论文改进汇总

工作指南与进度追踪

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第 I 部分

论文的审稿意见与修改方针

1

关于“Logarithmic vanishing theorems on weakly 1-complete Kähler manifolds”的审稿意见与修改方针

1.1. 审稿意见分析

[1] 本文在 Huang，Liu，Wan 和 Yang 的工作基础上，推广了弱 1 -完备 Kahler 流形的对数消没定理，他们建立了紧致 Kähler 流形的类似结果。主定理保证了某些上同调群在特定条件下的消失性，从而将紧致集合中已有的结果推广到更广泛的弱 1 -完备流形类。

[1] 这一页是翻译，下一页是原稿。

虽然这种扩展在数学上是正确的，但其方法紧紧地遵循了 Huang 等人使用的分析技术，特别是他们的 L^2 -方法和逼近定理的使用。事实上，**本文基本上将他们的论点应用到弱 1 -完备的 Kähler 流形上，而没有遇到重大的新挑战或在所使用的技术上引入实质性的创新** [2]。

[2] 对此，我并不完全赞同。首先，我的方法结合了 HLWY 与逼近定理，通过一致估计找到合适的逼近定理显然是有一定创新的，而且，这个估计定理最终需要通过结合 HLWY 发展的良层同构来从 Y_i 上转化为 X_i 上，这显然也是有着独特的思考在内，不能一味地否定我的独立思考成果，而将所有都归结于他人的方法与结果，这明显不合理！

从历史上看，消失定理在代数几何和复分析中都起到了至关重要的作用，最早可以追溯到经典的 Akizuki – 小平– Nakano 紧致 Kähler 流形上的消失定理，并得到了广泛的应用. 这些结果推广到非紧和弱 1 -完备情形，由 Norimatsu、埃斯诺和菲韦格等研究者开创，延续了这一趋势。这些推广旨在探索更广泛几何背景下的上同调行为，Huang 等人的工作代表了将这些思想推广到紧致 Kähler 流形上的关键发展。

本文的作者试图通过从紧致 Kähler 流形到弱 1 -完备 Kähler 流形来对这一正在进行的研究做出贡献。然而，这种转变显得相对直截了当，并没有遇到显著的额外困难。构造合适的 *Hermitian* 度量和使用 *Poincaré* 型度量的核心障碍在紧致情形下与以前的结果大致相同[3]。因此，虽然总体的贡献是正确的，但似乎缺乏新颖性和难度，这将不满足《微分几何及其应用》的典型预期。

总之，**尽管本文在技术上对现有结果进行了合理的扩展，但它并没有引入黄，刘，万和杨已经建立的新方法或见解** [4]。因此，我不确定这篇文章的贡献是否足以达到本刊的标准 [5]。

[3] 说的如此容易，那为何到现在还没人做出这方面的考虑呢？虽然碍于作者本人的思考深度和广度以及所掌握的知识有限性，没有办法创造具有独创性的方法来达成证明，但是能够就已有之方法来通过巧妙的组合运用来证明新发现的定理，这本就是创新。而且，就我所知，这里构造恰当的完备 Hermitian 度量也很巧妙地运用了 Hörmander 发展的 L^2 估计理论，将寻找完备的 Kähler 度量转化为构造合适的 Poincaré 度量，这使得难度大大降低，虽然这个方法是 HLWY 首先找到的，但是这无法改变我所设置的独特的 Poincré 度量-这一创新的有效性。

[4] 这个新方法和新见解在何处呢？有待求证！

[5] 不清楚这个期刊的标准是什么？如此高，匪夷所思????

1.2. 修改方针

1.2.1. 创新性

2

关于《L^q extension theorem for jets on weakly pseudoconvex Kähler manifolds》的审稿意见与修改方针

2.1. 审稿意见分析

1. **证明方法是否存在错误**

   在目前审阅的内容中，**数学上的证明**并没有明显的致命错误。然而，以下几点需要进一步注意：

   - **精确性和细节推导**：文中的某些证明依赖于之前文献中的技术和结果，比如 **Demailly** 和 **Popovici** 的工作。然而，在某些关键步骤中，推导过于简略，特别是在复杂几何和全纯分析的背景下。虽然方法正确，但一些中间步骤未解释清楚。这种简化可能会让读者难以理解推导过程，甚至可能遗漏一些边界情况或特殊情形。
   - **奇异性处理**：在弱伪凸 Kähler 流形上处理 jets 的扩展定理时，奇异性的处理方法较为复杂。如果论文在推导过程中对奇异性控制不严谨（比如度量奇点附近的估计不足），则可能导致最终结果不成立。这一点是需要特别检查的关键点。

2. **方法与参考文献过于相似的潜在问题**

   从你的描述中，该论文的主要结果是将参考文献中的方法推广到了 jets 的情形上。这里有几个需要注意的方面：

   - **延续原有方法的合法性**：如果你将参考文献中的定理应用于新的上下文（例如 jets 的扩展），那么证明的每一步都必须严格验证适用于新的情形。这包括：推广过程中所用的假设是否在新的几何背景下成立，jets 的特殊性质是否带来新的困难或条件，是否需要针对 jets 的性质引入新的技巧。如果这些方面没有充分考虑，则可能存在方法上的缺陷。
   - **新结果的独立性**：虽然使用已有的技术方法是学术研究中常见的做法，但为了避免与已有工作过于相

似，你的工作必须在某些方面有实质性的创新或推广。如果只是简单地套用原有方法而没有足够的扩展或创新，那么在学术上可能被认为贡献有限。

3. **潜在的致命错误或漏洞**

   要判断是否存在致命错误可能使论文作废，我们需要重点关注以下几个方面：

   - **基本假设的有效性**：在使用参考文献中的技术时，你是否重新检查了论文中的假设条件是否完全适用于 jets 情形？比如，在弱伪凸 Kähler 流形上的正则性条件、奇异度量的行为等，这些条件的有效性是整个证明体系的基础。如果基本假设不成立，整个证明将失效。
   - **技术细节的稳健性**：对于复杂的全纯分析问题，技术细节（如估计、解析延拓等）通常非常脆弱。如果你在论文中使用了与参考文献相同的方法，但没有处理 jets 中新增的复杂性，这可能会导致错误。例如，jets 在扩展过程中的范数控制问题，可能与原文的情况不同，简单延续原有方法可能无法处理这些新的技术问题。
   - **新情形下的定理适用性**：有时在推广某个定理时，新情形可能带来无法预见的障碍。如果你在证明过程中忽略了 jets 的特性（如更高阶微分的复杂性或几何结构的变化），可能导致证明结果不成立。

4. **结论**

   从目前的分析来看，**如果证明中的假设条件在推广过程中没有仔细检查，或者处理 jets 的特殊性时过于依赖原文的方法，而没有针对新问题作出调整**，这可能会成为论文的主要问题。如果只是形式上将已知结果应用于 jets 上，而未能有效解决扩展过程中的技术难题，那么论文可能会面临学术上的批评。

   建议你重点检查以下几个问题：

   - 证明的每一步是否严格适用于 jets 的情形？
   - 所有使用的假设条件在新情形下是否依然成立？
   - 是否有必要引入新的方法来处理 jets 的特殊性？

   如果在这些方面没有漏洞，论文应该是可以成立的，但

如果在处理这些问题时缺乏严谨性，可能会影响论文的有效性。

2.2. 修改方针

2.2.1. 创新性

第 II 部分
原论文附件

A
论文原件

A.1. 第一篇论文原稿

LOGARITHMIC VANISHING THEOREMS ON WEAKLY 1-COMPLETE KÄHLER MANIFOLDS

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ABSTRACT. In this paper, we prove a logarithmic vanishing theorem on weakly 1-complete kähler manifold, which is a generalization of Huang-Liu-Wan-Yang's result on compact kähler manifold. We first briefly introduce the local case of the theorem, which can be obtained as a corollary of the compact kähler case. Next, we prove the global case of the theorem. The difficulty here is how to find a continuous solution ψ from a sequence ψ_v of discrete solutions of equation $\varphi_v = \bar{\partial}\psi_v$, such that for each $v \in \mathbb{R}, \varphi_v = \bar{\partial}\psi$. In this paper, the continuous solution is given by using the approximation theorem. In the end, we show some direct applications of our main result.

1. INTRODUCTION

The concept of vanishing theorems plays a central role in the realm of algebraic geometry and complex analysis. These theorems, which assert the non-existence of certain cohomology groups under specific conditions, have been instrumental in the advancement of both theoretical understanding and practical applications.

Now we will focus on logarithmic vanishing theorems on weakly 1-complete Kähler manifolds, which is a generalization of logarithmic vanishing theorems on non-compact analytic spaces. Let X be a connected complex manifold of (complex) dimension n. X is called weakly 1-complete if there exists an exhaustion function Φ which is C^∞ and plurisubharmonic on X. We set $X_c := \{x \in X : \Phi(x) < c\}$ for every real number c, which will be called sublevel sets in X. Since S. Nakano established a vanishing theorem for positive bundles (cf. [10, 11]), there have been a lot of activities concerning analytic cohomology groups of weakly 1-complete manifolds (cf. [4, 8, 12, 14, 15, 16, 18, 19]). The purpose of these works is to treat the cohomology groups from differential geometric viewpoint based on the curvature conditions on vector bundles rather than the strong pseudoconvexity of the base manifold X. So they are regarded as natural generalizations of the results obtained for compact manifolds.

The basic properties of the sheaf of logarithmic differential forms and of the sheaves with logarithmic integrable connections on smooth projective manifolds were developed by Deligne in [2]. Esnault and Viehweg investigated in [5] the relations between logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds, and showed that many vanishing theorems follow the degeneration of certain Hodge to de Rham type spectral sequences.

Later the generalization of Akizuki–Nakano vanishing theorem on weakly pseudoconvex or weakly 1-complete Kähler manifolds are finished by Nakano [11, 12], Kazama [8], Abdelkader [1], Takegoshi [19], Ohsawa–Takegoshi [20] and so on. On the other hand, in [13] Norimatsu obtained the logarithmic vanishing theorem on compact Kähler manifold. In [6], Esnault and Viehweg studied the logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds. They obtain the logarithmic type vanishing theorems for the pair (X, D), here X is projective manifold and D is a simple normal crossing divisor. Their methods are based on the Hodge theory and the degeneration of Hodge to de Rham spectral sequence. Recently, in [7], Huang–Liu–Wan–Yang obtain

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the corresponding results on compact Kähler manifold by the standard anlalytic technique like L^2-method. And, in [22], Zou established a logarithmic type Akizuki-Nakano vanishing theorem for weakly pseudoconvex Kähler manifolds.

In this paper, we first follow the analytic method provided in Huang-Liu-Wan-Yang (2023) to prove the local vanishing theorems for the sheaves of logarithmic differential forms over a fixed sublevel set X_c, which is a relatively compact kähler manifold of X. Let us consider X_c and $Y_c = X_c - D$ where $D = \sum_{i=1}^{s} D_i$ is a simple normal crossing divisor in X. Suppose that E is a Hermitian vector bundle over X_c. We first, describe the key steps and main difficulties in our analytic approach. Let $h^E_{Y_c}$ and ω_{Y_c} be two smooth metrics on $E|_{Y_c}$ and Y_c respectively, then we need to show

(a) there is an L^2 fine resolution $(\Omega^{p,\bullet}_{(2)}(X_c, E, \omega_{Y_c}, h^E_{Y_c}), \overline{\partial})$ of the sheaf of logarithmic holomorphic differential forms $\Omega^p(\log D) \otimes \mathcal{O}(E)$ whenever the metrics $h^E_{Y_c}$ and ω_{Y_c} are chosen to be suitable;

(b) the desired curvature conditions for $h^E_{Y_c}$ and ω_{Y_c} can imply vanishing theorems for $(\Omega^{p,\bullet}_{(2)}(X_c, E, \omega_{Y_c}, h^E_{Y_c}), \overline{\partial})$ by using L^2-estimate.

The main difficulties arise from the construction of the Hermitian metric $h^E_{Y_c}$ and the Poincaré type metric ω_{Y_c} which are suitable for both (a) and (b).

It is well-known that various vanishing theorems are very important in complex analytic geometry and algebraic geometry. For instance, the Akizuki-Kodaira-Nakano vanishing theorem asserts that if L is a positive line bundle over a compact Kähler manifold M, then

$$H^q(M, \Omega^p_M \otimes L) = 0 \text{ for any } p + q \geqslant \dim M + 1. \tag{1.1}$$

The main purpose of this paper are to investigate logarithmic type Akizuki-Kodaira-Nakano vanishing theorems for the local pair (X_c, D) and the global pair (X, D), respectively.

First, we try to generalize Norimatsu, Esnalut–Viehweg and Huang–Liu–Wan–Yang's results to weakly 1-complete Kähler manifolds. More specifically, we first get the following vanishing theorem.

Theorem 1.1 (=Theorem 3.4)**.** *Let L be any nef holomorphic Hermitian line bundle on an n dimensional weakly 1-complete Kähler manifold X. Let N be a line bundle and $\Delta = \sum_{i=1}^{s} a_i D_i$ be an $\mathbb{R}$-divisor with $a_i \in [0,1]$ such that $N \otimes \mathcal{O}_{X_c}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. Then for each real number c and on the corresponding sublevel set X_c, we have the vanishing of cohomology groups,*

$$H^q(X_c, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \textit{for any} \quad p + q \geqslant n + k + 1.$$

Subsequently, we will apply the following approximation theorem to establish the global version of this theorem. Initially, we obtain

Lemma 1.2 (=Theorem 4.6)**.** *Let $X_1 \subset X_2$ be a pair of sublevel sets. For any holomorphic section $\varphi \in H^q(\overline{X_1}, \Omega^p(\log D) \otimes \mathscr{F})$, there exists a holomorphic section $\widetilde{\varphi} \in H^q(X_2, \Omega^p(\log D) \otimes \mathscr{F})$ such that for any $\varepsilon > 0$,*

$$\|\widetilde{\varphi} - \varphi\|_{X_1} < \varepsilon.$$

Later, we employ it to prove the following global logarithmic vanishing theorem.

Theorem 1.3 (=Theorem 4.7)**.** *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X. Let N be a line bundle such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle, where $\Delta = \sum_{i=1}^{s} a_i D_i$ $(a_i \in [0,1])$ is a $\mathbb{R}$-divisor.*

Then, we have the vanishing of cohomology groups,

$$H^q(X, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \textit{for any} \quad p + q \geqslant n + k + 1.$$

It is worth to point out that the setting in Theorem 1.1 is quite general and it has many straightforward applications in complex analytic geometry and complex algebraic geometry. The first application is the following log type Girbau's vanishing theorem.

Corollary 1.4. *Let X be a weakly 1-complete Kähler manifold of dimension n and D be a simple normal crossing divisor in X. If L is a nef line bundle and N is a k-positive line bundle over X, then*

$$H^q\left(X, \Omega_X^p(\log D) \otimes L \otimes N\right) = 0 \quad \text{for any } p+q \geqslant n+k+1.$$

In particular, we have the following corollary.

Corollary 1.5. *Let X be a weakly 1-complete Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $L \to X$ is an ample line bundle, then*

$$H^q\left(X, \Omega_X^p(\log D) \otimes L\right) = 0 \quad \text{for any } p+q \geqslant n+1.$$

The similar well-known result on compact Kähler manifolds is proved by Norimatsu [13] using analytic methods (see also Deligne-Illusie's proof [2] by the characteristic p methods). As an analogue to Corollary 1.5, we obtain the following log type Le Potier vanishing theorem for ample vector bundles.

Corollary 1.6. *Let X be a weakly 1-complete Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $E \to X$ is an ample vector bundle of rank r. Then,*

$$H^q\left(X, \Omega_X^p(\log D) \otimes E\right) = 0 \quad \text{for any } p+q \geqslant n+r.$$

As it is well known that the Kawamata-Viehweg-type vanishing theorems have played fundamental roles in algebraic geometry and complex analytic geometry. Another application of Theorem 1.1 is a log-type vanishing theorem for k-positive line bundles over weakly 1-complete Kähler manifolds, which generalizes a version of the Kawamata-Viehweg vanishing theorem over noncomplete manifolds.

Theorem 1.7. *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor. Suppose F is a line bundle over X and m is a positive real number such that $mF = L + D'$, where $D' = \sum_{i=1}^s v_i D_i$ is an effective normal crossing $\mathbb{R}$-divisor and L is a k-positive $\mathbb{R}$-line bundle. Then,*

$$H^q\left(X, \Omega^p(\log D) \otimes F \otimes \mathcal{O}\left(-\sum_{i=1}^s \left(1 + \left[\frac{v_i}{m}\right]\right) D_i\right)\right) = 0 \tag{1.2}$$

for $p+q \geqslant n+k+1$.

