Research Paper Comments

L^q-extension theorem for jets on weakly pseudoconvex kähler manifolds

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L^q-extension theorem for jets on weakly pseudoconvex kähler manifolds

1.1. Introduction

T. Ohsawa-K. Takegoshi established a remarkable extension theorem of holomorphic functions defined on a bounded pseudoconvex domain in $\mathbb{C}^n$ with growth control in [12]. Since then, many versions and variants of the L^2 extension theorems have been studied (see [2, 9, 10, 13, 14], etc.). These results lead to numerous applications in algebraic geometry and complex analysis.

One interesting problem is to study the L^2 extension theorem for jets. The first such result was given by D. Popovici [13], which generalized the L^2 extension theorems of Ohsawa-Takegoshi-Manivel to the case of jets of sections of a line bundle over a weakly pseudoconvex Kähler manifold. Then J.-P. Demailly [4] considered the extension from more general non-reduced varieties. Following a new method of B. Berndtsson-L. Lempert [1], G. Hosono [8] proved an L^2 extension theorem for jets with optimal estimate on a bounded pseudoconvex domain in $\mathbb{C}^n$ (see also [11]).

The idea of considering variable denominators was first introduced by J. McNeal-D. Varolin [10]. They obtained some results on weighted L^2 extension of holomorphic top forms with values in a holomorphic line bundle, where the weights used are determined by the variable denominators. Recently, X. Zhou-L. Zhu [14] proved an L^2 extension theorem for holomorphic sections of holomorphic line bundles equipped with singular metrics on weakly pseudoconvex Kähler manifolds. Furthermore, they obtained optimal constants corresponding to variable denominators.

The method of solving undetermined functions with ODEs was first used in [16]. From then on, a lot of spectacular works appear along this line, such as [6, 14, 15], etc. Several optimal L^2 extension theorems have been proved in this process.

The main goal of this paper[1] is to apply the methods of Zhou-Zhu [14] and Demailly [5] to L^q jet extension to slightly generalize the theorem 1.2 in [14]. As an application of our main theorem, we also obtain a corollary of a local $L^{\frac{2}{q}}$ extension theorem.

We make precise the setting for our work. Let X be an n-dimensional weakly pseudoconvex manifold with Kähler metric ω, and E a Hermitian holomorphic vector bundle of rank $m \geq 1$ over X. Assume that $s \in H^0(X, E)$ is transverse to the zero section. Set

$$Y := \{x \in X : s(x) = 0\}.$$

[1] An extended version of the theorem 1.2 in [14] is demonstrated using the existing method.

Furthermore, let L be a holomorphic line bundle equipped with a smooth Hermitian metric satisfying an appropriate positivity condition.

We denote by $\bigwedge^{r,s} T_X^*$ the bundle of differential forms of bidegree (r,s) on X, and $\mathcal{J}_Y$ the sheaf of germs of holomorphic functions on X which vanish on Y. For any integer $k \geq 0$, let $\mathcal{O}_X/\mathcal{J}_Y^{k+1}$ be the nonlocally free sheaf of k-jets which are "transversal" to Y. Fix a point $y \in X$ and a Stein neighborhood U in X of y. Then this gives rise to a surjective morphism

$$H^0(U, K_X \otimes L) \longrightarrow H^0\left(U, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$$

of local section spaces, and an arbitrary local lifting $\tilde{f} \in H^0(U, K_X \otimes L)$ of f. For any transversal k-jet $f \in H^0\left(U, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ and any weight function $\rho > 0$ on U, the pointwise ρ-weighted norm associated to the section s, was defined by [13, Definition 0.1.1]:

$$|f|^2_{s,\rho,(k)}(y) := |\tilde{f}|^2_L(y) + \frac{\left|\nabla^1 \tilde{f}\right|^2_L}{|\bigwedge^m(\mathrm{d}s)|_E^{2\frac{1}{m}}\rho^{2(m+1)}}(y) + \cdots + \frac{\left|\nabla^k \tilde{f}\right|^2_L}{|\bigwedge^m(\mathrm{d}s)|_E^{2\frac{k}{m}}\rho^{2(m+k)}}(y),$$

and the $L^2_{(k)}$ weighted norm by:

$$\|f\|^2_{s,\rho,(k)} = \int_Y \frac{|f|^2_{s,\rho,(k)}}{|\bigwedge^m(\mathrm{d}s)|^2_E}\mathrm{d}V_{Y,\omega}.$$

Here for $i = 0, \ldots, k$, $\nabla^i \widetilde{f}$ is constructed by induction as the projection of the $(1,0)$-part

$$\nabla^{1,0}\left(\nabla^k \tilde{f}\right) \in C^\infty\left(U, K_X \otimes L \otimes S^{j-1}N^*_{Y/X} \otimes T^*_X\right)$$

of $\nabla\left(\nabla^k \widetilde{f}\right)$ with the associated Chern connection ∇ to $C^\infty\left(U, K_X \otimes L \otimes S^j N^*_{Y/X}\right)$, induced by the surjective bundle morphism $K_X \otimes L \otimes T^*_X|_Y \to K_X \otimes L \otimes N^*_{Y/X}$.

It is worthwhile to notice that the norm $|f|^2_{s,\rho,(k)}$ of the k-jet f at the point $y \in Y$ is independent of the choice of the local lifting $\tilde{f}$. Moreover, one has the following notations [13, Notation 0.1.3]:

(a) For a transversal k-jet $f \in H^0\left(U, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$, denote $\nabla^j f := \left(\nabla^j \widetilde{f}\right)_{|U\cap Y}$, for all $j = 0, \ldots, k$ and an arbitrary lifting $\tilde{f} \in H^0(U, K_X \otimes L)$ of f.

(b) For every integer $k \geq 0$, and every open set $U \subset X$, set

$$J^k_U : H^0(U, K_X \otimes L) \longrightarrow H^0\left(U, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$$

as the cohomology group morphism induced by the projection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_Y^{k+1}$.

We refer to [13, pp. 2-5] for more details about the notations and the construction of relevant metrics on jets.

In [14, p. 136], Zhou-Zhu defined the *variable denominators*. Let $\mathfrak{R}$ be the class of functions defined by

$$\left\{ R \in C^\infty(-\infty, 0] : \begin{array}{c} R > 0,\ R' \le 0,\ \int_{-\infty}^{0} \frac{1}{R(t)} \mathrm{d}t < +\infty \\ \text{and } e^t R(t) \text{ is bounded above on } (-\infty, 0] \end{array} \right\}.$$

Denote $\int_{-\infty}^{0} \frac{1}{R(t)} \mathrm{d}t$ by C_R. Notice that the function $R(t)$ equals to the function $\frac{1}{c_A(-t)e^t}$ defined just before the main theorems in [16, p. 1143] when $A = 0$. With such preparation, our main theorem is as follows[2].

