Complexity Project: The Oslo Model

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Abstract:

The Oslo model is an algorithm to describe the evolution of a slowly driven ricepile. The model was used to investigate self-organised criticality and corrections to scaling from system size. The heights and the avalanche-sizes were analysed in detail for different system sizes. The heights were found to increase with a power law and tend to a constant value with time. On the other hand, the avalanche-size probabilities decreased with a power law and then decayed rapidly. Both observations show corrections to scaling in small system sizes.

Word count: 2483

1 Intorduction

The Oslo model is an example of self-organised criticality. The model was developed by adapting the one-dimensional BTW model which works on the sandpile metaphor. When the sandpile is slowly driven, by adding grains, the pile spontaneously organises itself into a critical state where a further grain could trigger an avalanche. Although, the Oslo model considers an experiment of a ricepile. The pile is driven from one end and open at the other. To model the dynamics of the ricepile more accurately, the critical gradient between the heights of the sites were chosen between 1 and 2 at random. This additional dynamic captures the fluctuation in the gradient along the profile of the ricepile.

2 Self-Organised Critical Model

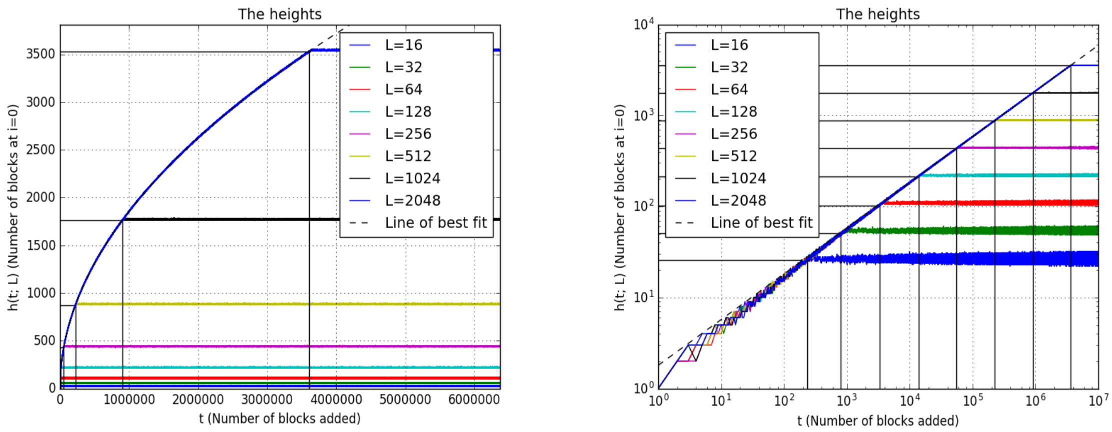
2.1 Implementation of the Oslo model.

Task 1:

The model was tested by changing the probabilities of the threshold slopes. At a probability of 1, the value of the threshold for the slope is only 1. The expected value of the heights would then be exactly the system size (L) once the system has reached the steady state, as predicted by the BTW model. The same can be found when changing the probability to 0. The threshold will be 2, producing a height of 2L in the steady state. The model conformed to these tests with no uncertainty in the heights. Further to this test, the time taken to reach the steady state can also be tested. This can be assumed to be the gradient multiplied by the area of the pile. Therefore, the predicted gradients should be $\frac{L^2}{2}$ and L^2. These approximately coincide with the measured values.

2.2 The height of the pile.

Task 2a:



(a) linear graph (b) log-log graph

Figure 1: The graphs show the height of the pile against the time. The height of the pile is the height of the first element of the system. This is also the only position that is driven, thereby grains are only added to it. The time is the number of grains added and the height is recorded after the system has completely relaxed (all the gradients are below their critical gradients). These figures show that there are two distinct configurations as the system increases with time. The first region increases with a power law and the second is constant.

There are clearly two distinct regions shown by different gradients in the heights. The Transient configuration is when the pile can grow without any grains being lost from the system. During this configuration, there is no dependence on the system size; therefore, the height of the pile will increase in the same way for all system sizes. The recurrent configuration occurs when the pile has reached the limit of its system size. Once a grain reaches the edge of the system, it falls off and is removed. In the steady state, the average number of grains added is equal to the average number of grains leaving. This produces an almost constant height. For an infinitely large system, this can be written as:

$$\langle h(t; L) \rangle = \langle z \rangle L \tag{2.1}$$

The cross-over time (t_c) is the time taken to reach the recurrent configuration. This is the first time where,

