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Ethan Lu

An Introduction to Complex Geometry

First Edition

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Simple Introduction

This book is my notes of Complex Geometry!

This book is a summary of the final examination review materials of complex analysis, mainly including the proof question types of the exam and various knowledge points, such as Riemann mapping theorem, generalized Schwarz lemma and so on. This book was written by me at the end of the semester and is for review only.

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Preface

As my first english book, i’m happy.

—— Ethan Lu
2023-01-11

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1 Differentiable Manifolds

Part I

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Proposition 1.1.1. (Proposition A.17 (Properties of the Subspace Topology).)

Let X be a topological space and let S be a subspace of X.[1]

(a) CHARACTERISTIC PROPERTY: *If Y is a topological space, a map $F : Y \to S$ is continuous if and only if the composition $\iota_S \circ F : Y \to X$ is continuous, where $\iota_S : S \hookrightarrow X$ is the inclusion map (the restriction of the identity map of X to S).*

(b) *The subspace topology is the unique topology on S for which the characteristic property holds.*

(c) *A subset $K \subseteq S$ is closed in S if and only if there exists a closed subset $L \subseteq X$ such that $K = L \cap S$.*

(d) *The inclusion map $\iota_S : S \hookrightarrow X$ is a topological embedding.*

(e) *If Y is a topological space and $F : X \to Y$ is continuous, then $F|_S : S \to Y$ (the restriction of F to S) is continuous.*

(f) *If $\mathscr{B}$ is a basis for the topology of X, then $\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$ is a basis for the subspace topology on S.*

(g) *If X is Hausdorff, then so is S.*

(h) *If X is first-countable, then so is S.*

(i) *If X is second-countable, then so is S.*

Proposition 1.1.2. (Proposition A.23 (Properties of the Product Topology).)

Suppose $X_1, \ldots, X_k$ are topological spaces, and let $X_1 \times \cdots \times X_k$ be their product space.

(a) CHARACTERISTIC PROPERTY: *If B is a topological space, a map $F : B \to X_1 \times \cdots \times X_k$ is continuous if and only if each of its component functions $F_i = \pi_i \circ F : B \to X_i$ is continuous.*

(b) *The product topology is the unique topology on $X_1 \times \cdots \times X_k$ for which the characteristic property holds.*

(c) *Each projection map $\pi_i : X_1 \times \cdots \times X_k \to X_i$ is continuous.*

(d) *Given any continuous maps $F_i : X_i \to Y_i$ for $i = 1, \ldots, k$, the product map $F_1 \times \cdots \times F_k : X_1 \times \cdots \times X_k \to Y_1 \times \cdots \times Y_k$ is continuous, where*

$$F_1 \times \cdots \times F_k (x_1, \ldots, x_k) = (F_1 (x_1), \ldots, F_k (x_k)).$$

(e) *If S_i is a subspace of X_i for $i = 1, \ldots, n$, the product topology and the subspace topology on $S_1 \times \cdots \times S_n \subseteq X_1 \times \cdots \times X_n$ coincide.*

(f) *For any $i \in \{1, \ldots, k\}$ and any choices of points $a_j \in X_j$ for $j \neq i$, the map $x \mapsto (a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k)$ is a topological embedding of X_i into the product space $X_1 \times \cdots \times X_k$.*

(g) *If $\mathscr{B}_i$ is a basis for the topology of X_i for $i = 1, \ldots, k$, then the collection*

$$\mathscr{B} = \{B_1 \times \cdots \times B_k : B_i \in \mathscr{B}_i\}$$

is a basis for the topology of $X_1 \times \cdots \times X_k$.

(h) *Every finite product of Hausdorff spaces is Hausdorff.*

(i) *Every finite product of first-countable spaces is first-countable.*

(j) *Every finite product of second-countable spaces is second-countable.*

Example 1.1.1. (Example 1.8 (Product Manifolds).)

Suppose $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$, respectively. The product space $M_1 \times \cdots \times M_k$ is shown to be a topological manifold of dimension $n_1 + \cdots + n_k$ as follows. ***It is Hausdorff and second-countable by Propositions A.17 (g), (i) and A.23 (h), (j), so only the locally Euclidean property needs to be checked.*** Given any point $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. ***The product map***

$$\varphi_1 \times \cdots \times \varphi_k : U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1 + \cdots + n_k}$***.*** Thus, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$.