Theorem 1.1.1 (Main Theorem)**.** *Let (X, ω) be a weakly pseudoconvex complex n-dimensional manifold possessing a Kähler metric ω, ψ be a plurisubharmonic function on X, E be a holomorphic vector bundle of rank m over X equipped with a smooth Hermitian metric $(1 \le m \le n)$, and s be a global holomorphic section of E. Assume that s is transverse to the zero section, and let*

$$Y := \{x \in X : s(x) = 0, \bigwedge^m (\mathrm{d}s)(x) \neq 0\}.$$

Let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L, which is written locally as $e^{-\varphi_L}$ for some function $\varphi_L \in L^1_{loc}$ with respect to a local holomorphic frame of L. Assume that $\frac{q}{2}\varphi_L + \left(1 - \frac{q}{2}\right)\varphi_\omega + \psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Let $0 < q \le 2$. Moreover, assume that the $(1,1)$-form

(i) $\frac{q}{2}\sqrt{-1}\Theta_L + \left(1 - \frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 + \sqrt{-1}\partial\bar{\partial}\psi \ge 0$ *holds on $X \backslash Y$,*

and that there is a continuous function $\alpha > 0$ on X such that the following two inequalities hold on $X \backslash Y$:

(ii) $\frac{q}{2}\sqrt{-1}\Theta_L + \left(1 - \frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 + \sqrt{-1}\partial\bar{\partial}\psi \ge \frac{\left\{\sqrt{-1}\Theta_E s, s\right\}_E}{\alpha|s|_E^2}$,

(iii) $\psi + m\log|s|^2 \le -2m\alpha$

Then, for every relatively compact open subset $\Omega \subset X$, and every k-jet $f \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ satisfying

$$C_f := \int_Y \frac{|f|^q_{s,\rho,(k)} e^{-\psi}}{|\bigwedge^m (ds)|_E^2} dV_{Y,\omega} < +\infty.$$

Furthermore, assume that there exists $F_1^{(k)} \in H^0(X, K_X \otimes L)$ such that $J^k F_1^{(k)} = f$ and

$$C_{F_1} := \int_\Omega \frac{\left|F_1^{(k)}\right|_L^q e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega} < +\infty.$$

[2] The main innovation of this paper is the extension of Theorem 1.2 from [14] to the jets case.

Then there exists $F^{(k)} \in H^0(X, K_X \otimes L)$ *such that* $J^k F^{(k)} = f$ *and*

$$\int_\Omega \frac{\left|F^{(k)}\right|_L^q e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega} \le C_{m,R}^{(k)} C_f,$$

where $C_{m,R}^{(k)} > 0$ *is a constant depending only on* m, R, k, E, *and* $\sup_\Omega \|i\Theta_L\|$.

Let p be a positive integer. If we take $q = \frac{2}{p}$ and replace L by $K_X^{p-1} \otimes L$, which is equipped with the metric $e^{(p-1)\varphi_\omega - \varphi_L}$, then we can get from Main Theorem the following corollary.

Corollary 1.1.1. *Assume that* $\frac{\varphi_L}{p} + \psi$ *is quasi-plurisubharmonic and* φ_L *is locally bounded above. Moreover, assume that*

(i) $\frac{\sqrt{-1}\Theta_L}{p} + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 \ge 0$ *holds on* $X \backslash Y$,

and that there is a continuous function $\alpha > 0$ *on* X *such that the following two inequalities hold on* $X \backslash Y$:

(ii) $\frac{\sqrt{-1}\Theta_L}{p} + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 \ge \frac{\left\{\sqrt{-1}\Theta_E s, s\right\}_E}{\alpha|s|_E^2}$,

(iii) $\psi + m\log|s|^2 \le -2m\alpha$.

For every relatively compact open subset $\Omega \subset X$, *and every* k*-jet* $f \in H^0(X, K_X^p \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$, *such that*

$$C_f := \int_Y \frac{(|f|_L)^{\frac{2}{p}} e^{-\psi}}{|\bigwedge^m(ds)|_E^2} dV_Y < +\infty.$$

Furthermore, assume that there exists a k*-jet* $F_1^{(k)} \in H^0(\Omega, K_X^p \otimes L)$ *such that* $J^k F_1^{(k)} = f$ *and*

$$C_{F_1} := \int_\Omega \frac{\left(\left|F_1^{(k)}\right|_L\right)^{\frac{2}{p}} e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_X < +\infty.$$

Then there exists a k*-jet* $F^{(k)} \in H^0(\Omega, K_X^p \otimes L)$, *such that* $J^k F^{(k)} = f$ *and*

$$\int_\Omega \frac{\left(|F^{(k)}|_L\right)^{\frac{2}{p}} e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_X \le C_{m,R}^{(k)} C_f,$$

where $C_{m,R}^{(k)} > 0$ *is a constant depending only on* m, R, k, E, *and* $\sup_\Omega \|i\Theta_L\|$.

1.2. Preliminaries

Lemma 1.2.1 (Baisc a priori inequality)**.** *Let* E *be a hermitian vector bundle on a complex manifold* X *equipped with a Kähler metric* ω. *Let* $\eta, \lambda > 0$ *be smooth functions on* X. *Then for every* *form* $u \in \mathcal{D}\left(X, \bigwedge^{p,q} T_X^\star \otimes E\right)$ *with compact support we have*

$$\left\|\left(\eta^{\frac{1}{2}}+\lambda^{\frac{1}{2}}\right) D''^{*} u\right\|^{2}+\left\|\eta^{\frac{1}{2}} D'' u\right\|^{2}+\left\|\lambda^{\frac{1}{2}} D' u\right\|^{2}+2\left\|\lambda^{-\frac{1}{2}} d'\eta \wedge u\right\|^{2}$$
$$\geqslant \left\langle\!\left\langle\left[\eta \mathrm{i}\Theta(E)-\mathrm{i} d' d''\eta-\mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda\right] u, u\right\rangle\!\right\rangle .$$

Lemma 1.2.2 (L^2-existence theorem with erro term)**.** *Let* (X,ω) *be a complete Kähler manifold equipped with a (non-necessarily complete) Kähler metric* ω*, and let* Q *be a Hermitian vector bundle over* X*. Assume that* τ *and* A *are smooth and bounded positive functions on* X *and let*

$$\mathrm{B} := \left[\tau\sqrt{-1}\Theta_Q - \sqrt{-1}\partial\bar{\partial}\tau - \sqrt{-1}A^{-1}\partial\tau\wedge\bar{\partial}\tau, \Lambda\right].$$

Assume that $\delta \geq 0$ *is a number such that* $\mathrm{B}+\delta\mathrm{I}$ *is semi-positive definite everywhere on* $\bigwedge^{n,q} T_X^* \otimes Q$ *for some* $q \geq 1$*. Then given a form* $g \in L^2\left(X, \bigwedge^{n,q} T_X^* \otimes Q\right)$ *such that* $\mathrm{D}''g = 0$ *and*

$$\int_X \left\langle (\mathrm{B}+\delta\mathrm{I})^{-1} g, g\right\rangle_Q dV_X < +\infty,$$

there exists an approximate solution $u \in L^2\left(X, \bigwedge^{n,q-1} T_X^* \otimes Q\right)$ *and a correcting term* $h \in L^2\left(X, \bigwedge^{n,q} T_X^* \otimes Q\right)$ *such that* $\mathrm{D}''u + \sqrt{\delta}h = g$ *and*

$$\int_X \frac{|u|_Q^2}{\tau+A} dV_X + \int_X |h|_Q^2 dV_X \leq \int_X \left\langle (\mathrm{B}+\delta\mathrm{I})^{-1} g, g\right\rangle_Q dV_X.$$

Proof. By lemma 1.2.1, lemma 1.2.2 can be obtained by almost the same arguments as in [5], where the term $\int_X \left\langle (\mathrm{B}+\delta\mathrm{I})^{-1} g, g\right\rangle_Q dV_X$ in the above inequality is written as $2\int_X \left\langle (\mathrm{B}+\delta\mathrm{I})^{-1} g, g\right\rangle_Q dV_X$. □

Lemma 1.2.3 (The property of psh function)**.** *Let* X *be a Stein manifold and* φ *be a plurisubharmonic function on* X*. Then there exists a decreasing sequence of smooth strictly plurisubharmonic functions* $\{\varphi_j\}_{j=1}^{+\infty}$ *such that* $\lim_{j\to+\infty}\varphi_j = \varphi$

Lemma 1.2.4 (Theorem 1.5 in [3])**.** *Let* X *be a Kähler manifold, and* Z *be an analytic subset of* X*. Assume that* Ω *is a relatively compact open subset of* X *possessing a complete Kähler metric. Then* $\Omega\backslash Z$ *carries a complete Kähler metric.*

Lemma 1.2.5 (Theorem 4.4.2 in [7])**.** *Let* Ω *be a pseudoconvex open set in* $\mathbb{C}^n$*, and* φ *be a plurisubharmonic function on* Ω*. For every* $h \in L^2_{(p,q+1)}(\Omega,\varphi)$ *with* $\bar{\partial}h = 0$ *there is a solution*