$$\langle inputflux \rangle = \langle outputflux \rangle. \tag{2.2}$$

As time is taken as the number of grains added, this is the number of grains required to reach the steady state. The cross-over time is, therefore, the area of the pile when the system has reached the steady state. In the limit of large systems, this is a triangle which is proportional to L^2. Due to the discrete number of blocks, a Riemann sum can be used to find that the actual area is

$$t_c = \langle z \rangle \frac{L^2}{2} \left(1 - \frac{1}{L}\right) \tag{2.3}$$

where $\langle z \rangle$ is the average gradient. The limits of the cross-over time can be found using the threshold slopes:

$$\frac{L^2}{2} \left(1 - \frac{1}{L}\right) < t_c < L^2(1 - \frac{1}{L}) \tag{2.4}$$

The theory corresponds to the actual values, with the measured exponent using large system sizes equaling 2.01 which is approximately 2.

Task 2b:

A data collapse can be performed using the scaling relationships found in the previous question. The height, during the recurrent configuration, scales with system size. By dividing the height by the system size, this will collapse the heights for different systems onto the same line. The time can then be divided by t_c, collapsing the cross-overtimes onto the same point. This can be written as an equation for the height in terms of a scaling function.

$$\frac{\tilde{h}(t; L)}{L} \propto \mathcal{F}\left(\frac{t}{t_c}\right) \tag{2.5}$$

Equation 2.5 can be rearranged to give the height in terms of the scaling relationship, the system size and the time. Note, that $\tilde{h}$ stands for the moving average of the height which is used to smooth out the data. The cross-over time scales as a power law of the system size (L^D), therefore this can be replaced in the equation.

$$\tilde{h}(t; L) \propto L\mathcal{F}\left(\frac{t}{L^D}\right) \tag{2.6}$$

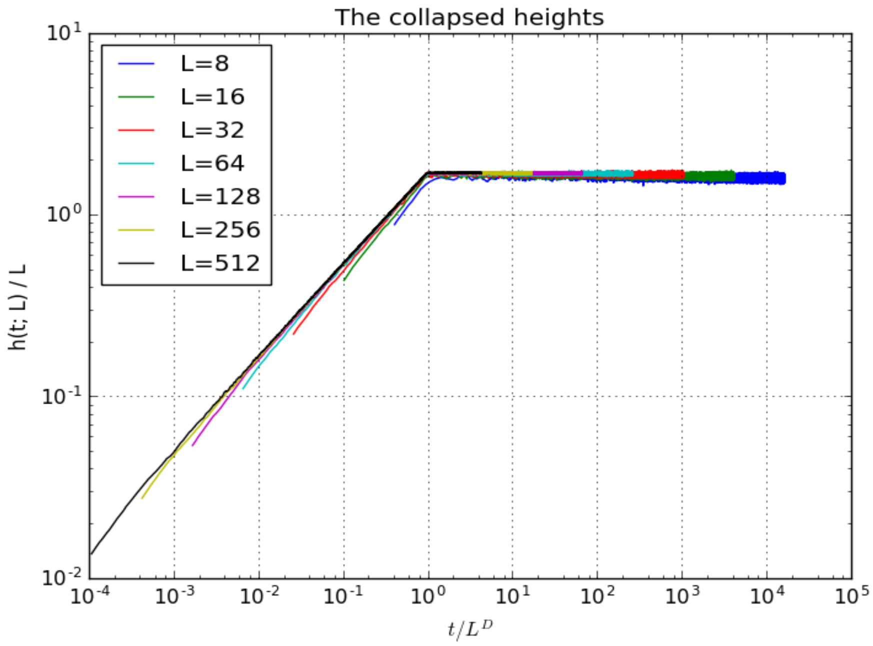


Figure 2: The graph shows that the scaling equation leads to each systems' line collapsing on top of each other. This is evidence of a data collapse of the heights, therefore showing that equation 2.6 is correct. The value of D used for the data collapse was 2. The small system sizes can be seen to deviate from the straight line of the larger system sizes in the transient configuration. This effect is from finite-size scaling.