Corollary 1.8. *Let X be a weakly 1-complete Kähler manifold $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s c_i D_i$ with $c_i > 0$ and $c_i \in \mathbb{R}$. Then,*

$$H^q\left(X, \Omega^p(\log D) \otimes \mathcal{O}_X\left(-\lceil D' \rceil\right)\right) = 0 \text{ for any } p+q < n-k$$

In particular, when $[D']$ is ample,

$$H^q\left(X, \Omega^p(\log D) \otimes \mathcal{O}_X\left(-\lceil D' \rceil\right)\right) = 0, \text{ for } p+q < n$$

The second variant is the following corollary.

Corollary 1.9. *Let X be a weakly 1-complete Kähler manifold and $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let [$[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s a_i D_i$ with $a_i > 0$ and $a_i \in \mathbb{R}$. If there exists a line bundle L over X and a real number b with $0 < a_j < b$ for all j and $bL = [D']$ as $\mathbb{R}$-line bundles. Then,*

$$H^q\left(X, \Omega^p(\log D) \otimes L^{-1}\right) = 0$$

for $p+q > n+k$ and $p+q < n-k$.

The third variant is the following corollary.

Corollary 1.10. *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X. Suppose there exist some real constants $a_i \geqslant 0$ such that $\sum_{i=1}^{s} a_i D_i$ is a k-positive $\mathbb{R}$-divisor, then for any nef line bundle L, we have*

$$H^q\left(X, \Omega_X^p(\log D) \otimes L\right) = 0 \text{ for any } p+q \geqslant n+k+1$$

The structure of this paper is as follows: In Section 2, we construct a complete Kähler metric of Poincaré type on every sublevel set in the weakly 1-complete Kähler manifold. Then, in Section 3, we apply the method from [7] to derive the local case of our main theorem. In Section 4, we establish the global version of the logarithmic vanishing theorem by obtaining the weak limit of a sequence of differential forms. Finally, in Section 5, we show some direct corollaries of the global theorem.

2. PRELIMINARIES

2.1. **The construction of the complete Kähler metric on the complement.** By [5], for a compact Kähler manifold (M, ω_0) with a simple normal crossing divisor D', there is a natural inclusion map $\tau : U = M \backslash D' \to M$. According to [23, P429, §3], we can choose a *special local coordinate chart* $(W; z_1, \cdots, z_n)$ of M such that the locus of D' is given by $z_1 \cdots z_k = 0$ and $U \cap W = W_r^* = (\Delta_r^*)^l \times (\Delta_r)^{n-l}$ where Δ_r (resp. Δ_r^*) is the (resp. punctured) open disk of radius r in the complex plane. Then we shall define a metric on the product $(\Delta_r^*)^l \times (\Delta_r)^{n-l}$ by

$$\omega_P = i\left(\sum_{j=1}^{l} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 \cdot \log^2 |z_j|^2} + \sum_{j=l+1}^{n} dz_j \wedge d\bar{z}_j\right). \tag{2.1}$$

which possesses the singularity of the Poincaré metric near the punctures (and away from the outer boundaries). Thus it is obvious that ω_P is not complete along the boundary of U.

Now we construct the complete Kähler metric from ω_0 on U, which is of Poincaré type.

Proposition 2.1. *There exists a Kähler metric on U which in special local coordinate chart is equivalent to the metric (2.1), in the sense that the two norms are mutually uniformly bounded.*

Proof. Let $[D_i']$ be the line bundle on M associated to D_i', σ_i a holomorphic section of $[D_i']$ which vanishes to first order on D_i', and $\|\cdot\|_i$ the norm from a C^∞ Hermitian metric $h^{[D']}$ on $[D_i']$ normalized so that $\|\sigma_i\|_i < 1$ on M. The desired metric is

$$\eta = \left(k\omega_0 - \frac{1}{2}\sum_{i=1}^{N} \partial\bar{\partial} \log\log^2 \|\sigma_i\|_i^2\right)$$

for k sufficiently large. In special local coordinate chart $(U; z_1, \cdots, z_n)$ of M in which D_i is defined by $z_i = 0$,

$$\|\sigma_i\|_i^2 = |z_i|^2 e^u$$

for some function $u \in C^\infty(U)$. Then

$$-\frac{1}{2}\partial\bar{\partial}\log\log^2 \|\sigma_i\|_i^2 = \frac{1}{(\log|z_i|^2 + u)^2}\left(\frac{dz_i}{z_i} + \partial u\right) \wedge \left(\frac{d\bar{z}_i}{z_i} + \bar{\partial} u\right) - \frac{1}{\log|z_i|^2 + u}\partial\bar{\partial} u. \tag{2.2}$$

It is now clear that η is positive near D', with singularities like (2.1); the term $k\omega_0$ is added to make η positive on all of U. It is also evident that the two metrics are equivalent if k is taken sufficiently large. □

Remark 2.2. Two nonnegative functions or Hermitian metrics f and g defined on $(\Delta_r^*)^l \times (\Delta_r)^{n-l}$ are said to be *equivalent* along D' if for any relatively compact subdomain V of U, there is a positive constant C such that $(1/C)g \leqslant f \leqslant Cg$ on $V \backslash D'$. In this case we shall use the notation $f \sim g$.

Then by [23, proposition 3.4], $U = M \backslash D'$, endowed with the metric η, is a *complete manifold of finite volume*.

2.2. **Local conversion of weakly 1-complete Kähler manifolds into compact Kähler manifolds.** Let (X, Φ) be a weakly 1-complete Kähler manifold with a fixed Kähler metric ω where Φ is a plurisubharmonic exhaustion function . Let $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor, i.e. every irreducible component D_i is smooth and all intersections are transverse.

Without loss of generality, we may assume Φ is positive. For any positive real number c, the sublevel set $X_c = \{x \in X : \Phi(x) < c\}$ is relatively compact in X and plurisubharmonic exhaustion with respect to plurisubharmonic exhaustion function $\Phi_c := \frac{1}{(c-\Phi)}$. Recall that there is a continuous plurisubharmonic exhaustion function in a weakly 1-complete domain. Set $\omega_c := \omega|_{X_c}$, then (X_c, ω_c, Φ_c) is again a weakly 1-complete Kähler manifold and thus we have an weakly 1-complete sublevel set (X_c, ω_c, Φ_c).

For a fixed positive real number c, we have a sublevel set (X_c, ω_c, Φ_c). We will focus on the sublevel set X_c, it is *relatively compact Kähler manifold*. Similar to Proposition 2.1, we set

$$\omega_{c,p} := (k_c\omega_c - \frac{1}{2}\sum \partial\bar{\partial}\log\log^2\|\sigma_i\|_i^2)$$

for large positive integer k_c which depends on X_c. In special local coordinate neighbourhood U where D_i is defined by $z_i = 0$, and $\|\sigma_i\|_i^2 = |z_i|^2 e^u$ for some function u that is smooth on U. Then

$$-\frac{1}{2}\partial\bar{\partial}\log\log^2\|\sigma_i\|_i^2 = \frac{1}{(\log|z_i|^2+u)^2}\left(\frac{dz_i}{z_i}+\partial u\right)\wedge\left(\frac{d\overline{z_i}}{\overline{z_i}}+\bar{\partial}u\right) - \frac{1}{\log|z_i|^2+u}\partial\bar{\partial}u.$$

It is clear that $\omega_{c,p}$ is positive on X_c and of Poincaré type along D provided k_c is sufficiently large. But $\omega_{c,p}$ is not complete along the boundary of X_c.

Now we provide the definition of the sheaves of logarithmic differential forms here, which is important for the proof. The sheaf of germs of differential p-forms on X with at most logarithmic poles along D denoted by $\Omega_X^p(\log D)$ is a subsheaf of $\Omega_X^p(*D)$. Its space of sections on any open subset V of X are

$$\Gamma(V, \Omega_X^p(\log D)) := \{\alpha \in \Gamma(V, \Omega_X^p \otimes \mathcal{O}_X(D)) \ \ \& \ \ \mathrm{d}\alpha \in \Gamma(V, \Omega_X^{p+1} \otimes \mathcal{O}_M(D))\}.$$

2.3. **Notations and Basic formulae.** Let $\pi : \mathscr{L} \to X$ ba a holomorphic line bundle over a complex manifold X and let $\{b_{ij}\}$ be a system of transition functions with respect to a coordinate cover $\{U_i\}_{i\in I}$ with holomorphic coordinates $(z_i^1, \cdots, z_i^n)$. We fix a hermitian metric $\{a_i\}_{i\in I}$ along the fibers of $\mathscr{L}$ with respect to $\{U_i\}_{i\in I}$ and assume that X is provided with a kähler metric ω, which is denoted by

$$\omega = \sum_{\alpha,\beta}^{n} g_{i,\alpha\bar{\beta}}\mathrm{d}z_i^\alpha \wedge \mathrm{d}\bar{z}_i^\beta.$$

Let $C^{p,q}(X, \mathscr{L})$ be the space of $\mathscr{L}$-valued smooth differential (p,q)-forms on X and let $C_0^{p,q}(X, \mathscr{L})$ be the space of the forms in $C^{p,q}(X, \mathscr{L})$ with compact supports. We express $\varphi = \{\varphi_i\}_{i\in I} \in C^{p,q}(X, \mathscr{L})$ as

$$\varphi_i = \frac{1}{p!q!}\sum_{\alpha_1,\cdots,\alpha_p\ \beta_1,\cdots,\beta_q} \varphi_{i,\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q}\mathrm{d}z_i^{\alpha_1}\wedge\cdots\wedge\mathrm{d}z_i^{\alpha_p}\wedge\mathrm{d}z_i^{\beta_1}\wedge\cdots\wedge\mathrm{d}z_i^{\beta_q}$$

With respect to $\{a_i\}_{i\in I}$ and ω, we set

$$\langle\varphi, \psi\rangle = a_i\sum_{A_p,B_q}\varphi_{i,A_p,B_q}\psi_i^{\overline{A_pB_q}} \tag{2.3}$$

where $A_p = \{\alpha_1, \cdots, \alpha_p\}$ and $B_q = \{\beta_1, \cdots, \beta_q\}$ with $1 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_p \leqslant n$ and $1 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_q \leqslant n$.

For real valued smooth function Φ on X, we put

$$\begin{cases} i) & \langle\varphi, \psi\rangle_\Phi = \langle\varphi, \psi\rangle e^{-\Phi}, \\ ii) & (\varphi, \psi)_\Phi = \displaystyle\int_X \langle\varphi, \psi\rangle_\Phi \mathrm{d}V, \ \text{for}\ \varphi, \psi \in C_0^{p,q}(X, \mathscr{L}). \end{cases} \tag{2.4}$$

In particular, we let

$$(\varphi, \psi) = (\varphi, \psi)_0, \quad \|\varphi\|^2 = (\varphi, \varphi), \quad \|\varphi\|_\Phi^2 = (\varphi, \varphi)_\Phi.$$

And we set the space of smooth L^2 integrable $\mathscr{L}$-valued (p, q)-forms with respect to $\| \ \|_\Phi$ denoted by $L^{p,q}(X, \mathscr{L}, h_\Phi^{\mathscr{L}})$. We denote by $\bar{\partial} : L^{p,q}(X, \mathscr{L}, h_\Phi^{\mathscr{L}}) \to L^{p,q+1}(X, \mathscr{L}, h_\Phi^{\mathscr{L}})$ the maximal closed extension of the original $\bar{\partial}$. Since $\bar{\partial}$ is a closed densely defined operator, the adjoint operator $\bar{\partial}_\Phi^*$(resp. $\bar{\partial}^*$) with respect to $(\varphi, \psi)_\Phi$ (resp. (φ, ψ)) can be defined. We denote the domain, the range, and the nullity of $\bar{\partial}$ by $D_{\bar{\partial}}^{p,q}, \mathrm{Im}(\bar{\partial}^{p,q}), \ker(\bar{\partial}^{p,q})$, respectively. Similarly $D_{\bar{\partial}_\Phi^*}^{p,q}, \mathrm{Im}(\bar{\partial}_\Phi^{*\,p,q}), \ker(\bar{\partial}_\Phi^{*\,p,q})$ are defined.

Definition 2.3 (Inner product)**.** For any non-negative integer μ and any $\varphi, \psi \in L^{p,q}(Y_2, \mathscr{L}, h_\mu^{\mathscr{L}})$, we define the inner product

$$(\varphi, \psi)_\mu := \int_{Y_2} \langle \varphi, \psi \rangle_{\tilde{\omega}_{Y_2}} h_\mu^{\mathscr{L}} dV = \int_{Y_2} \langle \varphi, \psi \rangle_{\tilde{\omega}_{Y_2}} h_{\mathscr{L}} e^{-\mu\psi} dV.$$

and $\|\varphi\|_\mu^2 = (\varphi, \varphi)_\mu$. We denote the adjoint operator of $\bar{\partial}$ in $L^{p,q}(Y_2, \mathscr{L}, h_\mu^{\mathscr{L}})$ by $\bar{\partial}_\mu^*$.

Theorem 2.4 (Sard's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^k function, where $k \geqslant \max\{n - m + 1, 1\}$. The set of critical values of f has measure zero in $\mathbb{R}^m$.*

Here, a point $x \in \mathbb{R}^n$ is called a critical point of f if the Jacobian matrix $Df(x)$ is not of full rank. The image of a critical point under f is called a critical value. Sard's theorem states that almost all values in the target space $\mathbb{R}^m$ are regular values, i.e., they are not critical values.

3. L^2-METHOD ON COMPLETE KÄHLER MANIFOLD

3.1. L^2-methods for the $\bar{\partial}$-equation.

Theorem 3.1 ($\bar{\partial}$-equation on complete Kähler manifolds)**.** *[3, ChapVIII, §3, Theorem 4.5] Let (X, ω) be a complete Kähler manifold. Let (E, h^E) be a Hermitian vector bundle of rank r over X, and assume that the curvature operator $B := [i\Theta(E, h^E), \Lambda_\omega]$ is semi-positive definite everywhere on $\bigwedge^{p,q} T_X^* \otimes E$, for some $q \geqslant 1$. Then for any (p, q)-form $g \in L^2(X, \bigwedge^{p,q} T_X^* \otimes E)$ satisfying $\bar{\partial} g = 0$ and $\int_X \langle B^{-1} g, g \rangle dV_\omega < +\infty$, there exists a $(p, q-1)$-form $f \in L^2(X, \bigwedge^{p,q-1} T_X^* \otimes E)$ such that $\bar{\partial} f = g$ and*

$$\int_X |f|^2 dV_\omega \leqslant \int_X \langle B^{-1} g, g \rangle dV_\omega.$$

Now we will follow the approach in [7] to acquire the L^2 resolution.

Let (X_c, ω_c) be the fixed sublevel set, we denote the metric of the restriction of line bundle L on $Y_c := X_c \backslash D$ by $h_{Y_c}^L$. The sheaf $\Omega_{(2)}^{p,q}(X_c, L, \omega_{c,p}, h_{Y_c}^L)$ over X_c is defined as follows. On any open subset U of X_c, the section space $\Gamma(U, \Omega_{(2)}^{p,q}(X_c, L, \omega_{c,p}, h_{Y_c}^L))$ over U consists of L-valued (p, q)-forms u with measurable coefficients such that the L^2 norms of both u and $\bar{\partial} u$ are integrable on any compact subset K of U. Here the integrability means that both $|u|^2_{\omega_{c,p} \otimes h_{Y_c}^L}$ and $|\bar{\partial} u|^2_{\omega_{c,p} \otimes h_{Y_c}^L}$ are integrable on $K \backslash D$.

Definition 3.2 (Fine sheaf)**.** Recall the sheaf $\mathcal{F}$ is called a **fine sheaf** if for any locally finite open covering $\{U_i\}$, there is a family of homomorphisms $\{f_i\}$, $f_i : \mathcal{F} \to \mathcal{F}$, such that

(1) supp $f_i \subset U_i$,
(2) $\sum_i f_i = 1$, i.e., $\sum_i f_i(s) = s$ for any section s.

Which can be viewed as the existence of the refinement of any open covering of X.

If the metric $\omega_{c,p}$ is of Poincaré type as in Section 2.2, then it is complete along the divisor D and is of finite volume (cf. [23, Proposition 3.4]). As a consequence, the sheaf $\Omega_{(2)}^{p,q}(X_c, L, \omega_{c,p}, h_{Y_c}^L)$ is a fine sheaf.

