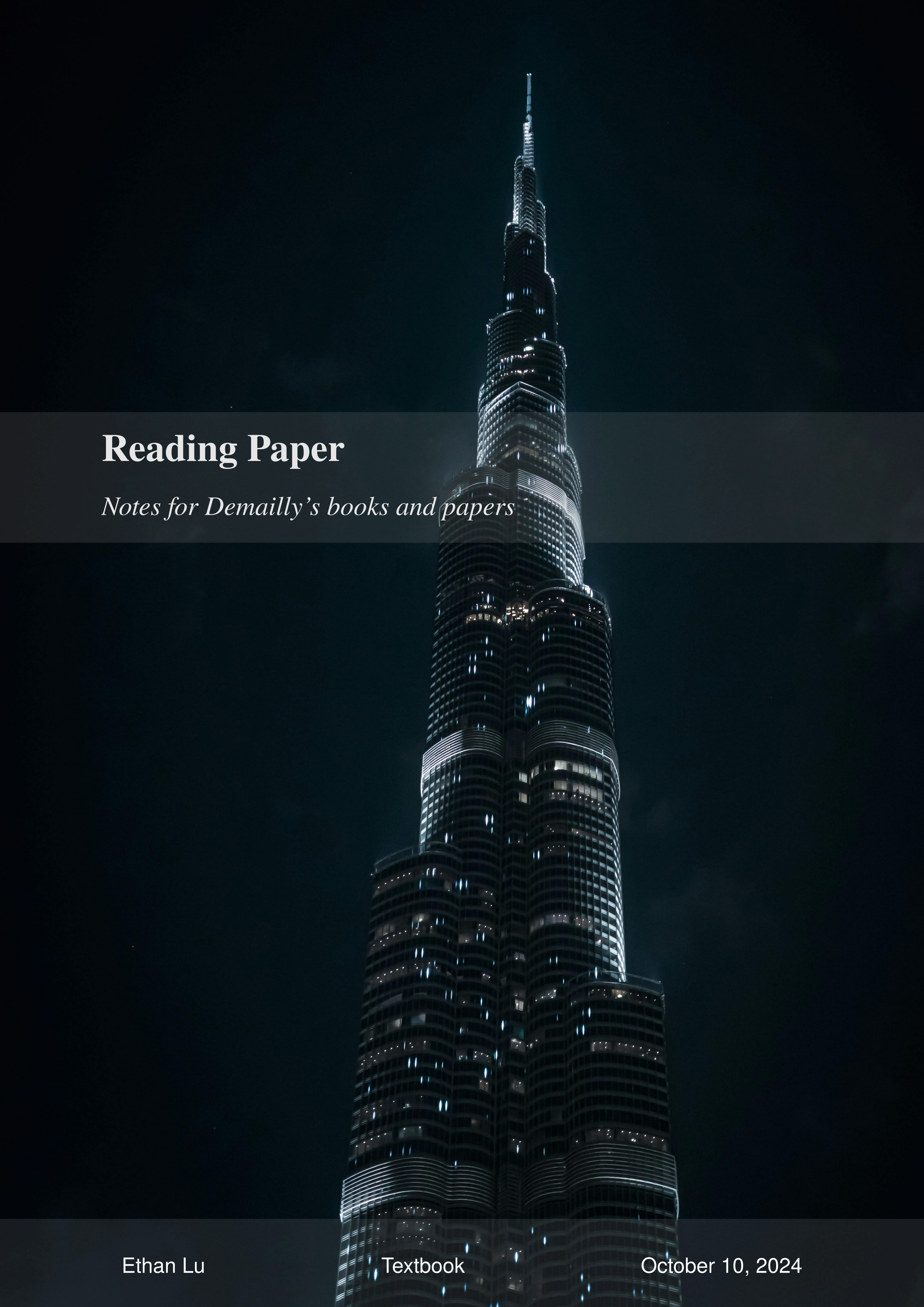

Reading Paper

Notes for Demailly's books and papers

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Contents

I
Proof of Opennes Conjecture

1 Notes for the Proof of Openness Conjecture

1.1. Organizing the proof strategy

1.1.1. The case 1

Lemma 1.1.1 (Lemma 2.3 (see [8]. *; see also [22]).] Let $F \in \mathcal{O}_n$ and $g_1, \ldots, g_s \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at the origin $o \in \mathbb{C}^n$. Assume that for any given neighborhood of o, $|F| \leq C\,|(g_1, \ldots, g_s)|$ does not hold for any constant C. Then there exists a germ of an analytic curve γ through o, satisfying $\gamma \cap \{F = 0\} \subseteq \{o\}$, such that $\frac{g_i}{F}\big|_\gamma$ is holomorphic on $\gamma \backslash o$ with 1.1.1 for any $i \in \{1, \ldots, s\}$, where $\widetilde{\frac{g_i}{F}}$ is the holomorphic extension of $\frac{g_i}{F}$ from $\gamma \backslash o$ to γ.*

Theorem 1.1.1 (cf [1]). *Let φ be a negative plurisubharmonic function on the unit polydisc $\Delta^n \in \mathbb{C}^n$. Suppose F is a holomorphic function on Δ^n, which satisfies*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty,$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. Then for some $r \in (0,1)$, there exists a number $p > 1$ such that

$$\int_{\Delta_r^n} |F|^2 e^{-p\varphi} d\lambda_n < +\infty.$$

Proof. As

$$\int_{\Delta' \times \Delta''} |F_j|^2 e^{-p_j \varphi} d\lambda_n \leqslant C \int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1}$$

and we already known that[1]

$$\int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1} \leqslant 2 \int_{H_j} |F|^2 e^{-\varphi} d\lambda_{n-1} = o\left(\frac{1}{|a_j|^2}\right),$$

thus we have

