TOPIC 3: MODELLING OF REAL-LIFE MIP PROBLEMS

The Ikunbelievable Group*

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1 Introduction

Lily has recently been accepted by the London School of Economics and Political Science, majored in Operational Research and her boyfriend, David, is set to begin his studies at Imperial College London. Eager to make the most of their summer vacation in Europe before the start of the school term, they decide to embark on a memorable 2-week trip, exploring famous European cities.

David, a music enthusiast, wants to visit Vienna, and Lily is keen on the Van Gogh Museum in Amsterdam. Considering their interests, they choose nine destinations: Vienna, Paris, Rome, Barcelona, Berlin, Amsterdam, Copenhagen, Zurich, and Budapest. They prefer traveling by train and plane, but due to Lily's past witness of a tragic plane crash, they will limit their plane trips.

To plan their itinerary, they must organize their routes considering time, cost, and visiting all desired destinations while accounting for varying train and plane ticket prices. Unsure of where to start, they seek assistance from the Ikunbelievable Group.

2 Problem Description

2.1 Input and Objective

1. Cities: For their graduation trip, they have chosen 10 representative cities, including London, Vienna, Paris, Barcelona, Berlin, Amsterdam, Copenhagen (København), Zurich, and Budapest.
2. Transportation Options: They will consider to take plane or train to travel between these cities.
3. Time: We will consider the time required to travel between cities by plane and train.
4. Price: Since ticket prices can vary from month to month, in early May, we calculated the average price of tickets from July 1 to July 14, from 8 AM to 8 PM.
5. Customization: To account for individual preferences, we will input different weights for cost and time, allowing each traveler to prioritize factors that are most important to them when selecting the optimal travel itinerary.

Their objective is to find the optimal travel route under the assumptions and the constraints described below.

2.2 Assumptions

We make the following assumptions to develop a comprehensive travel plan that caters to different individual preferences.

1. The travel time and cost between city A and city B are identical in both directions.
2. Two modes of transportation, train and plane, are considered, despite the existence of alternative choices.
3. The trip must begin and end in a specific city, in Lily and David's case, London, as they need to store their luggage at the school in London and take only essential items for the journey.
4. Considering factors like public health events and peak travel seasons, we averaged ticket prices between July 1st and 14th, from 8 AM to 8 PM, to determine ticket prices and minimize the impact of outliers.

2.3 Constraints

Our goal is to identify the best transportation options for various groups with diverse needs, enabling efficient travel across all cities. Therefore, some constraints are as followed:

1. Each city can only be visited once.

2. We set maximum limits for plane and train usage because some individuals may have constraints on taking planes or trains due to personal preferences, coupons for free food, or memberships with airlines or train companies providing extra luxury services.

2.4 Toy Example

Now we demonstrate a simple example to show the consequence of implementing different traveling routes.

2.4.1 Toy Example's Input

- Cities: For the trip in the summer vocation, traveling from Anhui Province to Guangdong Province, choose 3 representative cities, including Guangzhou, Shenzhen, and Foshan.
- Transportation Options: Only consider to take plane or train to travel between these cities.
- Time: Only consider the time required to travel between cities by plane and train.
- Price: The fares used were based on real prices observed around one month in advance, in June 2022.
- Customization: Only consider the total cost, and don't care about the time.

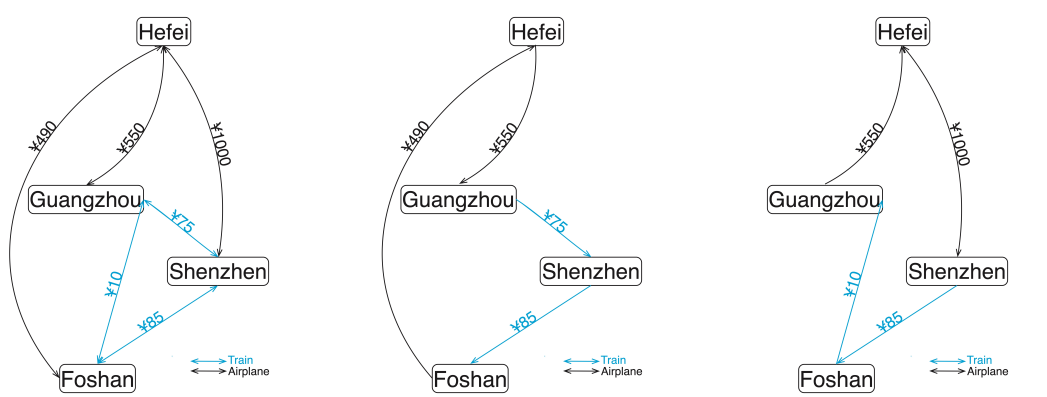
2.4.2 Toy Example's Assumptions

- The travel time and cost between city A and city B are identical in both directions.
- The trip starts and ends in Hefei.

