Lecture 13: Graphs I: Breadth First Search

Lecture Overview

- Applications of Graph Search
- Graph Representations
- Breadth-First Search

Recall:

Graph $G = (V, E)$

- V = set of vertices (arbitrary labels)
- E = set of edges i.e. vertex pairs (v, w)
  - ordered pair $\implies$ directed edge of graph
  - unordered pair $\implies$ undirected

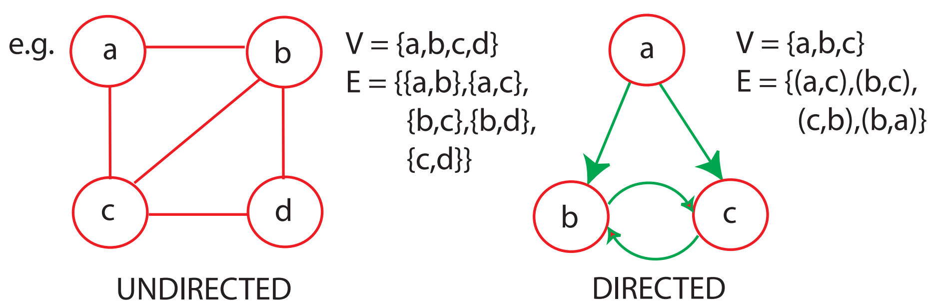


Figure 1: Example to illustrate graph terminology

Graph Search

"Explore a graph", e.g.:

- find a path from start vertex s to a desired vertex
- visit all vertices or edges of graph, or only those reachable from s

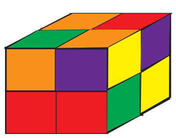
Applications:

There are many.

- web crawling (how Google finds pages)
- social networking (Facebook friend finder)
- network broadcast routing
- garbage collection
- model checking (finite state machine)
- checking mathematical conjectures
- solving puzzles and games

Pocket Cube:

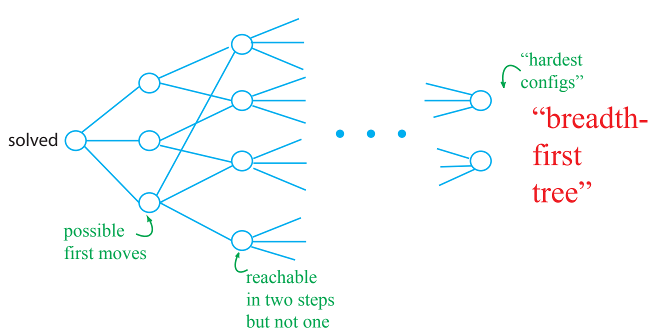
Consider a $2 \times 2 \times 2$ Rubik's cube



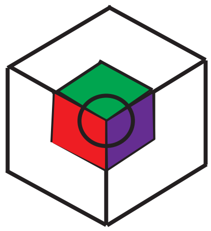
Configuration Graph:

- vertex for each possible state
- edge for each basic move (e.g., 90 degree turn) from one state to another
- undirected: moves are reversible

Diameter ("God's Number")
11 for $2 \times 2 \times 2$, 20 for $3 \times 3 \times 3$, $\Theta(n^2/\lg n)$ for $n \times n \times n$ [Demaine, Demaine, Eisenstat Lubiw Winslow 2011]



$\#$ vertices $= 8! \cdot 3^8 = 264{,}539{,}520$ where 8! comes from having 8 cubelets in arbitrary positions and 3^8 comes as each cubelet has 3 possible twists.



This can be divided by 24 if we remove cube symmetries and further divided by 3 to account for actually reachable configurations (there are 3 connected components).

Graph Representations: (data structures)

Adjacency lists:

Array *Adj* of $|V|$ linked lists

- for each vertex $u \in V$, $Adj[u]$ stores u's neighbors, i.e., $\{v \in V \mid (u, v) \in E\}$. (u, v) are just outgoing edges if directed. (See Fig. 2 for an example.)

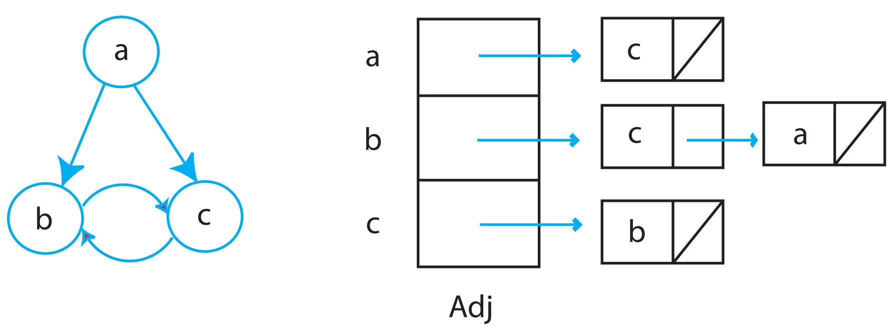


Figure 2: Adjacency List Representation: Space $\Theta(V + E)$

- in Python: *Adj* = dictionary of list/set values; vertex = any hashable object (e.g., int, tuple)
- advantage: multiple graphs on same vertices

Implicit Graphs:

Adj(u) is a function — compute local structure on the fly (e.g., Rubik's Cube). This requires "Zero" Space.

