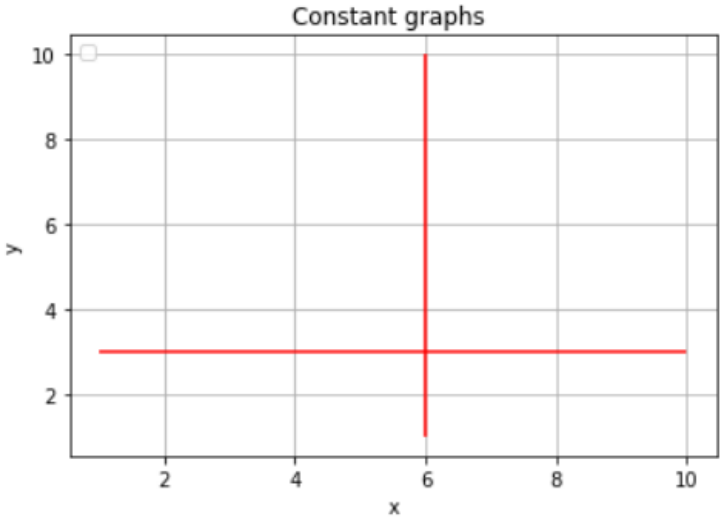
x = [-5,5]
m = [1,2,3,5]
b = [-2,3/4,2,1]
color = ['g', 'r', 'k', 'm']
for i in range(0,len(m)):
    y = m[i]*np.array(x) + b[i]

plt.plot(x,y,label='y=%sx+%s'
%(m[i],b[i]), color=color[i] )

plt.axis('auto')
plt.xlim(x)
plt.ylim(x)
plt.grid()
plt.xlabel('x')
plt.ylabel('y')
axis = plt.gca()
plt.legend(prop={'size':15})
plt.title('Generic plot.')
plt.show()
```

Constant Functions

$$f(x) = 3$$
$$x = 6$$




```
import matplotlib.pyplot as plt

plt.hlines(y = 3, xmin = 1, xmax = 10,
colors='r')
plt.vlines(x = 6, ymin = 1, ymax = 10,
colors='r')
plt.title('Constant graphs ')
plt.xlabel('x')
plt.ylabel('y')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

Functions Properties

Addition	$(f+g)(x) = f(x) + g(x)$
Subtraction	$(f-g)(x) = f(x) - g(x)$
Multiplication	$(f \cdot g)(x) = f(x)g(x)$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Exercises

1. $f(n) = n - 5$
   $g(n) = 4n + 2$
   Find $(f + g)(-8)$

2. $g(a) = 3a - 2$
   $h(a) = 4a - 2$
   Find $(g + h)(-10)$

3. $g(x) = x^2 - 2$
   $h(x) = 2x + 5$
   Find $g(-6) + h(-6)$

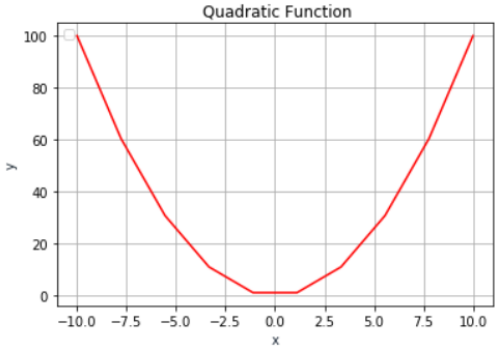
4. $h(t) = t + 5$
   $g(t) = 3t - 5$
   Find $(h \cdot g)(5)$

5. $h(n) = 2n - 1$
   $g(n) = 3n - 5$
   Find $h(0) \div g(0)$

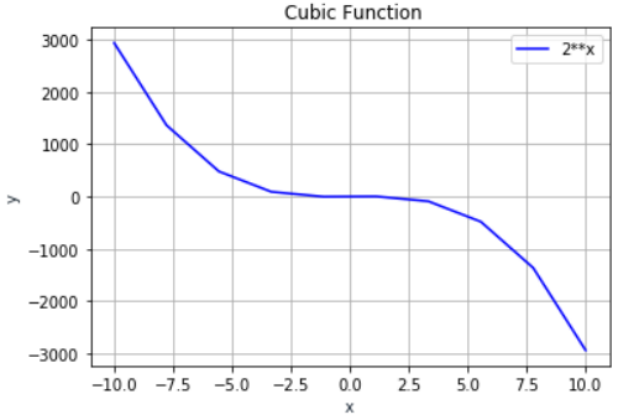
6. $g(t) = t - 3$
   $h(t) = -3t^3 + 6t$
   Find $g(1) + h(1)$

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

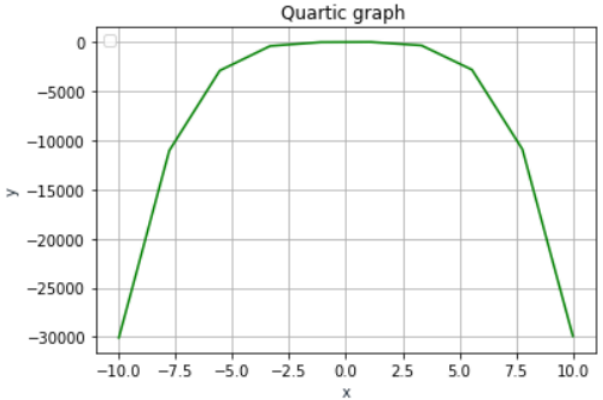
Continuous polynomial functions types




```
x = np.linspace(-10,10,10)
y1 = x**2
plt.plot(x, y1, '-r’)
plt.show()
```




```
x = np.linspace(-10,10,10)
y1 = -3*x**3+6*x
plt.plot(x, y1, '-b’)
plt.show()
```




```
x = np.linspace(-10,10,10)
y1 = -3*x**4+6*x+2*x
plt.plot(x, y1, '-b’)
plt.show()
```

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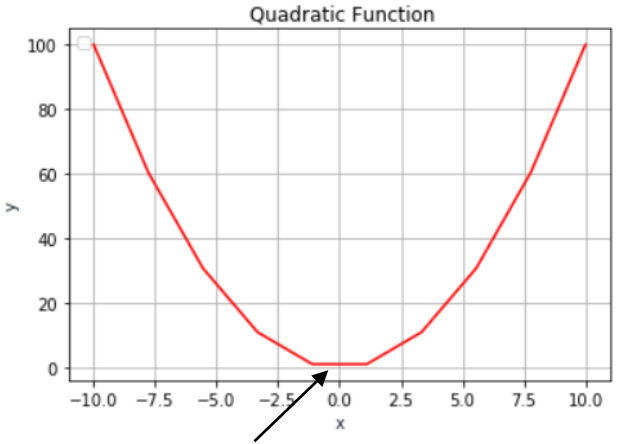
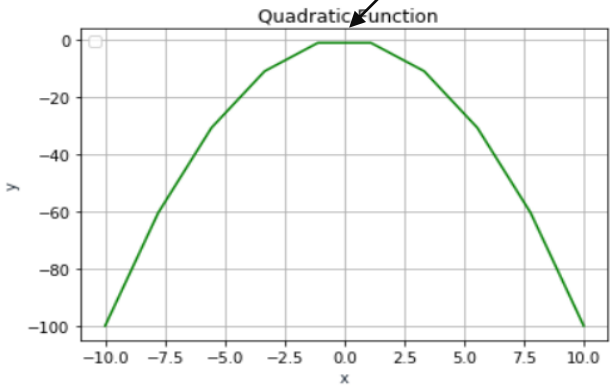
Maximum and minimum values



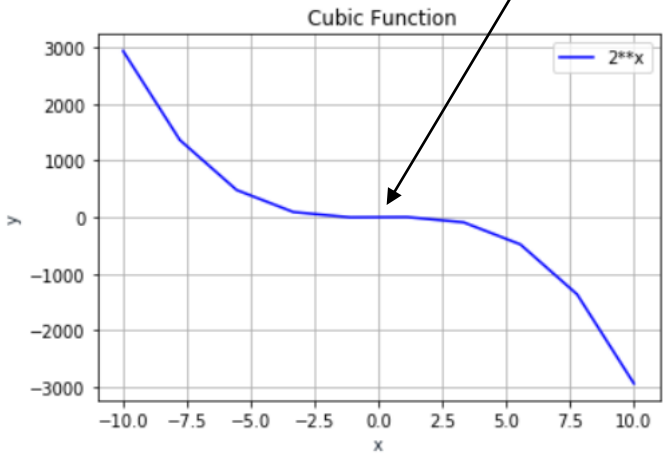

```
x = np.linspace(-10,10,1))
y1 = x**2
plt.plot(x, y1, '-r’)
plt.show()
```

Minimum value (0,0)

Maximum value (0,0)



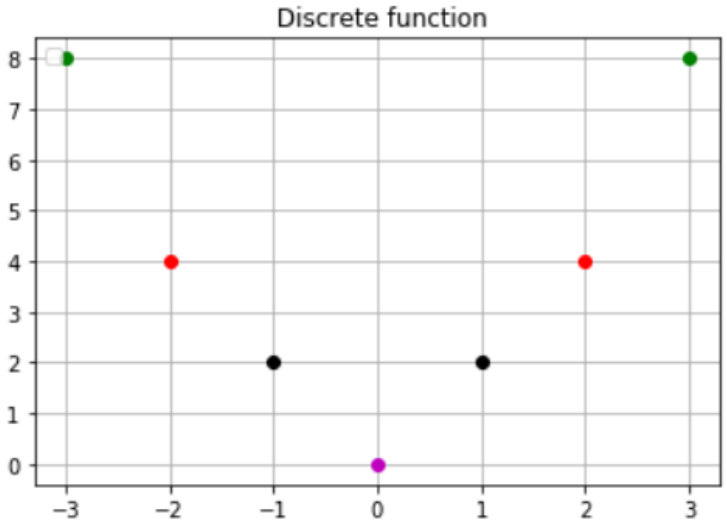
Inflection value (0,0)




```
x = np.linspace(-10,10,10)
y1 = -3*x**3+6*x
plt.plot(x, y1, '-b')
```

```
x = np.linspace(-10,10,10)
y1 = -x**2
plt.plot(x, y1, '-g’)
```

Discrete functions

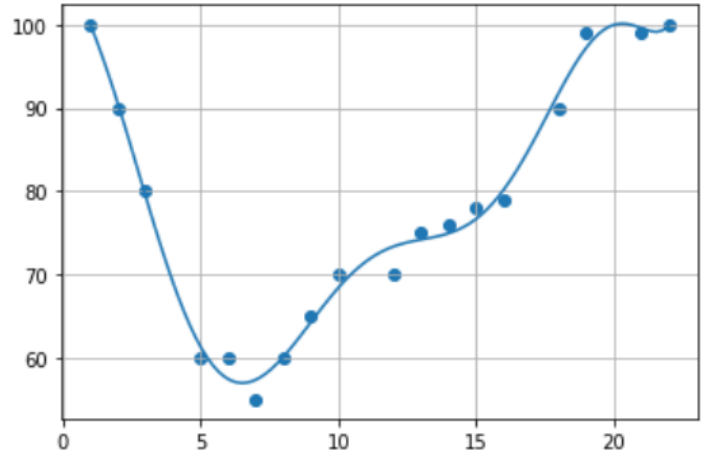



```
import matplotlib.pyplot as plt

x = [-3,-2,-1,0,1,2,3]
y = [8,4,2,0,2,4,8]
color = ['g', 'r', 'k', 'm', 'k', 'r',
'g']
for i in range(0,len(x)):
    plt.plot(x[i],y[i], 'o',
color=color[i] )


plt.title(' Discrete function')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

Polynomial regression




```
          8             7             6            5            4
9.893e-07 x - 8.394e-05 x + 0.002805 x - 0.04621 x + 0.3768 x
          3            2
 - 1.248 x + 0.6947 x - 7.237 x + 107.4
```

```
import numpy
import matplotlib.pyplot as plt

x =
[1,2,3,5,6,7,8,9,10,12,13,14,15,16,18,19,21,
22]
y =
[100,90,80,60,60,55,60,65,70,70,75,76,78,79,
90,99,99,100]

mymodel = numpy.poly1d(numpy.polyfit(x, y,
8))

myline = numpy.linspace(1, 22, 100)

plt.scatter(x, y)
plt.plot(myline, mymodel(myline))
plt.grid()
plt.show()
```

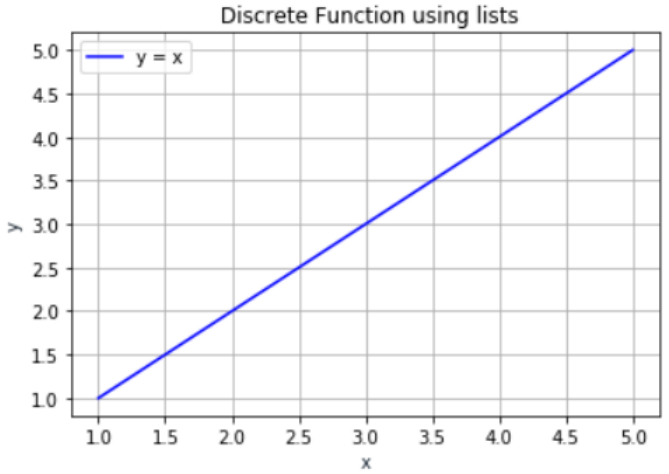
Exercises – Linear regression

The following example describes the expenditure (in dollars) on recreation per month by employees at a certain company, and their corresponding monthly incomes.

Expenditure ($)	Income ($)
2400	41200
2650	50100
2350	52000
4950	66000
3100	44500
2500	37700
5106	73500
3100	37500
2900	56700
1750	35600

What is the Slope, and the y-intercept ?
Write the linear regression equation that represents better the data in the table

Plotting discrete functions using **Lists**




```
import matplotlib.pyplot as plt
```

x = [1, 2 ,3 ,4 ,5]
y1 =[1 ,2, 3, 4, 5]

```
plt.plot(x, y1, '-b', label= 'y
= x')
plt.title(' Discrete function
using lists')
plt.xlabel('x')
plt.ylabel('y')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

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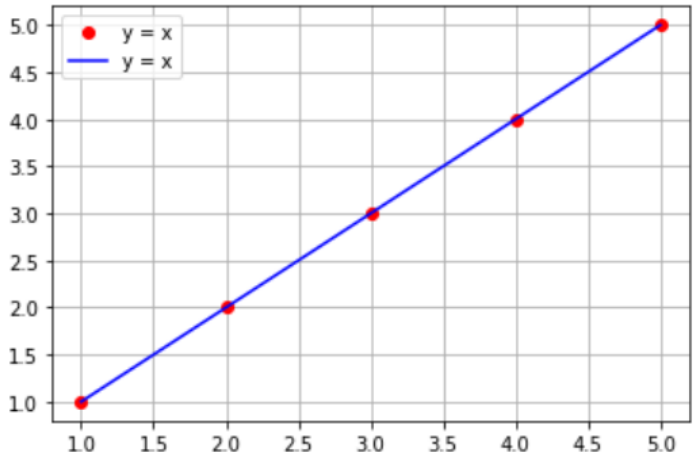
Plotting discrete functions using **Tuples**

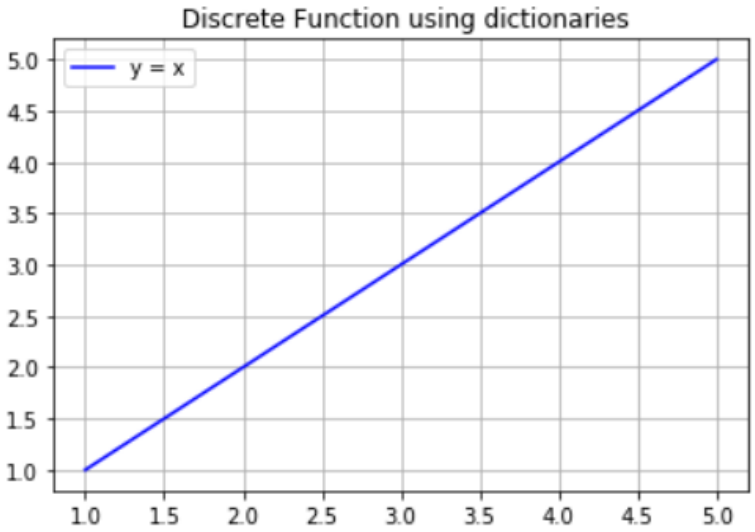



```
import matplotlib.pyplot as plt
```

x =(1,2,3,4,5)
y1 = (1,2,3,4,5)

```
plt.plot(x,y1,'or' ,label= 'y = x')
plt.plot(x,y1,'b', label= 'y = x’)
plt.title(' Discrete function using tuples')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

Plotting functions using **Dictionaries**




```
import matplotlib.pyplot as plt
```

```
x1 = {'x':[1,2,3,4,5], 'y1':[1,2,3,4,5] }
```

```
plt.plot('x', 'y1', '-b', label= 'y = x',
data=x1)
plt.title(' Discrete function using
dictionaries')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

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<Exponentials>

Exponentials logarithms and radicals

Exponentials

Exponent – identifies the multiplication factor.
Base – the number to be multiplied.

Exponent

$$5^2 = ? \rightarrow 5*5 = 25$$

Base

```
In [16]: print(5**2)
25
```

Logarithms

$$5^? = 5*5 = 25$$

```
In [14]: print(math.log(25,5))
2.0
```

$$\log_5 25 = 2$$ ← Exponent

Radicals

$$?^2 = 25$$

```
In [15]: print(np.sqrt(25))
5.0
```

$$\sqrt[2]{25} = 5$$ ← Base

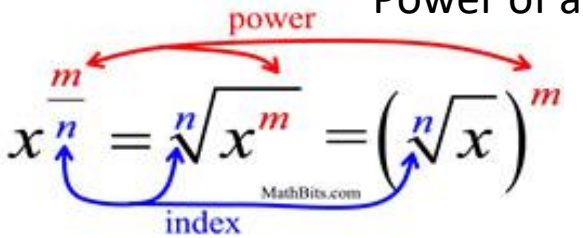
Exponentials - properties

Zero power $(x)^0 = 1$

Negative Exponents $x^{-a} = \frac{1}{x^a}$ *and* $\frac{1}{x^{-a}} = x^a$

Power of a power $(x^a)^b = x^{a*b}$

Rational Exponents



Quotient of powers $\frac{x^a}{x^b} = x^{a-b}$

Product of powers $x^a * x^b = x^{a+b}$

Power of a quotient $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Power of a product $(xy)^a = x^a . y^a$

Exponentials - examples

$3^1 = 3$

$3^0 = 1$

$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

$2x^{-2} = 2\left(\frac{1}{x^2}\right) = \frac{2}{x^2}$

$(2x)^{-2} = \frac{1}{(2x)^2} = \frac{1}{2^2x^2} = \frac{1}{4x^2}$

$16^{\frac{5}{2}} = \sqrt[2]{16^5} = 1024$

Be careful with the '-' signal

$(-3)^3 = (-3) * (-3) * (-3) = -27$

$-3^4 = -(3 * 3 * 3 * 3) = -81$

$(-3)^4 = (-3) * (-3) * (-3) * (-3) = 81$

$-3^3 = -(3 * 3 * 3) = -27$

Exponentials - exercises

1. $\frac{3^2}{3}$
2. $\frac{3nm^2}{3n}$
3. $\frac{4x^3y^4}{3xy^3}$
4. $(x^3y^4 \cdot 2x^2y^3)^2$
5. $2x(x^4y^4)^4$
6. $\frac{3^4}{3}$
7. $\frac{x^2y^4}{4xy}$
8. $\frac{xy^3}{4xy}$
9. $(u^2v^2 \cdot 2u^4)^3$
10. $\frac{3vu^5 \cdot 2v^3}{uv^2 \cdot 2u^3v}$

Exponentials - exercises

1. $\frac{3^2}{3}$ # 3

2. $\frac{3nm^2}{3n}$ # m^2

3. $\frac{4x^3y^4}{3xy^3}$ # $\frac{4x^2y}{3}$

4. $(x^3y^4 \cdot 2x^2y^3)^2$ # $4x^{10}y^{14}$

5. $2x(x^4y^4)^4$ # $2x^{17}2y^{16}$

6. $\frac{3^4}{3}$ # 27

7. $\frac{x^2y^4}{4xy}$ # $\frac{xy^3}{4}$

8. $\frac{xy^3}{4xy}$ # $\frac{y^2}{4}$

9. $(u^2v^2 \cdot 2u^4)^3$ # $8u^{18}v^6$

10. $\frac{3vu^5 \cdot 2v^3}{uv^2 \cdot 2u^3v}$ # 3uv


<Logarithms>

Exponentials logarithms and radicals

Exponentials

Exponent – identifies the multiplication factor.
Base – the number to be multiplied.

