Notes - Nozzle Theory
Notes and derivations of key equations with respect to nozzle theory and design. From study of Sutton text, Chapter 3

Nozzle Types

For a 15° half-angle conical nozzle,
$L[cone] = \frac{(r[e] - r[t])}{\tan \alpha}$

$$L_{cone} = \frac{r_e - r_t}{\tan \alpha} \quad \textbf{(1)}$$

Conical nozzles are simple to design and fabricate. The optimum divergence for a conical nozzle is a 15° half-angle based on axial momentum vs. mass.

Correction Factor for Conical Nozzle, α=15°
$\lambda = .983$

$$\lambda = 0.983 \quad \textbf{(2)}$$

For parabolic bell nozzles, as L decreases initial angle increases and exit angle decreases. Bell nozzles may be shorter axially and have fewer losses, but are more difficult to design and fabricate. Also note that shorter nozzles are less efficient. The benefits to a bell nozzle are likely not significant relative to the added costs to mass, design, and fabrication.

Basic Thermodynamics

Conservation of energy
$h[1] - h[2] = \frac{1}{2}\left(v[2]^2 - v[1]^2\right)$

$$h_1 - h_2 = -\frac{1}{2}\, v_1^2 + \frac{1}{2}\, v_2^2 \quad \textbf{(3)}$$

$h[1] - h[2] = c[p] \cdot (T[1] - T[2])$

$$h_1 - h_2 = c_p\, (T_1 - T_2) \quad \textbf{(4)}$$

Conservation of mass
$\dot{m} = \frac{A \cdot v}{V}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, m(t) = \frac{A\, v}{V} \quad \textbf{(5)}$$

Perfect gas law
$R = \frac{R[0]}{m[mol]}$

$$R = \frac{R_0}{m_{mol}} \quad \textbf{(6)}$$

$p \cdot V = R \cdot T$

$$p\, V = \frac{R\, T}{m_{mol}} \quad \textbf{(7)}$$

Specific heat ratio $k = \frac{c[p]}{c[v]}$

$$k = \frac{c_p}{c_v} \tag{8}$$

Therefore the following holds for isentropic flow between inlet (1) and exit (2)

$$\frac{T[1]}{T[2]} = \left(\frac{p[1]}{p[2]}\right)^{\frac{(k-1)}{k}}$$

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{k-1}{k}} \tag{9}$$

$$\frac{T[1]}{T[2]} = \left(\frac{V[2]}{V[1]}\right)^{k-1}$$

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{k-1} \tag{10}$$

Stagnation temperature (for adiabatic flow, stagnation temperature is constant)

$$T[0] = T + \frac{v^2}{2 \cdot c[p]}$$

$$T_0 = T + \frac{1}{2}\frac{v^2}{c_p} \tag{11}$$

Pressure ratio of stagnation pressure to local pressure. Neglecting velocity within the chamber, ideal chamber pressure is equal to stagnation pressure.

$$\frac{p[0]}{p} = \left(1 + \frac{v^2}{2 \cdot c[p] \cdot T}\right)^{\frac{k}{(k-1)}}$$

$$\frac{p_0}{p} = \left(1 + \frac{1}{2}\frac{v^2}{c_p T}\right)^{\frac{k}{k-1}} \tag{12}$$

$$\frac{p[0]}{p} = \left(\frac{V}{V[0]}\right)^{k}$$

$$\frac{p_0}{p} = \left(\frac{V}{V_0}\right)^{k} \tag{13}$$

Local velocity of sound
$a = \mathrm{sqrt}(k \cdot R \cdot T)$

$$a = \sqrt{k R T} \tag{14}$$

Mach number

$$M = \frac{v}{a}$$

$$M = \frac{v}{a} \quad \textbf{(15)}$$

$M = \mathrm{sqrt}\left(\frac{2}{k-1} \cdot \left(\frac{T[0]}{T} - 1 \right) \right)$

$$M = \sqrt{2} \sqrt{\frac{\frac{T_0}{T} - 1}{k-1}} \quad \textbf{(16)}$$

Expansion area ratio in terms of Mach

$\frac{A[2]}{A[1]} = \frac{M[1]}{M[2]} \cdot \mathrm{sqrt}\left(\left(\frac{1 + \frac{(k-1)}{2} \cdot M[2]^2}{1 + \frac{(k-1)}{2} \cdot M[1]^2} \right)^{\frac{(k+1)}{(k-1)}} \right)$

$$\frac{A_2}{A_1} = \frac{M_1 \sqrt{\left(\frac{1 + \frac{1}{2} (k-1) M_2^2}{1 + \frac{1}{2} (k-1) M_1^2} \right)^{\frac{k+1}{k-1}}}}{M_2} \quad \textbf{(17)}$$