$v \in L^2_{(p,q)}(\Omega, loc)$ of the equation $\bar\partial v = h$ such that

$$\int_\Omega \frac{|v|^2}{(1+|z|^2)^2} e^{-\varphi} dV \leq \int_\Omega |h|^2 e^{-\varphi} dV$$

Lemma 1.2.6 (Lemma 6.9 in [3]). *Let Ω be an open subset of $\mathbb{C}^n$ and Z be a complex analytic subset of Ω. Assume that v is a $(p, q-1)$-form with L^2_{loc} coefficients and h is a (p,q)-form with L^1_{loc} coefficients such that $\bar\partial v = h$ on $\Omega \backslash Z$ (in the sense of distribution theory). Then $\bar\partial v = h$ on Ω.*

Lemma 1.2.7 (Lagrange's inequality). *Let X be a complex manifold, E be a Hermitian vector bundle over X of rank m, and $\{\bullet,\bullet\}_E : \wedge^{p_1,q_1} T^*_X \otimes E \times \wedge^{p_2,q_2} T^*_X \otimes E \longrightarrow \wedge^{p_1+q_2,q_1+p_2} T^*_X$ be the sesquilinear product which combines the wedge product $(u,v) \mapsto u \wedge \bar v$ on scalar valued forms with the Hermitian inner product on E. Then for any smooth section s of E over X and any smooth section w of $T^*_X \otimes E$ over X,*

$$\sqrt{-1}\{w,s\}_E \wedge \{s,w\}_E \leq |s|^2_E \sqrt{-1}\{w,w\}_E. \qquad (1.1)$$

Proof. Since $\{\bullet,\bullet\}_E$ is a pointwise product, it's sufficient to prove (1.1) at every fixed point of X. Hence, we can regard T^*_X and E as vector spaces. Then s and w are regarded as elements in E and $T^*_X \otimes E$ respectively. If $s = 0$, (1.1) is trivial. If $s \neq 0$, without loss of generality, we can assume that $|s|_E = 1$. Then we choose $e_2, \cdots, e_m \in E$ such that $s, e_2, \cdots, e_m$ form an orthonormal basis of E. Now w can be written as

$$w_1 \otimes s + \sum_{j=2}^m w_j \otimes e_j,$$

for some $w_j \in T^*_X (1 \leq j \leq m)$. Then we have

$$\sqrt{-1}\{w,s\}_E \wedge \{s,w\}_E = \sqrt{-1} w_1 \wedge \bar w_1,$$

and

$$|s|^2_E \sqrt{-1}\{w,w\}_E = \sqrt{-1}\sum_{j=1}^m w_j \wedge \bar w_j \geq \sqrt{-1} w_1 \wedge \bar w_1.$$

Hence, (1.1) holds. The lemma is, thus, proved. □

1.3. Proof of the normal case theorem

In order to prove the main theorem, we should prove the following theorem firstly.

Theorem 1.3.1 (The case of L^2-extension). *Let (X, ω) be a weakly pseudoconvex complex n dimensional manifold possessing a Kähler metric ω, ψ be a plurisubharmonic function on X, E be a holomorphic vector bundle of rank m over X equipped with a smooth Hermitian metric $(1 \leq m \leq n)$, and s be a global holomorphic section of E. Assume that s is transverse to the zero section, and let*

$$Y := \{x \in X : s(x) = 0, \bigwedge^m (\mathrm{d}s)(x) \neq 0\}.$$

Let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L, which is written locally as $e^{-\varphi_L}$ for some function $\varphi_L \in L^1_{loc}$ with respect to a local holomorphic frame of L. Assume that $\frac{q}{2}\varphi_L + \left(1 - \frac{q}{2}\right)\varphi_\omega + \psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Moreover, assume that the $(1,1)$-form

(i) $\frac{q}{2}\sqrt{-1}\Theta_L + \left(1 - \frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 + \sqrt{-1}\partial\bar{\partial}\psi \geq 0$ *holds on $X \backslash Y$,*

and that there is a continuous function $\alpha > 0$ on X such that the following two inequalities hold on $X \backslash Y$:

(ii) $\frac{q}{2}\sqrt{-1}\Theta_L + \left(1 - \dfrac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 + \sqrt{-1}\partial\bar{\partial}\psi \geq \dfrac{\left\{\sqrt{-1}\Theta_E s, s\right\}_E}{\alpha|s|^2_E},$

(iii) $\psi + m\log|s|^2 \leq -2m\alpha$

Then, for every relatively compact open subset $\Omega \subset X$, and every k-jet $f \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ satisfying

$$C_f := \int_Y \frac{|f|^2_{s,\rho,(k)} e^{-\psi}}{|\bigwedge^m (ds)|^2_E} dV_{Y,\omega} < +\infty.$$

Then there exists $F^{(k)} \in H^0(X, K_X \otimes L)$ such that $J^k F^{(k)} = f$ and

$$\int_\Omega \frac{\left|F^{(k)}\right|^2_L}{e^{\psi + m\log|s|^2_E} R(\psi + m\log|s|^2_E)} dV_{X,\omega} \leq C^{(k)}_{m,R} C_f,$$

where $C^{(k)}_{m,R} > 0$ is a constant depending only on m, R, k, E, and $\sup_\Omega \|i\Theta_L\|$.

Proof. Without loss of generality, we can suppose that $C_R = 1$. Otherwise, we replace R with $C_R R$ in the proof.

If $f = 0$ on Y, then $F = 0$ satisfies the conclusion of Proposition 4.1. In the following proof, we assume that f is not 0 identically.

Since X is pseudoconvex, there exists a smooth strictly plurisubharmonic exhaustion function P on X. Instead of working on X

itself, we will work rather on the relatively compact subset $X_c \backslash Y$, where $X_c = \{P < c\}$ ($c = 1, 2, \cdots$, we choose P such that $X_1 \neq \emptyset$). By Lemma 1.2.4 $X_c \backslash Y$ ($c = 1, 2, \cdots$) are complete Kähler.

We will discuss for fixed c until the end of the proof.

Let $\zeta : (-\infty, 0) \longrightarrow (0, +\infty)$ be a smooth strictly increasing function, and $\chi : (-\infty, 0) \longrightarrow (0, +\infty)$ a smooth strictly decreasing function. Assume that $\chi(t) \geq -\frac{t}{2}$ for $t \in (-\infty, 0)$. We will find more assumptions about ζ and χ in the proof, by which we will get explicit ζ and χ in the end of this section.

Let $a \in (0, 1)$ and put $\sigma_\varepsilon = m \log\left(|s|^2 + \varepsilon^2\right) - a$ and $\sigma = m \log|s|^2 - a$. Since $|s| \leq 1$ on X, there exists a positive number $\varepsilon_a \in (0, 1)$ such that $\sigma_\varepsilon \leq -\frac{a}{2}$ on $\overline{X_c}$ for $\varepsilon \in (0, \varepsilon_a)$.

Assume that K_X is naturally equipped with the smooth metric e^{φ_ω} with respect to the dual frame of dz. Let $L_{a,\varepsilon}$ denote the line bundle L on $X_c \backslash Y$ equipped with the new metric $h_{a,\varepsilon} := e^{-\left(\frac{q}{2}\varphi_L + \left(1 - \frac{q}{2}\right)\varphi_\omega + \psi\right) - \sigma - \zeta(\sigma_\varepsilon)}$.