Exponent

$$5^2 = ? \rightarrow 5*5 = 25$$

Base

```
In [16]: print(5**2)
25
```

Logarithms

$$5^? = 5*5 = 25$$

```
In [14]: print(math.log(25,5))
2.0
```

$$\log_5 25 = 2$$ ← Exponent

Radicals

$$?^2 = 25$$

```
In [15]: print(np.sqrt(25))
5.0
```

$$\sqrt[2]{25} = 5$$ ← Base

Logarithms properties

$$\log_b 1 = 0$$
$$\log_b b = 1$$

```
In [4]: print(np.log(np.e))
1.0

In [5]: print(np.log10(10))
1.0

In [6]: print(np.log2(2))
1.0
```

Power: $\log_b a^n = n * \log_b a$

Product: $\log_b (a * c) = \log_b a + \log_b c$

Quotient: $\log_b \frac{a}{c} = \log_b a - \log_b c$

Inverse 1: $\log_b b^n = n$

Inverse 2: $b^{\log_b n} = n,\ n > 0$

One-to-One: $\log_b a = \log_b c$ if and only if $a = c$

- Definition of Common Logarithm:

Logarithms with a base of 10 are called common logarithms. It is customary to write $\log_{10} x \rightarrow \log x$

- Definition of Natural Logarithm:

Logarithms with the base of e ≈ 2.72 are called natural logarithms. It is customary to write $\log_e x = \ln x$

Logarithms – examples

Write each equation in its exponential form:

1. $2 = \log_7 x$
2. $3 = \log_{10}(x + 8)$
3. $\log_5 125 = x$

Rewrite into logarithms

1. $2^4 = 16$
2. $\sqrt{64} = 8$
3. $e^4 = 54.60$

Logarithms – examples

Write each equation in its exponential form

1. $2 = \log_7 x$ $\quad \#7^2 = x$
2. $3 = \log_{10}(x + 8)$ $\quad \#10^3 = x + 8$
3. $\log_5 125 = x$ $\quad \#\ 5^x = 125$

Rewrite into logarithms

1. $2^4 = 16$ $\quad \#\log_2 16 = 4$
2. $\sqrt{64} = 8$ $\quad \#\log_{64} 8 = \frac{1}{2}$
3. $e^4 = 54.60$ $\quad \#\log_e 54.60 = 4$

Logarithms – exercises

1. $0.6^{\sqrt{3}}$
2. $e^{3.2}$
3. $(1.005)^{400}$
4. $\log_4 64$
5. $\ln 1$
6. $\ln \sqrt{7}$

Evaluate without a calculator:

7. $\log_5 25$
8. $\log_3 \frac{1}{81}$
9. $\ln e^{-2}$

Use the change of base formula to evaluate the logarithms:

10. $\log_7 3$
11. $\log_2 \frac{1}{2}$
12. $\log_{15} 42$

Use the properties of logarithms to rewrite each expression into lowest terms:

13. $\log 10x$
14. $\ln \frac{xy}{z}$
15. $\log_4 4x^2$
16. $\log_3 \sqrt{x-2}$
17. $\ln \frac{\sqrt{3x}}{7}$

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Logarithms – exercises

1. $0.6^{\sqrt{3}}$ # 0.413
2. $e^{3.2}$ # 24.533
3. $(1.005)^{400}$ # 7.352
4. $\log_4 64$ # 3
5. $\ln 1$ # 0
6. $\ln \sqrt{7}$ # 0.973

Evaluate without a calculator:

7. $\log_5 25$ # 2
8. $\log_3 \frac{1}{81}$ # -4
9. $\ln e^{-2}$ # -2

Use the change of base formula to evaluate the logarithms:

10. $\log_7 3$ # 0.565
11. $\log_2 \frac{1}{2}$ # -1
12. $\log_{15} 42$ # 1.380

Use the properties of logarithms to rewrite each expression into lowest terms:

13. $\log 10x$ # $1 + \log x$
14. $\ln \frac{xy}{z}$ # $\ln x + \ln y - \ln z$
15. $\log_4 4x^2$ # $1 + 2\log_4 x$
16. $\log_3 \sqrt{x-2}$ # $\frac{1}{2}\log_3(x-2)$
17. $\ln \frac{\sqrt{3x}}{7}$ # $\frac{1}{2}\log 3x - \log 7$

Exponentials logarithms and radicals

Exponentials

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Base

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In [16]: print(5**2)
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```
In [14]: print(math.log(25,5))
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```

$$\log_5 25 = 2$$ ← Exponent

Radicals

$$?^2 = 25$$

```
In [15]: print(np.sqrt(25))
5.0
```

$$\sqrt[2]{25} = 5$$ ← Base

Radicals properties

If **n** is a positive integer greater than 1 and both **a** and **b** are positive real numbers then,

$$\sqrt[n]{a^n} = a$$

$$\sqrt[n]{ab} = \sqrt[n]{a}\ \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$$

Radicals – examples

1. $\sqrt{75}$

$\sqrt{25 * 3}$

$\sqrt{25} * \sqrt{3}$

$5\sqrt{3}$

2. $\sqrt{72}$

$\sqrt{9 * 8}$

$\sqrt{9} * \sqrt{8}$

$3\sqrt{8}$

$3\sqrt{4 * 2}$

$3\sqrt{4} * \sqrt{2}$

$3 * 2 * \sqrt{2}$

$6\sqrt{2}$

Radicals – exercises

1. $3\sqrt{12}$
2. $6\sqrt{128}$
3. $7\sqrt{128}$
4. $-8\sqrt{392}$
5. $-7\sqrt{63}$
6. $\sqrt{192n}$
7. $\sqrt{343b}$
8. $\sqrt{196v^2}$
9. $\sqrt{100n^3}$
10. $\sqrt{200a^3}$

Radicals – exercises

1. $3\sqrt{12}$ # $6\sqrt{3}$

2. $6\sqrt{128}$ # $48\sqrt{2}$

3. $7\sqrt{128}$ # $56\sqrt{2}$

4. $-8\sqrt{392}$ # $-112\sqrt{2}$

5. $-7\sqrt{63}$ # $-21\sqrt{7}$

6. $\sqrt{192n}$ # $8\sqrt{3n}$

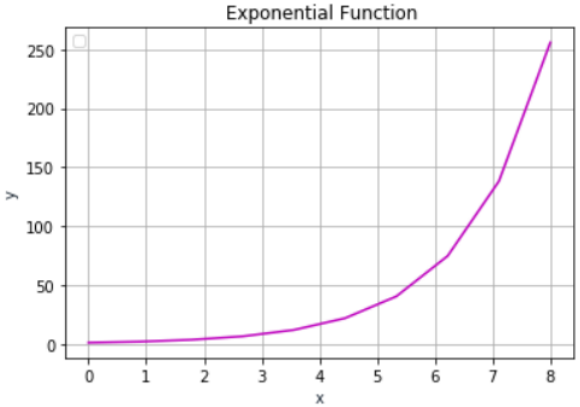
7. $\sqrt{343b}$ # $7\sqrt{7b}$

8. $\sqrt{196v^2}$ # 14v

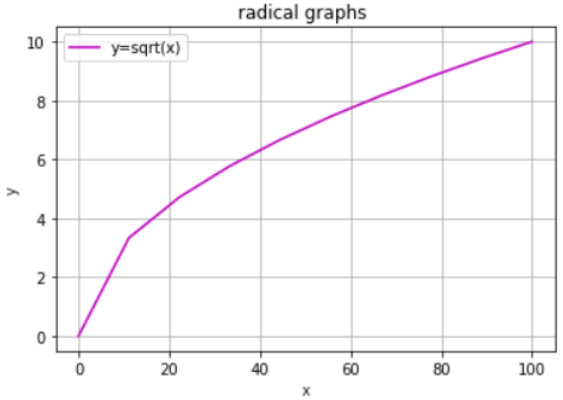
9. $\sqrt{100n^3}$ # $10n\sqrt{n}$

10. $\sqrt{200a^3}$ # $10a\sqrt{2a}$

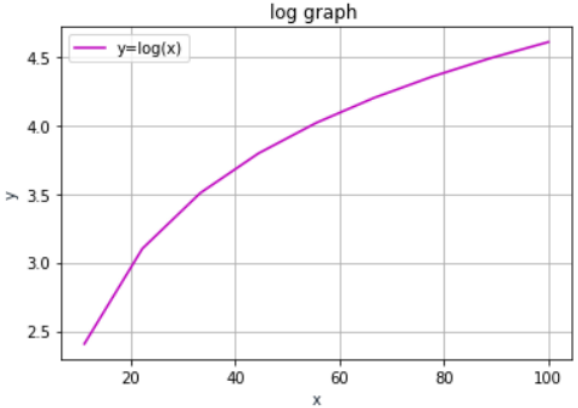
Exponents, logarithms and radicals




```
x = np.linspace(0,8,10)
y1 = 2**x -3
plt.plot(x, y1, ‘-m’)
plt.show()
```

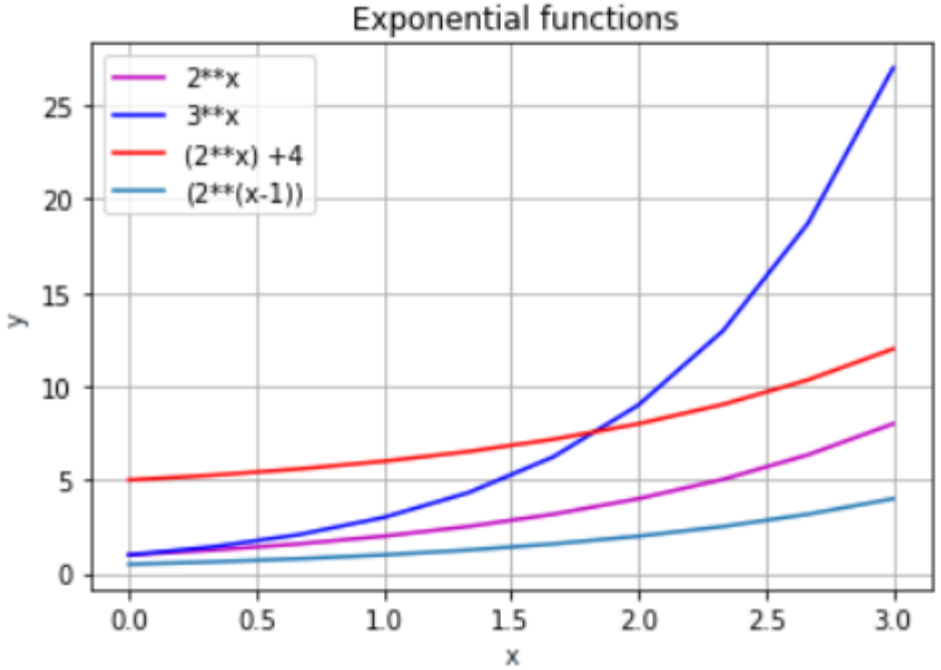



```
x = np.linspace(0,100,10)
y1 = np.sqrt(x)
plt.plot(x, y1,'-m', label='y=sqrt(x)')
plt.show()
```




```
x = np.linspace(0,100,10)
y1 = np.log(x)
plt.plot(x, y1,'-m', label='y=log(x)’)
plt.show()
```

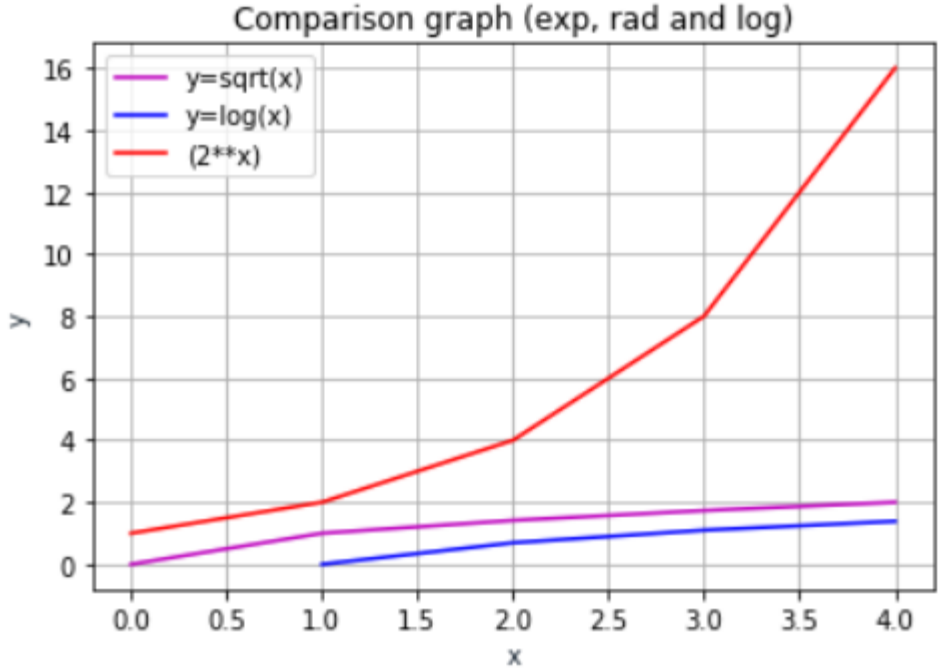
Multiple functions for comparisons




```
import matplotlib.pyplot as plt
import numpy as np
x = np.linspace(0,3,10)
y1 = 2**x
y2 = 3**x
y3 = (2**x) +4
y4 = (2**(x-1))

plt.plot(x,y1, 'm', label= '2**x')
plt.plot(x, y2, '-b', label=
'3**x')
plt.plot(x, y3, '-r', label=
'(2**x) +4')
plt.plot(x, y4, label= '(2**(x-
1))')
plt.title(' Exponential function')
plt.xlabel('x')
plt.ylabel('y')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

Multiple functions comparisons




```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0,4,5)
y1 = np.sqrt(x)
y2 = np.log(x)
y3 = 2**x

plt.plot(x, y1,'-m',
label='y=sqrt(x)')
plt.plot(x, y2, '-b', label=
'y=log(x)')
plt.plot(x, y3, '-r', label=
'(2**x)')

plt.title('Comparison graph (exp,
rad and log)')
plt.xlabel('x')
plt.ylabel('y')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

“

- *Make it work*
- *Make it Right*
- *Make it Fast*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Radicals – exercises

1. $3\sqrt{12}$ # $6\sqrt{3}$

2. $6\sqrt{128}$ # $48\sqrt{2}$

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7. $\sqrt{343b}$ # $7\sqrt{7b}$

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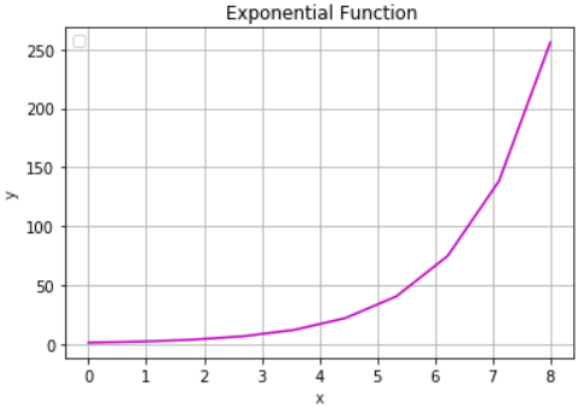
Exponents, logarithms and radicals

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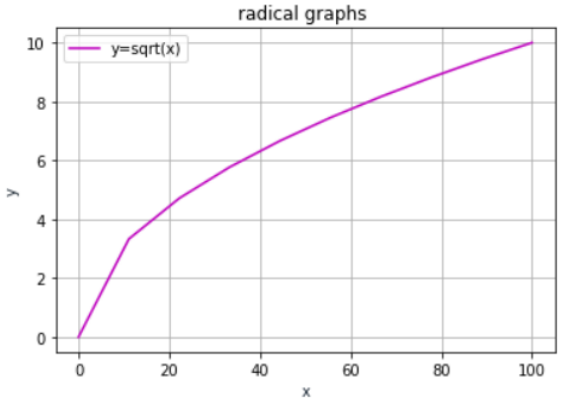
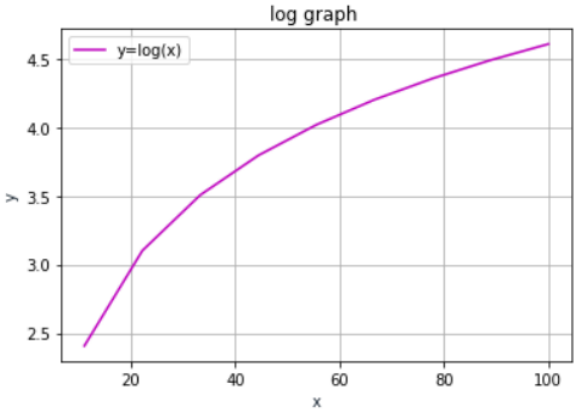



```
x = np.linspace(0,100,10)
y1 = np.sqrt(x)
plt.plot(x, y1,'-m', label='y=sqrt(x)')
plt.show()
```




```
x = np.linspace(0,100,10)
y1 = np.log(x)
plt.plot(x, y1,'-m', label='y=log(x)’)
plt.show()
```

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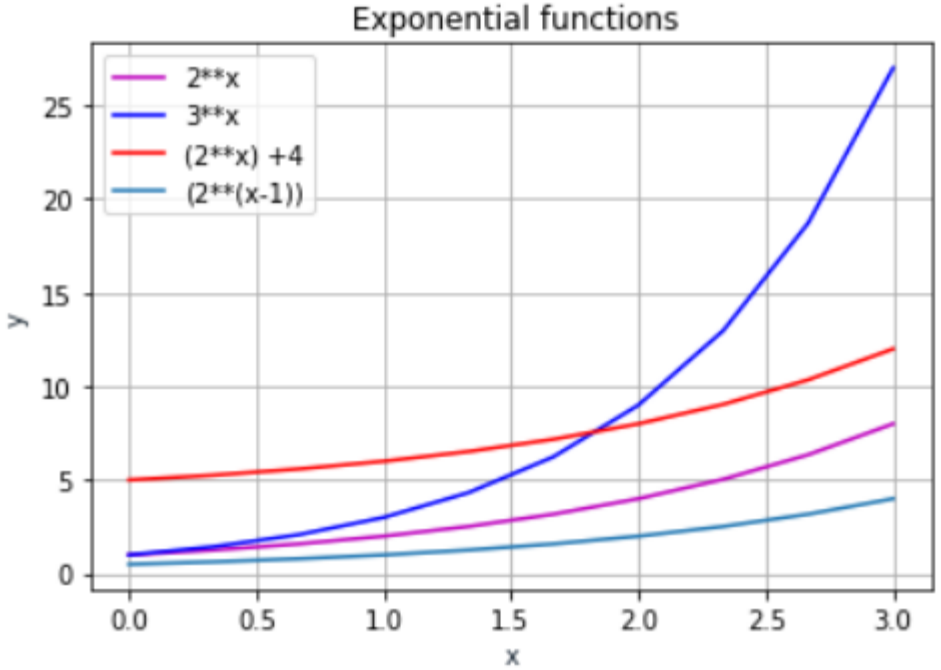
Multiple functions for comparisons

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plt.plot(x, y4, label= '(2**(x-
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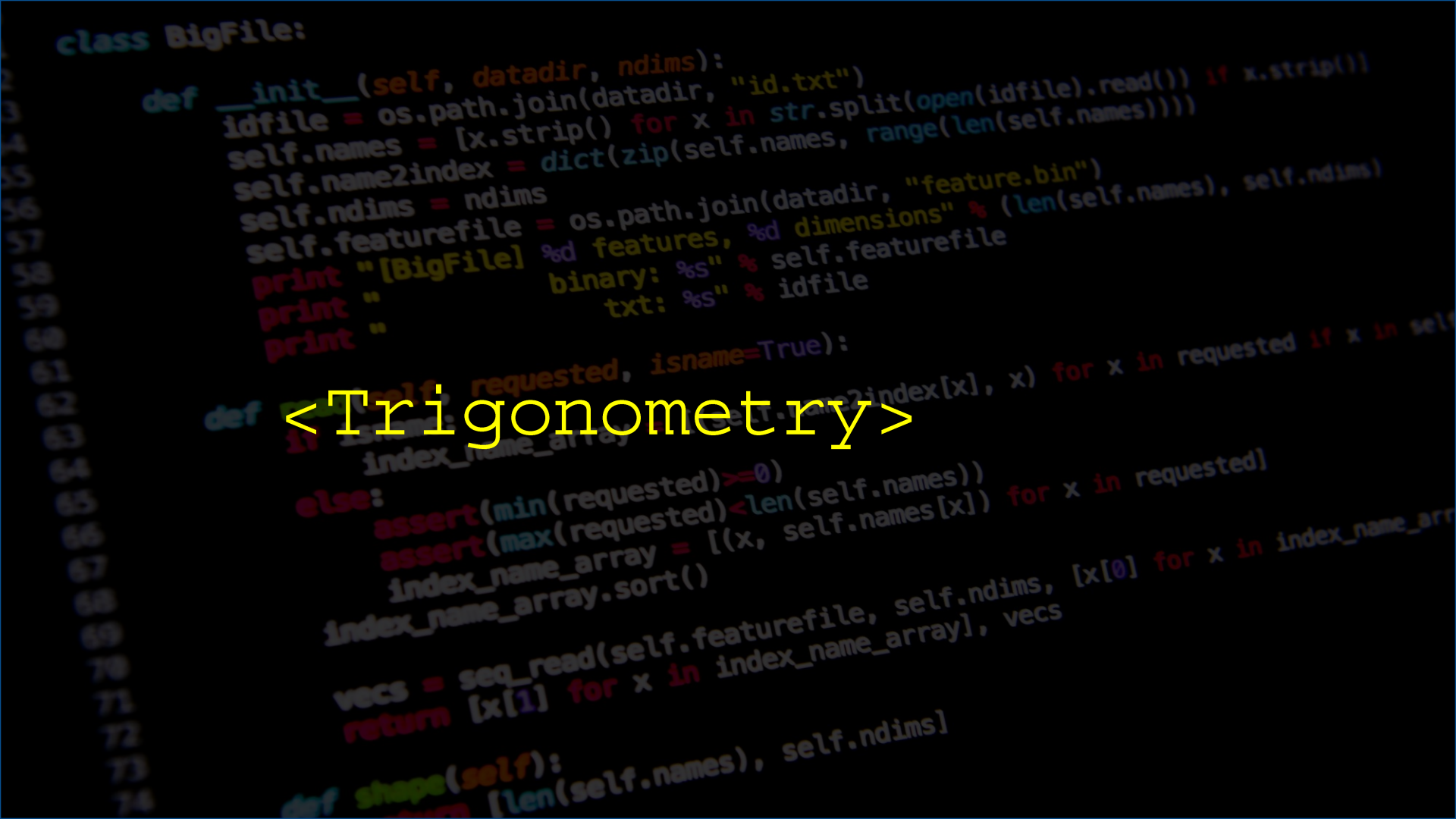
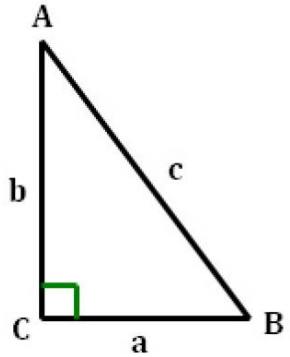
plt.title(' Exponential function')
plt.xlabel('x')
plt.ylabel('y')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```


<Trigonometry>

Trigonometric

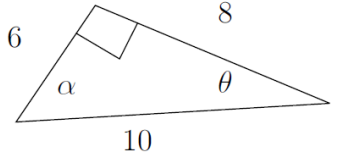


SOH-CAH-TOA

$$\sin = \frac{opposite}{hypoteneuse} \qquad \sin A = \frac{a}{c} \qquad \sin B = \frac{b}{c}$$

$$\cos = \frac{adjacent}{hypoteneuse} \qquad \cos A = \frac{b}{c} \qquad \cos B = \frac{a}{c}$$

$$\tan = \frac{opposite}{adjacent} \qquad \tan A = \frac{a}{b} \qquad \tan B = \frac{b}{a}$$



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$$

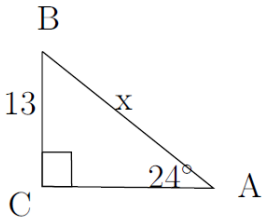
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{6}{8} = \frac{3}{4}$$