3.2. An L^2-Dolbeault isomorphism.

Theorem 3.3 (An L^2-type Dolbeault isomorphism)**.** *[7, Theorem 3.1] Let (X, ω) be a weakly pseudoconvex Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divosor in X. For a fixed real number c, let X_c be the sublevel set. Let $\omega_{c,p}$ be a smooth Kähler metric on Y_c which is of Poincaré type along D as in Section 2.2. For a line bundle L, there exists a smooth Hermitian metric $h^L_{Y_c,\alpha_c}$ on $L|_{Y_c}$ such that the sheaf $\Omega^p(\log D) \otimes \mathcal{O}(L)$ over X_c enjoys a fine resolution given by the L^2 Dolbeault complex $(\Omega^{p,*}_{(2)}(X_c, L, \omega_{c,p}, h^L_{Y_c,\alpha_c}), \overline{\partial})$, here α_c is a large positive constant depends on X_c. This is to say, we have an exact sequence of sheaves over X_c*

$$0 \to \Omega^p(\log D) \otimes \mathcal{O}(L) \to \Omega^{p,*}_{(2)}(X_c, L, \omega_{c,p}, h^L_{Y_c,\alpha_c}) \tag{3.1}$$

such that $\Omega^{p,q}_{(2)}(X_c, L, \omega_{c,p}, h^L_{Y_c,\alpha_c})$ is a fine sheaf for each $0 \leqslant p, q \leqslant n$. In particular, by Dolbeault isomorphism

$$H^q(X_c, \Omega^p(\log D) \otimes \mathcal{O}(L)) \cong H^{p,q}_{(2)}(Y_c, L, \omega_{c,p}, h^L_{Y_c,\alpha_c}). \tag{3.2}$$

Even though $\omega_{c,p}$ is not complete, but Y_c admit a complete Kähler metric. Indeed, let $\tilde{\omega} = \hat{\omega} + \omega_{c,p}$, here $\hat{\omega}$ is complete along the boundary of X_c like in the section 2.2. We know that $\tilde{\omega}$ is complete on Y_c. Hence we can still solve the certain $\overline{\partial}$-equation on Y_c thanks to Theorem 3.1. Now we slight modify Huang–Liu–Wan–Yang's approach in [7] to get the local vanishing.

3.3. The local logarithmic vanishing theorems.

Theorem 3.4 (Main Theorem)**.** *Let L be any nef holomorphic Hermitian line bundle on an n dimensional weakly 1-complete Kähler manifold X. Let N be a line bundle and $\Delta = \sum_{i=1}^{s} a_i D_i$ be an $\mathbb{R}$-divisor with $a_i \in [0, 1]$ such that $N \otimes \mathcal{O}_{X_c}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. Then for each real number c and on the corresponding sublevel set X_c, we have the vanishing of cohomology groups,*

$$H^q(X_c, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \textit{for any} \quad p + q \geqslant n + k + 1.$$

Same argument as in [7], with only minor alteration of the compact Kahler manifold, transitioning from X to X_c.

4. The Global Case

Now let us check the corresponding higher direct images. Let $f : X \to S$ be a proper surjective morphism from a Kähler manifold X to a reduced and irreducible complex space S. Let $W \subset S$ be any Stein open subset, we put $V = f^{-1}(W)$. Then V is a holomorphically convex Kähler manifold. Let $\mathcal{F}$ be a coherent sheaf on V. Then $f^* : H^q(V, \mathcal{F}) \to H^0(W, R^q f_* \mathcal{F})$ is an isomorphism of topological vector space for every $q \geqslant 0$. As a direct corollary of Theorem 3.4, we obtain

Corollary 4.1. *Let $f : X \to S$ be a proper holomorphic morphism from a Kähler manifold X of dimension n onto the reduced and irreducible complex space S. Let D be a simple normal crossing divisor for which $f|_D$ is proper. And let N be a line bundle such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive line bundle, where $\Delta = \sum_{i=1}^{s} a_i D_i$ $(a_i \in [0, 1])$ is a $\mathbb{R}$-divisor., then*

$$R^q f_* \left(\Omega^p_X(\log D) \otimes L \otimes N\right) = 0 \quad \textit{for any} \quad p + q \geqslant n + k + 1$$

As demonstrated previously, the local version of [7, Theorem4.1] on weakly 1-complete manifolds has been established. The remaining question pertains to its validity in the global case.

4.1. Basic notions.

By Sard's theorem 2.4, we can choose a sequence $\{c_v\}_{v=0,1,\ldots}$ of real numbers such that

i) $c_{v+1} > c_v > 0$ and $c_v \to +\infty$ as $v \to +\infty$,
ii) the boundary ∂X_{c_v} of $\{x \in X; \Phi(x) \leqq c_v\}$ is smooth for any $v \geqslant 0$.

Then we can choose a smooth convex increasing function $\rho(t)$ such that

i) $\rho(t)\colon (-\infty, +\infty) \to (-\infty, +\infty)$, ii)

$$\rho(t)=\begin{cases}0, & \text{if } t\leqslant \frac{1}{c_2-c_1},\\ >0, & \text{if } t>\frac{1}{c_2-c_1},\end{cases}$$

iii)

(4.1)

$$\begin{cases}\text{i). } \int_0^{+\infty}\sqrt{\rho''_\mu(t)}\mathrm{d}t=+\infty, \text{ for any } \mu\geqslant 1,\\ \text{ii). for every } \zeta<c_1 \text{ and any non-negative integer } v,\ \lim\limits_{\mu\to+\infty}\sup\limits_{t\in(-\infty,\zeta)}\left|\rho_\mu^{(v)}(t)-\rho^{(v)}(t)\right|=0,\end{cases}$$

where $\rho_\mu^{(v)}$ (resp. $\rho^{(v)}$) denotes the v-th derivative of ρ_μ (resp. ρ).

We set

$$\psi=\rho\left(\frac{1}{c_2-\Phi}\right),$$

which is a plurisubharmonic exhaustion function on X_2 and $\psi\equiv 0$ on X_1.

4.2. **Approximation Theorem.** The next lemma is very important in our proof. We will always work on the pairs (X_1,X_2) and (Y_1,Y_2).

Lemma 4.2 (Uniform estimate)**.** *Let $\mathscr{F}:=L\otimes N=L\otimes F\otimes\mathcal{O}_{X_2}(-[\Delta])$,where $F=N\otimes\mathcal{O}_{X_2}([\Delta])$ is a k-positive line bundle. And let L be any nef line bundle over X_2. There exists a positive constant C which is independent to μ such that for any $\mu\geqslant 1$ and $p+q\geqslant n+k+1$, we have the estimate*

$$\|\varphi\|^2_{\rho_\mu}\leqslant C(\|\overline{\partial}\varphi\|^2_{\rho_\mu}+\|\overline{\partial}^*_{\rho_\mu}\varphi\|^2_{\rho_\mu})$$

provided $\varphi\in D^{p,q}_{\overline{\partial}}\bigcap D^{p,q}_{\overline{\partial}^*_{\rho_\mu}}\subset L^{p,q}(Y_2,\mathscr{F},h^{\mathscr{F}}_{\rho_\mu})$. Here $D^{p,q}_{\overline{\partial}}$ is the domain of definition of $\overline{\partial}$ in $L^{p,q}(Y_2,\mathscr{F},h^{\mathscr{F}}_{\rho_\mu})$, and $D^{p,q}_{\overline{\partial}^*_{\rho_\mu}}$ is similar.

Proof. Note that $h^{\mathscr{F}}_{\alpha,\varepsilon,\tau}:=h^{\mathscr{F}}_{\rho;\alpha,\varepsilon,\tau}$. Since F is a k-positive $\mathbb{R}$-line bundle, there exist smooth metrics h^N and $h^{[D_i]}$ on N and $[D_i]$ respectively, such that the curvature form of the induced metric h^F on F

$$\sqrt{-1}\Theta(F,h^F)=\sqrt{-1}\Theta(N,h^N)+\sqrt{-1}\sum_{i=1}^s a_i\Theta([D_i],h^{[D_i]}) \tag{4.2}$$

is semipositive and has at least $n-k$ positive eigenvalues at each point of X_2.

Let $\{\lambda^j_{\omega_{c_2}}(h^F)\}_{j=1}^n$ be the eigenvalues of $\sqrt{-1}\Theta(F,h^F)$ with respect to ω_{c_2} such that $\lambda^j_{\omega_{c_2}}(h^F)\leqslant\lambda^{j+1}_{\omega_{c_2}}(h^F)$ for all j. Thus for any $j\geqslant k+1$ we have

$$\lambda^j_{\omega_{c_2}}(h^F)\geqslant\lambda^{k+1}_{\omega_{c_2}}(h^F)\geqslant\min_{x\in X_2}\left(\lambda^{k+1}_{\omega_{c_2}}(h^F)(x)\right)=:c_0>0.$$

Without loss of generality, we assume $\delta\in(0,1)$. Since L is nef, there exists a smooth metric h^L_δ on L such that

$$\sqrt{-1}\Theta(L,h^L_\delta)=-\sqrt{-1}\partial\overline{\partial}\log h^L_\delta>-\delta\omega_{c_2}. \tag{4.3}$$

Let σ_i be the defining section of D_i. Fix smooth metrics $h_{D_i}:=\|\cdot\|^2_{D_i}$ on line bundles $[D_i]$, such that $\|\sigma_i\|_{D_i}<\frac{1}{2}$. Write the curvature form of $[D_i]$ as $c_1(D_i)=\sqrt{-1}\Theta([D_i],h_{D_i})$. We define $h^\Delta:=\prod_{i=1}^s h^{a_i}_{D_i}$, then the curvature form of (Δ,h^Δ) is

$$-\sqrt{-1}\partial\overline{\partial}\log h^\Delta=-\sqrt{-1}\partial\overline{\partial}\log\prod_{i=1}^s h^{a_i}_{D_i}, \tag{4.4}$$

where $h_{D_i}^{a_i} := \|\sigma_i\|_{D_i}^{2\tau_i}(\log^2(\varepsilon\|\sigma_i\|_{D_i}^2))^{\alpha/2}$. Then let the induce metric of $\mathscr{F}$ be

$$h_{\alpha,\varepsilon,\tau}^{\mathscr{F}} := h^L \cdot h^F \cdot (h^{\Delta})^{-1} \cdot \prod_{i=1}^{s} \|\sigma_i\|_{D_i}^{2\tau_i}(\log^2(\varepsilon\|\sigma_i\|_{D_i}^2))^{\alpha/2}.$$

Here the constant $\alpha > 0$ is chosen to be large enough and the constants $\tau_i, \varepsilon \in (0,1]$ are to be determined later. Note that the smooth metric $h^F \cdot (h^{\Delta})^{-1}$ on $N = F \otimes \mathcal{O}_{X_2}(-[\Delta])$ is the same as h^N up to a globally defined function over X_2. Then the corresponding curvature form is

$$\begin{aligned}\sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) &= \sqrt{-1}\Theta(F, h^F) + \sqrt{-1}\Theta(L, h_\delta^L) + \sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) \\ &\quad + \sum_{i=1}^{s} \frac{\alpha c_1(D_i)}{\log\left(\varepsilon\|\sigma_i\|_{D_i}^2\right)} + \sqrt{-1}\sum_{i=1}^{s}\frac{\alpha\partial\log\|\sigma_i\|_{D_i}^2 \wedge \bar{\partial}\log\|\sigma_i\|_{D_i}^2}{\left(\log\left(\varepsilon\|\sigma_i\|_{D_i}^2\right)\right)^2}.\end{aligned} \tag{4.5}$$

Since $a_i \in [0,1]$, for a fixed large α, we can choose $\tau_1, \cdots, \tau_s \in (0,1]$ and ε such that $\tau_i - a_i, \varepsilon$ are small enough and

$$-\frac{\delta}{2}\omega_{c_2} \leqslant \sqrt{-1}\sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) \leqslant \frac{\delta}{2}\omega_{c_2}, \quad -\frac{\delta}{2}\omega_{c_2} \leqslant \sum_{i=1}^{s}\frac{\alpha c_1(D_i)}{\log\left(\varepsilon\|\sigma_i\|_{D_i}^2\right)} \leqslant \frac{\delta}{2}\omega_{c_2}. \tag{4.6}$$

Note that the constants τ_i and ε are thus fixed, and the choice of ε depends on α.

By (4.2), (4.3), (4.5) and (4.6), one has on Y_2

$$\sqrt{-1}\Theta\left(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) \geqslant \sqrt{-1}\Theta(F, h^F) - 2\delta\omega_{c_2}. \tag{4.7}$$

By the construction of ψ, we can define a *complete Kähler metric* of Y_2

$$\widetilde{\omega}_{Y_2} = \sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) + \kappa\delta\omega_{c_2} \tag{4.8}$$

and a new curvature form

$$\sqrt{-1}\Theta_\mu = \sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}) + \sqrt{-1}\partial\bar{\partial}\mu\psi, \tag{4.9}$$

where $\kappa > 2$ will be determined later.

Since $\widetilde{\omega}_{Y_2}$ is a complete Kähler metric on Y_2, the classical Bochner–Kodaira–Nakano identity shows

$$\Delta'' = \Delta' + [i\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}].$$

Let $\varphi \in \mathcal{C}_0^\infty(Y_2, \Lambda^{p,q}T^*Y \otimes \mathscr{F})$ be the set of smooth $\mathscr{F}$-valued (p,q)-forms with compact support over Y_2. We have

$$\|\bar{\partial}\varphi\|_\mu^2 + \|\bar{\partial}_\mu^*\varphi\|_\mu^2 \geqslant \int_{Y_2} \left\langle [i\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi \right\rangle_{\widetilde{\omega}_{Y_2}} h_{\rho_\mu}^{\mathscr{F}} e^{-\mu\psi} dV. \tag{4.10}$$

Since $\sqrt{-1}\Theta(F, h^F)$ is a semipositive (1,1) form, by (4.7) we see that on Y_2

$$\widetilde{\omega}_{Y_2} = \sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) + \kappa\delta\omega_{c_2} \geqslant (\kappa - 2)\delta\omega_{c_2} \tag{4.11}$$

and then

$$\delta\omega_{c_2} \leqslant \frac{1}{\kappa - 2}\widetilde{\omega}_{Y_2}. \tag{4.12}$$

This implies that

$$\sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) = \widetilde{\omega}_{Y_2} - \kappa\delta\omega_{c_2} \geqslant \frac{-2}{\kappa - 2}\widetilde{\omega}_{Y_2}. \tag{4.13}$$

We can diagonalize simultaneously the Hermitian forms $\omega_{c_2}, \widetilde{\omega}_{Y_2}$ and $\sqrt{-1}\Theta(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\varepsilon,\tau}), \sqrt{-1}\Theta_\mu$. Without lossing generality, we can choose local coordinate systems such that

$$\omega_{c_2} = \sqrt{-1}\sum_{j=1}^{n}\eta_j \wedge \bar{\eta}_j \quad \& \quad \widetilde{\omega}_{Y_2} = \sqrt{-1}\sum_{j=1}^{n}\eta'_j \wedge \bar{\eta}'_j,$$

$$\sqrt{-1}\Theta(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\varepsilon,\tau}) = \sqrt{-1}\sum_{j=1}^{n}\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)\eta_j \wedge \bar{\eta}_j = \sqrt{-1}\sum_{j=1}^{n}\lambda^j_{\widetilde{\omega}_{Y_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)\eta'_j \wedge \bar{\eta}'_j,$$

$$\sqrt{-1}\Theta_\mu = \sqrt{-1}\sum_{j=1}^{n}\gamma_j\eta'_j \wedge \bar{\eta}'_j.$$

Here

$$\eta'_j = \eta_j \cdot \sqrt{\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + \kappa\delta}$$

Then (4.13) will become

$$\sqrt{-1}\Theta(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\varepsilon,\tau}) \geqslant \frac{-2}{\kappa-2}\widetilde{\omega}_{Y_2} = \sum_j \frac{-2}{\kappa-2}\eta'_j \wedge \bar{\eta}'_j,$$

thus we have $\lambda^j_{\widetilde{\omega}_{Y_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geqslant \dfrac{-2}{\kappa-2}$. By (4.7) one has

$$\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geqslant \lambda^j_{\omega_{c_2}}(h^F) - 2\delta.$$

Hence for any $j \geqslant k+1$, we have

$$\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geqslant \min_{x\in X}\left(\lambda^{k+1}_{\omega_{c_2}}(h^F)(x)\right) - 2\delta = c_0 - 2\delta > 0.$$

Thus we have

$$\psi'_j = \frac{\psi_j}{\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + \kappa\delta} < 1.$$

It also implies that for $j \geqslant k+1$,

$$\begin{aligned}\gamma_j := \lambda^j_{\widetilde{\omega}_{Y_2}}(h^{\mathscr{F}}_{\alpha,\varepsilon,\tau}) &= \frac{\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)}{\lambda^j_{\omega_{c_2}}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + \kappa\delta} \\ &\geqslant \frac{c_0 - 2\delta}{c_0 - 2\delta + \kappa\delta} = \left(1 - \frac{\kappa\delta}{c_0 + (\kappa-2)\delta}\right).\end{aligned}$$

Let $\delta = \dfrac{c_0}{2(\kappa-2)n^2}$. Since $\gamma_j \in \left[\dfrac{-2}{\kappa-2}, 1\right)$, by (4.9) and [3, P334, Prop VI-8.3], we have

$$\begin{aligned}\langle[\sqrt{-1}\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi\rangle_{\widetilde{\omega}_{Y_2}} &\geqslant \left(\sum_{i=1}^{q}\gamma_i - \sum_{j=p+1}^{n}\gamma_j\right)|\varphi|^2 \\ &\geqslant \left[(q-k)\cdot\left(1 - \frac{\kappa\delta}{c_0 + (\kappa-2)\delta}\right) + k\cdot\frac{-2}{\kappa-2} - (n-p)\right]|\varphi|^2 \\ &= \left[(p+q-n-k) - \frac{2k}{\kappa-2} - \frac{(q-k)(\kappa\delta)}{c_0 + (\kappa-2)\delta}\right]|\varphi|^2 \\ &= \left[(p+q-n-k) - \frac{2k}{\kappa-2} - \frac{(q-k)}{(\kappa-2)}\cdot\frac{\kappa}{(2n^2+1)}\right]|\varphi|^2\end{aligned}$$

So we can arrange κ sufficiently large enough such that the operator $\langle[\sqrt{-1}\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi\rangle_{\widetilde{\omega}_{Y_2}}$ with respect to $\widetilde{\omega}_{Y_2}$ are all positive on the whole Y_2.