At larger times than t_c, the system is in the recurrent configuration and the height is constant. The argument is larger than 1 due to the time being greater than the cross-over time. From the previous analysis, the height is proportional to the system size in this configuration; therefore, the scaling function ($\mathcal{F}(x)$) must tend towards a constant value. When the argument (x) is small, the system is in the transient configuration and the height increases with time. This increase is a power law, therefore, $\mathcal{F}(x)$ must increase with a power law. As the height, in this configuration, does not depend on the system size, the L dependence in $\mathcal{F}(x)$ must cancel with L. This can only happen if the $\mathcal{F}(x)$ is a power law to $1/D$ corresponding to the D in $t_c \propto L^D$.

$$\mathcal{F}(x) \propto constant \qquad x \gg 1 \tag{2.7}$$

$$\mathcal{F}(x) \propto x^{\frac{1}{D}} \qquad x \ll 1 \tag{2.8}$$

This can be substituted into equation 2.6.

$$\tilde{h}(t;L) \propto L \left(\frac{t}{L^D}\right)^{\frac{1}{D}} \tag{2.9}$$

The system size cancels leaving the height scaling with time. Theoretically, for an infinite system, D becomes 2. Therefore, the equation for the height in the transient, as the system size goes to infinity, is

$$\tilde{h}(t; L) \propto t^{\frac{1}{2}}. \tag{2.10}$$

This value can be obtained from the log-log gradient of the transient region. The value found was 0.50 which is in conjunction with the theoretical value. Moment analysis could be performed to find a value with greater precision.

Task 2c:

The average height seems to scale linearly with system size. Although, if the average height is divided by the system size, the height tends to towards a value. This has isolated the correction to scaling of the height. Therefore, the height is no longer trivially equation 2.1. The average height can now be written with correction to scaling.

$$\langle h(t; L) \rangle = a_0 L \left(1 - a_1 L^{-\omega_1} + a_2 L^{-\omega_2} + ...\right) \tag{2.11}$$

The higher order terms ($i > 1$) can be neglected by assuming the corrections are small.

$$\langle h(t; L) \rangle = a_0 L \left(1 - a_1 L^{-\omega_1}\right) \tag{2.12}$$

The resulting equation can be rearranged, using logs, to find an equation which isolates a_0 and ω_1.

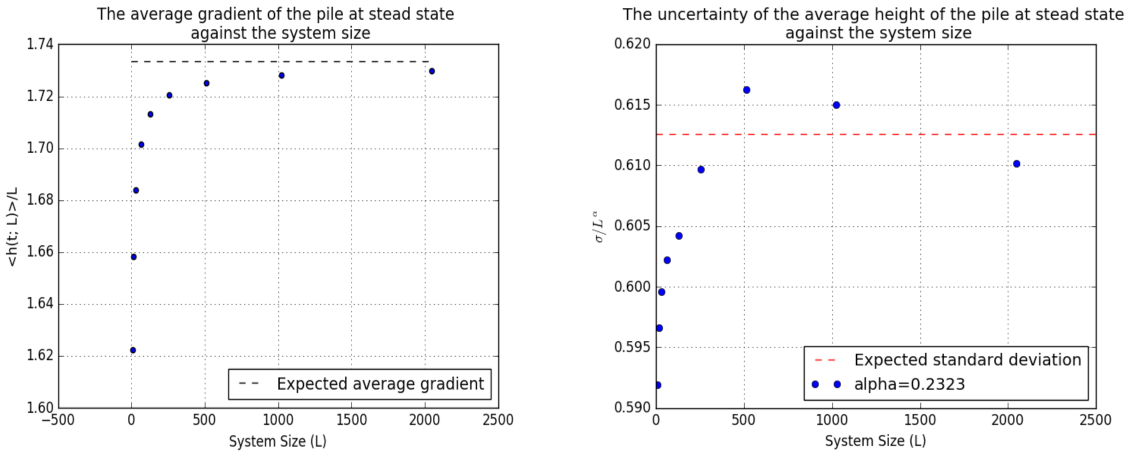
$$\log\left(1 - \frac{\langle h(t; L) \rangle}{a_0 L}\right) = -\omega_1 \log L + \log a_1 \tag{2.13}$$

This equation should produce a straight line with the correct value of a_0 and the given data for L and $\langle h(t; L) \rangle$. The correct value of a_0 can be found by minimising 1 - R-squared of a straight line fit, where R-squared is a statistic measure on the line of best fit. This method is better than using a fitting function because it reduces the uncertainty by only minimising one parameter. The gradient of the straight line found is ω_1. The value of a_0 was 1.73 and ω_1 was 0.64.

The standard deviation seems to scale with a power law of the system size. The exponent is less than 1 thereby making the σ_h tend to a value at an infinite system size. From the previous observation, the standard deviation requires a different correction to scaling formula due to it not being linearly dependent on system size. The adjusted correction to scaling formula can be proposed as:

$$\sigma_{\langle h \rangle} = b_0 L^{\alpha} \left(1 - b_1 L^{-\gamma_1}\right) \tag{2.14}$$

An approximation of the exponent can be found from the gradient of a log-log graph, and b_0 the intercept. The line was fitted using the last four points thereby removing some of the correction to scaling from the smaller system sizes. The standard deviation can then be divided by L^α to reveal the correction to scaling. α was 0.23.



