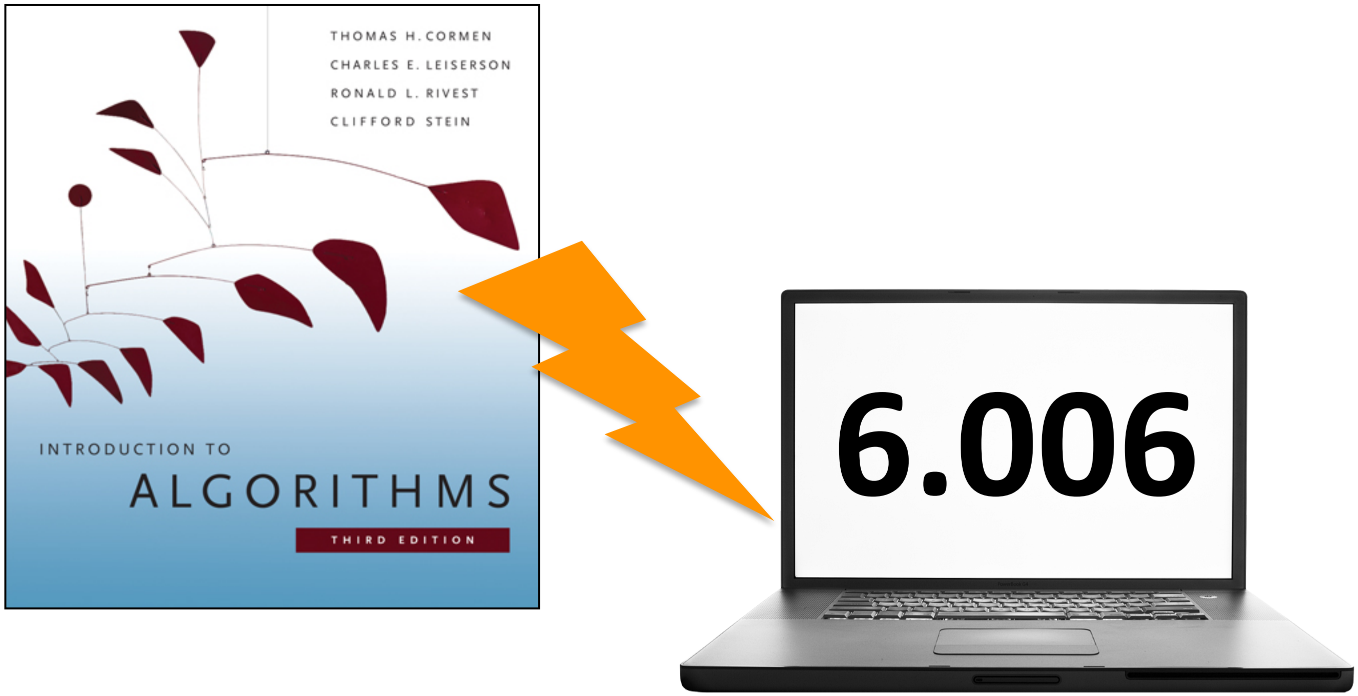
Lecture 4: Priority Queues, Heaps and Heapsort



Prof. Aleksander Mądry

Reading: CLRS 6.1-6.4

Priority Queue (PQ)

An ***abstract data structure*** (aka ***data type***) maintaining a set S of **elements**, each associated with a **key**, supporting the following operations:

insert(S, x) :	insert element x into set S
max(S) :	return element of S with largest key
extract_max(S) :	return element of S with largest key and remove it from S
increase_key(S, x, k) :	increase value of x's key to new value k (assumed to be $\geq$ the current key value)

Think: All the operations you would need to organize triage in an emergency room ➔ key value = severity of patient's condition

(Tons of applications in algorithms and across the whole CS too.)

Priority Queue (PQ)

An ***abstract data structure*** (aka ***data type***) maintaining a set S of **elements**, each associated with a **key**, supporting the following operations:

insert(S, x) :	insert element x into set S
max(S) :	return element of S with largest key
extract_max(S) :	return element of S with largest key and remove it from S
increase_key(S, x, k) :	increase value of x's key to new value k (assumed to be $\geq$ the current key value)

This abstraction specifies desired functionality/interface, but how to implement it?

Naïve way: Use an (unsorted) array and scan all elements to find max

➔ each insert and increase key takes **$\Theta(1)$ time**

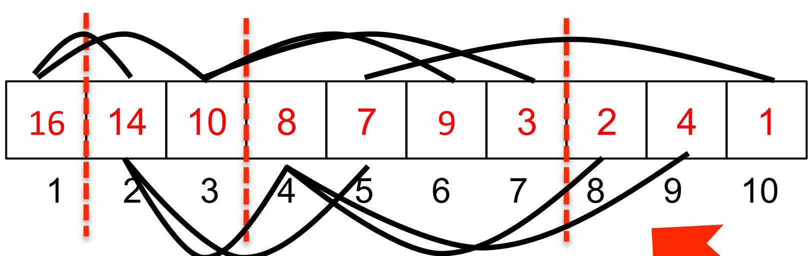
➔ all the other operations take **$\Theta(n)$ (worst-case) time**

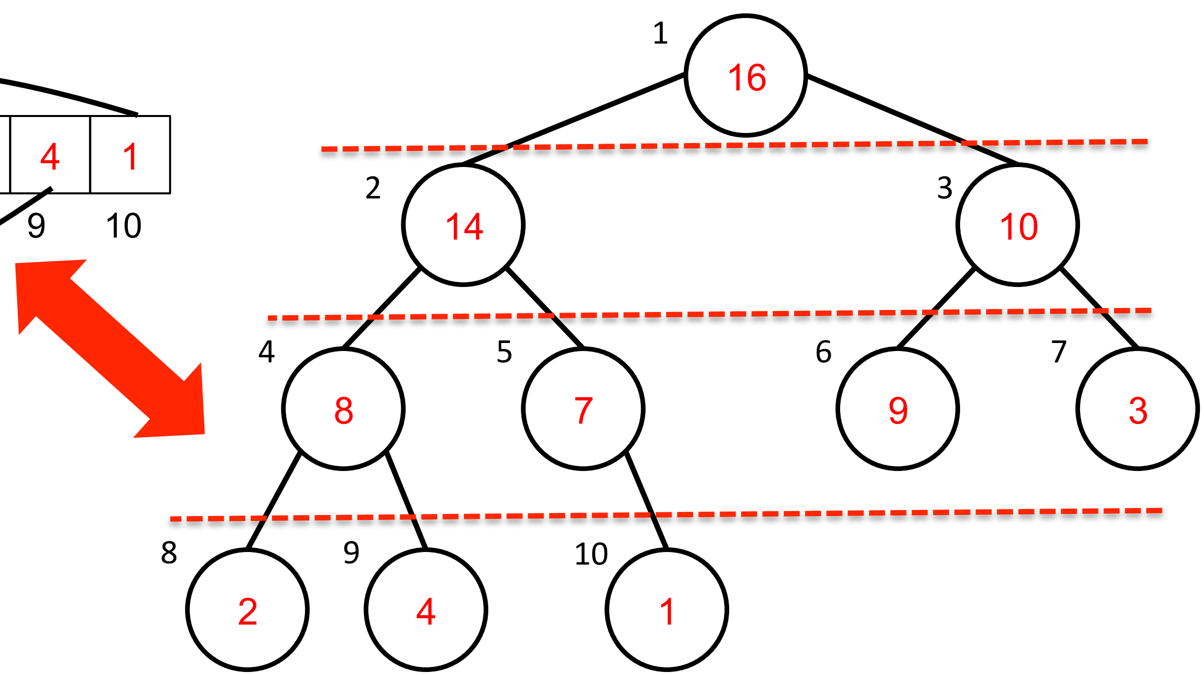
Can we do better?