Hence there exists a positive constant C_0 on Y_2, independent to μ, so that

$$\left\langle [i\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi \right\rangle_{\widetilde{\omega}_{Y_2}} \geqslant C_0|\varphi|^2.$$

So if we plug this back into formula (4.10) above, as a consequence, we get the desired uniform estimate

$$\|\varphi\|_\mu^2 \leqslant C(\|\overline{\partial}\varphi\|_\mu^2 + \|\overline{\partial}_\mu^*\varphi\|_\mu^2)$$

for any $\varphi \in \mathcal{C}_0^\infty(Y_2, \Lambda^{p,q}T_Y^* \otimes \mathscr{F})$. Since the metric $\widetilde{\omega}_{Y_2}$ is complete, the above estimate still holds provided $\varphi \in D_{\overline{\partial}}^{p,q} \bigcap D_{\overline{\partial}_\mu^*}^{p,q} \subset L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu;\alpha,\varepsilon,\tau}^{\mathscr{F}})$. □

Proposition 4.3 (Approximation Theorem). *If $p + q \geqslant n + k + 1$, then for any $f \in L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ with $\overline{\partial} f = 0$, then for any $\varepsilon > 0$, there exists an integer μ_0 and $\widetilde{f} \in L^{p,q}(Y_2, \mathscr{F}, h_{\rho_{\mu_0}}^{\mathscr{F}})$ with $\overline{\partial}\widetilde{f} = 0$ such that*

$$\left\| \widetilde{f}|_{\overline{Y_1}} - f \right\|_\rho^2 < \varepsilon. \tag{4.14}$$

Before proof, we provide some useful notions here.

Lemma 4.4 (Riesz representation theorem). *For a continuous linear functional φ on a Hilbert space V, there exists a unique $v \in V$ such that $\varphi(v) = \langle v, w\rangle$ for all $w \in V$. And*

$$\|v\| = \|\varphi\|.$$

Now we take the function $\{\rho_\mu\}_{\mu \geqslant 1}$ satisfying (4.1) and the following additional condition : There exists a constant C such that for every μ and $\varphi \in L^{p,q}\left(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}\right)$

$$\left\|\varphi|_{\overline{Y_1}}\right\|_\rho \leqslant C\|\varphi\|_{\rho_\mu}. \tag{4.15}$$

The existence of such functions has been proved (for any p, q) in [15], where B was assumed to be positive unnecessarily. We can easily obtain $\{\rho_\mu\}_{\mu \geqq 1}$ with required properties without any significant modifications.

Proof. First we have the equivalent proposition:

Proposition 4.5. *If $g \in L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ and $(g, \widetilde{f}|_{\overline{Y_1}})_\rho = 0$ for any $\widetilde{f} \in \bigcup_{\mu=1}^\infty L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$ with $\overline{\partial}\widetilde{f} = 0$, then $(g, f)_\rho = 0$ for any $f \in L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ with $\overline{\partial} f = 0$.*

Now we take the functions $\{\rho_\mu\}_{\mu \geqslant 1}$ satisfying the notions on the beginning of the subsection 4.1. According to (4.15), for any $v \in L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$ and for every $\mu \geqslant 1$, we have

$$\|v|_{\overline{Y_1}}\|_\rho \leqslant C\|v\|_{\rho_\mu}, \tag{4.16}$$

which introduces that when $v = (g, \cdot|_{\overline{Y_1}})_\rho$, then

$$\|(g, u|_{\overline{Y_1}})_\rho\|_\rho \leqslant C\|(g, u)_{\rho_\mu}\|_{\rho_\mu} \leqslant C\|g\|_\rho \cdot \|u\|_{\rho_\mu} \tag{4.17}$$

for any $u \in L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$.

It implies that $(g, \cdot|_{\overline{Y_1}})_\rho$ is a continuous linear functional on $L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$, by Riesz representation theorem there exists a $g_\mu \in L^{p,q}\left(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}\right)$ such that

$$(g_\mu, u)_{\rho_\mu} = \left(g, u|_{\overline{Y_1}}\right)_\rho \text{ for every } u \in L^{p,q}\left(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}\right)$$

and $\|g_\mu\|_{\rho_\mu} \leqq C\|g\|_\rho$. Here

$$(g, \cdot|_{\overline{Y_1}})_\rho(u) = (g, u|_{\overline{Y_1}})_\rho = (g_\mu, u)_{\rho_\mu},$$

and by (4.17)

$$\|g_\mu\|_{\rho_\mu} = \|(g, \cdot|_{\overline{Y_1}})_\rho\|_\rho \leqslant C\|g\|_\rho. \tag{4.18}$$

Since for every $\varphi \in C_0^{p,q}\left(Y_2 \backslash \overline{Y_1}, \mathscr{F}\right)$, we have $(g_\mu, \varphi)_{\rho_\mu} = \left(g, \varphi|_{\overline{Y_1}}\right)_\rho = 0 \quad (\varphi|_{\overline{Y_1}} \equiv 0)$, which implies that

$$\operatorname{Supp}\left(g_\mu, \cdot\right)_{\rho_\mu} \subseteq \overline{Y_1} \iff \operatorname{Supp}\left(g_\mu\right) \subseteq \overline{Y_1}$$

and

$$\left\| g_\mu|_{\overline{Y_1}} \right\|_\rho \leqslant C \cdot \left\| g_\mu \right\|_{\rho_\mu} \leqslant C^2 \|g\|_\rho. \tag{4.19}$$

According to (4.1) and $\operatorname{Supp}(g_\mu) \subseteq \overline{Y_1}, \mu \geqslant 1$, we have

$$\left(g_\mu|_{\overline{Y_1}}, v\right)_\rho \to (g, v)_\rho, \quad \mu \to +\infty \text{ for every } v \in C_0^{p,q}\left(\overline{Y_1}, \mathscr{F}\right).$$

Therefore $\left\{g_\mu|_{\overline{Y_1}}\right\}$ converges weakly to g in $L^{p,q}\left(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$.

On the other hand, since $g_\mu \perp N_{\bar\partial}^{p,q}$ in $L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$, then by

$$L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}) = \overline{R_{\bar\partial^*_{\rho_\mu}}^{p,q}} \oplus N_{\bar\partial}^{p,q} \tag{4.20}$$

we have $g_\mu \in \overline{R_{\bar\partial^*_{\rho_\mu}}^{p,q}}$, which implies that there exists a $w_\mu \in \mathcal{A}^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$ such that

$$g_\mu = \bar\partial^*_{\rho_\mu} w_\mu$$

and from uniform estimate lemma, we shall know that

$$\|w_\mu\|^2_{\rho_\mu} \leqslant 2\left(\|\bar\partial w_\mu\|^2_{\rho_\mu} + \|\bar\partial^* w_\mu\|^2_{\rho_\mu}\right) = 0 + 2\|g_\mu\|^2_{\rho_\mu}. \tag{4.21}$$

With (4.16) , (4.18) and (4.21), then we have

$$\|w_\mu|_{\overline{Y_1}}\|_\rho \overset{(4.16)}{\leqslant} C\|w_\mu\|_{\rho_\mu} \overset{(4.21)}{\leqslant} C \cdot \sqrt{2}\|g_\mu\|_{\rho_\mu} \overset{(4.18)}{\leqslant} C \cdot C\sqrt{2}\|g\|_\rho = \sqrt{2}C^2\|g\|_\rho.$$

Thus subsequence $\left\{w_{\mu_k}|_{\overline{Y_1}}\right\}_{k \geqq 1}$ of $\left\{w_\mu|_{\overline{Y_1}}\right\}_{\mu \geqq 1}$ converges weakly to some $w \in L^{p,q+1}\left(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$. For any $v \in C_0^{p,q}\left(\overline{Y_1}, \mathscr{F}\right)$, we have

$$(g, v)_\rho = \lim_{k \to +\infty} (g_{\mu_k}, v)_{\rho_{\mu_k}} \overset{\text{(I)}}{=} \lim_{k \to +\infty} \left(w_{\mu_k}, \bar\partial v\right)_{\rho_{\mu_k}} \overset{\text{(II)}}{=} \lim_{k \to +\infty} \left(w_{\mu_k}, \bar\partial v\right)_\rho \overset{\text{(III)}}{=} (w, \bar\partial v)_\rho.$$

Note that

(I) As $g_{\mu_k} = \bar\partial^*_{\rho_\mu} w_{\mu_k}$, then we have $(\bar\partial^*_{\rho_\mu} w_{\mu_k}, v)_{\rho_\mu} = (w_{\mu_k}, \bar\partial_\rho v)_{\rho_\mu}$,

(II) By (4.1), we have $\lim\limits_{k \to \infty} \rho_{\mu_k} = \lim\limits_{\mu \to \infty} \rho_\mu = \rho$,

(III) $\lim\limits_{k \to \infty} w_{\mu_k} = w$.

Since $\widetilde{\omega}_{Y_2}$ is a complete metric on $\overline{Y_1} \subset Y_2$, $C_0^{p,q}\left(\overline{Y_1}, \mathscr{F}\right)$ is dense in $D_{\bar\partial}^{p,q} \subset L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ with respect to the graph norm $\left(\|\varphi\|_\rho^2 + \|\bar\partial\varphi\|_\rho^2\right)^{1/2}$ (cf. [21, Theorem 1.1]). Thus $(g, v)_\rho = (w, \bar\partial v)_\rho$ for any $v \in D_{\bar\partial}^{p,q}$, whence $\bar\partial^*_\rho w = g$ in $L^{p,q}\left(Y_2, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$. Therefore, for every $f \in L^{p,q}\left(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$ with

$$\bar\partial f = 0 \qquad \& \qquad (g, f)_\rho = \left(\bar\partial^*_\rho w, f\right) = (w, \bar\partial f)_\rho = 0.$$

$\square$

Since we have the Dolbeault isomorphism from Theorem 3.3:

$$H^q(X_c, \Omega^p(\log D) \otimes \mathscr{F}) \cong H_{(2)}^{p,q}(Y_c, \mathscr{F}, \omega_{c,p}, h_{Y_c}^{\mathscr{F}}) = L^{p,q}(Y_c, \mathscr{F}, h_{\rho_c}^{\mathscr{F}}),$$

which implies the following approximation theorem:

Theorem 4.6 (Applicable Approximation Theorem)**.** *Let $X_1 \subset X_2$ be a pair of sublevel sets. For any holomorphic section $\varphi \in H^q(\overline{X_1}, \Omega^p(\log D) \otimes \mathscr{F})$, there exists a holomorphic section $\widetilde{\varphi} \in H^q(X_2, \Omega^p(\log D) \otimes \mathscr{F})$ such that for any $\varepsilon > 0$,*

$$\|\widetilde{\varphi} - \varphi\|_{X_1} < \varepsilon.$$

4.3. The global logarithmic vanishing theorem.

Theorem 4.7 (The global theorem). *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X. Let N be a line bundle such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle, where $\Delta = \sum_{i=1}^{s} a_i D_i$ $(a_i \in [0,1])$ is a $\mathbb{R}$-divisor.*
Then, we have the vanishing of cohomology groups,

$$H^q(X, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \textit{for any} \quad p+q \geqslant n+k+1.$$

Proof. By 4.1 ,for any pair (c_{v+2}, c_v) $(v \geqslant 0)$ and ρ^v, we choose a sequence of C^∞-strictly convex increasing functions $\left\{\rho_\mu^{v+2}\right\}_{\mu \geqslant 1}$ on $(-\infty, c_{v+2})$ satisfying the properties (4.1) and (4.15). The Approximation theorem holds for any pair (c_{v+2}, c_v) $(v \geqslant 0)$. We denote by $L^{p,q}_{\text{loc}}(X, \mathscr{F})$ (resp. $L^{p,q}_{\text{loc}}(X_v, \mathscr{F})$) the set of the locally square integrable (p,q) forms on X (resp. X_v) with values in $\mathscr{F}$. For $p \geqslant 1$, there is a natural isomorphism

$$\begin{aligned} & H^q\left(X, \Omega^p(\log D) \otimes \mathscr{F}\right) \\ \cong & \frac{\left\{f \in L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F}); \bar\partial f = 0\right\}}{\left\{f \in L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F}); f = \bar\partial g \text{ for some } g \in L^q_{\text{loc}}(X, \Omega^{p-1}(\log D) \otimes \mathscr{F})\right\}} \end{aligned} \tag{4.22}$$

Therefore, in order to prove $H^q(X, \Omega^p(\log D) \otimes \mathscr{F}) = 0$ for $p+q \geqslant n+k$, it suffices to show that for any $\varphi \in L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F})$ with $\bar\partial\varphi = 0$, there exists a $\psi \in L^q_{\text{loc}}(X, \Omega^{p-1}(\log D) \otimes \mathscr{F})$ such that $\varphi = \bar\partial\psi$. We set $\varphi_v = \varphi|_{X_v}$ for any $v \geqslant 0$. Then from Theorem 3.4 (local theorem) and (4.22), there exists a $\psi'_v \in L^{q-1}_{\text{loc}}(X_v, \Omega^p(\log D) \otimes \mathscr{F})$ with $\varphi_v = \bar\partial\psi'_v$ for every $v \geqslant 2$.

For any $v \geqslant 1$, let $L^q(X_v, \Omega^p(\log D) \otimes \mathscr{F})$ be the completion of $C^q_0(X_v, \Omega^p(\log D) \otimes \mathscr{F})$ by the norm $\| \ \|_{X_v}$ with respect to the original Kähler metric ω_v. Inductively, we choose a sequence $\{\psi_v\}_{v \geqslant 1}$ so that

$$\begin{cases} \text{i) } \psi_v \in L^q\left(X_v, \Omega^p(\log D) \otimes \mathscr{F}\right), \\ \text{ii) } \bar\partial\psi_v = \varphi_v, \\ \text{iii) } \|\psi_{v+1} - \psi_v\|^2_{X_{v-1}} < \dfrac{1}{2^v}. \end{cases} \tag{4.23}$$

First we set $\psi_1 = \psi'_2|_{X_1}$. Since $\varphi_2 = \bar\partial\psi'_2$ in $L^q_{\text{loc}}(X_2, \Omega^p(\log D) \otimes \mathscr{F})$,

$$\psi'_2|_{X_1} \in D^{p,q}_{\bar\partial} \subset L^q(X_1, \Omega^{p-1}(\log D) \otimes \mathscr{F})$$

and so $\bar\partial\psi_1 = \varphi_1$ on X_1. Suppose $\psi_1, \ldots, \psi_{v-1}$ are chosen. Then

$$(\psi'_{v+1} - \psi_{v-1})|_{X_{v-1}} \in L^q(X_{v-1}, \Omega^{p-1}(\log D) \otimes \mathscr{F}, \rho^{v-1}(\Phi))$$

and $\bar\partial(\psi'_{v+1} - \psi_{v-1})|_{X_{v-1}} = 0$.