Example 1.1.2. (Example 1.9 (Tori).)

For a positive integer n, the ***n*-torus** (plural: tori) is the product space $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. By the discussion above, it is a topological n-manifold. (The 2-torus is usually called simply the torus.)

Example 1.1.3. (Example 1.34 (Smooth Product Manifolds).)

If $M_1, \ldots, M_k$ are smooth manifolds of dimensions $n_1, \ldots, n_k$, respectively, we showed in Example 1.1.1 that the product space $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$. ***Any two such charts are smoothly compatible because, as is easily verified,***

$$(\psi_1 \times \cdots \times \psi_k) \circ (\varphi_1 \times \cdots \times \varphi_k)^{-1} = \left(\psi_1 \circ \varphi_1^{-1}\right) \times \cdots \times \left(\psi_k \circ \varphi_k^{-1}\right),$$

which is a smooth map. This defines a natural smooth manifold structure on the product, called the product smooth manifold structure. For example, this yields a smooth manifold structure on the n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Exercise 1.1. (Product Manifold)

Let M and N be differentiable manifolds and let $\{(U_\alpha, \mathbf{x}_\alpha)\}, \{(V_\alpha, \mathbf{y}_\alpha)\}$ be differentiable structure on M and N, respectively. Consider the cartesian product $M \times N$ and the mappings $\mathbf{z}_{\alpha\beta}(p, q) = (\mathbf{x}_\alpha(p), \mathbf{y}_\beta(q))$, $p \in U_\alpha, q \in V_\beta$.

(a) Prove that $\{(U_{\alpha\beta}, \mathbf{z}_{\alpha\beta})\}$ is a differentiable structure on $M \times N$ in which the projections $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ are differentiable. With this differentiable structure $M \times N$ is called the product manifold of M with N.

(b) Show that the product manifold $S^1 \times \cdots \times S^1$ of n circles S^1, where $S^1 \subset \mathbb{R}^2$ has the usual differentiable structure , is diffeomorphic to the n torus T^n of Example 4.9 (a).

Proof. For (a),

1. (Open Covering) As

$$\begin{aligned} \mathbf{z}_{\alpha\beta} \colon U_\alpha \times V_\beta &\to \mathbf{x}_\alpha(U_\alpha) \times \mathbf{y}_\beta(V_\beta) \subset M \times N \\ (p, q) &\mapsto (\mathbf{x}_\alpha(p), \mathbf{y}_\beta(q)) \end{aligned}$$

is a homeomorphism, there is a open covering of $M \times N$, i.e.

$$\bigcup_{\alpha,\beta} \mathbf{z}_{\alpha\beta}(U_\alpha, V_\beta) = \bigcup_{\alpha} \mathbf{x}_\alpha(U_\alpha) \times \bigcup_{\beta} \mathbf{y}_\beta(V_\beta) = M \times N.$$

2. (Atlas Compatibility) When $\mathbf{z}_{\alpha\beta}(U_\alpha, V_\beta) \bigcap \mathbf{z}_{\gamma\delta}(U_\gamma, V_\delta) \neq \emptyset$, one has

$$\mathbf{z}_{\gamma\delta}^{-1} \circ \mathbf{z}_{\alpha\beta}(p, q) = \mathbf{z}_{\gamma\delta}^{-1}(\mathbf{x}_\alpha(p), \mathbf{y}_\beta(q)) = (\mathbf{x}_\gamma^{-1} \circ \mathbf{x}_\alpha(p), \mathbf{y}_\delta^{-1} \circ \mathbf{y}_\beta(q))$$

is differentiable, i.e. any two charts of atlas are compatible.

Thus, by definition, with this differentiable structure, $M \times N$ is a differentiable manifold.