Set $\tau_\varepsilon = \chi(\sigma_\varepsilon)$ and let A_ε be a smooth positive function on $\overline{X_c}$, which will be determined later. Set $\mathrm{B}_\varepsilon = [\Theta_\varepsilon, \Lambda]$ on $X_c \backslash Y$, where

$$\Theta_\varepsilon := \tau_\varepsilon \sqrt{-1}\Theta_{L_{a,\varepsilon}} - \sqrt{-1}\partial\bar{\partial}\tau_\varepsilon - \sqrt{-1}\frac{\partial\tau_\varepsilon \wedge \bar{\partial}\tau_\varepsilon}{A_\varepsilon}.$$

Setting

$$\nu_\varepsilon := \frac{\{D's, s\}}{|s|^2 + \varepsilon^2}. \tag{1.2}$$

We want to find suitable ζ, χ and A_ε such that

$$\Theta_\varepsilon|_{X_c \backslash Y} \geq \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon \wedge \bar{\nu}_\varepsilon. \tag{1.3}$$

Simple calculations yield

$$\begin{aligned}
&\Theta_\varepsilon|_{X_c \backslash Y} \\
=&\chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1 - \frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + \sqrt{-1}\partial\bar{\partial}\sigma\right) \\
&+ \left(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon)\right)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon + \left(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{(\chi'(\sigma_\varepsilon))^2}{A_\varepsilon}\right)\sqrt{-1}\partial\sigma_\varepsilon \wedge \bar{\partial}\sigma_\varepsilon \\
=&\chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1 - \frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2\right) \\
&+ \left(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon)\right)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon + \left(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{(\chi'(\sigma_\varepsilon))^2}{A_\varepsilon}\right)\sqrt{-1}\partial\sigma_\varepsilon \wedge \bar{\partial}\sigma_\varepsilon.
\end{aligned} \tag{1.4}$$

Assume that the equalities

$$\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon) = 1 \tag{1.5}$$

and

$$\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{(\chi'(\sigma_\varepsilon))^2}{A_\varepsilon} = 0 \tag{1.6}$$

hold, we obtain that

$$\Theta_\varepsilon|_{X_c\backslash Y} \geq \chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2\right) + \sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon. \tag{1.7}$$

[3]

Furthermore, by (1.5) we can assume that $A_\varepsilon = \eta(\sigma_\varepsilon)$ for some smooth function $\eta : (-\infty, 0) \longrightarrow (0, +\infty)$ such that[4]

$$\chi\zeta'' - \chi'' - \frac{(\chi')^2}{\eta} = 0 \tag{1.8}$$

[3] We will provide the estimate for the last term.

[4] Here (1.8) is equivalent to the above (1.6).

Since it follows from Lemma 1.2.7 that

$$|s|^2\sqrt{-1}\sum_{i=1}^m ds^i \wedge d\bar{s}^i \geq \sqrt{-1}\left(\sum_{i=1}^m \bar{s}^i ds^i\right)\wedge\left(\sum_{i=1}^m s^i d\bar{s}^i\right)$$

, which can be stated as

$$|s|^2\sqrt{-1}\{D's, D's\} \geq \sqrt{-1}\{D's, s\}\wedge\{s, D's\}.$$

We obtain that on $X_c\backslash Y$,

$$\begin{aligned}
\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon &= \frac{m\sqrt{-1}\{D's, D's\}}{|s|^2+\varepsilon^2} - \frac{m\sqrt{-1}\{D's, s\}\wedge\{s, D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{m\sqrt{-1}\{\Theta_E s, s\}}{|s|^2+\varepsilon^2} \\
&\geqslant \frac{m\varepsilon^2}{|s|^2}\frac{\sqrt{-1}\{D's, s\}\wedge\{s, D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{m\sqrt{-1}\{\Theta_E s, s\}}{|s|^2+\varepsilon^2} \\
&= \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon - \frac{m\sqrt{-1}\{\Theta_E s, s\}}{|s|^2+\varepsilon^2}.
\end{aligned}$$

Then it follows from (1.7) that on $X_c\backslash Y$,

$$\begin{aligned}
\Theta_\varepsilon \geqslant &\left(\chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2\right) - \frac{m\sqrt{-1}\{\Theta_E s, s\}}{|s|^2+\varepsilon^2}\right) \\
&+ \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon.
\end{aligned}$$

Since $\chi(\sigma_\varepsilon) \geq m\alpha$ by the assumption $\chi(t) \geq -\frac{t}{2}$, it follows from the condition on $X\backslash Y$ in Theorem 1.1.1 that

$$\begin{aligned}
&\chi(\sigma_\varepsilon)(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2) - \frac{m\sqrt{-1}\{\Theta_E s, s\}}{|s|^2+\varepsilon^2} \\
&= \chi(\sigma_\varepsilon)(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2) - \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\frac{\sqrt{-1}\{\Theta_E s, s\}}{\alpha|s|^2} \\
&\geq \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + \left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + \sqrt{-1}\partial\bar{\partial}\sigma - \frac{\sqrt{-1}\{\Theta_E s, s\}}{\alpha|s|^2}\right) \\
&\geq 0 \quad \text{(By the assumption (ii) in Theorem 1.1.1.)}
\end{aligned}$$

on $X_c \backslash Y$. Hence, one obtain (1.3) as expected.

As a result, we have

$$B_\varepsilon \geq \left[\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon \wedge \bar{\nu}_\varepsilon, \Lambda\right] = \frac{m\varepsilon^2}{|s|^2} T_{\bar{\nu}_\varepsilon} T^*_{\bar{\nu}_\varepsilon} \tag{1.9}$$

on $X_c \backslash Y$ as an operator on $(n,1)$ forms, where $T_{\bar{\nu}_\varepsilon}$ denotes the operator $\bar{\nu}_\varepsilon \wedge \bullet$ and $T^*_{\bar{\nu}_\varepsilon}$ is its Hilbert adjoint operator.

1.3.1. Solving $\bar{\partial}$-equation on X_c with estimates.

With such preparation, we now argue by induction on $k \geq 0$. The case $k = 0$ is a special case of [14, Theorem 1.2]. Now, assume that the theorem has been proved for $k-1$, and we consider the short exact sequence of sheaves

$$0 \longrightarrow S^k N^*_{Y/X} \longrightarrow \mathcal{O}_X/\mathcal{J}_Y^{k+1} \longrightarrow \mathcal{O}_X/\mathcal{J}_Y^k \longrightarrow 0$$

Let $J^{k-1}f \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^k\right)$ be the image of $f \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ under the induced cohomology group morphism. By the induction hypothesis[5], there exists $F^{(k-1)} \in H^0\left(X, \ K_X \otimes L\right)$ such that

$$J^{k-1}F^{(k-1)} = J^{k-1}f,$$

$$\int_{X_c} \frac{\left|F^{(k-1)}\right|_L^2}{|s|^{2m} R\left(\psi + m\log|s|_E^2\right)} e^{-\psi} \mathrm{d}V_{X,\omega} \leq C_{m,R}^{(k-1)} \int_{Y_c} \frac{|f|^2_{s,\rho,(k-1)} e^{-\psi}}{|\Lambda^m(\mathrm{d}s)|^2} \mathrm{d}V_{Y,\omega}, \tag{1.10}$$

where $C_{m,R}^{(k-1)} > 0$ is a constant as in the statement of Theorem 1.3.1 and $Y_c := Y \cap X_c$. Thus, the image

$$J^{k-1}f - J^{k-1}F^{(k-1)} \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^k\right)$$

of $f - J^k F^{(k-1)} \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ vanishes. So we can view the jet $f - J^k F^{(k-1)}$ as a global holomorphic section (on Y) of the sheaf $K_X \otimes L \otimes S^k N^*_{Y/X} = K_X \otimes L \otimes S^k E^*_{|Y}$[6].