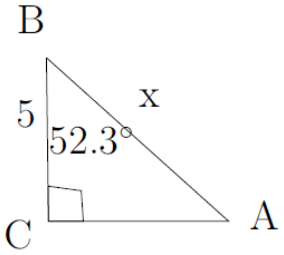
$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$$

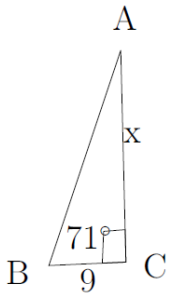
$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$

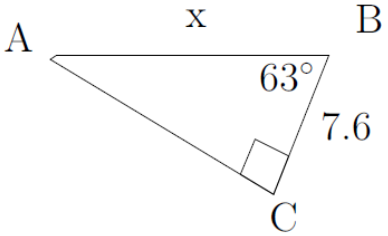
$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{8}{6} = \frac{4}{3}$$

Trigonometry – examples

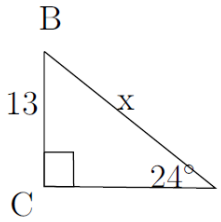






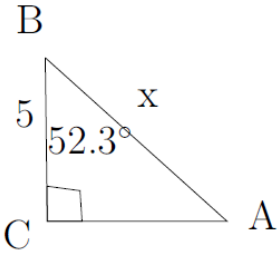


Trigonometry – examples



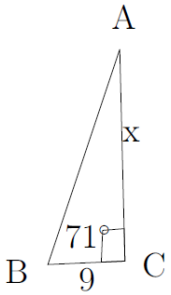
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

#31.96



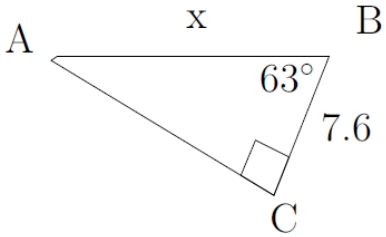
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

#8.17



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

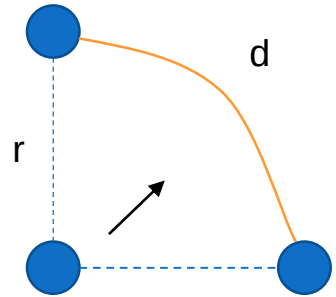
#26.1



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

#16.7

Radians and Degrees



Observer's perspective

Degrees to radians

360°= 2π radians
180°= π radians

Examples

$45° * \frac{\pi}{180}\ \text{rad} = \frac{\pi}{4}$

$150° * \frac{\pi}{180}\ \text{rad} = \frac{5\pi}{6}$

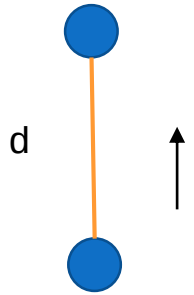
Radians to degrees

360°= 2π radians
180°= π radians

Examples

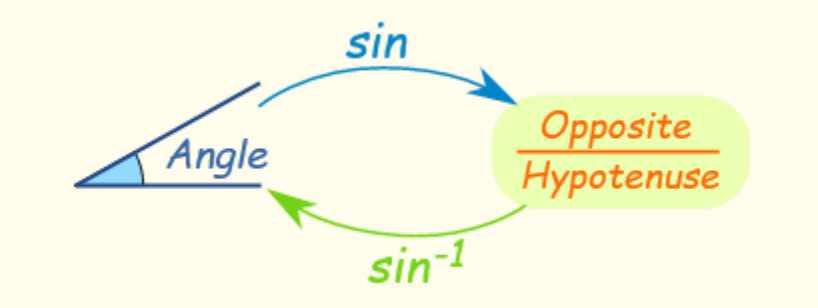
$\frac{\pi}{4}\ \text{rad} * \frac{180}{\pi\ \text{rad}} = 45°$

$60° * \frac{180}{\pi\ \text{rad}} = \frac{\pi}{3}$



Driver's perspective

Trigonometric



$$\sin^{-1}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)=\theta \quad \cos^{-1}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)=\theta \quad \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right)=\theta$$

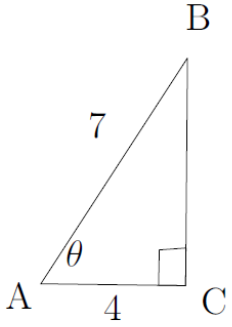
$$\sin A = 0.5$$
$$\sin^{-1}(0.5) = A$$
$$30^\circ = A$$

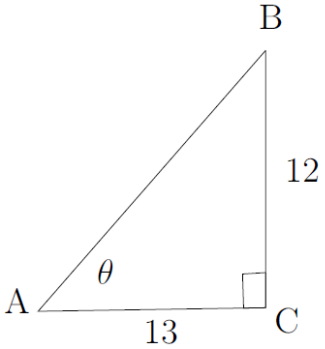
```
x= np.arcsin(0.5)
y = np.rad2deg(x)
```

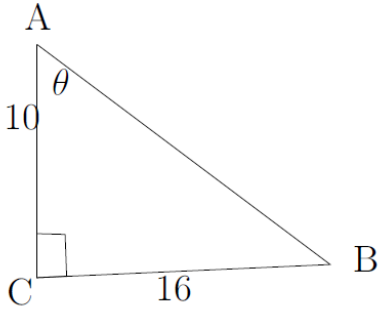
$$\cos B = 0.667$$
$$\cos^{-1}(0.667) = B$$
$$48^\circ = B$$

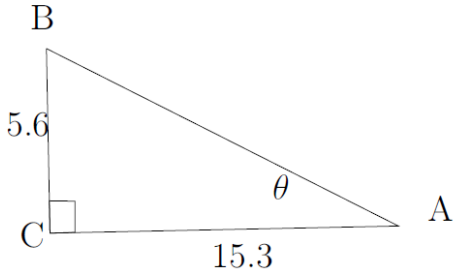
```
x= np.arccos(0.667)
y = np.rad2deg(x)
```

Exercises

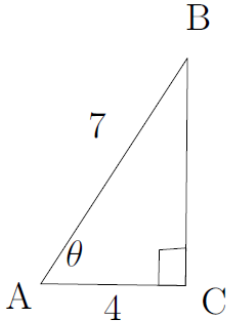






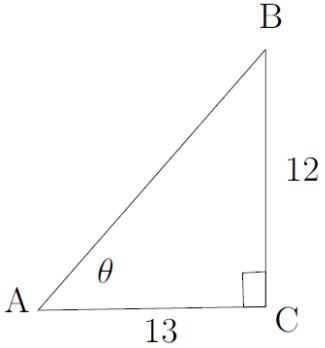


Exercises



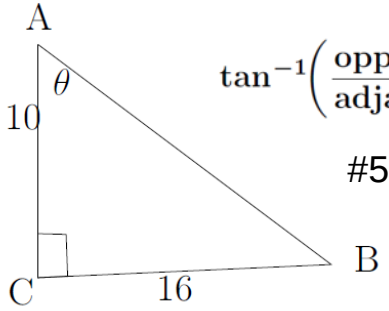
$$\cos^{-1}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)$$

#55.2



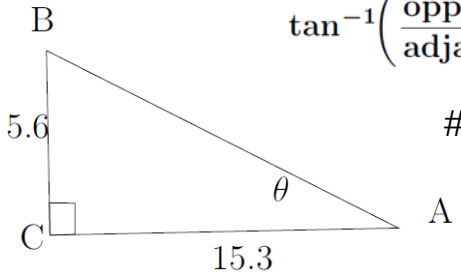
$$\tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

#42.7



$$\tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

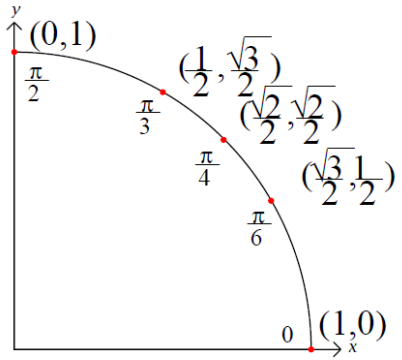
#57.99



$$\tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

#20.1

Most used angles in math located in the first quadrant



θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

```
import numpy as np
numpy.set_printoptions(precision=2)
numpy.set_printoptions(formatter={"fl
oat":'{:0.2f}'.format})
a = np.array([0,30,45,60,90])

print('Sine of different angles:')
# Convert to radians by multiplying
to pi/180
print(np.sin(a*np.pi/180), "\n")

print("Cosine values for angles in
array:")
print(np.cos(a*np.pi/180), "\n")

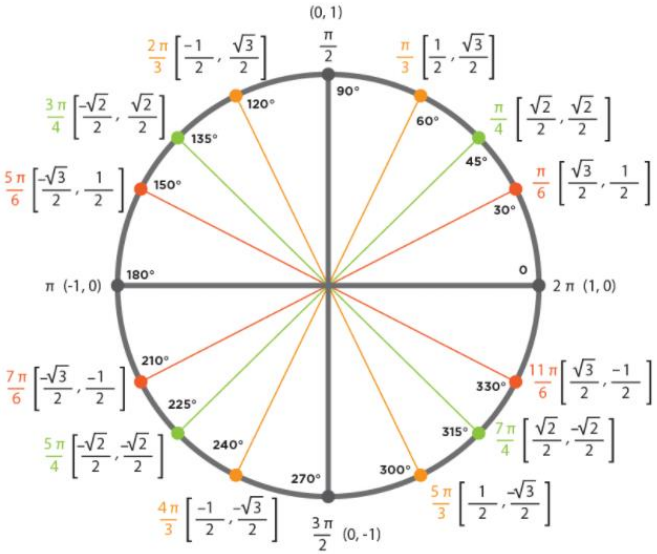
print('Tangent values for given
angles:')
print(np.tan(a*np.pi/180))
```

```
Sine of different angles:
[0.00 0.50 0.71 0.87 1.00]

Cosine values for angles in array:
[1.00 0.87 0.71 0.50 0.00]

Tangent values for given angles:
[0.00 0.58 1.00 1.73 16331239353195370.00]
```

Trigonometric relation in the circle



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$

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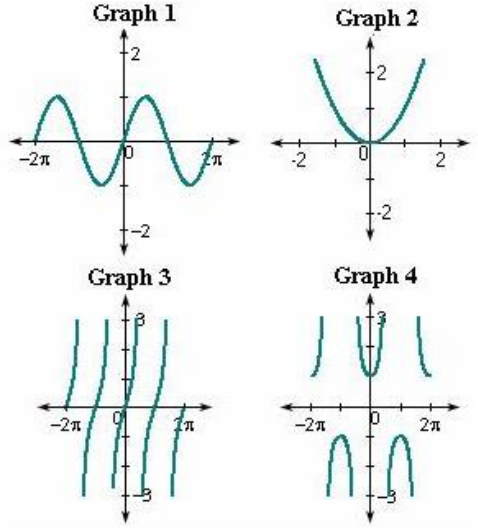
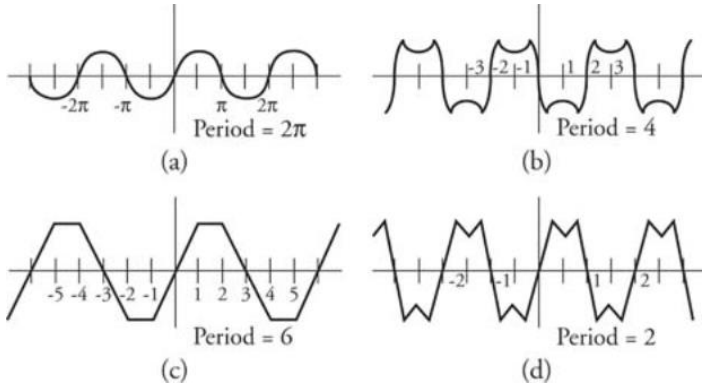
Periodic functions definition

A **periodic function** occurs when a specific horizontal shift, *P*, results in the original function; where $f(x + P) = f(x)$ for all values of x. When this occurs we call the horizontal shift the **period** of the function.





Periodic functions definition

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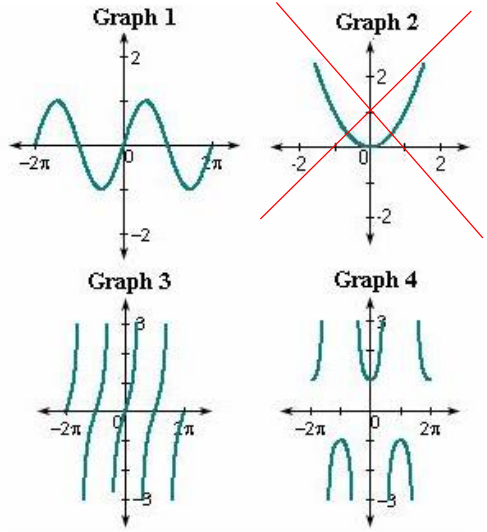
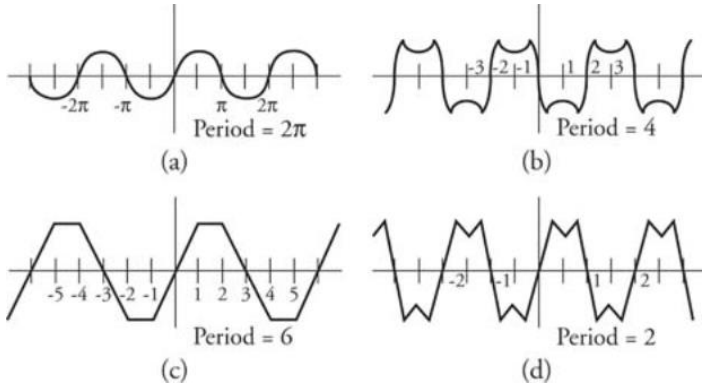
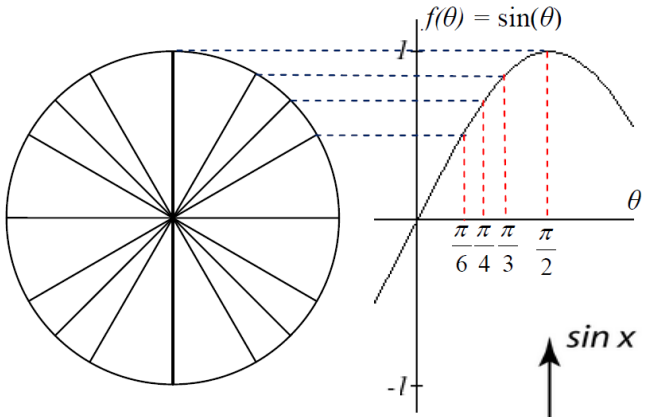
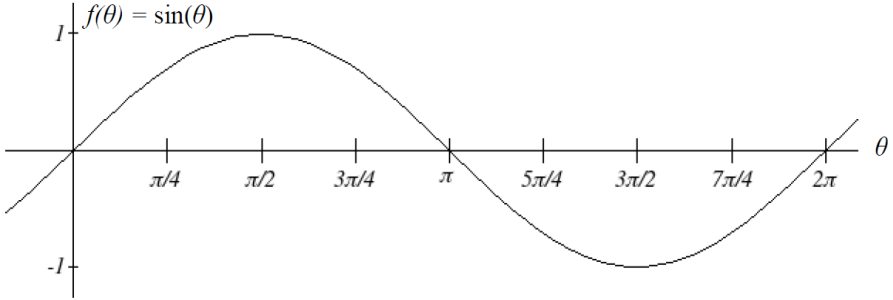


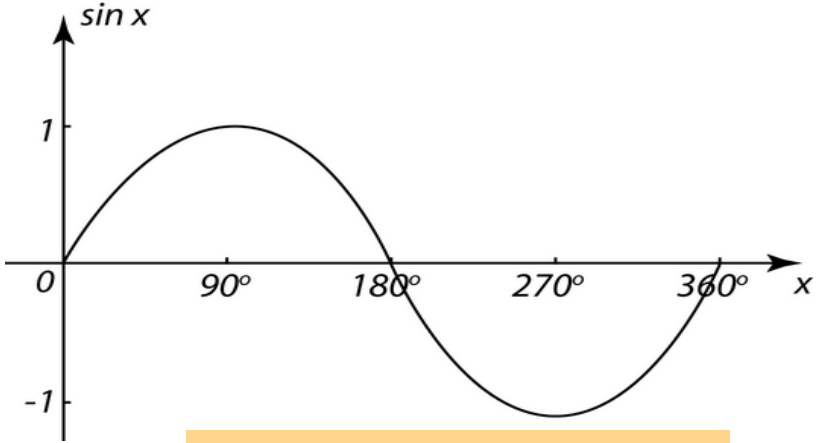


Periodic functions





radians representation

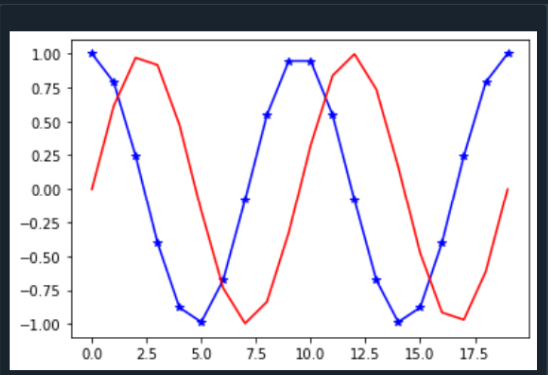


Degrees representation

C:\Users\Josifefe\Desktop\python\pythoncode5\readFile\test.py

temp.py | readJson.py | readfile.py | test.py

```
90  # x1 = np.array([[-1, 3],[4,2]])
91  # x2 = np.array([[1, 2],[3,4]])
92  # x4 = x1+x2
93  # x5 = x1*x2
94  # x3 =  np.dot(x1,x2)
95  # print(x3)
96  # print(x4)
97  # print(x5)
98
99  import matplotlib.pyplot as plt
100
101 values = np.linspace(-(2*np.pi),2*np.pi, 20)
102 cos_values = np.cos(values)
103 sin_values = np.sin(values)
104
105 plt.plot(cos_values, color = 'blue', marker= '*')
106 plt.plot(sin_values, color = 'red')
107 plt.show()
108
109
110
```



Explorador de variáveis | Ajuda | Figuras | Arquivos

Console 1/A

```
Desktop/python/pythoncode5/readFile/
test.py', wdir='C:/Users/Josifefe/
Desktop/python/pythoncode5/readFile')
2020-11-02 2020-11-03 2020-11-04
731 1
[[2. 2. 2. 2. 2. 2.]
 [2. 2. 2. 2. 2. 2.]]

In [156]:
```

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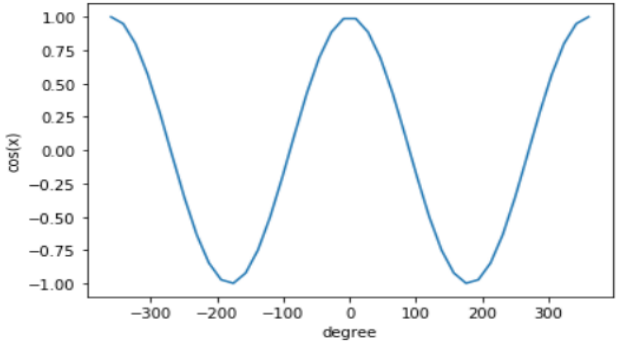
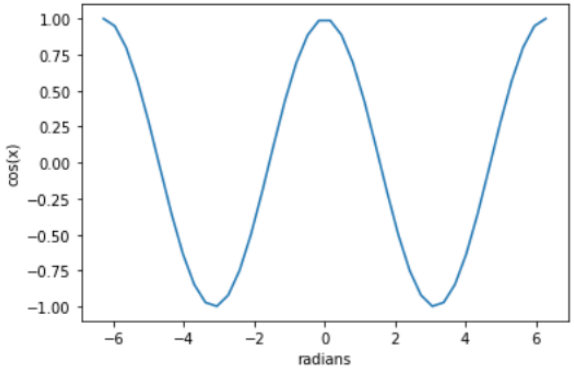
cos (x) in degrees




```
import numpy as np
import matplotlib.pyplot as plt
x = np.linspace(-2*np.pi, 2*np.pi, 40)
x1 = np.rad2deg(x)
y = np.cos(x)
plt.plot(x1,y)
plt.xlabel('degree')
plt.ylabel('cos(x)')
```

cos (x) in radians




```
import numpy as np
import matplotlib.pyplot as plt
x = np.linspace(-2*np.pi, 2*np.pi, 40)
y = np.cos(x)
plt.plot(x,y)
plt.xlabel('radians')
plt.ylabel('cos(x)')
```

Periodic functions

- Amplitude
- Period
- Phase shift

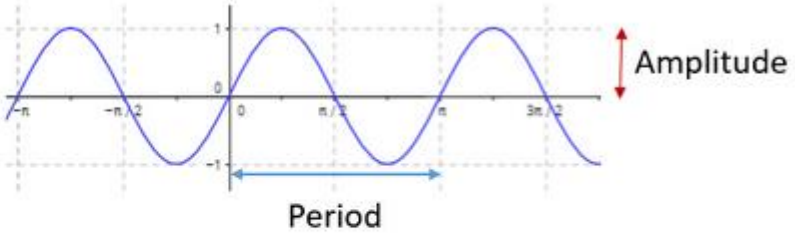
$$y = A\sin\left[B\left(x - C\right)\right] + D$$

$|A|$ is the amplitude

The period is $\frac{2\pi}{B}$

Phase (horizontal) shift is C

Vertical shift is D



The same applies for the Cosine Function.

$$y = 4 + \frac{1}{2}\sin(x)$$

$$y = -4\cos(3x - \pi)$$

Periodic functions – exercises spyder

$y = 4 + \frac{1}{2}\sin(x)$ Radians degrees samples

$y = -4\cos(3x - \pi)$

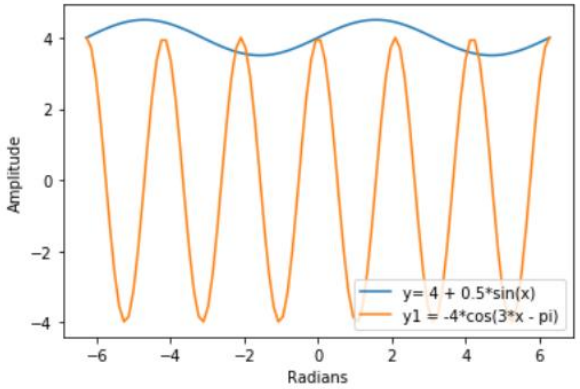
What is the period of these functions?
What is the amplitude?
How many samples (minimum amount) is needed to see the smoothing charts ?