For (b), define a mapping

$$\begin{aligned} \varphi : \mathbb{R}^n / G &\to \overbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}^{\text{n copies}} \\ (x_1, \ldots, x_n) &\mapsto \left(e^{2\pi i \frac{x_1}{m_1}}, \ldots, e^{2\pi i \frac{x_n}{m_n}}\right) \end{aligned}$$

It is obviously that φ is an immersion and submersion from its Jacobi matrix:

$$J(\varphi) = \begin{pmatrix} \frac{2\pi i}{m_1} e^{2\pi i \frac{x_1}{m_1}} & & \\ & \ddots & \\ & & \frac{2\pi i}{m_n} e^{2\pi i \frac{x_n}{m_n}} \end{pmatrix}$$

So φ is diffeomorphism. □

Exercise 1.2.

Prove that the tangent bundle of a differentiable manifold M is orientable (even though M may be not).

Proof. Choose a parametrization $\{(U_\alpha, X_\alpha)\}$ of M. it induce a parametrization of TM:

$$\begin{aligned} Y_\alpha : U_\alpha \times \mathbb{R}^n &\to \pi^{-1}(X_\alpha(U_\alpha)) \subset \mathrm{TM} \\ (x,t) &\mapsto (X_\alpha(x), dX_\alpha(t)) \end{aligned}$$

The corresponding coodinate transformation is

$$Y_\beta^{-1} Y_\alpha(x,t) = \left(X_\beta^{-1} X_\alpha(x), dX_\beta^{-1} dX_\alpha(t)\right) = \left(X_\beta^{-1} X_\alpha(x), d\left(X_\beta^{-1} X_\alpha\right)(t)\right)$$

To calculate its Jacobi matrix, we get

$$J\left(Y_\beta^{-1} Y_\alpha\right) = \begin{bmatrix} J & O \\ * & J \end{bmatrix}, \quad J = J\left(X_\beta^{-1} X_\alpha\right)$$

And its determination

$$\det J\left(Y_\beta^{-1} Y_\alpha\right) = (\det J)^2 > 0$$

So, TM is an orientable manifold. □

Exercise 1.3.

Prove that:

(a) A regular surface $S \subset \mathbb{R}^3$ is an orientable manifold if and only if there exists a differentiable mapping of $N : S \to \mathbb{R}^3$ with $N_p \perp T_p S, |N(p)| = 1, \forall p \in S$,

(b) the Möbius band (Example 4.9 (b)) is non–orientable.

Proof. (a) ($\Rightarrow$) Choose a parametrization $\{(X_\alpha, U_\alpha)\}$ of S. For any points $p \in X_\alpha(U_\alpha) \cap X_\beta(U_\beta) \subset S$, which $p = X_\alpha(x_1, x_2) = X_\beta(y_1, y_2)$. Since S is an orientable, thus $\det d\left(X_\beta^{-1} \circ X_\alpha\right) > 0, \quad \forall \alpha, \beta$ Define a mapping

$$N(p) = N(X_\alpha(x_1, x_2)) := \frac{\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}}{\left|\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right|}, \frac{\partial}{\partial x_j} \in T_p S = \mathbb{R}^2 \subset \mathbb{R}^3$$

Then

$$N(p) = N\left(X_\beta\left(y_1, y_2\right)\right)$$
$$= \frac{\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}}{\left|\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}\right|} = \frac{\det\left(d\left(X_\beta^{-1} \circ X_\alpha\right)\right)}{\left|\det\left(d\left(X_\beta^{-1} \circ X_\alpha\right)\right)\right|} \cdot \frac{\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}}{\left|\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right|} = \frac{\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}}{\left|\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right|}$$

$N(p)$ is well defined. ($\Leftarrow$) Suppose that $N(p)$ is a differentiable mapping as we known.

(b) We say that M is orientable if and only if there exists an atlas $A = \{(U_\alpha, \phi_\alpha)\}$ such that $\det\left(J\left(\phi_\alpha \circ \phi_\beta^{-1}\right)\right) > 0$, if it is defined.

Assume that the Möbius strip (band) is orientable. Then we would be able to define a map: $x \rightarrow \boldsymbol{n_x}$ that sends x to a unit vector normal to the surface in such a way that the map is continuous. Since M is two-dimensional and embedded in 3-space, this map is determined by the value at a single point (because you have two choices, one in each direction from the surface). Now observe that if you follow a loop around the strip, the value of $\boldsymbol{n_x}$ changes sign when you return to $\boldsymbol{x}$ from the other side.

Reference: The Möbius strip (band) is non-orientable.

□

4.