(a) Correction to scaling in the height

(b) Correction to scaling in σ_h

Figure 3: The correction to scaling in the height is also the gradient of the pile. This can be seen to tend to a constant value of a_0. This is only possible to see with very large system sizes. This clearly shows that the gradients and the heights contain corrections to scaling for small system sizes. At an infinite system size, the gradient will tend to a constant value. The standard deviation contains some correction to scaling but the uncertainty in these values is very large. The correction to scaling is not as pronounced as the height.

The average slope, in the recurrent configuration, can be calculated by dividing the height by the system size. This is the same as the correction to scaling of the heights. The gradient is found to tend to an average gradient as the system size tends to infinity. An accurate approximation of this value is a_0 which is 1.73. The standard deviation of the gradient can be found from the standard deviation of the heights ($\sigma_{\langle h\rangle}$) divided by the system size (L). As $\sigma_{\langle h\rangle}$ is proportional to L^α, the $\sigma_{\langle z\rangle}$ is proportional to $L^{\alpha-1}$. The value of $\alpha - 1$ is negative therefore $\sigma_{\langle z\rangle}$ tends to zero.

Task 2d:

The heights in the recurrent configuration tend to a constant value, with an uncertainty, due to the limit on the system size. This constant value can be approximated by a Gaussian distribution, with the width being the standard deviation of the heights. The equation for the probability is:

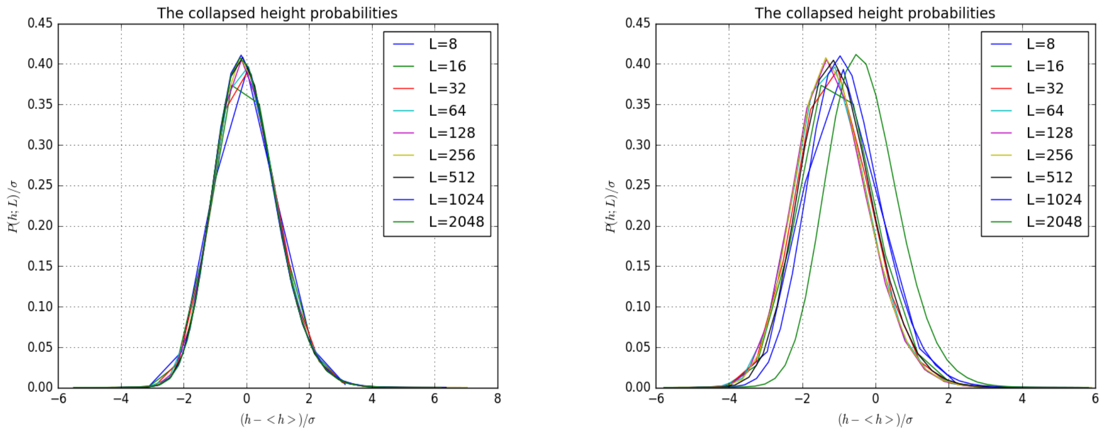
$$P(h; L) = \frac{1}{\sigma_h\sqrt{2\pi}} e^{\frac{-(h-\langle h\rangle)^2}{2{\sigma_h}^2}} \tag{2.15}$$

The Gaussian distribution can be transformed into a standard normal distribution by having a standard deviation of 1 and an average height of 0.

$$P(h;L) = \frac{1}{\sqrt{2\pi}} e^{\frac{-h^2}{2}} \tag{2.16}$$

The dependence of the probability can be multiplied by the standard deviation to remove its dependence, therefore scaling the y-axis. The exponential can be replaced with a scaling function in terms of the x-axis. Therefore, the average height needs to be taken away from the height and divided by the standard deviation to scale the x-axis.

$$P(h;L)\sigma_h \propto \mathcal{N}\left(\frac{h - \langle h \rangle}{\sigma_h}\right) \tag{2.17}$$