2.4.3 Toy Example's Constraints

- Each city can only be visited once.
- No limit to the number of times for taking trains and planes.

The ticket prices and travel routes are displayed with black lines representing airplane routes and blue lines indicating train routes. In Figure 1b, the optimal choice is priced at ¥1200. Figure 1c presents an alternative option that costs ¥1645, resulting in an increased expenditure of approximately ¥450.



(a) Ticket Prices (b) Optimal Choice (c) Choice With Higher Price

Figure 1: Toy Example

3 MIP MODEL

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two given graphs, each with their respective vertex and edge sets. Define a positive edge weight function $w : E_1 \cup E_2 \to \mathbb{R}^+$. We construct a new graph $G(V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The objective is to find a Hamiltonian cycle in G that minimizes the objective function. In order to achieve this, we introduce binary variables x_{ij}^k for edge selection from either G_1 or G_2.

Formally, let $x_{ij}^k \in \{0, 1\}$ be a binary variable such that:

$$x_{ij}^k = \begin{cases} 1, & \text{if edge } (i,j) \text{ from graph } G_k \text{ is selected,} \\ 0, & \text{otherwise.} \end{cases}$$

The goal is to determine the values of x_{ij}^k that lead to a Hamiltonian cycle with the minimum objective function value in the graph G. It can be written as:

$$\text{minimize } f = \sum_{k=1}^{2} \sum_{(i,j) \in E_k} w_{ij}^k x_{ij}^k$$

Subject to the following **constraints**:

1. Each vertex has a total degree of 2, with one incoming edge (in-degree) and one outgoing edge (out-degree) from either graph G_1 or G_2.

$$\sum_{k=1}^{2} \sum_{j \in V} x_{ij}^k = 2, \quad \forall i \in V$$

2. No subtours are allowed (subtour elimination constraint):

$$\sum_{(i,j) \in S \times (V \setminus S)} x_{ij}^k \geq 2, \quad \forall S \subset V, S \neq \emptyset, S \neq V, k \in \{1, 2\} \tag{1}$$

Here, S is a subset of V, and $(V \setminus S)$ is the complement of S in V.equation 1 ensures that there are no smaller cycles within the Hamiltonian cycle.

3. The number of edges selected from G_1 and G_2 are no more than N_1 and N_2 respectively.

$$\sum_{(i,j) \in E_1} x_{ij}^1 \leq N_1 \quad \text{and} \quad \sum_{(i,j) \in E_2} x_{ij}^2 \leq N_2, \quad \forall i, j \in V$$

VARIABLES AND PARAMETERS

- $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ store the information of airplane's and train's transportation modes.
- The vertices V_1 and V_2 in each graph represent cities in different countries.
- k is an index representing the transportation mode in the given graphs. Specifically, $k = 1$ corresponds to the airplane transportation mode in graph G_1, and $k = 2$ corresponds to the train transportation mode in graph G_2.
- If there is an edge between the vertices, it indicates that travel between the cities is accessible using either mode of transportation. We can represent an edge between vertex i and j as $(i, j) \in E_k$, where $k \in \{1, 2\}$.
- The edges store $cost$ and $time$, which represent the one-way ticket price and travel time, respectively. So the weight function w is defined as:

$$w_{ij}^k = \alpha \cdot \text{cost}_{ij}^k + \beta \cdot \text{time}_{ij}^k, \quad where \quad \alpha + \beta = 1$$

where α and β are the weights of the cost and time attributes respectively.

Therefore, our objective function can be further written as:

$$\text{minimize } f = \sum_{k=1}^{2} \sum_{(i,j) \in E_k} (\alpha \cdot \text{cost}_{ij}^{k} + \beta \cdot \text{time}_{ij}^{k}) x_{ij}^{k}$$

4 Computational Results

To find how different constraints and weight functions influence the consequence, we write a Python solver with held-kalp ("See Held & Karp (1962) for more information") and the raw data and code are shown in the Appenix.