Proof. For simplicity, assume (without loss of generality) that in the following, by ”chart” I mean “connected chart”.

By definition, a manifold M is orientable iff you can find a covering of M by coherently oriented charts. Meaning that the determinant of the derivative of the transition functions between overlapping charts (i.e. $\det(J)$) are all positive. Next, note that given two charts $(U, f), (V, g)$ of M of opposite orientation (i.e. the transition function is of negative determinant), the charts $(U, f^*), (V, g)$ are of the same orientation, where f^* is defined by changing the sign of the first coordinate of f. Call (U, f^*) the “modified” chart (U, f).

Thirdly, note that a manifold M is orientable iff given any covering of M by charts, it is always possible to modify them (in the above sense) to obtain a coherently oriented covering of M.

($\Longleftarrow$) Clearly.

($\Longrightarrow$) Let $\{(U_i, f_i)\}$ be any covering of M by charts. Take a coherently oriented covering of M by charts $\{(V_j, g_j)\}$. For each U_i, chose a V_j that intersects it. Are U_i and V_j of the same orientation? If so, do nothing. If not, modify (U_i, f_i) to (U_i, f_i^*). Note that for two charts to have the same orientation is an equivalence relation on the set of all charts of a manifold. Clearly then, this process makes $\{(U_i, f_i)\}$ into a coherently oriented covering.

Now let M be 2 dimensional. Let $X = ([0, 1]x]-1, 1[)/(0, t)\ (1, -t)$ be the Möbius band, and $f : X \longrightarrow M$ be an embedding. Consider $\{(U_i, \phi_i)\}$ a covering of $f(X)$ by charts lying entirely in $f(X)$ ($f(X)$ is open in M). Since the “pullback charts” $(f^{-1}(U_i), \phi_i \circ f)$ make up a covering of the nonorientable manifold X, it is impossible to modify the covering $\{(U_i, \phi_i)\}$ to make it coherently oriented. Then just extend $\{(U_i, \phi_i)\}$ to a covering of the whole of M. This covering either cannot be modified to be coherently oriented, so M is nonorientable.

Reference: $\mathbb{R}P^n$ is non-orientable.

□

5.

Proof. (a) $\tilde{\varphi} : P^2 \to \mathbb{R}^4$ is a immersion. Pf: Since

$$d\tilde{\varphi}_{[p]} = d\varphi_p = J_p(F) = \begin{pmatrix} 2x & y & z & 0 \\ -2y & x & 0 & z \\ 0 & 0 & x & y \end{pmatrix}^T$$

Let $p = (0, 0, 1)$ For symmetry of sphere $\mathbb{S}^2$.

$$\Rightarrow J_p(F) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T$$

rank $dF_p = \dim T_p\mathbb{S}^2 = 2$, So $\tilde{\varphi}$ is a immersion.

(b) $\tilde{\varphi}$ is injective; together with (a) and the compactness of P^2, this implies that $\tilde{\varphi}$ is an embedding. Pf: If $\tilde{\varphi}([p]) = \tilde{\varphi}([q])$, where $p = (x, y, z), q = (x', y', z')$, it means

$$\begin{cases} x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 = 1 \\ x^2 - y^2 = x^2 - y'^2 \\ xy = x'y' & (*1) \\ xz = x'z' & (*2) \\ yz = y'z' & (*3) \end{cases}$$

From $(*1) \times (*2)$ we obtained

$$x^2yz = x^2y'z'$$

If $yz \neq 0$, by $(*3)$ we get

$$x = \pm x'$$

By the continuity of the equations, when $y \to 0$ or $z \to 0$, it still holds. Similarly, we can get $y = \pm y', z = \pm z'$. Thus, $p = -q \in [p]$. □

6.

Exercise 1.4.

Show that the mapping $G\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$G(x, y) = ((r\cos y + a)\cos x, (r\cos y + a)\sin x, r\sin y\cos\frac{x}{2}, r\sin x\sin\frac{x}{2}),$$

with $(x, y) \in \mathbb{R}^2$ induces an embedding of the klein bottle into $\mathbb{R}^4$.

Proof. To show that G induces an embedding of the Klein bottle into $\mathbb{R}^4$, we need to show that G is an immersion and a homeomorphism onto its image.

