Notes Series

Shilong.Lu

Notes Of Self-learning

Complex Geometry Notes

First Edition

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Simple Introduction

This book is my notes of Complex Geometry!

This book is a summary of the final examination review materials of complex analysis, mainly including the proof question types of the exam and various knowledge points, such as Riemann mapping theorem, generalized Schwarz lemma and so on. This book was written by me at the end of the semester and is for review only.

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Preface

As my first english book, i'm happy.

—- Ethan Lu

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Part I

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1.1.1 Hartogs' Phenomenon and Hartogs' Theorem

Example 1.1.1.

Let

$$G = \{(z,w) \in \mathbb{C}^2 : |z| < 1, \beta < 1, \beta < |w| < 1\}$$
$$\bigcup \{(z,w) \in \mathbb{C}^2 : |z| < \alpha < 1, |w| < 1\}.$$

then every holomorphic function on G can be expanded to double cyclindrical domain $\{(z,w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\}$.

Solution. Now let we see the Figure 1 and let S denote the shadow part of $\mathbb{R}^2$. Define map φ by

$$\varphi\colon \mathbb{C}^2 \to \mathbb{R}^2; (z,w) \mapsto (|z|, |w|),$$

then $G = \varphi^{-1}(S)$. Next we will show that the conclusion above is true.

Setp I. Taking Laurent expansion of $f(z,w)$. For every fixed $|z| < 1$, $f(z,w)$ can be expressed as Laurent series

$$f(z,w) = \sum_{v=-\infty}^{+\infty} a_v(z) w^v.$$

,where $a_v(z)$ is holomorphic on $D_z(0,1)$. For fixed $|z| < 1$, φ^{-1} transfers into an single variable complex function $\varphi^{-1}_{|z|<1}(|w|)$, $|w| < 1$, then f is holomorphic about w and posses Laurent series expansion on $D_w(0,1)$. Owing to $a_v(z)$ being holomorphic on $D_z(0,1)$, when $|z| < \alpha$, the Laurent series has no term with negative power, in other words, $a_v(z) = 0, \forall v < 0$.

Because $a_v(z)$ is holomorphic on $D_z(0,1)$ and $a_v(z) = 0, \forall v < 0$, then according to Identity theorem, we yield $a_v(z) \equiv 0$ on $D_z(0,1)$. It is clear that $f(z,w)$ is holomorphic on $\{(z,w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\}$.

Step II. The expression of $f(z, w)$ analytic continuation is obtained by using Cauchy Integral theorem. On step I, we have shown that $f(z, w)$ is holomorphic on $\{(z, w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\}$, then on step II, our purpose is to ascertain the expression of the expanded function.

Let $\beta' < \beta < 1$. Define a function by using Cauchy Integral theorem, we gain

$$\widetilde{f}(z, w) = \frac{1}{2\pi i} \int_{|\xi|=\beta'} \frac{f(z, \xi)}{\xi - w} \mathrm{d}\xi. \tag{1.1}$$

Where $\widetilde{f}(z, w)$ is holomorphic function on $\{(z, w) \in \mathbb{C}^2 \mid |z| < 1, |w| < \beta'\}$. In particular, $\widetilde{f}(z, w) = f(z, w)$ on $\{(z, w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\}$. So $\widetilde{f}(z, w)$ dose the expanded function what we find. $\square$

Lemma 1.1.1.

Let $U \subset \mathbb{C}^n$ be an open subset and let $V \subset \mathbb{C}$ be an open neighbourhood of the boundary of $B_\varepsilon(0) \subset \mathbb{C}$. Assume that $f : V \times U \to \mathbb{C}$ is a holomorphic function. Then

$$g(z) := g(z_1, \cdots, z_n); = \int_{|\xi|=\varepsilon} f(\xi, z_1, \cdots, z_n) \mathrm{d}\xi$$

is holomorphic function on U.

Theorem 1.1.1. Hartogs' Theorem

Suppose $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ and $\varepsilon' = (\varepsilon'_1, \cdots, \varepsilon'_n)$ are given such that for all i one has $\varepsilon'_i < \varepsilon$. If $n > 1$, then any holomorphic map $f : B_\varepsilon(0) \backslash \overline{B_{\varepsilon'}(0)} \to \mathbb{C}$ can be uniformly extended to a holomorphic map $\widetilde{f} : B_\varepsilon(0) \to \mathbb{C}$.