(Max) Heap

- **Data structure** implementing a priority queue
- It is an **array** that:
  - ➔ we visualize as a (nearly complete) **binary tree**
  - ➔ satisfies **Max Heap Property (MHP)**:

    Key of a node is $\geq$ than the keys of its children
- (**Min Heap** defined analogously)





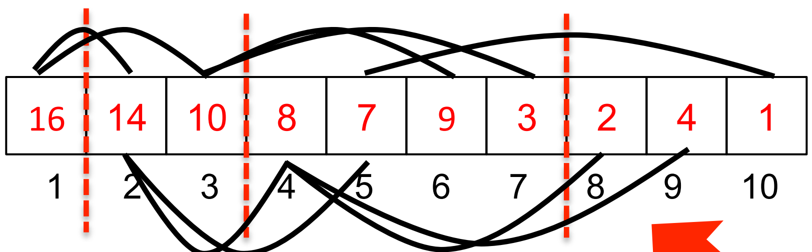
Important fact:
Height of the tree is always O(log n)

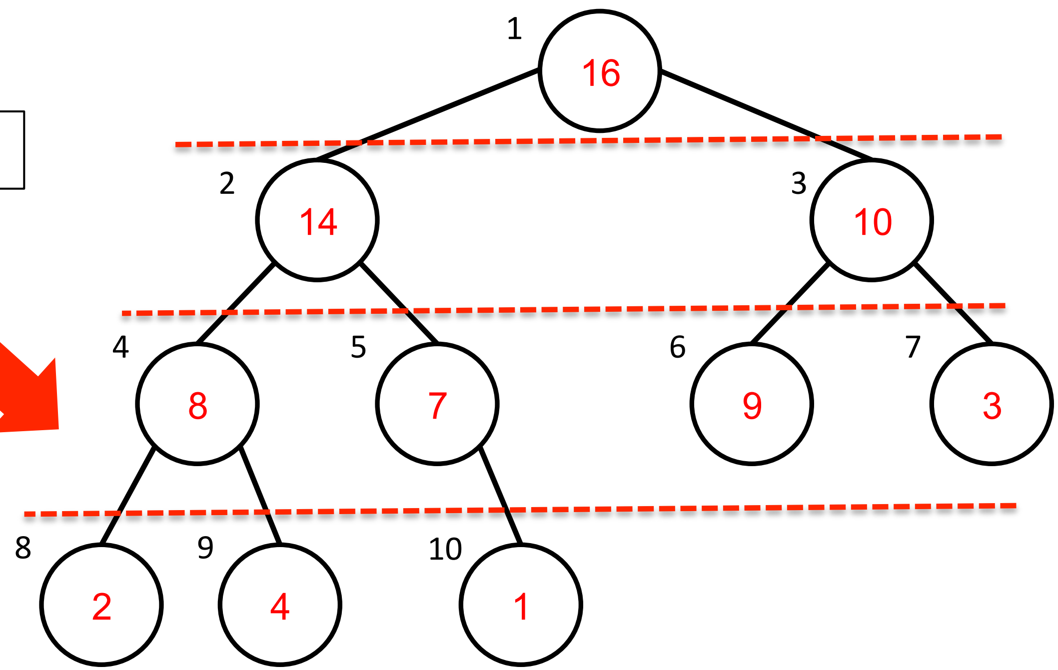
Mapping Tree to a Heap

root: first element in the array (**i=1**)
parent(i): **floor(i/2)** returns index of node's parent
left(i): **2i** returns index of node's *left* child
right(i): **2i+1** returns index of node's *right* child

(**Note:** No pointers needed!)

Important detail: We index elements starting from **i=1** here





Why Heaps?

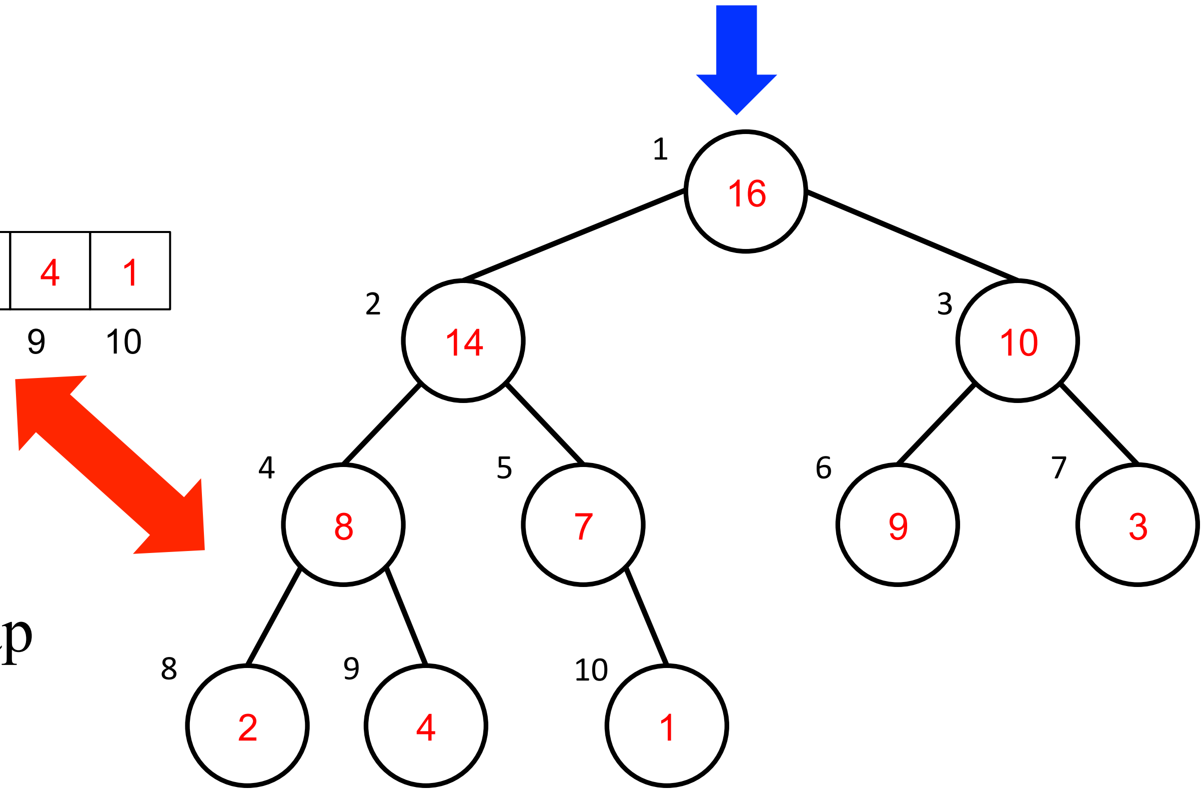
Key consequence of Max Heap Property:

Root/first element is always the max → can do max(S) in **Θ(1) time**!

(**Note:** the array is not sorted though!)

But: How to maintain the Max Heap Property property after insert/extract_max/increase_key?

16	14	10	8	7	9	3	2	4	1
1	2	3	4	5	6	7	8	9	10



In fact:

How to build a Max Heap to begin with?