First, we will show that G is an immersion. To do this, we need to show that $DG(x, y)$ is injective for all $(x, y) \in \mathbb{R}^2$. We have already computed the matrix $DG(x, y)$ in the previous answer, so we only need to check that its determinant is nonzero. The derivative of G at $(x, y) \in \mathbb{R}^2$ is the 4×2 matrix

$$DG(x, y) = \begin{pmatrix} -(r\cos y + a)\sin x & (r\cos y + a)\cos x & a - r\sin y \sin\frac{x}{2} & \frac{1}{2}r\cos x\cos\frac{x}{2} \\ -(r\cos y + a)\cos x & a - (r\cos y + a)\sin x & 0 & \frac{1}{2}r\cos x\sin\frac{x}{2} \\ 0 & -r\cos y\cos\frac{x}{2} & -\frac{1}{2}r\sin y\sin\frac{x}{2} & r\sin y\cos\frac{x}{2} \\ 0 & -r\sin y\sin\frac{x}{2} & \frac{1}{2}r\sin x\cos\frac{x}{2} & \frac{1}{2}r\sin x\sin\frac{x}{2} \end{pmatrix}$$

Then the determinant of $\det DG(x, y)$ is

$$\det DG(x, y) = -\frac{1}{2}r^2\sin^2 x \sin y \neq 0,$$

so $DG(x, y)$ is always invertible (i.e. G is non-degenerate). Therefore, G is an immersion.

Next, we will ***show that G is a homeomorphism onto its image***. To do this, we need to show that ***G is bijective, continuous, and has a continuous inverse***.

To show that G is bijective, we need to show that distinct points in the domain of G are mapped to distinct points in the image of G, and that every point in the image of G is the image of some point in the domain of G.

The first part follows from the fact that the horizontal and vertical components of $G(x, y)$ depend only on x and y, respectively, while the other two components involve both x and y in a nontrivial way. Specifically, if $(x1, y1) \neq (x2, y2)$, then there must be at least one component of $G(x1, y1)$ that is different from the corresponding component of $G(x2, y2)$.

The second part follows from the fact that every point on the Klein bottle can be parametrized by $(x, y) \in [0, 2\pi) \times [0, 2\pi)$, which is precisely the domain of G. Therefore, G is bijective. To show that G is continuous, we need to show that each component of G is a continuous function of (x, y). We can see that this is the case, since each component is a sum and product of trigonometric functions, which are all continuous.

To show that G^{-1} is continuous, we need to show that $G^{-1}(U)$ is open in $\mathbb{R}^2$ for every open set U in the image of G. To do this, we will use the inverse function theorem, which states that if $\det DG(x, y) \neq 0$ for all $(x, y) \in U$, then G is a local diffeomorphism on U and G^{-1} is continuous on $G(U)$. By our earlier computation, we know that $\det DG(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$, so we don't need to worry about singular points.

Therefore, we have shown that G induces an embedding of the Klein bottle into $\mathbb{R}^4$. □

7.

Proof. □

8.

Proof. Pf: Let $\varphi : U \subset M_1 \to \varphi(U) \subset V_\alpha \cap V_\beta \subset M_2$, V_α, V_β are charts of M_2. Since φ is a local diffeomorphisma, $\left\{\varphi^{-1}\left(U \cap X_\alpha\left(V_\alpha\right)\right), X_\alpha \circ \varphi\right\}$ is a parametrization of M_1. Since

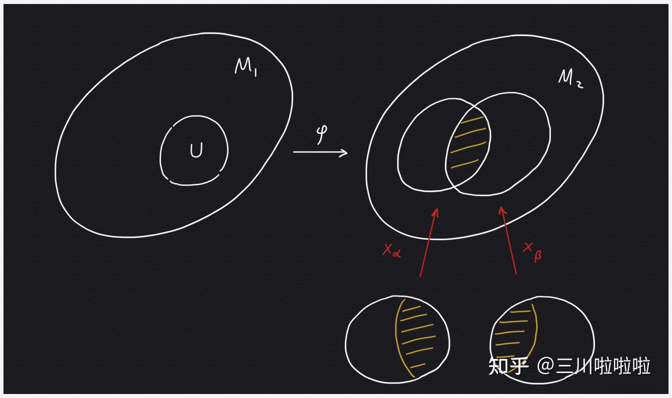

Figure 1.1: exercise 8

M_2 is orientable, i.e. $\det\left(d\left(X_\beta^{-1} \circ X_\alpha\right)\right) > 0$. Therefore, for $p \in (U_1 \cap X_\alpha(V_\alpha)) \cap (U_2 \cap X_\beta(V_\beta)) \neq \varnothing$