Using the results in [13, p12], one can construct an extension $\tilde{f} \in C^\infty\left(X, K_X \otimes L\right)$ of the holomorphic k-jet $f \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ by means of a partition of unity, satisfying

$$\bar{\partial}\tilde{f} = 0 \quad \text{ on } Y,$$

and

$$|\bar{\partial}\tilde{f}| = O\left(|s|^{k+1}\right) \quad \text{ in a neighbourhood of } Y.$$

Set[7]

$$G_\varepsilon^{(k-1)} := \theta\left(\frac{\varepsilon^2}{|s|^2 + \varepsilon^2}\right)\left(\tilde{f} - F^{(k-1)}\right) \in C^\infty\left(X, K_X \otimes L\right),$$

where $0 < \varepsilon < \varepsilon_a$, and $\theta : \mathbb{R} \to [0,1]$ is a C^∞ function such that $\theta \equiv 0$ on $(-\infty, \frac{l}{2}]$; $\theta \equiv 1$ on $\left[1 - \frac{l}{2}, +\infty\right)$, and $|\theta'| \leq \frac{1+l}{1-l}$ on $\mathbb{R}$.

Then it suffices to solve the equation[8]

[5] By theorem 1.3.1, there exists $F^{(k-1)} \in H^0\left(X, K_X \otimes L\right)$ such that $J^{k-1}F^{(k-1)} = J^{k-1}f$ and

$$\int_\Omega \frac{\left|F^{(k-1)}\right|_L^2}{e^{\psi + m\log|s|_E^2} R(\psi + m\log|s|_E^2)} dV_{X,\omega} \leq C_{m,R}^{(k-1)} C_f,$$

where $C_{m,R}^{(k-1)} > 0$ is a constant depending only on m, R, k, E, and $\sup_\Omega \|i\Theta_L\|$.

[6] The explanation:

1. Short Exact Sequence and Structure: The given short exact sequence

$$0 \longrightarrow S^k N^*_{Y/X} \longrightarrow \mathcal{O}_X/\mathcal{J}_Y^{k+1} \longrightarrow \mathcal{O}_X/\mathcal{J}_Y^k \longrightarrow 0$$

shows that the difference between the sheaves $\mathcal{O}_X/\mathcal{J}_Y^{k+1}$ and $\mathcal{O}_X/\mathcal{J}_Y^k$ lies in sections of $S^k N^*_{Y/X}$.

2. Constructing the Section: Given $f \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$, its image under the induced map is $J^{k-1}f$ in $H^0(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^k)$. By the induction hypothesis, we find $F^{(k-1)} \in H^0(X, K_X \otimes L)$ such that $J^{k-1}F^{(k-1)} = J^{k-1}f$.
Now consider $f - J^k F^{(k-1)}$. This difference has a vanishing $(k-1)$-th jet, i.e., it lies in the kernel of the projection to $\mathcal{O}_X/\mathcal{J}_Y^k$. Therefore, it can be viewed as a section of $S^k N^*_{Y/X}$ via the injection in the short exact sequence.

3. Identifying the Sheaf: Since $N_{Y/X}$ is the normal bundle, its symmetric power $S^k N^*_{Y/X}$ describes jets of order k. Moreover, as $N_{Y/X} \cong E_{|Y}$, we identify $S^k N^*_{Y/X}$ with $S^k E^*_{|Y}$. Thus, $f - J^k F^{(k-1)}$ becomes a holomorphic section of the sheaf $K_X \otimes L \otimes S^k E^*_{|Y}$.

Conclusion: The structure of the exact sequence allows us to interpret the jet $f - J^k F^{(k-1)}$ as a section of $K_X \otimes L \otimes S^k N^*_{Y/X} = K_X \otimes L \otimes S^k E^*_{|Y}$.

$$\bar{\partial} u_\varepsilon = \bar{\partial} G_\varepsilon^{(k-1)}, \tag{1.11}$$

with the extra condition $\frac{|u_\varepsilon|^2}{|s|^{2m}} \in L^1_{\text{loc}}$ in a neighbourhood of Y. This condition guarantees that u_ε, as well as all its jets of orders $\leq k$, vanishes on Y. By direct calculations, one has

$$\bar{\partial} G_\varepsilon^{(k-1)} = g_\varepsilon^{(1)} + g_\varepsilon^{(2)},$$

where

$$g_\varepsilon^{(1)} = -\frac{\varepsilon^2}{|s|^2+\varepsilon^2} \cdot \theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \bar{v}_\varepsilon \wedge \left(\tilde{f} - F^{(k-1)}\right),$$
$$g_\varepsilon^{(2)} = \theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \bar{\partial}\left(\tilde{f} - F^{(k-1)}\right).$$

Recall that v_ε is given in (1.2).

In this situation, $g_\varepsilon^{(2)}$ turns out to have no contribution in the limit since it converges uniformly to 0 on every compact set when ε tends to 0 . Actually, Supp $\left(g_\varepsilon^{(2)}\right) \subset \{|s| < \sqrt{2}\varepsilon\}$ and $\left|g_\varepsilon^{(2)}\right| = O\left(|s|^{k+1}\right)$ because of $|\bar{\partial}\widetilde{f}| = O\left(|s|^{k+1}\right)$ in a neighbourhood of Y as we have previously shown.

Then

$$\int_{X_c \setminus Y} \left\langle B_\varepsilon^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)}\right\rangle_L |s|^{-2m} e^{-\psi} \mathrm{d}V_{X,\omega} = O(\varepsilon),$$

provided that B_ε is locally uniformly bounded below in a neighbourhood of Y. Otherwise, we shall solve the approximate equation $\bar{\partial}u + \sqrt{\delta}h = g_\varepsilon$ with $\delta > 0$ small (see Lemma 1.2.2 and [5, Remark 3.2] for more details). One can remove the extra error term $\sqrt{\delta}h$ by putting $\delta \longrightarrow 0$ at the end. Since there is no essential difficulty during this procedure, for the purpose of simplicity, we will assume to have the desired lower bound for B_ε and the estimate of $g_\varepsilon^{(2)}$ as above.

Next, we turn to estimate the term involving $g_\varepsilon^{(1)}$ on $X_c \setminus Y$. By (1.9),

$$\left\langle B_\varepsilon^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\right\rangle_{L_{a,\varepsilon}} \leq \frac{|s|^2}{m\varepsilon^2} \cdot \left|\theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \frac{\varepsilon^2}{|s|^2+\varepsilon^2}\left(\tilde{f} - F^{(k-1)}\right)\right|^2_{L_{a,\varepsilon}}.$$

In [13, p17], Popovici showed that on every compact set,

$$\frac{\left|\left(\tilde{f} - F^{(k-1)}\right)(\varepsilon s, z')\right|_L^2}{\varepsilon^{2k}} \longrightarrow \left|\nabla^k \left(f - J^k F^{(k-1)}\right)(z')\right|_L^2, \quad (\varepsilon \to 0).$$

Then using a partition of unity $\{\xi_p\}_{p=1}^{p_0}$ around $\overline{X_c} \setminus Y$ and the Fubini theorem, we obtain

[7] Since we hardly know $\tilde{f}$ away from Y , we take a truncation with support in a tubular neighbourhood of Y.