Periodic functions – exercises spyder

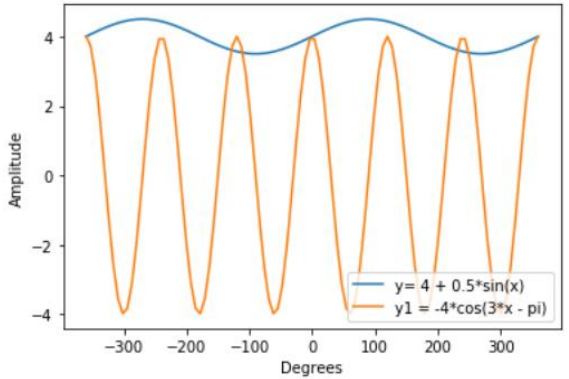
$$y = 4 + \frac{1}{2}\sin(x)$$

$$y = -4\cos(3x - \pi)$$

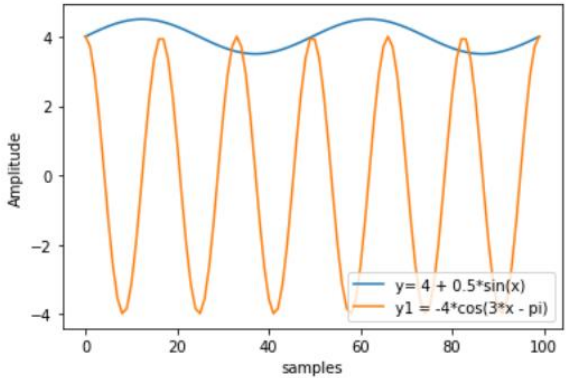
Radians



degrees



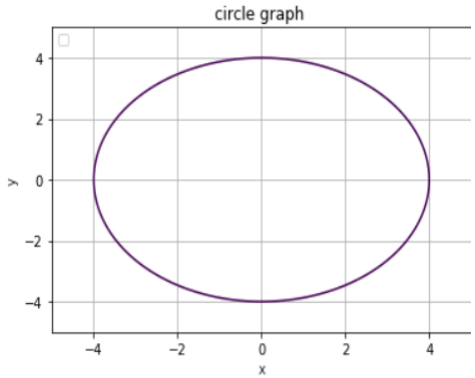
Samples

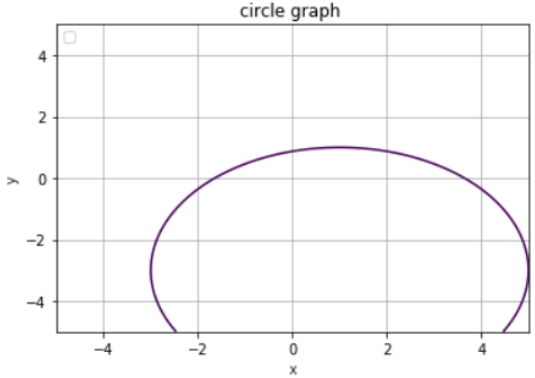


What is not a function ?

Circle equation

X^2 +Y^2 = 16
(X-1)**2+(Y+3)**2 =16






```
import matplotlib.pyplot as plt
import numpy as np
x = np.linspace(-5,5,100)
y = np.linspace(-5,5,100)

X, Y = np.meshgrid(x,y)
f = X**2+Y**2 -16
fig, ax = plt.subplots()
ax.contour(X,Y,f,[0])
plt.title('circle graph ')
plt.xlabel('x', color='#1C2836')
plt.ylabel('y', color='#1C2833')
plt.legend(loc='upper left')
plt.grid()
plt.show()
```

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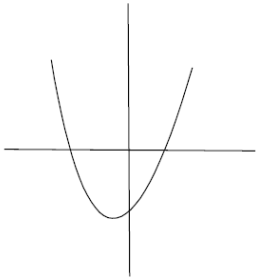
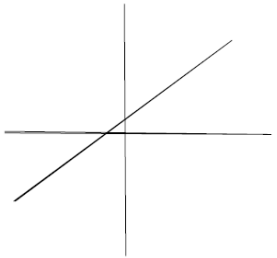
What are functions here ?

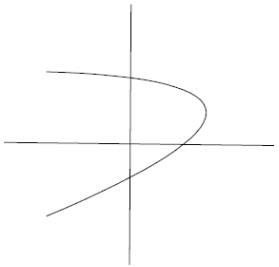
a)



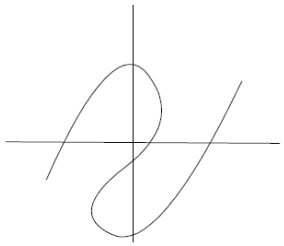
c)



b)

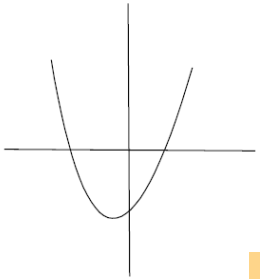


d)



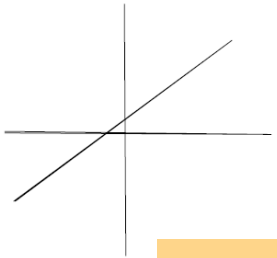
What are functions here ?

a)



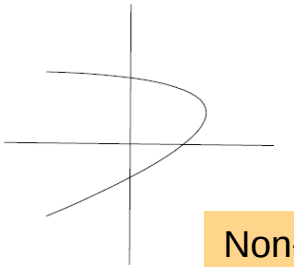
Function

c)



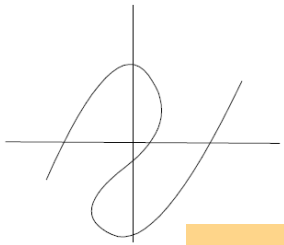
Function

b)



Non-Function

d)



Non-Function

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https://www.smashingmagazine.com/2010/02/applying-mathematics-to-web-design/

“

- *Make it work*
- *Make it Right*
- *Make it Fast*


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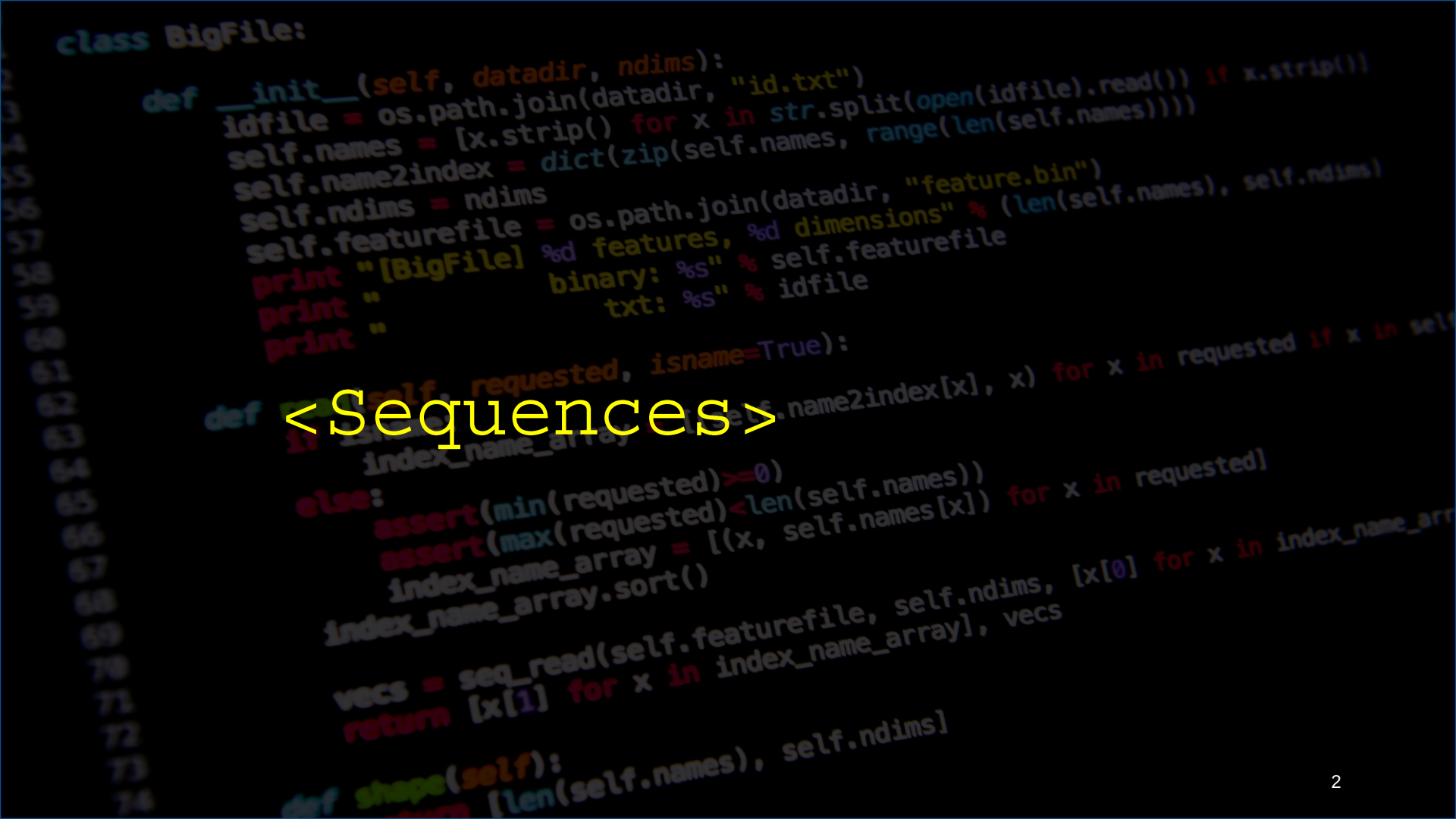

<Sequences>

Sequences represent **ordered lists** of elements.

1, 3, 5, 7, 9, …

-1, 1, -1, 1, -1, …

2, 5, 10, 17, 26, …

0.25, 0.5, 0.75, 1, 1.25 …

3, 9, 27, 81, 243, …

Notation:

A = {3, 5, 7, ...}

Strings

Finite sequences are also called **strings**, denoted by $a_1 a_2 a_3 ... a_n$.

The **length** of a string S is the number of terms that it consists of.

The **empty string** contains no terms at all. It has length zero.

Sequences classification

Finite Sequence has a fixed number of terms.

A = {2, 4, 6, 8}

A sequence that has infinitely many terms is called an **infinite sequence**.

A = {2, 4, 6, 8,…}

Monotonic sequence if its terms are nondecreasing .

$$a_1 \leq a_2 \leq a_3 \ldots\ldots \leq a_n \leq \ldots.$$

Or if its terms are nonincreasing .

$$a_1 \geq a_2 \geq a_3 \ldots \geq a_n \ldots.$$

What do we want to know about sequences:

What is the N element? (without calculate every element)
How many elements the sequence has?
Evaluate the summation of the sequence
Evaluate the product of a sequence

Is the sequence converge to a finite value or not ?

We use the notation a_n to denote the element represented by integer n. We call a_n a term of the sequence.

1, 3, 5, 7, 9, …	$a_n = 2n - 1$
-1, 1, -1, 1, -1, …	$a_n = (-1)^n$
2, 5, 10, 17, 26, …	$a_n = n^2 + 1$
0.25, 0.5, 0.75, 1, 1.25 …	$a_n = 0.25n$
3, 9, 27, 81, 243, …	$a_n = 3^n$

Arithmetic progression

Explicit formula for arithmetic sequence:

$$a_n = a_1 + (n - 1)d$$

First term

Common difference

Recursive formula for an arithmetic sequence:

$$a_1 = \#$$
$$a_n = a_{n-1} + d$$

Example: 5, 2, -1, -4, …

$a_n = a_1 + (n - 1)d$

Explicit Formula Substitute the values:

$a_n = 5 + (n – 1)(- 3)$

Explicit formula is: $a_n = -3n + 8$

The Recursive Formula is:

$a1 = 5$

$a_n = a_{n-1} – 3$

Example: 5, 2, -1, -4, …

$a_n = a_1 + (n - 1)d$

Explicit Formula Substitute the values:

$a_n = 5 + (n – 1)(- 3)$

Explicit formula is: $a_n = -3n + 8$

The Recursive Formula is:

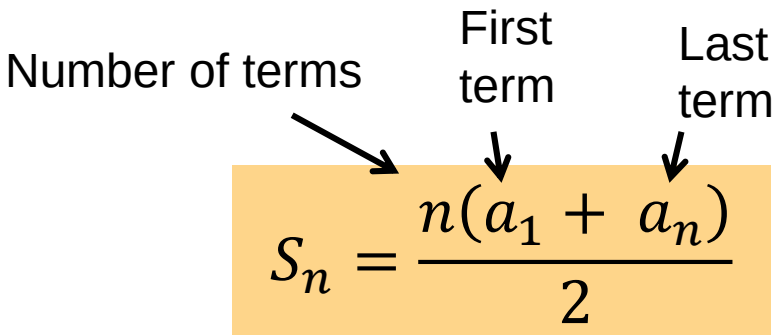
$a1 = 5$

$a_n = a_{n-1} – 3$

Summation

Example: 5, 2, -1, -4, …

The sum of the first n terms of a series…denoted S_n



#2

Geometric progression

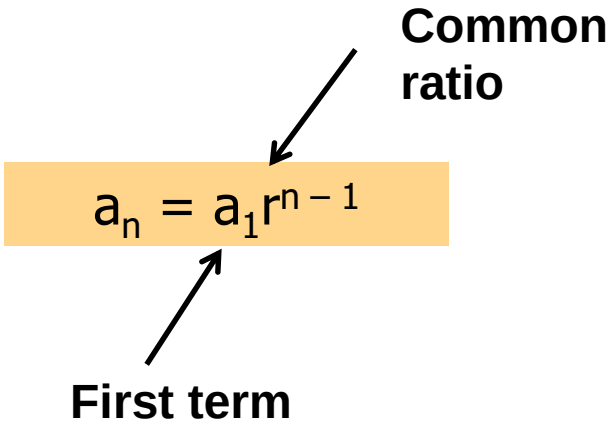
A ***geometric progression, or geometric sequence***, is a sequence of numbers in which each consecutive term is found by multiplying the previous by a non-zero constant.
$G = (a, ax, ax^2, ..., ax^n)$ for some $x \neq 0$

A sequence is geometric if the ratios of consecutive terms are the same.

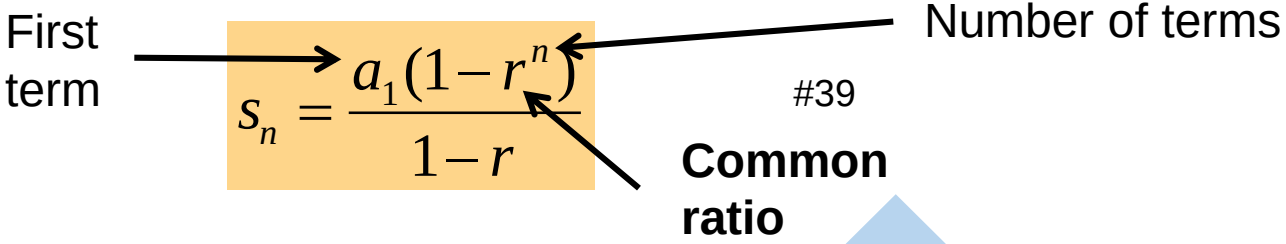
$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = ... = r$$

Geometric progression

Example : 3, 9, 27



The sum of the first n terms of a geometric series is denoted as S_n



#39

Summations

$$\sum_{j=m}^{n} a_j = a_m + am_{+1} + am_{+2} + \ldots + an$$

The variable j is called the index of summation, running from its lower limit m to its upper limit n. We could as well have used any other letter to denote this index. For example:

$$\sum_{i=1}^{5} a_i = 1 + 2 + 3 + 4 + 5 = 15$$

Summations

How can we express the sum of the first 100 terms of the sequence $\{a_n\}$ with $a_n = n$ for $n = 1, 2, 3, \ldots$?

We write it as $\sum_{j=1}^{1000} j$

What is the value of $\sum_{j=1}^{6} j$?

It is $1 + 2 + 3 + 4 + 5 + 6 = 21$

What is the value of $\sum_{j=1}^{20} j$ or $\sum_{j=1}^{100} j$?

That's a lot of work.

Summations

How can we express the sum of the first 100 terms of the sequence $\{a_n\}$ with $a_n = n$ for $n = 1, 2, 3, ...$? We can use the following equation from Friedrich Gauss to get the sum:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \longrightarrow \sum_{j=1}^{100} j = \frac{100 * (100+1)}{2} = 5050$$

In other words, it is the same equation of the summation of arithmetic sequences:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_n = \frac{100*(1+100)}{2} = 5050$$

Find the sum: $\sum_{n=1}^{12} 4(0.3)^n$

Write out a few terms:

$$\sum_{n=1}^{12} 4(0.3)^n = 4(0.3)^1 + 4(0.3)^2 + 4(0.3)^3 + \ldots + 4(0.3)^{12}$$

$$a_1 = 4(0.3) \rightarrow r = 0.3 \text{ and } n = 12$$

$$\sum_{n=1}^{12} 4(0.3)^n = a_1\left(\frac{1-r^n}{1-r}\right) = 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \approx 1.714$$

If the index began at i = 0, you would have to adjust your formula:

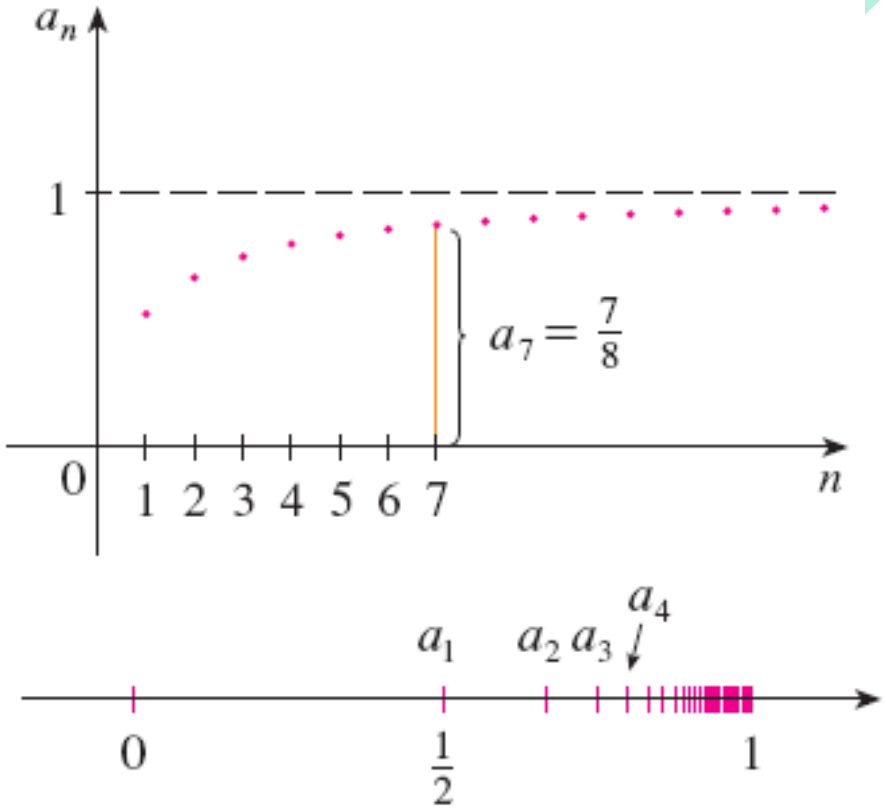
$$\sum_{i=0}^{12} 4(0.3)^n = 4(0.3)^0 + \sum_{n=1}^{12} 4(0.3)^n = 4 + \sum_{n=1}^{12} 4(0.3)^n \approx 4 + 1.714 \approx 5.714$$

Convergence

Let’s take a look in the following sequence:

$$a_n = \frac{n}{n+1}$$

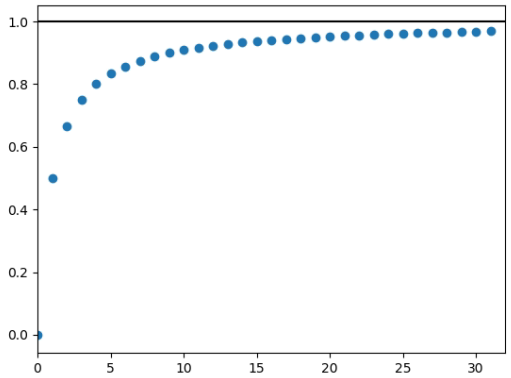
From either figure, it appears that the terms of the sequence are approaching 1 as n becomes large. We say that the sequence is converging to 1.