By Approximation Theorem 4.6, for any $\varepsilon > 0$, there exists a $g \in L^q(X_{v+1}, \Omega^{p-1}(\log D) \otimes \mathscr{F}, \rho^{v+1}_{\mu_0}(\Phi))$ such that

$$\|g|_{X_{v-1}} - (\psi'_{v+1} - \psi_{v-1})|_{X_{v-1}}\|^2_{\rho^{v-1}} < \varepsilon$$

and $\bar\partial g = 0$. Since $\| \ \|_{\rho^{v-1}, X_{v-2}}$ and $\| \ \|_{X_{v-2}}$ are equivalent norms on $L^q(X_{v-2}, \Omega^{p-1}(\log D) \otimes \mathscr{F})$, we may assume

$$\|g|_{X_{v-2}} - (\psi'_{v+1} - \psi_{v-1})|_{X_{v-2}}\|^2_{X_{v-2}} < \frac{1}{2^{v-1}}.$$

We set $\psi_v = (\psi'_{v+1} - g)|_{X_v}$. Then we have

$$\begin{cases} \text{i) } \psi_v \in D^{p-1,q}_{\bar\partial} \subset L^q(X_v, \Omega^{p-1}(\log D) \otimes \mathscr{F}), \\ \text{ii) } \varphi_v = \bar\partial\psi_v, \\ \text{iii) } \|\psi_v - \psi_{v-1}\|^2_{X_{v-2}} > \dfrac{1}{2^{v-1}}. \end{cases}$$

Thus $\{\psi_v\}_{v \geqslant 1}$ has been chosen. From (4.23), for any v, $\{\psi_\mu\}_{\mu \geqslant v+1}$ converges with respect to the norm $\| \ \|_{X_v}$ and clearly the limit is the same as the restriction of $\lim\limits_{\mu \geqslant \eta} \psi_\mu$ for any $\eta \geqslant v+2$. Thus we can define an element ψ of $L^q_{\text{loc}}(X, \Omega^{p-1}(\log D) \otimes \mathscr{F})$ by $\psi = \lim_{v \to +\infty} \psi_v$. Since δ is a closed operator in $L^q(X_v, \Omega^{p-1}(\log D) \otimes \mathscr{F})$ for every $v \geqslant 1$, we have

$$\varphi_v = \bar{\partial}\psi \quad \text{in} \quad L^q(X_v, \Omega^p(\log D) \otimes \mathscr{F}) \quad (v \geqslant 1).$$

Hence we have $\varphi = \bar{\partial}\psi$ in $L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F})$. □

5. Applications

As applications of Theorem 4.7, we obtain Corollary 1.4. In particular, for $k = 0$ and N a trivial line bundle, one can deduce Corollary 1.5 As an analogue to Corollary 1.5, we obtain a log-type Le Potier vanishing theorem for ample vector bundles, that is, Corollary 1.6 .

Proof of Corollary 1.6. Let $\pi : \mathbb{P}(E^*) \to X$ be the projective bundle of E and $\mathcal{O}_E(1)$ be the tautological line bundle. One can check that $\pi^* D$ is also a simple normal crossing divisor, and for $p \geqslant 0$, one has

$$\pi_* \left(\Omega^p_{\mathbb{P}(E^*)}(\log \pi^* D) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(1) \right) = \Omega^p_X(\log D) \otimes \mathcal{O}_X(E)$$

and

$$R^q \pi_* \left(\Omega^p_{\mathbb{P}(E^*)}(\log \pi^* D) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(m) \right) = 0 \quad q > 0, m \geqslant 1$$

From [17, Lemma 5.28], we have

$$H^q\left(X, \Omega^p_X(\log D) \otimes E\right) \cong H^q\left(\mathbb{P}(E^*), \Omega^p_{\mathbb{P}(E^*)}(\log \pi^* D) \otimes \mathcal{O}_E(1)\right)$$

Hence, Corollary 1.6 follows from Corollary 1.5. □

By using the same strategy as in the proof of Theorem 4.7, we also obtain several log type Nakano vanishing theorems for vector bundles on X. For instance, we have the following proposition.

Proposition 5.1. *Let E be a vector bundle of $\operatorname{rank} r$ and L be a line bundle on X.*

(1) If E is Nakano positive (resp. Nakano semi-positive) and L is nef (resp. ample), then for any $q \geqslant 1$

$$H^q\left(X, \Omega^n_X(\log D) \otimes E \otimes L\right) = 0.$$

(2) If E is dual-Nakano positive (resp. dual-Nakano semi-positive) and L is nef (resp. ample), then for any $p \geqslant 1$,

$$H^n\left(X, \Omega^p_X(\log D) \otimes E \otimes L\right) = 0.$$

(3) If E is globally generated and L is ample, then for any $p \geqslant 1$,

$$H^n\left(X, \Omega^p_X(\log D) \otimes E \otimes L\right) = 0.$$

Indeed, the vector bundle $E \otimes L$ in Proposition 5.1 is either *Nakano positive* or *dual Nakano positive* (e.g., [9]). Hence, the proof is very similar to (but simpler than) that in Theorem 4.7.

Following, we will present several straightforward applications of Theorem 4.7 over weakly 1-complete Kähler manifolds, which are also closely related to a number of classical vanishing theorems in algebraic geometry.

Proof of Theorem 1.7. Let

$$N = F \otimes \mathcal{O}_X\left(-\sum_{i=1}^{s} \left(1 + \left[\frac{v_i}{m}\right]\right) D_i \right)$$

and

$$\Delta = \sum_i \left(1 + \left[\frac{v_i}{m}\right] - \frac{v_i}{m}\right) D_i.$$

We have that

$$N \otimes \mathcal{O}_X([\Delta]) = \frac{1}{m} L, \tag{5.1}$$

which is a k-positive $\mathbb{R}$-line bundle. Hence, we can apply Theorem 4.7 to complete the proof. □

Proof of Corollary 1.8. Let

$$N = \mathcal{O}_X(-D) \otimes \lceil D' \rceil \quad \text{and} \quad \Delta = \sum_i (1 + c_i - \lceil c_i \rceil) D_i.$$

It is easy to see that

$$N \otimes \mathcal{O}_X([\Delta]) = [D'], \tag{5.2}$$

which is a k-positive $\mathbb{R}$-line bundle. By using Theorem 4.7, one has

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-D) \otimes \lceil D' \rceil) = 0 \tag{5.3}$$

for any $p + q \geqslant n + k + 1$. By Serre duality and the isomorphism

$$(\Omega_X^p(\log D))^* \cong \Omega_X^{n-p}(\log D) \otimes \mathcal{O}_X(-K_X - D), \tag{5.4}$$

we see that (5.3) is equivalent to

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-\lceil D' \rceil)) = 0$$

for any $p + q < n - k$. The proof is complete. □

Proof of Corollary 1.9. Let b' be a real number such that $\max_j a_j < b' < b$, and set

$$N = L^{-1}, \quad \Delta = \frac{D'}{b'} = \sum_{j=1}^{s} \frac{a_j}{b'} D_j.$$

Let

$$F = L^{-1} \otimes \mathcal{O}_X([D]) = L^{-1} + \frac{D'}{b'} = \frac{b - b'}{bb'} D'.$$

It is easy to see that F is a k-positive $\mathbb{R}$-line bundle and the coefficients of Δ are in $[0, 1]$. By Theorem 4.7, we obtain

$$H^q\left(X, \Omega^p(\log D) \otimes L^{-1}\right) = 0 \text{ for } p + q > n + k.$$

On the other hand, we can set

$$N = L \otimes \mathcal{O}_X(-D), \quad \Delta = \sum_{j=1}^{s} \left(1 - \frac{a_j}{2b}\right) D_j, \quad \text{and} \quad F = N \otimes \mathcal{O}_X([D]) = \frac{D'}{2b}$$

It is easy to see that F is a k-positive $\mathbb{R}$-line bundle and the coefficients of Δ are in $[0, 1]$. By Theorem 4.7 again, we get

$$H^q(X, \Omega^p(\log D) \otimes L \otimes \mathcal{O}_X(-D)) = 0, \quad \text{for } p + q > n + k$$

By Serre duality and the isomorphism (5.4), we have

$$H^q\left(X, \Omega^p(\log D) \otimes L^{-1}\right) = 0$$

for any $p + q < n - k$. □

Proof of Corollary 1.10. We can set $N = \mathcal{O}_X$ and $\Delta = \frac{1}{1+\sum_{i=1}^{s} a_i} \sum_{i=1}^{s} a_i D_i$. Then $N \otimes \mathcal{O}([\Delta]) = \mathcal{O}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle.

□

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A.2. 第二篇论文原稿

L^Q-EXTENSION THEOREM FOR JETS ON WEAKLY PSUDOCONVEX KÄHLER MANIFOLDS

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ABSTRACT. By studying the the variable denominators introduced by X. Zhou and L. Zhu, we generalize the theorem 1.2 in their paper to the case of jets, which mainly depends on the method from D. Popovici for the L^2 extension theorem for jets. As a direct corollary, we also give a corollary of a local $L^{\frac{2}{q}}$ extension theorem.

1. INTRODUCTION

T. Ohsawa-K. Takegoshi established a remarkable extension theorem of holomorphic functions defined on a bounded pseudoconvex domain in $\mathbb{C}^n$ with growth control in [12]. Since then, many versions and variants of the L^2 extension theorems have been studied (see [2, 9, 10, 13, 14], etc). These results lead to numerous applications in algebraic geometry and complex analysis.

One interesting problem is to study the L^2 extension theorem for jets. The first such result was given by D. Popovici [13], which generalized the L^2 extension theorems of Ohsawa-Takegoshi-Manivel to the case of jets of sections of a line bundle over a weakly pseudoconvex Kähler manifold. Then J.-P. Demailly [4] considered the extension from more general non reduced varieties. Following a new method of B. Berndtsson-L. Lempert [1], G. Hosono [8] proved an L^2 extension theorem for jets with optimal estimate on a bounded pseudoconvex domain in $\mathbb{C}^n$ (see also [11]).

The idea of considering variable denominators was first introduced by J. McNeal-D. Varolin [10]. They obtained some results on weighted L^2 extension of holomorphic top forms with values in a holomorphic line bundle, where the weights used are determined by the variable denominators. Recently, X. Zhou-L. Zhu [14] proved an L^2 extension theorem for holomorphic sections of holomorphic line bundles equipped with singular metrics on weakly pseudoconvex Kähler manifolds. Futhermore, they obtained optimal constants corresponding to variable denominators.

The method of solving undetermined functions with ODEs was first used in [16]. From then on, a lot of spectacular works appear along this line, such as [6, 14, 15], etc. Several optimal L^2 extension theorems have been proved in this process.

The main goal of this paper is to apply the methods of Zhou-Zhu [14] and Demailly [5] to L^q jet extension to slightly generalize the theorem 1.2 in [14]. As an application of our main theorem, we also obtain a corollary of a local $L^{\frac{2}{q}}$ extension theorem .

We make precise the setting for our work. Let X be an n-dimensional weakly pseudoconvex manifold with Kähler metric ω, and E a Hermitian holomorphic vector bundle of rank $m \geq 1$ over X. Assume that $s \in H^0(X,E)$ is transverse to the zero section. Set

$$Y := \{x \in X : s(x) = 0\}.$$

Futhermore, let L be a holomorphic line bundle equipped with a smooth Hermitian metric satisfying an appropriate positivity condition.

We denote by $\Lambda^{r,s}T_X^*$ the bundle of differential forms of bidegree (r,s) on X, and $\mathcal{J}_Y$ the sheaf of germs of holomorphic functions on X which vanish on Y. For any integer $k \geq 0$, let $\mathcal{O}_X/\mathcal{J}_Y^{k+1}$ be the nonlocally free sheaf of k-jets which are "transversal" to Y. Fix a point $y \in X$ and a Stein

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neighborhood U in X of y. Then this gives rise to a surjective morphism

$$H^0(U, K_X\otimes L) \longrightarrow H^0\left(U, K_X\otimes L\otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$$

of local section spaces, and an arbitrary local lifting $\tilde{f}\in H^0(U,K_X\otimes L)$ of f. For any transversal k-jet $f\in H^0\left(U,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ and any weight function $\rho>0$ on U, the pointwise ρ-weighted norm associated to the section s, was defined by [13, Definition 0.1.1]:

$$|f|^2_{s,\rho,(k)}(y) := |\tilde{f}|^2_L(y) + \frac{\left|\nabla^1\tilde{f}\right|^2_L}{|\Lambda^m(\,\mathrm{d}s)|_E^{2\frac{1}{m}}\rho^{2(m+1)}}(y)+\cdots+\frac{\left|\nabla^k\tilde{f}\right|^2_L}{|\Lambda^m(\,\mathrm{d}s)|_E^{2\frac{k}{m}}\rho^{2(m+k)}}(y),$$

and the $L^2_{(k)}$ weighted norm by:

$$\|f\|^2_{s,\rho,(k)} = \int_Y \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\,\mathrm{d}s)|^2_E}\,\mathrm{d}V_{Y,\omega}.$$

Here for $i=0,\dots,k, \nabla^i\tilde{f}$ is constructed by induction as the projection of the $(1,0)$-part

$$\nabla^{1,0}\left(\nabla^k\tilde{f}\right)\in C^\infty\left(U,K_X\otimes L\otimes S^{j-1}N^*_{Y/X}\otimes T^*_X\right)$$

of $\nabla\left(\nabla^k\tilde{f}\right)$ with the associated Chern connection ∇ to $C^\infty\left(U,K_X\otimes L\otimes S^jN^*_{Y/X}\right)$, induced by the surjective bundle morphism $K_X\otimes L\otimes T^*_X\big|_Y\to K_X\otimes L\otimes N^*_{Y/X}$.

It is worthwhile to notice that the norm $|f|^2_{s,\rho,(k)}$ of the k-jet f at the point $y\in Y$ is independent of the choice of the local lifting $\tilde{f}$. Moreover, one has the following notations [13, Notation 0.1.3]:

(a) For a transversal k-jet $f\in H^0\left(U,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$, denote $\nabla^jf:=\left(\nabla^j\tilde{f}\right)_{|U\cap Y}$, for all $j=0,\dots,k$ and an arbitrary lifting $\tilde{f}\in H^0(U,K_X\otimes L)$ of f.

(b) For every integer $k\geq 0$, and every open set $U\subset X$, set

$$J^k_U: H^0(U,K_X\otimes L)\longrightarrow H^0\left(U,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$$

as the cohomology group morphism induced by the projection $\mathcal{O}_X\to\mathcal{O}_X/\mathcal{J}_Y^{k+1}$. We refer to [13, pp. 2-5] for more details about the notations and the construction of relevant metrics on jets.

In [14, p. 136], Zhou-Zhu defined the variable denominators. Let $\mathfrak{R}$ be the class of functions defined by

$$\left\{R\in C^\infty(-\infty,0]: \begin{array}{c} R>0,\ R'\leq 0,\ \int_{-\infty}^0\frac{1}{R(t)}\mathrm{d}t<+\infty \\ \text{and } e^tR(t) \text{ is bounded above on } (-\infty,0]\end{array}\right\}.$$

Denote $\int_{-\infty}^0\frac{1}{R(t)}\mathrm{d}t$ by C_R. Notice that the function $R(t)$ equals to the function $\frac{1}{c_A(-t)e^t}$ defined just before the main theorems in [16, p. 1143] when $A=0$. With such preparation, our main theorem is as follows.

Theorem 1.1 (Main Theorem)**.** *Let (X,ω) be a weakly pseudoconvex complex n-dimensional manifold possessing a Kähler metric ω,ψ be a plurisubharmonic function on X, E be a holomorphic vector bundle of rank m over X equipped with a smooth Hermitian metric $(1\leq m\leq n)$, and s be a global holomorphic section of E. Assume that s is transverse to the zero section, and let*

$$Y:=\{x\in X: s(x)=0, \Lambda^m(\mathrm{d}s)(x)\neq 0\}.$$

Let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L, which is written locally as $e^{-\varphi_L}$ for some function $\varphi_L\in L^1_{loc}$ with respect to a local holomorphic frame of L. Assume that $\frac{q}{2}\varphi_L+\left(1-\frac{q}{2}\right)\varphi_\omega+\psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Let $0<q\leq 2$. Moreover, assume that the $(1,1)$-form

(i) $\frac{q}{2}\sqrt{-1}\Theta_L+\left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar\partial\varphi_\omega+m\sqrt{-1}\partial\bar\partial\log|s|^2+\sqrt{-1}\partial\bar\partial\psi\geq 0$ *holds on $X\backslash Y$,*

and that there is a continuous function $\alpha>0$ on X such that the following two inequalities hold on $X\backslash Y$:

(ii) $\frac{q}{2}\sqrt{-1}\Theta_L + \left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar{\partial}\varphi_\omega + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 + \sqrt{-1}\partial\bar{\partial}\psi \geq \frac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha|s|_E^2}$,

(iii) $\psi + m\log|s|^2 \leq -2m\alpha$

Then, for every relatively compact open subset $\Omega \subset X$*, and every* k*-jet* $f \in H^0\left(X, K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ *satisfying*

$$C_f := \int_Y \frac{|f|^q_{s,\rho,(k)} e^{-\psi}}{|\Lambda^m(ds)|_E^2} dV_{Y,\omega} < +\infty.$$

Furthermore, assume that there exists $F_1^{(k)} \in H^0(X, K_X \otimes L)$ *such that* $J^k F_1^{(k)} = f$ *and*

$$C_{F_1} := \int_\Omega \frac{\left|F_1^{(k)}\right|_L^q e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega} < +\infty.$$

Then there exists $F^{(k)} \in H^0(X, K_X \otimes L)$ *such that* $J^k F^{(k)} = f$ *and*

$$\int_\Omega \frac{\left|F^{(k)}\right|_L^q e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega} \leq C_{m,R}^{(k)} C_f,$$

where $C_{m,R}^{(k)} > 0$ *is a constant depending only on* $m, R, k, E,$ *and* $\sup_\Omega \|i\Theta_L\|$.