[8] Here u_ε is used to construct the erro term between $F^{(k)}$ and $F^{(k-1)}$. Later, we will see that it is the key to end the proof.

$$\begin{aligned}\int_{X_c\setminus Y}\left\langle \mathrm{B}_\varepsilon^{-1}g_\varepsilon^{(1)},g_\varepsilon^{(1)}\right\rangle_{L_{a,\varepsilon}}e^{-\psi}\mathrm{d}V_X &\le \frac{e^a(1+l)^2}{m(1-l)^2}\sum_{p=1}^{p_0}\int_{X_c\cap\left\{\sqrt{\frac{l}{2-l}}\varepsilon<|s|<\sqrt{\frac{2-l}{l}}\varepsilon\right\}}\frac{\varepsilon^2\xi_p|\tilde f-F^{(k-1)}|_L^2e^{-\psi}\mathrm{d}V_X}{\left(|s|^2+\varepsilon^2\right)^2|s|^{2m-2}}\\
&\le\sum_{p=1}^{p_0}\left(\int_{\left\{z\in\mathbb{C}^m:\sqrt{\frac{l}{2-l}}\varepsilon<|z|<\sqrt{\frac{2-l}{l}}\varepsilon\right\}}\frac{\varepsilon^2(\sqrt{-1})^{m^2}\bigwedge^m(dz)\wedge\bigwedge^m(d\bar z)}{\left(|z|^2+\varepsilon^2\right)^2|z|^{2m-2}}\right.\\
&\left.\times\frac{e^a(1+l)^2}{m(1-l)^2}\int_{Y_c}\frac{\xi_p\left|\nabla^k\left(\tilde f-J^kF^{(k-1)}\right)\right|_L^2}{|\wedge^m(ds)|^2}e^{-\psi}\mathrm{d}V_{Y,\omega}\right)\\
&\to\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J^kF^{(k-1)}\right)\right|_L^2}{|\wedge^m(ds)|^2}e^{-\psi}\mathrm{d}V_{Y,\omega}\quad(\varepsilon\to0),\end{aligned}$$

where

$$C_{m,k}:=\int_{\left\{z\in\mathbb{C}^m:\sqrt{\frac{l}{2-l}}<|z|<\sqrt{\frac{2-l}{l}}\varepsilon\right\}}\frac{(\sqrt{-1})^{m^2}\bigwedge^m(dz)\wedge\bigwedge^m(d\bar z)}{\left(|z|^2+1\right)^2|z|^{2m-2}},$$

which depends only on m and k. It may be worthwhile to note that

$$\left|\nabla^k\left(f-J^kF^{(k-1)}\right)\right|_L=\left|f-J^kF^{(k-1)}\right|_L,$$

where $f-J^kF^{(k-1)}\in H^0\left(Y,K_X\otimes L\otimes S^kN_{Y/X}^*\right)$. Then, one has

$$\int_{X_c\setminus Y}\left\langle B_\varepsilon^{-1}g_\varepsilon^{(1)},g_\varepsilon^{(1)}\right\rangle_{L_{a,\varepsilon}}e^{-\psi}\mathrm{d}V_X\le\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J^kF^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{~d}s)|^2}e^{-\psi}\ \mathrm{d}V_{Y,\omega},$$

when ε is small enough. By using Lemma 1.2.2[9] with $\delta=0$, we can solve (1.11), i.e., there exists $u_{c,a,\varepsilon}\in L^2\left(X_c\setminus Y,K_X\otimes L_{a,\varepsilon}\right)$ such that

$$\bar\partial u_{c,a,\varepsilon}=\bar\partial G_\varepsilon^{(k-1)}=g_\varepsilon^{(1)}+g_\varepsilon^{(2)}$$

on $X_c\setminus Y$ and

$$\int_{X_c\setminus Y}\frac{\left|u_{c,a,\varepsilon}\right|_L^2e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}e^{-\psi}\mathrm{d}V_X\underset{10}{\le}\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J^kF^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{~d}s)|^2}e^{-\psi}\ \mathrm{d}V_{Y,\omega}+O(\varepsilon).\tag{1.12}$$

Since $\sigma,\zeta\left(\sigma_\varepsilon\right),\tau_\varepsilon+A_\varepsilon$ are all bounded above on $\overline{X_c}$ for each fixed ε, the inequality (1.12) implies that $u_{c,a,\varepsilon}\in L^2\left(X_c,K_X\otimes L\right)$. As (1.11), (1.12) and G_ε^{k-1} is smooth, Lemma 1.2.6[10] gives that

$$\bar\partial u_{c,a,\varepsilon}=\bar\partial G_\varepsilon^{(k-1)}=g_\varepsilon^{(1)}+g_\varepsilon^{(2)}\tag{1.13}$$

extends across Y and

[9] L^2 existence theorem with erro term $\delta=0$.

Lemma 1.3.1 (Lemma 6.9 in [3]). *Let Ω be an open subset of $\mathbb{C}^n$ and Z be a complex analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and h is a (p,q)-form with L^1_{loc} coefficients such that $\bar\partial v=h$ on $\Omega\setminus Z$ (in the sense of distribution theory). Then $\bar\partial v=h$ on Ω.*

$$\int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|_L^2 e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} e^{-\psi} \mathrm{d}V_X \le \frac{e^a(1+l)^2}{m(1-l)^2} C_{m,k} \int_{Y_c} \frac{\left|\nabla^k \left(f - J^k F^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi} \,\mathrm{d}V_{Y,\omega} + O(\varepsilon). \tag{1.14}$$

The k-jet extension[11] of f to X_c is then given by

$$F_{c,a,\varepsilon}^{(k)} := \left(G_\varepsilon^{(k-1)} - u_{c,a,\varepsilon}\right) + F^{(k-1)}.$$

Then $F_{c,a,\varepsilon}^{(k)}$ is holomorphic on X_c, thanks to (1.13), $F^{(k-1)} \in H^0(X, K_X \otimes L)$ as well as the ellipticity of the operator $\bar\partial$ in bidegree $(n,0)$. So $u_{c,a,\varepsilon}$ is also smooth on X_c. Locally, near an arbitrary point of Y, all partial derivatives of order $s \le k$ of $F_{c,a,\varepsilon}^{(k)}$ are prescribed by f.

By the variant of Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\langle \kappa_1 + \kappa_2 + \kappa_3, \kappa_1 + \kappa_2 + \kappa_3 \rangle \\ &\le (1+l)\,\langle \kappa_1+\kappa_2, \kappa_1+\kappa_2\rangle + (1+\frac{1}{l})\,\langle \kappa_3,\kappa_3\rangle \\ &\le (1+l)^2\,\langle \kappa_1,\kappa_1\rangle + \frac{(1+l)^2}{l}\,\langle\kappa_2,\kappa_2\rangle + (1+\frac{1}{l})\,\langle\kappa_3,\kappa_3\rangle \end{aligned} \tag{1.15}$$

for any inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{H}$.

Then for some sufficiently small ε, $R(\sigma_\varepsilon) \le R(\sigma)$, $R\left(m \log |s|^2\right) \le R(\sigma)$, the induction hypothesis (1.10), (1.14) and (1.15) give the estimate on the relatively compact open subset X_c,

[11] Here is the key point of the proof. The k-th term is represented by the $k-1$ term, which is also the core step of the induction. In above we set

$$G_\varepsilon^{(k-1)} := \theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)\left(\tilde f - F^{(k-1)}\right),$$

and we have known that

$$\begin{aligned} \bar\partial u_\varepsilon = \bar\partial G_\varepsilon^{(k-1)} &\sim g_\varepsilon^{(1)} \\ &= -\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\cdot\theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)\bar v_\varepsilon \\ &\wedge \left(\tilde f - F^{(k-1)}\right). \end{aligned}$$