Convergence

Let's take a look in the following sequence:

$$a_n = \frac{n}{n+1}$$




```
0 0.0
1 0.5
2 0.6666666666666666
3 0.75
4 0.8
5 0.8333333333333334
6 0.8571428571428571
7 0.875
8 0.8888888888888888
9 0.9
n = 10, result is 0.9.
```

```
import matplotlib.pyplot as plt

x = 1.0
n = 50
data = []
for i in range(n):
    x1 = i/(i + 1)

    if abs(x1 - x) < 1.e-2:
        print('n = {i}, result is
{x}.'.format(i=i, x=x))
        last_index = i
        break
    x = x1
    data.append([i,x])
    print(i, x)
else:
    print('No convergence within {n}
iterations.'.format(n=n))
    last_index = n
plt.scatter(*zip(*data))
plt.hlines(y=1,xmin=0,xmax=last_index)
plt.xlim(0, last_index)
```

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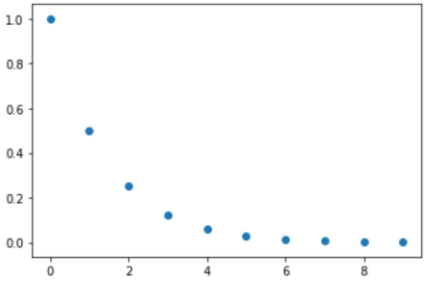
Sequences Exercise

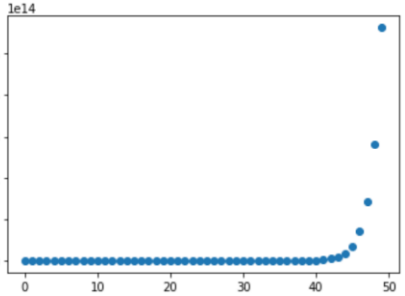
Check if the sequence converges using python:

f(x) = ar^n $0 < |r| < 1.$

r = 0.5



r = 2




```
import matplotlib.pyplot as plt

x = 0
n = 50
data = []
for i in range(n):
    x1 = 0.5**i
    print(i, x)
    if abs(x1 - x) < 1.e-3:
        print('n = {i}, result is
{x}.'.format(i=i, x=x))
        last_index = i
        break
    x = x1
    data.append([i,x])
else:
    print('No convergence within {n}
iterations.'.format(n=n))
    last_index = n
plt.scatter(*zip(*data))
```

It is possible to have for and else connected. Check more information on the link below

List of lists

Plotting list of lists

Converge to 0 when $0 < |r| < 1.$

https://book.pythontips.com/en/latest/for_-_else.html

Approximations using series

Power Series for Elementary Functions

Function	*Interval of Convergence*
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n (x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1^*$

* *The convergence at $x = \pm 1$ depends on the value of k.*

“

- *Make it work*
- *Make it Right*
- *Make it Fast*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Periodic functions

- Amplitude
- Period
- Phase shift

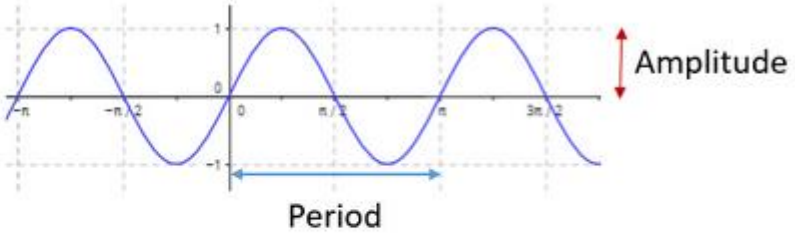
$$y = A\sin\left[B\left(x - C\right)\right] + D$$

$|A|$ is the amplitude

The period is $\frac{2\pi}{B}$

Phase (horizontal) shift is C

Vertical shift is D



The same applies for the Cosine Function.

$$y = 4 + \frac{1}{2}\sin(x)$$

$$y = -4\cos(3x - \pi)$$

Periodic functions – exercises spyder

$y = 4 + \frac{1}{2}\sin(x)$ Radians degrees samples

$y = -4\cos(3x - \pi)$

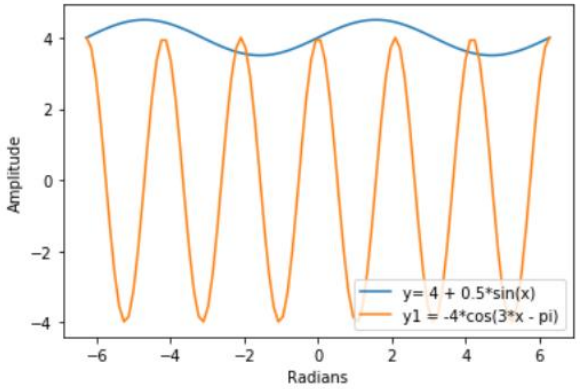
What is the period of these functions?
What is the amplitude?
How many samples (minimum amount) is needed to see the smoothing charts ?

Periodic functions – exercises spyder

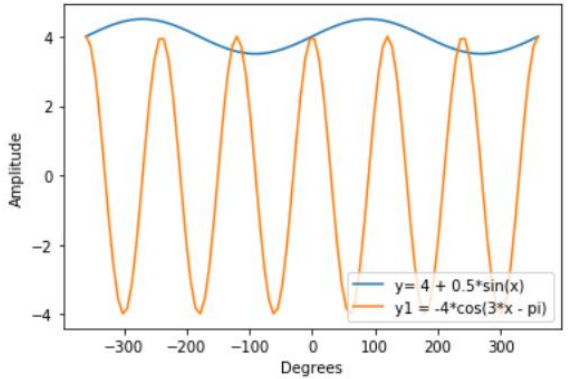
$$y = 4 + \frac{1}{2}\sin(x)$$

$$y = -4\cos(3x - \pi)$$

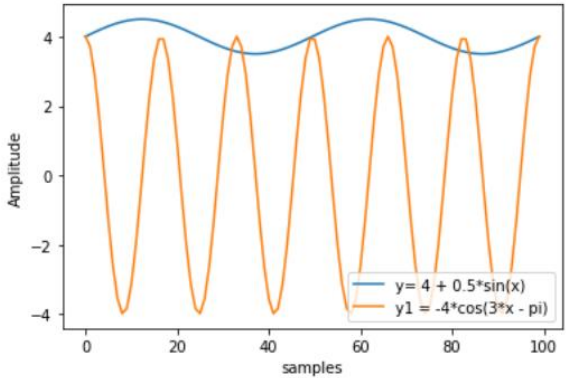
Radians



degrees

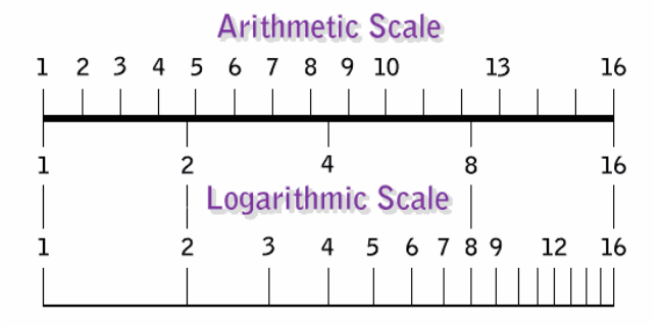


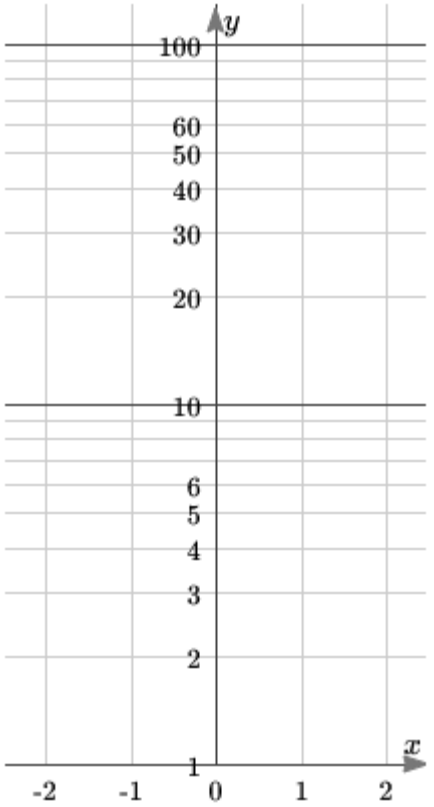
Samples




<Logarithms>

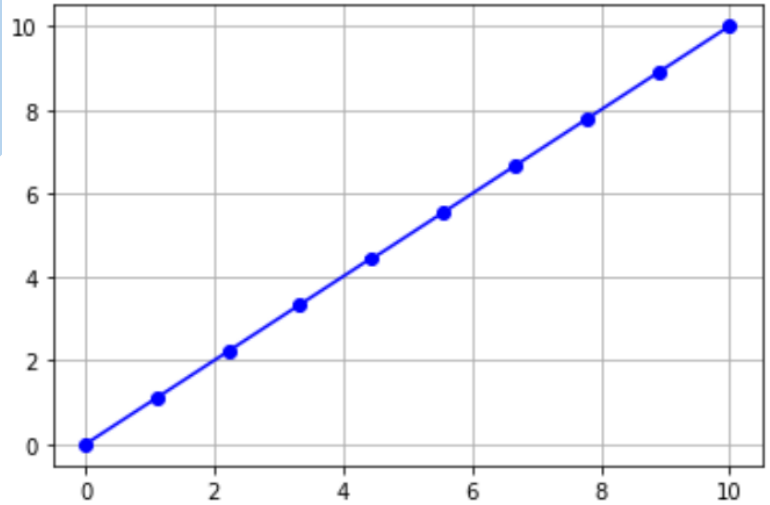
Log scale





Semilogarithmic axes.

y = x

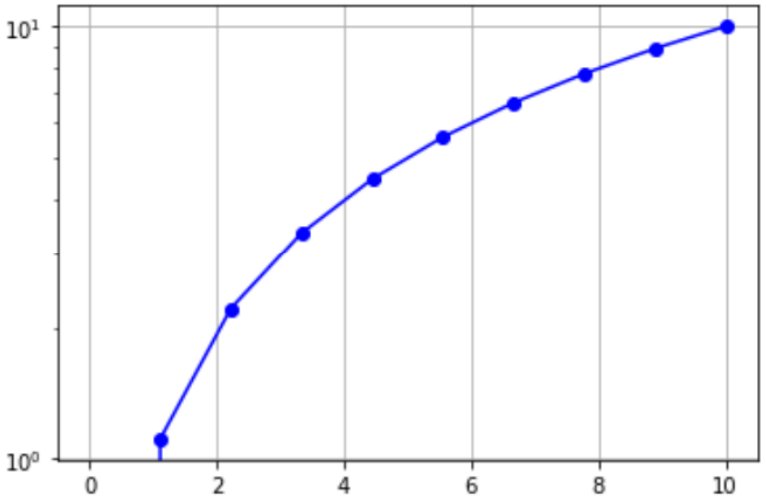


y = x

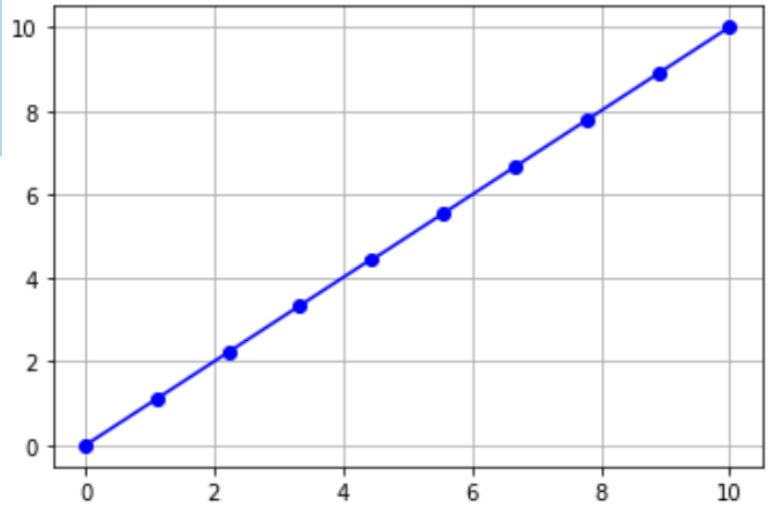
```
import numpy as np
x = np.linspace(0,10,10)
y = x
plt.plot(x, y)
plt.grid()
```

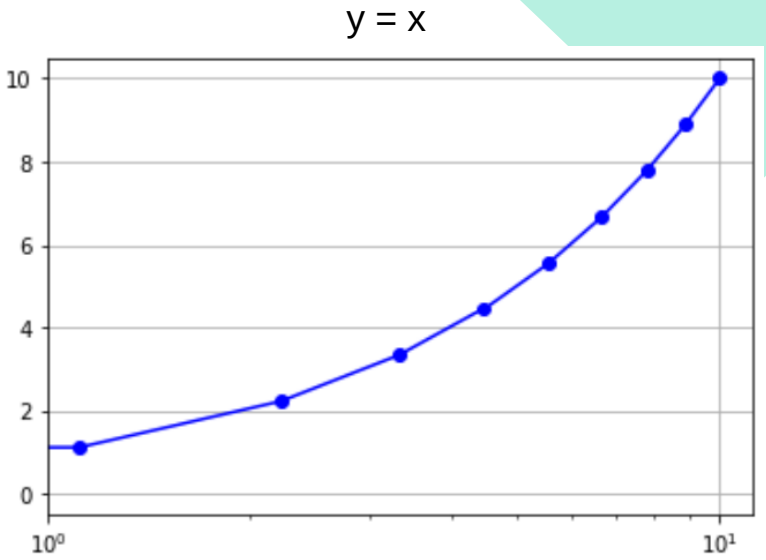
Same function

y = x




```
import numpy as np
x = np.linspace(0,10,10)
y = x
plt.plot(x, y)
plt.semilogy()
plt.grid()
```



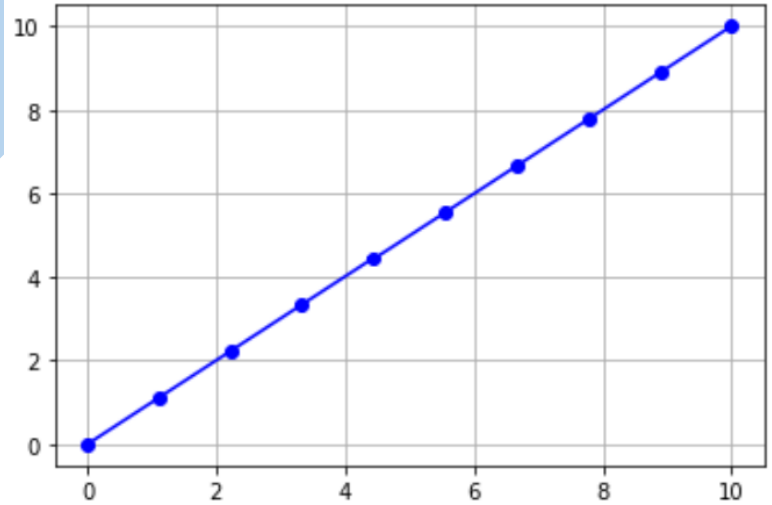



```
import numpy as np
x = np.linspace(0,10,10)
y = x
plt.plot(x, y)
plt.grid()
```

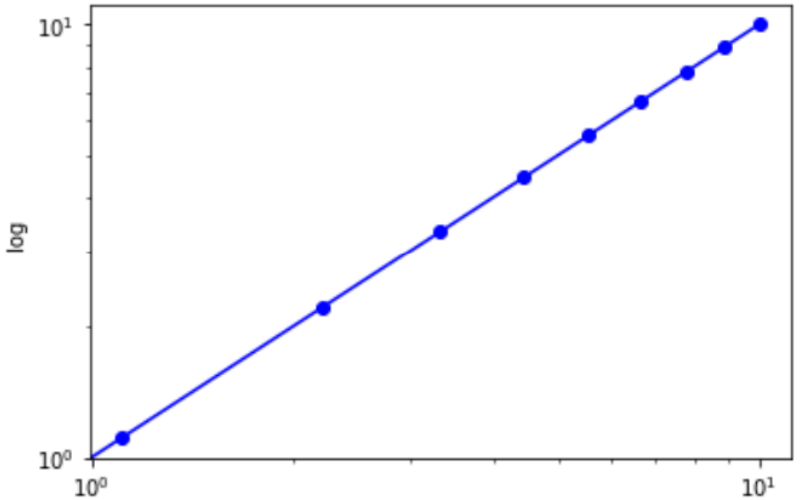
Same function

```
import numpy as np
x = np.linspace(0,10,10)
y = x
plt.plot(x, y)
plt.semilogx()
plt.grid()
```

y = x



y = x




```
import numpy as np
x = np.linspace(0,10,10)
y = x
plt.plot(x, y)
plt.grid()
```

Same function

```
import numpy as np
x = np.linspace(0,10,10)
y = x
plt.yscale('log')
plt. xscale('log')
plt.grid()
```

Exercises

(a) The streptococci bacteria population N at time t (in months) is given by $N = N_0 e^{2t}$ where N_0 is the initial population. If the initial population was 100, how long does it take for the population to reach one million?

(b) The formula for the amount of energy E (in joules) released by an earthquake is

$$E = 1.74 \times 10^{19} \times 10^{1.44M}$$

where M is the magnitude of the earthquake on the Richter scale.

i. The Newcastle earthquake in 1989 had a magnitude of 5 on the Richter scale. How many joules were released?

ii. In an earthquake in San Francisco in the 1900's the amount of energy released was double that of the Newcastle earthquake. What was its Richter magnitude?

Plot the function using linear scale and log scale and tell what is the best way to understand the what is happening with the data

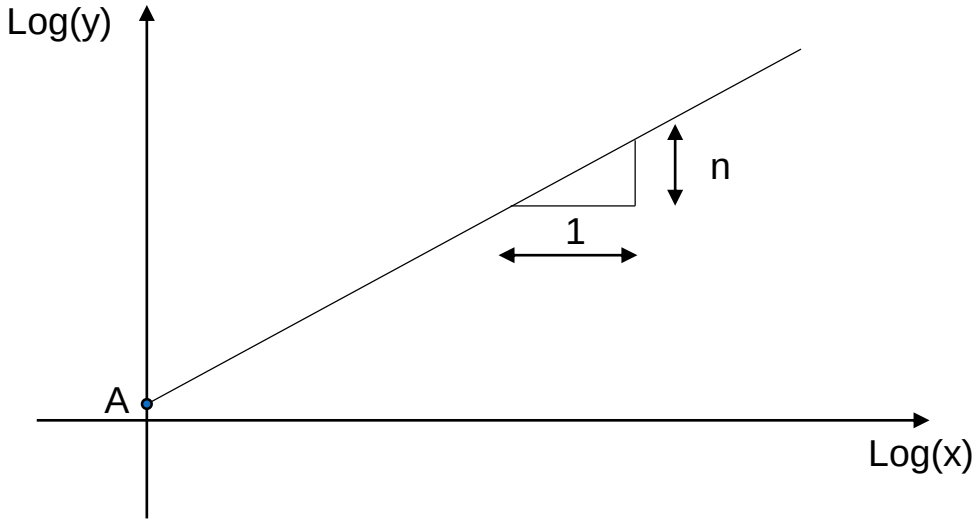
(a) 4.6054 months

(b) i. 2.76×10^{26} Joules ii. 5.2 on the Richter scale.

How to find the A and n in the function

$y = Ax^n$

We change y to loglog scale
so we could find A and n easily



Consider the data

x	2	30	70	100	150
y	4.24	16.4	25.1	30.0	36.7

$$n = \frac{\log(y_2) - \log(y_1)}{\log(x_2) - \log(x_1)} = \frac{1.56 - 0.63}{2.18 - 0.3} = 0.5$$

$$\log_{10}(A) = 0.48.$$

$$A = 10^{0.48} = 3.0$$

$$y = 3x^{\frac{1}{2}}$$

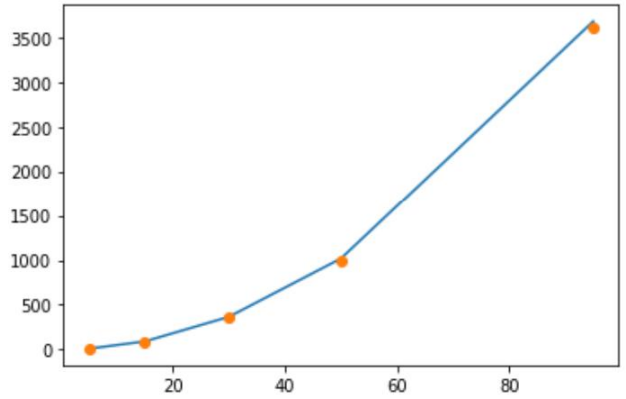
```
import matplotlib.pyplot as plt
import numpy as np
x = [2, 30, 70, 100, 150]
y = [4.24, 16.4, 25.1, 30.0, 36.7]

logx = np.log10(x)
logy = np.log10(y)
print(np.poly1d(np.polyfit(logx, logy, 1)))
plt.plot(logx,logy, 'o')
```

Exercises

The data below obeys a power law, $y = Ax^n$. Obtain the equation and select the correct statement.

x	5	15	30	50	95
y	10	90	360	1000	3610



Answer: n = 2


<Statistics>

Statistics

Consider the following two sets of data:

A: 10, 30, 50
B: 20, 30, 40

The mean of both two data sets is 30. But, the distance of the observations from the mean in data set A is larger than in the data set B. Thus, the mean of data set B is a better representation of the data set than is the case for set A.

Commonly used methods to interpreted data are **mean**, **median**, **mode**, **geometric mean**, etc.

Mean: Summing up all the observation and dividing by number of observations. Mean of 20, 30, 40 is

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3} = \frac{\sum_{i=1}^{3} x_i}{3} \qquad \frac{20+30+40}{3} = 30.$$

Statistics

Median: The middle value in an ordered sequence of observations. For example:

{9, 3, 6, 7, 5} → {3, 5, 6, 7, 9} → middle value 6.
{9, 3, 6, 7, 5, 2} → {2, 3, 5, 6, 7, 9} → average of the two middle values from the sorted sequence, $\frac{5+6}{2} = 5.5$.

Mode: The value that is observed most frequently. The mode is undefined for sequences in which no observation is repeated. For example:

{1, 2, 2, 3, 4, 7, 9} → mode is 2.

Statistics

Geometric mean: Product of all the observation and finding the root to the power of the number of observations. Geometric mean of 2, 4, 8 is $\sqrt[3]{2 * 4 * 8} = 4$

$$\sqrt[3]{x_1 \cdot x_2 \cdot x_3} = \left(\prod_{i=1}^{3} x_i \right)^{\frac{1}{3}}$$

Harmonic mean: Dividing the number of observations by the reciprocal of all the observation. Harmonic mean of 1, 4, 4 is $\frac{3}{\frac{1}{1}+\frac{1}{4}+\frac{1}{4}} = \frac{3}{1.5} = 2$.

$$H = \frac{3}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}} = \frac{3}{\sum_{i=1}^{3} x_i^{-1}}$$

Variability (or dispersion) measures how much a data is spread around a central value.

Commonly used methods: *variance*, *standard deviation*, *interquartile range*, *etc*.

Variance: average of the squares of the deviations of the observations from their mean. For example:

Variance of 5, 7, 3 → Mean is $\frac{5+7+3}{3} = 5$ and the variance is $\frac{(5-5)^2+(7-5)^2+(3-5)^2}{3-1} = 4$

$$S^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3-1}$$

Standard Deviation: Square root of the variance. The standard deviation of the above example is $\sqrt{4} = 2$.

Quartile: Division of the data in four regions that cover the total range of observed values. In other words, the first quartile (Q1) is the first 25% of the data. The second quartile (Q2) is from 25% to 50% of the data. The third quartile (Q3) is from 50% to 75% of the data. For example:

Consider the data: {3, 6, 7, 11, 13, 22, 30, 40, 44, 50, 52, 61, 68, 80, 94}

Divide the data in 4 equal groups. In this case, the group consists of 4 elements:

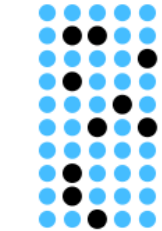
Q1 Q2 Q3
{3, 6, 7, 11, 13, 22, 30, 40, 44, 50, 52, 61, 68, 80, 94}

The first quartile is Q1 = 11. The second quartile is Q2 = 40 (this is also the median). The third quartile is Q3 = 61.