$$\begin{aligned}\int_{\Delta' \times \Delta''} |F_j|^2 e^{-p_j \varphi} d\lambda_n &\leqslant C \int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1} \\ &\leqslant C \int_{H_j} |F|^2 e^{-\varphi} d\lambda_{n-1} = o\left(\frac{1}{|a_j|^2}\right).\end{aligned}$$

[1] see (3.7) in original paper.

Then by $F_j|_{H_j} = F|_{H_j}$ and negativeness of φ, we have

$$\int_{\Delta'\times\Delta''} |F_j|^2 d\lambda_n \leqslant \int_{\Delta'\times\Delta''} |F_j|^2 e^{-p_j\varphi} d\lambda_n = o\left(\frac{1}{|a_j|^2}\right),$$

which is to say

$$\int_{\Delta'\times\Delta''} |F_j|^2 d\lambda_n = o\left(\frac{1}{|a_j|^2}\right).$$

We choose r_0 small enough such that[2]

$$\int_{\Delta_{r_0}} |F|^2 e^{-\varphi} d\lambda_1 < +\infty.$$

Step 1: Theorem 1.1.1 for the dimension one case. We first prove Theorem 1.1.1 for the dimension one case, which is elementary but instructive. Our proof for the general dimension case is quite similar. Actually the proofs for both cases are parallel.

We choose r_0 small enough such that $\int_{\Delta_{r_0}} |F|^2 e^{-\varphi} d\lambda_1 < +\infty$[3]. Then there exist complex numbers $a_j \to 0$ $(j \to +\infty)$ such that

$$\left|F\left(a_j\right)\right|^2 e^{-\varphi\left(a_j\right)} = o\left(\frac{1}{\left|a_j\right|^2}\right).$$

As $\left|F\left(a_j\right)\right|^2 e^{-\varphi\left(a_j\right)} < +\infty$, then one can find $p_j > 1$ small enough such that[4]

$$\left|F\left(a_j\right)\right|^2 e^{-p_j\varphi\left(a_j\right)} \leq 2\left|F\left(a_j\right)\right|^2 e^{-\varphi\left(a_j\right)} = o\left(\frac{1}{\left|a_j\right|^2}\right). \tag{1.1}$$

Using movable (respect to a_j) the Ohsawa-Takegoshi L^2 extension theorem on Δ, we obtain holomorphic functions F_j on Δ such that $F_j\big|_{a_j} = F\left(a_j\right)$ and[5]

$$\int_{\Delta} \left|F_j\right|^2 e^{-p_j\varphi} d\lambda_1 \leq \mathbf{C}\left|F\left(a_j\right)\right|^2 e^{-p_j\varphi\left(a_j\right)}, \tag{1.2}$$

where $\mathbf{C}$ is a universal constant.