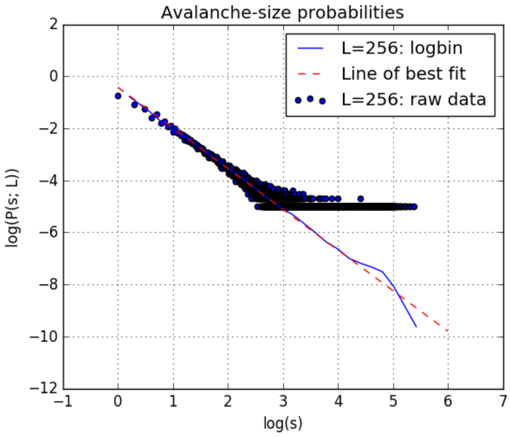
(a) Actual values

(b) Functional form

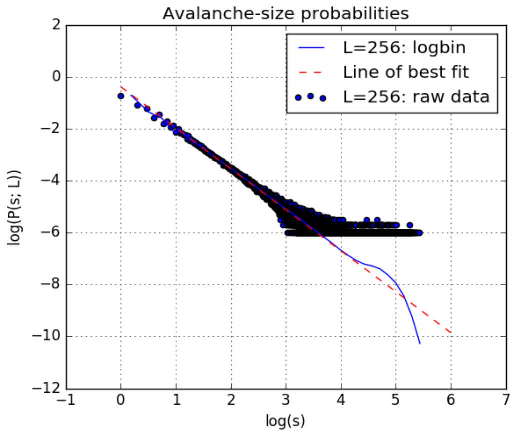
Figure 4: The graphs show the collapsed height probabilities using a Gaussian approximation for the shape of the graphs. (a) uses the actual values found for σ and $\langle h \rangle$. (b) uses the functional form of σ and $\langle h \rangle$ found in the analysis of Task 2c, but ignoring i=1 as well. This additional assumption allows corrections of scaling to be seen in the data collapse. The widths of the distributions are similar for system size but the position on the x-axis change. The higher system sizes are closer to being centred around zero because they are less effected by the missing correction to scaling. There is a more defined shape for the larger system sizes due to their greater range of values of height.

2.3 The avalanche-size probability.

Task 3a:



(a) a=1.6, N=10^5



(b) a=1.4, N=10^7

Figure 5: The log-log graph shows the avalanche-size probabilities against the avalanche-sizes. The raw data and the logbin data are plotted on the same graph. The logbin line shows that probabilities are linear with the avalanche-sizes on a log-log graph until a particular point where the probability decreases rapidly. This rapid decrease is due to finite-size scaling. Increasing the number of total avalanches and decreasing 'a' gives a greater resolution for the log bin line.

The avalanche-size probability decreases with larger avalanche-sizes. If a grain is introduced into a site, it will have a probability of falling. This probability is large enough so that there is a reasonable chance of an avalanche. Although larger avalanches are more complex, the probability of a larger avalanche is a multiplication of small avalanche probabilities. The overall probability will decrease with an increasing number of avalanches. This means that it is intuitive that a small avalanche-size is more likely than a large one. On a log-log scale, the decrease in probability is linear with the avalanche-size; therefore, the probability scales with a power law to the avalanche-size. At very large avalanche-sizes, the probability decays rapidly which is due to the finite-size scaling of the system. The binned probabilities allow these relationships to be seen clearly and compared to different system sizes.

The values for the total number of avalanches used (N) and the base for the log binning (a) are chosen to optimise the smoothness and the resolution of the line. The larger N, the more avalanches there are to log bin and the log binning improves. Values of 'a' that are too small make a jerky line but too large removes resolution. The value of N was chosen by the maximum value that could be achieved in a reasonable time period, 10^7, and 'a' was found by sight, 1.4.

Task 3b:

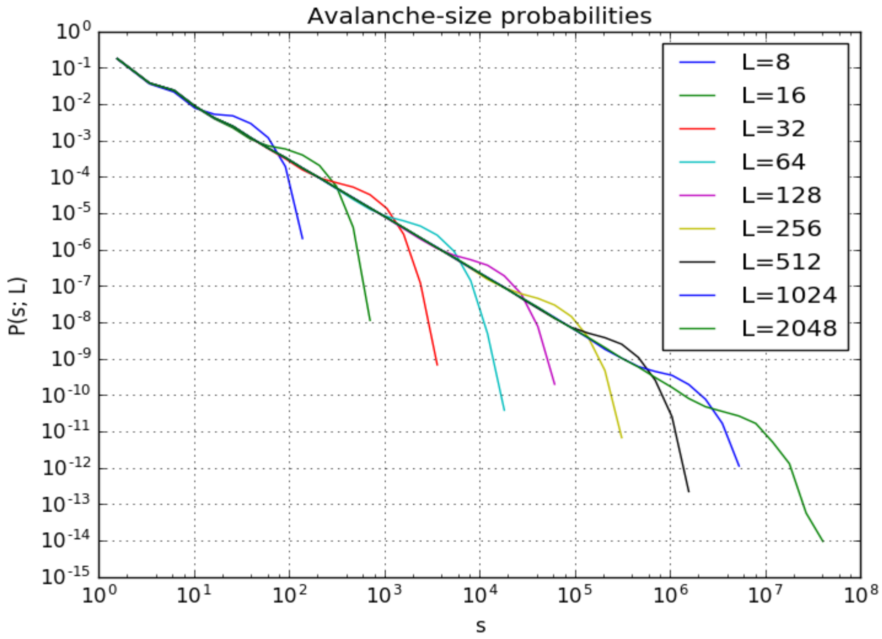


Figure 6: The log-log graph shows the avalanche-size probabilities against the avalanche-sizes for multiple system sizes. The bump in the data can be seen to scale with the system size. The larger the system size, the later the rapid decrease in probability occurs.

The data was processed by first removing any values that occur before the cross-over time; this analysis is only performed in the recurrent configuration. The raw avalanches were log binned, including zero avalanches for normalisation. The log binning was used to extract the information of the noisy tail present in the probability seen in figure 5. This can improve statistics on the region. The bins are exponentially increased in length and the number of avalanches are counted within that region.

Figure 6 show that there are two regions for the avalanche-size probabilities. The regions are separated by the cut-off avalanche-size (s_c). This cut-off scales with system size and can be assumed using finite-size scaling ansatz that this is a power law ($s_c \propto L^D$). Qualitatively, there is a theoretical limit on the avalanche-size which is where the whole system empties. Each grain can topple at most L times before dropping out of the system, therefore $s_c \leq L^3$. For each system size, there has to be a zero avalanche probability past this value. In this theoretical limit, D would equal 3. The maximum avalanche found is proportional to this limit; therefore, the s_c will scale as a power law.

The first region of the scaling function does not depend on system size, although there may be corrections to scaling. In this region, the avalanche-probabilities follow a power law relationship shown by the straight line on a log-log graph. An infinitely large system would only follow this line. In the second region, there is a rapid decay towards the maximal avalanche possible for that system. There is now a dependence on the system size. The bumps are a clear distinction of finite-size scaling because these are due to grains leaving the system in a system spanning avalanche. Grains are only able to leave a system with a finite system size.

Task 3c:

The avalanche-size probability can be tested if they are consistent with the finite-size scaling ansatz which is

$$\tilde{P}_N(s;L) \propto s^{-\tau_s}\mathcal{G}\left(s/L^D\right) \tag{2.18}$$

An approximation of τ_s can, therefore, be found from the gradient of the log-log graph. The distinctive bumps can be vertically aligned by multiplying each avalanche-size probability with s^{τ_s}. τ_s was 1.58. By assuming that the largest avalanche-size, for each system, is proportionate to the cut-off avalanche size, the D coefficient can be approximated. The D coefficient is, therefore, found from the gradient of the log-log plot of the last points against the system sizes. The bumps can be horizontally aligned by scaling the avalanche-size with the cutoff avalanche-size $s_c(L) \propto L^D$. D was found to be 2.19. These values will have a large error associated with them.