Isentropic Flow Through Nozzles

States: Inlet (1), Throat (t),Arbitrary Point (y), Exit (2)

Velocity

Exit velocity

$v[2] = \mathrm{sqrt}\left(2 \cdot (h[1] - h[2]) + v[1]^2\right)$

$$v_2 = \sqrt{v_1^2 + 2\, h_1 - 2\, h_2} \quad \textbf{(18)}$$

Velocity at any two points in the nozzle

$v[2] = \mathrm{sqrt}\left(\frac{2 \cdot k}{k-1} \cdot R \cdot T[1] \cdot \left(1 - \left(\frac{p[2]}{p[1]} \right)^{\frac{(k-1)}{k}} \right) + v[1]^2 \right)$

$$v_2 = \sqrt{\frac{2\, k\, R\, T_1 \left(1 - \left(\frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right)}{k-1} + v_1^2} \quad \textbf{(19)}$$

Assuming isentropic conditions, T~= T_0 so exit velocity may be solved without needing inlet velocity

$v[2] = \mathrm{sqrt}\left(\frac{2 \cdot k}{k-1} \cdot R \cdot T[1] \cdot \left(1 - \left(\frac{p[2]}{p[1]} \right)^{\frac{(k-1)}{k}} \right) \right) - \mathrm{sqrt}\left(\frac{2 \cdot k}{k-1} \cdot R[0] \cdot T[0] \cdot \left(1 \right.\right.$

$$-\left(\frac{p[2]}{p[1]}\right)^{\frac{(k-1)}{k}}\Bigg)\Bigg)$$

$$v_2 = \sqrt{2}\sqrt{\frac{k\,R\,T_1\left(1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right)}{k-1}} \to \sqrt{2}\sqrt{\frac{k\,R_0\,T_0\left(1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right)}{k-1}} \quad \textbf{(20)}$$

Maximum theoretical exit velocity in a vacuum

$$v[2][\max] = \mathrm{sqrt}\left(\frac{2\cdot k\cdot R\cdot T[0]}{k-1}\right)$$

$$v_{2_{\max}} = \sqrt{2}\sqrt{\frac{k\,R\,T_0}{k-1}} \quad \textbf{(21)}$$

Throat Condition

The maximum gas flow per unit area occurs at the throat where there is a unique gas pressure ratio which is only a function of the ratio of specific heats k. This pressure ratio is found by setting M = 1

$$\frac{p[t]}{p[1]} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

$$\frac{p_t}{p_1} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \quad \textbf{(22)}$$

The throat pressure for which the isentropic mass flow rate is a maximum is called the critical pressure. Critical pressure ratio is the pressure ratio where the flow is accelerated to a velocity equal to the local velocity of sound in the fluid. Typical values of this critical pressure ratio range between 0.53 and 0.57 of the inlet pressure.

At the point of critical pressure, M=1

$$V[t] = V[1]\cdot\left(\frac{(k+1)}{2}\right)^{\frac{1}{k-1}}$$

$$V_t = V_1\left(\frac{1}{2}\,k+\frac{1}{2}\right)^{\frac{1}{k-1}} \quad \textbf{(23)}$$

$$T[t] = \frac{2\cdot T[1]}{k+1}$$

$$T_t = \frac{2\,T_1}{k+1} \quad \textbf{(24)}$$

$$v[t] = \mathrm{sqrt}(k\cdot R\cdot T[t])$$

$$v_t = \sqrt{k\,R\,T_t} \quad \textbf{(25)}$$

Max mass flow rate for a given inlet pressure (p1), inlet temperature (T1), and throat area (At)

$$\dot{m} = \frac{A[t]\cdot v[t]}{V[t]}$$

$$\frac{d}{dt}\, m(t) = \frac{A_t\, v_t}{V_t} \tag{26}$$

$$\dot{m} = \frac{A[t]\cdot p[1]\cdot k\cdot \mathrm{sqrt}\left(\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\right)}{\mathrm{sqrt}(k\cdot R\cdot T[1])}$$

$$\frac{d}{dt}\, m(t) = \frac{A_t\, p_1\, k \sqrt{\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}}{\sqrt{k\, R\, T_1}} \tag{27}$$