By inequality (1.1) and negativeness of φ, we obtain that

$$\int_{\Delta} \left|F_j\right|^2 d\lambda_1 = o\left(\frac{1}{\left|a_j\right|^2}\right). \tag{1.3}$$

By contradiction, assume that Theorem 1.1.1 does not hold for $n = 1$; that is to say, $\int_{\Delta_r} |F|^2 e^{-p_j\varphi} d\lambda = +\infty$ for any $r > 0$ and any $j \in \{1, 2, \ldots\}$.

[2] The reason is that from the assumption of theorem 1.1.1, we have known that F is a holomorphic function on Δ^n, which satisfies

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty,$$

thus we can find a small enough r_0 such that

$$\int_{\Delta_{r_0}} |F|^2 e^{-\varphi} d\lambda_1 < +\infty.$$

[3] It has already been explained above.

[4] As

$$\begin{aligned} & e^{-p_j\varphi(a_j)} \\ = & [e^{-\varphi(a_j)}]^{p_j} \\ = & e^{-\varphi(a_j)} \cdot [e^{-\varphi(a_j)}]^{p_j-1}, \end{aligned}$$

$p_j > 1$ $(p_j - 1 > 0)$ and the negativeness of φ $(-\varphi > 0)$, there exists p_j small enough such that $p_j - 1 \leqslant \ln 2$, thus we have

$$[e^{-\varphi(a_j)}]^{p_j-1} \leqslant e^{p_j-1} \leqslant 2.$$

[5] This is because $|F(a_j)|^2 \cdot e^{-p_j(a_j)}$ is a constant.

Assertion. Since $\{F = 0\} \cap \Delta_{r_0} \subseteq \{o\}$, then it follows from inequality (1.2) that one can derive that F/F_j *is unbounded*. Otherwise, the boundedness would imply the finiteness of the integral of $|F|^2 e^{-p_j\varphi}$, according to inequality (1.2). This contradicts the assumption. Then there exists a holomorphic function h_j on Δ_{r_0} satisfying

(1) $F_j|_{\Delta_{r_0}} = F|_{\Delta_{r_0}} h_j$,
(2) $h_j(o) = 0$,
(3) $h_j(a_j) = 1$.

According to Lemma 2.1[6], it follows that

$$\frac{C_1}{|a_j|^2} \leq \int_{\Delta_{r_0}} |F_j|^2 \, d\lambda_1,$$

which contradicts equality (1.3), where $C_1 > 0$ is independent of j. We have thus proved Theorem 1.1.1 for $n = 1$.

Step 2: Theorem 1.1.1 for n. By induction on the dimension n, one may assume that Theorem 1.1.1 holds for $n - 1$.

We prove Theorem 1.1.1 for the general dimension n by contradiction. Assume that Theorem 1.1.1 for n is not true, therefore, for some negative psh function φ, there exists a holomorphic function F such that

$$\int_{\Delta^n_{r_0}} |F|^2 e^{-\varphi} d\lambda_n < +\infty, \tag{1.4}$$

for some $r_0 > 0$, and

$$\int_{\Delta^n_r} |F|^2 e^{-p\varphi} d\lambda_n = +\infty, \tag{1.5}$$

for any $r \in (0, r_0)$ and $p > 1$. That is to say, the germ of the holomorphic function F is in $\mathcal{I}(\varphi)_o$ but not in $\mathcal{I}_+(\varphi)_o$.

By Lemma 2.4[7] and equality 1.5, it follows that there exists a germ of an analytic curve γ through o satisfying $\left\{F|_\gamma = 0\right\} \subseteq \{o\}$ such that for any germ of holomorphic function g in $\mathcal{I}_+(\varphi)_o$, there exists a holomorphic function h_g on γ satisfying

$$h_g(o) = 0 \quad \text{and} \quad g|_\gamma = F|_\gamma h_g. \tag{1.6}$$

This plays a similar role with the assertion in the proof for dimension 1.

Using the local parametrization of γ (see [5]), without loss of generality, one may assume that γ and $\Delta' \times \Delta''$ are as in Section 2.1.