Let p be a positive integer. If we take $q = \frac{2}{q}$ and replace L by $K_X^{p-1} \otimes L$, which is equipped with the metric $e^{(p-1)\varphi_\omega - \varphi_L}$, the we can get from Main Theorem the following corollary.

Corollary 1.2. *Assume that* $\frac{\varphi_L}{p} + \psi$ *is quasi-plurisubharmonic and* φ_L *is locally bounded above. Moreover, assume that*

(i) $\frac{\sqrt{-1}\Theta_L}{p} + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 \geq 0$ *holds on* $X\backslash Y$,

and that there is a continuous function $\alpha > 0$ *on* X *such that the following two inequalities hold on* $X\backslash Y$ *:*

(ii) $\frac{\sqrt{-1}\Theta_L}{p} + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2 \geq \frac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha|s|_E^2}$,

(iii) $\psi + m\log|s|^2 \leq -2m\alpha$.

For every relatively compact open subset $\Omega \subset X$*, and every* k*-jet* $f \in H^0(X, K_X^p \otimes L \otimes \mathcal{O}_X/\mathcal{J}_Y^{k+1})$ *, such that*

$$C_f := \int_Y \frac{(|f|_L)^{\frac{2}{p}} e^{-\psi}}{|\Lambda^m(ds)|_E^2} dV_Y < +\infty.$$

Furthermore, assume that there exists a k*-jet* $F_1^{(k)} \in H^0(\Omega, K_X^p \otimes L)$ *such that* $J^k F_1^{(k)} = f$ *and*

$$C_{F_1} := \int_\Omega \frac{\left(\left|F_1^{(k)}\right|_L\right)^{\frac{2}{p}} e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_X < +\infty.$$

Then there exists a k*-jet* $F^{(k)} \in H^0(\Omega, K_X^p \otimes L)$ *, such that* $J^k F^{(k)} = f$ *and*

$$\int_\Omega \frac{(|F^{(k)}|_L)^{\frac{2}{p}} e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_X \leq C_R \frac{(2\pi)^m}{m!} C_f.$$

2. Preliminaries

Lemma 2.1 (Baisc a priori inequality)**.** *Let* E *be a hermitian vector bundle on a complex manifold* X *equipped with a Kähler metric* ω*. Let* $\eta, \lambda > 0$ *be smooth functions on* X*. Then for every form* $u \in \mathcal{D}\left(X, \Lambda^{p,q} T_X^\star \otimes E\right)$ *with compact support we have*

$$\begin{aligned}\left\|\left(\eta^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\right) D''^* u\right\|^2 + \left\|\eta^{\frac{1}{2}} D'' u\right\|^2 + \left\|\lambda^{\frac{1}{2}} D' u\right\|^2 + 2\left\|\lambda^{-\frac{1}{2}} d'\eta \wedge u\right\|^2 \\ \geqslant \langle\langle [\eta \mathrm{i}\Theta(E) - \mathrm{i} d'd''\eta - \mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda]\, u, u\rangle\rangle.\end{aligned}$$

Lemma 2.2 (L^2-existence theorem). *Let (X,ω) be a complete Kähler manifold equipped with a (non-necessarily complete) Kähler metric ω, and let Q be a Hermitian vector bundle over X. Assume that τ and A are smooth and bounded positive functions on X and let*

$$\mathrm{B} := \left[\tau\sqrt{-1}\Theta_Q - \sqrt{-1}\partial\bar{\partial}\tau - \sqrt{-1}A^{-1}\partial\tau\wedge\bar{\partial}\tau, \Lambda\right].$$

Assume that $\delta \geq 0$ is a number such that $\mathrm{B}+\delta\mathrm{I}$ is semi-positive definite everywhere on $\bigwedge^{n,q} T_X^*\otimes Q$ for some $q\geq 1$. Then given a form $g \in L^2\left(X, \bigwedge^{n,q} T_X^*\otimes Q\right)$ such that $\mathrm{D}''g = 0$ and

$$\int_X \langle(\mathrm{B}+\delta\mathrm{I})^{-1}g, g\rangle_Q\, dV_X < +\infty,$$

there exists an approximate solution $u \in L^2\left(X, \bigwedge^{n,q-1} T_X^*\otimes Q\right)$ and a correcting term $h \in L^2(X, \bigwedge^{n,q} T_X^*\otimes Q)$ such that $\mathrm{D}''u + \sqrt{\delta}h = g$ and

$$\int_X \frac{|u|_Q^2}{\tau + A} dV_X + \int_X |h|_Q^2 dV_X \leq \int_X \langle(\mathrm{B}+\delta\mathrm{I})^{-1}g, g\rangle_Q\, dV_X.$$

Proof. By lemma 2.1, lemma 2.2 can be obtained by almost the same arguments as in [5], where the term $\int_X \langle(\mathrm{B}+\delta\mathrm{I})^{-1}g, g\rangle_Q\, dV_X$ in the above inequality is written as $2\int_X \langle(\mathrm{B}+\delta\mathrm{I})^{-1}g, g\rangle_Q\, dV_X$. □

Lemma 2.3 (The property of psh function). *Let X be a Stein manifold and φ be a plurisubharmonic function on X. Then there exists a decreasing sequence of smooth strictly plurisubharmonic functions $\{\varphi_j\}_{j=1}^{+\infty}$ such that $\lim_{j\to+\infty}\varphi_j = \varphi$*

Lemma 2.4 (Theorem 1.5 in [3]). *Let X be a Kähler manifold, and Z be an analytic subset of X. Assume that Ω is a relatively compact open subset of X possessing a complete Kähler metric. Then $\Omega\backslash Z$ carries a complete Kähler metric.*

Lemma 2.5 (Theorem 4.4.2 in [7]). *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$, and φ be a plurisubharmonic function on Ω. For every $h \in L^2_{(p,q+1)}(\Omega,\varphi)$ with $\bar{\partial}h = 0$ there is a solution $v \in L^2_{(p,q)}(\Omega, loc)$ of the equation $\bar{\partial}v = h$ such that*

$$\int_\Omega \frac{|v|^2}{\left(1+|z|^2\right)^2} e^{-\varphi} dV \leq \int_\Omega |h|^2 e^{-\varphi} dV$$

Lemma 2.6 (Lemma 6.9 in [3]). *Let Ω be an open subset of $\mathbb{C}^n$ and Z be a complex analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and h is a (p,q)-form with L^1_{loc} coefficients such that $\bar{\partial}v = h$ on $\Omega\backslash Z$ (in the sense of distribution theory). Then $\bar{\partial}v = h$ on Ω.*

Lemma 2.7 (Lagrange's inequality). *Let X be a complex manifold, E be a Hermitian vector bundle over X of rank m, and $\{\bullet,\bullet\}_E : \wedge^{p_1,q_1} T_X^*\otimes E \times \wedge^{p_2,q_2} T_X^*\otimes E \longrightarrow \wedge^{p_1+q_2,q_1+p_2} T_X^*$ be the sesquilinear product which combines the wedge product $(u,v)\mapsto u\wedge\bar{v}$ on scalar valued forms with the Hermitian inner product on E. Then for any smooth section s of E over X and any smooth section w of $T_X^*\otimes E$ over X,*

$$\sqrt{-1}\{w,s\}_E\wedge\{s,w\}_E \leq |s|_E^2\sqrt{-1}\{w,w\}_E. \tag{2.1}$$

Proof. Since $\{\bullet,\bullet\}_E$ is a pointwise product, it's sufficient to prove (2.1) at every fixed point of X. Hence, we can regard T_X^* and E as vector spaces. Then s and w are regarded as elements in E and $T_X^*\otimes E$ respectively. If $s = 0$, (2.1) is trivial. If $s \neq 0$, without loss of generality, we can assume that $|s|_E = 1$. Then we choose $e_2,\cdots,e_m \in E$ such that $s, e_2,\cdots,e_m$ form an orthonormal basis of E. Then w can be written as

$$w_1\otimes s + \sum_{j=2}^m w_j\otimes e_j,$$

for some $w_j \in T_X^*(1\leq j\leq m)$. Then

$$\sqrt{-1}\{w,s\}_E\wedge\{s,w\}_E = \sqrt{-1}w_1\wedge\bar{w}_1,$$

and

$$|s|_E^2\sqrt{-1}\{w,w\}_E=\sqrt{-1}\sum_{j=1}^{m}w_j\wedge\bar{w}_j\geq\sqrt{-1}w_1\wedge\bar{w}_1.$$

Hence, (2.1) holds. The lemma is, thus, proved. □

3. PROOF OF THE NORMAL CASE THEOREM

In order to prove the main theorem, we should prove the following theorem firstly.

Theorem 3.1 (The case of L^2-extension). *Let (X,ω) be a weakly pseudoconvex complex n dimensional manifold possessing a Kähler metric ω, ψ be a plurisubharmonic function on X, E be a holomorphic vector bundle of rank m over X equipped with a smooth Hermitian metric $(1\leq m\leq n)$, and s be a global holomorphic section of E. Assume that s is transverse to the zero section, and let*

$$Y:=\{x\in X:s(x)=0,\Lambda^m(\mathrm{d}s)(x)\neq 0\}.$$

Let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L, which is written locally as $e^{-\varphi_L}$ for some function $\varphi_L\in L^1_{loc}$ with respect to a local holomorphic frame of L. Assume that $\frac{q}{2}\varphi_L+\left(1-\frac{q}{2}\right)\varphi_\omega+\psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Moreover, assume that the $(1,1)$-form

(i) $\frac{q}{2}\sqrt{-1}\Theta_L+\left(1-\frac{q}{2}\right)\sqrt{-1}\partial\bar\partial\varphi_\omega+m\sqrt{-1}\partial\bar\partial\log|s|^2+\sqrt{-1}\partial\bar\partial\psi\geq 0$ *holds on $X\backslash Y$,*

and that there is a continuous function $\alpha>0$ on X such that the following two inequalities hold on $X\backslash Y$:

(ii) $\frac{q}{2}\sqrt{-1}\Theta_L+\left(1-\dfrac{q}{2}\right)\sqrt{-1}\partial\bar\partial\varphi_\omega+m\sqrt{-1}\partial\bar\partial\log|s|^2+\sqrt{-1}\partial\bar\partial\psi\geq\dfrac{\{\sqrt{-1}\Theta_E s,s\}_E}{\alpha|s|_E^2}$,

(iii) $\psi+m\log|s|^2\leq -2m\alpha$

Then, for every relatively compact open subset $\Omega\subset X$, and every k-jet $f\in H^0\left(X,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ satisfying

$$C_f:=\int_Y\frac{|f|^2_{s,\rho,(k)}e^{-\psi}}{|\Lambda^m(ds)|_E^2}dV_{Y,\omega}<+\infty.$$

Then there exists $F^{(k)}\in H^0(X,K_X\otimes L)$ such that $J^kF^{(k)}=f$ and

$$\int_\Omega\frac{\left|F^{(k)}\right|_L^2}{e^{\psi+m\log|s|_E^2}R(\psi+m\log|s|_E^2)}dV_{X,\omega}\leq C^{(k)}_{m,R}C_f,$$

where $C^{(k)}_{m,R}>0$ is a constant depending only on m,R,k,E, and $\sup_\Omega\|i\Theta_L\|$.

Proof. Without loss of generality, we can suppose that $C_R=1$. Otherwise, we replace R with C_RR in the proof.

If $f=0$ on Y, then $F=0$ satisfies the conclusion of Proposition 4.1. In the following proof, we assume that f is not 0 identically.

Since X is pesudoconvex, there exists a smooth strictly plurisubharmonic exhaustion function P on X. Instead of working on X itself, we will work rather on the relatively compact subset $X_c\backslash Y$, where $X_c=\{P<c\}$ $(c=1,2,\cdots$, we choose P such that $X_1\neq\emptyset)$. By Lemma 2.4 $X_c\backslash Y$ $(c=1,2,\cdots)$ are complete Kähler.

We will discuss for fixed c until the end of the proof.

Let $\zeta:(-\infty,0)\longrightarrow(0,+\infty)$ be a smooth strictly increasing function, and $\chi:(-\infty,0)\longrightarrow(0,+\infty)$ a smooth strictly decreasing function. Assume that $\chi(t)\geq-\frac{t}{2}$ for $t\in(-\infty,0)$. We will find more assumptions about ζ and χ in the proof, by which we will get explicit ζ and χ in the end of this section.

Let $a\in(0,1)$ and put $\sigma_\varepsilon=m\log\left(|s|^2+\varepsilon^2\right)-a$ and $\sigma=m\log|s|^2-a$. Since $|s|\leq 1$ on X, there exists a positive number $\varepsilon_a\in(0,1)$ such that $\sigma_\varepsilon\leq-\frac{a}{2}$ on $\overline{X_c}$ for $\varepsilon\in(0,\varepsilon_a)$.

Assume that K_X is natuarally equipped with the smooth metric e^{φ_ω} with respect to the dual frame of dz. Let $L_{a,\varepsilon}$ denote the line bundle L on $X_c\backslash Y$ equipped with the new metric $h_{a,\varepsilon} := e^{-\left(\frac{q}{2}\varphi_L+(1-\frac{q}{2})\varphi_\omega+\psi\right)-\sigma-\zeta(\sigma_\varepsilon)}$.

Set $\tau_\varepsilon = \chi(\sigma_\varepsilon)$ and let A_ε be a smooth positive function on $\overline{X_c}$, which will be determined later. Set $\mathrm{B}_\varepsilon = [\Theta_\varepsilon, \Lambda]$ on $X_c\backslash Y$, where

$$\Theta_\varepsilon := \tau_\varepsilon\sqrt{-1}\Theta_{L_{a,\varepsilon}} - \sqrt{-1}\partial\bar{\partial}\tau_\varepsilon - \sqrt{-1}\frac{\partial\tau_\varepsilon\wedge\bar{\partial}\tau_\varepsilon}{A_\varepsilon}.$$

Set

$$\nu_\varepsilon := \frac{\{D's, s\}}{|s|^2+\varepsilon^2}. \tag{3.1}$$

We want to find suitable ζ, χ and A_ε such that

$$\Theta_\varepsilon|_{X_c\backslash Y} \geq \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon. \tag{3.2}$$

Simple calculations yield

$$\begin{aligned}
&\Theta_\varepsilon|_{X_c\backslash Y}\\
=&\chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + \sqrt{-1}\partial\bar{\partial}\sigma\right)\\
&+\left(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon)-\chi'(\sigma_\varepsilon)\right)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon + \left(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon)-\chi''(\sigma_\varepsilon)-\frac{\left(\chi'(\sigma_\varepsilon)\right)^2}{A_\varepsilon}\right)\sqrt{-1}\partial\sigma_\varepsilon\wedge\bar{\partial}\sigma_\varepsilon\\
=&\chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2\right)\\
&+\left(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon)-\chi'(\sigma_\varepsilon)\right)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon + \left(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon)-\chi''(\sigma_\varepsilon)-\frac{\left(\chi'(\sigma_\varepsilon)\right)^2}{A_\varepsilon}\right)\sqrt{-1}\partial\sigma_\varepsilon\wedge\bar{\partial}\sigma_\varepsilon.
\end{aligned} \tag{3.3}$$

Assume that the equalities

$$\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon)-\chi'(\sigma_\varepsilon) = 1 \tag{3.4}$$

and

$$\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon)-\chi''(\sigma_\varepsilon)-\frac{\left(\chi'(\sigma_\varepsilon)\right)^2}{A_\varepsilon} = 0 \tag{3.5}$$

hold, we obtain that

$$\Theta_\varepsilon|_{X_c\backslash Y} \geq \chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2\right) + \sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon. \tag{3.6}$$