$$\begin{aligned}
&\int_{X_c} \frac{\left|F^{(k)}_{c,a,\varepsilon}\right|^2_L}{e^{\sigma}R(\sigma)} e^{-\psi}\mathrm{d}V_X \\
\leq{}& (1+l)^2 \int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|^2_L}{e^{\sigma}R(\sigma)} e^{-\psi}\mathrm{d}V_X + \frac{(1+l)^2}{l}\int_{X_c} \frac{\left|F^{(k-1)}\right|^2_L}{e^{\sigma}R(\sigma)} e^{-\psi}\mathrm{d}V_X + (1+\frac{1}{l})\int_{X_c} \frac{\left|G^{(k-1)}_{\varepsilon}\right|^2_L}{e^{\sigma}R(\sigma)} e^{-\psi}\mathrm{d}V_X \\
\leq{}& (1+l)^2 \left(\sup_{X_c} \frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right) \int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|^2_L e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} e^{-\psi}\mathrm{d}V_X \\
&+ \frac{(1+l)^2}{l} e^a \int_{X_c} \frac{\left|F^{(k-1)}\right|^2_L}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi}\mathrm{d}V_X + (1+\frac{1}{l})e^a \int_{X_c} \frac{\theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)^2 \left|\tilde f - F^{(k-1)}\right|^2_L}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi}\mathrm{d}V_X \\
\leq{}& (1+l)^2 \left(\sup_{X_c} \frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right) \int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|^2_L e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} e^{-\psi}\mathrm{d}V_X \\
&+ \frac{(1+l)^2}{l} e^a \int_{X_c} \frac{\left|F^{(k-1)}\right|^2_L}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi}\mathrm{d}V_X + (1+\frac{1}{l})C_1 e^a \int_{X_c} \frac{1}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi}\mathrm{d}V_X \\
\leq{}& \frac{(1+l)^4 e^a}{m(1-l)^2} \left(\sup_{X_c} \frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right) C_{m,k} \int_{Y_c} \frac{\left|\nabla^k\left(f - J^k F^{(k-1)}\right)\right|^2_L}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\,\mathrm{d}V_Y \\
&+ \frac{(1+l)^2}{l} e^a C^{(k-1)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k-1)}}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\,\mathrm{d}V_Y + C_2 \int_{-\infty}^{2m\log\varepsilon + C_3} \frac{1}{R(t)}\mathrm{d}t + O(\varepsilon) \\
\leq{}& e^a C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\,\mathrm{d}s)|^2} e^{-\psi}\,\mathrm{d}V_Y + \frac{(1+l)^4 e^a}{m(1-l)^2} C_{m,k} \int_{Y_c} \frac{\left|\nabla^k\left(J^k F^{(k-1)}\right)\right|^2_L}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\,\mathrm{d}V_Y \\
&+ C_2 \int_{-\infty}^{2m\log\varepsilon + C_3} \frac{1}{R(t)}\mathrm{d}t + O(\varepsilon),
\end{aligned} \tag{1.16}$$

where $C'^{(k)}_{m,R} = \frac{(1+l)^4}{m(1-l)^2} C_{m,k} + \frac{(1+l)^2}{l} C^{(k-1)}_{m,R}$ and C_1, C_2, C_3 are all positive numbers independent of ε. Here in (16), we also assume that

$$\frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)} = 1 \tag{1.17}$$

on X_c. We will solve (1.17) together with (1.5) and (1.8) in above.

1.3.2. Final jet extensions on Ω via limits

As $\sup_{t\leq 0}\left(e^t R(t)\right) < \infty$, applying Montel's theorem and (1.16) to extract a weak limit of $\left\{F^{(k)}_{c,a,\varepsilon}\right\}_{\varepsilon>0}$ as $\varepsilon \to 0$[12], we get a holomorphic L-valued n-form $F^{(k)}_{c,a}$ on X_c such that $J^k_{X_c} F^{(k)}_{c,a} = f$ and

[12] First Limitation.

$$\int_{X_c} \frac{\left|F^{(k)}_{c,a}\right|^2_L}{e^{\sigma}R(\sigma)} e^{-\psi}\mathrm{d}V_X \leq e^a C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)}}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\,\mathrm{d}V_Y + \frac{(1+l)^4 e^a}{m(1-l)^2} C_{m,k} \int_{Y_c} \frac{\left|\nabla^k\left(J^k F^{(k-1)}\right)\right|^2_L}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\,\mathrm{d}V_Y.$$

In other words,

$$\int_{X_c} \frac{\left|F_{c,a}^{(k)}\right|_L^2 e^{-\psi} \mathrm{d}V_X}{|s|^{2m} R\left(m\log|s|^2 - a\right)} \leq C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)} e^{-\psi}\, \mathrm{d}V_Y}{|\wedge^m(\,\mathrm{d}s)|^2} + \frac{(1+l)^4 C_{m,k}}{m(1-l)^2} \int_{Y_c} \frac{\left|\nabla^k\left(J^k F^{(k-1)}\right)\right|_L^2 e^{-\psi}\, \mathrm{d}V_Y}{|\wedge^m(\,\mathrm{d}s)|^2} \tag{1.18}$$

Since R is continuous decreasing on $(-\infty, 0]$, $\sup_{t\leq 0}\left(e^t R(t)\right) < \infty$, similarly as before, we use Montel's theorem and extract a weak limit of $\left\{F_{c,a}^{(k)}\right\}_{a>0}$ as $a \to 0$[13], to obtain a holomorphic L-valued n-form $F_c^{(k)}$ on X_c from (1.18) such that $J^k_{X_c} F_c^{(k)} = f$ and

[13] Second Limitation.

$$\int_{X_c} \frac{\left|F_{c}^{(k)}\right|_L^2 e^{-\psi}}{|s|^{2m} R\left(m\log|s|^2\right)} \mathrm{d}V_X \leq C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)} e^{-\psi}}{|\wedge^m(\,\mathrm{d}s)|^2}\, \mathrm{d}V_Y + \frac{(1+l)^4 C_{m,k}}{m(1-l)^2} \int_{Y_c} \frac{\left|\nabla^k\left(J^k F^{(k-1)}\right)\right|_L^2}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y. \tag{1.19}$$

As Popovici [13, Sections 0.4-0.6] has shown that the last term in the right-hand side of (1.19) can be controlled uniformly, a slight modification of his proof in [13, Section 0.4] in terms of the variable denominators introduced by [14, p136] can complete the proof of Theorem 1.3.1. Indeed, one just needs to modify the first and second inequalities in [13, p22], respectively, as

$$\begin{aligned}
&\frac{\sum_{|\alpha|=k}\left|\frac{\frac{\partial^\alpha F^{(k-1)}}{\partial z'^\alpha}(0,z'')}{\alpha!}\right|^2 e^{-2\varphi(0,z'')-2A|z''|^2}}{|\wedge^m(\,\mathrm{d}s)\,(0,z'')|^{2\frac{(m+k)}{m}}} \\
&\leq \text{ Const } \cdot \frac{2(m+k)}{\rho^{2(m+k)}} e^{2\left(\varepsilon(\rho)+A\rho^2\right)} \sup_{(z',z'')\in U_j} \frac{|s\,(z',z'')|^{2m} R\left(m\log s\,(z',z'')^2\right)}{|\wedge^m(\,\mathrm{d}s)\,(0,z'')|^{2\frac{(m+k)}{m}}} \\
&\times \int_{z'\in B'(0,\rho)} \frac{\left\|F^{(k-1)}\,(z',z'')\right\|^2}{|s\,(z',z'')|^{2m} R\left(m\log s\,(z',z'')^2\right)} \mathrm{d}\lambda\,(z'),
\end{aligned}$$

and

$$\int_{Y_c} \frac{\left|\nabla^k\left(J^k F^{(k-1)}\right)\right|^2}{|\wedge^m(\,\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y \leq D_{m,k} N M(c) \frac{1}{\rho^{2(m+k)}} e^{2\left(\varepsilon(\rho)+A\rho^2\right)} \int_{\Omega'} \frac{\left\|F^{(k-1)}\right\|^2}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi} \mathrm{d}V_{X,\omega},$$

where

$$M(c) := \sup_{(z',z'')\in\Omega'} \frac{|s\,(z',z'')|^{2m} R\left(m\log s\,(z',z'')^2\right)}{|\wedge^m(\,\mathrm{d}s)\,(0,z'')|^{2\frac{m+k}{m}}}.$$

and $D_{m,k} :=$ Const $\cdot 2(m+k)$. Notice that the smoothness of the function R on $(-\infty, 0]$ ensures that one can get the suprema on U_j and Ω', respectively. We refer to [13, Section 0.4] for more explanations about the above notations. Then as a result, we get a holomorphic L-valued n-form $F_c^{(k)}$ on Ω such that $J^k_\Omega F_c^{(k)} = f$ and

$$
\begin{aligned}
\int_\Omega \frac{\left|F_c^{(k)}\right|_L^2 e^{-\psi}}{|s|_E^{2m} R\left(m \log |s|_E^2\right)} \mathrm{d}V_{X,\omega} \leq \int_{X_c} \frac{\left|F_c^{(k)}\right|_L^2 e^{-\psi}}{|s|_E^{2m} R\left(m \log |s|_E^2\right)} \mathrm{d}V_{X,\omega} &\leq C_{m,R}^{(k)} \int_{Y_c} \frac{|f|_{s,\rho,(k)}^2}{|\bigwedge^m(\mathrm{d}s)|_E^2} e^{-\psi}\,\mathrm{d}V_{Y,\omega} \\
&\leq C_{m,R}^{(k)} \int_{Y} \frac{|f|_{s,\rho,(k)}^2}{|\bigwedge^m(\mathrm{d}s)|_E^2} e^{-\psi}\,\mathrm{d}V_{Y,\omega},
\end{aligned}
$$

where $C_{m,R}^{(k)} > 0$ is a constant depending only on m, k, E, R and $\sup_\Omega \|i\Theta(L)\|$.