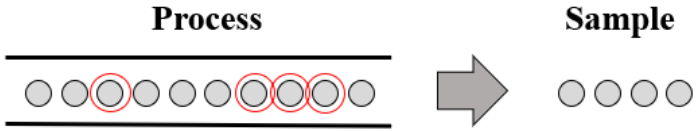
Inter-quartile Range: Difference between Q3 and Q1. Considering the last case, 61 – 40 = 21.

Statistics

Sampling: selection of a subset of individuals from a statistical population to estimate characteristics of the whole population.



Random Sample
(pick randomly
from list)

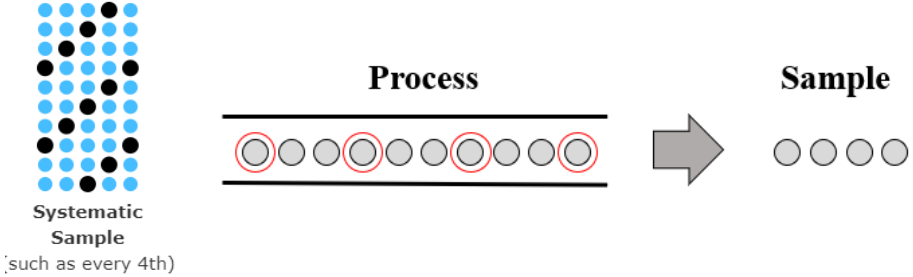


Random sample: pick the individuals randomly from the population. It is the best way to avoid bias. The results are better when you pick more individuals. For example, nationwide opinion polls survey around 2000 people, and the results are nearly as good (within about 1%) as asking everyone.

```
Sample = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
print("With list:", random.sample(Sample, 5))
```

Statistics

Systematic sample: select individual following a specific rule such as picking an individual every Nth



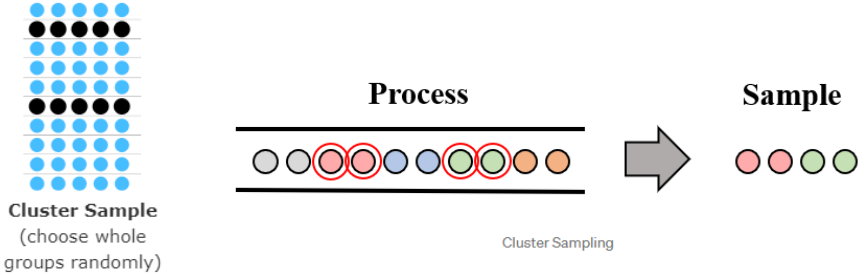
mathsisfun.com/data/sampling

N = step = 2

```
systematic_sample =  []
sample = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 ,14, 15]
step = 2
print(np.mean(sample))
size = int(len(sample)/step)
for i in range(0,size):
    systematic_sample.append(sample[i*step])
systematic_mean = np.mean(systematic_sample)
print(systematic_mean)
```

Statistics

Cluster sample: divide the individuals in many groups and select randomly whole groups. For example, divide the city in regions and pick randomly 3 regions to survey everyone there.



mathsisfun.com/data/sampling

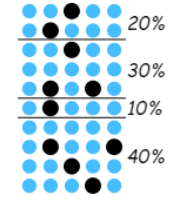
```
systematic_sample =  []
sample = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 ,14, 15]
step = 3
print(np.mean(sample))
size = int(len(sample)/step)
for i in range(0,size):
    systematic_sample.append(sample[i*step])
    systematic_sample.append(sample[i*step]+1)

systematic_mean = np.mean(systematic_sample)
print(systematic_mean)
```

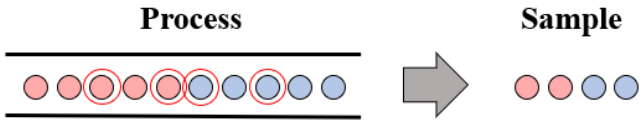
Statistics

Stratified sample: divide the individuals in specific categories and pick them according to its proportional in the sample.

.



Stratified Sample
(randomly, but in ratio to group size)




```
import random as rd
def split_list(alist, wanted_parts=1):
    length = len(alist)
    return [ alist[i*length // wanted_parts: (i+1)*length
// wanted_parts]
             for i in range(wanted_parts) ]

systematic_sample =  []
sample = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 ,14,
15]
A, B, C = split_list(sample, 3)
total= split_list(sample, 3)
stratified_list =[]
print('mean all values:', np.mean(sample) )
splited_list = [A, B, C]

for i in range(0,len(splited_list)):
    stratified_list.append(
splited_list[i][random.randint(0,4)])

systematic_mean = np.mean(stratified_list)
print('stratified mean', systematic_mean)
```

Statistics - Python

```
import numpy as np
from scipy import stats
from scipy.stats import gmean
from scipy.stats import hmean
import random

numbers = [ 1, 1, 2, 2, 3, 3, 4, 1, 2 ]
print(numbers)
Mean = np.mean(numbers)
Median = np.median(numbers)
Mode = int(stats.mode(numbers)[0])    # index 1 gives frequency
print("Mean = ", Mean, "\nMedian = ", Median, "\nMode = ", Mode)

Geometric_Mean = gmean(numbers)
Harmonic_mean = hmean(numbers)
print("Geometric Mean = ", Geometric_Mean, "\nHarmonic Mean = ",
Harmonic_mean)

Variance = np.var(numbers)
Standard_Deviation = np.std(numbers)
print("Variance = ", Variance, "\nStandard deviation = ",
Standard_Deviation)

Q1 = np.percentile(numbers, 25)
Q2 = np.percentile(numbers, 50)
Q3 = np.percentile(numbers, 75)
print("Q1 = ", Q1, "\nQ2 = ", Q2, "\nQ3 = ", Q3) # 1 1 1 2 2 2 3 3
```

Output:

```
[1, 1, 2, 2, 3, 3, 4, 1, 2]
Mean =  2.111111111111111
Median =  2.0
Mode =  1
Geometric Mean =  1.8761425449123872
Harmonic Mean =  1.6615384615384616
Variance =  0.9876543209876544
Standard deviation =  0.9938079899999066
Q1 =  1.0
Q2 =  2.0
Q3 =  3.0
```

Exercises

1. Ten observations x_i are given:

$$4,\ 7,\ 2,\ 9,\ 12,\ 2,\ 20,\ 10,\ 5,\ 9$$

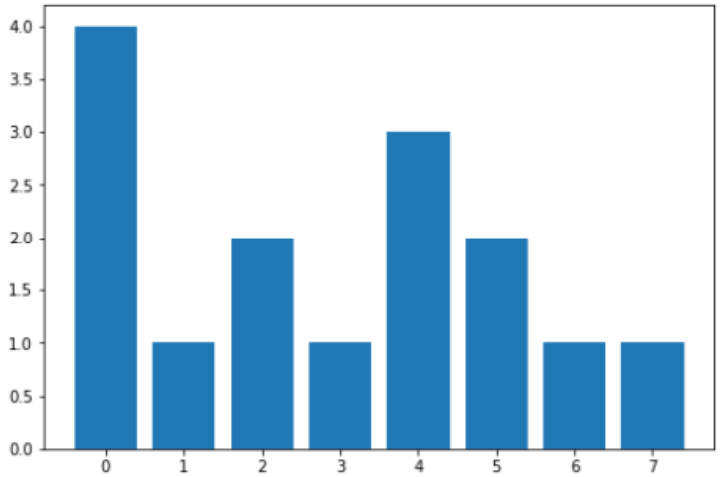
Determine the median, upper, and lower quartile and the inter-quartile range.

2. Four observations x_i are given:

$$2,\ 5,\ 10,\ 11$$

Determine the mean, empirical variance, and empirical standard deviation.

Exercises



3. Find the median of the data in Figure

4. Find the standard deviation of the data in Figure

Exercises

Use Pisa data available in https://data.oecd.org/pisa/mathematics-performance-pisa.htm#indicator-chart

Calculate Portugal's mode, median, variance and standard deviation for boys and girls over the years for 4 countries

country	AUS		CAN		CZE	
	Boys	girls	Boys	girls	Boys	girls
mode						
Median						
Variance						
Std deviation						

```
print(np.var(a, ddof=1))
print(np.std(b, ddof=1))
```

Exercises – Solution

```
a = [2,2,4,5,7,9,9,10,12,20]
print(np.median(a))
print(np.percentile(a, 25, interpolation='lower'))
print(np.percentile(a, 75, interpolation='higher'))

df = pd.DataFrame(a)
print(df.quantile(0.25, interpolation='lower'))
print(df.quantile(0.75, interpolation='higher'))
```

The observations are ordered according to size:

$$2,\ 2,\ 4,\ 5,\ 7,\ 9,\ 9,\ 10,\ 12,\ 20$$

```
8.0
4
10
0    4
Name: 0.25, dtype: int64
0    10
Name: 0.75, dtype: int64
```

The median is the mean of the two "middle" observations, i.e.

$$x(0.5) = \frac{7+9}{2} = 8$$

Similarly the lower and upper quartiles are

$$x(0.25) = \frac{2+4}{2} = 3\ ,\ x(0.75) = \frac{10+12}{2} = 11$$

The inter-quartile range thus becomes $11 - 3 = 8$.

Exercises – Solution

Mean is:

$$\bar{x} = \frac{2+5+10+11}{4} = 7$$

The empirical variance is

$$s^2 = \frac{(2-7)^2+(5-7)^2+(10-7)^2+(11-7)^2}{4-1} = 18$$

The empirical standard deviation thus becomes

$$s = \sqrt{18} \approx 4.24$$

```
import statistics as st
print(st.variance(b))
```

```
import numpy as np
print(np.var(a, ddof=1))
print(np.std(b, ddof=1))
```

“

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- *Make it Right*
- *Make it Fast*


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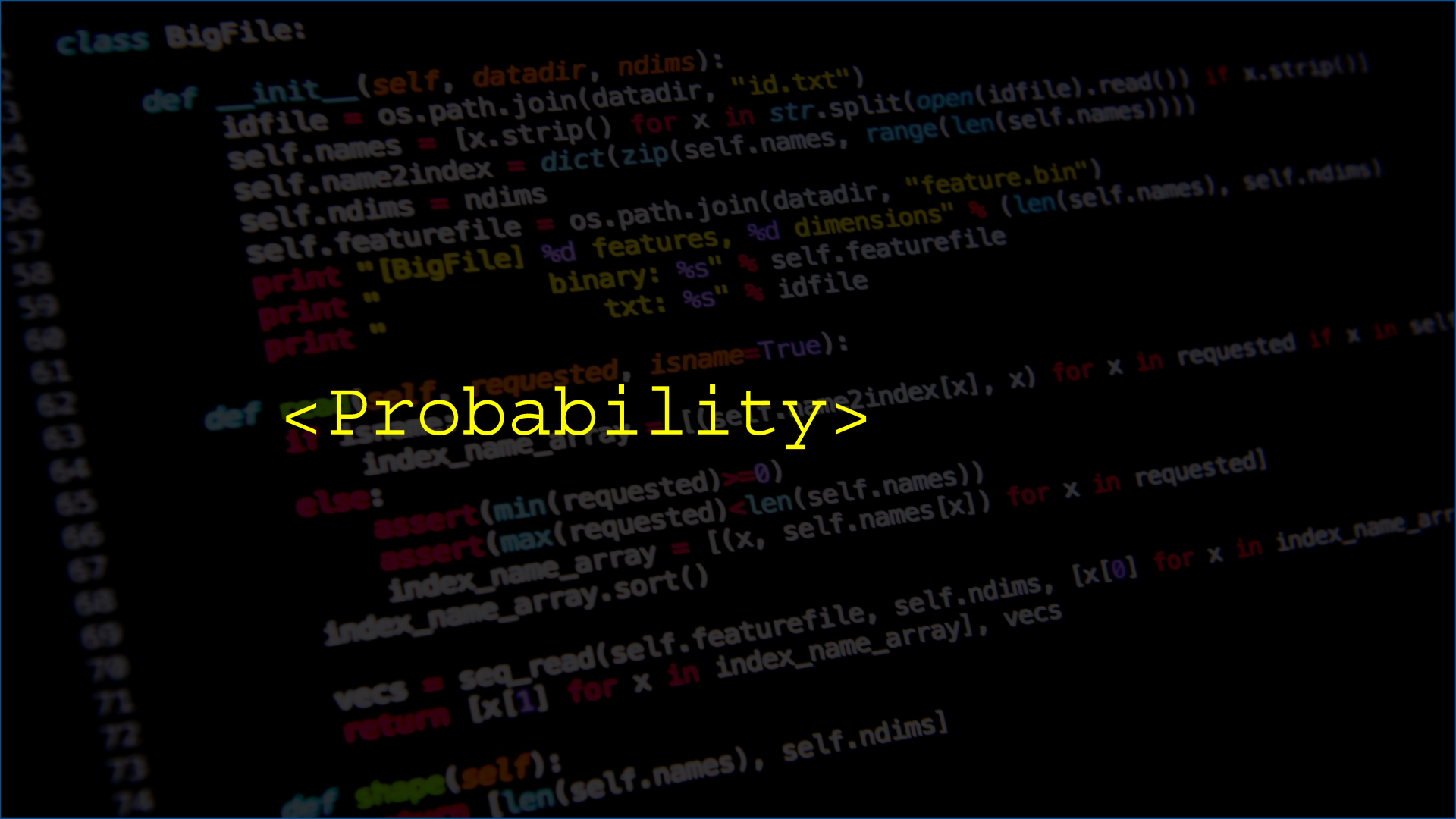

<Probability>

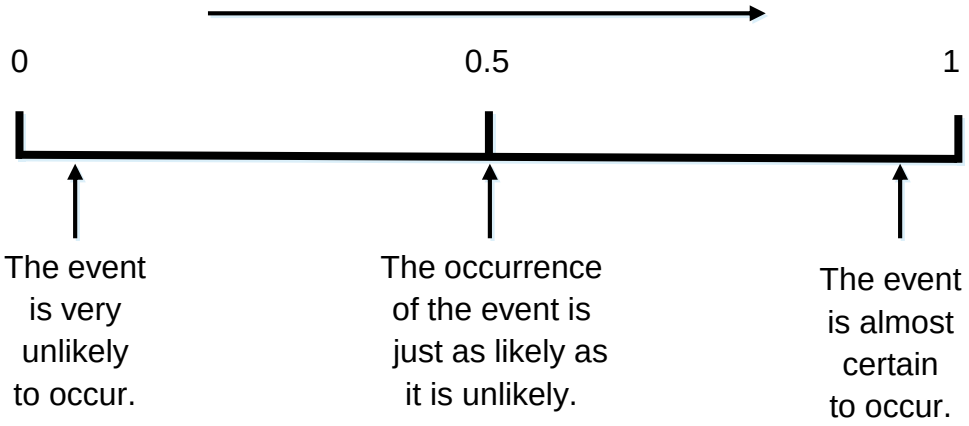
What are the *chances* that sales will increase if we increase prices?

What is the *likelihood* a new assembly method will increase productivity?

What are the odds that a new investment will be profitable?

- Basic Probability Concepts:
  - Sample Spaces and Events,
  - Simple Probability,
  - Joint Probability,
- Conditional Probability

Discrete probability



$$\text{Probability event} = \frac{number\ of\ ways\ it\ can\ happen}{Total\ number\ of\ outcomes}$$

$Total\ number\ of\ outcomes$ = Sample Space

- Two Conditions:
- Value is between 0 and 1.
- Sum of the probabilities of all events must be 1.

Probability

Sample Space

Deck of 52 cards

- 13 *ranks*: 2, 3, . . . , 9, 10, J, Q, K, A
- 4 *suits*: ♡, ♠, ♢, ♣,

What is the probability of picking the 5 of Clubs from this deck?

$$P = \frac{1}{52}$$ ← Single probability

Python example calculating simple probability

Simulation repetition

Sample Space

random.choice Sorts one element of each list

Event

Bar chart

Calculate event ocurrences

```
import numpy as np
import random
import matplotlib.pyplot as plt

simulation_number = 10000
club =0;spade=0;diamont=0;heart=0;percentages =[]
for i in range(0, simulation_number):

    card_points =[1,12,11,10,2,3,4,5,6,7,8,9,10]
    card_signs =['Heart','CLUB','DIAMOND','SPADE']
    random_point = random.choice(card_points)
    random_sign = random.choice(card_signs)
    random_card = random_point,random_sign
    # print (random_card)
    ##Condition

    if (random_card[0] == 7 and random_card[1] == 'CLUB'):
        club+=1;
X = ['CLUB']
percentages.append(np.round(club/simulation_number*100,2))
fig = plt.figure()
ax = fig.add_axes([0,0,1,1])
ax.bar(X,percentages)
print(sum(percentages), '%')
```

Example:

A bag contains fifteen balls distinguishable only by their colours; ten are blue and five are red. I reach into the bag with both hands and pull out two balls (one with each hand) and record their colours.

What is the *sample space*?

Example:

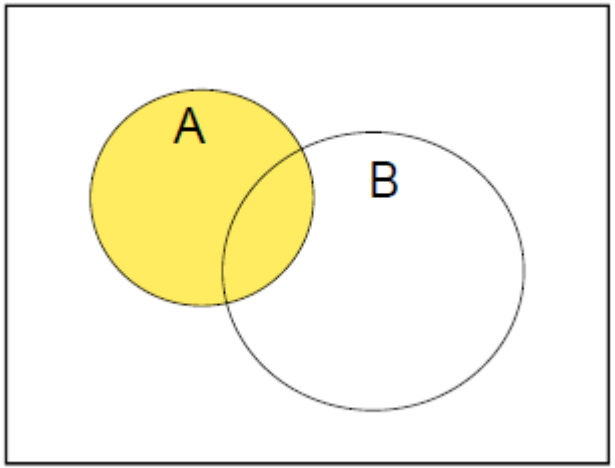
A bag contains fifteen balls distinguishable only by their colours; ten are blue and five are red. I reach into the bag with both hands and pull out two balls (one with each hand) and record their colours.

What is the *sample space*?

$$\{(B, B), (B, R), (R, B), (R, R)\}$$

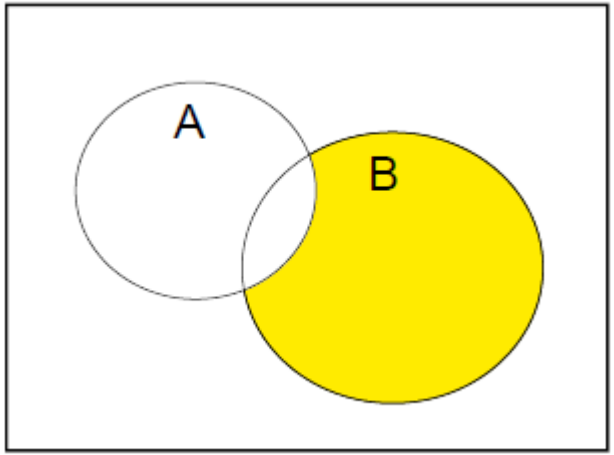
Probability in Venn Diagrams

A = Houses with white doors
B = Houses with 2 bedrooms



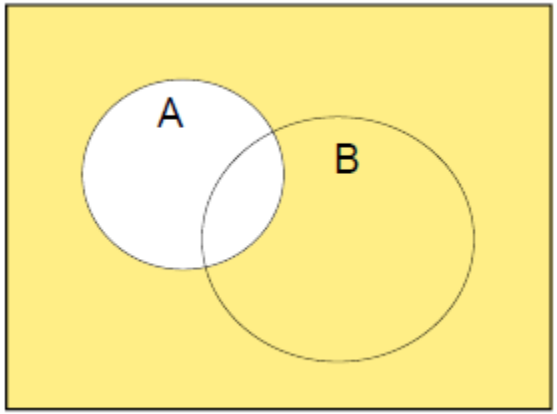
Probability

A = Houses with white doors
B = Houses with 2 bedrooms



Probability

A = Houses with white doors
B = Houses with 2 bedrooms

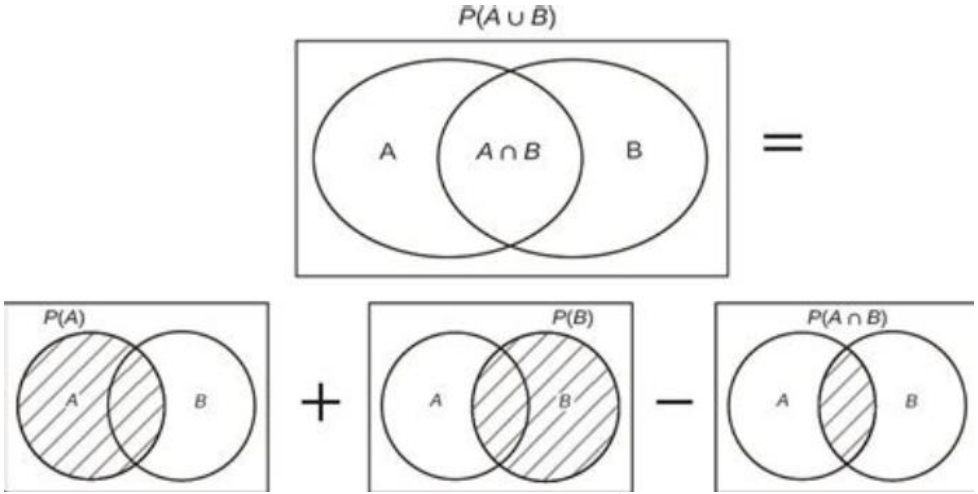


Addition Law

The addition law provides a way to compute the probability of event *A*, or *B*, or both *A* and *B* occurring

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is an OR probability problem!