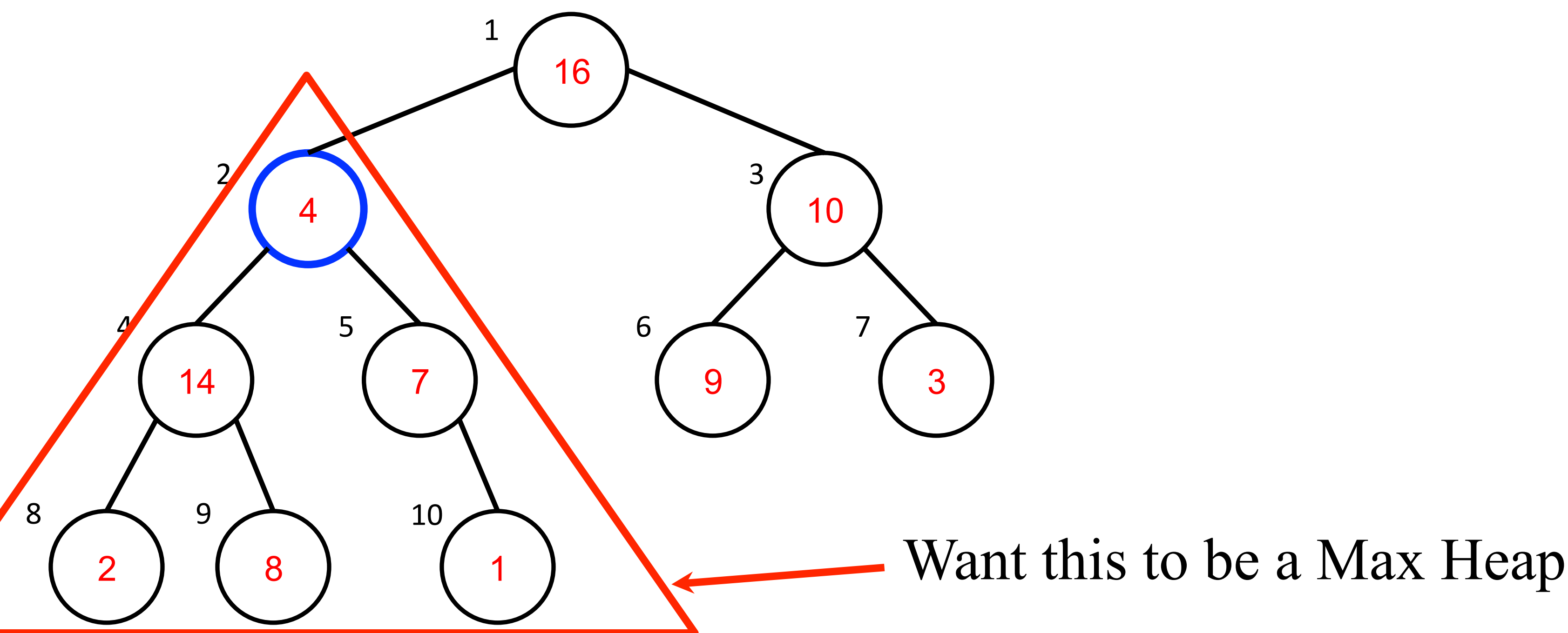
Key Primitive

max_heapify(A[i]): Corrects a **single** violation of Max Heap Property in a subtree rooted at **i** **only**

How to implement it?

➔Assume that the trees rooted at left(i) and right(i) are Max Heaps

➔If element A[i] violates the MHP, correct violation by "trickling" this element down the tree, making the subtree rooted at i a Max Heap (**Important:** Always swap with the larger of two children. **Why?**)



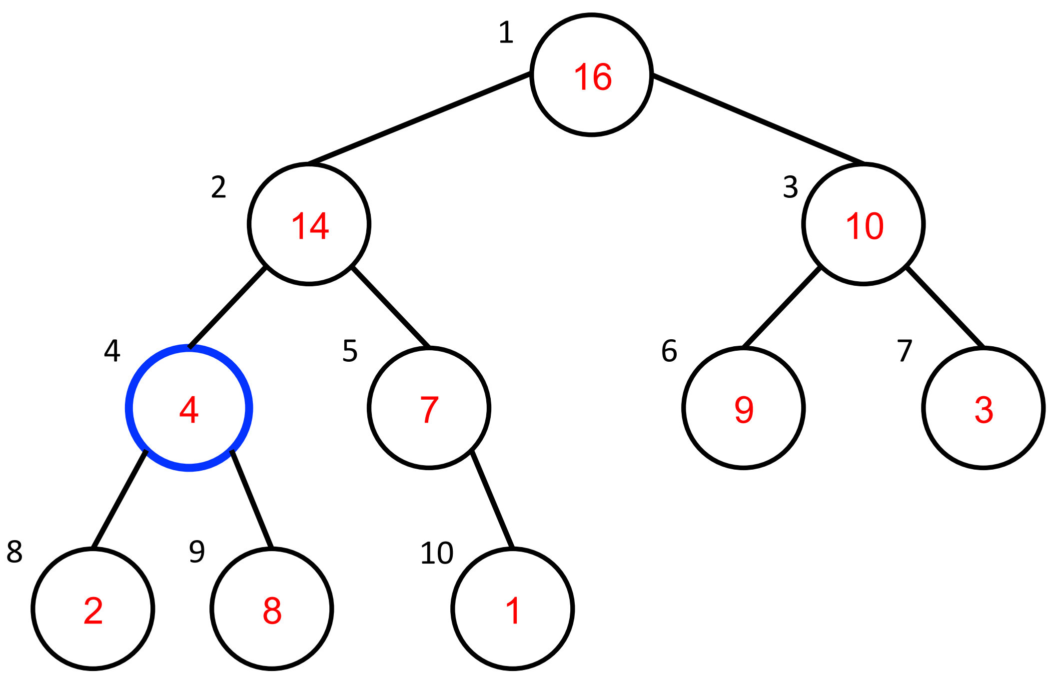
Key Primitive

max_heapify(A[i]): Corrects a **single** violation of Max Heap Property in a subtree rooted at **i** **only**

In other words:

➔ Assume that the trees rooted at left(*i*) and right(*i*) are Max Heaps

➔ If element A[*i*] violates the MHP, correct violation by "trickling" this element down the tree, making the subtree rooted at i a Max Heap (**Important:** Always swap with the larger of two children)



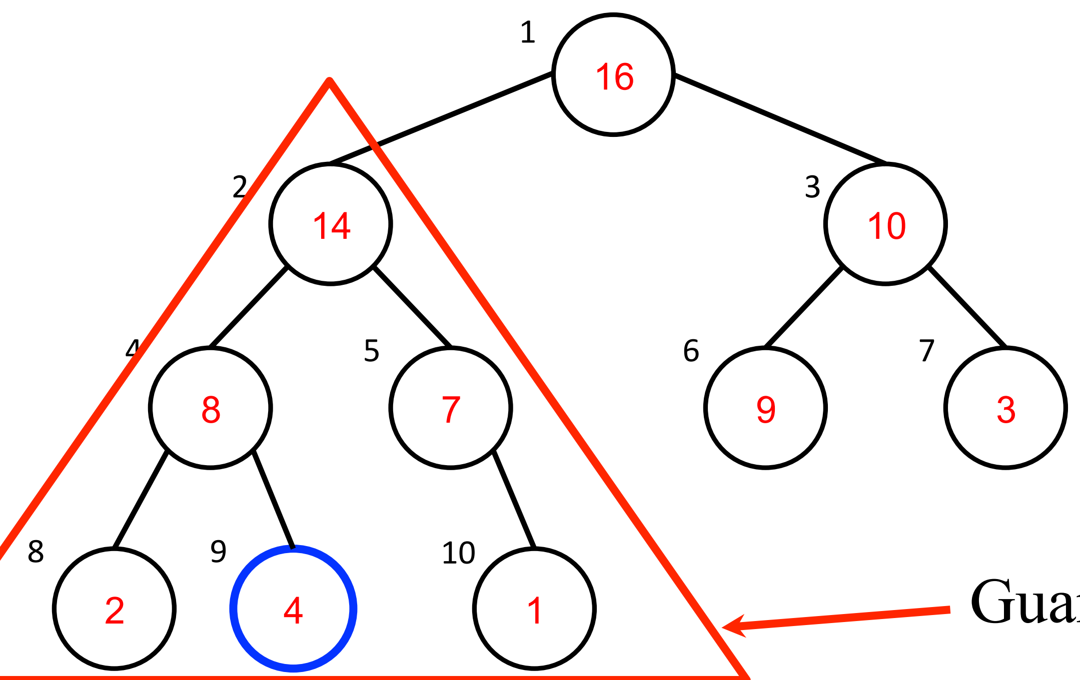
Key Primitive

max_heapify(A[i]): Corrects a **single** violation of Max Heap Property in a subtree rooted at **i** **only**

In other words:

➔ Assume that the trees rooted at left(i) and right(i) are Max Heaps

➔ If element A[i] violates the MHP, correct violation by "trickling" this element down the tree, making the subtree rooted at i a Max Heap (**Important:** Always swap with the <u>larger</u> of two children)



Run time?