By inequality (1.4), it follows that there exist hyperplanes $H_j := H_{a_j} = \left\{z' = a_j\right\}$ that satisfy $a_j \to 0 (j \to \infty)$ and

$$\int_{H_j} |F|^2 e^{-\varphi} d\lambda_{n-1} = o\left(\frac{1}{|a_j|^2}\right). \tag{1.7}$$

[6] Let $f \not\equiv 0$ be a holomorphic function on the disc Δ_r of radius r containing the origin o in $\mathbb{C}$. Let h_a be a holomorphic function on Δ_r, which satisfies $h_a(o) = 0$ and $h_a(b) = 1$ for any $b^k = a$ (k is a positive integer), where $a \in \Delta_r$ whose norm is small enough. Then we have

$$\int_{\Delta_r} |f|^2 \, |h_a|^2 \, d\lambda_1 > C_1 |a|^{-2}$$

where C_1 is a positive constant independent of a and h_a.

[7] Assume that $F \in \mathcal{O}_n$ is a holomorphic function on some neighborhood V of o that is not a germ of

$$\mathcal{I}\left(\psi_{j_1}\right)_o = \left(\cup_{j=1}^{\infty} \mathcal{I}\left(\psi_j\right)\right)_o.$$

Then there exists a germ of an analytic curve (γ, o) such that

$$\left.\frac{\widetilde{g o \gamma}}{F o \gamma}\right|_o = 0$$

holds for any germ (g, o) of $\mathcal{I}\left(\psi_{j_1}\right)_o$, where $\frac{\widetilde{g o \gamma}}{F o \gamma}$ is the holomorphic extension of $\frac{g o \gamma}{F o \gamma}$ from $\gamma \backslash o$ to γ.

[8] It may be the reason that from the truth of one case in (1.1), we can obtain that when $a_j \to 0$, we have (1.7), thus for any given j, by dominated convergence theorem, (1.8) is true.

According to *the induction assumption* and *the Lebesgue dominated convergence theorem*, it follows that for any given j, there exists a small enough $p_j > 1$ such that[8]

$$\int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1} \leq 2\int_{H_j} |F|^2 e^{-\varphi} d\lambda_{n-1} = o\left(\frac{1}{|a_j|^2}\right). \qquad (1.8)$$

Using movably (respect to $j \to \infty$) *the Ohsawa-Takegoshi L^2 extension theorem* on $\Delta' \times \Delta''$, we obtain a holomorphic function $F_j := F_{a_j}$ on $\Delta' \times \Delta''$ for each j such that $F_j|_{H_j} = F|_{H_j}$, and

$$\int_{\Delta' \times \Delta''} \left|F_j\right|^2 e^{-p_j\varphi} d\lambda_n \leq \mathbf{C}\int_{H_j} |F|^2 e^{-p_j\varphi} d\lambda_{n-1},$$

where $\mathbf{C}$ is a universal constant. By inequality (1.8) and negativeness of φ, it follows that

$$\int_{\Delta' \times \Delta''} \left|F_j\right|^2 d\lambda_n = o\left(\frac{1}{|a_j|^2}\right) \qquad (1.9)$$

Note that $F_j|_{H_j} = F|_{H_j}$ and $\left(F_j, o\right) \in \mathcal{I}_+(\varphi)_o$ but $(F, o) \notin \mathcal{I}_+(\varphi)_o$. According to equality (1.6) and Lemma 2.2[9], it follows that

$$\int_{\Delta' \times \Delta''} \left|F_j\right|^2 d\lambda_n \geq \frac{C_2}{|a_j|^2}$$

where $C_2 > 0$ is independent of j, which contradicts equality (1.9). We have thus proved Theorem 1.1.1 for n. The proof of Theorem 1.1.1 is now complete.

□

[9] Let F be a holomorphic function on $\Delta' \times \Delta''$ satisfying $F|_\gamma \not\equiv 0$. Denote by hyperplane $H_a := \{z' = a\}$ near o. Let F_a be the holomorphic extension of $F|_{H_a}$ on $\Delta' \times \Delta''$ such that there exists a holomorphic function h_a on γ satisfying

(1) $F_a|_\gamma = F|_\gamma\, h_a$,
(2) $h_a(o) = 0$.