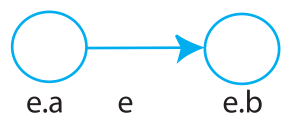
Object-oriented Variations:

- object for each vertex u
- u.neighbors = list of neighbors i.e. $Adj[u]$

In other words, this is method for implicit graphs

Incidence Lists:

- can also make edges objects



- u.edges = list of (outgoing) edges from u.
- advantage: store edge data without hashing

Breadth-First Search

Explore graph level by level from s

- level 0 = $\{s\}$
- level i = vertices reachable by path of i edges but not fewer

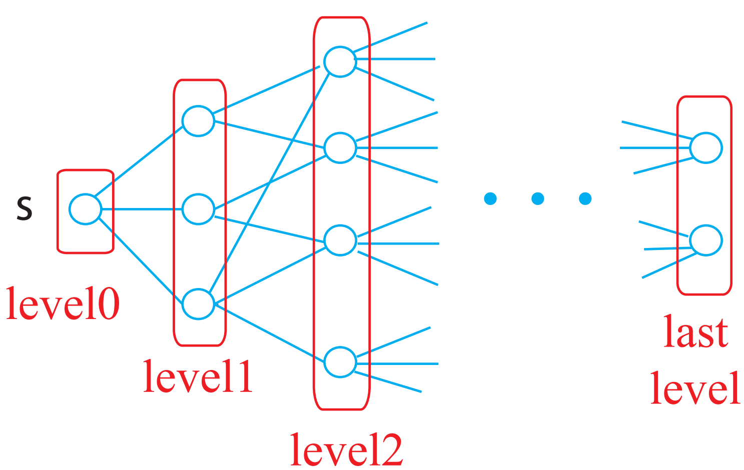


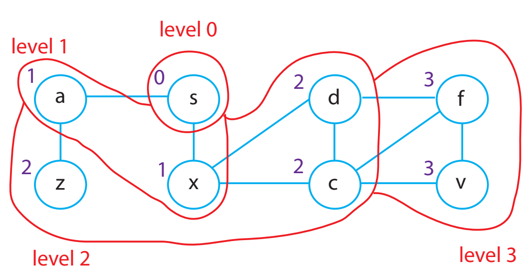
Figure 3: Illustrating Breadth-First Search

- build level $i > 0$ from level $i-1$ by trying all outgoing edges, but ignoring vertices from previous levels

Breadth-First-Search Algorithm

```
BFS (V,Adj,s):                                  See CLRS for queue-based implementation
    level = { s: 0 }
    parent = {s : None }
    i = 1
    frontier = [s]                              # previous level, i − 1
    while frontier:
        next = [ ]                              # next level, i
        for u in frontier:
            for v in Adj [u]:
                if v not in level:              # not yet seen
                    level[v] = i                ♯ = level[u] + 1
                    parent[v] = u
                    next.append(v)
        frontier = next
        i + =1
```

Example



frontier$_0$ = {s}
frontier$_1$ = {a, x}
frontier$_2$ = {z, d, c}
frontier$_3$ = {f, v}
(not x, c, d)

Figure 4: Breadth-First Search Frontier

Analysis:

- vertex V enters next (& then frontier) only once (because level$[v]$ then set)

  base case: $v = s$

- $\Longrightarrow$ Adj[v] looped through only once

$$\text{time} \;= \sum_{v \in V} |Adj[V]| = \begin{cases} |E| \text{ for directed graphs} \\ 2|E| \text{ for undirected graphs} \end{cases}$$

- $\Longrightarrow$ $O(E)$ time

- $O(V+E)$ ("LINEAR TIME") to also list vertices unreachable from v (those still not assigned level)

Shortest Paths:

cf. L15-18

- for every vertex v, fewest edges to get from s to v is

$$\begin{cases} \text{level}[v] \text{ if } v \text{ assigned level} \\ \infty \qquad \text{else (no path)} \end{cases}$$

- parent pointers form <u>shortest-path tree</u> = union of such a shortest path for each v
$\Longrightarrow$ to find shortest path, take v, parent[v], parent[parent[v]], etc., until s (or None)