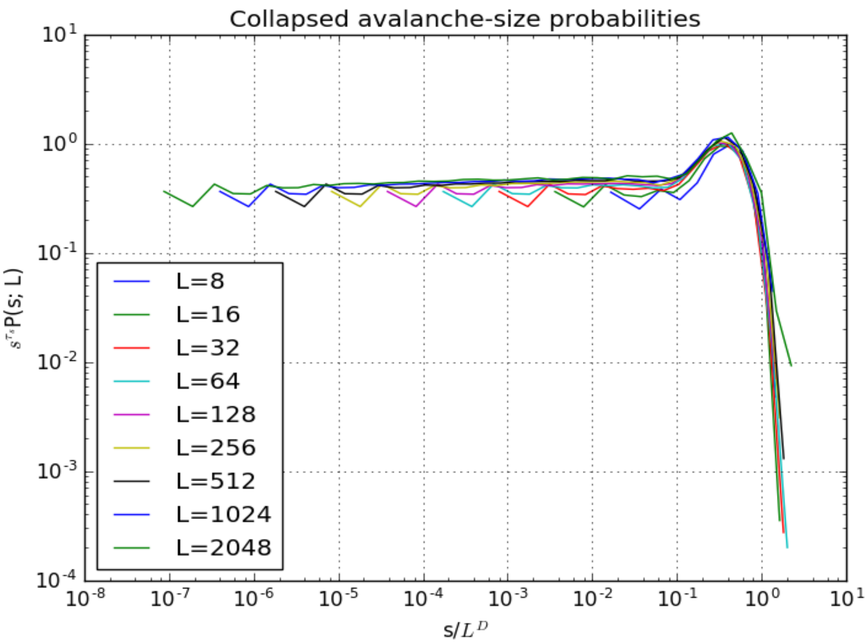


Figure 7: The graph shows the data collapse of the avalanche-size probabilities using scaling ansatz. The curves collapse on the graph for the scaling function showing that the ansatz is a good approximation. The values found do not perfectly collapse the avalanche-size probabilities but that can be improved by moment analysis.

The lines collapse onto the same curve thus providing evidence that the avalanche-size probability satisfies the simple finite-size scaling ansatz proposed in equation 2.18. The small size systems are further away from the largest system which is from larger corrections to scaling in smaller systems.

A scaling relationship can be derived between D and τ_s. There are two constraints which need to be applied to the avalanche-size probabilities: the probability must be normalised and have a diverging first moment. The average average-size moment can be calculated using a sum over an avalanche of zero to infinity. Note, a zero avalanche is included for normalisation.

$$\langle s^k \rangle = \sum_{s=0}^{\infty} s^k P(s; L) \tag{2.19}$$

Using substitution of $u = s/L^D$, finite-size ansatz for the probability and approximating the sum as an integral, the equation can be rewritten as

$$\langle s^k \rangle = L^{D(1+k-\tau_s)} \int_0^{\infty} u^{k-\tau_s} \mathcal{G}(u) du \tag{2.20}$$

The use of integral instead of a sum can only be approximated for infinitely large system sizes. Therefore, as the system size tends to infinity, the integral in equation 2.20 will converge to a constant value.

$$\langle s^k \rangle \propto L^{D(1+k-\tau_s)} \tag{2.21}$$

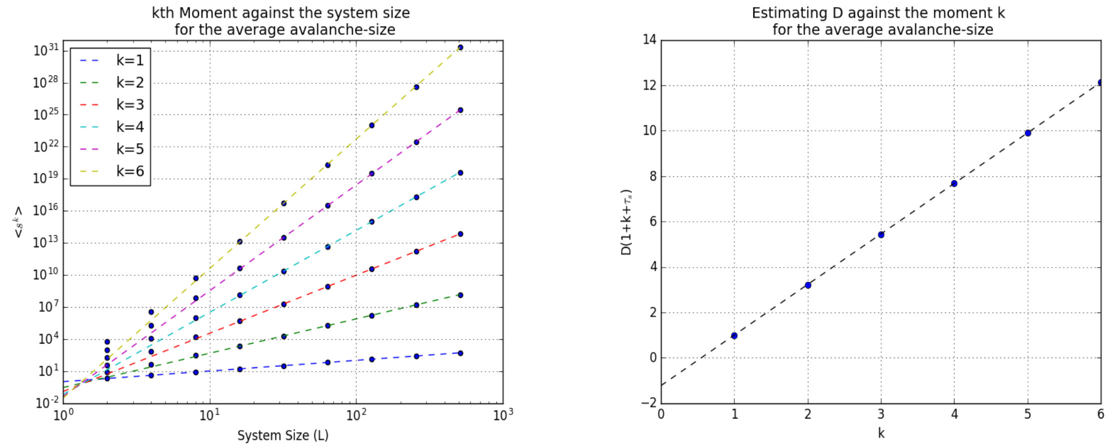
For the first moment, the exponent of L is $D(2 - \tau_s)$. Considering that the system is in the recurrent configuration, the average influx is equal to average outflux of grains. Therefore, if a grain is added, a grain will on average fall to the end of the system to maintain this steady state. This can be clearly shown in the BTW model where a grain added will always drop out the system, causing an avalanche of L. The averaged first moment is, therefore, equal to L. Combining these,

$$D(2 - \tau_s) = 1 \tag{2.22}$$

The values found for D and τ_s, from the approximation, give a value of 0.92 for the scaling relationship. The difference in the number could be due to corrections to scaling from smaller system sizes.

Task 3d:

The k'th moment will scale almost linearly on a log-log graph. Although, at small system sizes there will be corrections to scaling. These are caused by approximating the sum as an integral and the entire integral not converging to a constant value. These lower system sizes should not be included when finding the gradient of the moment, on a log-log graph.