Furthermore, by (3.4) we can assume that $A_\varepsilon = \eta(\sigma_\varepsilon)$ for some smooth function $\eta : (-\infty, 0) \longrightarrow (0, +\infty)$ such that

$$\chi\zeta''-\chi''-\frac{\left(\chi'\right)^2}{\eta} = 0 \tag{3.7}$$

Since it follows from Lemma 2.7 that

$$|s|^2\sqrt{-1}\sum_{i=1}^m ds^i\wedge d\bar{s}^i \geq \sqrt{-1}\left(\sum_{i=1}^m \bar{s}^i ds^i\right)\wedge\left(\sum_{i=1}^m s^i d\bar{s}^i\right)$$

, which can be stated as

$$|s|^2\sqrt{-1}\{D's, D's\} \geq \sqrt{-1}\{D's, s\}\wedge\{s, D's\}.$$

We obtain that on $X_c\backslash Y$,

$$\begin{aligned}\sqrt{-1}\partial\bar\partial\sigma_\varepsilon &= \frac{m\sqrt{-1}\{D's,D's\}}{|s|^2+\varepsilon^2} - \frac{m\sqrt{-1}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\ &\geqslant \frac{m\varepsilon^2}{|s|^2}\frac{\sqrt{-1}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\ &= \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}.\end{aligned}$$

Then it follows from (3.6) that on $X_c\backslash Y$,

$$\Theta_\varepsilon \geqslant \left(\chi(\sigma_\varepsilon)\left(\frac{q}{2}\sqrt{-1}\partial\bar\partial\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar\partial\varphi_\omega + \sqrt{-1}\partial\bar\partial\psi + m\sqrt{-1}\partial\bar\partial\log|s|^2\right) - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\right) + \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon.$$

Since $\chi(\sigma_\varepsilon)\geq m\alpha$ by the assumption $\chi(t)\geq -\frac{t}{2}$, it follows from the condition on $X\backslash Y$ in Theorem 1.1 that

$$\begin{aligned}&\chi(\sigma_\varepsilon)(\frac{q}{2}\sqrt{-1}\partial\bar\partial\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar\partial\varphi_\omega + \sqrt{-1}\partial\bar\partial\psi + m\sqrt{-1}\partial\bar\partial\log|s|^2) - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\ &= \chi(\sigma_\varepsilon)(\frac{q}{2}\sqrt{-1}\partial\bar\partial\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar\partial\varphi_\omega + \sqrt{-1}\partial\bar\partial\psi + m\sqrt{-1}\partial\bar\partial\log|s|^2) - \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\frac{\sqrt{-1}\{\Theta_E s,s\}}{\alpha|s|^2}\\ &\geq \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\left(\frac{q}{2}\sqrt{-1}\partial\bar\partial\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar\partial\varphi_\omega + \sqrt{-1}\partial\bar\partial\psi + \sqrt{-1}\partial\bar\partial\sigma - \frac{\sqrt{-1}\{\Theta_E s,s\}}{\alpha|s|^2}\right)\\ &\geq 0 \quad \text{By the assumption (ii) in Theorem 1.1.}\end{aligned}$$

on $X_c\backslash Y$. Hence one obtain (3.2) as expected.

As a result, we have

$$B_\varepsilon \geq \left[\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon,\Lambda\right] = \frac{m\varepsilon^2}{|s|^2}T_{\bar\nu_\varepsilon}T^*_{\bar\nu_\varepsilon} \tag{3.8}$$

on $X_c\backslash Y$ as an operator on $(n,1)$ forms, where $T_{\bar\nu_\varepsilon}$ denotes the operator $\bar\nu_\varepsilon\wedge\bullet$ and $T^*_{\bar\nu_\varepsilon}$ is its Hilbert adjoint operator.

3.1. **Solving $\bar\partial$ equation on X_c with estimates.** With such preparation, we now argue by induction on $k\geq 0$. The case $k=0$ is a special case of [14, Theorem 1.1]. Now, assume that the theorem has been proved for $k-1$, and we consider the short exact sequence of sheaves

$$0\longrightarrow S^kN^*_{Y/X}\longrightarrow \mathcal{O}_X/\mathcal{J}_Y^{k+1}\longrightarrow \mathcal{O}_X/\mathcal{J}_Y^k\longrightarrow 0$$

Let $J_X^{k-1}f\in H^0\left(X,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^k\right)$ be the image of $f\in H^0\left(X,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ under the induced cohomology group morphism. By the induction hypothesis, there exists $F^{(k-1)}\in H^0(X, K_X\otimes L)$ such that

$$J_X^{k-1}F^{(k-1)} = J_X^{k-1}f,\quad \int_{X_c}\frac{\left|F^{(k-1)}\right|_L^2}{|s|^{2m}R\left(\psi+m\log|s|_E^2\right)}e^{-\psi}\mathrm{d}V_{X,\omega} \leq C^{(k-1)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k-1)}e^{-\psi}}{|\Lambda^m(\mathrm{d}s)|^2}\mathrm{d}V_{Y,\omega}, \tag{3.9}$$

where $C^{(k-1)}_{m,R}>0$ is a constant as in the statement of Theorem 3.1 and $Y_c := Y\cap X_c$. Thus, the image $J_X^{k-1}f - J_X^{k-1}F^{(k-1)}\in H^0\left(X,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^k\right)$ of $f-J_X^kF^{(k-1)}\in H^0\left(X,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ vanishes. So we can view the jet $f-J_X^kF^{(k-1)}$ as a global holomorphic section (on Y) of the sheaf $K_X\otimes L\otimes S^kN^*_{Y/X} = K_X\otimes L\otimes S^kE^*_{|Y}$.

Using the results in [13, p12], one can construct an extension $\tilde f\in C^\infty(X,K_X\otimes L)$ of the holomorphic k-jet $f\in H^0\left(X,K_X\otimes L\otimes\mathcal{O}_X/\mathcal{J}_Y^{k+1}\right)$ by means of a partition of unity, satisfying

$$\bar\partial\tilde f = 0 \quad \text{on } Y,$$

and

$$|\bar{\partial}\tilde{f}| = O\left(|s|^{k+1}\right) \quad \text{in a neighbourhood of } Y.$$

Set

$$G_{\varepsilon}^{(k-1)} := \theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)\left(\tilde{f} - F^{(k-1)}\right) \in C^{\infty}(X, K_X \otimes L),$$

where $0 < \varepsilon < \varepsilon_a$, and $\theta : \mathbb{R} \to [0,1]$ is a C^{∞} function such that $\theta \equiv 0$ on $\left(-\infty, \frac{l}{2}\right]$, $\theta \equiv 1$ on $\left[1-\frac{l}{2}, +\infty\right)$, and $\left|\theta'\right| \leq \frac{1+l}{1-l}$ on $\mathbb{R}$. Then it suffices to solve the equation

$$\bar{\partial}u_{\varepsilon} = \bar{\partial}G_{\varepsilon}^{(k-1)}, \tag{3.10}$$

with the extra condition $\frac{|u_{\varepsilon}|^2}{|s|^{2m}} \in L^1_{\text{loc}}$ in a neighbourhood of Y. This condition guarantees that u_{ε}, as well as all its jets of orders $\leq k$, vanishes on Y. By direct calculations, one has

$$\bar{\partial}G_{\varepsilon}^{(k-1)} = g_{\varepsilon}^{(1)} + g_{\varepsilon}^{(2)},$$

where

$$g_{\varepsilon}^{(1)} = -\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\cdot\theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)\bar{v}_{\varepsilon}\wedge\left(\tilde{f} - F^{(k-1)}\right),$$
$$g_{\varepsilon}^{(2)} = \theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)\bar{\partial}\left(\tilde{f} - F^{(k-1)}\right).$$

Recall that v_{ε} is given in (3.1).

In this situation, $g_{\varepsilon}^{(2)}$ turns out to have no contribution in the limit since it converges uniformly to 0 on every compact set when ε tends to 0 . Actually, Supp $\left(g_{\varepsilon}^{(2)}\right) \subset \{|s| < \sqrt{2}\varepsilon\}$ and $\left|g_{\varepsilon}^{(2)}\right| = O\left(|s|^{k+1}\right)$ because of $|\bar{\partial}\tilde{f}| = O\left(|s|^{k+1}\right)$ in a neighbourhood of Y as we have previously shown.

Then

$$\int_{X_c \setminus Y} \langle B_{\varepsilon}^{-1} g_{\varepsilon}^{(2)}, g_{\varepsilon}^{(2)}\rangle_L |s|^{-2m} e^{-\psi} \mathrm{d}V_{X,\omega} = O(\varepsilon),$$

provided that B_{ε} is locally uniformly bounded below in a neighbourhood of Y. Otherwise, we shall solve the approximate equation $\bar{\partial}u + \sqrt{\delta}h = g_{\varepsilon}$ with $\delta > 0$ small (see Lemma 2.2 and [5, Remark 3.2] for more details). One can remove the extra error term $\sqrt{\delta}h$ by putting $\delta \to 0$ at the end. Since there is no essential difficulty during this procedure, for the purpose of simplicity, we will assume to have the desired lower bound for B_{ε} and the estimate of $g_{\varepsilon}^{(2)}$ as above. Next, we turn to estimate the term involving $g_{\varepsilon}^{(1)}$ on $X_c \setminus Y$. By (3.8),

$$\langle B_{\varepsilon}^{-1} g_{\varepsilon}^{(1)}, g_{\varepsilon}^{(1)}\rangle_{L_{a,\varepsilon}} \leq \frac{|s|^2}{m\varepsilon^2}\cdot\left|\theta'\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\left(\tilde{f} - F^{(k-1)}\right)\right|^2_{L_{a,\varepsilon}}.$$

In [13, p17], Popovici showed that on every compact set,

$$\frac{\left|\left(\tilde{f} - F^{(k-1)}\right)(\varepsilon s, z')\right|^2_L}{\varepsilon^{2k}} \longrightarrow \left|\nabla^k\left(f - J_X^k F^{(k-1)}\right)(z')\right|^2_L, \quad (\varepsilon \to 0).$$

Then using a partition of unity $\{\xi_p\}_{p=1}^{p_0}$ around $\overline{X_c}\setminus Y$ and the Fubini theorem, we obtain

$$\begin{aligned}\int_{X_c\setminus Y}\left\langle \mathrm{B}_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\right\rangle_{L_{a,\varepsilon}} e^{-\psi}\mathrm{d}V_X &\le \frac{e^a(1+l)^2}{m(1-l)^2}\sum_{p=1}^{p_0}\int_{X_c\cap\left\{\sqrt{\frac{l}{2-l}}\varepsilon<|s|<\sqrt{\frac{2-l}{l}}\varepsilon\right\}}\frac{\varepsilon^2\xi_p|\tilde{f}-F^{(k-1)}|_L^2e^{-\psi}\mathrm{d}V_X}{\left(|s|^2+\varepsilon^2\right)^2|s|^{2m-2}}\\ &\le\sum_{p=1}^{p_0}\left(\int_{\left\{z\in\mathbb{C}^m:\sqrt{\frac{l}{2-l}}\varepsilon<|z|<\sqrt{\frac{2-l}{l}}\varepsilon\right\}}\frac{\varepsilon^2(\sqrt{-1})^{m^2}\Lambda^m(dz)\wedge\Lambda^m(d\bar{z})}{\left(|z|^2+\varepsilon^2\right)^2|z|^{2m-2}}\right.\\ &\qquad\left.\times\frac{e^a(1+l)^2}{m(1-l)^2}\int_{Y_c}\frac{\xi_p\left|\nabla^k\left(\tilde{f}-J_X^kF^{(k-1)}\right)\right|_L^2}{|\wedge^m(ds)|^2}e^{-\psi}\mathrm{d}V_{Y,\omega}\right)\\ &\to\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J_X^kF^{(k-1)}\right)\right|_L^2}{|\wedge^m(ds)|^2}e^{-\psi}\mathrm{d}V_{Y,\omega}\quad(\varepsilon\to0),\end{aligned}$$

where

$$C_{m,k}:=\int_{\left\{z\in\mathbb{C}^m:\sqrt{\frac{l}{2-l}}<|z|<\sqrt{\frac{2-l}{l}}\varepsilon\right\}}\frac{(\sqrt{-1})^{m^2}\Lambda^m(dz)\wedge\Lambda^m(d\bar{z})}{\left(|z|^2+1\right)^2|z|^{2m-2}},$$

which depends only on m and k. It may be worthwhile to note that

$$\left|\nabla^k\left(f-J_X^kF^{(k-1)}\right)\right|_L=\left|f-J_X^kF^{(k-1)}\right|_L$$

, where $f-J_X^kF^{(k-1)}\in H^0\left(Y,K_X\otimes L\otimes S^kN_{Y/X}^*\right)$. Then, one has

$$\int_{X_c\setminus Y}\left\langle B_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\right\rangle_{L_{a,\varepsilon}} e^{-\psi}\mathrm{d}V_X\le\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J_X^kF^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{d}s)|^2}e^{-\psi}\,\mathrm{d}V_{Y,\omega},$$

when ε is small enough. By using Lemma 2.2 with $\delta=0$, we can solve (3.10), i.e., there exists $u_{c,a,\varepsilon}\in L^2\left(X_c\setminus Y,K_X\otimes L_{a,\varepsilon}\right)$ such that

$$\bar\partial u_{c,a,\varepsilon}=\bar\partial G_\varepsilon^{(k-1)}=g_\varepsilon^{(1)}+g_\varepsilon^{(2)}$$

on $X_c\setminus Y$ and

$$\int_{X_c\setminus Y}\frac{\left|u_{c,a,\varepsilon}\right|_L^2e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}e^{-\psi}\mathrm{d}V_X\le\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J_X^kF^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{d}s)|^2}e^{-\psi}\,\mathrm{d}V_{Y,\omega}+O(\varepsilon).\tag{3.11}$$

Since $\sigma,\zeta(\sigma_\varepsilon),\tau_\varepsilon+A_\varepsilon$ are all bounded above on $\overline{X_c}$ for each fixed ε, the inequality (3.11) implies that $u_{c,a,\varepsilon}\in L^2(X_c,K_X\otimes L)$. As (3.10), (3.11) and G_ε^{k-1} is smooth, Lemma 2.6 gives that

$$\bar\partial u_{c,a,\varepsilon}=\bar\partial G_\varepsilon^{(k-1)}=g_\varepsilon^{(1)}+g_\varepsilon^{(2)}\tag{3.12}$$

extends across Y and

$$\int_{X_c}\frac{\left|u_{c,a,\varepsilon}\right|_L^2e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}e^{-\psi}\mathrm{d}V_X\le\frac{e^a(1+l)^2}{m(1-l)^2}C_{m,k}\int_{Y_c}\frac{\left|\nabla^k\left(f-J_X^kF^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{d}s)|^2}e^{-\psi}\,\mathrm{d}V_{Y,\omega}+O(\varepsilon).\tag{3.13}$$

The jet extension of f to X_c is then given by

$$F_{c,a,\varepsilon}^{(k)}:=G_\varepsilon^{(k-1)}-u_{c,a,\varepsilon}+F^{(k-1)}.$$

Then $F_{c,a,\varepsilon}^{(k)}$ is holomorphic on X_c, thanks to (3.12), $F^{(k-1)}\in H^0(X,K_X\otimes L)$ as well as the ellipticity of the operator $\bar\partial$ in bidegree $(n,0)$. So $u_{c,a,\varepsilon}$ is also smooth on X_c. Locally, near an arbitrary point of Y, all partial derivatives of order $s\le k$ of $F_{c,a,\varepsilon}^{(k)}$ are prescribed by f.

By the variant of Cauchy-Schwarz inequality, we have

$$\begin{aligned}\langle\kappa_1+\kappa_2+\kappa_3,\kappa_1+\kappa_2+\kappa_3\rangle&\le(1+l)\langle\kappa_1+\kappa_2,\kappa_1+\kappa_2\rangle+(1+\frac{1}{l})\langle\kappa_3,\kappa_3\rangle\\ &\le(1+l)^2\langle\kappa_1,\kappa_1\rangle+\frac{(1+l)^2}{l}\langle\kappa_2,\kappa_2\rangle+(1+\frac{1}{l})\langle\kappa_3,\kappa_3\rangle\end{aligned}\tag{3.14}$$

for any inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle), \kappa_1, \kappa_2, \kappa_3 \in \mathcal{H}$.