Proof. Let $\varepsilon = (1, \cdots, 1)$ and $\exists \delta > 0$ such that

$$V = \{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i \neq 1| < 1}\} \bigcup \{z \in \mathbb{C}^n \mid 1 - \delta < |z_2| < 1, |z_{i \neq 2| < 1}\}.$$

is contained in $B_\varepsilon(0) \backslash \overline{B_{\varepsilon'}(0)}$. So f is holomorphic on V. Thus, for any $w := (z_2, \cdots, z_n)$ with $|z_j| < 1$, $j = 2, \cdots, n$, there exists a holomorphic function $f_w(z_1) := f(z_1; z_2, \cdots, z_n)$ on annulus $1 - \delta < |z_1| < 1$.

Remark. *For the Lemma 1.1.1, $V \subset \mathbb{C}^n$ is open subset and let $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i \neq 1}| < 1\} \subset \mathbb{C}$ be an neighbourhood of the boundary of $B_1(0) \subset \mathbb{C}$. Because f is holomorphic on $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i \neq 1}| < 1\} \times V$, so $g(z_1; z_2, \cdots, z_n) := f(z_1, \cdots, z_n) := f_w(z_1)$ is holomorphic on $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i \neq 1}| < 1\}$.*

Now due to $f_w(z_1)$ is holomorphic on $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i \neq 1}| < 1\} \subset \mathbb{C}$, then $f_w(z_1)$ can be expanded to Laurent series by $f_w(z_1) = \sum_{n=-\infty}^{+\infty} a_n(w) z_1^n$ with the coefficient

$$a_n(w) = \frac{1}{2\pi i}\int_{|\xi|=1-\delta/2} \frac{f(\xi)}{\xi^{n+1}} \mathrm{d}\xi.$$

By Lemma 1.1.1 ,$a_n(w)$ is holomorphic for w in the unit polydisc of $\mathbb{C}^{n-1}$.

On the other hand, the function $f_w : z_1 \mapsto f_w(z_1)$ is holomorphic on the unit disc for fixed w such that $1 - \delta < |z_1| < 1$.

Remark. *On above description, we have shown that $f_w : z_1 \mapsto f_w(z_1)$ is holomorphic on $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i\neq 1}| < 1\}$, so it's obviously that f_w is also holomorphic on*

$$\{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, 1 - \delta < |z_2| < 1, |z_{i\neq 1,2}| < 1\}$$
$$\subset \{z \in \mathbb{C}^n \mid 1 - \delta < |z_1| < 1, |z_{i\neq 1}| < 1\}.$$

Thus, $a_n(w) = 0$, for $n < 0$ and $1 - \delta < |z_2| < 1$.

Remark. *$a_n(w)$ is holomorphic for w on the unit disc of $\mathbb{C}^n$, so $a_n(w) = 0, \forall n < 0$.*

By the Identity theorem , we show that $a_n(w) \equiv 0$ for $n < 0$. But then we define the holomorphic extension $\widetilde{f}$ of f by the power series $\sum_{n=0}^{\infty} a_n(w) z_1^n$. (without terms of negative power)

Remark. *Also, we could use the Cauchy Integral theorem, which is equivalent to that.*

This series converges uniformly ,as $a_n(w)$ are holomorphic and attain maximum at the boundary.

Remark. *For $\widetilde{f} = \sum_{n=0}^{\infty} a_n(w) z_1^n, 1 - \delta < |z_1| < 1$, if we want the series converges uniformly, it just need $a_n(w)$ are bounded on $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_2| < 1, |z_{i\neq 2}| < 1\}$. By $a_n(w)$ be holomorphic on $\{z \in \mathbb{C}^n \mid 1 - \delta < |z_2| < 1, |z_{i\neq 2}| < 1\}$, then use Maximum princple, $a_n(w)$ attain their maximum at the boundary.*

So the convergence of the Laurent series on the annulus yields the uniformly convergence everywhere. Clearly, the holomorphic function given by the series (the power series) glues with f to give the disired holomorphic function. □

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