➔ **Θ(1)** at each level

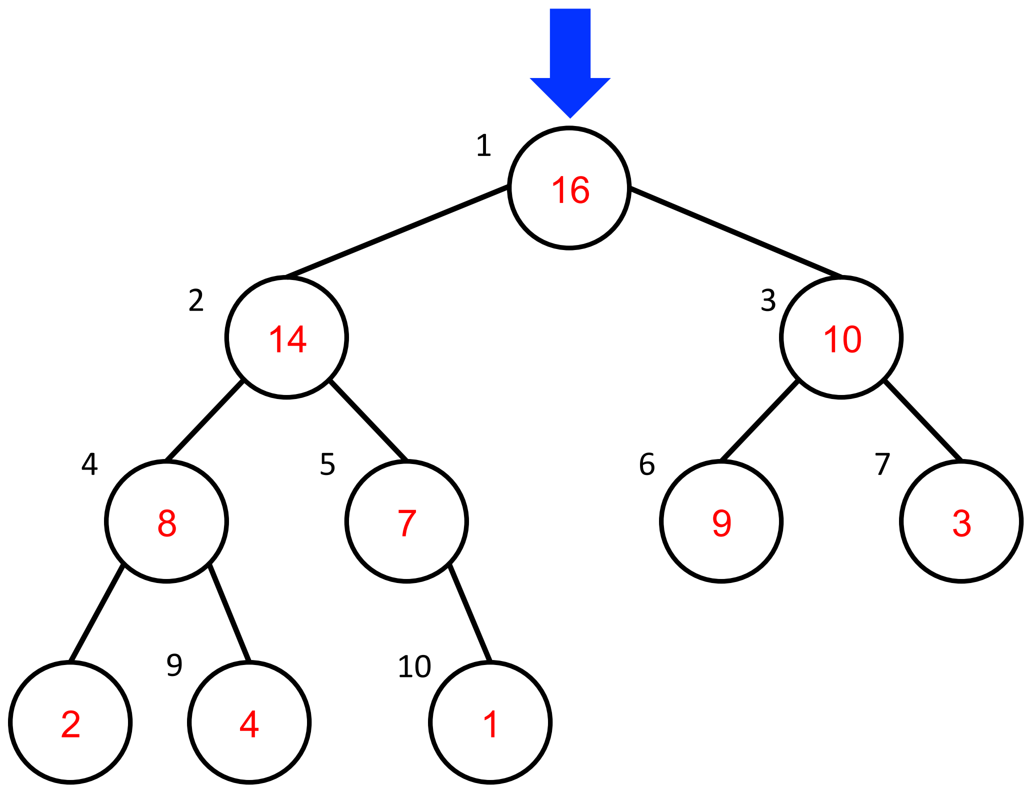
➔ total: **O(subtree height)**

= Θ (log n) (worst case)

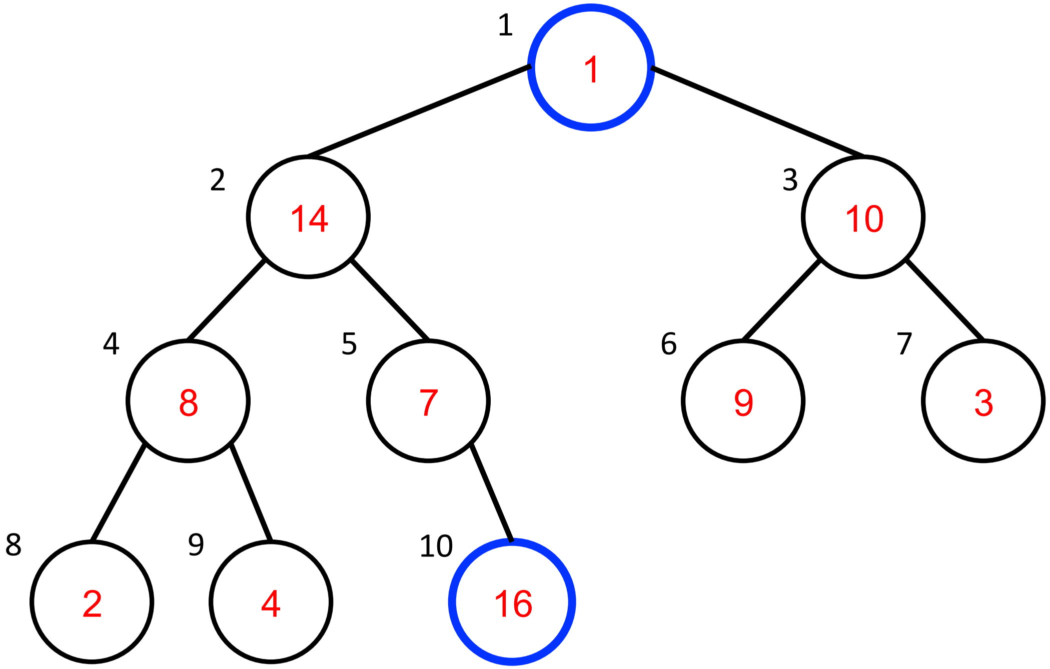
(**Recall:** tree is balanced)

