Lecture Note for Topology

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Mathematics is the queen of science.

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CHAP

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CH 1 Topology

1.1 Topics

1. *An important fact is that a topological space is a T_1–space iff every point is closed.*

2. *The intersection of connected sets may not be connected!*

   An illustrative instance is the set of intersection points between two curves, which comprises a finite and discrete space. It is non-connected.

3. The key trait of a compact space A is that if $\{\bigcup U_i\}$ is an open cover for A, then there must exist a *finite open subcover* $\{\bigcup_{j\in I} U_j\}$, where I is a finite index set.

4. The union and intersection operations of topological spaces remain compact.

5. The product space is Hausdorff (T_2) and path-connected, which is consistent with the generators.

6. !!! Let C be a closed and bounded subset of a metric space (X, d), where we give X its metric topology. *Sometimes C is not always compact.*

   For instance, if X is an infinite set with the metric d where $\mathrm{d}\left(x_1, x_2\right)$ is 1 if $x_1 \neq x_2$ and 0 if $x_1 = x_2$. Then X is bounded and closed but only finite subsets of X are compact. In particular, X is closed and bounded but not compact.

7. Let $f : X \to Y$ be a continuous surjective map of topological spaces such that X is Hausdorff. Then Y is not Hausdorff.

   Take an instance. Let $X = Y$ be any set with at least two points but give X the discrete topology and give Y the indiscrete topology. Let $f(x) = x$ for all $x \in X$. Then f is continous , X is a T_2–space but Y is not a T_2–space.

8. When prove the *connectedness* of a space, we use *proof by contradiction* in general. In fact, we can always assume that X is disconnected, then there must exist two non-empty disjoint ($U \cap V = \emptyset$) open sets U and V such that $U \cup V = X$.

9. Every path-connected space is connected space, but the converse is not true.

Problem 1.1.1 Let (X, τ) be a topological space. Then the following statements are equivalent.

(i) X is T_1–space,

(ii) Each singleton subset of X is closed,

(iii) Each subset A of X is the intersection of its open supersets.

Proof. (i)→(ii): Let $x \in X$ and X is T_1–space. Then for any $y \in X$ and $y \neq x$, there exists a neighborhood V of y such that $x \notin V$, i.e. $V \cap \{x\} = \emptyset$. Thus, $y \notin \overline{\{x\}}$, then $\{x\} = \overline{\{x\}}$. (Here, we use the fact that $y \notin \{x\}^d$ for $V \cap (\{x\} - \{y\}) = V \cap \{x\}$. It is clear that $y \notin \{x\}$ and $\overline{\{x\}} = \{x\} \cup \{x\}^d$.) Then $\{x\}$ is closed.

(ii)→(i): Let $\{x\}$ be closed for any $x \in X$. Then for any $y \in X$ and $y \neq x$, one abtains that $y \notin \{x\} = \overline{\{x\}}$. It suffices to show that $y \in \{x\}^c$, which means that X is T_1–space.

(ii)→(iii): Suppose any singleton subset of X is closed and let $A \subseteq X$. We can write

$$A = \bigcup_{x \in A} \{x\}.$$

Let $y \in X$ and $y \notin A$, then $y \neq x$. By assumption, $\{y\}$ is closed. Thus $\{y\}^c$ is open. As $A \subseteq \{y\}^c$ for any $y \notin A$, thus $A = \bigcap_{y \notin A} \{y\}^c$, i.e. A is the intersection of its open supersets.

(iii)→(i): According to the above, one has $A = \bigcap_{y \notin A} \{y\}^c$, where $A \subseteq \{y\}^c$. For any $x \in A$, we gain $x \in \{y\}^c$ and $y \notin A$ i.e. $y \neq x$, which suffice to show that X is T_1–space. □

Problem 1.1.2 An infinite set with **co-finite topology** is T_1 but not T_2.

Proof. Suppose the contrary is true, i.e. X is T_2. Then for any $a, b \in X$, there are two open subsets U, V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

$$U^c \cup V^c = (U \cap V)^c = \emptyset^c = X$$

The left hand side is the union of two finite sets, but the right hand side is an infinite set X, which is a contradiction. Thus X is not T_2. □

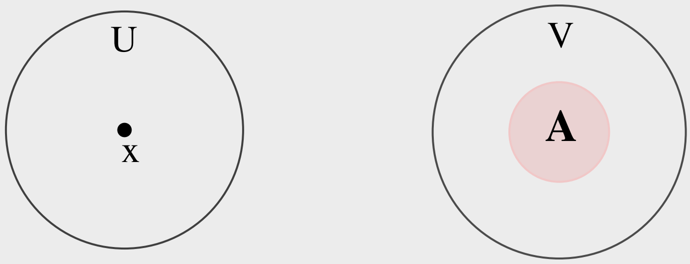

Figure 1.1: (Regular Space)

Regular Space A regular T_1–space is T_3–space.