After evaluating unconstrained air and train travel options, we present three recommendations based on varying cost and time priorities. Our objective function f incorporates α and β as weights. In Table 1 ($\alpha = 1, \beta = 0$) highlights the least expensive route, while Table 2 ($\alpha = 0, \beta = 1$) showcases the quickest option. Table 3 ($\alpha = 0.3, \beta = 0.7$) offers a balanced choice, taking only 20 minutes longer than the fastest plan but saving 1,300 yuan, making it a potentially superior option after extensive iterations.

Segment	Transportation Mode	Cost (¥)	Time (hours)
London → Barcelona	Airplane	532.0	2.5
Barcelona → Paris	Airplane	590.0	2.0
Paris → Rome	Airplane	292.0	2.25
Rome → Vienna	Airplane	344.0	1.45
Vienna → Budapest	Train	175.0	2.63
Budapest → Berlin	Train	420.0	14.0
Berlin → Copenhagen	Train	240.0	5.0
Copenhagen → Amsterdam	Train	469.0	16.0
Amsterdam → Zurich	Train	525.0	10.0
Zurich → London	Airplane	616.0	1.83
Total		4203.0	57.66

Table 1: $\alpha = 1, \beta = 0, N_1 = N_2 = 10$

Segment	Transportation Mode	Cost (¥)	Time (hours)
London → Paris	Airplane	459.0	1.05
Paris → Budapest	Airplane	1171.0	0.83
Budapest → Vienna	Airplane	794.0	0.83
Vienna → Rome	Airplane	344.0	1.45
Rome → Barcelona	Airplane	634.0	2.0
Barcelona → Zurich	Airplane	760.0	1.92
Zurich → Copenhagen	Airplane	855.0	1.75
Copenhagen → Berlin	Airplane	317.0	1.0
Berlin → Amsterdam	Airplane	967.0	1.4
Amsterdam → London	Airplane	940.0	1.1
Total		7241.0	13.33

Table 2: $\alpha = 0, \beta = 1, N_1 = N_2 = 10$

Since Lily has witnessed a plane crash, she hopes to limit the air travels within N times. However, none of the former results fits Lily's constraint of flight number. If we limit the flights within 5 times, the calculation shows that no matter what alpha is, the optimal route remains the same, which is shown in Table 4 when $N_1 = 4, N_2 = 10$.

Segment	Transportation Mode	Cost (¥)	Time (hours)
London → Paris	Airplane	459.0	1.05
Paris → Zurich	Airplane	608.0	1.3
Zurich → Barcelona	Airplane	760.0	1.92
Barcelona → Rome	Airplane	634.0	2.0
Rome → Vienna	Airplane	344.0	1.45
Vienna → Budapest	Airplane	794.0	0.83
Budapest → Berlin	Airplane	537.0	1.5
Berlin → Copenhagen	Airplane	317.0	1.0
Copenhagen → Amsterdam	Airplane	559.0	1.5
Amsterdam → London	Airplane	940.0	1.1
Total		5952.0	13.65

Table 3: $\alpha = 0.3, \beta = 0.7, N_1 = N_2 = 10$

Segment	Transportation Mode	Cost (¥)	Time (hours)
London → Copenhagen	Airplane	669.0	2.0
Copenhagen → Barcelona	Airplane	1214.0	3.0
Barcelona → Rome	Airplane	634.0	2.0
Rome → Budapest	Airplane	489.0	1.67
Budapest → Vienna	Train	175.0	2.63
Vienna → Zurich	Train	630.0	8.0
Zurich → Berlin	Train	455.0	9.0
Berlin → Amsterdam	Train	599.0	6.3
Amsterdam → Paris	Train	697.0	3.5
Paris → London	Train	1148.0	2.0
Total		6710.0	40.3

Table 4: $\alpha = 0.3, \beta = 0.7, N_1 = 4, N_2 = 10$

According to the calculation above, we recommend Lily to limit their times of flight into 6 times and consider $\alpha = 0.7, \beta = 0.3$ to meet the balance between cost and time, along with the fear for taking the plane.

5 Conclusion

We have presented an approach to find an optimal travel route for Lily and David, considering different transportation modes: airplanes and trains. We have formulated the problem as a Graph Structural Mixed Integer Problem, where nodes and edges represent information about the two transportation modes. A binary variable, x, is used to control whether a particular mode of transportation is chosen or not. We defined an objective function based on weighted costs and times to optimize the overall travel experience.

