Reading Paper

Notes for some books and papers

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Textbook

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New Topics

1 Notes for some new topics

1.1. Perverse Sheaf and Intersection Cohomology

1.1.1. Poincaré Duality

Definition 1.1.1 (cap product)**.** *On an n-manifold X, the* cap product *is*

$$C^i(X) \times C_n(X) \xrightarrow{\frown} C_{n-i}(X),$$

where C_i and C^i denote the (simplicial/singular) i-(co)chains on X with $\mathbb{Z}$- coefficients.

The cap product is defined as follows: if $a \in C^{n-i}(X), b \in C^i(X)$, and $\sigma \in C_n(X)$ then

$$a(b \frown \sigma) = (a \smile b)(\sigma).$$

The cap product is compatible with the boundary maps, thus it descends to a map

$$H^i(X;\mathbb{Z}) \times H_n(X;\mathbb{Z}) \xrightarrow{\frown} H_{n-i}(X;\mathbb{Z}).$$

The following statement lies at the heart of algebraic and geometric topology. For a modern proof see, e.g., [1, Section 3.3]:

Theorem 1.1.1 (Poincaré Duality)**.** *Let X be a closed, connected, oriented topological n-manifold with fundamental class $[X]$. Then capping with $[X]$ gives an isomorphism*

$$H^i(X;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(X;\mathbb{Z})$$

for all integers i.

As a consequence of Theorem 1.1.1 one gets a non-degenerate pairing

$$H_i(X,\mathbb{C}) \otimes H_{n-i}(X;\mathbb{C}) \longrightarrow \mathbb{C}.$$

In particular, the Betti numbers[1] of X in complementary degrees coincide, i.e.,

$$\dim_{\mathbb{C}} H_i(X;\mathbb{C}) = \dim_{\mathbb{C}} H_{n-i}(X;\mathbb{C}).$$

Note that the existence of Hodge structures on the cohomology of complex projective manifolds leads to an important consequence that the odd Betti numbers of a complex projective manifold are even.

[1] It is known as the rank of the corresponding homology groups.

1.1.2. Understanding Why the Odd Betti Numbers of a Complex Projective Manifold are Even?

For a complex projective manifold, the odd Betti numbers are always even. This can be understood through a combination of complex geometry and topological properties. Let's break this down in detail:

1. **Definition of Betti Numbers:** Betti numbers, denoted as b_k, quantify the topology of a manifold by representing the rank of the k-th homology group $H_k(M,\mathbb{Z})$ (or the k-th cohomology group $H^k(X;\mathbb{Z})$). They indicate the number of k-dimensional "holes" or independent cycles in the manifold. For instance, b_0 represents the number of connected components, b_1 represents the number of independent loops, and so on.
2. **Complex Projective Manifolds:** A complex projective manifold is a complex manifold that can be embedded into complex projective space. These manifolds have a rich structure and are inherently Kähler manifolds, meaning they have a compatible triple structure of a complex structure, a symplectic structure, and a Riemannian metric.
3. **Hodge Decomposition:** For a Kähler manifold M, the complex de Rham cohomology group $H^k(M,\mathbb{C})$ can be decomposed into a direct sum of Hodge components:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$

   Here, $H^{p,q}(M)$ denotes the space of harmonic forms of type (p,q), and $h^{p,q} = \dim H^{p,q}(M)$ are the Hodge numbers.
4. **Relation Between Betti Numbers and Hodge Numbers:** The k-th Betti number b_k is related to the Hodge numbers $h^{p,q}$ by the following formula:

$$b_k = \sum_{p+q=k} h^{p,q}$$

5. **Symmetry of Hodge Numbers:** For Kähler manifolds, there is a fundamental symmetry in the Hodge numbers:

$$h^{p,q} = h^{q,p}$$

   This symmetry implies that the Hodge components $H^{p,q}$ and $H^{q,p}$ appear in pairs.
6. **Implication for Odd Betti Numbers:** Due to the symmetry $h^{p,q} = h^{q,p}$, the sum of Hodge numbers for odd k (such as b_1, b_3, etc.) will always be an even number because each nonzero $h^{p,q}$ has a matching $h^{q,p}$. Thus, the odd Betti numbers must be even.

7. **Example:** Consider the complex projective space $\mathbb{CP}^n$. The Hodge numbers are as follows:
   - $h^{0,0} = 1$
   - $h^{1,1} = 1$
   - $h^{2,2} = 1$ (if $n \geq 2$)
   - All other $h^{p,q} = 0$.

   The Betti numbers calculated are:
   - $b_0 = h^{0,0} = 1$
   - $b_2 = h^{1,1} = 1$
   - $b_4 = h^{2,2} = 1$ (for $n \geq 2$)
   - The odd Betti numbers $b_1 = b_3 = 0$.