For a supersonic nozzle the ratio between the
throat and any downstream area at which a pressure py prevails can
be expressed as a function of the pressure ratio and the ratio of
specific heats

$$\frac{A[t]}{A[y]} = \frac{V[t]v[y]}{V[y]v[t]}$$

$$\frac{A_t}{A_y} = \frac{V_t\, v_y}{V_y\, v_t} \tag{28}$$

$$\frac{A[t]}{A[y]} = \left(\frac{(k+1)}{2}\right)^{\frac{1}{k-1}}\cdot\left(\frac{p[y]}{p[1]}\right)^{\frac{1}{k}}\cdot \mathrm{sqrt}\left(\frac{k+1}{k-1}\cdot\left(1-\left(\frac{p[y]}{p[1]}\right)^{\frac{(k-1)}{k}}\right)\right)$$

$$\frac{A_t}{A_y} = \left(\frac{1}{2}\, k + \frac{1}{2}\right)^{\frac{1}{k-1}} \left(\frac{p_y}{p_1}\right)^{\frac{1}{k}} \sqrt{\frac{(k+1)\left(1-\left(\frac{p_y}{p_1}\right)^{\frac{k-1}{k}}\right)}{k-1}} \tag{29}$$

When $py = p2$ then

$$\frac{A[y]}{A[t]} = \frac{A[2]}{A[t]}$$

$$\frac{A_y}{A_t} = \frac{A_2}{A_t} \tag{30}$$

Thrust & Thrust Coefficient

Instantaneous thrust

$$F = \dot{m}\cdot v[2] + p[2]\cdot A[2]$$

$$F = \left(\frac{d}{dt}\, m(t)\right) v_2 + p_2\, A_2 \tag{31}$$

Ideal thrust equation (k is constant)

$$F = A[t]\cdot p[t]\cdot \mathrm{sqrt}\left(\frac{2\cdot k^2}{k-1}\cdot\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\cdot\left(1-\left(\frac{p[2]}{p[1]}\right)^{\frac{(k-1)}{k}}\right)\right) + p[2]\cdot A[2]$$

$$F = A_t p_t \sqrt{2} \sqrt{\frac{k^2 \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}} \left(1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right)}{k-1}} + p_2 A_2 \qquad \textbf{(32)}$$

Thrust coefficient (dimensionless)

$$C[F] = \frac{v[2]^2 A[2]}{o[1] \cdot A[t] \cdot V[2]} + \frac{p[2] \cdot A[2]}{p[1] \cdot A[t]}$$

$$C_F = \frac{v_2^2 A_2}{o_1 A_t V_2} + \frac{p_2 A_2}{p_1 A_t} \qquad \textbf{(33)}$$

$$F = C[F] \cdot A[t] \cdot p[1]$$

$$F = C_F A_t p_1 \qquad \textbf{(34)}$$

Characteristic Velocity & Specific Impulse

Characteristic velocity is used to compare propellants or chamber designs.

$$cstar = \frac{p[1] \cdot A[t]}{\dot{m}}$$

$$cstar = \frac{p_1 A_t}{\frac{d}{dt} m(t)} \qquad \textbf{(35)}$$

$$cstar = \frac{I[s] \cdot g[0]}{C[F]}$$

$$cstar = \frac{\mathrm{I}[s]\, g_0}{C_F} \qquad \textbf{(36)}$$

$$cstar = \frac{c}{C[F]}$$

$$cstar = \frac{c}{C_F} \qquad \textbf{(37)}$$

$$cstar = \frac{\mathrm{sqrt}(k \cdot R \cdot T[1])}{k \cdot \mathrm{sqrt}\left(\frac{2}{k+1^{\frac{(k+1)}{(k-1)}}}\right)}$$

$$cstar = \frac{1}{2} \frac{\sqrt{k R T_1}\,\sqrt{2}}{k \sqrt{\frac{1}{k+1}}} \qquad \textbf{(38)}$$

Influence of Chamber Geometry

When the chamber has a cross section that is larger than about four times the throat area, the chamber velocity v1, can be neglected.

See Table 3.2 on page 146 to show how chamber diameter vs throat diameter affects pt, F, and Is. Long story short: aim for $\frac{A[1]}{A[t]} > 4$

Nozzle Performance

Correction Factors

Performance Parameters

Each performance parameter, such as F, Is, c, v2, and/or , should be accompanied by a clear definition of the conditions under which it applies. Not all the items below apply to every one of the parameters.

a. Chamber pressure; also, for slender chambers, the location where this pressure prevails or is measured (e.g., at nozzle entrance).

b. Ambient pressure or altitude or space (vacuum).

c. Nozzle expansion area ratio and whether this is an optimum.

d. Nozzle shape and exit angle (see Table 3.3 and Fig. 3.12).

e. Propellants, their composition or mixture ratio.

f. Key assumptions and corrections made in the calculations of the theoretical performance: for example, was frozen or shifting equilibrium used in the analysis? (This is described in Chapter
5.)

g. Initial ambient temperature of propellants, prior to start.