Then for some sufficiently small $\varepsilon, R(\sigma_\varepsilon) \le R(\sigma), R\left(m\log|s|^2\right) \le R(\sigma)$, the induction hypothesis (3.9), (3.13) and (3.14) give the estimate on the relatively compact open subset X_c,

$$\begin{aligned}
&\int_{X_c} \frac{\left|F^{(k)}_{c,a,\varepsilon}\right|^2_L}{e^\sigma R(\sigma)} e^{-\psi} \mathrm{d}V_X \\
&\le (1+l)^2 \int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|^2_L}{e^\sigma R(\sigma)} e^{-\psi} \mathrm{d}V_X + \frac{(1+l)^2}{l} \int_{X_c} \frac{\left|F^{(k-1)}\right|^2_L}{e^\sigma R(\sigma)} e^{-\psi} \mathrm{d}V_X + (1+\frac{1}{l}) \int_{X_c} \frac{\left|G^{(k-1)}_\varepsilon\right|^2_L}{e^\sigma R(\sigma)} e^{-\psi} \mathrm{d}V_X \\
&\le (1+l)^2 \left(\sup_{X_c} \frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right) \int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|^2_L e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} e^{-\psi} \mathrm{d}V_X \\
&+ \frac{(1+l)^2}{l} e^a \int_{X_c} \frac{\left|F^{(k-1)}\right|^2_L}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi} \mathrm{d}V_X + (1+\frac{1}{l}) e^a \int_{X_c} \frac{\theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right)^2 \left|\tilde{f} - F^{(k-1)}\right|^2_L}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi} \mathrm{d}V_X \\
&\le (1+l)^2 \left(\sup_{X_c} \frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right) \int_{X_c} \frac{\left|u_{c,a,\varepsilon}\right|^2_L e^{-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} e^{-\psi} \mathrm{d}V_X \qquad (3.15) \\
&+ \frac{(1+l)^2}{l} e^a \int_{X_c} \frac{\left|F^{(k-1)}\right|^2_L}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi} \mathrm{d}V_X + (1+\frac{1}{l}) C_1 e^a \int_{X_c} \frac{1}{|s|^{2m} R\left(m\log|s|^2\right)} e^{-\psi} \mathrm{d}V_X \\
&\le \frac{(1+l)^4 e^a}{m(1-l)^2} \left(\sup_{X_c} \frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right) C_{m,k} \int_{Y_c} \frac{\left|\nabla^k\left(f - J^k_X F^{(k-1)}\right)\right|^2_L}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y \\
&+ \frac{(1+l)^2}{l} e^a C^{(k-1)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k-1)}}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y + C_2 \int_{-\infty}^{2m\log\varepsilon + C_3} \frac{1}{R(t)} \mathrm{d}t + O(\varepsilon) \\
&\le e^a C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y + \frac{(1+l)^4 e^a}{m(1-l)^2} C_{m,k} \int_{Y_c} \frac{\left|\nabla^k\left(J^k_X F^{(k-1)}\right)\right|^2_L}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y \\
&+ C_2 \int_{-\infty}^{2m\log\varepsilon + C_3} \frac{1}{R(t)} \mathrm{d}t + O(\varepsilon),
\end{aligned}$$

where $C'^{(k)}_{m,R} = \frac{(1+l)^4}{m(1-l)^2} C_{m,k} + \frac{(1+l)^2}{l} C^{(k-1)}_{m,R}$ and C_1, C_2, C_3 are all positive numbers independent of ε. Here in (16), we also assume that

$$\frac{(\tau_\varepsilon + A_\varepsilon)\, e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)} = 1 \qquad (3.16)$$

on X_c. We will solve (3.16) together with (3.4) and (3.7) in above.

3.2. **Passing to the limits to get the final jet extension on Ω.** As $\sup_{t\le 0}\left(e^t R(t)\right) < \infty$, applying Montel's theorem and (3.15) to extract a weak limit of $\left\{F^{(k)}_{c,a,\varepsilon}\right\}_{\varepsilon>0}$ as $\varepsilon \to 0$, we get a holomorphic L-valued n-form $F^{(k)}_{c,a}$ on X_c such that $J^k_{X_c} F^{(k)}_{c,a} = f$ and

$$\int_{X_c} \frac{\left|F^{(k)}_{c,a}\right|^2_L}{e^\sigma R(\sigma)} e^{-\psi} \mathrm{d}V_X \le e^a C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y + \frac{(1+l)^4 e^a}{m(1-l)^2} C_{m,k} \int_{Y_c} \frac{\left|\nabla^k\left(J^k_X F^{(k-1)}\right)\right|^2_L}{|\Lambda^m(\mathrm{d}s)|^2} e^{-\psi}\, \mathrm{d}V_Y.$$

In other words,

(3.17)

$$\int_{X_c} \frac{\left|F^{(k)}_{c,a}\right|^2_L e^{-\psi} \mathrm{d}V_X}{|s|^{2m} R\left(m\log|s|^2 - a\right)} \le C'^{(k)}_{m,R} \int_{Y_c} \frac{|f|^2_{s,\rho,(k)} e^{-\psi}\, \mathrm{d}V_Y}{|\Lambda^m(\mathrm{d}s)|^2} + \frac{(1+l)^4 C_{m,k}}{m(1-l)^2} \int_{Y_c} \frac{\left|\nabla^k\left(J^k_X F^{(k-1)}\right)\right|^2_L e^{-\psi}\, \mathrm{d}V_Y}{|\Lambda^m(\mathrm{d}s)|^2}$$

Since R is continuous decreasing on $(-\infty,0]$, $\sup_{t\le 0}\left(e^t R(t)\right)<\infty$, similarly as before, we use Montel's theorem and extract a weak limit of $\left\{F_{c,a}^{(k)}\right\}_{a>0}$ as $a\to 0$, to obtain a holomorphic L-valued n-form $F_c^{(k)}$ on X_c from (3.17) such that $J^k_{X_c}F_c^{(k)}=f$ and

$$\int_{X_c}\frac{\left|F_c^{(k)}\right|_L^2 e^{-\psi}}{|s|^{2m}R\left(m\log|s|^2\right)}\mathrm{d}V_X\le C'^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}e^{-\psi}}{|\Lambda^m(\mathrm{d}s)|^2}\,\mathrm{d}V_Y+\frac{(1+l)^4C_{m,k}}{m(1-l)^2}\int_{Y_c}\frac{\left|\nabla^k\left(J^k_XF^{(k-1)}\right)\right|_L^2}{|\Lambda^m(\mathrm{d}s)|^2}e^{-\psi}\,\mathrm{d}V_Y. \tag{3.18}$$

As Popovici [13, Sections 0.4-0.6] has shown that the last term in the right-hand side of (3.18) can be controlled uniformly, a slight modification of his proof in [13, Section 0.4] in terms of the variable denominators introduced by [14, p136] can complete the proof of Theorem 3.1. Indeed, one just needs to modify the first and second inequalities in [13, p22], respectively, as

$$\begin{aligned}
&\frac{\sum_{|\alpha|=k}\left|\frac{\frac{\partial^\alpha F^{(k-1)}}{\partial z'^\alpha}(0,z'')}{\alpha!}\right|^2 e^{-2\varphi(0,z'')-2A|z''|^2}}{|\Lambda^m(\mathrm{d}s)(0,z'')|^{2\frac{(m+k)}{m}}}\\
&\le \text{Const}\cdot\frac{2(m+k)}{\rho^{2(m+k)}}e^{2(\varepsilon(\rho)+A\rho^2)}\sup_{(z',z'')\in U_j}\frac{\left|s\left(z',z''\right)\right|^{2m}R\left(m\log s\left(z',z''\right)^2\right)}{|\Lambda^m(\mathrm{d}s)(0,z'')|^{2\frac{(m+k)}{m}}}\\
&\times\int_{z'\in B'(0,\rho)}\frac{\left\|F^{(k-1)}\left(z',z''\right)\right\|^2}{|s(z',z'')|^{2m}R\left(m\log s(z',z'')^2\right)}\mathrm{d}\lambda\left(z'\right),
\end{aligned}$$

and

$$\int_{Y_c}\frac{\left|\nabla^k\left(J^k_XF^{(k-1)}\right)\right|^2}{|\Lambda^m(\mathrm{d}s)|^2}e^{-\psi}\,\mathrm{d}V_Y\le D_{m,k}NM(c)\frac{1}{\rho^{2(m+k)}}e^{2(\varepsilon(\rho)+A\rho^2)}\int_{\Omega'}\frac{\left\|F^{(k-1)}\right\|^2}{|s|^{2m}R\left(m\log|s|^2\right)}e^{-\psi}\mathrm{d}V_{X,\omega},$$

where

$$M(c):=\sup_{(z',z'')\in\Omega'}\frac{\left|s\left(z',z''\right)\right|^{2m}R\left(m\log s\left(z',z''\right)^2\right)}{|\Lambda^m(\mathrm{d}s)(0,z'')|^{2\frac{m+k}{m}}}.$$

and $D_{m,k}:=\text{Const}\cdot 2(m+k)$. Notice that the smoothness of the function R on $(-\infty,0]$ ensures that one can get the suprema on U_j and Ω', respectively. We refer to [13, Section 0.4] for more explanations about the above notations. Then as a result, we get a holomorphic L-valued n-form $F_c^{(k)}$ on Ω such that $J^k_\Omega F_c^{(k)}=f$ and

$$\begin{aligned}
\int_\Omega\frac{\left|F_c^{(k)}\right|_L^2e^{-\psi}}{|s|_E^{2m}R\left(m\log|s|_E^2\right)}\mathrm{d}V_{X,\omega}&\le\int_{X_c}\frac{\left|F_c^{(k)}\right|_L^2e^{-\psi}}{|s|_E^{2m}R\left(m\log|s|_E^2\right)}\mathrm{d}V_{X,\omega}\le C^{(k)}_{m,R}\int_{Y_c}\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|_E^2}e^{-\psi}\,\mathrm{d}V_{Y,\omega}\\
&\le C^{(k)}_{m,R}\int_Y\frac{|f|^2_{s,\rho,(k)}}{|\Lambda^m(\mathrm{d}s)|_E^2}e^{-\psi}\,\mathrm{d}V_{Y,\omega},
\end{aligned}$$

where $C^{(k)}_{m,R}>0$ is a constant depending only on m,k,E,R and $\sup_\Omega\|i\Theta(L)\|$.

3.3. **Step 4. Solving ordinary differential equations.** We have already proved Theorem 1.1 , provided that there exist appropriate χ,η,ζ satisfying some assumptions. Now, we will come to use these assumptions about χ,η,ζ to get their explicit expressions.

Notice that (3.4), (3.7) and (3.16) are equivalent to the following system of ordinary differential equations defined on $(-\infty,0)$:

$$\begin{cases}
\chi(t)\zeta'(t)-\chi'(t)=1,\\
(\chi(t)+\eta(t))e^{\zeta(t)}=R(t),\\
\dfrac{\left(\chi'(t)\right)^2}{\chi(t)\zeta''(t)-\chi''(t)}=\eta(t).
\end{cases}$$

Moreover, we have assumed that ζ, χ and η are all smooth on $(-\infty, 0)$ and that $\zeta > 0, \chi > 0, \eta > 0, \zeta' > 0, \chi' < 0$ and $\chi(t) \geq -\frac{t}{2}$ on $(-\infty, 0)$. In the proof of Theorem 1.1, we have assumed that $C_R = \int_{-\infty}^{0} \frac{1}{R(t)} \mathrm{d}t = 1$.

Following the argument of solving undetermined functions with ODEs introduced in [14, Section 4, pp. 151-153], we get

$$\begin{cases} \zeta = -\log\left(1 - \int_{-\infty}^{t} \dfrac{1}{R(t_1)} \mathrm{d}t_1\right), \\ \chi = \dfrac{-t - \int_t^0 \left(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} \mathrm{d}t_1\right) \mathrm{d}t_2}{1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} \mathrm{d}t_1}, \\ \eta = \left(1 - \int_{-\infty}^{t} \dfrac{1}{R(t_1)} \mathrm{d}t_1\right) R(t) + \dfrac{t + \int_t^0 \left(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} \mathrm{d}t_1\right) \mathrm{d}t_2}{1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} \mathrm{d}t_1}, \end{cases}$$

and

$$\chi' + \frac{1}{2} = \left(\frac{-\frac{1}{2}\left(\lambda_1'\right)^2 + \lambda_1 \lambda_1''}{\left(\lambda_1'\right)^2}\right) \leq 0.$$

It is easy to verify all the previous assumptions about ζ, χ and η. In the end, we have proven the L^2-extension theorem 3.1. □

Now we will show that the main theorem which is the case of L^q-extension is true later.

4. Proof of the main theorem

From now on, we will denote $F^{(k)}$ in theorem 3.1 by $F_1^{(k)}$. K_X is naturally equipped with the smooth metric e^{φ_ω} with respect to the dual frame of dz. Let L' be the line bundle L equipped with the new metric $e^{-\varphi_{L'}}$, where $\varphi_{L'} := (2-q)\log|F_1|_L + \varphi_L$. Then the assumptions in the theorem imply that

(i) $\sqrt{-1}\Theta_{L'} + \sqrt{-1}\partial\bar{\partial}\sigma \geq 0$,
(ii) $\sqrt{-1}\Theta_{L'} + \sqrt{-1}\partial\bar{\partial}\sigma \geq \frac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha |s|_E^2}$.

Since the k-jet $f \in H^0\left(X, K_X \otimes L' \otimes \mathcal{O}_X / \mathcal{J}_Y^{k+1}\right)$ satisfies

$$\int_Y \frac{|f|^2_{L',s,\rho,(k)} e^{-\psi}}{|\Lambda^m(ds)|_E^2} dV_{Y,\omega} = C_f < +\infty,$$

by Theorem 3.1, there exists $F_2^{(k)}$ on X with values in $K_X \otimes L'$, such that $J^k F_2^{(k)} = f$ on Ω and

$$\int_\Omega \frac{\left|F_2^{(k)}\right|_L^2 e^{-\psi}}{\left(\left|F_1^{(k)}\right|_L\right)^{2-q} |s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega}$$
$$= \int_\Omega \frac{\left|F_2^{(k)}\right|_{L'}^2 e^{-\psi}}{|s|^{2m} R(\psi + m\log|s|^2)} dV_{X,\omega} \leq C_{m,R}^{(k)} \int_Y \frac{|f|^2_{L',s,\rho,(k)} e^{-\psi}}{|\Lambda^m(ds)|_E^2} dV_{Y,\omega} = C_{m,R}^{(k)} C_f.$$

Then Hölder's inequality gives that

$$\begin{aligned}
C_{F_2^{(k)}} &:= \int_\Omega \frac{\left(\left|F_2^{(k)}\right|_L\right)^q e^{-\psi}}{|s|^{2m}R(\psi+m\log|s|^2)} dV_{X,\omega} \\
&\leq \left(\int_\Omega \frac{\left|F_2^{(k)}\right|_L^2 e^{-\psi}}{\left(\left|F_1^{(k)}\right|_{L'}\right)^{2-q}|s|^{2m}R(\psi+m\log|s|^2)} dV_{X,\omega}\right)^{\frac{q}{2}} \left(\int_\Omega \frac{\left(\left|F_1^{(k)}\right|_L\right)^q e^{-\psi}}{|s|^{2m}R(\psi+m\log|s|^2)} dV_{X,\omega}\right)^{1-\frac{q}{2}} \\
&\leq \left(C_{m,R}^{(k)}C_f\right)^{\frac{q}{2}} \left(C_{F_1^{(k)}}\right)^{1-\frac{q}{2}}.
\end{aligned}$$

We can then repeat the same argument with $F_1^{(k)}$ replaced by $F_2^{(k)}$, etc, and get a sequence of holomorphic extensions $\left\{F_s^{(k)}\right\}_{s=1}^{+\infty}$ of f and a sequence $\left\{C_{F_s^{(k)}}\right\}_{s=1}^{+\infty}$ such that

$$C_{F_{s+1}^{(k)}} \leq \left(C_{m,R}^{(k)}C_f\right)^{\frac{q}{2}} \left(C_{F_s^{(k)}}\right)^{1-\frac{q}{2}}, \quad s=1,2,\cdots. \tag{4.1}$$

If $C_{F_s}^{(k)} \leq C_{m,R}^{(s)}C_f$ for some $C_{F_s}^{(k)}$, then we finish the proof since $F_s^{(k)}$ can be regarded as the desired k-jet extension $F^{(k)}$ in the conclusion.

If $C_{F_s^{(k)}} > C_{m,R}^{(k)}C_f$ for any k, then $C_{F_{s+1}^{(k)}} < C_{F_s^{(k)}}$ for any s. Since φ_L is locally bounded above and $e^\sigma R(\sigma)$ is bounded above, applying MonteLs theorem and extracting weak limits of $\left\{F_s^{(k)}\right\}_{s=1}^{+\infty}$, we can get from (4.1) a k-jet $F^{(k)}$ on X with values in $K_X \otimes L$, such that $J^kF^{(k)} = f$ on Ω and

$$\int_\Omega \frac{\left(|F^{(k)}|_L\right)^q e^{-\psi}}{|s|^{2m}R(\psi+m\log|s|^2)} dV_X \leq C_{m,R}^{(k)}C_f.$$

Theorem 1.1 is, thus, proved.

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