Then we have

$$\int_{\Delta' \times \Delta''} |F_a|^2 \, d\lambda_n \geq \frac{C_2}{|a|^2}$$

where C_2 is a positive constant independent of a, H_a and F_a.

1.2. Some topics may be benifit for the future research

1.2.1. Multiplier ideal sheaf $\mathcal{J}(\varphi)$

Definition 1.2.1 (Multiplier ideal sheaf)**.** *Multiplier ideal sheaf $\mathcal{J}(\varphi)$ is considered as the subsheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally integrable (/summable).*

It is worth to point out that it is a coherent algebraic sheaf *over X and satisfies*

$$H^q(X, K_X \otimes L \otimes \mathcal{J}(\varphi)) = 0$$

for all $q \geq 1$ if the curvature of L is positive in the sense of current.

The two efficient ways to solve the singularities are *Multiplier ideal sheaf* by Nadel (Which can be seen as a generalization of Kawamata-Viehweg's vanishing theorem.) and *The theory of positive currents* by Lelong. Currents can be seen as generalization of **algebraic cycles**.

1.2.2. Plurisubharmonic functions

Definition 1.2.2 (Plurisubharmonic functions). *A function $u : \Omega \longrightarrow [-\infty, +\infty\,[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if*

(a) u is upper semicontinuous;

(b) for every complex line $L \subset \mathbb{C}^n$, $u_{\upharpoonright \Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \complement\Omega)$, the function u satisfies the mean value inequality[10]

$$u(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u\left(a + e^{\mathrm{i}\theta}\xi\right) d\theta.$$

The set of psh functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

[10] A vital characteristic, which is equivalent to the integrability of the inegral.

Theorem 1.2.1. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh if and only if the hermitian form*

$$Hu(a)(\xi) = \sum_{1 \leqslant j,k \leqslant n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k(a)} \xi_j \bar{\xi}_k$$

is semi-positive at every point $a \in \Omega$.

1.2.3. Positive Currents

A current of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathscr{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element Θ in the dual space $\mathscr{D}'_p(M) := (\mathscr{D}^p(M))'$ of the space $\mathscr{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$\langle \Theta, \alpha \rangle = \int_M \Theta \wedge \alpha, \quad \alpha \in \mathscr{D}^p(M). \tag{1.10}$$

A basic example is the current of integration $[S]$ over a compact oriented submanifold S of M :

$$\langle [S], \alpha \rangle = \int_S \alpha, \quad \deg \alpha = p = \dim_{\mathbb{R}} S \tag{1.11}$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S] = 0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta \in \mathscr{D}'_p(M)$. The current Θ is said to be closed if $d\Theta = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathscr{D}'^{p,q}(X) = \mathscr{D}'_{n-p,n-q}(X), \quad n = \dim X, \tag{1.12}$$

[2] P. Lelong. "Intégration sur un ensemble analytique complexe". In: *Bulletin de la Société Mathématique de France* 85 (1957), pp. 239–262. DOI: 10.24033/bsmf.1488. URL: http://www.numdam.org/articles/10.24033/bsmf.1488/

the space of currents of bidegree (p,q) and bidimension $(n-p,n-q)$ on X. According to [2], a current Θ of bidimension (p,p) is said to be (weakly) positive if for every choice of smooth $(1,0)$-forms $\alpha_1,\ldots,\alpha_p$ on X the distribution

$$\Theta \wedge (\mathrm{i}\alpha_1 \wedge \bar{\alpha}_1) \wedge \ldots \wedge (\mathrm{i}\alpha_p \wedge \bar{\alpha}_p)$$

is a positive measure.

1.3. Perverse Sheaf and Intersection Cohomology

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2. test2
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2.2. test2

2.3. Intersection coHomology and Perverse Sheaves

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Test the long title

Bibliography

[1] Q. Guan and X. Zhou. “A proof of Demailly’s strong openness conjecture”. In: *Annals of mathematics* (2015), pp. 605–616.

[2] P. Lelong. “Intégration sur un ensemble analytique complexe”. In: *Bulletin de la Société Mathématique de France* 85 (1957), pp. 239–262. DOI: 10.24033/bsmf.1488. URL: http://www.numdam.org/articles/10.24033/bsmf.1488/.