(a) Moments

(b) Finding D and τ_s

Figure 8: Graph (a) shows that the moment of the average avalanche-size increases with system size. A line of best fit has been plotted through each moment. Only the last four points were chosen to find the line because the last points were assumed to have less corrections to scaling. At low system sizes, the $\langle s^k \rangle$ found deviates from the line of best fit. (b) finds D and τ_s using the gradients of the moment. A linear regression can be used to find the gradient D.

Figure 8 is consistent with the prediction of the scaling in small system sizes. The lower system sizes overestimate the value for $\langle s^k \rangle$. The higher moments, exaggerate the effect giving larger corrections to scaling in smaller system sizes. If the small system sizes are included in the moment analysis, the gradient of each moment will be an underestimate.

The gradient $(D(1 + k - \tau_s))$ of each moment, from the log-log graph of figure 8, can be used to find the coefficients D and τ_s.

$$k \log\langle s \rangle \propto D(1 + k - \tau_s) \log L \tag{2.23}$$

By plotting the gradient against k, a straight line should be found.

$$D(1 + k - \tau_s) = mk + c \tag{2.24}$$

The straight line can be used to find D, the gradient, and τ_s, $1 - \frac{c}{D}$. These values should have minimal correction to scaling.

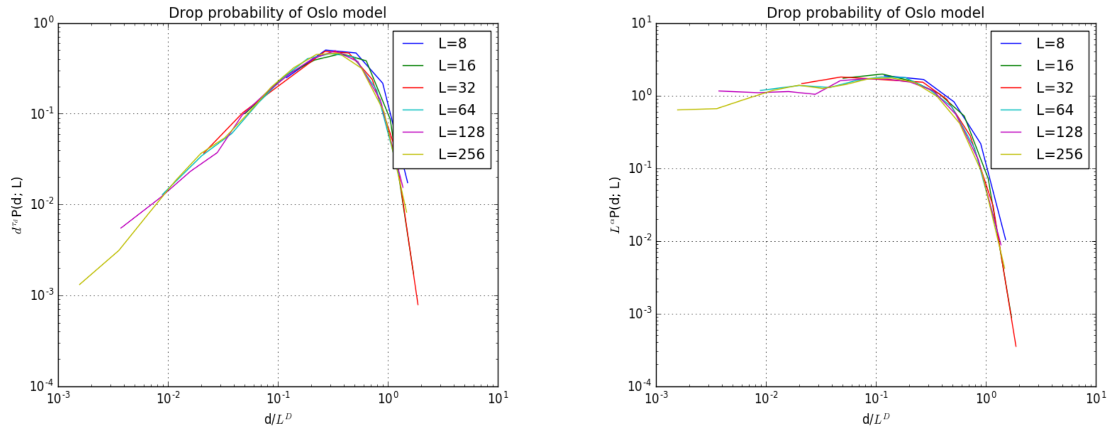
The moment analysis found $D = 2.23$ and $\tau_s = 1.57$. These are closer to the theoretical values of $\frac{9}{4}$ and $\frac{14}{9}$, respectively. D and τ_s also conforms to the scaling relation, giving a value of 0.97. These are much closer to the theoretical values than the values found in Task 3c are. The values found earlier include correction to scaling. The disparity in the results shows that there are corrections to scaling in the numerical data.

3 Bonus section

3.1 Outflux of grains in the Oslo model.

BONUS TASK 1:

The drop-size (output flux) was found for the Oslo model. Two forms of the data collapse were performed on the data.



(a) τ_d=1.0 and D=1.25

(b) α=1.2 and D=1.25

Figure 9: The graphs above show the two options to collapse the drop size. (a) uses the finite-scaling ansatz which was used by the avalanche-size probabilities. (b) uses the system size which was used in the height analysis. The values for τ_d, α and D were found by sight.

4 Conclusion

The heights and the avalanche-sizes have been analysed for the Oslo model. The height of the pile was found to have two regions: the transient and recurrent. In the recurrent region, the height scaled linearly with system size. The cross-over time was found to scale as a power law with the system size. These relationships enabled the heights for different systems to be collapsed onto a single line. Further analysis of the average height showed correction to scaling in small system sizes. This led to the average gradient of the pile tending to a constant value at an infinite system size. The avalanche-size probabilities were found to scale with finite-size scaling ansatz. Values for D and τ_s were roughly estimated. Moment analysis was used to find these values again and provided evidence that corrections to scaling exist, from the system size, in the avalanche-size probabilities.

References

[1] K. Christensen and N. R. Maloney, *Complexity & Networks: Complexity*, Imperial College Press, London, 2005.