We analyzed various scenarios, giving different weights to cost and time attributes, as well as considering the constraint of limiting the number of flights due to Lily's fear of flying. Computational results show that when the number of flights is limited to 6 and the weight of cost is 0.7, while the weight of time is 0.3, we achieve a balanced travel plan that accommodates Lily's concerns and still provides a satisfactory travel experience for the group.

The research presented here can be extended in the future. As the complexity of the optimization algorithm itself is an NP-hard problem, the time required for the solver to find a solution may increase exponentially when the number of cities is too large. Therefore, optimizing the algorithm or finding local optimal solutions offers room for improvement. Moreover, measuring the concept of cost-effectiveness remains vague and is only derived from our observations after thousands of experiments. The difficulty in defining cost-effectiveness lies in its subjective nature, as it varies from person to person. However, our results are helpful to Lily and her friends, and we wish them a great time on their journey.

References

Michael Held and Richard M. Karp. A dynamic programming approach to sequencing problems. *Journal of the Society for Industrial and Applied Mathematics*, 10(1):196–210, 1962. doi: 10.1137/0110015. URL https://doi.org/10.1137/0110015.

A Appendix

	London	Vienna	Paris	Rome	Barcelona	Berlin	Amsterdam	København	Zurich	Budapest
London	0	545	459	769	532	559	940	669	616	296
Vienna	545	0	868	344	742	880	1367	931	3456	794
Paris	459	868	0	292	590	635	921	718	608	1171
Rome	769	344	292	0	634	820	1819	1109	690	489
Barcelona	532	742	590	634	0	840	1104	1214	760	728
Berlin	559	880	635	820	840	0	967	317	N/A	537
Amsterdam	940	1367	921	1819	1104	967	0	559	1049	1269
København	669	931	718	1109	1214	317	559	0	855	613
Zurich	616	3456	608	690	760	N/A	1049	855	0	2650
Budapest	296	794	1171	489	728	537	1269	613	2650	0

Table 5: Travel cost (¥) of Airplane between cities in July

	London	Vienna	Paris	Rome	Barcelona	Berlin	Amsterdam	København	Zurich	Budapest
London	0	1351	1148	N/A	2541	1015	753	1680	1155	3150
Vienna	1351	0	1072	365	1778	625	552	565	630	175
Paris	1148	1072	0	770	1386	691	697	1078	1120	980
Rome	N/A	365	770	0	N/A	1036	974	N/A	840	1750
Barcelona	2541	1778	1386	N/A	0	2249	2073	2940	2450	4060
Berlin	1015	625	691	1036	2249	0	599	240	455	420
Amsterdam	753	552	697	974	2073	599	0	469	525	595
København	1680	565	1078	N/A	2940	240	469	0	2002	665
Zurich	1155	630	1120	840	2450	455	525	2002	0	1050
Budapest	3150	175	980	1750	4060	420	595	665	1050	0

Table 6: Travel cost (¥) of Train between cities in July

	London	Vienna	Paris	Rome	Barcelona	Berlin	Amsterdam	København	Zurich	Budapest
London	0	2.25	1.05	2.6	2.5	2	1.1	2	1.83	2.67
Vienna	2.25	0	2	1.45	2.5	1.1	1.8	1.5	1.3	0.83
Paris	1.05	2	0	2.25	2	1.8	1.3	2	1.3	0.83
Rome	2.6	1.45	2.25	0	2	2	2.25	2.5	1.58	1.67
Barcelona	2.5	2.5	2	2	0	2.55	2.3	3	1.92	2.67
Berlin	2	1.1	1.8	2	2.55	0	1.4	1.5	N/A	1.5
Amsterdam	1.1	1.8	1.3	2.25	2.3	1.4	0	1.67	1.75	2.08
København	2	1.5	2	2.5	3	1	1.5	0	N/A	2
Zurich	1.83	1.3	1.3	1.58	1.92	N/A	1.67	N/A	0	1.67
Budapest	2.67	0.83	0.83	1.67	2.67	1.5	2.08	2	1.67	0