Guaranteed to be a Max Heap now

Implementing Extract_Max

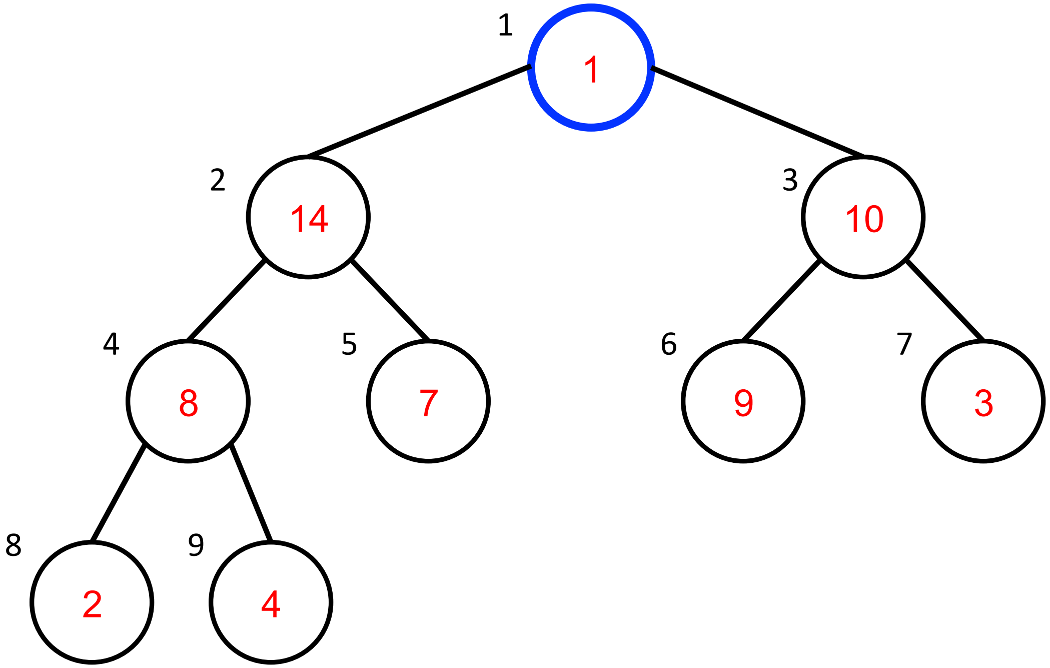


Implementing Extract_Max



- Swap the root with the last element of the heap

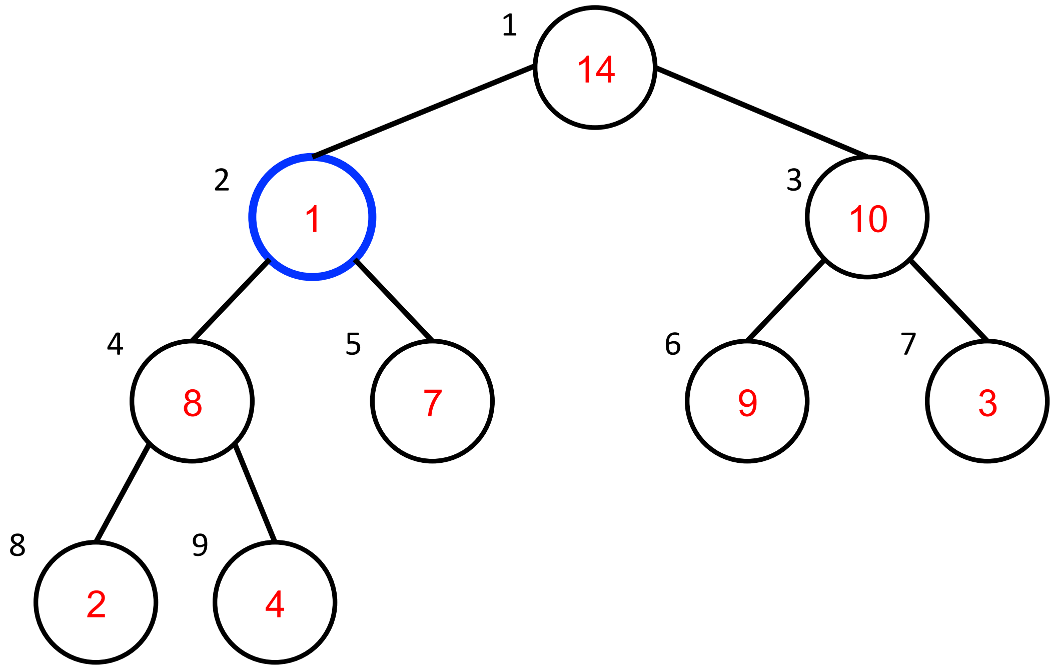
Implementing Extract_Max



- Swap the root with the last element of the heap
- Now we can remove it from the heap

  (decrease the heap size by one)
- **To fix MHP:** Max_heapify the root

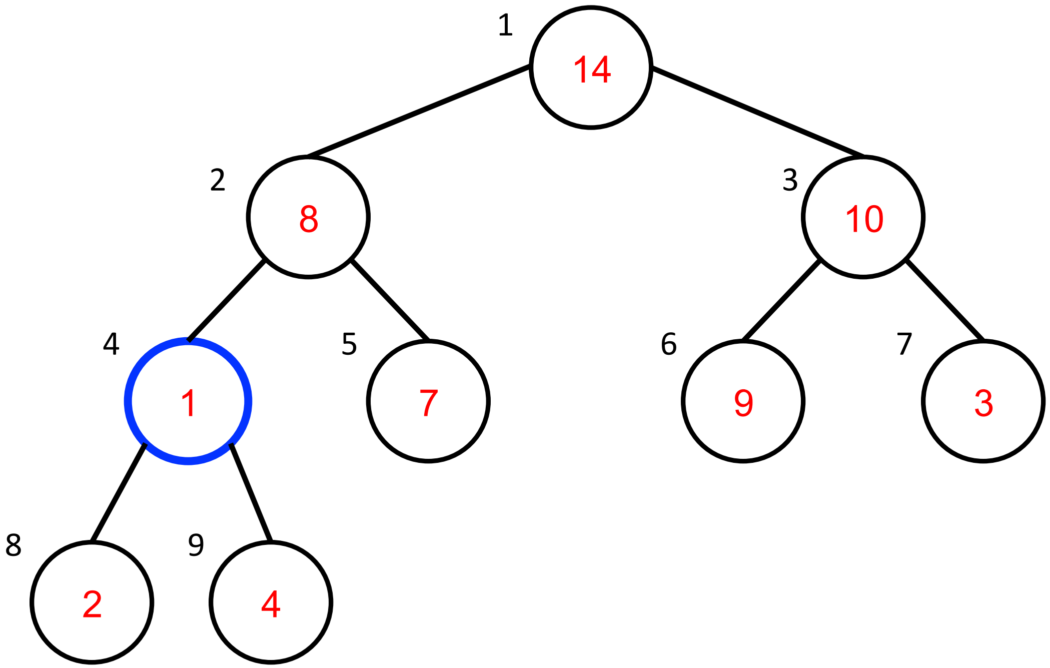
Implementing Extract_Max



- Swap the root with the last element of the heap
- Now we can remove it from the heap

  (decrease the heap size by one)
- **To fix MHP:** Max_heapify the root

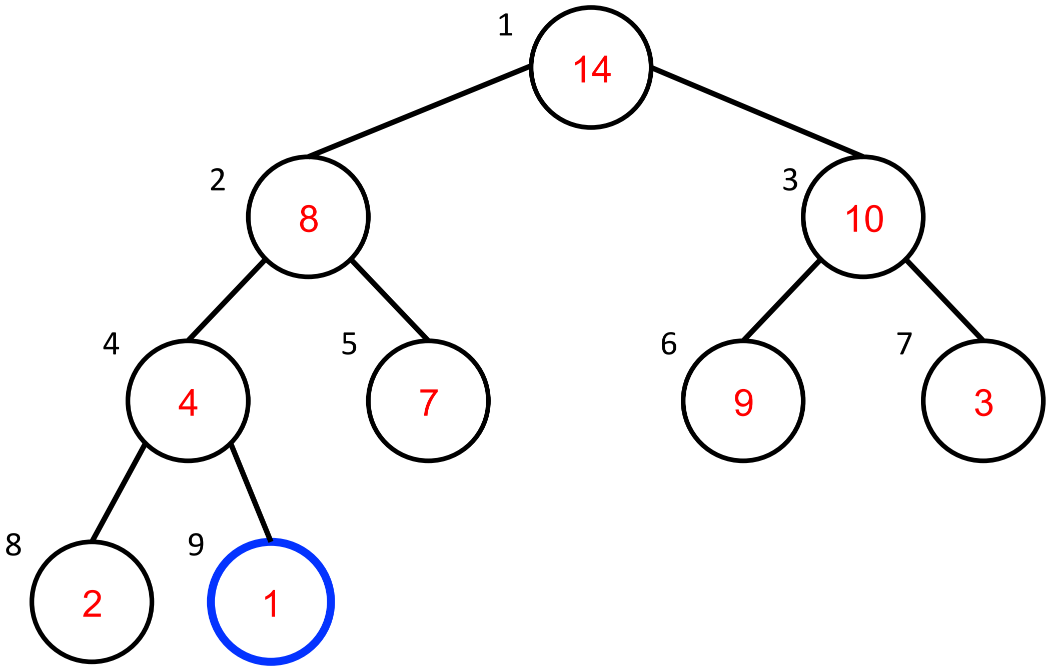
Implementing Extract_Max



- Swap the root with the last element of the heap
- Now we can remove it from the heap

  (decrease the heap size by one)
- **To fix MHP:** Max_heapify the root

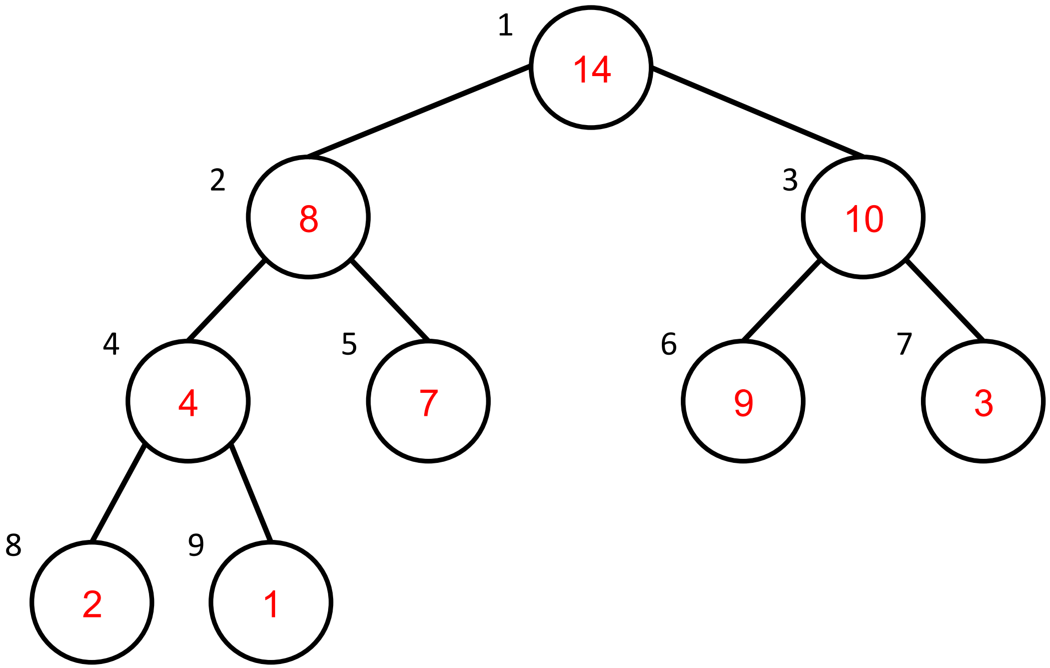
Implementing Extract_Max



- Swap the root with the last element of the heap
- Now we can remove it from the heap