The Probability of a Compound Event, A or B:

$$P(A \cup B) = \frac{\text{Numer of Event Outcomes from Either A or B}}{\text{Total Outcomes in the Sample Space}}$$

e.g. P(Red Card *or* Ace)

$$= \frac{4 \text{ Aces} + 26 \text{ Red Cards} - 2 \text{ Red Aces}}{52 \text{ Total Number of Cards}} = \frac{28}{52} = \frac{7}{13}$$

```
##Condition

if ((random_card[1] == 'Heart') or(random_card[1] == 'DIAMOND') or
    (random_card[0] == 1 and random_card[1] == 'SPADE') or
    (random_card[0] == 1 and random_card[1] == 'CLUB')):
    club+=1;
```

Probability

Sample Space

Deck of 52 cards

- 13 *ranks*: 2, 3, . . . , 9, 10, J, Q, K, A
- 4 *suits*: ♡, ♠, ♢, ♣,

What is the probability of picking the 5 of Clubs from this deck?

$$P = \frac{1}{52}$$

Single probability

```
import numpy as np
import random
import matplotlib.pyplot as plt

simulation_number = 10000
club =0;spade=0;diamont=0;heart=0;percentages =[]
for i in range(0, simulation_number):

    card_points =[1,12,11,10,2,3,4,5,6,7,8,9,10]
    card_signs =['Heart','CLUB','DIAMOND','SPADE']
    random_point = random.choice(card_points)
    random_sign = random.choice(card_signs)
    random_card = random_point,random_sign
    # print (random_card)
    ##Condition

    if (random_card[0] == 7 and random_card[1] == 'CLUB'):
        club+=1;
X = ['CLUB']
percentages.append(np.round(club/simulation_number*100,2))
fig = plt.figure()
ax = fig.add_axes([0,0,1,1])
ax.bar(X,percentages)
print(sum(percentages), '%')
```

https://docs.python.org/3/library/random.html

Joint probability of A and B

```
##Condition

if (random_card[0] == 7 and random_card[1] == 'CLUB'):
    club+=1;
if (random_card[0] == 7 and random_card[1] == 'SPADE' ):
    spade+=1;
```

- This is an AND probability problem!

$$P(A \cap B) = \frac{\textit{Number of Event Outcomes from both A and B}}{\textit{Total Number of Possible Outcomes in Sample Space}}$$

e.g. P(7 of Clubs and 7 of spade)

$$= \frac{2}{52 \text{ Total Number of Cards}} = \frac{1}{26}$$

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Multiplication Law for Independent Events

The concept of independent events is not related to the simultaneous occurrence of the events, but it is only concerned with the influence of the occurrence of one event on another.

If the probability of event *A* is not changed by the existence of event *B*, we would say that events *A* and *B* are independent.

$$P(A \cap B) = P(A) \times P(B)$$

Experiment	Outcomes
Toss a coin	Head, tail
Inspection a part	Defective, non-defective
Conduct a sales call	Purchase, no purchase
Roll a die	1, 2, 3, 4, 5, 6
Play a football game	Win, lose, tie

Exercises

1. A fair coin is tossed, and a fair die is thrown. Write down sample spaces for

   (a) the toss of the coin;

   (b) the throw of the die;

   (c) the combination of these experiments.

Exercises

1. A fair coin is tossed, and a fair die is thrown. Write down sample spaces for

   (a) the toss of the coin;

   (b) the throw of the die;

   (c) the combination of these experiments.

(a) $\{Head, Tail\}$

(b) $\{1, 2, 3, 4, 5, 6\}$

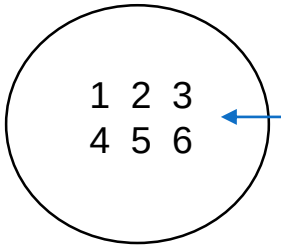
(c) $\{(1 \cap Head), (1 \cap Tail), \ldots, (6 \cap Head), (6 \cap Tail)\}$

Exercises

2. What is the chance of rolling a “4” with a dice?

$$Number\ of\ ways\ it\ can\ happen = 1$$
$$Total\ number\ of\ outcomes = 6 \Longrightarrow P = \frac{1}{6} = 17\%$$

Sample Space



Dice possibilities

3. What is the chance of getting 2 heads from two coins?

$$Number\ of\ ways\ it\ can\ happen = 1$$
$$Total\ number\ of\ outcomes = 4 \Longrightarrow P = \frac{1}{4} = 25\%$$

(head, head) (head, tail)
(tail, head) (tail, tail)

Plot the bar chart with the probability of each possible outcome

Exercises

4 . Find the probability of the sum of 2 dices been > 7 or odd ?

$$Number\ of\ ways\ it\ can\ happen = 26$$
$$Total\ number\ of\ outcomes = 36$$
$$P = \frac{26}{36} = 75\%$$

Dice possibilities

(1,2)(1,4)(1,5)(1,6)
(2,1)(2,3)(2,5)
(3,2)(3,4),(3,5) (3,6)
(4,1)(4,3),(4,4),(4,5),(4,6),
(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,6),(6,2),(6,3),(6,4),(6,5)

Plot the bar chart with the probability of each possible outcome

Exercises

5 . Find the probability of the sum of 2 dices been > 7 or even ?
Explain why it is different from the exercise 3

66,6%

Plot the bar chart with the probability of each possible outcome

Conditional Probability

The Probability of Event A given that Event B has occurred:

```
card_points =[1]
```

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

```
if (random_card[0] == 1 and (random_card[1] == 'Heart' or random_card[1] == 'DIAMOND')):
    club+=1
```

e.g. P(Red Card *given that it is* an Ace) = $\frac{2 \text{ Red Aces}}{4 \text{ Aces}} = \frac{1}{2}$

“

- *Make it work*
- *Make it Right*
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<Matrices>

Matrices Fundamentals

Elements designated by subscripted lower case letters.

E.G. For first row: $a_{11}, a_{12}, a_{13}, a_{14}, \cdots$

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 9 & 3 & 8 & 6 \\ 0 & 4 & 7 & 1 \end{bmatrix}$$

Then

$a_{11} = 1$	$a_{12} = 0$	$a_{13} = 2$	$a_{14} = 3$
$a_{21} = 9$	$a_{22} = 3$	$a_{23} = 8$	$a_{24} = 6$
$a_{31} = 0$	$a_{32} = 4$	$a_{33} = 7$	$a_{34} = 1$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Positions = a_{ij}

Types :

$A = [2 \quad 3]$

Row matrix

rows m = 1
columns n = 2

```
print(np.array([2,3]))    [2 3]
print(np.matrix('2 3'))   [[2 3]]
```

$A = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Column matrix

rows m = 2
columns n = 1

```
print(np.array([[2],[4]]))
print(np.matrix('2; 4'))
```

$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 7 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 7 & 9 \end{bmatrix}$

Rectangular matrices

rows m = 2
columns n = 2

rows m = 3
columns n = 2

Square matrix

$A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$

rows m = 2 = n

Symmetric matrix

$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 7 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$\sigma_{ij} = \sigma_{ji}$

Diagonal matrix

$I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

```
print(np.diag(x))
```

Unit matrix

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

```
b = np.eye(3)
print(b)
```

Matrices operations and properties

Equal

$$A = \begin{bmatrix} 5 & 9 & 5 \\ 3 & -7 & 4 \\ 6 & 8 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 9 & 5 \\ 3 & -7 & 4 \\ 6 & 8 & 3 \end{bmatrix}$$

A = B
Same elements
Same dimensions

Transpose

$$\sigma_{ij} = \sigma_{ji}$$

$$A = \begin{bmatrix} 5 & 9 & 5 \\ 3 & -7 & 4 \\ 6 & 8 & 3 \end{bmatrix} \longrightarrow B = A^T = \begin{bmatrix} 5 & 3 & 6 \\ 9 & -7 & 8 \\ 5 & 4 & 3 \end{bmatrix}$$

Matrices addition

$$A = \begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 7+1 & 3+5 & -1+6 \\ 2-4 & -5-2 & 6+3 \end{bmatrix}$$

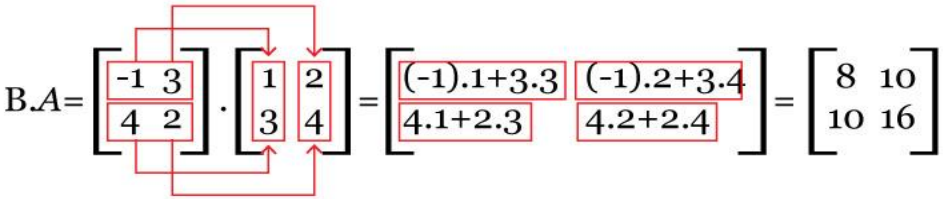
Commutative properties

A +B = B+A

Associative Properties

A+(B+C) = (A+B) +C

Matrices multiplication



AB ≠ BA
If AB = 0, A and/or B can be ≠ 0
If AB = AC, B can be ≠ C

- Inner dimensions must be equal for the matrices to be conformable for multiplication

- First distributive law :
A(B+C) = AB * AC
- Second Distributive law:
(A +B)C = AC * BC
- Associative Law :
A(BC) = (AB)C
- Transpose of product

$$(AB)^T = B^T A^T$$

```
x1 = np.array([[-1, 3],[4,2]])
x2 = np.array([[1, 2],[3,4]])
x4 = x1+x2
x5 = x1*x2
x3 =  np.dot(x1,x2)
print(x3)
print(x4)
print(x5)
```

```
[[ 8 10]
 [10 16]]
[[0 5]
 [7 6]]
[[-1  6]
 [12  8]]
```

CAUTIONS:

II. If $AB = 0$, neither A nor B are necessarily $= 0$

$$\text{Let } A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

then $AB = 0$, but neither A nor $B = 0$

III. If $AB = AC$, B does not necessarily equal C.

$$\text{Let } A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 10 \\ -2 & -5 \end{bmatrix}$$

then $AB = AC$, but $B \neq C$

Matrices determinants

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

```
a = [[0,3],[4,0]]
np.linalg.det(a)
```

Inverse matrices:

The matrix A is ***invertible*** if there exists a matrix A_1 such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

```
inverse_array = np.linalg.inv(a)
```

Inverse of a 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{da - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Representing equations using matrices

$$1x + 2y = 3$$
$$6x - 3y = 8$$

$$\begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

1. Coefficients are placed in one matrix
2. Variables are placed in another matrix
3. Constants are placed in a third matrix

Solving using cramer's method

$$ax + by = c$$
$$dx + ey = f$$

Let's eliminate y

$$aex + bey = ce$$
$$bdx + bey = bf$$

$$aex - bdx = ce - bf$$
$$x(ae - bd) = ce - bf$$

$$y = \frac{af - cd}{ae - bd} \qquad x = \frac{ce - bf}{ae - bd}$$

Determinants

$$y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

$$1x + 2y = 3$$
$$6x - 3y = 8$$

```
A = np.matrix('1 2;6 -3')
A1 = np.matrix('3 2;8 -3')
A2 = np.matrix('1 3;6 8')
det1 = np.linalg.det(A)
det2 = np.linalg.det(A1)
det3 = np.linalg.det(A2)
print ('x = ', det2/det1)
print('y = ', det3/det1)
```

```
x,y =  [1.66666667 0.66666667]
x =  1.6666666666666665
y =  0.6666666666666667
```

Solving by using matrices inverses

The inverse matrix is the opposite of multiplication:

$$A^{-1}A = AA^{-1} = I$$

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

```
import numpy as np
A = np.matrix('1 2;6 -3')
B = np.array([3, 8])
X = np.linalg.inv(A).dot(B)
print('x,y = ',X)
```

```
x,y =  [1.66666667 0.66666667]
x =  1.6666666666666665
y =  0.6666666666666667
```

Solving using scipy or numpy libraries

$$1x + 2y = 3$$
$$6x - 3y = 8$$

$$\begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

```
Import scipy as la
Import numpy as np

x = np.linalg.solve(M,b)
y  = la.linalg.solve(M,b)
```

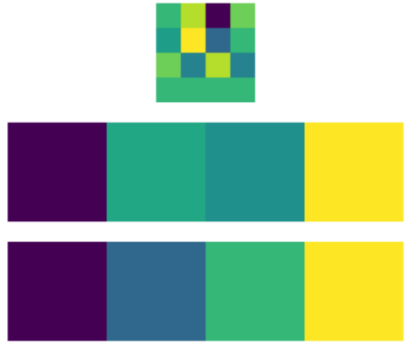
```
x,y =  [1.66666667 0.66666667]
x =  1.6666666666666665
y =  0.6666666666666667
```

numpy.linalg for more linear algebra functions.
Note that although **scipy.linalg** imports most of them, identically named functions from scipy.linalg may offer more or slightly differing functionality.

$$
\begin{aligned}
w + 3x - 5y + 2z &= 0 \\
4x - 2y + z &= 6 \\
2w - x + 3y - z &= 5 \\
w + x + y + z &= 10
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 3 & -5 & 2 \\ 0 & 4 & -2 & 1 \\ 2 & -1 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} 0 \\ 6 \\ 5 \\ 10 \end{bmatrix}
$$

```
l = []; l1 = []
fig,ax = plt.subplots(3,1)
M = np.array ([[1,3,-5, 2],
[0,4,-2,1], [2,-1,3,-
1],[1,1,1,1]])
b = np.array ([0,6,5,10])
l.append(b)
x = np.linalg.solve(M,b)
print(x)
l1.append(x)
ax[0].axis('off')
ax[0].imshow(M)
ax[1].axis('off')
ax[1].imshow(l)
ax[2].imshow(l1)
ax[2].axis('off')
```



https://matplotlib.org/3.3.3/api/_as_gen/matplotlib.pyplot.imshow.html

Other Methods of Matrix Solution

- Gaussian elimination
- For large matrices with many zeros, sparse matrix methods can be used

1. Solve

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Find the inverses

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

3. Find the inverses

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

.

4. Find the numbers a and b that give the inverse of $5 * \text{eye}(4) - \text{ones}(4,4)$:

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of $6 * \text{eye}(5) - \text{ones}(5,5)$?

5. Use `matplotlib.pyplot.plot` to produce a plot of the functions $f(x) = e^{-x/10}\sin(\pi x)$ and $g(x) = xe^{-x/3}$ over the interval $[0, 10]$. Include labels for the x- and y-axes, and a legend explaining which line is which plot. Save the plot as a `.jpg` ("Jpeg") file and send me a copy with your work.

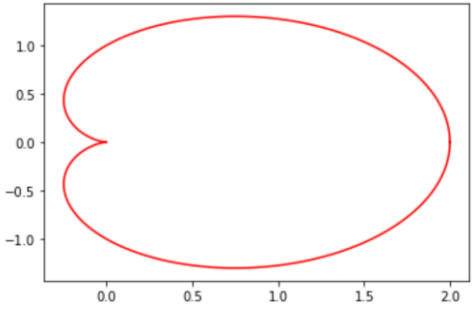
6. The function `np.random.rand` can be used to construct and fill vectors and matrices with random numbers. Use the help facility in Python to learn how to construct a 10×10 matrix named `M` filled with random numbers.

Plot the result using matplotlib

7. The shape of a limaçon can be defined parametrically as

$$r = r_0 + \cos\theta$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

When $r_0 = 1$, this curve is called a cardioid. Use this definition to plot the shape of a limaçon for $r_0 = 0.8$, $r_0 = 1.0$, and $r_0 = 1.2$. Be sure to use enough points that the curve is closed and appears smooth (except for the cusp in the cardioid). Use a legend to identify which curve is which.



8. Solve matrix exercise from the 4th list

“

- *Make it work*
- *Make it Right*
- *Make it Fast*


O futuro profissional
começa aqui
iscte
INSTITUTO
UNIVERSITÁRIO
DE LISBOA
emprego
digital
UPskill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DE LISBOA
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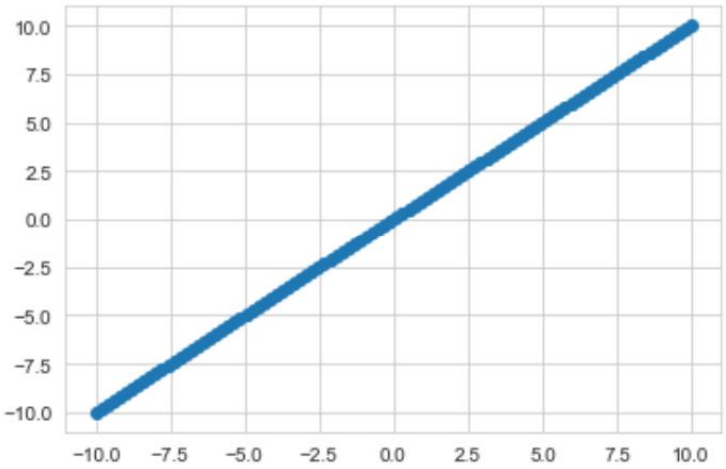
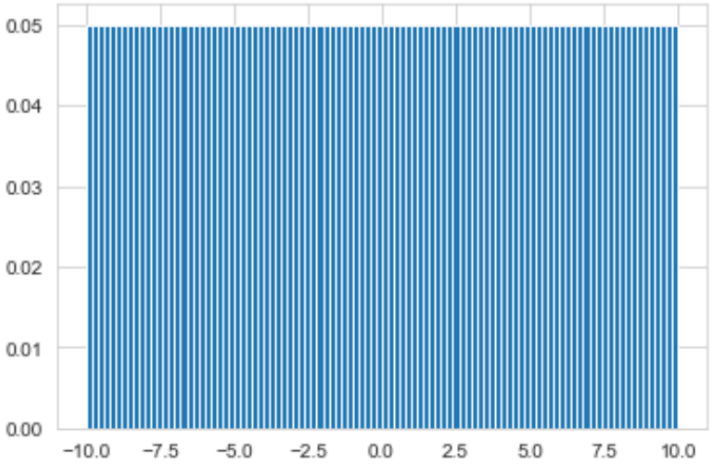
<Distributions>

A **histogram** is an approximate representation of the distribution of numerical data.

Histograms give a rough sense of the density of the underlying distribution of the data, and often for density estimation: estimating the probability density function of the underlying variable. The total area of a histogram used for probability density is always normalized to 1

Histogram from a linear function



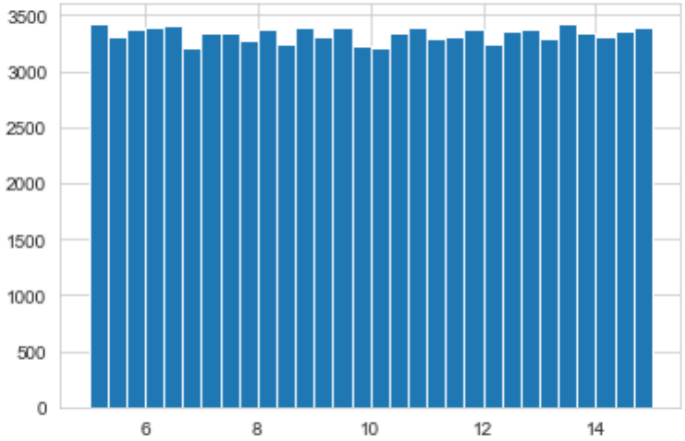



```
x = np.linspace(-10, 10, 100)
y = x
plt.hist(y, 100, density=1)
plt.show()
plt.plot(x,y,'o')
plt.show()
```

Uniform distribution

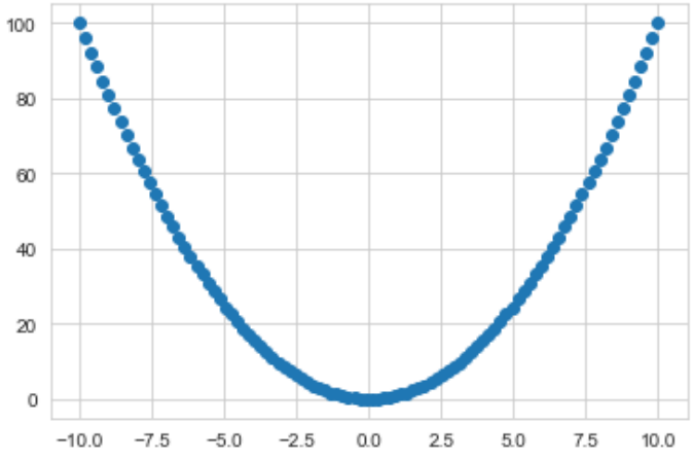
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

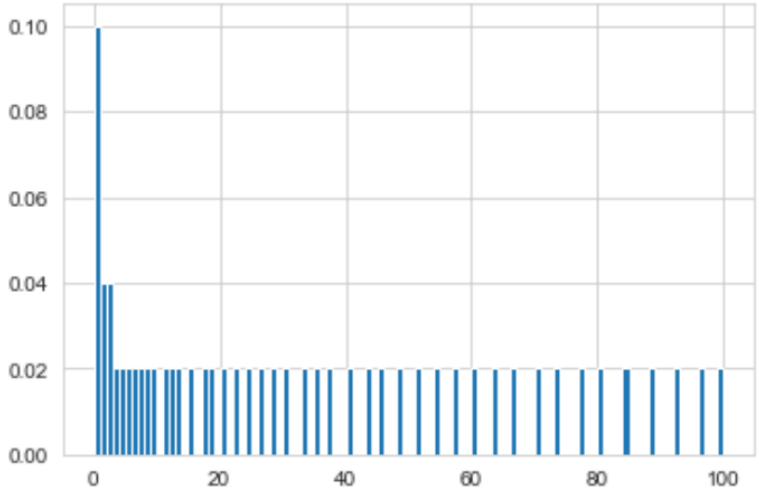
Uniform Distribution




```
from scipy.stats import uniform
import matplotlib.pyplot as plt
data = uniform.rvs(size = 100000, loc = 5, scale=10)
 ax = plt.hist(data, bins = 30)
ax.set(xlabel = 'interval')
plt.show()
```

Histogram from a quadratic function



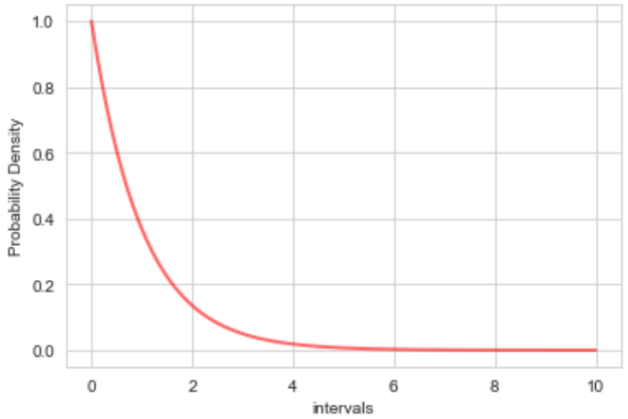



```
x = np.linspace(-10, 10, 100)
y = x**2
plt.hist(y, 100,density=1)
plt.show()
plt.plot(x,y,'o')
plt.show()
```