Problem 1.1.3 Every path-connected space is connected.

Proof. (Proof by contradiction.) Assume that a path-connected space X is disconnected. Then there exist two non-empty disjoint subsets U and V in X such that $U \cup V = X$.

For X is path-connected, there exist a continous function $f : [0, 1] \to X$ with $f(0) = a, f(1) = b, \forall a, b \in X$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are two non-empty disjoint open subsets in $[0, 1]$ such that

$$[0, 1] = f^{-1}(U) \bigcup f^{-1}(V),$$

i.e. $[0, 1]$ is dis-connected. It is contradictive with the fact that $[0, 1]$ is connected. So X is connected.

□

PROBLEMS

1. Any metrizable space is second-countable.[1]

   Take a discrete uncountable space. It's metrizable (for instance, with the metric where all distances $d(x, y) = 1$ for $x \neq y$) but not second-countabale.

2. Countable union of path-connected space is path-connected.

3. Is *any dense subset of the Cantor set is uncountable*? That's wrong, for instance, we take the union of boundary points of all the inetrvals in $[0, 1]$ that we're using to define C.

4. Be aware of the identity mapping between X equipped with one topology and X equipped with another topology! Because for $\mathbb{R}$, the *lower limit topology* is finer than the *standard topology*, and for *product space*, the *box topology* is finer than the *product topology*.

5. *connectedness* : If topological space X admits nontrival partition into open sets.

   compact : If every open cover possesses a finite subcover.

6. The *diameter* of a subset A of a metric space (X, d) is $\sup\{d(x, y) \mid (x, y) \in A \times A\}$.

7. The *torus* is an orientable surface and it can be embedded without self-intersection into $\mathbb{R}^3$. The Klein bottle is a non-orientable surface which cannot be embedded without self-intersection into $\mathbb{R}^3$.

8. Given any topological space X, one abtains another topological space $C(X)$ with complement topology?

   That's wrong! For example,

9. There are topological spaces with countably many points, which have uncountably many open sets? $\surd$

   That's true. For example, *countable set with the discrete topology*.

10. The number of points of a finite Hausdorff space is always a *prime power*?

    That's wrong! For instance, 6 *-element set with the discrete topology*. (Note that *finite discrete topological space is always Hausdorff.*) This is a Hausdorff space whose number of points is not a prime power.

11. $\mathbb{R}$ with the Zariski topology is a compact topological space? $\surd$

    That's true! *proof* :

12. $\mathbb{R}$ with the Zariski topology is a connected topological space? $\surd$

    That's true! *No subset of* $\mathbb{R}$ *is both finite and has a finite complement-so the above holds here.*

13. Let $\mathbb{Z}$ be endowed with the topology where *the open sets are the unions of residue classes*. Then $f : \mathbb{Z} \to \mathbb{Z}, n \mapsto n + (-1)^n$ is an homeomorphism? $\surd$

    The function f maps uions of residue classes to unions of residue classes, so it is an homeomorphism.

Problem 1.1.4 If $\{A_i : i \in N\}$ is a collection of path-connected subsets of a space (X, τ), and $\bigcap_{i\in N} A_i \neq \emptyset$ (There is atleast one common point!) then $A = \bigcup_{i\in N} A_i$ is path-connected. In other words, countable union of path-connected sets is path-connected.

Proof. Let $x, y \in A$, where $x \in A_{i_1}, y \in A_{i_2}$. Let $z \in \bigcap_{i\in N} A_i$, then $z \in A_{i_1}, z \in A_{i_2}$.

For A_{i_1} is path-connected, there exists a continuous function $f : [0, 1] \to A_{i_1}$ that maps x to z with $f(0) = x, f(1) = z$. Similarily, there exists another continuous function $g : [0,1] \to A_{i_2}$ that maps y to z with $g(0) = y, g(1) = z$.

Define function h by

$$h(t) = \begin{cases} f(2t), & 0 \leqslant t < \frac{1}{2}, \\ g(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then h is continuous. Thus A is path-connected. □

Problem 1.1.5 – ($\star\star\star$) Let $\mathbb{R}^2$ be endowed with the usual topology. Either prove or disprove that $[0,1[\times]0,1[$ and $[0,1[\times[0,1]$ are homeomorphic subspaces of $\mathbb{R}^2$.

Proof. □

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Bibliography

[1] John L Kelley. *General topology*. Courier Dover Publications, 2017.