  (decrease the heap size by one)
- **To fix MHP:** Max_heapify the root

Implementing Extract_Max



- Swap the root with the last element of the heap
- Now we can remove it from the heap

  (decrease the heap size by one)
- **To fix MHP:** Max_heapify the root
- Done!

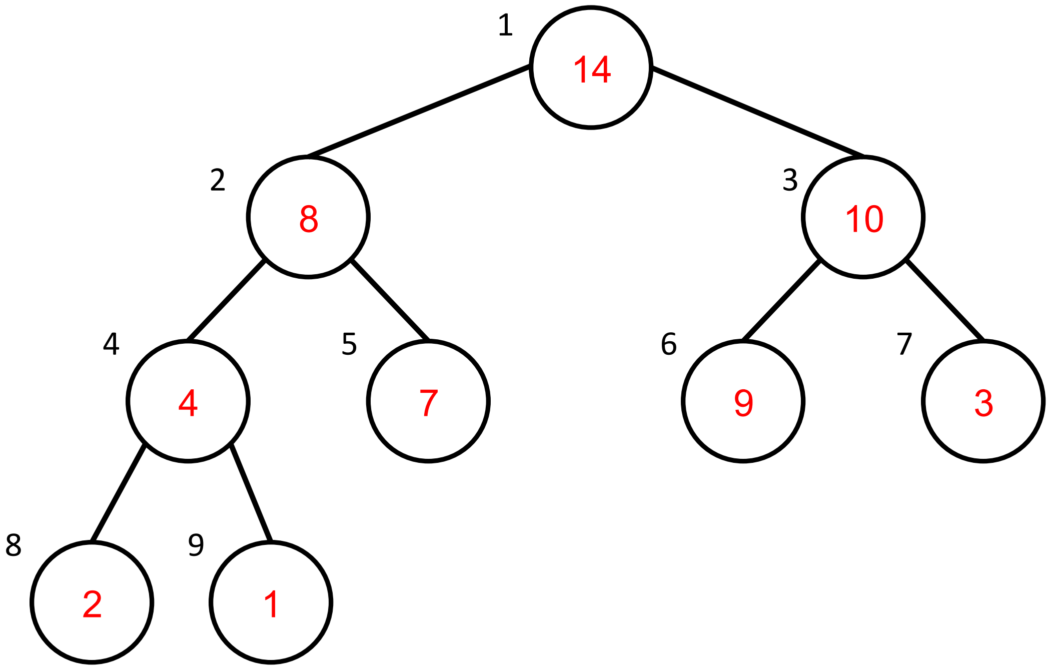
Run time?

➔ **$\Theta(1)$** (swapping) **+ $\Theta(1)$** (removal) **+ O (log n)** (max_heapify)

➔ total: **$\Theta(\log n)$ (worst case)**

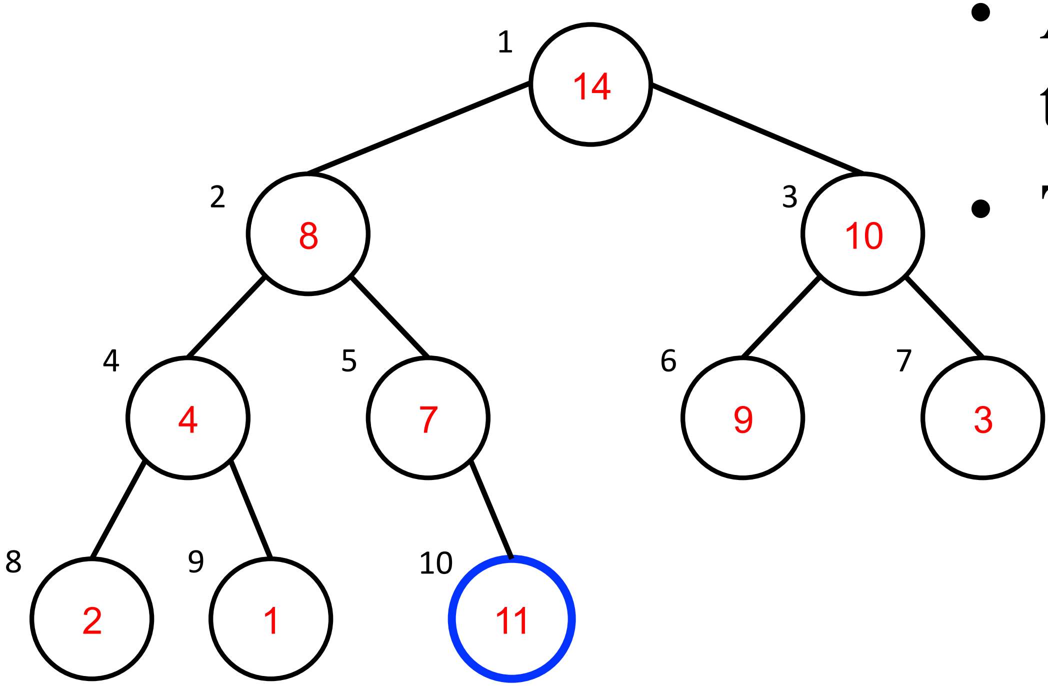
Implementing Insert

(**in a sense:** "reversing" the extract_max)



Implementing Insert

(in a sense: “reversing” the extract_max)



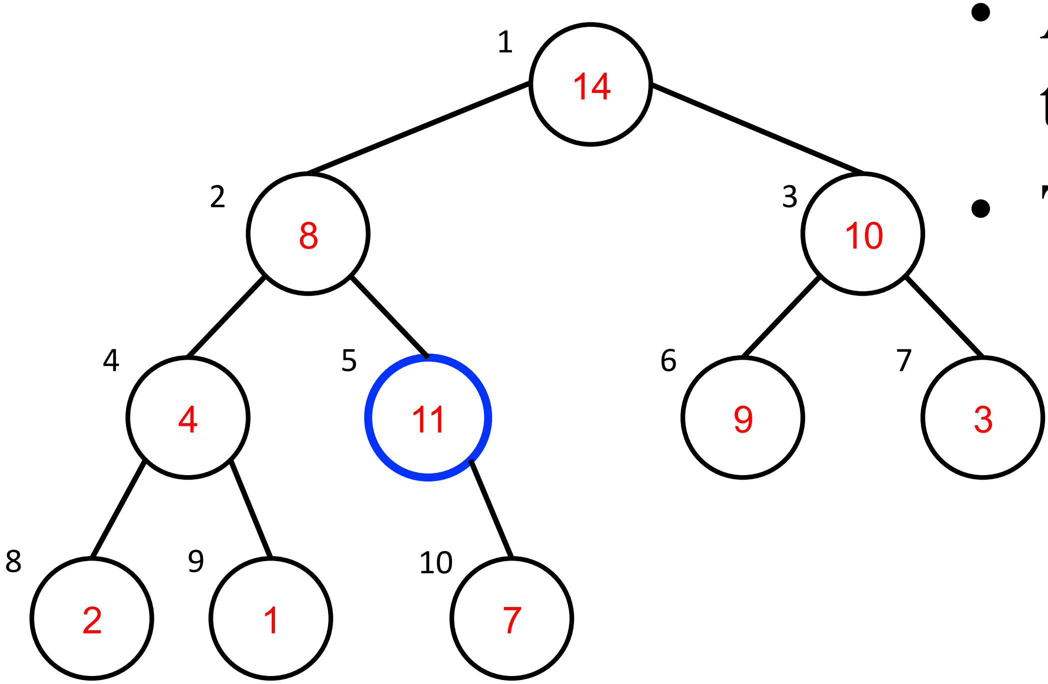
- Add the new element as the last one
- **To fix MHP:**