Exponential distribution

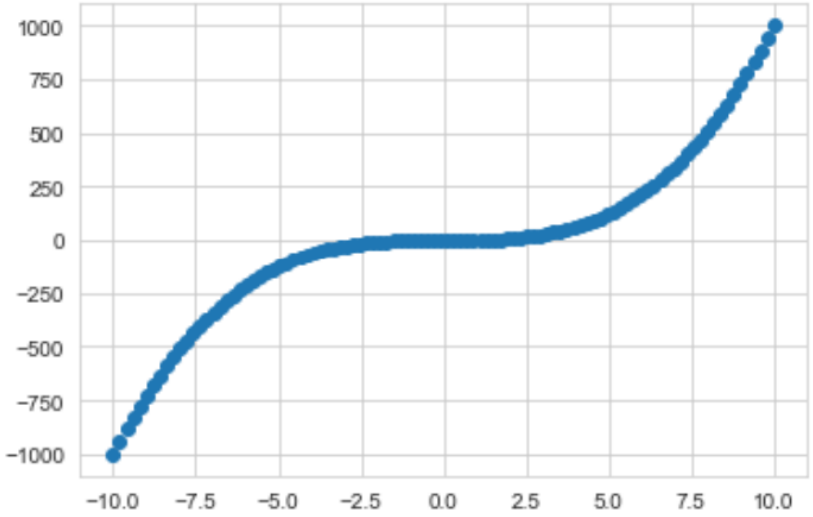
$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

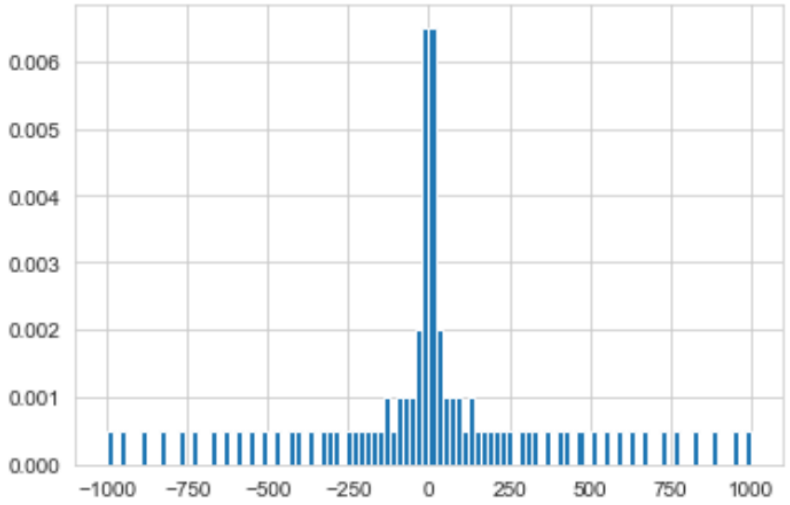
Exponential Distribution Pdf




```
import numpy as np
import matplotlib.pyplot as plt
from  scipy.stats import expon
 x = np.linspace(0.001,10, 100)
pdf = expon.pdf(x)
plt.plot(x, pdf , 'r-', lw=2, alpha=0.6,
label='expon pdf')
plt.xlabel('intervals')
plt.ylabel('Probability Density')
plt.show()
```

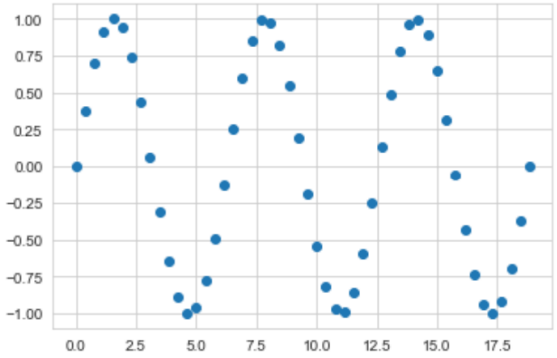
Histogram from a cubic function

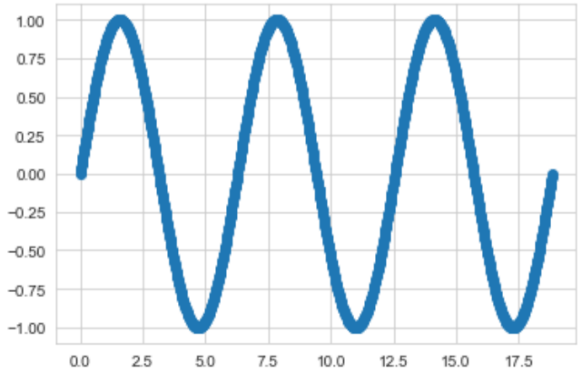




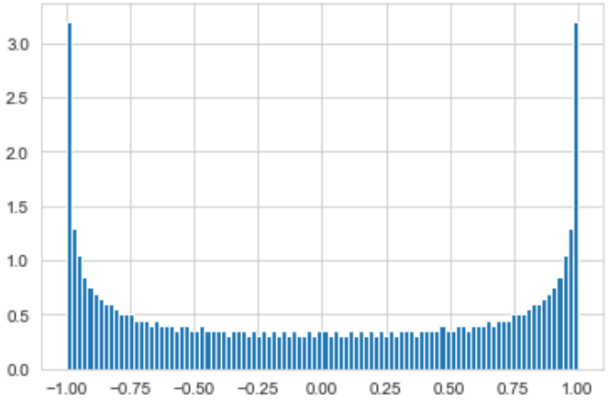

```
x = np.linspace(-10, 10, 100)
y = x**3
plt.hist(y, 100, density=1)
plt.show()
plt.plot(x,y,'o')
plt.show()
```

Histogram from a sin function






```
n = 50
```




```
n = 1000
x = np.linspace(0,6*np.pi,n)
sig1 = np.sin(x)
plt.hist(sig1, 100)
plt.show()
plt.plot(x,sig1,'o')
plt.show()
```

Gaussian Distribution

def An Normal random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

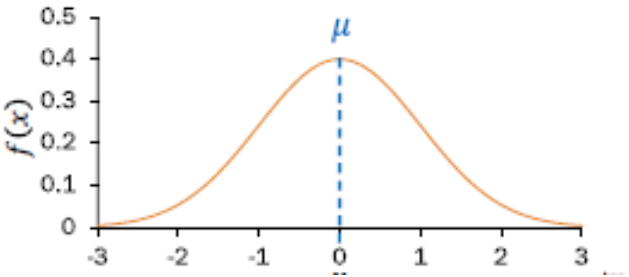
PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation	$E[X] = \mu$
Variance	$\mathrm{Var}(X) = \sigma^2$

Other names: Gaussian random variable

mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



https://docs.scipy.org/doc/numpy-1.15.0/reference/routines.random.html

Gaussian Distribution

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Support: $(-\infty, \infty)$

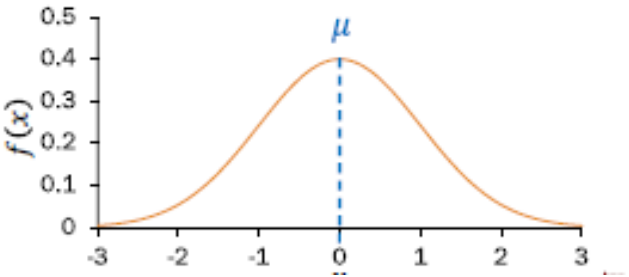
PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation	$E[X] = \mu$
Variance	$\mathrm{Var}(X) = \sigma^2$

Other names: Gaussian random variable

mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



https://docs.scipy.org/doc/numpy-1.15.0/reference/routines.random.html

Gaussian Distribution

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of X is defined as:

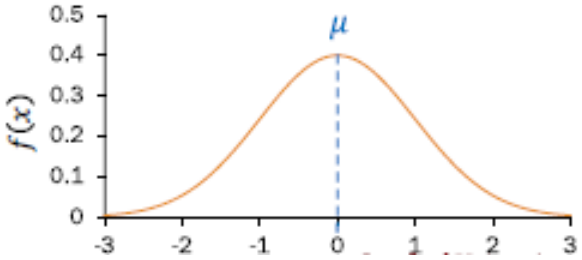
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

normalizing constant

exponential tail

symmetric around μ

variance σ^2 manages spread



https://docs.scipy.org/doc/numpy-1.15.0/reference/routines.random.html

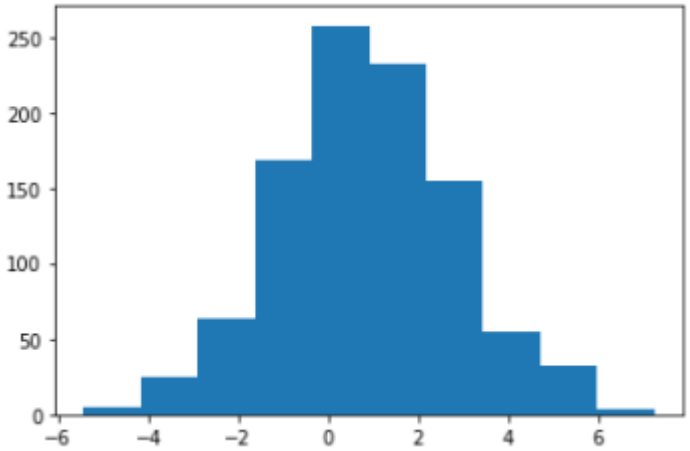
Gaussian Distribution

loc - (Mean) where the peak of the bell exists.
scale - (Standard Deviation) how flat the graph distribution should be.
size - The shape of the returned array

```
from numpy import random
import matplotlib.pyplot as plt

x = random.normal(loc=1, scale=2,
size=1000)

plt.hist(x)
plt.show()
```



https://docs.scipy.org/doc/numpy-1.15.0/reference/routines.random.html

Gaussian Distribution

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

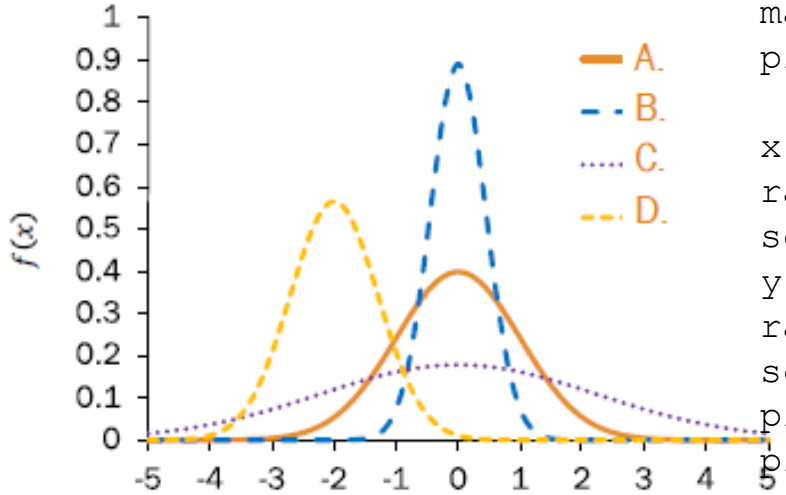
Match PDF to distribution:

$\mathcal{N}(0, 1)$

$\mathcal{N}(-2, 0.5)$

$\mathcal{N}(0, 5)$

$\mathcal{N}(0, 0.2)$

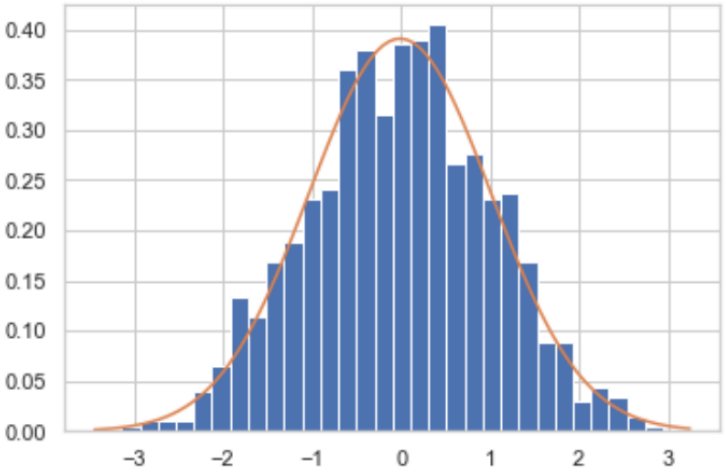



```
from numpy import random
import matplotlib.pyplot as plt

x = random.normal(loc=1, scale=2, size=1000)
y = random.normal(loc=1, scale=0.5, size=1000)
plt.hist(x)
plt.hist(y)
plt.show()
```

https://docs.scipy.org/doc/numpy-1.15.0/reference/routines.random.html

Gaussian Distribution




```
from scipy.stats import norm
import numpy as np
import matplotlib.pyplot as plt

data = np.random.normal(loc=0,
size=1000)
mean,std=norm.fit(data)
plt.hist(data, bins=30,
density=True)
xmin, xmax = plt.xlim()
x = np.linspace(xmin, xmax, 100)
y = norm.pdf(x, mean, std)
plt.plot(x, y)
plt.show()
```

https://docs.scipy.org/doc/scipy/reference/stats.html

Central Limit Theorem

General Idea: Regardless of the population distribution model, as the sample size increases, the **sample mean** tends to be normally distributed around the population mean, and its standard deviation shrinks as n increases.

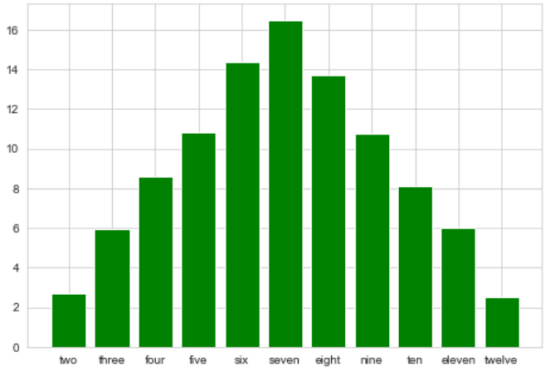
Certain conditions must be met to use the CLT.

Independent Samples Test

- **"Randomization":** Each sample should represent a random sample from the population, or at least follow the population distribution.

Large Enough Sample Size

- Sample size *n* should be large enough so that $np \geq 10$ and $nq \geq 10$



Dice exercise

Population Random Selection

Information About the data

Analyzed 100 data

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import random as rd

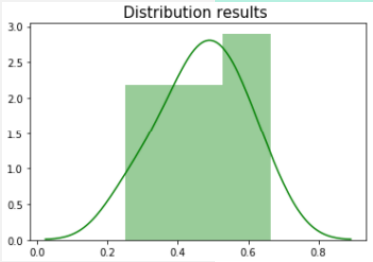
def random_samples(population, sample_qty, sample_size):
    sample_means = []
    for i in range(sample_qty):
        sample = rd.sample(population, sample_size)
        sample_mean = np.array(sample).mean()
        sample_means.append(sample_mean)
    return sample_means

uniform = np.random.rand(100000)

print("mean =", np.mean(uniform))
plt.figure(figsize=(7,4))
plt.title("Distribution", fontsize=15)
plt.hist(uniform,100)

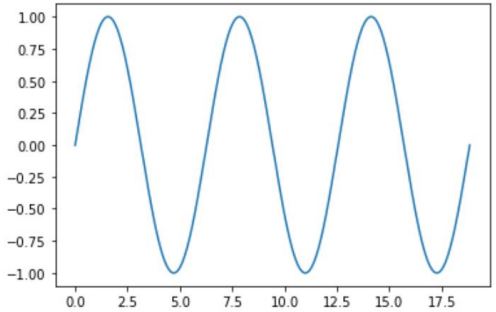
samples_from_normal = random_samples(list(uniform), 10, 10)
plt.figure()
plt.title("Distribution Results", fontsize=15)
sns.distplot(samples_from_normal, hist=False)
```

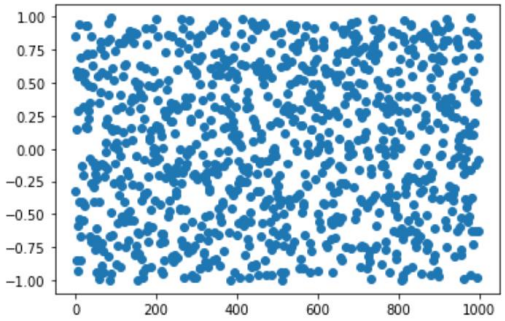
```
mean = 0.49965595000635993
```

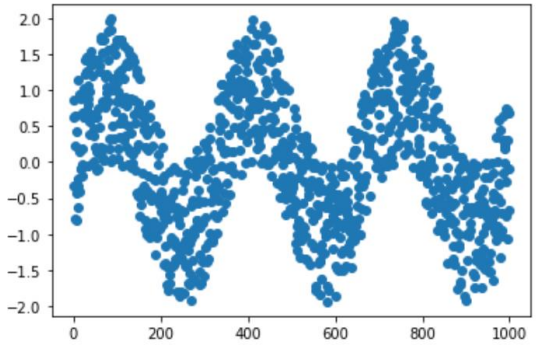


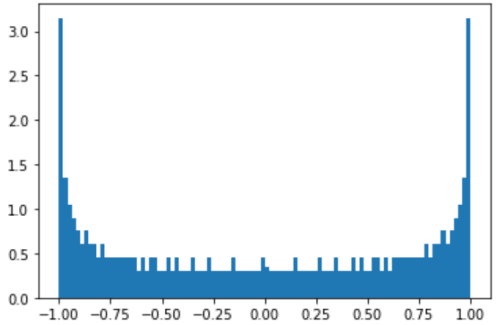
Data generation 100000 data

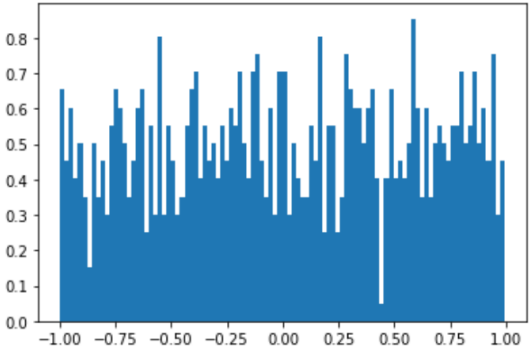
Adding a sin and a random number to see gaussian distribution

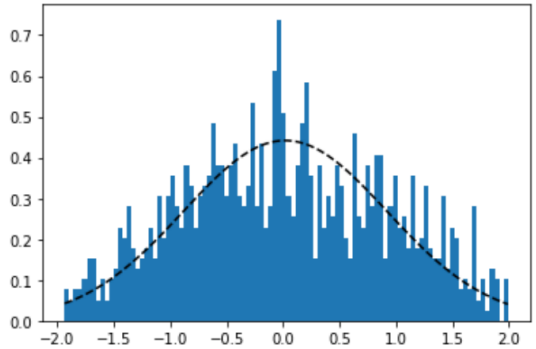












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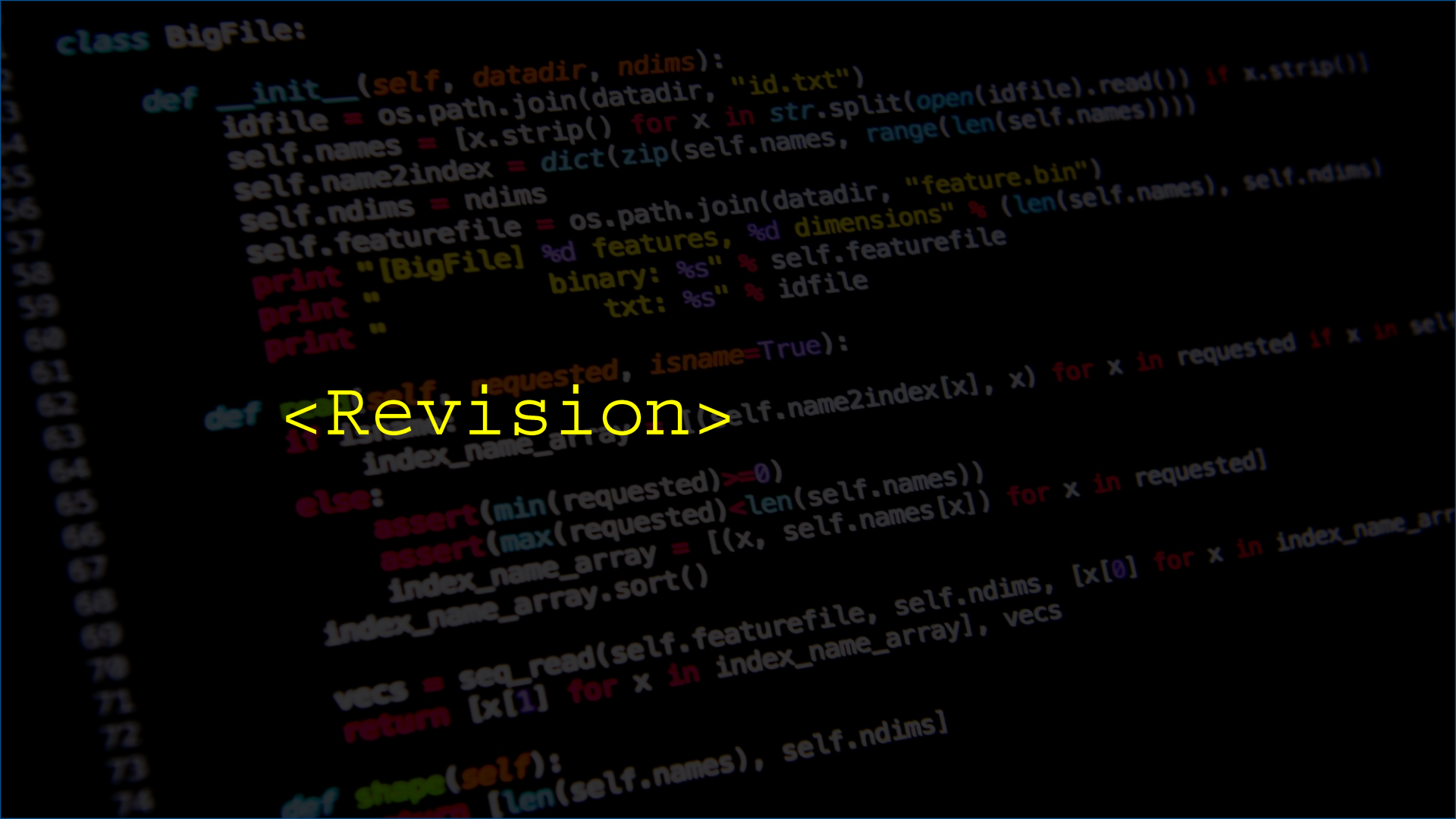
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- *Make it work*
- *Make it Right*
- *Make it Fast*


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