1.3.3. Solving ordinary differential equations

We have already proved Theorem 1.1.1 , provided that there exist appropriate χ, η, ζ satisfying some assumptions. Now, we will come to use these assumptions about χ, η, ζ to get their explicit expressions.

Notice that (1.5), (1.8) and (1.17) are equivalent to the following system of ordinary differential equations defined on $(-\infty, 0)$:

$$
\begin{cases}
\chi(t)\zeta'(t) - \chi'(t) = 1, \\
(\chi(t) + \eta(t))e^{\zeta(t)} = R(t), \\
\dfrac{(\chi'(t))^2}{\chi(t)\zeta''(t) - \chi''(t)} = \eta(t).
\end{cases}
$$

Moreover, we have assumed that ζ, χ and η are all smooth on $(-\infty, 0)$ and that $\zeta > 0, \chi > 0, \eta > 0, \zeta' > 0, \chi' < 0$ and $\chi(t) \geq -\frac{t}{2}$ on $(-\infty, 0)$. In the proof of Theorem 1.1.1, we have assumed that $C_R = \int_{-\infty}^0 \frac{1}{R(t)}\mathrm{d}t = 1$.

Following the argument of solving undetermined functions with ODEs introduced in [14, Section 4, pp. 151-153], we get

$$
\begin{cases}
\zeta = -\log\left(1 - \displaystyle\int_{-\infty}^t \frac{1}{R(t_1)}\mathrm{d}t_1\right), \\
\chi = \dfrac{-t - \int_t^0 \left(\int_{-\infty}^{t_2} \frac{1}{R(t_1)}\mathrm{d}t_1\right)\mathrm{d}t_2}{1 - \int_{-\infty}^t \frac{1}{R(t_1)}\mathrm{d}t_1}, \\
\eta = \left(1 - \displaystyle\int_{-\infty}^t \frac{1}{R(t_1)}\mathrm{d}t_1\right) R(t) + \dfrac{t + \int_t^0 \left(\int_{-\infty}^{t_2} \frac{1}{R(t_1)}\mathrm{d}t_1\right)\mathrm{d}t_2}{1 - \int_{-\infty}^t \frac{1}{R(t_1)}\mathrm{d}t_1},
\end{cases}
$$

and

$$
\chi' + \frac{1}{2} = \left(\frac{-\frac{1}{2}(\lambda_1')^2 + \lambda_1\lambda_1''}{(\lambda_1')^2}\right) \leq 0.
$$

It is easy to verify all the previous assumptions about ζ, χ and η. In the end, we have proven the L^2-extension theorem 1.3.1. □

Now we will show that the main theorem which is the case of L^q-extension is true later.

1.4. Proof of the main theorem

From now on, we will denote $F^{(k)}$ in theorem 1.3.1 by $F_1^{(k)}$. K_X is naturally equipped with the smooth metric e^{φ_ω} with respect to the dual frame of dz. Let L' be the line bundle L equipped with the new metric $e^{-\varphi_{L'}}$, where $\varphi_{L'} := (2-q)\log|F_1|_L + \varphi_L$. Then the assumptions in the theorem imply that

(i) $\sqrt{-1}\Theta_{L'} + \sqrt{-1}\partial\bar{\partial}\sigma \geq 0$,

(ii) $\sqrt{-1}\Theta_{L'} + \sqrt{-1}\partial\bar{\partial}\sigma \geq \frac{\left\{\sqrt{-1}\Theta_E s, s\right\}_E}{\alpha|s|_E^2}$.

Since the k-jet $f \in H^0\left(X, K_X \otimes L' \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ satisfies

$$\int_Y \frac{|f|^2_{L',s,\rho,(k)} e^{-\psi}}{|\bigwedge^m(\mathrm{d}s)|_E^2} dV_{Y,\omega} = C_f < +\infty,$$

by Theorem 1.3.1, there exists $F_2^{(k)}$ on X with values in $K_X \otimes L'$, such that $J^k F_2^{(k)} = f$ on Ω and

$$\int_\Omega \frac{\left|F_2^{(k)}\right|_L^2 e^{-\psi}}{\left(\left|F_1^{(k)}\right|_L\right)^{2-q} |s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega}$$
$$= \int_\Omega \frac{\left|F_2^{(k)}\right|_{L'}^2 e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega} \leq C_{m,R}^{(k)} \int_Y \frac{|f|^2_{L',s,\rho,(k)} e^{-\psi}}{|\bigwedge^m(\mathrm{d}s)|_E^2} dV_{Y,\omega} = C_{m,R}^{(k)} C_f.$$

Then Hölder's inequality gives that

$$C_{F_2^{(k)}} := \int_\Omega \frac{\left(\left|F_2^{(k)}\right|_L\right)^q e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega}$$
$$\leq \left(\int_\Omega \frac{\left|F_2^{(k)}\right|_L^2 e^{-\psi}}{\left(\left|F_1^{(k)}\right|_{L'}\right)^{2-q} |s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega}\right)^{\frac{q}{2}} \left(\int_\Omega \frac{\left(\left|F_1^{(k)}\right|_L\right)^q e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega}\right)^{1-\frac{q}{2}}$$
$$\leq \left(C_{m,R}^{(k)} C_f\right)^{\frac{q}{2}} \left(C_{F_1^{(k)}}\right)^{1-\frac{q}{2}}.$$

We can then repeat the same argument with $F_1^{(k)}$ replaced by $F_2^{(k)}$, etc., and get a sequence of holomorphic extensions $\left\{F_s^{(k)}\right\}_{s=1}^{+\infty}$ of f and a sequence $\left\{C_{F_s^{(k)}}\right\}_{s=1}^{+\infty}$ such that

$$C_{F_{s+1}^{(k)}} \leq \left(C_{m,R}^{(k)} C_f\right)^{\frac{q}{2}} \left(C_{F_s^{(k)}}\right)^{1-\frac{q}{2}}, \quad s = 1, 2, \cdots. \tag{1.20}$$

(I) If $C_{F_s}^{(k)} \leq C_{m,R}^{(k)} C_f$ for some $C_{F_s}^{(k)}$, then we finish the proof since $F_s^{(k)}$ can be regarded as the desired k-jet extension $F^{(k)}$ in the conclusion.

(II) If $C_{F_s^{(k)}} > C_{m,R}^{(k)} C_f$ for any k, then $C_{F_{s+1}^{(k)}} < C_{F_s^{(k)}}$ for any s. Since φ_L is locally bounded above and $e^{\sigma} R(\sigma)$ is bounded above, applying Montel's theorem and extracting weak limits of $\left\{F_s^{(k)}\right\}_{s=1}^{+\infty}$, we can get from (1.20) a k-jet $F^{(k)}$ on X with values in $K_X \otimes L$, such that $J^k F^{(k)} = f$ on Ω and

$$\int_{\Omega} \frac{\left(|F^{(k)}|_L\right)^q e^{-\psi}}{|s|^{2m} R(\psi + m \log |s|^2)} dV_X \leq C_{m,R}^{(k)} C_f.$$

Theorem 1.1.1 is, thus, proved.

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