  “Promote” the new element up the tree

  (“reversed” max_heapify)

Implementing Insert

(in a sense: "reversing" the extract_max)

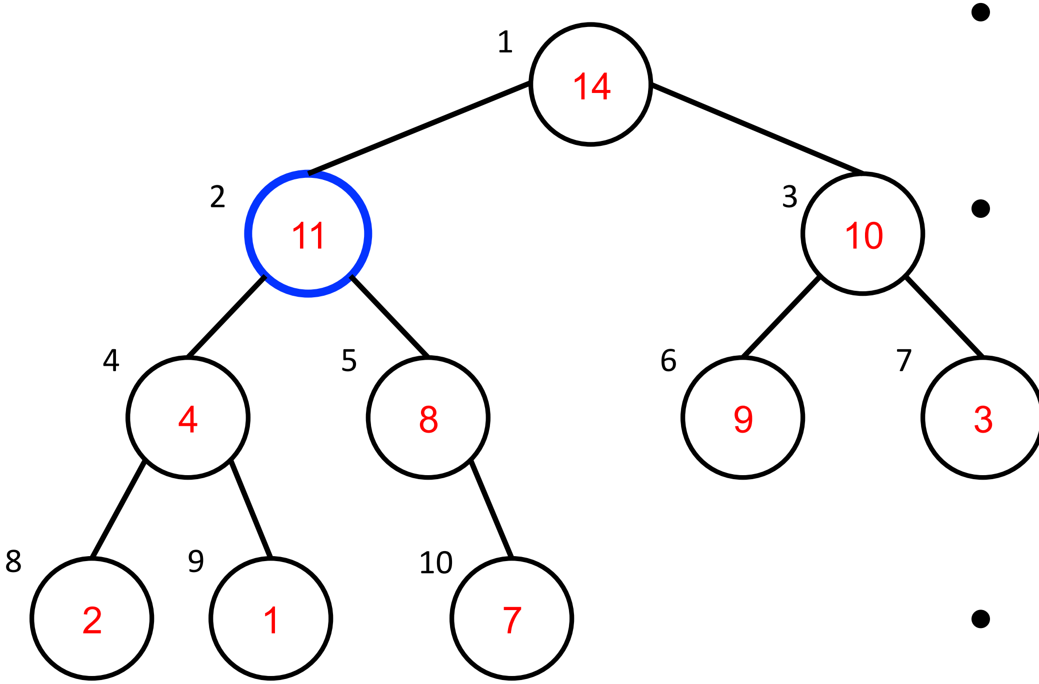


- Add the new element as the last one
- **To fix MHP:**

  "Promote" the new element up the tree

  ("reversed" max_heapify)

Implementing Insert
(in a sense: “reversing” the extract_max)



- Add the new element as the last one
- **To fix MHP:**

  “Promote” the new element up the tree

  (“reversed” max_heapify)
- Done!

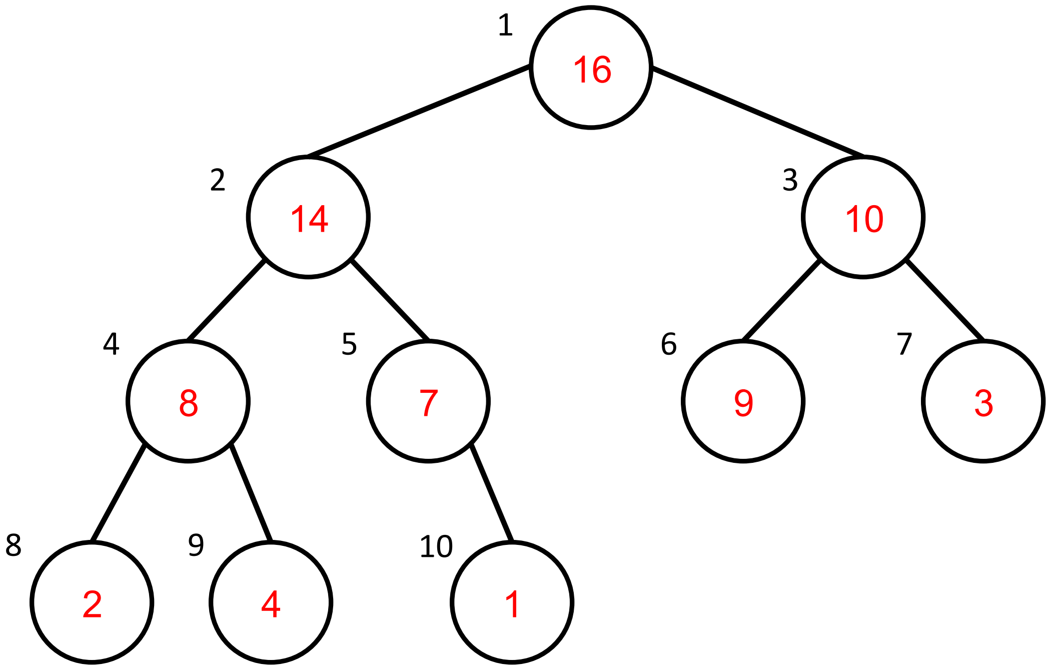
Run time?

➔ **Θ(1)** (addition) **+ O (log n)** (promotion up the tree)

➔ total: **Θ(log n) (worst case)**

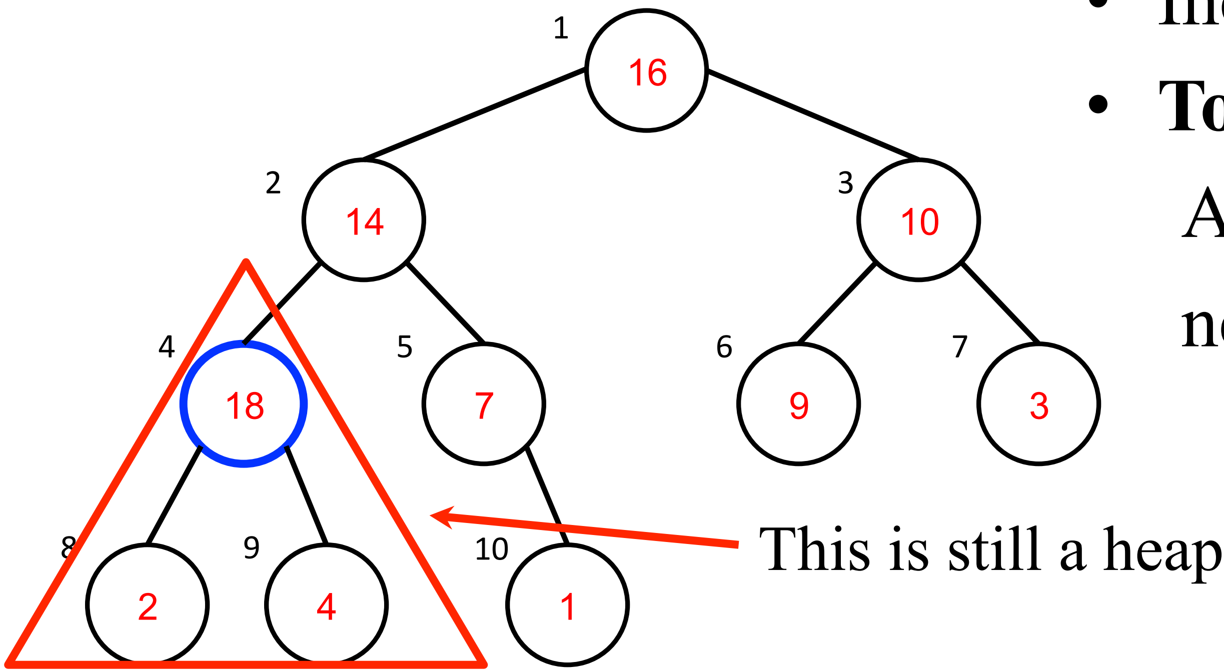
Implementing Increase_key

(Similar to Insert)