Table 7: Travel time (in hours) of Airplane between cities in July

	London	Vienna	Paris	Rome	Barcelona	Berlin	Amsterdam	København	Zurich	Budapest
London	0	14	2	N/A	11	11	3.9	27	8	24
Vienna	14	0	11	17.5	35	12	14	17.6	8	2.63
Paris	2	11	0	10.5	6.75	8.8	3.5	14.5	4.5	13.42
Rome	N/A	17.5	10.5	0	N/A	25	32	N/A	8	27
Barcelona	11	35	6.75	N/A	0	23.3	20	35	15	32
Berlin	11	12	8.8	25	23.3	0	6.3	5	9	14
Amsterdam	3.9	14	3.5	32	20	6.3	0	16	10	21
København	27	17.6	14.5	N/A	35	5	16	0	18.4	21
Zurich	8	8	4.5	8	15	9	10	18.4	0	11
Budapest	24	2.63	13.42	27	32	14	21	21	11	0

Table 8: Travel time (in hours) of Train between cities in July

The solver script is also available on https://github.com/Beautifuldog01/CW2/tree/main

```
import itertools
import pandas as pd
import numpy as np
from tqdm import tqdm
import os
import csv


def read_and_process_csv(file_name):
    matrix = []

    with open(file_name, 'r') as file:
        reader = csv.reader(file)
        for row in reader:
            new_row = [np.nan if cell == '' else float(cell) for cell in
                row]
            matrix.append(new_row)

    matrix = np.array(matrix, dtype=float)
    n = matrix.shape[0]
    matrix[np.isnan(matrix)] = 0

    for i in range(n):
        for j in range(i + 1, n):
            matrix[i, j] = matrix[j, i]

    # matrix_nonzero = matrix[matrix > 0]
    # matrix_min = np.min(matrix_nonzero)
    # matrix_max = np.max(matrix_nonzero)
    # matrix[matrix > 0] = (matrix[matrix > 0] - matrix_min) / (
        matrix_max - matrix_min)

    matrix[matrix == 0] = 1000

    return matrix


def combined_cost_and_transport(path, flight_df, train_df, max_flights,
    max_trains):
    total_cost = 0
    transport = []
    flight_count = 0
    train_count = 0
    for i in range(len(path) - 1):
        flight_cost = flight_df.loc[path[i], path[i + 1]]
        train_cost = train_df.loc[path[i], path[i + 1]]
        if flight_count < max_flights and (train_count >= max_trains or
            flight_cost < train_cost):
            total_cost += flight_cost
            transport.append("Airplane")
            flight_count += 1
        elif train_count < max_trains:
            total_cost += train_cost
            transport.append("Train")
            train_count += 1
        else:
            return float('inf'), transport
    return total_cost, transport


def tsp_bruteforce_combined(flight_data, train_data, start, max_flights,
    max_trains):
```

```
    cities = flight_data.columns.tolist()
    cities.remove(start)
    min_cost = float('inf')
    min_path = None
    min_transport = None
    for path in tqdm(list(itertools.permutations(cities)), desc="
        Calculating progress"):
        path = (start,) + path + (start,)
        cost, transport = combined_cost_and_transport(path, flight_data,
            train_data, max_flights, max_trains)
        if cost < min_cost:
            min_cost = cost
            min_path = path
            min_transport = transport

    return min_path, min_cost, min_transport


cities = ['London', 'Vienna', 'Paris', 'Rome', 'Barcelona', 'Berlin', '
    Amsterdam', 'Copenhagen', 'Zurich', 'Budapest']


def read_month_data(directory):
    data = {}
    for file_name in os.listdir(directory):
        if file_name.endswith(".csv"):
            data_type = file_name[:-4]
            file_path = os.path.join(directory, file_name)
            data[data_type] = read_and_process_csv(file_path)
    return data


july_data_path = "/Users/badudu/Documents/MTH203/CW2/code/data/July"
august_data_path = "/Users/badudu/Documents/MTH203/CW2/code/data/August"

july_data = read_month_data(july_data_path)
# august_data = read_month_data(august_data_path)

flight_cost_df = pd.DataFrame(july_data['flight_costs'], columns=cities,
    index=cities)
train_cost_df = pd.DataFrame(july_data['train_costs'], columns=cities,
    index=cities)
flight_time_df = pd.DataFrame(july_data['flight_time'], columns=cities,
    index=cities)
train_time_df = pd.DataFrame(july_data['train_time'], columns=cities,
    index=cities)


def read_original_data(file_name):
    data = []

    with open(file_name, 'r') as file:
        reader = csv.reader(file)
        for row in reader:
            new_row = [np.nan if cell == '' else float(cell) for cell in
                row]
            data.append(new_row)

    data_df = pd.DataFrame(data, columns=cities, index=cities)

    n = data_df.shape[0]
    for i in range(n):
        for j in range(i + 1, n):
            data_df.iloc[i, j] = data_df.iloc[j, i]
```

```
    return data_df