   This example shows that odd Betti numbers are zero (which is even) for $\mathbb{CP}^n$.
8. **Conclusion:** In summary, the reason the odd Betti numbers of a complex projective manifold are even is due to the Hodge decomposition and the inherent symmetry of Hodge numbers on Kähler manifolds.

Remark. In the diagram, δ is labeled as a **meridian**, and η is labeled as a **longitude**. The reason why the homology class of δ vanishes can be explained from the perspective of algebraic topology.

1. **Meridian as a Boundary**: From the diagram, the meridian δ appears to be the boundary of a region. In homology theory, any curve that forms the boundary of a region has a **trivial homology class** (i.e., it vanishes). This is because a boundary does not represent a closed, independent cycle—it is merely the edge of a higher-dimensional region. In other words, since δ bounds some region within X, it is a boundary, and hence its homology class must vanish.

2. **Boundaries and Homology in Algebraic Topology**: In homology theory, the boundary of a higher-dimensional object always has a zero homology class. For example, in the case of a surface, if a loop (like the meridian δ) is the boundary of a region, its homology class is trivial because it does not represent a free, closed cycle but rather a boundary.

3. **Betti Number and Hodge Decomposition**: The passage also mentions that δ's homology class vanishes, and this is related to the fact that the first Betti number b_1 of X is odd. According to Hodge theory, if the first Betti number is odd, a complete Hodge decomposition cannot exist. This implies that certain homology classes in $H^1(X;\mathbb{C})$ cannot be fully decomposed into pure $(1,0)$ and $(0,1)$ components. This is connected to the fact that δ's homology class vanishes in the homology of X.

Summary:

1. The meridian δ is the boundary of some region, and by the

fundamental property of homology, **any boundary has a trivial homology class**.

2. This follows from basic algebraic topology, where boundaries do not contribute to non-trivial homology classes.
3. Additionally, the fact that X has an odd first Betti number implies that a full Hodge decomposition is not possible for $H^1(X;\mathbb{C})$, which further supports why the homology class of δ is trivial.

1.1.3. Lefschetz Hyperplane Section Theorem

A map $f : X \to Y$ is called **homotopy equivalence** if there is a map $g : Y \to X$ such that $fg \cong \mathbb{I}$ and $gf \cong \mathbb{I}$. It is an equivalent relation and X and Y are homotopy equivalent if they are the deformation retracts of the third space Z containing them. In general, we can take Z as the mapping Cylinder M_f of any homotopy equivalence $f : X \to Y$. As we know that M_f deformation retracts to Y, it suffices to prove that M_f also deformation retracts to its other end X.

1.1.4. Hard Lefschetz Theorem

Let X be a nonsingular complex projective variety of complex dimension n, and let H be a generic hyperplane. The intersection $X \cap H$ yields a homology class $[X \cap H] \in H_{2n-2}(X;\mathbb{Z})$, and its Poincaré dual is a degree-two cohomology class, denoted by $[H] \in H^2(X;\mathbb{Z})$. The Lefschetz operator is the map

$$L : H^i(X;\mathbb{C}) \xrightarrow{\smile [H]} H^{i+2}(X;\mathbb{C})$$

defined by taking the cup product with $[H]$. Then the following important result holds:

Theorem 1.1.2 (Hard Lefschetz Theorem)**.** *The map*

$$L^i : H^{n-i}(X;\mathbb{C}) \xrightarrow{\smile [H]^i} H^{n+i}(X;\mathbb{C})$$

is an isomorphism, for all integers $i \geq 0$.

No, the statement as written is not generally correct. Let's carefully analyze the situation.

The "residue" of a function $f(z,w)$ with respect to w at a root b_i is related to the partial derivative of f with respect to w. Specifically, for a fixed z, if $f(z,w)$ has roots $w = b_1, b_2, \ldots, b_d$, then the residue of f at $w = b_i$ is given by:

$$\operatorname*{Res}_{w=b_i}\left(\frac{1}{f(z,w)}\right) = \frac{1}{\frac{\partial f}{\partial w}(z,b_i)}.$$

The residue you seem to be referring to might be the **sum of the residues**. The sum of the residues of a meromorphic function

$\frac{1}{f(z,w)}$ over all its poles $w = b_1, \ldots, b_d$ (roots of $f(z,w) = 0$) is given by:

$$\sum_{i=1}^{d} \operatorname*{Res}_{w=b_i} \left(\frac{1}{f(z,w)} \right).$$

Using the residue theorem, this sum is typically related to the behavior of $f(z,w)$ at infinity or some global property, but it is not simply the sum of the roots $b_1 + b_2 + \cdots + b_d$.

Important Note:

If $f(z,w)$ is a polynomial in w of degree d, then ***Viète's formulas*** imply that the sum of the roots $b_1 + b_2 + \cdots + b_d$ is given by the coefficient of w^{d-1} (up to a sign) in $f(z,w)$, divided by the leading coefficient of w^d. This is unrelated to residues.

Bibliography

[1] A. Hatcher. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002.