Implementing Increase_key

(Similar to Insert)

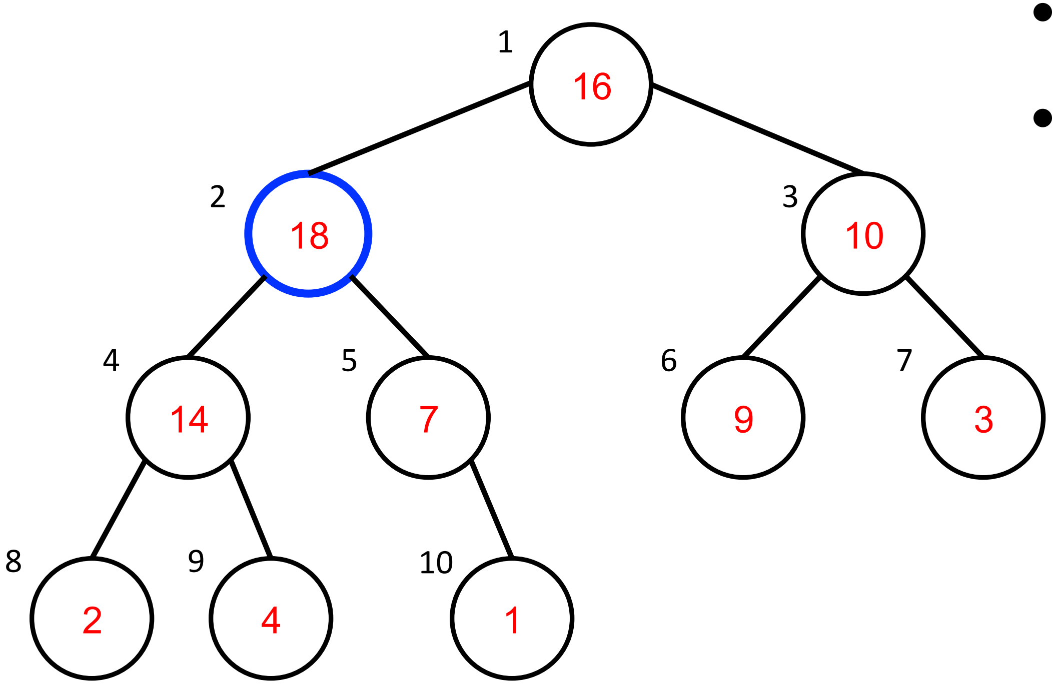


- Increase the key value
- **To fix MHP:**

  Again, "promote" the new element up the tree

Implementing Increase_key

(Similar to Insert)

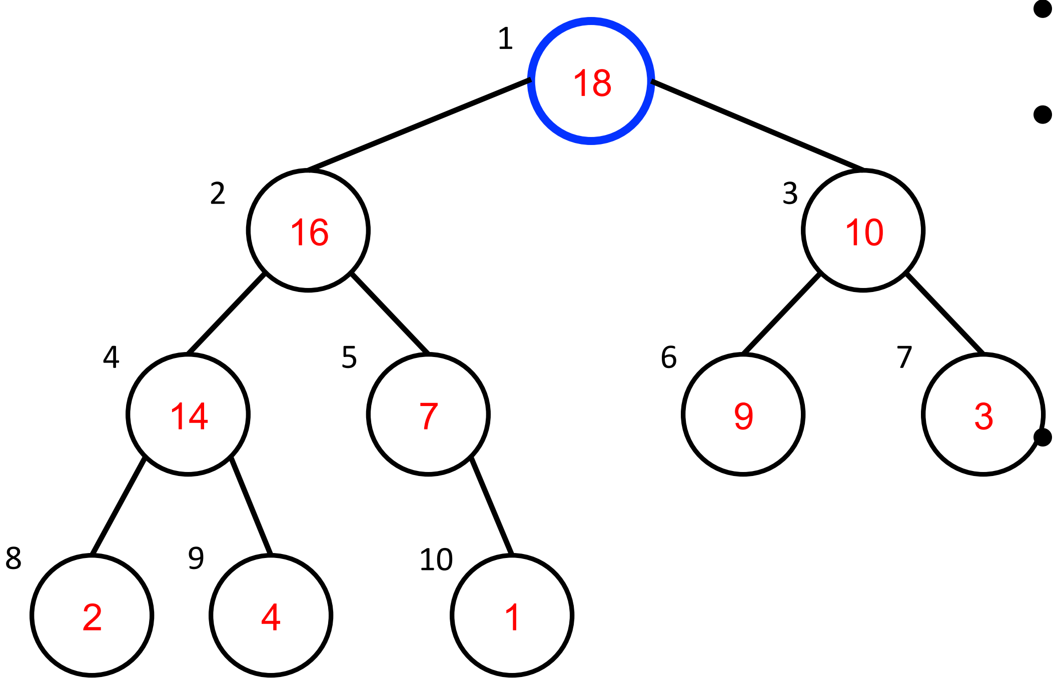


- Increase the key value
- **To fix MHP:**

  Again, "promote" the new element up the tree

Implementing Increase_key

(Similar to Insert)



- Increase the key value
- **To fix MHP:**
  Again, "promote" the new element up the tree
- Done!

Run time?

➔ **Θ(1)** (key value increase) **+ O (log n)** (promotion up the tree)

➔ total: **Θ(log n) (worst case)**

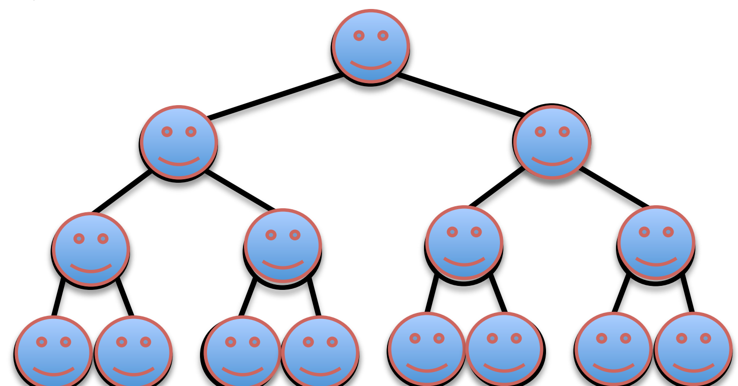
How to build a heap from a scratch?

Simple way:
➔ Start with an empty heap
➔ Insert all the **n** elements into it
➔ Total time: **Θ(n log n) (worst-case)**

Better way:
➔ Use divide & conquer! (see blackboard)
➔ **T(n) = 2 T(n/2)** (conquer) + **O(log n)** (<u>in-place</u> divide & combine)
➔ Total time: **Θ(n)** (by Master Theorem)

Iterative (and in-place) way:

build_max_heap(A):
    for i=n downto 1
        do max_heapify(A,i)



O(n log n)

➔ Total time? **At first glance:** **~~Θ(n log n)~~** (cost of **n** max_heapify)
Actually: **Θ(n)** (see blackboard)

Cool application: Sorting

Heapsort: Sorting using a heap/priority queue

➔ Build a heap out of all elements

➔ Extract_max all elements one-by-one in an (inversely) sorted order!

➔ Total time: **Θ(n log n) (worst-case)**

This is a different algorithm than Merge sort!

In particular: Heapsort is actually an in-place algorithm (once we unravel the implementation of the heap)