flight_cost_original_df = read_original_data(os.path.join(july_data_path,
     'flight_costs.csv'))
train_cost_original_df = read_original_data(os.path.join(july_data_path,
    'train_costs.csv'))
flight_time_original_df = read_original_data(os.path.join(july_data_path,
     'flight_time.csv'))
train_time_original_df = read_original_data(os.path.join(july_data_path,
    'train_time.csv'))


def get_cost_and_time(optimal_path, optimal_transport, flight_cost_matrix
    , train_cost_matrix, flight_time_matrix,
                      train_time_matrix):
    cost_list = []
    time_list = []

    for i in range(len(optimal_path) - 1):
        city1 = optimal_path[i]
        city2 = optimal_path[i + 1]
        transport = optimal_transport[i]

        if transport == "Airplane":
            cost = flight_cost_matrix.loc[city1, city2]
            time = flight_time_matrix.loc[city1, city2]
        else:  # transport == "Train"
            cost = train_cost_matrix.loc[city1, city2]
            time = train_time_matrix.loc[city1, city2]

        cost_list.append(cost)
        time_list.append(time)

    return cost_list, time_list


from functools import lru_cache


def held_karp_modified(flight_df, train_df, start_city, max_flights,
    max_trains):
    cities = flight_df.columns.tolist()
    cities.remove(start_city)
    cities_set = frozenset(cities)

    @lru_cache(maxsize=None)
    def dp(city, visited_cities, remaining_flights, remaining_trains):
        if visited_cities == cities_set:
            if remaining_flights > 0:
                return flight_df.loc[city, start_city], [start_city]
            else:
                return train_df.loc[city, start_city], [start_city]

        min_cost = float('inf')
        min_path = []

        for next_city in cities:
            if next_city not in visited_cities:
                new_visited_cities = visited_cities | {next_city}
                flight_cost = flight_df.loc[city, next_city]
                train_cost = train_df.loc[city, next_city]

                if remaining_flights > 0:
```

```
                    cost_flight, path_flight = dp(next_city,
                        new_visited_cities, remaining_flights - 1,
                                              remaining_trains)
                    cost_flight += flight_cost

                    if cost_flight < min_cost:
                        min_cost = cost_flight
                        min_path = path_flight + [next_city]

                if remaining_trains > 0:
                    cost_train, path_train = dp(next_city,
                        new_visited_cities, remaining_flights,
                        remaining_trains - 1)
                    cost_train += train_cost

                    if cost_train < min_cost:
                        min_cost = cost_train
                        min_path = path_train + [next_city]

        return min_cost, min_path

    min_cost, optimal_path = dp(start_city, frozenset(), max_flights,
        max_trains)
    optimal_path = [start_city] + optimal_path[::-1]

    # Determine transport mode
    transport = []
    for i in range(len(optimal_path) - 1):
        if max_flights > 0 and (flight_df.loc[optimal_path[i],
            optimal_path[i + 1]] <= train_df.loc[
            optimal_path[i], optimal_path[i + 1]] or max_trains == 0):
            transport.append("Airplane")
            max_flights -= 1
        else:
            transport.append("Train")
            max_trains -= 1

    return optimal_path, min_cost, transport


alpha = 0.35
beta = 1 - alpha
optimal_li = []
flight_df = flight_cost_df * alpha + flight_time_df * beta * 500
train_df = train_cost_df * alpha + train_time_df * beta * 500
start_city = cities[0]
max_flights = 10
max_trains = 10
optimal_path, min_cost, optimal_transport = tsp_bruteforce_combined(
    flight_df, train_df, start_city, max_flights,
                                                                    max_trains
                                                                        )
cost_list, time_list = get_cost_and_time(optimal_path, optimal_transport,
     flight_cost_original_df,
                                            train_cost_original_df,
                                                flight_time_original_df,
                                                train_time_original_df)
optimal_li.append([optimal_path, optimal_transport, sum(cost_list), sum(
    time_list)])
print("Optimal path:", optimal_path)
print("Transportation mode:", optimal_transport)
print("Costs for each segment:", cost_list)
print("Times for each segment:", time_list)
```