$\det\left(d\left(\left(X_\beta \circ \varphi\right)^{-1} \circ X_\alpha \circ \varphi\right)\right) = \det\left(d\left(\varphi^{-1} X_\beta^{-1} \circ X_\alpha \circ \varphi\right)\right) = \det\left(d\left(X_\beta^{-1} \circ X_\alpha\right)\right) > 0.$

$\square$

Exercise 1.5. (Exer 9.)

Let $G \times M \to M$ be a properly discontinuous action of a group G on a differentiable manifold M.

a) Prove that the manifold M/G (Example 4.8) is oriented if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.

b) Use a) to show that the projective plane $P^2(\mathbb{R})$, the Klein bottle and the Mobius band are non-orientable.

c) Prove that $P^2(\mathbb{R})$ is orientable if and only if n is odd.

Proof. a) if part: Let (U_α, x_α) be an orientation of M that is preserved by all the diffeomorphisms of G, i.e.

$$W = U_\beta \cap g(U_\alpha) \neq \varnothing \Rightarrow \det\left(x_\beta^{-1} \circ g \circ x_\alpha\right) > 0$$

We claim that $(\pi(U_\alpha), \pi \circ x_\alpha)$ is an orientation of M/G. Indeed,

$$\pi(U_\alpha) \cap \pi(U_\beta) \neq \varnothing \Rightarrow \det\left((\pi \circ x_\beta)^{-1} \circ (\pi \circ x_\alpha)\right) = \det\left(x_\beta^{-1} \circ g \circ x_\alpha\right) > 0$$

for some $g \in G$. Only if part: We know the atlas of M/G is induced from M, hence the conclusion follows from the reverse of the "if part".

b) Let $G = \{Id, A\}$ where A is the antipodal map. Recall that

$$\begin{aligned} \text{Projective 2- space } P^2(\mathbb{R}) &= S^2/G, \text{ where } S^2 = 2-\dim \text{sphere} \\ \text{Klein bottle } K &= \mathbb{T}^2/G, \text{ where } \mathbb{T}^2 = 2-\dim \text{torus} \\ \text{Mobius band } M &= C/G, \text{ where } C = 2-\dim \text{cylinder} \end{aligned}$$

Clearly, $S^2, \mathbb{T}^2, C$ are orientable $2-$ dim manifols, but A reverse the orientation of $\mathbb{R}^3$, hence $S^2, \mathbb{T}^2, C$. The conclusion follows from a).

c) We've the following equivalence:

$$\begin{aligned}
&P^n(\mathbb{R}) \text{ is orientable } \Leftrightarrow A \text{ preserves the orientation of } S^n (\text{ by } a)) \\
&\Leftrightarrow A \text{ preserves the orientation of } \mathbb{R}^{n+1} \\
&(\text{The orientation is induced from } \mathbb{R}^{n+1}) \\
&\Leftrightarrow (n+1) \text{ is even} \\
&\Leftrightarrow n \text{ is odd}
\end{aligned}$$

□

10.

Proof. □

11. $(\mathbb{R}, \mathbf{x}_1), \mathbf{x}_1 : x \mapsto x, (\mathbb{R}, \mathbf{x}_2), \mathbf{x}_2 : x \mapsto x^3$.

(a) $id : (\mathbb{R}, \mathbf{x}_1) \to (\mathbb{R}, \mathbf{x}_2)$ is not a diffeomorphism.

(b) $f : (\mathbb{R}, \mathbf{x}_1) \to (\mathbb{R}, \mathbf{x}_2)$ is a diffeomorphism, where $f(x) = x^3$.

Proof. (a) It is not differentiable at $x = 0$ for $\mathbf{x}_2^{-1} \circ id : x \mapsto \sqrt[3]{x}$.

(b) $\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1(x) = x$. Obviously, it is a diffeomorphism. □

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